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SOME STOCHASTIC CONSIDERATIONS UPON EMPIRICAL FUNCTIONS OF VARIOUS TYPES[1]

By Tosio KITAGAWA

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Introduction

In spite of current normal regression theory developed along general formulations of the modern statistical inferences, there still remain undeveloped aspects of so-called empirical functions. General formulations of empirical functions will require more profound insights into their original and their uses than usually treated. We must take into our considerations at least the following three aspects from more general points of view than usually treated in current theories: (1°) formulations of fundamental mathematical models for object (population) function to be infered about ; (2°) procedures of our samplings and those of observations upon realised sample functions ; (3°) procedures of statistical inferences about population function.

As to the first point, our situations may be recognised to be quite similar to those which we treated in introducing random integrations. There are two fundamental distinctions concerning our mathematical models: type (a) our object of statiatical inferences is a deterministic function which is how-

(1) This paper was communicated by the author at the Branch-Meeting of Japanese Math. Soc. held at Kumamoto University on Feb. 28, 1933.

ever unknown to us; type (b) in this case a family of individual functions is assumed to be given with suitably defined probability meansure in itself. Our object of statistical inferences is in reality a stochastic process, and any concurrence-function may be considered as a sample function from this stochastic process.

As to the second point, our procedures of observations can be broadly classified into the following two cases: (D) observations on concurrence function are made merely at some discrete points of a certain finite number; (C) observations on concurrence functions are performed continuously in a certain interval.

Whether our observations may of type (D) or of the type (C), stochastic scheme will be always required so far as our observations involve some error terms.

As the third point, it is to be noted that there are various procedures of statistical inferences, which can only be systematically treated from the standpoints of successive process of statistical inferences, and which will consequently belong to a realm of successive design of experiments.

In this preliminary paper on these problems, we shall take into our consideration all the four types of empirical functions, (AD), (AC), (BD) and (CD), but we shall restrict ourselves with simplest procedures of statistical inferences.

Part I is devoted to the discussions on the empirical functions themselves, while Part II to those on the derivative of them. In both of these two Parts certain generalisation of classical one-dimensional t distributions into multi-dimensional ones will be indicated, and some normal distributions associated with infinite sequences of stochastic variables will be involved.

The preparation of numerical tables concerning some of these distribution functions which will make our theoretical results to be available in practical applications will be postponed to another occassion.

Part I. Statistical inferences concerning empirical functions

§ 1. Statistical inferences concerning empirical functions of the type (AD)

In this paragraph we shall make the following Assumptions:

Assumption (1°). *Let $g(t)$ be a deterministic function defined over the interval $0 \leqq t \leqq 1$, and let it be squarely integrable in the sense of Lebesque over the interval, that is, let $g(t)$ belong to $L^2(0, 1)$.*

Assumption (2°). *Let a set of n points $0 \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_n \leqq 1$ be assigned. Our observations for the function $g(t)$, which is determined but unknown to us, are assumed to be made exclusively at these points.*

Assumption (3°). *The r_i observations are made at each point t_i ($i = 1, 2, \cdots, n$), and yield us the values $\{y_{ij}\}$ ($i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, r_i$), which can be written as*

$$y_{ij} = g(t_i) + \varepsilon_{ij}, \tag{1.01}$$

where the $(r_1 + r_2 + \cdots + r_n)$ error terms $\{\varepsilon_{ij}\}$ are mutually independently distributed according the normal distribution $N(0, \sigma^2)$, where σ^2 are common to all of them, being unknown to us.

It is well-known that under these Assumptions there exists a system of orthogonal functions $\{\varphi_\nu(t_i; n)\}$ ($\nu = 0, 1, 2, \cdots, n-1$) such that

$$\sum_{i=1}^{n} \varphi_\mu(t_i; n)\, \varphi_\nu(t_i; n) = \delta_{\mu\nu}, \tag{1.02}$$

where $\delta_{\mu\nu}$ mean Kronecker symbols for any pair μ, ν such that $\mu, \nu = 1, 2, \cdots, n$. $\varphi_\nu(t_i; n)$ may be defined as a polynomial in t_i of the ν th degree, and hence it can be prolongated to the function defined for all values of t, w ich we denote by $\varphi_\nu(t; n)$. Under these circumstances let us define an empirical function $\hat{g}_m(t)$ for the unknown function $g(t)$ by

$$\hat{g}_m(t) \equiv \sum_{\nu=0}^{m-1} \varphi_\nu(t; n)\, \hat{g}_\nu, \tag{1.03}$$

where

$$\hat{g}_\nu \equiv \sum_{i=1}^{n} \bar{y}_{i\cdot}\, \varphi_\nu(t_i; n), \tag{1.04}$$

and m is a positive integer such that $m \leqq n$.

To discuss the problems of statistical inferences concerning empirical function, it will be adequate to introduce various norms in the function space $L^2(0, 1)$ which are convenient for our purposes. For example, we may and we shall introduce

Definition 1.1. The overall L^2-norm of a function $f(t)$ belonging to $L^2(0, 1)$ is defined by

$$\|f\| = \left\{ \int_0^1 |f(t)|^2\, dt \right\}^{1/2} \tag{1.05}$$

Definition 1.2. Let $D: 0 \leqq \tau_1 \leqq \tau_2 \leqq \cdots \leqq \tau_k \leqq 1$ be any assigned division of the interval $[0, 1]$. Then the $L^2(D)$-morm of a function $f(t)$ belonging to $L^2(0, 1)$ is defined by

$$\|f\|_D = \mathrm{Max}\, \{\|f\|_{I_1}, \|f\|_{I_2}, \cdots, \|f\|_{I_{k+1}}\}, \tag{1.06}$$

where for $h = 1, 2, \cdots, k+1$

$$\|f\|_{I_h}^2 = (\tau_h - \tau_{h-1})^{-1} \int_{\tau_{h-1}}^{\tau_h} |f(t)|^2\, dt, \tag{1.07}$$

putting $\tau_0 = 0$, and $\tau_{k+1} = 1$.

