On the statistical inferences in finite populations by two sample theory

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He has introduced following assumptions in dealing with these problems:

**Assumption I.** Considering a grand population II, the finite population \((N)\) in his consideration can be recognized as random sample of size \(N\) drawn independently from II.

**Assumption II.** The finite population \((N)\), as a sample from the grand population II, can be recognized as consisting of two independent random samples \(O_1 : (x_1, \ldots, x_n)\) and \(O_2 : (y_1, \ldots, y_{N-n})\) of sizes \(n\) and \(N-n\) respectively, which are drawn from the grand population II.

**Assumption III.** The grand population II is distributed normally according to \(N(\bar{\mu}, \sigma^2)\).

Putting

\[
(x.01) \quad \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}, \quad s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}
\]

\[
(1.02) \quad \bar{y} = \frac{\sum_{j=1}^{N-n} y_j}{(N-n)}, \quad s'^2 = \frac{\sum_{j=1}^{N-n} (y_j - \bar{y})^2}{(N-n)}
\]

\[
(1.03) \quad \bar{x} = \left\{ n \bar{x} + (N - n) \bar{y} \right\} / N, \quad S^2 = \left\{ \frac{n}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 + \frac{N-n}{n} \sum_{j=1}^{N-n} (y_j - \bar{x})^2 \right\} / N,
\]

what we have known from actual sampling are some or all of \((1.01)\), and what we want to infer are some or all of \((1.03)\). The statistics \((1.02)\) are auxiliary ones which are indispensable in his formulation, but what are unknown to us.

T. Kitagawa\(^{(1)}\) developped following theorems, which will be of use in giving confidence intervals associated with statistical inferences about finite populations:

**Theorem 1.1.** Under the Assumptions I, II and III, for any assigned \(\alpha\), \(0 < \alpha < 1\), the confidence interval for \(\bar{x}\) with confidence-coefficient \(1 - \alpha\), which takes the form of \((\bar{x} - As, \bar{x} + As)\) is given by

\(^{(1)}\) Prof. Kitagawa informed me after the preparation of this paper that he found the same result as to Theorem 1.1 was already established by E. S. Pearson \([3]\) which was not accessible in Japan until 1951.
Where $t_{n-1}(\alpha)$ means $\alpha$-significance level of $t$-distribution with $n - 1$ degrees of freedom such that $\Pr\{ \lvert t \rvert > t_{n-1}(\alpha) \} = \alpha$.

Theorem 1.2. Under the same hypothesis to Theorem 1.1, the confidence interval for $S^2$ by means of $s^2$ with confidence-coefficient $1 - \alpha$, which takes the form of $(n/N)s^2$, $B s^2$, is given by

\begin{equation}
B = \frac{n}{N} \left\{ 1 + \frac{N - n}{n - 1} F_{n-1}^{N-n}(\alpha) \right\},
\end{equation}

where $F_{n-1}^{N-n}(\alpha)$ means $\alpha$-significance level of $F$-distribution with pair of degrees of freedom $N - n$ and $n - 1$ such that $\Pr\{ F > F_{n-1}^{N-n}(\alpha) \} = \alpha$.

He extends these two theorems to various directions. For example, the confidence interval for the differences of two means $\bar{x}_1$ and $\bar{x}_2$, and those for the ratios of two variances $S_1^2$ and $S_2^2$ of two finite populations can be obtained in quite similar ways. For this purpose he considers two sets of two independent samples

$O^{(1)}_i : (x_{i1}, \ldots, x_{in_i})$ and $O^{(2)}_i : (y_{i1}, \ldots, y_{iN_i})$, $(i = 1, 2)$

drawn respectively from the population $\Pi_i$, and he puts for $i = 1, 2$

\begin{align}
(1.06) \quad \bar{x}_i &= \frac{\sum_{j=1}^{n_i} x_{ij}}{n_i}, \quad s_i = \frac{\sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2}{n_i}, \\
(1.07) \quad \bar{y}_i &= \frac{\sum_{k=1}^{N_i} y_{ik}}{(N_i - n_i)}, \quad s_i^2 = \frac{\sum_{k=1}^{N_i} (y_{ik} - \bar{y}_i)^2}{(N_i - n_i)}, \\
(1.08) \quad \xi_i &= \left\lceil n_i \bar{x}_i + (N_i - n_i) \bar{y}_i \right\rceil /N_i, \\
(1.09) \quad S_i^2 &= \left\{ \frac{\sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2 + \sum_{k=1}^{N_i - n_i} (y_{ik} - \bar{y}_i)^2}{N_i} \right\} /N_i.
\end{align}

As in the case of one system of two samples, following theorems(2) give him confidence intervals associated with statistical inferences about two finite populations:

