Note on the Estimation of the Mean Value of the Stochastic Process

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OF THE STOCHASTIC PROCESS

By Akio Kudô

§ 1. Introduction.

The main purpose of this paper is to discuss some combination of the theory
of the linear estimation of the mean value function of the stochastic process
due to U. GRENENDER [1] and the notion of the random integration of type A
introduced by T. KITAGAWA [1]. In § 2 we shall discuss the estimation of
the mean value fnnction of the stochastic process under the condition that
the continuous observation of the realization of the process in our concern
is not available. § 3 is devoted to the discussion of the special case when
the process is assumed to be stationary, and in this section we shall give
certain estimations of the mean value which is assumed to be constant, and
this hype of estimation has some connection with the computor of the in-
tegration such as recently planned in Japan. (IZAKI [1])

In preparation of this paper the author is deeply indebt to the valuable
criterions and the warmest encouragement given by PROF. T. KITAGAWA.

§ 2. Combination with the notion of the random integration of type a.

In his eminent paper, U. GRENANDER [1] discussed the linear estimation
of the mean value of the stochastic process. The estimate of the mean value
given by him is represented in the form of an integration of the realization of
the stochastic process in our concern by certain function of bounded variation,
i.e. in the following form

$$(2.1) \qquad m^* = \int_0^T x(t)\, dF(t)\,,$$

where $x(t)$ is a realization of the process in our concern. And the necessary
and sufficient condition to $F(t)$ that m^* should be the unbiased estimate of
minimum variance is given by him. In his argument, the mean value func-
tion of the process is assumed to be constant. But his result can be obviously
generalized to a fairly general case when the mean value function is a linear
form of known functions.

And hence in this section without essential loss of generality we may and
we shall assume that the mean value function of the process in our consern
be constant and its variance fumction is 1. On the other hand, he assumes
that a continuous observation of the realization of the process is available
to the statisticians. However, it may not always be possible, and even if it
be possible, the numerical value of this integration may be perhaps neither

directly nor strictly calculatable. Hence, we should adopt either of the
following two methods, one of which is to aporoximate the vlaue of the
integration with a finite Riemann sum, while the other is to estimate it
by making use of the method of random integration of type A introduced
by T. Kitagawa [1]. And hence, some modifications to the formulations of
the theory of statistical inference about stochastic process due to Grenander
[1] seems to the author to be necessary, at least in view of its applications.

In this paper, we shall not here concern ourselves with the approximation
with finite Riemann sum, because in this case we may consider the problem
as the linear estimation for the process with an integral valued parameter.

Concerning the method of random integration of type A we shall enun-
ciate the following

Theorem 2.1. Let $x(t)$ be a stochastic process with unknown constant
mean value $m = E(x(t))$ and with correlation function

$$(2.2) \qquad r(s, t) = E(x(s) - m)(x(t) - m),$$

which is known to us. Let us assume that any realization of our process
can be observed at any number of any time points over the time interval
(O, T) and let us draw N time points of observation with the help of some
chance mechanism in such a manner that these time points will be distributed
over the interval (O, T) mutually independently in the same distribution
function $F(t)$. Here let us define an estimation of the mean value of our
process as follows,

$$(2.3) \qquad m^* = \frac{1}{N} \sum_{i=1}^{N} x(t_i).$$

Then it is an unbiased estimation and its variance is given by

$$(2.4) \qquad D^2(m^*) = \left(1 - \frac{1}{N}\right) \int_0^T \int_0^T r(s, t)\, dF(t)\, dF(s) + \frac{1}{N}.$$

Especially when we can choose the distribution function $F(t)$ satisfying
the relation

$$(2.5) \qquad \int_0^T r(s, t)\, dF(t) = \text{constant}$$

then the variance of this estimate is asymptotically equal to the variance
of the unbiased estimate of minimum variance obtained by the conctiucns
observation of the realization as N becomes infinite, and this convergence
being of the order N.

Proof. In view of the independency of $\{t_i\}$ we have

$$(2.6) \qquad D^2(m^*) = \frac{1}{N^2} \int_0^T \cdots \int_0^T \sum_{i,j=1}^{N} r(t_i, t_j)\, dF(t_i) \cdots dF(t_N)$$

$$= \frac{1}{N^2} \left\{ 2 \sum_{\substack{i,j=1 \\ i>j}}^{N} \int_0^T \int_0^T r(t_i, t_j) \, dF(t_i) \, dF(t_j) + N \right\}$$

$$= \int_0^T \int_0^T r(s, t) \, dF(s) \, dF(t)$$

$$+ \frac{1}{N} \left\{ 1 - \int_0^T \int_0^T r(s, t) \, dF(t) \, dF(s) \right\}.$$

§ 3. Discussion on the stationary case.

