Successive process of statistical inferences, (4)

Kitagawa, Toshio
Mathematical Institute, Kyushu University

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SUCCESSIVE PROCESS OF STATISTICAL INFERENCES, (4)[1]

Tosio KITAGAWA

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Introduction

In this fourth paper we directed our methods of successive process of statistical inferences to somewhat particular problems. In Part IX we shall discuss certain applications of ranges to successive processes of statistical inferences, which will be useful specially in statistical controls of quality such as control chart methods. Indeed it seems reasonable both from theoretical and from practical points of view that we should appeal to successive process of statistical inferences.

On the other hand it may be also urgent demands to simplify as far as possible any calculations involved in statistical analysis. In Part X we shall proceed to discuss fiducial inferences due to R. A. FISHER from the view points of successive process of statistical inferences. Here we shall restrict ourselves with an enunciation of certain two sample formulation to the famous Behrens-Fisher test in order that the test may be suitably interpretated from our view points. This formulation may be recognised as being along the lines due to BARNARD [1]and STEIN [1] . It is to be noted that there may be perhaps any other interprepations from the general point of view of successive process of statistical inferences.

Part IX. Applications of ranges to successive processes of statistical inferences

§ 1. Modified t-test in the two sample theory

We shall first observe

(1) Parts IX and X were communicated by the author at the Annual Meeting of Math. Soc. of Japan held at Tokyo Univ. in June 6, 1952.

Theorem 9.1. *Let O_{n_1} and O_{n_2} be two independent random samples of sizes n_1 and n_2 respectively drawn from the same normal population. Let the sample means of O_{n_i} be $\bar{x}_i$ $(i = 1, 2)$. Let $W = \bar{W}(m, n)$ be the mean range defined as mean value of m mutually independent sample ranges where each range is defined for a sample of size n. Let us assume that w, $\bar{x}_1$ and $\bar{x}_2$ are mutually independent. Let $1-\alpha$ be an arbitrarily assigned confidence-coefficient. Then the equation*

$$(1.01) \qquad Pr.\{\bar{x}_1 - Bw < \bar{x}_2 < \bar{x}_1 + Bw\} = 1 - \alpha$$

determines a constant B in an approximate form:

$$(1.02) \qquad B \cong (n_1^{-1} + n_2^{-1})^{1/2} t_\nu(\alpha)/C(m, n),$$

where the constant $C(m, n)$ and the ν degrees of freedom of t-distribution are those by means of which $\bar{w}(m, n)$ can be approximated as $C(m,n)\chi_\nu \nu^{-1/2}$, where χ_ν means the chi-distribution with the ν degrees of freedom, and $t_\nu(\alpha)$ denotes the significance level of t-distribution with the ν degrees of freedom.

Proof: The assertion that a mean range $\bar{w}(m, n)$ can be approximated by $C(m, n)\ \chi_\nu \nu^{-1/2}$ is established in English authors(1) such as PATNAIK [1]. Writing $\chi_\nu \nu^{-(1/2)}$ by s_ν, we shall have

$$(1.03) \qquad (\bar{x}_2 - \bar{x}_1)/\bar{w}(m, n) = (\bar{x}_2 - \bar{x}_1)/C(m, n)\, s_\nu = \frac{(n_1^{-1} + n_2^{-1})^{1/2}}{C(m, n)} \cdot \frac{\bar{x}_2 - \bar{x}_1}{s_\nu (n_1^{-1} + n_2^{-1})^{1/2}},$$

where the second factor of the right-hand side is approximately distributed as a t-distribution with the ν degrees of freedom.

For the sake of applications we have constructed the following Table I, which yields us the values of $\varphi(m, n, \alpha) \equiv t_\nu(\alpha)/C(m, n)$. The α-points of t-distribution with the ν degrees of freedom, $t_\nu(\alpha)$, were calculated from MERRINGTON, M [1] by quadratic interpolations, except for the low values of d.f. ν for which interpolations of higher orders were employed. Here it is to be noted that the ordinary t-distributions defined for the positive integral values of degrees of freedom can be generalised for any positive number ν as that whose probability density function is

$$(1.04) \qquad \frac{1}{\sqrt{\nu}\, B\left(\frac{\nu}{2}, \frac{1}{2}\right)} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}} dt.$$

§ 2. Successive poolings of data in control charts

Let O_i $(i = 1, 2, \cdots\cdots, m)$ be a sequence of independent random samples of size n, and let $\{\bar{x}_i\}$ and $\{w_i\}$ $(i = 1, 2, \cdots, m)$ be sequences of sample means and sample ranges defined for each of these samples O_i. In the terminology of control chart method due to SHEWHART [1] , the sequences $\{\bar{x}_i\}$ and $\{\bar{w}_i\}$ define $\bar{x}$-chart and R-chart, so far as each sample O_i means independent sample from each of successive lots from the same supply. When we have obtained these m samples, we can make use of the pooled data, that are, the mean range $\bar{w}(m, n)$ and the pooled mean $\bar{x}_{12\cdots m}$ defined as follows:

$$(2.01) \qquad \bar{w}(m, n) = (w_1 + w_2 + \cdots\cdots + w_m)/m$$

$$(2.02) \qquad \bar{x}_{12\cdots m} = (\bar{x}_1 + \bar{x}_2 + \cdots\cdots + \bar{x}_m)/m .$$