Furthermore let us introduce the following notations:

$$(1.08) \qquad \delta_m(t) \equiv g(t) - \hat{g}_m(t)$$

$$(1.09) \qquad L(t\,;\,t_i,\,m) \equiv \sum_{\nu=0}^{m-1} \varphi_\nu(t_i\,;\,n)\,\varphi_\nu(t\,;\,n)$$

$$(1.10) \qquad \bar{\varepsilon}_i. \equiv r_i^{-1} \sum_{j=1}^{r_i} \varepsilon_{ij}$$

$$(1.11) \qquad s_i^2 \equiv (r_i - 1)^{-1} \sum_{j=1}^{r_i} (\varepsilon_{ij} - \bar{\varepsilon}_i.)^2,$$

and

$$(1.12) \qquad s^2 \equiv R^{-1} \sum_{i=1}^{n} \sum_{j=1}^{r_i} (\varepsilon_{ij} - \bar{\varepsilon}_i.)^2,$$

$$(1.13) \qquad R \equiv \sum_{i=1}^{n} (r_i - 1),$$

which play their important rôles in the following discussions.

Now we shall readily observe by direct calculations

Lemma 1.1. *We have*

$$(1.14) \qquad \|g - \hat{g}_m\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{\varepsilon}_i.\,\bar{\varepsilon}_j.\,A_{ij} + 2\sum_{i=1}^{n} B_i\,\bar{\varepsilon}_i. + C$$

and

$$(1.15) \qquad \|g - \hat{g}_m\|_{I_h}^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{\varepsilon}_i.\,\bar{\varepsilon}_j.\,A_{ij}^{(h)} + 2\sum_{i=1}^{n} B_i^{(h)}\,\bar{\varepsilon}_i. + C^{(h)},$$

where we define

$$(1.16) \qquad A_{ij} = \int_0^1 L(t\,;\,t_i,\,m)\,L(t\,;\,t_j,\,m)\,dt$$

$$(1.17) \qquad A_{ij}^{(h)} = (\tau_h - \tau_{h-1})^{-1} \int_{\tau_{h-1}}^{\tau_h} L(t\,;\,t_i,\,m)\,L(t\,;\,t_j,\,m)\,dt$$

$$(1.18) \qquad B_i = \int_0^1 \delta_m(t)\,L(t\,;\,t_i,\,m)\,dt$$

$$(1.19) \qquad B_i^{(h)} = (\tau_h - \tau_{h-1})^{-1} \int_{\tau_{h-1}}^{\tau_h} \delta_m(t)\,L(t\,;\,t_i,\,m)\,dt$$

$$(1.20) \qquad C = \int_0^1 \delta_m^2\,(t)\,dt$$

and

$$(1.21) \qquad C^{(h)} = (\tau_h - \tau_{h-1})^{-1} \int_{\tau_{h-1}}^{\tau_h} \delta_m(t)^2\,dt\,.$$

Lemma 1.2. *Under the Assumptions* (1°) ~ (3°) *of this paragraph, we have, for each assigned positive number* x,

(1.22) $$\Pr.\{\|g-\hat{g}\|^2/s^2<x\} = \iint\cdots\int_{\mathfrak{D}_n^{(1)}} \frac{\Gamma\left(\frac{R+n-1}{2}\right) dt_1\, dt_2 \cdots dt_n}{\pi^{n/2}\,\Gamma\left(\frac{R-1}{2}\right)\left(1+\sum_{i=1}^{n} t_i^2\right)^{(R+n-1)/2}},$$

where the domain of integration $\mathfrak{D}_n^{(1)}$ *is defined by*

(1.23) $$\mathfrak{D}_n^{(1)}:\ \left|R\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\, r_i^{-1/2} r_j^{-1/2} t_i t_j + 2R^{1/2}\sum_{i=1}^{n} B_i r_i^{-1/2} t_i + C\right| \leqq x,$$

and also

(1.24) $$\Pr.\{\|g-\hat{g}\|_D^2<x\} = \iint\cdots\int_{\varepsilon_{n,k}^{(1)}} \prod_{i=1}^{n} \frac{r_i^{1/2}\exp\{-(2\sigma^2)^{-1} r_i \eta_i^2\}}{(2\pi\sigma^2)^{1/2}}\, d\eta_1\, d\eta_2 \cdots d\eta_n,$$

where the domain of integration $\varepsilon_{n,k}^{(1)}$ *is defined as the one which satisfies all the following* $(k+1)$ *inequalities simultaneously*:

(1.25) $$\varepsilon_{n,k}^{(1)}:\ \left|\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}^{(h)} \eta_i \eta_j + 2\sum_{i=1}^{n} B_i^{(h)} \eta_i + C^{(h)}\right| > x$$

for $h=1, 2, \cdots, k+1$.

