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ON THE STATISTICAL INFERENCES IN FINITE POPULATIONS BY TWO SAMPLE THEORY

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§ 1. **Introduction.** In May, 1951, at the Annual Meeting of the Mathematical Society of Japan, T. KITAGAWA [1] has established a method of statistical inferences on finite populations from the view point of two sample theory.

He has introduced following assumptions in dealing with these problems:

Assumption I. Considering a grand population Π, the finite population (N) in his consideration can be recognized as random sample of size N drawn independently from Π.

Assumption II. The finite population (N), as a sample from the grand population Π, can be recognized as consisting of two independent random samples $O_1 : (x_1, \cdots, x_n)$ and $O_2 : (y_1, \cdots, y_{N-n})$ of sizes n and $N-n$ respectively, which are drawn from the grand population Π.

Assumption III. The grand population Π is distributed normally according to $N(\xi, \sigma^2)$.

Putting

$$(1.01) \quad \bar{x} = \sum_{i=1}^{n} x_i/n, \qquad s^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2/n$$

$$(1.02) \quad \bar{y} = \sum_{j=1}^{N-n} y_j/(N-n), \qquad s'^2 = \sum_{j=1}^{N-n} (y_j - \bar{y})^2/(N-n)$$

$$(1.03) \quad \tilde{x} = \left\{ n\bar{x} + (N-n)\bar{y} \right\}/N, \quad S^2 = \left\{ \sum_{i=1}^{n} (x_i - \tilde{x})^2 + \sum_{j=1}^{N-n} (y_j - \tilde{x})^2 \right\}/N,$$

what we have known from actual sampling are some or all of (1.01), and what we want to infer are some or all of (1.03). The statistics (1.02) are auxiliary ones which are indispensable in his formulation, but what are unknown to us.

T. KITAGAWA(1) developped following theorems, which will be of use in giving confidence intervals associated with statistical inferences about finite populations:

Theorem 1.1. *Under the Assumptions* I, II *and* III, *for any assigned* α, $0 < \alpha < 1$, *the confidence interval for* $\tilde{x}$ *with confidence-coefficient* $1-\alpha$, *which takes the form of* $(\bar{x} - As, \bar{x} + As)$ *is given by*

(1) Prof. Kitagawa informed me after the preparation of this paper that he found the same result as to Theorem 1.1 was already established by E. S. PEARSON [3] which was not accessible in Japan until 1951.

(1.04) $$A = \frac{t_{n-1}(\alpha)}{\sqrt{n-1}}\sqrt{\frac{N-n}{N}},$$

where $t_{n-1}(\alpha)$ *means* α*-significance level of t-distribution with* $n-1$ *degrees of freedom such that* $\Pr.\{|t| > t_{n-1}(\alpha)\} = \alpha$.

Theorem 1.2. *Under the same hypothesis to Theorem* 1.1, *the confidence interval for* S^2 *by means of* s^2 *with confidence-coefficient* $1-\alpha$, *which takes the form of* $\{(n/N)s^2, Bs^2\}$ *is given by*

(1.05) $$B = \frac{n}{N}\left\{1 + \frac{N-n}{n-1}F_{n-1}^{N-n}(\alpha)\right\},$$

where $F_{n-1}^{N-n}(\alpha)$ *means* α*-significance level of F-distribution with pair of degrees of freedom* $N-n$ *and* $n-1$ *such that* $\Pr.\{F > F_{n-1}^{N-n}(\alpha)\} = \alpha$.

He extends these two theorems to various directions. For examples, the confidence interval for the differences of two means $\tilde{x}_1$ and $\tilde{x}_2$, and those for the ratios of two variances S_1^2 and S_2^2 of two finite populations can be obtained in quite similar ways. For this purpose he considers two sets of two independent samples

$$O_1^{(i)} : (x_{i1}, \cdots, x_{in_i}) \text{ and } O_2^{(i)} : (y_{i1}, \cdots, y_{i,N_i-n_i}), \ (i = 1, 2)$$

drawn respectively from the population Π_i, and he puts for $i = 1, 2$

(1.06) $$\bar{x}_i = \sum_{j=1}^{n_i} x_{ij}/n_i, \qquad s_i^2 = \sum_{j=1}^{n_i}(x_{ij} - \bar{x}_i)^2/n_i$$