Under these circumstances, let us assume that a random sample of size n_2, O_{n_2}, will be obtained independently from these m samples. Let $\bar{y}$ be the sample mean defined for O_{n_2}. Then we shall observe

Theorem 9.2. *Under the general hypothesis to this paragraph* 2, *let us assume that production is statistical controlled and that each lot has the same normal distribution.*

Let $1-\alpha$ *be arbitrarily assigned confidence-coefficient. Then the equation*

$$(2.03) \qquad Pr.\{|\bar{y} - \bar{x}_{12\cdots m}| \leqq C\bar{w}(m, n)\} = 1-\alpha$$

determines a constant C *in an approximate form*

$$(2.04) \qquad C \cong t_\nu(\alpha)\,\{(mn)^{-1} + n_2^{-1}\}^{1/2}/C(m, n),$$

where the meanings of ν, $C(m, n)$ *and* $t_\nu(\alpha)$ *are those defined in Theorem* 9.1.

To proceed to successive procedures of poolings which we have discussed in Part III in our paper [1] , we should take into consideration the sequences of successive means $\{\bar{x}_{12\cdots i}\}$ $(i = 2, \cdots\cdots, m)$ defined as

$$(2.05) \qquad \bar{x}_{12\cdots i} = (N_{i-1}\bar{x}_{12\cdots i-1} + n_i\bar{x}_i)/(N_{i-1} + n_i), \qquad (i = 2, 3, \cdots\cdots, m),$$

where $N_{i-1} = n_1 + n_2 + \cdots + n_{i-1}$, and of successive mean ranges $\{\bar{w}(i, n)\}$ $(i = 2, \cdots, m)$ such that

$$(2.06) \qquad \bar{w}(i, n) = (w_1 + w_2 + \cdots\cdots + w_i)/i, \qquad (i = 2, 3, \cdots\cdots, m).$$

Our problem will turn out to calculate the probability

$$(2.07) \qquad Pr.\left\{\frac{|\bar{x}_1 - \bar{x}_2|}{\bar{w}(2, n)} < A_2,\ \frac{|\bar{x}_{12} - \bar{x}_3|}{\bar{w}(3, n)} < A_3 \cdots\cdots,\ \frac{|\bar{x}_{12\cdots m-1} - \bar{x}_m|}{\bar{w}(m, n)} < A_m\right\}.$$

Since $\{\bar{w}(i, n)\}$ $(i = 2, 3, \cdots\cdots, m)$ are not mutually independent, this probability can not be obtained directly from the chi-approximation of each mean range.

On the other hand the similar problem which will treat the successive poolings of sample means in stead of the simultaneously successive poolings of sample means and sample mean ranges may be solved extremely simply. In view of Part III, § 1 in our previous paper [1] , we shall readily observe

Theorem 9.3. *Let O_k $(k = 1, 2, \cdots\cdots, m)$ be m independent random samples of size n_k from the same normal population $N(\xi, \sigma^2)$, where ξ and σ^2 are unknown to us. Let us denote by w_k the sample range defined for O_k. Let ε_m be the domain in the $n_1 + n_2 + \cdots\cdots + n_m$ -dimensional sample space such that*

$$(2.08) \quad \varepsilon_m : |\bar{x}_1 - \bar{x}_2| < A_2 w_2, |\bar{x}_{12} - \bar{x}_3| < A_3 w_3, \cdots\cdots, |\bar{x}_{12\cdots m-1} - \bar{x}_m| < A_m w_m,$$

where $\bar{x}_{12\cdots i}$ $(i = 2, 3, \cdots\cdots, m-1)$ are those successively pooled means defined as in (2.01).
Then we have

$$(2.09) \qquad Pr.\{\varepsilon_m\} = \prod_{i=1}^{m} Pr.\{|\bar{x}_i - \bar{x}_{12\cdots i-1}| < A_i w_i\}$$

$$\simeq \prod_{i=1}^{m} Pr.\left\{\frac{|\bar{x}_i - \bar{x}_{12\cdots i-1}|}{C(1, n_i)\, s_{\nu(1, n_i)}} < A_i\right\}.$$

Proof: This follows from the transformation $u_i = \bar{x}_i - \bar{x}_{12\cdots i-1}$ $(i = 2, 3, \cdots\cdots, m)$ which yields us .

$$(2.10) \qquad \sum_{i=1}^{m} n_i (\bar{x}_i - \xi)^2 = N_m (\bar{x}_1 - \xi + \frac{n_2}{N_2} u_2 + \cdots\cdots + \frac{n_k}{N_k} u_k)^2$$

$$+ \sum_{j=2}^{m} N_{j-1} n_j N_j^{-1} u_j^2 .$$

After integrating with respect to $\bar{x}_1$ in $-\infty < \bar{x}_1 < \infty$, we reach the first equation in (2.08) , which yields the approximate evaluations in view of §1.

We have also constructed Tables II and III by means of Table I for the sake of applications of Theorem 9.2 and 9.3.