Theorem 1.3. Under the Assumptions I, II and III, when we have $\sigma_1^2 = \sigma_2^2$, the confidence interval for $\bar{x}_2 - \bar{x}_1$ by means of $\bar{x}_1$, $\bar{x}_2$, $s_1^2$ and $s_2^2$ with confidence-coefficient $1 - \alpha$, which takes the form of $(\bar{x}_2 - \bar{x}_1 - As, \bar{x}_2 - \bar{x}_1 + As)$, where

\begin{equation}
s^2 = (n_1 s_1^2 + n_2 s_2^2) / (n_1 + n_2)
\end{equation}

is given by

(2) See Kitagawa [1] Theorems 6.3 and 6.4. Slight misprints concerning $s^2$, $b_i$ and $c_i$ $(i = 1, 2)$ should be corrected as we enunciate in (1.10) (1.13) and (1.14).
(1.11) \[ A = \frac{t_{m+n_2-2}(\alpha)}{\sqrt{n_1+n_2-2}} \sqrt{\frac{n_1}{n_1} \cdot \frac{N_1-n_1}{N_1} + \frac{n_1+n_2}{n_2} \cdot \frac{N_2-n_2}{N_2}}. \]

**Theorem 1.4.** Under the Assumptions I, II and III, the confidence interval for the ratio \( S_2^2/S_1^2 \) by means of \( s_1^2 \) and \( s_2^2 \) with confidence-coefficient \( 1 - \alpha \), which takes the form of \( B \left( \frac{s_2^2}{s_1^2} \right), C \left( \frac{s_2^2}{s_1^2} \right) \) can be given by

\[ 1 - \alpha = \Pr\left\{ B \left( \frac{s_2^2}{s_1^2} \right) < \frac{S_2^2}{S_1^2} < C \left( \frac{s_2^2}{s_1^2} \right) \right\} \]

\[ = \int h_{N_2-n_2-1}(G) h_{N_1-n_1, n_1-1}(L) dG dL, \]

where \( h_{i}(H) \) means the F-distribution with degrees of freedom \((i, j)\) and he has put

\[ b_1 = B \cdot \frac{n_1 N_2}{n_2 N_1} \cdot \frac{N_1-n_1}{N_2-n_2} \cdot \frac{n_2-1}{n_1-1}, \quad b_2 = \frac{n_2-1}{n_2-n_2} \left( \frac{n_1 N_2}{n_2 N_1} B - 1 \right) \]

\[ c_1 = C \cdot \frac{n_1 N_2}{n_2 N_1} \cdot \frac{N_1-n_1}{N_2-n_2} \cdot \frac{n_2-1}{n_1-1}, \quad c_2 = \frac{n_2-1}{n_2-n_2} \left( \frac{n_1 N_2}{n_2 N_1} C - 1 \right). \]

In this paper we shall study the effects of non-normality, when we remove the normality assumption III in these theorems.

\section{2. The effects of non-normality.}

Instead of the Assumption III of normality, we shall put the following

**Assumption III'.** The grand population II is distributed according to the Gram-Charlier Type A, that is, according to

\[ f(x) \, dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \{ 1 + a_3 H_3(z) \}
\]

\[ + a_4 H_4(z) + a_5 H_5(z) + a_6 H_6(z) \} \, dz, \]

where \( z = (x - \bar{x})/\sigma \) is the standardized variate, and we have put

\[ a_3 = \frac{\mu_3}{3! \sigma^3} = \frac{1}{6} \beta_1, \quad a_4 = \frac{1}{4} \left( \frac{\mu_4}{\sigma^4} - 3 \right) = \frac{1}{24} \beta_2 - 3 \]

\[ a_5 = \frac{\mu_5 - 10 \mu_2 \mu_3}{120 \sigma^5}, \quad a_6 = \frac{1}{720} (\beta_4 - 15 \beta_2 + 30). \]

The joint distribution of \( \bar{x} \) and \( s \) drawn from this population was obtained in the previous paper [2]. By the use of the result and following Kitagawa's formulation, we can derive the following

**Theorem 2.1.** Under the Assumptions I, II and III', and neglecting the higher powers of \( a_3, a_4, a_5 \) and \( a_6 \), the distribution of \( \tau = \frac{\bar{x} - \bar{x}}{s} \sqrt{\frac{N}{N-n}} \) is given as follows:
for any assigned \( \tau_0 (\geq 0) \), putting \( u = (1 + \tau_0^2)^{-1} \) and \( m = N - n \),

\[
(2.03) \quad \Pr \{ | \tau | > \tau_0 \} = \frac{1}{2} \left[ 1 + \frac{1}{2} \left( \beta_4 - 15\beta_2 + 30 \right) P_6(u) + \beta_1 P_3,3(u) \right],
\]

\[
(2.04) \quad \Pr \{ | \tau | > \tau_0 \} = \frac{1}{2} \Pr \{ | \tau | > \tau_0 \} + a_2 P_3(u) + a_3 P_5(u),
\]

where \( I_u(p, q) \) means the ratio of the Incomplete Beta Functions \( I_u(p, q) = B_u(p, q)/B(p, q) \), and

