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# CONVERGENCE OF INTEGRAL AND ITS APPLICATIONS

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## 1. Introduction

It is a matter of common knowledge that if $f(t)$ be a continuous function defined over the closed interval $0 \leqq t \leqq l$, the following sum (1) converges to a finite value as $\max(t_k - t_{k-1}) \to 0$.

$$(1) \qquad S_{(1)} = \sum_{k=1}^{n} f(\xi_k)(t_k - t_{k-1}) ,$$

where $t_0 = 0$, $t_n = l$ and $n-1$ $t$-points are randomly chosen in the interval $0 \leqq t \leqq l$, and $\xi_k$ is randomly drawn from the sub-interval $t_{k-1} \leqq \xi_k \leqq t_k$.

The random variable defined by the sum (1) will converge in probability as merely $n \to \infty$. We will investigate the aspect of convergence and its applications will be described.

## 2. Mean value

Since the random variables $(t_1, t_2, \cdots, t_{n-1})$ and $(\xi_1, \xi_2, \cdots, \xi_n)$ are mutually independent and $(t_1, t_2, \cdots, t_{n-1})$ are the order statistics, the mean value of random variable defined by the sum (1) is given by the integral

$$(2) \quad E(S_{(1)}) = \frac{(n-1)!}{l^{n-1}} \int \cdots\cdots \int_{0 \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_{n-1} \leqq l} \left( \prod_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{d\xi_k}{t_k - t_{k-1}} \right) \times \left[ \sum_{k=1}^{n} (t_k - t_{k-1}) f(\xi_k) \right] dt_1\, dt_2 \cdots\cdots dt_{n-1} .$$

This integration is easily performed and its result is given by

$$(3) \qquad E(S_{(1)}) = \int_0^l f(t)\, dt .$$

This equation shows that the sum (1) is an unbiased estimator.

## 3. Variance

Similary the variance of random variable (1) is given by

$$(4) \quad var.(S_{(1)}) = \frac{(n-1)!}{l^{n-1}} \int \cdots\cdots \int_{0 \leqq t_1 \leqq \cdots \leqq t_{n-1} \leqq l} \prod_{k=1}^{n} \left( \int_{t_{k-1}}^{t_k} \frac{d\xi_k}{t_k - t_{k-1}} \right) \times \left[ \sum_{k=1}^{n} (t_k - t_{k-1}) \{ f(\xi_k) \}^2 + \sum_{i \neq j} (t_i - t_{i-1})(t_j - t_{j-1}) f(\xi_i) f(\xi_j) \right]$$

$$\times\, dt_1\, dt_2 \cdots\cdots dt_{n-1} - \left\{ \int_0^l f(t)\, dt \right\}^2$$

$$= \frac{(n-1)!}{l^{n-1}} \int \cdots\cdots\cdots\cdots\cdots \int_{0 \leqq t_1 \leqq \cdots \leqq t_{n-1} \leqq l} \sum_{k=1}^{n} \left[ (t_k - t_{k-1}) \int_{t_{k-1}}^{t_k} \{ f(t) \}^2 dt \right.$$

$$\left. - \left\{ \int_{t_{k-1}}^{t_k} f(t)\, dt \right\}^2 \right] dt_1\, dt_2 \cdots\cdots dt_{n-1},$$

but in this case, under the condition that $f(t)$ is continuous, it is difficult that the integral is led to a simple formula. But if $f(t)$ be differentiable, we shall have

$$(5) \qquad var.(S_{(1)}) = 2 \left\{ \frac{1}{l} \int_0^l \left( \frac{df}{dt} \right)^2 dt \right\} \frac{l}{\left( \frac{n}{l} \right)^3} + O\left( \frac{1}{n^4} \right).$$

Hence the variance of random variable (1) is proportional to the average of square of tangent of the function $f(t)$ and the length of interval $l$, and inversely proportional to the cube of number of the divided strata per unit length.