Table 1. $\varphi_{m,n}(\alpha) : (m, n, \alpha) \to \varphi_{m,n}(\alpha) = t_\nu(\alpha)/C(m, n)$

$$Pr.\left\{\frac{|x|}{\bar{w}(m, n)} > \varphi_{m,n}(\alpha)\right\} = \alpha$$

(I) $\alpha = 0.5\%$

Size of sample n	Number of samples, m				
	1	2	3	4	5
3	9.5370	2.5979	2.5116	2.2403	2.1011
4	3.4268	2.0570	1.7746	1.6542	1.5881
5	2.3003	1.6330	1.4661	1.3919	1.3517
6	1.8536	1.4113	1.2907	1.2430	1.2137
7	1.5948	1.2695	1.1833	1.1438	1.1215
8	1.4276	1.1502	1.1053	1.0727	1.1385
9	1.3099	1.1032	1.0457	1.0188	1.0033
10	1.2224	1.0389	0.9993	0.9593	0.9627

(II) $\alpha = 1\%$

Size of sample n	Number of samples, m				
	1	2	3	4	5
3	6.2309	2.5915	2.1450	1.9551	1.8536
4	2.6704	1.7597	1.5581	1.4703	1.4213
5	1.8904	1.4243	1.3032	1.2488	1.2181
6	1.5568	1.2441	1.1579	1.1195	1.0973
7	1.3599	1.1269	1.0628	1.0333	1.0161
8	1.2301	1.0255	0.9956	0.9700	0.9572
9	1.1376	0.9869	0.9440	0.9237	0.9119
10	1.0644	0.9405	0.9037	0.8861	0.8758

(III) $\alpha = 2.5\%$

Size of sample n	Number of samples, m				
	1	2	3	4	5
3	3.6239	1.9632	1.7083	1.5960	1.5344
4	1.8971	1.4018	1.2824	1.2276	1.1980
5	1.4314	1.1622	1.0884	1.0546	1.0352
6	1.2125	1.0290	0.9754	0.9500	0.9369
7	1.0792	0.9399	0.8993	0.8811	0.8702
8	0.9891	0.8604	0.8461	0.8303	0.8214
9	0.9235	0.8315	0.8044	0.7917	0.7838
10	0.8733	0.7950	0.7715	0.7603	0.7505

(IV) $\alpha = 5\%$

Size of sample n	Number of samples, m				
	1	2	3	4	5
3	2.4175	1.5558	1.4027	1.3334	1.2946
4	1.4395	1.1518	1.0775	1.0433	1.0238
5	1.1338	0.9707	0.9242	0.9024	0.8898
6	0.9801	0.8676	0.8331	0.8172	0.8085
7	0.8840	0.7972	0.7716	0.7593	0.7522
8	0.8176	0.7327	0.7273	0.7169	0.7110
9	0.7686	0.7103	0.6928	0.6836	0.6792
10	0.7303	0.6808	0.6651	0.6581	0.6537

Table 2. $\Psi_{m,n}(\alpha)$: $(m, n, \alpha) \to \Psi_{m,n}(\alpha) = \varphi_{m,u}(\alpha)\sqrt{\frac{1}{mn}+\frac{1}{n}}$

$$Pr.\left\{\frac{|\bar{x}_m - y_1|}{w(m, n)} > \Psi_{m,n}(\alpha)\right\} = \alpha$$

(I) 0.5%

Size of sample n	Number of samples, m				
	1	2	3	4	5
3	7.7878	1.8110	1.6744	1.4461	1.3289
4	2.4231	1.2596	1.0246	0.9551	0.8698
5	1.4548	0.8944	0.7571	0.6960	0.6622
6	1.0702	0.7057	0.6084	0.5674	0.5428
7	0.8525	0.5877	0.5164	0.4833	0.4643
8	0.7138	0.4980	0.4512	0.4240	0.4409
9	0.6175	0.4504	0.4025	0.3797	0.3664
10	0.5467	0.4024	0.3649	0.3392	0.3239

(II) 1%

Size of sample n	Number of samples, m				
	1	2	3	4	5
3	5.0881	1.8325	1.4300	1.2620	1.1723
4	1.8883	1.0776	0.8996	0.8219	0.7785
5	1.1956	0.7801	0.6730	0.6244	0.5967
6	0.8988	0.6221	0.5438	0.5110	0.4907
7	0.7269	0.5217	0.4638	0.4367	0.4207
8	0.6151	0.4441	0.4065	0.3834	0.3707
9	0.5363	0.4029	0.3633	0.3442	0.3330
10	0.4760	0.3643	0.3300	0.3133	0.3034

(III) 2.5%

Size of sample n	Number of samples, m				
	1	2	3	4	5
3	2.9592	1.3882	1.1389	1.0302	0.9704
4	1.3415	0.8584	0.7404	0.6863	0.6562
5	0.9053	0.6366	0.5621	0.5273	0.5071
6	0.7000	0.5145	0.4598	0.4336	0.4190
7	0.5769	0.4351	0.3925	0.3723	0.3603
8	0.4946	1.3726	0.3454	0.3282	0.3181
9	0.4353	0.3395	0.3096	0.2951	0.2862
10	0.3905	0.3079	0.2818	0.2688	0.2500

(IV) 5%

Size of sample n	Number of samples, m				
	1	2	3	4	5
3	1.9741	1.1001	0.9351	0.8607	0.8188
4	1.0179	0.7053	0.6221	0.5832	0.5608
5	0.7171	0.5317	0.4773	0.4512	0.4359
6	0.5659	0.4338	0.3927	0.3730	0.3616
7	0.4725	0.3690	0.3368	0.3209	0.3114
8	0.4088	0.3173	0.2969	0.2834	0.2754
9	0.3623	0.2900	0.2667	0.2548	0.2480
10	0.3266	0.2637	0.2429	0.2327	0.2264