The last Lemma 1.3 yields us consequently the following

Theorem 1.1. *Let* H_0 *be a null hypothesis that* $\delta_m(t)\equiv 0$ *in* $0\leqq t\leqq 1$. *Then we have, for each assigned positive* x,

(1.26) $$\Pr.\{\|g-\hat{g}\|^2/s^2<x\} = \iint\cdots\int_{\mathfrak{D}_n^{(0)}} \frac{\Gamma\left(\frac{R+n-1}{2}\right) dt_1\, dt_2 \cdots dt_n}{\pi^{n/2}\,\Gamma\left(\frac{R-1}{2}\right)\left(1+\sum_{i=1}^{n} t_i^2\right)^{(R+n-1)/2}},$$

where the domain $\mathfrak{D}_n^{(0)}$ *is defined by*

(1.27) $$\mathfrak{D}_n^{(0)}:\ \left|\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\, r_i^{-1/2} r_j^{-1/2} t_i t_j\right| \leqq x R^{-1},$$

and also

(1.28) $$\Pr.\{\|g-\hat{g}\|_D^2/s^2<x\} = \iint\cdots\int_{\varepsilon_{n,k+1}^{(0)}} \frac{\Gamma\left(\frac{R+n-1}{2}\right) dt_1\, dt_2 \cdots dt_n}{\pi^{n/2}\,\Gamma\left(\frac{R-1}{2}\right)\left(1+\sum_{i=1}^{n} t_i^2\right)^{(R+n-1)/2}},$$

where the domain $\varepsilon^{(0)}_{n,k+1}$ *is defined as the one which satisfies all the following* $(k+1)$ *inequalities simultaneously*

$$\varepsilon^{(0)}_{n,k+1}: \sum_{i=1}^{n}\sum_{j=1}^{n} A^{(h)}_{ij} r_i^{-1/2} r_j^{-1/2} t_i t_j \leqq x R^{-1} \tag{1.29}$$

for $h = 1, 2, \cdots, k+1$.

Now let us define the norm for a continuous function $g(t)$ in $0 \leqq t \leqq 1$ by its maximum value over the interval, and let $D^{(k)}$: $0 \leqq t^{(k)}_1 \leqq t^{(k)}_2 \leqq \cdots \leqq t^{(k)}_{n_k}$ be a sequence of division of the interval such that $\max_{\nu}(t^{(k)}_{\nu} - t^{(k)}_{\nu-1})$ tends to zero as k becomes infinite. Then we have readily

Theorem 1.2. *For a continuous function* $g(t)$ *in* $0 \leqq t \leqq 1$, *we have*

$$\lim_{k\to\infty} \| g - \hat{g} \|_{D^{(k)}}^2 = \operatorname*{Max}_{0 \leqq t \leqq 1} |g(t) - \hat{g}(t)|^2 \tag{1.30}$$

and, for each assigned positive number x,

$$\begin{aligned} &\Pr.\{\operatorname*{Max}_{0 \leqq t \leqq 1} |g(t) - \hat{g}(t)|^2 \leqq x\} \\ &= \iint_{\varepsilon^{(1)}_{n,\infty}} \cdots \int \prod_{i=1}^{n} \frac{r_i^{1/2} \exp\{-r_i \eta_i^2/2\sigma^2\}}{(2\pi)^{1/2}\sigma} d\eta_1 d\eta_2 \cdots d\eta_n, \end{aligned} \tag{1.31}$$

where the domain $\varepsilon^{(1)}_{n,\infty}$ is defined as the one in the n-dimensional space $(\eta_1, \eta_2, \cdots, \eta_n)$ such that the following inequality hold true for all values of t in $0 \leqq t \leqq 1$:

$$\varepsilon^{(1)}_{n,\infty}: \left|\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(t)\eta_i\eta_j + 2\sum_{i=1}^{n} b_i(t)\eta_i + c(t)\right| \leqq x, \tag{1.32}$$

where

$$a_{ij}(t) = L(t; t_i, m) L(t; t_j, m) \tag{1.33}$$

$$b_i(t) = L(t, t_i; m)\,\delta_m(t) \tag{1.34}$$

and

$$c(t) = \delta_m^2(t). \tag{1.35}$$

Furthermore we have also

$$\begin{aligned} &\Pr.\{\operatorname*{Max}_{0 \leqq t \leqq 1} |g(t) - \hat{g}(t)|^2/s^2 \leqq x\} \\ &= \iint_{\mathfrak{S}^{(1)}_{n,\infty}} \cdots \int \frac{\Gamma\left(\dfrac{R+n-1}{2}\right) dt_1 dt_2 \cdots dt_n}{\pi^{n/2}\,\Gamma\left(\dfrac{R-1}{2}\right)\left(1 + \sum_{i=1}^{n} t_i^2\right)^{(R+n-1)/2}}, \end{aligned} \tag{1.36}$$

where the domain $\mathfrak{S}^{(1)}_{n,\infty}$ is defined as the one in the n-dimensional space $(t_1, t_2, \cdots, t_n)$ such that the following inequality hold true for all values of t in $0 \leqq t \leqq 1$:

$$\mathfrak{S}^{(1)}_{n,\infty}: \left|R\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(t) r_i^{-1/2} r_j^{-1/2} t_i t_j + 2R^{1/2}\sum_{i=1}^{n} b_i(t) r_i^{-1/2} t_i + c(t)\right| \leqq x. \tag{1.37}$$

From the view points of practical applications it will be of some uses to give the following

Remark 1. In our formulation the order m of the approximate function $\hat{g}(t)$ may be selected such as $0 \leqq m \leqq n$. It is to be noted that the bias $\delta_m(t)$ becomes smaller as m becomes larger, while the variance of $\hat{g}(t)$ becomes larger. The higher orders does not necessarily suit our purposes.