GRENANDER proved in his paper [1] § 5, when the process is assumed to have the constant mean value and be stationary, i.e. its correlation function is of the form

$$(3.1) \qquad\qquad r(s, t) = r(s - t) = \int_{-\infty}^{+\infty} e^{i\lambda(s-t)} \, dG(\lambda)$$

and is purely non-deterministic, then under certain conditions the equi-distributed estimate defined as follows

$$(3.2) \qquad\qquad \mu_T = \frac{1}{T} \int_0^T x(t) \, dt$$

is an unbiased and asymptotically efficient estimate in the class of linear estimates, and here let us denote its variance by $D^2(\mu_T)$, then we have the assertion

$$(3.3) \qquad\qquad D^2(\mu_T) \sim \frac{\pi}{T} G'(0),$$

where $G'(0)$ is the spectral density at the origin of the stochastic process in our concern.

On the other hand, let us now take N time points of observation $t_1, t_2, \cdots, t_N$, which are uniformly and mutually independently distributed over the time interval (O, T) and let us define an unbiased estimate as follows

$$(3.4) \qquad\qquad \bar{\mu}_{T, N} = \frac{1}{N} \sum_{i=1}^{N} x(t_i).$$

Concerning the relation between the equi-distributed estimate and the estimate above defined, let us now ennunciate the following

Theorem 3. 1. Between $D^2(\mu_{T_1})$ and $D^2(\bar{\mu}_{T_2, N})$, which are the variance of μ_{T_1} and $\bar{\mu}_{T_2, N}$ respectively, there exists an asymptotical relation

$$(3.5) \qquad\qquad \frac{D^2(\mu_{T_1})}{D^2(\bar{\mu}_{T_2, N})} \sim \frac{\pi G'(0)}{\pi G'(0) \left(1 - \dfrac{1}{N}\right) + \dfrac{T^2 T_1}{N^2}} \cdot \frac{T_2}{T_1}.$$

Here let us now consider the following estimations, which have some connetions with the computor of the integration recently being planned in Japan. (see IZAKI [1]).

First let us draw time points $t_1, t_2, t_3, \cdots$ sequentially in such a way as they may be regarded as a POISSON process i.e. the probability law of t_k when t_{k-1} is assigned is given in the following formulae;

(3.6) $P_r(t_k \leq t_k \leq t_k + dt_k / t_{k-1}) = a\, e^{-a(t_k - t_{k-1})}\, dt_k$.

Further let us define a random variable M by the following relation

(3.7) $t_{N-1} \leq T \leq t_M$.

Here we note that a is the average number of the time points in a unit interval.

Here let us define two estimations of the mean value of our stochastic process as follows;

First let us define the following estimation m^*,

(3.8) $$m^* = \frac{1}{M-1} \sum_{i=1}^{M-1} x(t_i),$$

where M is defined as follows: First we take series of time points of observations in the above mentioned manner and if M defined by (3.7) is not equal to one we take this number as M of (3.8), and on the other hand if $M = 1$ we repeat the drawing of the time points until it becomes $M \neq 1$, and take this number as M of (3.8).

Another estimation of the mean value of our process is defined as follows

(3.9) $$m^{**} = \frac{1}{T} \sum_{k=1}^{M} (t_k - t_{k-1})\, x(\xi_k),$$

where M is a number defined by (3.7) and $t_0 = 0$, $t_M = T$ and ξ_k ($k = 1, 2, \cdots, M$) are time points of observations uniformly and mutually independently distributed over the interval (t_k, t_{k-1}) respectively.

Here let us remark that this estimation of the mean value may be obtained when we estimate it by making use of the computor of the integration above mentioned.

Concerning these two estimations we have following

Theorem 3.2.

(3.10) $E(m^*) = E(m^{**}) = m$

(3.11) $D^2(m^*) = D^2(\mu_T) + \dfrac{1 - D^2(\mu_T)}{e^{aT} - 1} \sum_{N=1}^{\infty} \dfrac{(aT)^N}{N!\,N} = D^2(\mu_T) + O\left(\dfrac{1}{aT}\right)$

(3.12) $D^2(m^{**}) = D^2(\mu_T) + O\left(\dfrac{1}{aT}\right)$,

where we set

(3.13) $$D^2(\mu_T) = \frac{1}{T^2} \int_0^T \int_0^T r(s, t)\, ds\, dt.$$

Especially when our process is assumed to be stationary and the correlation function $r(t)$ is analytic in $t > 0$ and expressible in the following form,

$$(3.14) \qquad r(t) = 1 + c\, t^\alpha + \sum_{k=\alpha+1}^{\infty} c_k\, t^k, \quad \text{for} \quad t > 0$$

where α is some positive integer.
Then the variance is given in the following

$$(3.15) \quad D^2(m^{**}) = D^{\cdot}(\mu_T) - 2\, \iota\,!\, c\, e^{-aT}\, T^\alpha \sum_{N=1}^{\infty} \frac{N(a\,T)^{N-1}}{(\alpha + N + 1)\,!} + O\left(\frac{1}{a^{\alpha+2}\,T}\right)$$

and

$$(3.16) \qquad e^{-aT}\, T^\alpha \sum_{N=1}^{\infty} \frac{N(a\,T)^{N-1}}{(\alpha + N + 1)\,!} = O\left(\frac{1}{a^{\alpha+1}\,T}\right).$$

Proof. The unbiasedness of m^* and m^{**} is clear and we shall here give the calculation of the variance of m^{**}. The calculation of the variance of m^* can be done in the same way as that of m^{**}.