(1.07) $$\bar{y}_i = \sum_{k=1}^{N_i-n_i} y_{ik}/(N_i - n_i), \qquad s_i'^2 = \sum_{k=1}^{N_i-n_i}(y_{ik} - \bar{y}_i)^2/(N_i - n_i)$$

(1.08) $$\tilde{x}_i = \{n_i\bar{x}_i + (N_i - n_i)\bar{y}_i\}/N_i$$

(1.09) $$S_i^2 = \left\{\sum_{j=1}^{n_i}(x_{ij} - \tilde{x}_i)^2 + \sum_{k=1}^{N_i-n_i}(y_{ik} - \tilde{x}_i)^2\right\}/N_i.$$

As in the case of one system of two samples, following theorems(2) give him confidence intervals associated with statistical inferences about two finite populations:

Theorem 1.3. *Under the Assumptions* I, II *and* III, *when we have* $\sigma_1^2 = \sigma_2^2$, *the confidence interval for* $\tilde{x}_2 - \tilde{x}_1$ *by means of* $\bar{x}_1$, $\bar{x}_2$, s_1^2 *and* s_2^2 *with confidence-coefficient* $1-\alpha$, *which takes the form of* $(\bar{x}_2 - \bar{x}_1 - As, \bar{x}_2 - \bar{x}_1 + As)$, *where*

(1.10) $$s^2 = (n_1 s_1^2 + n_2 s_2^2)/(n_1 + n_2)$$

is given by

(2) See Kitagawa [1] Theorems 6.3 and 6.4. Slight misprints concerning s^2, b_i and c_i $(i = 1, 2)$ should be corrected as we enunciate in (1.10) (1.13) and (1.14).

$$(1.11) \quad A = \frac{t_{n_1+n_2-2}(\alpha)}{\sqrt{n_1+n_2-2}} \sqrt{\frac{n_1+n_2}{n_1} \cdot \frac{N_1-n_1}{N_1} + \frac{n_1+n_2}{n_2} \cdot \frac{N_2-n_2}{N_2}} .$$

Theorem 1 4. *Under the Assumptions* I, II *and* III, *the confidence interval for the ratio* S_2^2/S_1^2 *by means of* s_1^2 *and* s_2^2 *with confidence-coefficient* $1-\alpha$, *which takes the form of* $\{B(s_2^2/s_1^2), C(s_2^2/s_1^2)\}$ *can be given by*

$$(1.12) \quad 1-\alpha = \Pr.\left\{B\frac{s_2^2}{s_1^2} < \frac{S_2^2}{S_1^2} < C\frac{s_2^2}{s_1^2}\right\}$$

$$= \iint\limits_{b_1 L + b_2 < G < c_1 L + c_2} h_{N_2-n_2,\, n_2-1}(G)\, h_{N_1-n_1,\, n_1-1}(L)\, dG\, dL ,$$

where $h_{ij}(H)$ *means the F-distribution with degrees of freedom* (i, j) *and he has put*

$$(1.13) \quad b_1 = B \cdot \frac{n_1 N_2}{n_2 N_1} \cdot \frac{N_1-n_1}{N_2-n_2} \cdot \frac{n_2-1}{n_1-1}, \quad b_2 = \frac{n_2-1}{N_2-n_2}\left(\frac{n_1 N_2}{n_2 N_1}B - 1\right)$$

$$(1.14) \quad c_1 = C \cdot \frac{n_1 N_2}{n_2 N_1} \cdot \frac{N_1-n_1}{N_2-n_2} \cdot \frac{n_2-1}{n_1-1}, \quad c_2 = \frac{n_2-1}{N_2-n_2}\left(\frac{n_1 N_2}{n_2 N_1}C - 1\right).$$

In this paper we shall study the effects of non-normality, when we remove the normality assumption III in these theorems.

§ 2. The effects of non-normality.