Remark 2. Any domain approximate to $\mathfrak{G}_{n,\infty}$ and hence any approximate value to the integral (2.36) can be obtained by suitable subdivision of the interval $0 \leqq t \leqq 1$.

§ 2. Statistical inferences concerning empirical functions of the type (BD).

Here we shall consider the stochastic process $x(t)$ such that

$$x(t) = \sum_{\nu=1}^{\infty} \lambda_\nu^{-1/2} z_\nu \varphi_\nu(t) \tag{2.01}$$

with convergence in the mean for every t in $0 \leqq t \leqq 1$, where $\{z_\nu\}$, $\{\varphi_\nu(t)\}$ and $\{\lambda_\nu\}$ satisfy the following conditions:

(1°) $\{z_\nu\}$ ($\nu = 1, 2, 3, \cdots$) are a sequence of mutually independent stochastic variables, and each z_ν is distributed according to the normal distribution $N(a_\nu, \sigma_1^2)$.

(2°) Functions $\varphi_\nu(t)$ are the eigen functions of the integral equation

$$\varphi(t) = \lambda \int_0^1 r(t, s)\, \varphi(s)\, ds \tag{2.02}$$

and λ_ν the corresponding eigen-values.

(3°) The sequence $\{\varphi_\nu(t)\}$ ($\nu = 1, 2, 3, \cdots$) form a complete orthogonal normalised system (CONS) in the space $L^2(0, 1)$.

Now our objects of observations $g(t)$ are sample functions of this stochastic process, and we shall assume the same Assumptions (1°) $\sim$ (3°) as in § 1. Then we may and we shall proceed quite similarly as in § 1 except that our probability field becomes now the infinite product space composed of $(\varepsilon_{11}, \varepsilon_{12}, \cdots, \varepsilon_{1r_1}, \varepsilon_{21}, \cdots, \varepsilon_{2r_2}, \cdots, \varepsilon_{n1}, \cdots, \varepsilon_{nr_n}, z_1, z_2, z_3, \cdots, z_n, \cdots)$.

Indeed we may and we shall write now

$$\hat{g}(t) = \sum_{i=1}^{n} \bar{y}_{i.} L(t\,;\, t_i, m), \tag{2.03}$$

expand $L(t\,;\, t_i, m)$ by means of CONS $\{\varphi_\nu(t)\}$

$$L(t\,;\, t_i, m) = \sum_{\nu=1}^{\infty} l(\nu\,;\, i, m)\, \varphi_\nu(t). \tag{2.04}$$

Consequently we shall have

$$(2.05)\quad \hat{g}_m(t) = \sum_{\mu=1}^{\infty} \varphi_\mu(t) \sum_{i=1}^{n} l(\mu\,;\, i, m)\{\bar{\varepsilon}_i. + \sum_{\nu=1}^{\infty} \lambda_\nu^{-1/2} \varphi_\nu(t_i)\, z_\nu\}$$

with the mean convergence at every point t in $0 \leqq t \leqq 1$. We shall have also immediately

Theorem 1.3. *Under the hypothesis to this paragraph, we have*

$$(2.06)\quad \hat{g}_m(t) - g(t) = \sum_{\mu=1}^{\infty} \varphi_\mu(t) \{\sum_{i=1}^{n} d_{\mu i}\, \bar{\varepsilon}_i. + \sum_{\nu=1}^{\infty} e_{\mu\nu}\, z_\nu\},$$

$$\equiv \sum_{\mu=1}^{\infty} \varphi_\mu(t)\, l_\mu(\bar{\varepsilon}, z), \quad \text{say}$$

where

$$(2.07)\quad d_{\mu i} = l(\mu\,;\, i)$$

$$(2.08)\quad e_{\mu\nu} = \sum_{i=1}^{n} l(\mu\,;\, i)\, \lambda_\nu^{-1/2} \varphi_\nu(t_i) \qquad (\mu \neq \nu).$$

$$= \sum_{i=1}^{n} l(\mu\,;\, i)\, \lambda_\nu^{-1/2} \varphi_\nu(t_i) - \lambda_\nu^{-1/2} \qquad (\mu = \nu).$$

Furthermore we have

$$(2.09)\quad \|\hat{g}_m - g\|^2 \equiv \int_0^1 |\hat{g}_m(t) - g(t)|^2\, dt$$

$$= \sum_{\mu=1}^{\infty} [\sum_{i=1}^{n} d_{\mu i}\, \bar{\varepsilon}_i \cdot + \sum_{\nu=1}^{\infty} e_{\mu\nu}\, z_\nu]^2 = \sum_{\mu=1}^{\infty} l(\bar{\varepsilon}, z)^2,$$

and

$$(2.10)\quad \|\hat{g}_m - g\|_{I_h}^2 \equiv (\tau_h - \tau_{h-1})^{-1} \int_{\tau_{h-1}}^{\tau_h} |\hat{g}_m(t) - g(t)|^2\, dt$$

$$= \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} l_\mu(\bar{\varepsilon}, z)\, l_\nu(\varepsilon, z)\, a^{(h)}(\mu, \nu),$$

where

$$(2.11)\quad a^{(h)}(\mu, \nu) \equiv (\tau_h - \tau_{h-1})^{-1} \int_{\tau_{h-1}}^{\tau_h} \varphi_\mu(t)\, \varphi_\nu(t)\, dt.$$