Table 3. $\Psi'_{m,n}(\alpha)$: $(m, n, \alpha) \to \Psi'_{m,n}(\alpha) = \varphi_{m,n}(\alpha)\sqrt{\frac{1}{(m-1)n} + \frac{1}{n}}$

(I) 0.5%

Size of sample n	Number of samples, m			
	2	3	4	5
3	2.1214	1.7760	1.4935	1.3563
4	1.4545	1.0867	0.9551	0.8878
5	1.0328	0.8030	0.7188	0.6759
6	0.8149	0.6454	0.5860	0.5540
7	0.6786	0.5478	0.4992	0.4739
8	0.5751	0.4786	0.4379	0.4500
9	0.5200	0.4269	0.3921	0.3739
10	0.4646	0.4870	0.3503	0.3404

(II) 1%

Size of sample n	Number of samples, m			
	2	3	4	5
3	2.1162	1.5168	1.3034	1.1965
4	1.2443	0.9541	0.8489	0.7945
5	0.9008	0.7138	0.6449	0.6091
6	0.7183	0.5790	0.5277	0.5009
7	0.6024	0.4920	1.4510	1.4294
8	0.5128	0.4311	0.3960	0.3784
9	0.4652	0.3854	0.3555	0.3398
10	0.4206	0.3500	0.3236	0.3096

(III) 2.5%

Size of sample n	Number of samples, m			
	2	3	4	5
3	1.6031	1.2080	1.0640	0.9905
4	0.9912	0.7853	0.7088	0.6697
5	0.7350	0.5061	0.5446	0.5176
6	0.5941	0.4877	0.4478	0.4276
7	0.5024	0.4163	0.3845	0.3677
8	0.4302	0.3664	0.3390	0.3247
9	0.3920	0.3284	0.3047	0.2921
10	0.3555	0.2988	0.2776	0.2653

(IV) 5%

Size of sample n	Number of samples, m			
	2	3	4	5
3	1.2705	0.9919	0.8889	0.8357
4	0.8144	0.6598	0.6023	0.5723
5	0.6139	0.5062	0.4660	0.4449
6	0.5009	0.4166	0.3852	0.3690
7	0.4261	0.3572	0.3314	0.3179
8	0.3664	0.3149	0.2927	0.2810
9	0.3348	0.2828	0.2631	0.2531
10	0.3045	0.2576	0.2403	0.2311

Part X. Fiducial inferences from the view point of successive processes of statistical inferences

§ 1. Fiducial inferences

After reading our previous paper [1], Prof. R. A. FISHER gave me a letter which, referring to his previous paper in 1935, FISHER [1], said, "In section II I seem to be following very much the method you recommend." Since then we have re-read his famous papers on fiducial inferences, and in conclusions it seems to us to be indispensable both for fiducial arguments and for successive processes of statistical inferences to make clear the inter-relationships between these two formulations.

In his theory of fiducial inferences, FISHER [1] emphasised two points of view. The first is to distinguish fiducial probability statements from "those that would be derived by the method of inverse probability, from any postulated knowledge of the distribution of μ in the different populations which might have been sampled." The second is to emphasise also that "statements similar to those of fiducial probability can only represent the true state of knowledge derivable from the sample, if the statistics used contain the whole of the relevant information which the sample provides." To the author of the present paper it seems adequate to emphasise that these two points of view concern themselves deeply with the two sample theoretic formulations. In current text-books on mathematical statistics the two sample theoretic ideas of R. A. FISHER developed in Part II in his paper [1] are not duly treated, and in some of them his fiducial probability theory is scarcely mentioned.

It is well known that such unfortunate circumstances have been derived partly from current discussions concerning Behrens-Fisher's test. If R. A. FISHER would formulate this test of significance more accurately and more throughly from the stand point of the two sample theory, the test would deserve its relevant interpretations and its due circulations. The author of the present paper dares to say that the agreements and disputes between R. A. FISHER and J. NEYMAN seem to derive from the following circumstances.

(1°) R. A. FISHER and J. NEYMAN agree with each other in the sense that both of them reject any postulated knowledge of the distribution of parameters in different populations which might have been sampled.

(2°) R. A. FISHER and J. NEYMAN differ from each other in the sense that the fiducial inferences of the former concern themselves deeply with the two sample theoretic formulation while the theories of inductive behaviours of the latter deal entirely with inferences from a sample to its parent population. I say here "deeply", because the real features of fiducial arguments depending upon the two sample formulations are not necessarily apparent. Statistical inferences from a sample to its parent population in the sense of FISHER [1] ought to be recognised as the limiting case of the two sample formulation when the size of the second sample n_2 becomes infinity as we have emphasized in Part I in our previous paper (1).

In our point of view there may be at least two interpretations of Behrens-Fisher's test from our successive processes of statistical inferences. One of our interpretations is essentially due to the argument of BARNARD [1]. It seems to us, however, necessary to make concrete formulations to this argument along the method of STEIN [1]. The other interpretations may be said to be a method of multiple assertions in two sample formulations. Here the sufficiency of estimations play its essential role as R. A. FISHER emphasized it. In the following § 2 we shall give the first interpretation. The second will be discussed in another occasion.