Instead of the Assumption III of normality, we shall put the following

Assumption III'. *The grand population* Π *is distributed according to the* GRAM-CHARLIER *Type A, that is, according to*

$$(2.01) \quad f(x)\,dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \{1 + a_3 H_3(z) + a_4 H_4(z) + a_5 H_5(z) + a_6 H_6(z)\}\, dz$$

where $z = (x-\xi)/\sigma$ *is the standardized variate, and we have put*

$$(2.02) \quad a_3 = \frac{\mu_3}{3!\,\sigma^3} = \frac{1}{6}\sqrt{\beta_1}, \qquad a_4 = \frac{1}{4!}\left(\frac{\mu_4}{\sigma^4} - 3\right) = \frac{1}{24}(\beta_2 - 3)$$

$$a_5 = \frac{\mu_5 - 10\mu_2\mu_3}{120\sigma^5}, \qquad a_6 = \frac{1}{720}(\beta_4 - 15\beta_2 + 30).$$

The joint distribution of $\bar{x}$ and s drawn from this population was obtained in the previous paper [2]. By the use of the result and following KITAGAWA's formulation, we can derive the following

Theorem 2.1. *Under the Assumptions* I, II *and* III', *and neglecting the higher powers of* a_3, a_4, a_5 *and* a_6, *the distribution of* $\tau = \frac{\bar{x}-\bar{\bar{x}}}{s}\sqrt{\frac{N}{N-n}}$ *is given as follows*:

for any assigned $\tau_0 (\geq 0)$, *putting* $u = (1+\tau_0^2)^{-1}$ *and* $m = N - n$,

$$(2.03)\quad \Pr.\{|\tau| > \tau_0\} = \mathrm{I}_u\left(\frac{n-1}{2}, \frac{1}{2}\right) + (\beta_2 - 3)\,\mathrm{P}_4(u) + \frac{1}{10}(\beta_4 - 15\beta_2 + 30)\,\mathrm{P}_6(u) + \beta_1 \mathrm{P}_{3,3}(u),$$

$$(2.04)\quad \Pr.\{\tau > \tau_0\} = \frac{1}{2}\Pr.\{|\tau| > \tau_0\} + \alpha_3 \mathrm{P}_3(u) + \alpha_5 \mathrm{P}_5(u),$$

where $\mathrm{I}_u(p,q)$ *means the ratio of the Incomplete Beta Functions* $\mathrm{I}_u(p,q) = \mathrm{B}_u(p,q)/\mathrm{B}(p,q)$, *and*

$$(2.05)\quad \mathrm{P}_4(u) = -\frac{n-1}{24n\,m\,N}\Big[n\{2m^2 - 2nm - n(n-3)\}\,\mathrm{I}_u\left(\frac{n-1}{2}, \frac{1}{2}\right)$$
$$- 2\{(2n+4)m^2 - 2n(n-1)m - n^2(n-1)\}\,\mathrm{I}_u\left(\frac{n+1}{2}, \frac{1}{2}\right)$$
$$+ \{(2n+8)m^2 - 2n(n-2)m - n^2(n+1)\}\,\mathrm{I}_u\left(\frac{n+3}{2}, \frac{1}{2}\right)\Big]$$

$$(2.06)\quad \mathrm{P}_6(u) = \frac{n-1}{72n^2m^2N^3}\Big[n\{8(2n-1)m^5 + 30n(n-1)m^4$$
$$- 15n^2(n-1)^2m^2 + n^4(n-3)(n-5)\}\,\mathrm{I}_u\left(\frac{n-1}{2}, \frac{1}{2}\right)$$
$$- 3(n-1)\{16(n+2)m^5 + 10n(3n+8)m^4$$
$$+ 60n^2m^3 - 15n^3(n-1)m^2 + n^5(n-3)\}\,\mathrm{I}_u\left(\frac{n+1}{2}, \frac{1}{2}\right)$$
$$+ 3\{8(n+4)(2n-3)m^5 + 10n(3n^2+13n-20)m^4$$
$$+ 120n^2(n-1)m^3 - 15n^3(n-1)^2m^2 + n^5(n^2-1\}\,\mathrm{I}_u\left(\frac{n+3}{2}, \frac{1}{2}\right)$$
$$- \{16(n+6)(n-2)m^5 + 30n(n^2+7n-12)m^4 + 180n^2(n-1)m^3$$
$$- 15n^3(n-1)^2m^2 + n^5(n+1)(n+3)\}\,\mathrm{I}_u\left(\frac{n+5}{2}, \frac{1}{2}\right)\Big]$$