We have also

Theorem 1.4. *The distribution function of the stochastic variable $l_\mu(\bar{\varepsilon}, z)$ is normal with the mean m_μ and the variance $\sigma_\mu^2 \equiv \sigma_{\mu,\mu}$ where*

$$(2.12)\quad m_\mu = \sum_{\nu=1}^{\infty} e_{\mu\nu}\, a_\nu$$

$$(2.13)\quad \sigma_\mu^2 \equiv \sigma_{\mu,\mu} = \sigma^2 \sum_{i=1}^{n} d_{\mu i}^2\, r_i^{-1} + \sigma^2 \sum_{\nu=1}^{\infty} e_{\mu\nu}^2,$$

while the covariance between $l_\mu(\bar{\varepsilon}, z)$ and $l_\nu(\bar{\varepsilon}, z)$ is given by

$$(2.14)\quad \sigma_{\mu,\nu} \equiv \sigma^2 \sum_{i=1}^{y} d_{\mu i}\, d_{\nu i}\, r_i^{-1} + \sigma_1^2 \sum_{\zeta=1}^{\infty} e_{\mu\zeta}\, e_{\nu\zeta},$$

provided that the convergences of the right-hand sides of (2.12) $\sim$ (2.14) *are secured.*

In view of these two Lemmas and Theorem 1.2, our problems of statistical inferences concerning the empirical functions of the type (BD) may be reduced to establish to introduce a certain class of distribution functions which may be called the infinite dimensional central and non-central t-distribution and to establish allied Theorems concerning them distributions.

§ 3. Statistical inferences concerning empirical functions type (AC)

Various observations such as frequently used in meterology, seismology and biological sciences belong to a realm of continuous observations. The essential aspects depend upon how we may define the stochastic dependency of error functions of our observations, and it remains unsolved to define them in accordance with real situations with which we shall meet the problem will demand some detailed analysis of mechanisms of our measuring processes. Formal theories may be, however, developed along the lines which we have adopted in the previous paragraphies. Here we shall restrict ourselves with brief descriptions of these analogous matter.

Now we shall make the following Assumptions:

Assumption (1°) *Let* $g(t)$ *be a deterministic function, defined over the interval* $0 \leqq t \leqq 1$, *and let it be assumed to belong to* $L^2(0, 1)$.

Assumption (2°) *Let* $\{\varphi_\nu(t)\}$, $\{\lambda_\nu\}$ *and* $\{Z_\nu\}$ *be defined as in* § 2.

Assumption (3°) *Our continuous observations yield us an observation function* $f(t)$ *which may be recognised as a sample function of the stochastic process* $x(t)$ *defined in* (2.01).

Assumption (4°) *The expansions of* $g(t)$ *and* $f(t)$ *with respect to the* CONS $\{\varphi_\nu(t)\}$ *are given by*

$$(3.01) \qquad g(t) \sim \sum_{\nu=1}^{\infty} a_\nu \lambda_\nu^{-1/2} \varphi_\nu(t)$$

$$(3.02) \qquad f(t) \sim \sum_{\nu=1}^{\infty} \alpha_\nu \lambda_\nu^{-1/2} \varphi_\nu(t),$$

where α_ν *are sample values of the stochastic variables* Z_ν *respectively and such that*

$$(3.03) \qquad E\{Z_\nu\} = b_\nu \qquad (\nu = 1, 2, 3, \cdots).$$

Now we shall observe

Lemma 1.4. *Under the Assumptions* (1°) $\sim$ (4°) *in this paragraph, let* $\{f_i(t)\}$ $(i = 1, 2, \cdots, m)$ *be a sample of* m *independent observation functions on an unknown function* $g(t)$

$$(3.04) \qquad f_i(t) \sim \sum_{\nu=1}^{\infty} \alpha_{i\nu} \lambda_\nu^{-1/2} \varphi_\nu(t),$$

and let us define

$$(3.05) \qquad \bar{f}(t) \equiv m^{-1} \sum_{i=1}^{m} f_i(t) \sim \sum_{\nu=1}^{\infty} \bar{\alpha}_\nu \lambda_\nu^{1/2} \varphi_\nu(t)$$

$$(3.06) \qquad s^2(t) \equiv (m-1)^{-1} \sum_{i=1}^{m} (f_i(t) - \bar{f}(t))^2 .$$

Then we have

$$(3.07) \qquad \|\bar{f} - g\|^2 \equiv \int_0^l |\bar{f}(t) - g(t)|^2\, dt = \sum_{\nu=1}^{\infty} \lambda_\nu^{-1} (\bar{\alpha}_\nu - a_\nu)^2$$

$$(3.08) \qquad \|s\|^2 \equiv \int_0^l s^2(t)\, dt = \sum_{\nu=1}^{\infty} \lambda_\nu^{-1} (m-1)^{-1} \sum_{i=1}^{m} (\alpha_{i\nu} - \bar{\alpha}_\nu)^2$$