§ 2. Application of Barnard-Stein method to an interpretation of Behrens-Fisher test

(1°) Let O_{m_0} : $(x_1, x_2, \cdots\cdots, x_{m_0})$ and O_{n_0} : $(y_1, y_2, \cdots\cdots, y_{n_0})$ be two independent random samples drawn from normal populations $N(\xi_1, \sigma_1^2)$

and $N_2(\xi_2, \sigma_2^2)$ respectively, where ξ_i and σ_i^2 $(i = 1, 2)$ are unknown to us. Let us define as usually the sample means $\bar{x}_1$ and $\bar{y}_1$

$$(2.01) \qquad \bar{x}_1 = \sum_{i=1}^{m_0} x_i/m_0, \qquad \bar{y}_1 = \sum_{i=1}^{n_0} y_i/n_0,$$

and estimates of variances

$$(2.02) \quad s_1^2 = \sum_{i=1}^{m_0} (x_i - \bar{x}_1)^2/(m_0 - 1), \qquad s_2^2 = \sum_{i=1}^{n_0} (y_i - \bar{y}_1)^2/(n_0 - 1).$$

(2°) Having obtained these estimates, we shall make additional independent random samplings from each of our populations, which we denote by O'_{m-m_0} : $(x_{m_0+1}, x_{m_0+2}, \cdots\cdots, x_m)$ and O'_{n-n_0} : $(y_{n_0+1}, y_{n_0+2}, \cdots\cdots, y_n)$ respectively. Here m and n are defined in the following manners

$$(2.03) \qquad m = max\left\{\left[\frac{1}{k}\frac{m_0+n_0}{m_0}s_1^2\right]+1,\ m_0+1\right\}$$

$$(2.04) \qquad n = max\left\{\left[\frac{1}{k}\frac{m_0+n_0}{n_0}s_2^2\right]+1,\ n_0+1\right\},$$

where k is an assigned positive number whose meaning will be explained afterwards.

Since m and n depend upon s_1^2 and s_2^2 respectively, they are stochastic variables. We shall denote by $E_{m,n}$ the event that these stochastic variables become assigned values m and n respectively.

(3°) Now let us define a sequence $\{a_i\}$ such that

$$(2.051) \qquad \begin{aligned} &(1^\circ) \quad \sum_{i=1}^{m} a_i = 1 \\ &(2^\circ) \quad s_1^2 \sum_{i=1}^{m} a_i^2 = m_0 k/(m_0+n_0) \\ &(3^\circ) \quad a_1 = a_2 = \cdots\cdots = a_{m_0}. \end{aligned}$$

Similarly we may and we shall define a sequence $\{b_j\}$ such that

$$(2.052) \qquad \begin{aligned} &(1^\circ) \quad \sum_{j=1}^{n} b_j = 1 \\ &(2^\circ) \quad s_2^2 \sum_{j=1}^{n} b_j^2 = n_0 k/(m_0+n_0) \\ &(3^\circ) \quad b_1 = b_2 = \cdots\cdots = b_{n_0}. \end{aligned}$$

The existence of such sequences follows from the fact that, under the conditions (1°) and (3°) in (2.051), min. $s_1^2 \sum a_i^2 = m^{-1} s_1^2$ is not greater than $m_0 k (m_0 + n_0)^{-1}$, according to the definition of m in (2.03).

(4°) Let us introduce the statistics

(2.061) $$t_1 = \left(\sum_{i=1}^{m} a_i x_i - \xi_1\right) \Big/ s_1 \left\{\sum_{i=1}^{m} a_i^2\right\}^{1/2}$$

and

(2.062) $$t_2 = \left(\sum_{j=1}^{n} b_j y_j - \xi_2\right) \Big/ s_2 \left\{\sum_{j=1}^{n} b_j^2\right\}^{1/2}.$$

For the sake of simplicity, let us put

(2.07) $$c_1 = m_0^{1/2}/(m_0 + n_0)^{1/2}, \quad c_2 = -n_0^{1/2}/(m_0 + n_0)^{1/2}.$$

After these preparations, we shall now observe

Theorem 10.1. *Let τ_1 and τ_2 $(\tau_1 < \tau_2)$ be arbitrarily assigned real numbers.*

Then we have under the hypothesis $(1^\circ) \sim (4^\circ)$ *in this paragraph,*

(2.08) $$Pr.\{\tau_1 k^{-(1/2)} < c_1 t_1 + c_2 t_2 < \tau_2 k^{-(1/2)}\} = \iint\limits_{\tau_1 k^{-(1/2)} < c_1 t_1 + c_2 t_2 < \tau_2 k^{-(1/2)}} g_{m_0 - 1}(t_1)\, g_{n_0 - 1}(t_2)\, d t_1\, d t_2,$$

where $g_{\nu_i - 1}(t_i)$ $(\nu_1 = m_0, \nu_2 = n_0)$ mean the probability density functions of t-distribution with the $\nu_i - 1$ degrees of freedom, that are,

(2.09) $$g_{\nu-1}(t) = \frac{\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu-1}{2}\right)\sqrt{\nu - 1}\sqrt{\pi}} \left(1 + \frac{t^2}{\nu - 1}\right)^{-(\nu/2)}.$$

Proof : Since any possible values of m and n satisfy the conditions $m \geqq m_0 + 1$ and $n \geqq n_0 + 1$, and $E_{m,n}$ and $E_{m',n'}$ are mutually exclusive provided that $|m - m'| + |n - n'| > 0$, we can write