\[
(2.05) \quad P_4(u) = -\frac{n-1}{24n m N} \left[ n \left( 2m^2 - 2nm - n(n-3) \right) I_u \left( \frac{n-1}{2}, \frac{1}{2} \right) 
- 2 \left( 2n+4 \right) m^2 - 2n(n-1) m - n^2(n-1) \right] I_u \left( \frac{n+1}{2}, \frac{1}{2} \right) 
+ \left\{ (2n+8) m^2 - 2n(n-2) m - n^2(n+1) \right\} I_u \left( \frac{n+3}{2}, \frac{1}{2} \right)
\]

\[
(2.06) \quad P_6(u) = \frac{n-1}{72n^2 m^2 N^3} \left[ n \left( 8(2n-1) m^2 + 30n(n-1) m^4 
- 15n^2(n-1)^2 m^2 + n^4(n-3)(n-5) \right) I_u \left( \frac{n-1}{2}, \frac{1}{2} \right) 
- 3(n-1) \left( 16(n+2) m^2 + 10n(3n+8) m^4 
+ 60n^2 m^3 - 15n^2(n-1) m^2 + n^5(n-3) \right) I_u \left( \frac{n+1}{2}, \frac{1}{2} \right) 
+ 3 \left( 8(n+4)(2n-3) m^2 + 10n(3n^2+13n-20) m^4 
+ 120n^2(n-1) m^3 - 15n^2(n-1)^2 m^2 + n^5(n-1) \right) I_u \left( \frac{n+3}{2}, \frac{1}{2} \right) 
- 16(n+6)(n-2) m^5 + 30n(n^2+7n-12) m^4 + 180n^2(n-1) m^3 
- 15n^3(n-1)^2 m^2 + n^5(n+1)(n+3) \right\} I_u \left( \frac{n+5}{2}, \frac{1}{2} \right)\]

\[
(2.07) \quad P_{3,3}(u) = -\frac{n-1}{72n^2 m^2 N^3} \left[ n \left( 2(2n^2-3n+4) m^2 + 6n(2n^2-4n+5) m^4 
+ n^2(13n^2-26n+15) m^3 + 3n^2(2n^3-5n^2+5) m^2 
+ n^4(n-3)(n-5)(m-1) \right) I_u \left( \frac{n-1}{2}, \frac{1}{2} \right) 
- 3 \left( 2(2n^2-3n+16) m^2 + 2n(6n^3-8n^2-5n+40) m^4 
+ n^2(13n^3-6n^2-25n+60) m^3 + n^4(6n^3+n^2-16n+15) m^2 
+ n^5(n-1)(n-3)(m-1) \right) I_u \left( \frac{n+1}{2}, \frac{1}{2} \right)\]
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\[\begin{align*} +3 \cdot 2^2 (2n^3 - 3n^2 - 8n + 48) m^5 + 2n (6n^3 - 4n^2 - 33n + 100) m^4 \\
+ n^2 (13n^3 + 14n^2 - 65n + 120) m^3 + n^3 (6n^3 + 17n^2 - 16n + 15) m^2 \\
+n^4 (n^2 - 1) (m - 1) \cdot I_u \left( \frac{n + 3}{2} , \frac{1}{2} \right) \\
- \{ 2 (2n^3 - 3n^2 - 20n + 96) m^5 + 6n (2n^3 - 23n + 60) m^4 \\
+ n^2 (13n^3 + 34n^2 - 105n + 180) m^3 + 3n^3 (2n^3 + 11n^2 + 5) m^2 \\
+n^4 (n + 1) (n + 3) (m - 1) \cdot I_u \left( \frac{n + 5}{2} , \frac{1}{2} \right) \}
\end{align*}\]

\[\begin{align*}
(2.08) \quad P_3(u) &= -\frac{1}{\sqrt{2\pi n m N}} \left\{ (2n - 1) m + n (n - 2) \cdot I_u \left( \frac{n - 1}{2} , 1 \right) \\
- (n - 1) (2m + n) \cdot I_u \left( \frac{n + 1}{2} , 1 \right) \right\}
\end{align*}\]

\[\begin{align*}
(2.09) \quad P_5(u) &= \frac{1}{\sqrt{2\pi n m^3 N^5}} \left\{ 3 (2n^2 - 2n + 1) m^4 + 10n (n - 1) (2n - 1) m^3 \\
+ 15n^2 (n - 1)^2 m^2 - n^4 (n - 2) (n - 4) \cdot I_u \left( \frac{n - 1}{2} , 1 \right) \\
- 2 (n - 1) \cdot 6 (n - 2) m^4 + 5n (4n - 5) m^3 + 15n^2 (n - 1) m^2 \\
- n^4 (n - 2) \cdot I_u \left( \frac{n + 1}{2} , 1 \right) \\
+ (n - 1) \cdot 6 (n - 4) m^4 + 20n (n - 2) m^3 \\
+ 15n^2 (n - 1)^2 m^2 - n^4 (n + 1) \cdot I_u \left( \frac{n + 3}{2} , 1 \right) \right\}
\end{align*}\]

**Corollary.** Under the same hypothesis to Theorem 2.1, if we use the same confidence interval for \( \bar{x} \) obtained by Theorem 1.1, the confidence-coefficient is approximately given by

\[\begin{align*}
(2.10) \quad 1 - \alpha - (\beta_2 - 3) P_4(u_0) - \frac{1}{10} (\beta_4 - 15\beta_2 + 30) P_6(u_0) - \beta_1 P_{3,3}(u_0),
\end{align*}\]

where we have put \( u_0 = \left\lfloor 1 + \frac{t_{n-1}^2(\alpha)}{n-1} \right\rfloor \).