Lemma 1.5. *Under the hypothesis to Lemma 1.4, the simultaneons characteristic function of the stochastic variables* $\|\bar{f} - g\|^2$ *and* $\|s\|^2$ *is given by*

$$(3.09) \qquad \begin{aligned} &E[\exp\{i(\tau_1 \|\bar{f} - g\|^2 + \tau_2 \|s\|^2)\}] \\ &= E[\exp\{i\tau_1 \|\bar{f} - g\|^2\}]\, E[\exp\{i\tau_2 \|s\|^2\}], \\ &\equiv h(\tau_1 ; b-a)\, q_m(\tau_2), \quad \text{say,} \end{aligned}$$

$$(3.10) \qquad E[\exp\{i\tau_1 \|\bar{f} - g\|^2\}] = \prod_{\nu=1}^{\infty} h_{\nu, m}(\tau_1 ; b-a)$$

$$(3.11) \qquad E[\exp\{i\tau_2 \|s\|^2\}] = \prod_{\nu=1}^{\infty} q_{\nu, m}(i\tau_2)$$

with

$$(3.12) \qquad h_{\nu, m}(\tau_1 ; b-a) \equiv \frac{\exp\{i\tau_1^2 (b_\nu - a_\nu)^2 + 2(b_\nu - a_\nu)^2 \tau_1^2 \sigma^2 \lambda_\nu^{-1} m^{-1} (2i\tau_1 \lambda_\nu^{-1} m^{-1} \sigma^2 - 1)^{-1}\}}{(1 - 2i\tau_1 \sigma^2 \lambda_\nu^{-1} m^{-1})^{1/2}}$$

and

$$(3.13) \qquad q_{\nu, m}(\tau_2) \equiv (1 - 2i\tau_2 (m-1) \sigma^2 \lambda_\nu^{-1})^{-m/2} .$$

The proof of Lemma 1.4 is immediate, and that of Lemma 1.5 can be obtained by the elegant method due to KAC [1] (see specially § 1).

Finally we shall reach the following theorem which yields us at least theoretically the way how to test the null hypothesis H_0: $b = a$, which means that all the equalities $a_\nu = b_\nu$ ($\nu = 1, 2, 3, \cdots$) hold true.

Theorem 1.4. *Under the Assumptions* (1°) $\sim$ (4°) *and the null hypothesis* H_0: $b = a$, *the probability that* $\|\bar{f} - g\|^2 / \|s\|^2$ *will exceed any assigned positive number* x *is given by*

$$(3.14) \qquad \Pr.\{(\|\bar{f} - g\|^2 / \|s\|^2) > x\} = \int_x^{\infty} k_m(u)\, du$$

where

$$(3.15) \qquad k_m(u) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} h_m(\tau ; 0)\, q'_m(-u\tau)\, d\tau .$$

The proof can be obtained from a well-known Theorem due to CRAMÉR concerning the d.f. of the ratio two independent stochastic variables, in view of Lemma 1.4.

§ 4. Statistical inferences concerning empirical functions of the type (BC).
Our formulation in this paragraph may be the combination of those adopted in § 2 and § 4. Indeed our objects of statistical inferrnces belong to a stochastic process $x(t)$, while our observations will always involve some error function of the form $f(t) - g(t)$ in the termologies in § 3. Consequently our sample function $a(t)$ may be, under these circumstances, written as the sum of the form

$$(4.01) \qquad a(t) = g(t) + \varepsilon(t),$$

where $g(t)$ is a sample function of the stochastic process, while $\varepsilon(t)$ denotes an error function which is also a sample function drawn from another stochastic process $y^{(t)}$.

In view of the discussions hither to developed in § 1 ~ § 3, it may be now evident that our statistical inferences concerning empirical functions of the type (BC), will be treated by the orthogonal expansions of these two stochastic processes.

Part II. Differentiation of empirical functions

§ 1. Differentiation of empirical functions of the type (AD).
Under the Assumption to § 1 of Part I, we may and we shall define a derivative function $\hat{g}(t)$ as an empirical approximation to the given function $g^{(t)}$ by

$$(1.01) \qquad \hat{g}'(t) = \sum_{i=1}^{n} \bar{y}_i . L'(t ; t_i, m) .$$

In our formulation this approximative devivative function $\hat{g}(t)$ may be treated quite similarly from the probabilistic standpoint, provided that we shall use the norm of functions such as

$$(1.02) \qquad \| g' - \hat{g}' \|^2 = \int_0^l |g'(t) - \hat{g}'(t)|^2 dt$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{\varepsilon}_i . \bar{\varepsilon}_j . D_{ij} + 2 \sum_{i=1}^{n} E_i \overline{\varepsilon_i .} + F$$

$$(1.03) \qquad \| g' - \hat{g}' \|_h = \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{\varepsilon}_i . \varepsilon_j . D_{ij}^{(h)} + 2\sum_{i=1}^{n} E_i^{(h)} \bar{\varepsilon}_i . + F^{(h)},$$

where we define

$$(1.04)\qquad D_{ij}=\int_0^l L'(t\,;\,t_i,m)\,L'(t\,;\,t_j,m)\,dt$$

$$(1.05)\qquad E_i=\int_0^l L'(t\,;\,t_i,m)\,\delta_m'(t)\,dt$$

$$(1.06)\qquad F=\int_0^l \{\delta_m'(t)\}^2\,dt$$

$$(1.07)\qquad D_{ij}^{(h)}=(\tau_h-\tau_{h-1})^{-1}\int_{\tau_{h-1}}^{\tau_h} L'(t\,;\,t_i,m)\,L'(t\,;\,t_j,m)\,dt$$

$$(1.08)\qquad E_{ij}^{(h)}=(\tau_h-\tau_{h-1})^{-1}\int_{\tau_{h-1}}^{\tau_h} L'(t\,;\,t_i,m)\,\delta_m'(t)\,dt$$

$$(1.09)\qquad F_{ij}^{(h)}=(\tau_h-\tau_{h-1})^{-1}\int_{\tau_{h-1}}^{\tau_h} [\delta_m'(t)]^2\,dt\,,$$

and

$$(1.10)\qquad \delta_m'(t)\equiv g'(t)-\sum_{i=1}^n g(t_i)\,L'(t\,;\,t_i,m)\,.$$