(2.10) $$Pr.\{\tau_1 k^{-(1/2)} < c_1 t_1 + c_2 t_2 < \tau_2 k^{-(1/2)}\} = \sum_{m=m_0+1}^{\infty} \sum_{n=n_0+1}^{\infty} Pr.\{E_{m,n}, k^{-)1/2)} \tau_1 < c_1 t_1 + c_2 t_2 < \tau_2 k^{-(1/2)}\}.$$

To calculate each summand of the right-hand side, let us notice the following assertion that (1°) for each assigned set of values of s_1 and s_2, the variables X and Y difined by

(2.11) $$X = \left(\sum_{i=1}^{m} a_i x_i - \xi_1\right) \Big/ \left\{\sum_{i=1}^{m} a_i^2\right\}^{1/2}, \quad Y = \left(\sum_{j=1}^{n} b_j y_j - \xi_2\right) \Big/ \left\{\sum_{j=1}^{n} b_j^2\right\}^{1/2}$$

are distributed according to $N(0, \sigma_1^2 s_1^2)$ and $N(0, \sigma_2^2 s_2^2)$ respectively and that (2°) X and Y are mutully independent.

To prove (1°) and (2°), let us first notice that $\{a_i\}$ $(i = 1, 2, \cdots\cdots, m)$, $\{b_j\}$ $(j = 1, 2, \cdots\cdots, n)$, m and n are certain functions of s_1 and s_2. Consequently we should write $a_i = a_i(s_1, s_2)$, $b_j = b_j(s_1, s_2)$, $m = m(s_1, s_2)$ and $n = n(s_1, s_2)$, and hence

$$(2.12) \qquad X = \sum_{i=m_0+1}^{m(s_1, s_2)} a_i(s_1, s_2)(\bar{x}_i - \xi_1)/s_1 \left\{ \sum_{i=1}^{m(s_1, s_2)} a_i(s_1, s_2) \right\}^{1/2}$$
$$- \left\{ \sum_{i=1}^{m} a_i(s_1, s_2) \right\} (\bar{x}_i - \xi_1)/s_1 \left\{ \sum_{i=1}^{m(s_1, s_2)} a_i^2(s_1, s_2) \right\}^{1/2},$$

according to the condition (3°) in (2.051), and similarly for Y. For each assigned set of values s_1 and s_2, $\{x_i\}$ $(i \geqq m_0 + 1)$, $\{y_j\}$ $(j \geqq n_0 + 1)$, $\bar{x}_1$ and $\bar{y}_1$ are mutually independently distibuted according to normal distribution.

Due to these facts direct calculations of means and variances yield us what was to be proved.

After these preparatory remarks, we can now give

$$(2.13) \qquad Pr. \{E_{m,n}, k^{-(1/2)}\tau_1 < c_1 t_1 + c_2 t_2 < k^{-(1/2)}\tau_2\}$$
$$= \iint_{E_{m,n}} f_{m_0-1}(s_1, \sigma_1) f_{n_0-1}(s_2, \sigma_2)\, ds_1\, ds_2$$
$$\cdot \iint_{\tau_1 < c_1 x + c_2 y < \tau_2} \frac{s_1 s_2}{2\pi \sigma_1 \sigma_2} exp \left\{ -\frac{1}{2} \left(\frac{s_1^2 x^2}{\sigma_1^2} + \frac{s_2 y^2}{\sigma_2^2} \right) \right\} dx\, dy,$$

where we put

$$(2.14) \quad f_{m_0-1}(s_1, \sigma_1) \equiv \frac{(m_0 - 1)^{\frac{m_0-1}{2}}}{2^{\frac{m_0-1}{2}-1} \Gamma\left(\frac{m_0-1}{2}\right) \sigma_1^{m_0-1}} s_1^{m_0-2} exp \left\{ -\frac{(m_0-1) s_1^2}{2\sigma_1^2} \right\}$$

and similarly for $f_{n_0-1}(s_2, \sigma_2)$.

Due to the special forms of the integral with respect to (x, y) which are independent of m and n, we may write, in view of (2.10),

$$(2.15) \qquad Pr. \{k^{-(1/2)}\tau_1 < c_1 t_1 + c_2 t_2 < k^{-(1/2)}\tau_2\}$$
$$= \int_0^\infty \int_0^\infty f_{m_0-1}(s_1, \sigma_1) f_{n_0-1}(s_2, \sigma_2)\, ds_1\, ds_2$$
$$\cdot \iint_{k^{-(1/2)}\tau_1 < c_1 x + c_2 y < k^{-(1/2)}\tau_2} \frac{s_1 s_2}{2\pi \sigma_1 \sigma_2} exp \left\{ -\frac{1}{2} \left(\frac{s_1^2 x^2}{\sigma_1^2} + \frac{s_2^2 y^2}{\sigma_2^2} \right) \right\} dx\, dy$$
$$= \iint_{k^{-(1/2)}\tau_1 < c_1 x + c_2 y < k^{-(1/2)}\tau_2} \left(\int_0^\infty \frac{s_1}{\sqrt{2\pi}\, \sigma_1} exp \left\{ -\frac{1}{2} \frac{s_1^2 x^2}{\sigma_1^2} \right\} f_{m_0-1}(s_1, \sigma_1)\, ds_1 \right)$$
$$\left(\int_0^\infty \frac{s_2}{\sqrt{2\pi}\, \sigma_2} exp \left\{ -\frac{1}{2} \frac{s_2^2 y}{\sigma_2^2} \right\} f_{n_0-1}(s_2, \sigma_2)\, ds_2 \right) dx\, dy,$$

which proves (2.08), rewriting x and y by t_1 and t_2 respectively.