The evaluation of the values of \( P_4(u_0) \), \( P_6(u_0) \) and \( P_{3,3}(u_0) \) can easily be done by the use of the PEARSON's Tables of the Incomplete Beta Functions. For examples in the case \( \alpha = 0.05 \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( n )</th>
<th>( P_4(u_0) )</th>
<th>( P_6(u_0) )</th>
<th>( P_{3,3}(u_0) )</th>
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in the case $\alpha = 0.01$

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The effects of non-normality seems to be small in these cases.

**Theorem 2.2.** Under the Assumptions I, II and III', and neglecting the higher powers of $a_3$, $a_4$, $a_5$ and $a_6$, the distribution of

$$F = \left\{ s^2 + \frac{n}{N} (\bar{x} - \bar{y})^2 \right\} \cdot \frac{n-1}{n s^2}$$

is given as follows:

For any assigned $F_0 (\geq 0)$, putting $u = \left(1 + \frac{N-n}{n-1} F_0\right)^{-1}$

$$\Pr. \{ F > F_0 \} = I_u \left(\frac{n-1}{2}, \frac{N-n}{2}\right) + (\beta_2 - 3) Q_4(u)$$

$$+ \frac{1}{10} (\beta_4 - 15 \beta_2 + 30) Q_6(u) + \beta_1 Q_{5,3}(u),$$

which turns to

$$\Pr. \left\{ \frac{n}{N} \leq \frac{S^2}{s^2} \leq \frac{n}{N} \left(1 + \frac{N-n}{n-1} F_0\right) \right\}$$

$$= 1 - \left\{ I_u \left(\frac{n-1}{2}, \frac{N-n}{2}\right) + (\beta_2 - 3) Q_4(u) \right.$$  

$$+ \frac{1}{10} (\beta_4 - 15 \beta_2 + 30) Q_6(u) + \beta_1 Q_{5,3}(u) \right\},$$

where

$$Q_4(u) = \frac{n-1}{8nN(N-n+2)} \left[ (n-1)N^2 - 2nN + (n+1) \right]$$

$$+ (n+1) \left\{ I_u \left(\frac{n-1}{2}, \frac{N-n}{2}\right) - 2 \right\}$$

$$\frac{1}{N} (n-1)N^2 - 2N - (n+1) \right\} I_u \left(\frac{N-n}{2}, \frac{N-n}{2}\right)$$

$$+ \frac{1}{N} \right\} I_u \left(\frac{n+3}{2}, \frac{N-n}{2}\right) \right]\]$$

$$Q_6(u) = \frac{5(n-1)}{24n^2N(N-n+2)(N-n+4)} \left[ (N-1)!(n-1)^2N^3 \right.$$  

$$- 2(n-1)^2(n-2)N^2 + (4n^3 - 13n^2 - 2n + 3)N$$

$$\]$$
\[-2n (n + 1)(n - 3) \{ I_u \left( \frac{n - 1}{2}, \frac{N - n}{2} \right) \}
\]
\[-3 (n - 1)(N + 1) \{ (n - 1) N^3 - 2 (n^2 - 2n + 3) N^2 \}
\]
\[+ (4n^2 - 3n - 9) N - 2n (n + 1) \{ I_u \left( \frac{n + 1}{2}, \frac{N - n}{2} \right) \}
\]
\[+ 3 \{(n - 1)^2 N^4 - (2n^3 - 7n^2 + 16n - 11) N^3 \}
\]
\[- (2n^3 - 11n^2 + 40n - 39) N^2 + (10n^3 - 7n^2 - 32n + 45) N
\]
\[- 6n (n + 1)^2 \{ I_u \left( \frac{n + 3}{2}, \frac{N - n}{2} \right) \}
\]
\[- \{(n - 1)^2 N^4 - (2n^3 - 7n^2 + 20n - 15) N^3 \}
\]
\[- (6n^3 - 15n^2 + 64n - 71) N^2 + (18n^3 + 17n^2 - 76n + 105) N
\]
\[- 10n (n + 1)(n + 3) \{ I_u \left( \frac{n + 5}{2}, \frac{N - n}{2} \right) \}
\]