## 4. A special case

Here, we will consider a special case such that the interval $0 \leqq t \leqq l$ is divided into $n$ strata of the equal length $l/n$ by $t_k\ (k = 1, 2, \cdots\cdots, n-1)$. A sample $(\xi_1, \xi_2, \cdots\cdots, \xi_n)$ is such that each $\xi_k$ is randomly drawn from the sub-interval $(k-1)\frac{l}{n} \leqq t \leqq k\frac{l}{n}$. Thus we shall have the finite sum of the following form corresponding to the equation (1)

$$(6) \qquad S_{(2)} = \frac{l}{n} \sum_{k=1}^{n} f(\xi_k).$$

Also in this case, its mean value is the unbiased estimator[1], and when $f(t)$ be differentiable, then the variance is given by

$$(7) \qquad var.(S_{(2)}) = \frac{1}{12} \left\{ \frac{1}{l} \int_0^l \left( \frac{df}{dt} \right)^2 dt \right\} \frac{l}{\left( \frac{l}{n} \right)^3} + O\left( \frac{1}{n^5} \right).$$

It is worthly of notice that the equations (7) and (5) are the similar formula.

## 5. Some applications

The errors in the integrator, correlator etc. will be estimated by the results of (3), (5) etc. and this paper deals with integrator which was just made at our laboratory. The integration by this computor is carried out in such a way that the integrand is substituted by a step function, hence it appears that the integration is just the evaluation of the $S_{(1)}$ (eq. (1)), but strictly speaking, the calculation is performed in such a way that the point $t$ is drawn sequentially, so it should be treated as process. If the random variable $t$ is Poisson's process, we shall have the same formulae with eq. (3) and (5) as follows.

Since $t$ is the Poisson's process, we shall have

$$P_r\,(t_k \leqq t_k \leqq t_k + dt_k \,|\, t_{k-1}) = a\, e^{-a(t_k - t_{k-1})}\, dt_k\,,$$

where $a$ is the mean value of the points taken in a unit length. When a sample $(t_1, t_2, \cdots\cdots, t_{n-1})$ was taken in the interval $0 \leqq t \leqq l$ and $t'_n$ in the interval $l \leqq t' \leqq \infty$, the integration is performed.

While, it may be permissible that $\xi$ is mutually independent of $t$ in our case.

Thus we shall have the following equation corresponding to equation (2)

$$(8) \qquad a^n \int \cdots\cdots\cdots\cdots \int_{0 \leqq t_1 \leqq \cdots \leqq t_{n-1} \leqq l \leqq t'_n < \infty} \left[ \prod_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{d\,\xi_k}{t_k - t_{k-1}} \right] \times \left[ \sum_{k=1}^{n} (t_k - t_{k-1})\, f(\xi_k) \right] dt_1\, dt_2 \cdots\cdots dt_{n-1}\, dt'_n\,.$$

And by comparison with equation (8) and (2), we shall easily have

$$(9) \qquad \frac{(a\,l)^{n-1}}{(n-1)\,!}\, e^{-al} \int_0^l f(t)\, dt\,,$$

but in this case we must treat so as $n$ is a random variable, then the mean value is

$$(10) \qquad E\,(S_{(1)}) = \sum_{n=1}^{\infty} \frac{(a\,l)^{n-1}}{(n-1)\,!}\, e^{-al} \int_0^t f(t)\, dt = \int_0^l f(t)\, dt\,.$$

Simiraly, if the integrand $f(t)$ be differentiable, we shall have

$$(11) \quad var\,(S_{(1)}) = 2 \left\{ \frac{1}{l} \int_0^l \left( \frac{df}{dt} \right)^2 dt \right\} l^4 \sum_{n=1}^{\infty} \frac{1}{n^3}\, \frac{(a\,l)^{n-1}}{(n-1)\,!}\, e^{-al} + O\left( \frac{1}{n^4} \right) = 2 \left\{ \frac{1}{l} \int_0^l \left( \frac{df}{dt} \right)^2 dt \right\} l^4 E\left[ \left( \frac{1}{n+1} \right)^3 \right] + O\left( \frac{1}{n^4} \right),$$

and since we can estimate the value of $\{\quad\}$ in the equation (11) by an appropriate expedient, the estimation of error in the integrator will be performed numerically.

## References

( 1 ) Tosio KITAGAWA: Random Integrations, Bull. Math. Stat. Vol. 4, No. 1 ~ 2, (1950).

Added in proof: To prove (5) and (7), we have assumed that $f(t)$ be analytic in $0 \leq t \leq l$, and expansible in a power series of t whose convergence-radius $R$ is greater than $l$:

$$f(t) = \sum_{k=0}^{\infty} a_n t^n, \qquad |t| < R.$$