Theorem 10.1 shows one of the inferential behaviours which corresponds to an axact application of two dimensional t-distributions. Here it is to be noted that

$$(2.16)\qquad c_1 t_1 + c_2 t_2 = \left\{\left(\sum_{i=1}^{m} a_i x_i - \sum_{j=1}^{n} b_j y_j - (\xi_1 - \xi_2\right)\right\} k^{-(1/2)},$$

and hence that we may make use of the relation (2.08) in statistical inferences about the population difference $\xi_1 - \xi_2$. To any assigned α, $0 < \alpha < 1$, and to each pair of m_0 and n_0, we can uniquely determine $t_{m_0-1,\, n_0-1}(\alpha)$ such that

$$(2.17)\qquad \iint_{D_{m_0,\, n_0}(\alpha)} g_{m_0-1}(t_1)\, g_{n_0-1}(t_2)\, d\,t_1\, d\,t_2 = \alpha,$$

where the domain of integration $D_{m_0,\, n_0}(\alpha)$ in (t_1, t_2) is defined by

$$(2.18)\quad D_{m_0,\, n_0}(\alpha) : | m_0^{1/2}(m_0+n_0)^{-(1/2)} t_1 - n_0^{1/2}(m_0+n_0)^{-(1/2)} t_2 | \geqq t_{m_0-1,\, n_0-1}(\alpha).$$

To give a connection between this inferential behaviour and Behrens-Fisher's test, we shall adopt approximate values for $\{a_i\}$ and $\{b_j\}$ by defining, for the set $E_{m,n}$,

$$(2.19)\qquad a_i = m^{-1},\quad b_j = n^{-1},\qquad (i = 1, 2, \cdots, m;\quad j = 1, 2, \cdots, n).$$

If we shall make use of exact solutions $\{a_i\}$ and $\{b_j\}$ satisfying (2.051) and (2.052), we obtain

$$(2.20)\quad c_1 \equiv m_0^{1/2}(m_0 + n_0)^{-(1/2)} = s_1 \left(\sum_{i=1}^{m} a_i^2\right)^{1/2} \left(s_1 \sum_{i=1}^{m} a_i^2 + s_2^2 \sum_{j=1}^{n} b_j^2\right)^{-(1/2)}$$

$$(2.21)\quad c_2 \equiv - n_0^{1/2}(m_0 + n_0)^{-(1/2)} = - s_2 \left(\sum_{j=1}^{n} b_j^2\right)^{1/2} \left(s_1 \sum_{i=1}^{m} a_i^2 + s_2^2 \sum_{j=1}^{n} b_j^2\right)^{-(1/2)}$$

Now the approximate solutions (2.19) will give us approximately

$$(2.22)\qquad c_1 \cong \frac{s_1}{m^{1/2}} \left(\frac{s_1^2}{m} + \frac{s_2^2}{n}\right)^{-(1/2)} = \cos\theta, \quad say,$$

$$(2.23)\qquad c_2 \cong - \frac{s_2}{n^{1/2}} \left(\frac{s_1^2}{m} + \frac{s_2^2}{n}\right)^{-(1/2)} = -\sin\theta$$

$$(2.24)\qquad c_1 t_1 + c_2 t_2 \cong t_1 \cos\theta - t_2 \sin\theta$$

$$(2.25)\qquad \sum_{i=1}^{m} a_i x_i = \bar{x}, \qquad \sum_{j=1}^{n} b_j y_j = \bar{y}.$$

On the other hand, since we have

$$(2.26)\qquad c_1 t_1 + c_2 t_2 = \left(\sum_{i=1}^{m} a_i x_i - \sum_{j=1}^{n} b_j y_j\right) k^{-(1/2)}$$

for exact solutions $\{a_i\}$ and $\{b_j\}$, we shall reach the following approximate relations:

$$\begin{aligned}(2.27)\qquad \alpha &= Pr.\{| c_1 t_1 + c_2 t_2 | \geqq t_{m_0-1,\, n_0-1}(\alpha)\}\\ &= Pr.\{| t_1 \cos\theta - t_2 \sin\theta | \geqq t_{m_0-1,\, n_0-1}(\alpha)\}\\ &= Pr.\{| \bar{x} - \bar{y} - (\xi_1 - \xi_2) | \geqq k^{1/2}\, t_{m_0-1,\, n_0-1}(\alpha)\}.\end{aligned}$$

Thus in order that we may obtain a confidence interval for the difference of the population means $\xi_1 - \xi_2$, with an approximate confidence-coefficient $1 - \alpha$ whose length is l, it will suffice us to define $k^{1/2} = l/2t_{m_0-1, n_0-1}(\alpha)$. It is useful to enunciate our results just obtained in the following

