On the weighted power function of some testing hypotheses

Washio, Yasutoshi
Kyushu University

https://doi.org/10.5109/12964

出版情報：統計数理研究. 6 (1/2), pp.11-15, 1955-12. Research Association of Statistical Sciences
バージョン：

ON THE WEIGHTED POWER FUNCTION OF SOME TESTING HYPOTHESES

By

Yasutoshi WASHIO

(*Kyusyu University*)

§ 1. **Introduction.** In testing the hypothesis, it is often seen that there does not exist any uniformly most powerful test against a given composite alternative hypothesis. Therefore, various additional criteria to select a test with a reasonable optimum property have been considered by many authors.

In this paper we shall give a test which maximizes the area (or volume) circumscribed by the power function of a test on the surface of the given alternative hypothesis. It is obvious that if there exists a uniformly most powerful test against a given alternative, our test must coincide with it. In general, if an adequate weight on the alternative hypothesis is considered, this method is applied to the problem maximizing the weighted power function. Such a weight may not be defined for all situations, but there may be some circumstances in practical situations where relative weights can be associated with each of possible alternative hypotheses at least in some rough sense, that is to say, a class of possible weight functions can be assigned.

In preparation of this paper, the author is much indebted to Professor T. KITAGAWA for suggestions and enrouregements.

§ 2. **Notations and lemmas.** Let $X=(X_1, \cdots, X_n)$ be a random variable, $x=(x_1, \cdots, x_n)$ be an observed value of X. We assume that the type of the joint density function $f(x;\theta)$ of X is known, but some of components of $\theta=(\theta_1, \cdots, \theta_k)$, say $\theta_1, \cdots, \theta_s$ $(s \leq k)$, are unknown to us. Let R be the n-dimensional sample space of possible x's, and Ω be the k-dimensional parameter space of possible θ's. We wish to test the hypothesis H_0 that the joint density function of X is some $f(x;\theta)$ with $\theta \in \omega_0$ at the level of significance α against the alternative H_1 that the joint density function of X is some $f(x;\theta)$ with $\theta \in \omega_1$, where ω_0 and ω_1 are the disjoint subsets of Ω.

By a test φ of H_0 we shall mean a function φ from R to the interval $[0, 1]$, such that $\varphi(x)=$ probability of rejecting H_0 when x is the observed value of X. For a test φ, the power function of φ is defined by

$$\beta(\theta, \varphi)=\int_R \varphi(x) f(x;\theta)\, dx\,, \tag{2.1}$$

where $dx=dx_1 \cdots dx_n$. The envelope power function $\beta^*(\theta)$ is defined by

$$\beta^*(\theta)=\sup_{\varphi \in \Phi_\alpha} \beta(\theta, \varphi)\,, \tag{2.2}$$

where Φ_α is the class of all tests φ with $\beta(\theta, \varphi) \leq \alpha$ for all $\theta \in \omega_0$. A. WALD defined a test φ_0 to be most stringent[1] when φ_0 satisfies the following condition:

$$\sup_{\theta \in \omega_1}[\beta^*(\theta) - \beta(\theta, \varphi_0)] = \min_{\varphi \in \Phi_\alpha} \sup_{\theta \in \omega_1}[\beta^*(\theta) - \beta(\theta, \varphi)] . \tag{2.3}$$

In the sequel, we shall consider a test φ which maximizes the integral

$$\int_{\omega_1} \beta(\theta, \varphi)\, d\theta , \quad \varphi \in \Phi_\alpha . \tag{2.4}$$

This test is equivalent to the test which minimizes

$$\int_{\omega_1} [\beta^*(\theta) - \beta(\theta, \varphi)]\, d\theta , \quad \varphi \in \Phi_\alpha , \tag{2.5}$$

if the above integral exists.

For this purpose we confine ourselves to the case where ω_1 is a finite subset of Ω, and where the following relation holds true:

$$\int_{\omega_1} \left(\int_R \varphi(x) f(x ; \theta)\, dx \right) d\theta = \int_R \varphi(x) \left(\int_{\omega_1} f(x ; \theta)\, d\theta \right) dx . \tag{2.6}$$

Lemma 1. *In order to test a null hypothesis $H_0 : \theta = \theta_0$ against an alternative hypothesis $H_1 : \theta \in \omega_1$, let the test φ_0 be defined by*

$$\varphi_0(x) = 1 , \quad \textit{when} \quad \int_{\omega_1} f(x ; \theta)\, d\theta \geq k f(x ; \theta_0) , \tag{2.7}$$

$$\varphi_0(x) = 0 , \quad \textit{when} \quad \int_{\omega_1} f(x ; \theta)\, d\theta < k f(x ; \theta) , \tag{2.8}$$

where k is a constant. Then for any test φ which satisfies

$$\int_R \varphi(x) f(x ; \theta_0)\, dx \leq \int_R \varphi_0(x) f(x ; \theta_0)\, dx , \tag{2.9}$$

we have

$$\int_{\omega_1} \beta(\theta, \varphi)\, d\theta \leq \int_{\omega_1} \beta(\theta, \varphi_0)\, d\theta . \tag{2.10}$$

This lemma is essentially the same as Neyman-Pearson's Lemma and, using the relation (2.6), is proved by the similar way.