$$(2.07)\quad \mathrm{P}_{3,3}(u) = \frac{n-1}{72n^2m^2N^3}\Big[n\{2(2n^2-3n+4)m^5 + 6n(2n^2-4n+5)m^4$$
$$+ n^2(13n^2-26n+15)m^3 + 3n^2(2n^3-5n^2+5)m^2$$
$$+ n^4(n-3)(n-5)(m-1)\}\,\mathrm{I}_u\left(\frac{n-1}{2}, \frac{1}{2}\right)$$
$$- 3\{2(2n^3-3n^2+16)m^5 + 2n(6n^3-8n^2-5n+40)m^4$$
$$+ n^2(13n^3-6n^2-25n+60)m^3 + n^3(6n^3+n^2-16n+15)m^2$$
$$+ n^5(n-1)(n-3)(m-1)\}\,\mathrm{I}_u\left(\frac{n+1}{2}, \frac{1}{2}\right)$$

$$+3\{2(2n^3-3n^2-8n+48)m^5+2n(6n^3-4n^2-33n+100)m^4$$
$$+n^2(13n^3+14n^2-65n+120)m^3+n^3(6n^3+17n^2-16n+15)m^2$$
$$+n^5(n^2-1)(m-1)\}\,\mathrm{I}_u\left(\frac{n+3}{2},\ \frac{1}{2}\right)$$
$$-\{2(2n^3-3n^2-20n+96)m^5+6n(2n^3-23n+60)m^4$$
$$+n^2(13n^3+34n^2-105n+180)m^3+3n^3(2n^3+11n^2+5)m^2$$
$$+n^5(n+1)(n+3)(m-1)\}\,\mathrm{I}_u\left(\frac{n+5}{2},\ \frac{1}{2}\right)\Big]$$

$$(2.08)\quad \mathrm{P}_3(u) = -\frac{1}{\sqrt{2\pi n m N}}\Big[\{(2n-1)m+n(n-2)\}\,\mathrm{I}_u\left(\frac{n-1}{2},\ 1\right) - (n-1)(2m+n)\,\mathrm{I}_u\left(\frac{n+1}{2},\ 1\right)\Big]$$

$$(2.09)\quad \mathrm{P}_5(u) = \frac{1}{\sqrt{2\pi n^3 m^3 N^5}}\Big[\{3(2n^2-2n+1)m^4+10n(n-1)(2n-1)m^3$$
$$+15n^2(n-1)^2m^2-n^4(n-2)(n-4)\}\,\mathrm{I}_u\left(\frac{n-1}{2},\ 1\right)$$
$$-2(n-1)\{6(n-2)m^4+5n(4n-5)m^3+15n^2(n-1)m^2$$
$$-n^4(n-2)\}\,\mathrm{I}_u\left(\frac{n+1}{2},\ 1\right)$$
$$+(n-1)\{6(n-4)m^4+20n(n-2)m^3$$
$$+15n^2(n-1)m^2-n^4(n+1)\}\,\mathrm{I}_u\left(\frac{n+3}{2},\ 1\right)\Big].$$