Theorem 10.2. *Let $O_{m_0} : (x_1, x_2, \cdots\cdots, x_{m_0})$ and $O_{n_0} : (y_1 y_2, \cdots\cdots, y_{n_0})$ be two independent random samples drawn from normal populations $N(\xi_1, \sigma_1^2)$ and $N(\xi_2, \sigma_2^2)$ respectively, where ξ_i and σ_i $(i = 1, 2)$ are unknown to us. Let us define $\bar{x}_1, \bar{y}_1, s_1^2, s_2^2$ and m and n as given in* (2.03) and (2.04). *Let $O'_{m-m_0} : (x_{m_0+1}, \cdots\cdots, x_m)$ and $O'_{n-m_0} : (y_{n_0+1}, \cdots\cdots, y_n)$ be two independent random samples defined as in* (2°). *Let us define $\bar{x}, \bar{y}$, t_1, t_2 and $\cos\theta$ (hence $\sin\theta$) by*

$$(2.28) \qquad \bar{x} = m^{-1}\sum_{i=1}^{m} x_i, \qquad \bar{y} = n^{-1}\sum_{j=1}^{n} y_j$$

$$(2.29) \qquad t_1 = m^{1/2}(\bar{x} - \xi_1)\, s_1^{-1}, \qquad t_2 = n^{1/2}(\bar{y} - \xi_2)\, s_2^{-1}$$

$$(2.30) \qquad \cos\theta = s_1 m^{-(1/2)} (s_1^2 m^{-1} + s_2^2 n^{-1}).$$

Then we have

$$(2.31) \quad Pr.\{|t_1 \cos\theta - t_2 \sin\theta| < \tau\}$$
$$= \sum_{m=m_0+1}^{\infty} \sum_{n=n_0+1}^{\infty} \iint_{E_{m,n}} f_{m_0-1}(s_1, \sigma_1) f_{n_0-1}(s_2, \sigma_2)\, ds_1 ds_2$$
$$\cdot \iint_{D_{m,n}(s_1, s_2)} \frac{m^{1/2} n^{1/2}}{2\pi\sigma_1\sigma_2} \exp\left\{-\frac{1}{2}\left(\frac{m(\bar{x}-\xi_1)^2}{\sigma_1^2} + \frac{n(\bar{y}-\xi_2)^2}{\sigma_2^2}\right)\right\} dx\, dy,$$

where $f_{\nu-1}$ $(\nu = m_0, n_0)$ and $E_{m,n}$ are as such defined in (2.14) *and* (2°) *respectively and the domain of integration in $(\bar{x}, \bar{y})$ is defined by*

$$(2.32) \qquad D_{m,n}(s_1, s_2) : |(\bar{x} - \xi_1) - (y - \xi_2)| < \tau (s_1^2 m^{-1} + s_2^2 n^{-1})^{1/2}.$$

Further we have an approximate evaluation to the effect that

$$(2.23) \qquad Pr.\{|t_1 \cos\theta - t_2 \sin\theta| < \tau\}$$
$$\cong \iint_{|c_1 t_1 + c_2 t_2| < \tau} g_{m_0-1}(t_1)\, g_{n_0-1}(t_2)\, dt_1 dt_2,$$

where c_1 and c_2 are such as defined in (2.07).

It will be sufficient to show how the approximate evaluation given in (2.33) may be obtained from (2.31).

For any m and n except when $m = m_0$ or $n = n_0$ we have, in $E_{m,n}$,

$$(2.34) \qquad m \geqq k^{-1} m_0^{-1}(m_0 + n_0)\, s_1^2 \geqq m - 1$$

$$(2.35) \qquad n \geqq k^{-1} n_0^{-1}(m_0 + n_0)\, s_2^2 \geqq n - 1,$$

which yield us that

$$(2.36) \qquad s_1^2 (m-1)^{-1} + s_2^2 (n-1)^{-1} \geqq k \geqq s_1^2 m^{-1} + s_2^2 n^{-1},$$

and that

$$(2.37) \qquad \begin{aligned} &\{k^{-1} m_0^{-1} (m_0 + n_0) s_1^2 + 1\} \sigma_1^{-2} (\bar{x} - \xi_1)^2 \\ &\quad + \{k^{-1} n_0^{-1} (m_0 + n_0) s_2^2 + 1\} \sigma_2^2 (\bar{y} - \xi_2)^2 \\ &\geqq m (\bar{x} - \xi_1)^2 \sigma_1^{-2} + n (\bar{y} - \xi_2)^2 \sigma_2^{-2} \\ &\geqq k^{-1} (m_0 + n_0) m_0^{-1} (\bar{x} - \xi_1)^2 \sigma_1^{-2} \\ &\quad + k^{-1} (m_0 + n_0) n_0^{-1} (\bar{y} - \xi_2)^2 \sigma_2^2 . \end{aligned}$$

Let us now define X and Y by

$$(2.38) \qquad \begin{aligned} X &= k^{-(1/2)} m_0^{-(1/2)} (m_0 + n_0)^{1/2} (\bar{x} - \xi_1), \\ Y &= k^{-(1/2)} n_0^{-(1/2)} (m_0 + n_0)^{1/2} (\bar{y} - \xi_2), \end{aligned}$$

and let us replace $s_1^2 m^{-1} + s_2^2 n^{-1}$ and $m (\bar{x} - \xi_1)^2 \sigma_1^{-2} + n (\bar{y} - \xi_2)^2 \sigma_2^{-2}$ by k and the last terms in (2.37) respectively for all $m \geqq m_0$ and $n \geqq n_0$. Then we shall readily reach (2.33), which was to be proved.

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