Making use of Lemma 1, we have the following lemma after LEHMANN-STEIN [1] and MIYASAWA [3].

---

1) LEHMANN, E. L. [2],

Lemma 2. *In order to test the null hypothesis H_0: $\theta \in \omega_0$, against the alternative hypothesis H_1 : $\theta \in \omega_1$, let $\mu(\theta)$ be a probability distribution on ω_0, and let a test φ_0 be defined by*

$$\varphi_0(x)=1, \quad \text{when} \quad \int_{\omega_1} f(x;\theta)d\theta \geqq k\int_{\omega_0} f(x;\theta)d\mu(\theta), \tag{2.11}$$

$$\varphi_0(x)=0, \quad \text{when} \quad \int_{\omega_1} f(x;\theta)d\theta < k\int_{\omega_0} f(x;\theta)d\mu(\theta), \tag{2.12}$$

where k is a constant. Suppose that

$$\varphi_0 \in \Phi_\alpha \tag{2.13}$$

and

$$\beta(\mu, \varphi_0)\equiv\int_{\omega_0}\beta(\theta, \varphi_0)d\mu(\theta)=\alpha. \tag{2.14}$$

Then for any φ in Φ_α we have

$$\int_{\omega_1}\beta(\theta, \varphi)d\theta \leqq \int_{\omega_1}\beta(\theta, \varphi_0)d\theta. \tag{2.15}$$

Remark. If our test obtained by the above lemmas is independent of any ω_1 in some subset Ω_0 of $\Omega-\omega_0$, then our test is the uniformly most powerful test against the alternative $\theta \in \Omega_0$.

§ 3. Applications.

Example 1. When $(X_1, \cdots, X_n)$ is a sample from a normal population with unknown mean ξ and variance 1, we want to test the hypothesis H_0 : $\xi=0$ against the alternative H_1 : $\xi \in \omega_1 \equiv \{\xi;\ a \leqq |\xi| \leqq b\}$, where a and b are given numbers such that $0<a<b$.

This can be solved as follows. The joint density function $f(x;\xi)$ of $X=(X_1, \cdots, X_n)$ is given by

$$f(x;\xi)=(2\pi)^{-n/2}\exp\Big[-\frac{1}{2}\sum_{i=1}^{n}(x_i-\xi)^2\Big]. \tag{3.1}$$

The inequality (2.7) of Lemma 1 becomes as follows

$$\frac{\int_a^b\Big\{\exp\Big[-\frac{1}{2}\sum_{i=1}^{n}(x_i+\xi)^2\Big]+\exp\Big[-\frac{1}{2}\sum_{i=1}^{n}(x_i-\xi)^2\Big]\Big\}d\xi}{\exp\Big[-\frac{1}{2}\sum_{i=1}^{n}x_i^2\Big]} \geqq k, \tag{3.2}$$

which is equivalent to the relation

$$\int_a^b\Big\{\exp\Big[-n\xi\bar{x}-\frac{1}{2}n\xi^2\Big]+\exp\Big[n\xi\bar{x}-\frac{1}{2}n\xi^2\Big]\Big\}d\xi \geqq k. \tag{3.3}$$

The integrand in (3.3) is a function of ξ and $|\bar{x}|$, say $\phi(\xi, |\bar{x}|)$, and it can be shown easily that for fixed $\xi(a \leq \xi \leq b)$, $\phi(\xi, |\bar{x}|)$ is a monotone increasing function of $|\bar{x}|$.

Hence we can readily see that the left hand side of (3.3) is a monotone increasing function depends only upon $|\bar{x}|$. Therefore the relation (3.3) is equivalent to the relation

$$(3.4) \qquad |\bar{x}| \geq k'.$$

Making use of Lemma 1, our test φ_0 required is given by

$$(3.5) \qquad \varphi_0(x) = 1, \quad \text{when} \quad |\bar{x}| \geq c,$$

$$(3.6) \qquad \varphi_0(x) = 0, \quad \text{when} \quad |\bar{x}| < c,$$

where c is determined by the level of significance α, that is, by the equation

$$(3.7) \qquad (n/2\pi)^{1/2} \int_{|\bar{x}| \geqq c} \exp\left[-\frac{n}{2}\bar{x}^2\right] d\bar{x} = \alpha.$$

In this case our that φ_0 is the equal tail region test of the normal distribution. Since φ_0 is independent of a and b, we can conclude by the remark of § 2 that as long as the alternative is taken as above φ_0 is the uniformly most powerful test against the alternative $\xi \in \omega_1 \equiv \{\xi; |\xi| \geq a, a > 0\}$.