It is a well-known fact that any approximation to derivative function is far delicate than that to the original function. The numerical evaluations of the values of the constants defined in (1.02) $\sim$ (1.10) will show how and why this difficulty derive from. In a comparison with those of $\{L(t\,;t_i,m)\}$, the behaviours of $L(t\,;\,t_i,m)$ will show some acute change near the point $t=t_i$, and it may be realidy seen that D_{ij} and $D_{ij}^{(h)}$ are not so large as A_{ij} and $A_{ij}^{(h)}$ respectively. Similar discrepancies may occur between $g'(t)$ and $\hat{g}'(t)$ even when $\hat{g}(t)$ themselves is a fairly good approximate function to $g(t)$.

Summaring our discussions, we shall enunciate

Theorem 2.1. *Under the Assumptions in § 1 of Part I, the distribution functions of the stochastic variables* $\|g'-\hat{g}'\|^2$, $\|g'-\hat{g}'\|_D^2$, $\|g'-\hat{g}'\|/s^2$, *and* $\|g'-\hat{g}'\|_D^2/s^2$ *can be evaluated as those of* $\|g-\hat{g}\|$, $\|g-\hat{g}\|_D^2$, $\|g-\hat{g}\|^2/s^2$, *and* $\|g-\hat{g}\|_D^2/s^2$ *by means of* $\{D_{ij}\}$, $\{E_i)$, $\{F\}\{D_{ij}^{(h)}\}$, $\{E_i^{(h)}\}$ *and* $\{F^{(h)}\}$ *instead of* $\{A_{ij}\}$, $\{B_i\}$, $\{C\}$, $\{A_{ij}^{(h)}\}$, $\{B_i^{(h)}\}$ *and* $\{C^{(h)}\}$ *respectively.*

§ 2. Differentiation of empirical functions due to continuous observation

Besides the possibilities that we may develop some considerations similar to that of § 1, as will be verified by the arguments in § 2 $\sim$ § 4 in Part I, we shall notice here that there will be certain characteristic problems associated with continuous observations. Here we remind the autocorrelation function of the error function defined in § 4 of Part I. Then for the empirical functions of the type (AC), we shall have an approximate derivative at a point t, provided that (1°) the deterministic function $g(t)$ itsself is defferentiable at the point t; (2°) the autocorrelation function $r(s,t)$ possesses its generalised second derivative at $s=t=t_0$.

That is to say, under these assumptions (1°) ~ (2°), we shall have

$$(2.01)\qquad \lim_{\Delta+\to 0} E\{(\hat{g}(t+\Delta t)-\hat{g}(t))(\Delta t)^{-1}\} = g'(t)$$

and

$$(2.02)\qquad \lim_{\Delta+\to 0} \sigma^2\{(\hat{g}(t)-\hat{g}(t))(\Delta t)^{-1}\} = \left(\frac{\partial r(s,t)}{\partial s}\right)_{s=t},$$

which, however, yield us no special interests, since they are direct consequences in the theories of regular random functions due to MOYAL [1] and of "fonctions aleatoires du second ordre" due to LOEVE [1].

From the practical point of view, however, our problems of finding an approximate derivative at a point still remain to be very difficult to approach, since there are two types of limiting process in (2.01). To overcome such difficulties must be a problem worth while to be trated systematically from the point view of stochastic analysis as developed in a paper of mine [1].

The following procedure which also appeals to the limiting processes of two kinds will be of some interest in making the convergence more easy. In fact we may propose the procedure defined as follows:

(1°) Let a sample of n non-negative numbers $(\tau_1, \tau_2, \cdots, \tau_n)$ be drawn from a parent population whose probability density function is given by

$$(2.03)\qquad p(\tau) = \lambda^{-1}\exp\{-\lambda\tau\}$$

for $\tau \geqq 0$, vanishing in $-\infty < \tau < 0$.

(2°) Then by a random mechanism either of positive and negative signs will be associated with each of τ_i independently with the probabilities 1/2, that is, each τ_i becomes $\varepsilon_i\tau_i$, where $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ are mutually independent and such as $\Pr.\{\varepsilon_i = 1\} = \Pr.\{\varepsilon_i = -1\} = 1/2$ $(i = 1, 2, \cdots, n)$.

(3°) Let us now define

$$(2.04)\qquad y_{\lambda_n}(t) \equiv \frac{1}{n}\sum_{i=1}^{n}\frac{y(t+\varepsilon_i\tau_i)-y(t)}{\varepsilon_i\tau_i}.$$

(4°) Finally let us define the sequence of functions $\{y_{\lambda_k, n_k}(t)\}$ for certain assigned sequence $\{(\lambda_k, n_k\}$ $(k = 1, 2, 3, \cdots)$.