(2.15) \[Q_{3, 3}(u) = \frac{24n^2 N^2 (N - n + 2)(N - n + 4)}{(N - 1)(3n^2
\]
\[- 6n + 5) N^3 - (3n^3 - 18n^2 + 25n - 20) N^2 + (15n^3 - 49n^2
\]
\[+ 11n + 15) N - 10n (n + 1)(n - 3) \{ I_u \left( \frac{n - 1}{2}, \frac{N - n}{2} \right) \}
\]
\[- 3 (N + 1) \{(3n^2 - 6n + 5) N^3 - (3n^2 - 16n^2 + 29n - 30) N^2
\]
\[+ (17n^3 - 23n^2 - 15n + 45) N - 10n (n^2 - 1) \{ I_u \left( \frac{n + 1}{2}, \frac{N - n}{2} \right) \}
\]
\[- 3 \{(3n^2 - 6n + 5) N^4 - (3n^3 - 23n^2 + 51n - 55) N^3
\]
\[+ 5 (2n^3 + 9n^2 - 28n + 39) N^2 + (47n^3 - 11n^2 - 133n + 225) N
\]
\[- 30n (n + 1)^2 \{ I_u \left( \frac{n + 3}{2}, \frac{N - n}{2} \right) \}
\]
\[- \{(3n^2 - 6n + 5) N^4 - (3n^3 - 27n^2 + 67n - 75) N^3
\]
\[+ (6n^3 + 89n^2 - 252n + 355) N^2 + 5 (19n^3 + 21n^2 - 73n + 105) N
\]
\[- 50n (n + 1)(n + 3) \{ I_u \left( \frac{n + 5}{2}, \frac{N - n}{2} \right) \}
\]

\[Q_{3, 3}(u) = \frac{N - n}{n - 1} \left\{ 1 + \frac{N - n}{n - 1} F_{n-1}^{s-n}(\alpha) \right\}^{-1}
\]

Corollary. Under the same hypothesis to Theorem 2.2, if we use the same confidence interval for \( S^2 \) obtained by Theorem 1.2, the confidence-coefficient is approximately given by

\[1 - \alpha - (\beta_2 - 3) Q_1(u_3) - \frac{1}{10} (\beta_4 - 15\beta_2 + 30) Q_6(u_6) - \beta_1 Q_{3, 3}(u_3),
\]

where we have put \( u_3 = \left\{ 1 + \frac{N - n}{n - 1} F_{n-1}^{s-n}(\alpha) \right\}^{-1} \).
in the case $\alpha = 0.05$

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<th>$N = 99$, $n = 5$</th>
<th>$Q_4(\mu_0)$</th>
<th>$Q_6(\mu_0)$</th>
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in the case $\alpha = 0.01$

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<td>$N = 100$, $n = 16$</td>
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<td>-0.0080</td>
<td>0.0057</td>
</tr>
<tr>
<td>$N = 99$, $n = 25$</td>
<td>0.0116</td>
<td>-0.0082</td>
<td>0.0067</td>
</tr>
</tbody>
</table>

In these cases, the effects of non-normality due to kurtosis seems to be too large to neglect.

Following Kitagawa's Theorem 1.3 which gives the confidence interval for the difference of two means $\bar{x}_1$ and $\bar{x}_2$ that are recognized the population means of two finite populations, by means of sample means $\bar{x}_1$, $\bar{x}_2$ and sample variances $s_1^2$ and $s_2^2$, we can derive the following

**Theorem 2.3.** Under the Assumptions I, II and III', when we have

$$\sigma_1^2 = \sigma_2^2 = \sigma^2$$

neglecting the higher powers of $a_3^{(o)}$, $a_4^{(o)}$, $a_5^{(o)}$ and $a_6^{(o)}$, ($i = 1, 2$), the distribution of

$$\tau = \bar{x}_2 - \bar{x}_1 - (\bar{x}_2 - \bar{x}_1) / s$$

is given as follows:

for any assigned $\tau_0 (\geq 0)$, putting $u = (1 + \tau_0^2)^{-1}$

$$\text{Pr.} | | \tau | | > \tau_0 \approx L_u \left( \frac{n_1 + n_2 - 2}{2}, \frac{1}{2} \right) + \frac{\mu_3^{(1)} \mu_3^{(2)}}{\sigma^6} \text{Q}_{3,3}(\mu_0)(u)$$

$$+ \sum_{i=1}^{2} (\beta_2^{(i)} - 3) \text{P}_4^{(i)}(u) + \sum_{i=1}^{2} \frac{1}{10} (\beta_4^{(i)} - 15 \beta_2^{(i)} + 30) \text{P}_6^{(i)}(u)$$

$$+ \sum_{i=1}^{2} \beta_4^{(i)} \text{P}_3^{(i)}(u),$$

where we have put $s^2 = (n_1 s_1^2 + n_2 s_2^2) / (n_1 + n_2)$

$$\text{Pr.} \left| \text{I}_u \left( \frac{n_1 + n_2 - 2}{2}, \frac{1}{2} \right) + \text{I}_u \left( \frac{n_1 + n_2 + 2}{2}, \frac{1}{2} \right) \right| \leq 2 \cdot \text{Q}_4(\mu_0)(u)$$