Example 2. When $(X_1, \cdots, X_n)$ is a sample from a normal population with unknown mean ξ and variance 1, we want to test the hypothesis $H_0: \xi \in \omega_0 \equiv \{\xi; |\xi| \leq \rho\}$ against the alternative $H_1: \xi \in \omega_1 \equiv \{\xi; a \leq |\xi| \leq b\}$, where ρ, a and b are given numbers such that $0 < \rho < a < b$.

The solution is as follows. The joint density function $f(x; \xi)$ of X is given by (3.1). We consider a probability distribution $\mu(\theta)$ on ω_0 which assigns probability 1/2 to each of the points $\xi = \pm\rho$, then the inequality (2.11) of Lemma 2 is written as

$$(3.8) \qquad \frac{\int_a^b \exp\left[-\frac{1}{2}\sum_{i=1}^n (x_i - \xi)^2\right] d\xi + \int_{-b}^{-a} \exp\left[-\frac{1}{2}\sum_{i=1}^n (x_i - \xi)^2\right] d\xi}{\frac{1}{2}\left\{\exp\left[-\frac{1}{2}\sum_{i=1}^n (x_i - \rho)^2\right] + \exp\left[-\frac{1}{2}\sum_{i=1}^n (x_i + \rho)^2\right]\right\}} \geq k.$$

By a simple calculation, the relation (3.8) is equivalent to the relation

$$(3.9) \qquad \int_a^b \left\{\exp\left[-\frac{1}{2}n\xi^2\right] \frac{\exp[n\xi\bar{x}] + \exp[-n\xi\bar{x}]}{\exp[n\rho\bar{x}] + \exp[-n\rho\bar{x}]}\right\} d\xi \geq k'.$$

Similarly to Example 1, we can easily verify that the left hand side of (3.9) is a monotone increasing function depends only upon $|\bar{x}|$. Then (3.9) is equivalent to

$$(3.10) \qquad |\bar{x}| \geq k''.$$

Therefore our test φ_0 which is determined by (2.11) and (2.12) is given by

$$\varphi_0(x) = 1, \quad \text{when} \quad |\bar{x}| \geqq k'', \tag{3.11}$$

$$\varphi_0(x) = 0, \quad \text{when} \quad |\bar{x}| < k''. \tag{3.12}$$

As to the power function $\beta(\xi, \varphi_0)$ of φ_0 the relation

$$\beta(\xi, \varphi_0) = \beta(-\xi, \varphi_0) \tag{3.13}$$

holds, and $\beta(\xi, \varphi_0)$ is a monotone increasing function of $|\xi|$. Therefore if we determine k'' in (3.11) and (3.12) such that

$$\beta(\rho, \varphi_0) = (n/2\pi)^{1/2} \int_{|\bar{x}| \geqq k''} \exp[-n(\bar{x}-\rho)^2/2]\, d\bar{x} = \alpha, \tag{3.14}$$

then $\varphi_0 \in \Phi_\alpha$ and $\beta(\mu, \varphi_0) = \alpha$.

From these facts we observe that the test φ_0 above defined satisfies the conditions of Lemma 2, and hence φ_0 is the required one.

As we have seen in Example 1, a test φ_0 is also the equal tail region test. Since φ_0 is independent of a and b, we can say by the remark of § 2 that as long as the alternative is taken as above our test φ_0 is the uniformly most powerful test against the alternative $\xi \in \omega_1 \equiv \{\xi; |\xi| \geqq a\}$, where a is any number such that $a > \rho$.

Here it is to be noted that our test φ_0 coincides with that obtained by MIYASAWA [3] as an example of the most stringent test against the alternative $\xi \in \omega_1 \equiv \{\xi; |\xi| \geqq a\}$.

References

[1] LEHMANN, E L. and STEIN, C., *Most powerful test of composite hypotheses.* **I.** *Normal distributions*, Ann. Math. Stat., **19** (1948), 495–516.

[2] LEHMANN, E. L., *Some principles of the theory of testing hypotheses*, Ann. Math. Stat., **21** (1950), 1–26.

[3] MIYASAWA, K., *Most stringent tests and invariant tests*, Mem. Fac. Sci., Kyūsyū Univ., Ser. A, 8 (1953), 57–88.