The conditional variance of m^* when $\{t_k\}$ and $\{\xi_k\}$ are assigned is given in the following

$$(3.17) \qquad \frac{1}{T^2} \sum_{k,l=1}^{N} (t_k - t_{k-1})(t_l - t_{l-1})\, r(\xi_k \xi_l)$$

and in view of the conditions to $\{t_k\}$ and $\{\xi_k\}$ we are lead to the calculation of the variance as follows,

$$(3.18) \quad T^2 D^2(m^{**}) = \sum_{N=1}^{\infty} \int_{0 \le t_1 \le t_2 \le \cdots \le t_{N-1} \le T \le t'_N < \infty} \int$$

$$\left\{ \sum_{k,l=1}^{N} (t_k - t_{k-1})(t_l - t_{l-1})\, r(\xi_k \xi_l) \right\}$$

$$\left\{ \prod_{k=1}^{N} \int_{t_{k-1}}^{t_k} \frac{d\xi_k}{t_k - t_{k-1}} \right\} \left\{ \prod_{k=1}^{N-1} a\, e^{-a(t_k - t_{k-1})} \, dt_k \right\} a\, e^{-a(t_{N'} - t_{N-1})} \, dt'_N$$

$$= e^{-aT} \sum_{N=1}^{\infty} a^{N-1} \int_{0 \le t_1 \le t_2 \le \cdots \le t_{N-1} \le T} \int$$

$$\left\{ 2 \sum_{\substack{k,l=1 \\ k>l}}^{N} \int_{t_{k-1}}^{t_k} \int_{t_{l-1}}^{t_l} r(s,t)\, ds\, dt + \sum_{k=1}^{N} (t_k - t_{k-1})^2 \right\}$$

$$dt_1\, dt_2 \qquad dt_{N-1}$$

$$= e^{-aT} \sum_{N=1}^{\infty} a^{N-1} \frac{T^{N-1}}{(N-1)\,!} \int_0^T \int_0^T r(s,t)\, ds\, dt$$

$$+ e^{-aT} \sum_{N=1}^{\infty} a^{N-1} \int_{0 \le t_1 \le t_2 \le \cdots \le t_{N-1} \le T} \int$$

$$\left\{ \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} (1 - r(s,t))\, ds\, dt \right\} dt_1\, dt_2 \cdots dt_{N-1}$$

$$= \int_0^T \int_0^T r(s,t)\, ds\, dt + I$$

On the other hand, we have

(3.18) $$\int_{0 \le t_1 \le t_2 \le \cdots \le t_{N-1} \le T} \int \sum_{k=1}^{N} (t_k - t_{k-1})^n \, dt_1 \, dt_2 \cdots dt_{N-1}$$

$$= \frac{(n)! \, N}{(n+N-1)!} \, T^{n+N-1}.$$

Hence we have in view of the inequality $|1 - r(s, t)| \le 2$,

(3.19) $$I \le 2T^2 e^{-aT} \sum_{N=1}^{\infty} \frac{(aT)^{N-1}}{(N+1)!} (2N) \le 4T^2 \sum_{N=1}^{\infty} \frac{(aT)^{N-1}}{N!}$$

$$= 4\frac{T}{a} e^{-aT} (e^{aT} - 1) = O\left(\frac{T}{a}\right).$$

Hence (3.15) is proved.

And on the other hand, if the condition (3.14) is fulfilled, we have

(3.20) $T^2 D^2(m^{**}) = \int_0^T \int_0^T r(s, t) \, ds \, dt$

$$- 2e^{-aT} \sum_{N=1}^{\infty} a^{N-1} \frac{c}{(\alpha+1)(\alpha+2)} \int_{0 \le t_1 \le t_2 \le \cdots \le t_{N-1} \le T} \int \sum_{k=1}^{N} (t_k - t_{k-1})^{\alpha+2} \, dt_1 \cdots dt_N,$$

$$+ I_2$$

$$= \int_0^T \int_0^T r(s, t) \, ds \, dt - 2c \, \alpha! \, e^{-aT} \sum_{N=1}^{\infty} \frac{N}{(\alpha+N+1)!} a^{N-1} T^{\alpha+N+1}$$

$$+ I_2$$

$$= T^2 D^2(\mu_T) + I_1 + I_2$$

and

(3.21) $$|I_1| \le 2|c| \, \alpha! \, e^{-aT} a^{-(\alpha+1)} T \sum_{N=1}^{\infty} \frac{(aT)^{\alpha+N}}{(N+\alpha)!} = O\left(\frac{T}{a^{\alpha+1}}\right)$$

and hence

(3.22) $$|I_2| = O\left(\frac{T}{a^{\alpha+2}}\right)$$

which is to be proved.

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