$$- 2 \cdot \text{Q}_6(\mu_0)(u)$$

$$+ 2 \cdot \text{Q}_{2,3}(\mu_0)(u)$$

$$+ \sum_{i=1}^{2} \beta_2^{(i)} \text{P}_3^{(i)}(u)$$

$$+ \sum_{i=1}^{2} \beta_4^{(i)} \text{P}_3^{(i)}(u),$$

$$\frac{1}{24} \left[ \text{I}_u \left( \frac{n_1 + n_2 - 2}{2}, \frac{1}{2} \right) + \text{I}_u \left( \frac{n_1 + n_2 + 2}{2}, \frac{1}{2} \right) \right] \text{I}_u \left( \frac{n_1 + n_2}{2}, \frac{1}{2} \right),$$

$$- 2 \cdot \text{Q}_6(\mu_0)(u) + 2 \cdot \text{Q}_{2,3}(\mu_0)(u)$$

$$+ \sum_{i=1}^{2} \beta_2^{(i)} \text{P}_3^{(i)}(u)$$

$$+ \sum_{i=1}^{2} \beta_4^{(i)} \text{P}_3^{(i)}(u),$$

$$\frac{1}{24} \left[ \text{I}_u \left( \frac{n_1 + n_2 - 2}{2}, \frac{1}{2} \right) + \text{I}_u \left( \frac{n_1 + n_2 + 2}{2}, \frac{1}{2} \right) \right]$$
Statistical Inferences in Finite Populations by Two Sample Theory

\[ -B_4^{(i)}\left(\frac{n_1 + n_2 - 2}{2}\right) I_u\left(\frac{n_1 + n_2 - 2}{2}, \frac{1}{2}\right) - 2(n_1 + n_2 - 1) I_u\left(\frac{n_1 + n_2 - 2}{2}, \frac{1}{2}\right) + \left(n_1 + n_2\right) I_u\left(\frac{n_1 + n_2 + 2}{2}, \frac{1}{2}\right) \]

\[ + C_4^{(i)}\left(\frac{n_1 + n_2 - 4}{2}\right) I_u\left(\frac{n_1 + n_2 - 2}{2}, \frac{1}{2}\right) - 2(n_1 + n_2 - 2) I_u\left(\frac{n_1 + n_2 + 2}{2}, \frac{1}{2}\right) \]

\[ + C_4^{(i)}\left(\frac{n_1 + n_2}{2}\right) I_u\left(\frac{n_1 + n_2 + 2}{2}, \frac{1}{2}\right) \]

\[ \frac{1}{72} [A_6^{(i)} E - B_6^{(i)} F + C_6^{(i)} G - D_6^{(i)} H] \]

\[ \frac{1}{72} [A_3^{(i)} E + B_3^{(i)} F + C_3^{(i)} G - D_3^{(i)} H] \]

\[ \frac{1}{36} [B_3 F - C_3 G + D_3 H] \]

where, putting \( K_i = n_i N_i (N_i - n_i) \) for \( i, j = 1, 2, (i \neq j) \)

\[ A_4^{(i)} = 3(n_i - 1)^2/n_i \]

\[ B_4^{(i)} = 6K_i (N_i - n_i) (n_i - 1) [n_i N_i (K_1 + K_2)] \]

\[ C_4^{(i)} = K_i^2 (n_i + n_i - 2) [n_i^2 - n_i (N_i - n_i) + (N_i - n_i)^2]/[n_i N_i (N_i - n_i) (K_1 + K_2)^2] \]

\[ A_6^{(i)} = 15(n_i - 1)^3/n_i^2 \]

\[ B_6^{(i)} = 45K_i (N_i - n_i) (n_i - 1)^2 [n_i^2 N_i (K_1 + K_2)] \]

\[ C_6^{(i)} = 15K_i^2 (N_i - n_i)^2 (n_i - 1) [n_i^2 N_i^2 (K_1 + K_2)^2] \]

\[ D_6^{(i)} = K_i^3 (n_i + n_i - 2) [n_i^2 + (N_i - n_i)^2]/[n_i^2 N_i^2 (K_1 + K_2)^2] \]

\[ A_3^{(i)} = 3(n_i - 1)(3n_i^2 - 6n_i + 5)/n_i^2 \]

\[ B_3^{(i)} = 9K_i (N_i - n_i) (n_i - 1) (n_i^2 - 4n_i + 5) [n_i^2 N_i (K_1 + K_2)] \]

\[ C_3^{(i)} = 3K_i^2 (n_i - 1)(2n_i^3 - 2n_i^2 - 5)(N_i - n_i)^2/[n_i^2 N_i^2 (K_1 + K_2)^2] \]

\[ D_3^{(i)} = K_i^3 (n_i + n_i - 2)[n_i (N_i - n_i)(N_i - n_i)]/[n_i^2 N_i^2 (K_1 + K_2)^2] \]

\[ B_3 = 9(N_1 - n_1)(N_2 - n_2)(n_1 - 1) (n_2 - 1)/(K_1 + K_2) \]

\[ C_3 = 3K_1 (N_1 - 2n_1)(N_1 - n_1)(n_2 - 1) + 2K_2 (N_2 - 2n_2)(N_1 - n_1)(n_1 - 1)/(K_1 + K_2)^2 \]