The detailed discussions will be postponed to another occasion. It is to be noted that, for each fixed pair (λ, n), we shall have

$$(2.05)\qquad E\{y_{\lambda,n}(t)\} = (2\lambda)^{-1}\int_{-\infty}^{\infty}(g(t+\tau)-g(t))\exp\{-\lambda|\tau|\}\tau^{-1}d\tau$$

$$(2.06)\qquad \sigma^2\{y_{\lambda,n}(t)\}$$
$$= (n)^{-1}(2\lambda)^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\tau_1\tau_2)^{-1}\exp\{-\lambda(|\tau_1|+|\tau_2|)\}\rho(\tau_1, \tau_2; t)\,d\tau_1 d\tau_2,$$

where we put

$$(2.07)\qquad \rho(\tau_1, \tau_2; t) \equiv r(t+\tau_1, t+\tau_2) - r(t, t+\tau_2) - r(t+\tau_1, t) + r(t, t).$$

§ 3. Functional operations on empirical functions of the type (B) and statistical inferences.

So far we have discussed in § 1 ~ § 2 of this Part II, there is no essential connection between the differentiation and the autocovariance function $r(s, t)$ of the sbochastic process. In the case when all eigenfunctions $\varphi_\nu(t)$ are continuously differentiable, over $0 \leqq t \leqq 1$ and that derivative functions $\varphi^{\nu'(t)\nu}$ may be expansible into series by means of the original CONS system $\{\varphi_\nu(t)\}$, the problem will become extremely complicated by the fact that all the sample functions are not necessarily differentiable. Such difficulties are well known in functional analysis, and will lead us to the employments of some approximate bounded operations in stead of the differentiations which are unbounded operations in current function spaces. Thus N. WIENER [1] makes uses of certain approximate differentiation (see *loc. cit.* p. 119 ~ 120). From the general points of views it may be noticed that some deep analysis can only be developed by bringing the autocovariance function $r(s, t)$ and functional operations into intrinsic connections. Thus in dealing with linear translatable functional operations on empirical functions some generalised exponential analysis such as expansions into FOURIER series, representations by FOURIER integrals, and that due to CAUCHY series due to the author [2] will be reasoneablely adopted, simply because there is the simple relation that $De^{\lambda x} = \lambda e^{\lambda x}$ for every λ and all x. This principle has been generalised to some theory of operational calculus by the present author [3] in which some systematic treatments of CAUCHY-DELSARTE series were developed. For the present purpose we shall introduce

Definition 2.1. A linear operation T which transforms each of eigenfunctions $\{\varphi_\nu(t)\}$ such that $T\varphi_\nu(t) = \rho_T(\lambda_\nu)\varphi_\nu(t)$ $(\nu = 1, 2, 3, \cdots)$ with the conditions that the sequence $\{\rho_T(\lambda_\nu)\}$ $(\nu = 1, 2, 3, \cdots)$ is bounded is called a bounded operation for the stochastic process $x(t)$ introduced in § 2 of Part I, associated with its auto covariance function $r(s, t)$.

In virtue of this Definition the situations become transparently simple. Indeed we shall observe immediately

Theorem 2.2. *Let T be a bounded linear operation introduced in Definition* 2.1.

Further under the Assumptions enunciated in § 2 *of Part I, let us define*

$$(3.01)\qquad s_\tau(t) \equiv (m-1)^{-1} \sum_{i=1}^{m} (Tf_i(t) - T\bar{f}(t))^2.$$

Then we have

$$(3.02)\qquad |T\bar{f} - Tg = \sum_{\nu=1}^{\infty} (\bar{\alpha}_\nu - a_\nu)^2 \lambda_\nu^{-1} \rho_T^2(\lambda_\nu)$$

and

$$(3.03) \qquad s_T^2 = \sum_{\nu=1}^{\infty} (m-1)^{-1} \sum_{\iota=1}^{m} (\alpha_{i\nu} - \bar{\alpha}_\nu)^2 \lambda_\nu^{-1} \rho_T^2(\lambda_\nu).$$

Moreover the simultaneous characteristic function of thestochastic variables $|TF - tg|$ and $|s_T^2|$ can be obtained quite similarly as in Lemma 1.4, merely by replacing the factors λ_ν^{-1} by $\lambda_\nu^{-1}\rho_T^2(\lambda_\nu)$, and by the uses of this simultaneous characteristic function we can make statistical inferences about $H_0(T)$: $(a_\nu - b_\nu)\rho(\lambda_\nu) = 0$ ($\nu = 1, 2, 3, \cdots$) quite similarily as in Theorem 1.3.

References

MOYAL, J. E.: [1] *Stochastic processes and statistical physics* (Sympothium on stochastic processes), Journ. Roy. Stat. Soc., Ser. B, 11 (1949).

LOEVE, M.: [1] *Fonctions aleatoires du second ordre*, (P. Levy: Processus stochastiques et mouvement brownien) (1948).

KITAGAWA, T.: [1] *Random integrations*, Bull. Math. Stat., 4 (1951).

KITAGAWA, T.: [2] *On the theory of linear tranalatable functional equations*, Jap. Journ. Math. 13 (1937).

KITAGAWA, T.: [3] *A formulation of the operational calculus on the family of mutually permutable operations*, Jap. Journ. Math. 14 (1938).

KITAGAWA, T.: [4] *Linear stochastic translatable functional equations and stochastic Canchy series*, Jap. Journ. Math. 22 (1952).

WIENER, N.: [1] *The solution of a difference equation by trigonsmetrical integrals*, Jour. Math. Phys. M.I.T. 4 (1926).

KAC, M.: [1] *On some connections between probability theory and differential and integral equations*, in Proceedings of the second Berkeley Symposium on Mathematical Statistics and Probability, Univ. of Calif. Press, Berekeley. 1951.