\[ D_3 = K_1 K_2 (N_1 - 2n_1)(N_2 - 2n_2)(n_1 + n_2 - 2)/(K_1 + K_2)^3 \]

\[ E = I_u\left(\frac{n_1 + n_2 - 2}{2}, \frac{1}{2}\right) - 3I_u\left(\frac{n_1 + n_2}{2}, \frac{1}{2}\right) \]

\[ + 3I_u\left(\frac{n_1 + n_2 + 2}{2}, \frac{1}{2}\right) - I_u\left(\frac{n_1 + n_2 + 4}{2}, \frac{1}{2}\right) \]
\[ F = (n_1 + n_2 - 2) I_u \left( \frac{n_1 + n_2 - 2}{2}, \frac{1}{2} \right) - 3 \left( n_1 + n_2 - \frac{2}{3} \right) I_u \left( \frac{n_1 + n_2 - 2}{2}, \frac{1}{2} \right) \]
\[ + 3 \left( n_1 + n_2 + \frac{2}{3} \right) I_u \left( \frac{n_1 + n_2 + 2}{2}, \frac{1}{2} \right) - (n_1 + n_2 + 2) I_u \left( \frac{n_1 + n_2 + 4}{2}, \frac{1}{2} \right) \]
\[ G = (n_1 + n_2 - 2)(n_1 + n_2 - 4) I_u \left( \frac{n_1 + n_2 - 2}{2}, \frac{1}{2} \right) \]
\[ - 3 (n_1 + n_2 - 2)(n_1 + n_2 - 4) I_u \left( \frac{n_1 + n_2}{2}, \frac{1}{2} \right) \]
\[ + 3 (n_1 + n_2)(n_1 + n_2 - 2) I_u \left( \frac{n_1 + n_2 + 2}{2}, \frac{1}{2} \right) \]
\[ - (n_1 + n_2)(n_1 + n_2 + 2) I_u \left( \frac{n_1 + n_2 + 4}{2}, \frac{1}{2} \right) \]
\[ H = (n_1 + n_2 - 4)(n_1 + n_2 - 6) I_u \left( \frac{n_1 + n_2 - 2}{2}, \frac{1}{2} \right) \]
\[ - 3 (n_1 + n_2 - 2)(n_1 + n_2 - 4) I_u \left( \frac{n_1 + n_2}{2}, \frac{1}{2} \right) \]
\[ + 3 (n_1 + n_2 - 2)(n_1 + n_2) I_u \left( \frac{n_1 + n_2 + 2}{2}, \frac{1}{2} \right) \]
\[ - (n_1 + n_2)(n_1 + n_2 + 2) I_u \left( \frac{n_1 + n_2 + 4}{2}, \frac{1}{2} \right). \]

**Corollary.** Under the same hypothesis to Theorem 2.3, if we use the same confidence interval for \( \bar{x}_2 - \bar{x}_1 \) obtained by Theorem 1.3, the confidence-coefficient is approximately given by

\[
(2.22) \quad 1 - \alpha = \frac{\mu_3^{(1)} \mu_3^{(2)}}{\sigma^6} Q_{3,3}(\nu_0) - \sum_{i=1}^3 (\beta_2^{(i)} - 3) P_{4;1}(\nu_0) \]
\[
- \sum_{i=1}^2 \frac{1}{10} (\beta_4^{(i)} - 15\beta_2^{(i)} + 30) P_{6;3}(\nu_0) - \sum_{i=1}^3 \beta_1^{(i)} P_{3,3;1}(\nu_0) \]

where we have put \( \nu_0 = \left\{ 1 + \frac{\theta_1^2 + n_2^2}{n_1 + n_2 - 2} \right\}^{-1} \).

The evaluation of the values of \( Q_{3,3;3}(\nu_0) \), \( P_{4;1}(\nu_0) \), \( P_{6;3}(\nu_0) \) and \( P_{3,3;1}(\nu_0) \) can easily be done by the use of the PEASON’S Tables of the Incomplete Beta Functions. For examples:

- in the case \( N_1 = N_2 = 100, n_1 = n_2 = 10 \),

<table>
<thead>
<tr>
<th></th>
<th>( \alpha = 0.05 )</th>
<th>( \alpha = 0.01 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_{4;1}(\nu_0) = P_{4;2}(\nu_0) )</td>
<td>-0.0003</td>
<td>0.0004</td>
</tr>
<tr>
<td>( P_{6;1}(\nu_0) = P_{6;2}(\nu_0) )</td>
<td>0.00004</td>
<td>0.0003</td>
</tr>
<tr>
<td>( P_{3,3;1}(\nu_0) = P_{3,3;2}(\nu_0) )</td>
<td>0.0023</td>
<td>0.0016</td>
</tr>
<tr>
<td>( Q_{3,3;3}(\nu_0) )</td>
<td>-0.0047</td>
<td>-0.0034</td>
</tr>
</tbody>
</table>
in the case \( N_1 = N_2 = 100, n_1 = 5, n_2 = 15, \)

<table>
<thead>
<tr>
<th>( P_4^{(1)}(u_0) )</th>
<th>( P_4^{(2)}(u_0) )</th>
<th>( P_6^{(1)}(u_0) )</th>
<th>( P_6^{(2)}(u_0) )</th>
<th>( P_3^{(1)}(u_0) )</th>
<th>( P_3^{(2)}(u_0) )</th>
<th>( Q_3^{(0)}(u_0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = 0.05 )</td>
<td>( -0.0028 )</td>
<td>0.0030</td>
<td>0.0013</td>
<td>0.0006</td>
<td>0.0004</td>
<td>0.0022</td>
</tr>
<tr>
<td>( \alpha = 0.01 )</td>
<td>( -0.0012 )</td>
<td>0.0012</td>
<td>0.0006</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.0018</td>
</tr>
</tbody>
</table>

The effects of non-normality seems to be negligible small in these cases.

**Theorem 2.4** Under the Assumptions I, II and III, and neglecting the higher powers of \( a_i^{(0)}, a_i^{(1)}, a_i^{(2)} \) and \( a_6^{(0)} \), the simultaneous probability density of

\[
F_i = \frac{N_i s_i^2 - n_i s_i^2}{n_i s_i^2} \cdot \frac{n_i - 1}{N_i - n_i}, \quad (i = 1, 2)
\]
is given by

\[
(2.23) \quad p(F_1, F_2) = h_{N_1-n_1, n_1-1}(F_1) h_{N_2-n_2, n_2-1}(F_2) \times \left\{ 1 + \sum_{i=1}^{2} (\beta_i^{(0)} - 3) p_i^{(0)} + \sum_{i=1}^{2} \frac{1}{10} (\beta_i^{(0)} - 15\beta_i^{(0)} + 30) p_i^{(0)} + \sum_{i=1}^{2} \beta_i^{(0)} p_i^{(0)} \beta_i^{(0)} \right\},
\]

where, putting \( G_i = \left( 1 + \frac{N_i - n_i}{n_i - 1} F_i \right)^{-1} \) and omitting the suffix \( i \) in both sides of the following equations

\[
(2.24) \quad p_i = \frac{N - 1}{8n N(N-n+2)} \left[ (n-1)! (n-1) N - (n+1)! \right] - 2(N+1) G \cdot (n-1) N - (n+1)! + \frac{N + 1}{n + 1} G^2 \cdot (n-1) N^2 + 2(n-2) N - 3(n+1)! \]

\[
(2.25) \quad p_0 = \frac{5(N-1)}{24n^2 N^2 (N-n+2)(N-n+4)} \left[ (n-1)! (n-1)^2 N^3 - 2(n-1)^2(n-2) N^2 + (4n^2 - 13n^2 - 2n + 3) N - 2n(n+1)(n-3)! - 3(N+1)G \cdot (n-1) N^3 - 2(n-2) N^2 + (4n^2 - 3n - 9) N - 2n(n+1)! + \frac{3(N+1)}{n+1} G^2 \cdot (n-1)^2 N^4 - (2n^3 - 7n^2 + 16n - 11) N^3 - (2n^3 - 11n^2 + 40n - 39) N^2 + (10n^3 - 7n^2 - 32n + 45) N - 6n(n+1)^2! \right]
\]
\[ -\frac{(N+1)(N+3)}{(n+1)(n+3)} G^3(n-1)^2N^4 - (2n^3 - 7n^2 + 20n - 15)N^3 \\
- (6n^3 - 15n^2 + 64n - 71)N^2 + (18n^3 + 17n^2 - 76n + 105)N \\
- 10n(n+1)(n+3) \]

\[ p_{3,3} = \frac{N-1}{24n^2N^2(N-n+2)(N-n+4)} \left[ (n-1)^2(3n^2 - 6n + 5)N^3 \\
- (3n^3 - 18n^2 + 25n - 20)N^2 + (15n^3 - 49n^2 + 11n + 15)N \\
- 10n(n+1)(n+3) \right] \\
- 3(N+1)G^2(3n^2 - 6n + 5)N^3 - (3n^2 - 16n^2 + 29n - 30)N^2 \\
+ (17n^3 - 23n^2 - 15n + 45)N - 10n(n^2 - 1) \]

\[ + \frac{3(N+1)}{n+1} G^2(3n^2 - 6n + 5)N^3 - (3n^3 - 23n^2 + 51n - 55)N^3 \\
+ (47n^3 - 11n^2 - 133n + 225)N \\
- 30n(n+1)^2 \] 

\[ - \frac{(N+1)(N+3)}{(n+1)(n+3)} G^3(3n^2 - 6n + 5)N^4 - (3n^2 - 27n^2 + 67n - 75)N^3 \\
+ (6n^2 + 89n^2 - 252n + 355)N^2 + 5(19n^2 + 21n^2 - 73n + 105)N \\
- 50n(n+1)(n+3) \].

§ 3. Acknowledgement In conclusion, the author would like to express his great indebtedness to Prof. T. Kitagawa for having suggested the problem and for helpful advice and criticism.

References