Corollary. *Under the same hypothesis to Theorem* 2.1, *if we use the same confidence interval for* $\bar{x}$ *obtained by Theorem* 1.1, *the confidence-coefficient is approximately given by*

$$(2.10)\quad 1-\alpha-(\beta_2-3)\,\mathrm{P}_4(u_0)-\frac{1}{10}(\beta_4-15\beta_2+30)\,\mathrm{P}_6(u_0)-\beta_1\mathrm{P}_{3,3}(u_0),$$

where we have put $u_0 = \left\{1+\frac{t^2_{n-1}(\alpha)}{n-1}\right\}^{-1}.$

The evaluation of the values of $\mathrm{P}_4(u_0)$, $\mathrm{P}_6(u_0)$ and $\mathrm{P}_{3,3}(u_0)$ can easily be done by the use of the PEARSON's Tables of the Incomplete Beta Functions. For examples

in the case $\alpha = 0.05$

	$\mathrm{P}_4(u_0)$	$\mathrm{P}_6(u_0)$	$\mathrm{P}_{3,3}(u_0)$
$N=100$, $n=5$	-0.0050	0.0088	0.0156
$N=100$, $n=10$	-0.0018	0.0015	0.0145
$N=100$, $n=25$	0.0004	-0.0003	0.0083
$N=1000$, $n=25$	-0.0009	0.0001	0.0087

in the case $\alpha = 0.01$

	$P_4(u_0)$	$P_6(u_0)$	$P_{3,3}(u_0)$
$N=100,\quad n=5$	-0.0020	0.0042	0.0056
$N=100,\quad n=10$	0.0001	0.0020	0.0082
$N=100,\quad n=25$	-0.0002	0.0002	0.0062
$N=1000,\quad n=25$	-0.0008	0.0005	0.0063

The effects of non-normality seems to be small in these cases.

Theorem 2.2. *Under the Assumptions* I, II *and* III', *and neglecting the higher powers of* a_3, a_4, a_5 *and* a_6, *the distribution of*

$$F = \left\{ s'^2 + \frac{n}{N}(\bar{x} - \bar{y})^2 \right\} \cdot \frac{n-1}{n\, s^2}$$

is given as follows:

For any assigned $F_0 (\geq 0)$, *putting* $u = \left(1 + \frac{N-n}{n-1} F_0\right)^{-1}$

$$(2.11)\quad \Pr.\{F > F_0\} = I_u\left(\frac{n-1}{2}, \frac{N-n}{2}\right) + (\beta_2 - 3)\, Q_4(u) + \frac{1}{10}(\beta_4 - 15\beta_2 + 30)\, Q_6(u) + \beta_1 Q_{3,3}(u),$$

which turns to

$$(2.12)\quad \Pr.\left\{\frac{n}{N} \leq \frac{S^2}{s^2} \leq \frac{n}{N}\left(1 + \frac{N-n}{n-1} F_0\right)\right\}$$
$$= 1 - \left\{ I_u\left(\frac{n-1}{2}, \frac{N-n}{2}\right) + (\beta_2 - 3)\, Q_4(u) + \frac{1}{10}(\beta_4 - 15\beta_2 + 30)\, Q_6(u) + \beta_1 Q_{3,3}(u)\right\},$$

where

$$(2.13)\quad Q_4(u) = \frac{n-1}{8nN(N-n+2)}\Big[\{(n-1)N^2 - 2nN + (n+1)\}\, I_u\left(\frac{n-1}{2}, \frac{N-n}{2}\right)$$
$$- 2\{(n-1)N^2 - 2N - (n+1)\}\, I_u\left(\frac{n+1}{2}, \frac{N-n}{2}\right)$$
$$+ \{(n-1)N^2 + 2(n-2)N - 3(n+1)\}\, I_u\left(\frac{n+3}{2}, \frac{N-n}{2}\right)\Big]$$

$$(2.14)\quad Q_6(u) = \frac{-5(n-1)}{24n^2N^2(N-n+2)(N-n+4)}\Big[(N-1)\{(n-1)^2N^3$$
$$- 2(n-1)^2(n-2)N^2 + (4n^3 - 13n^2 - 2n + 3)N$$

$$- 2n(n+1)(n-3)\} \mathrm{I}_u\left(\frac{n-1}{2}, \frac{N-n}{2}\right)$$
$$- 3(n-1)(N+1)\{(n-1)N^3 - 2(n^2-2n+3)N^2$$
$$+ (4n^2-3n-9)N - 2n(n+1)\} \mathrm{I}_u\left(\frac{n+1}{2}, \frac{N-n}{2}\right)$$
$$+ 3\{(n-1)^2 N^4 - (2n^3-7n^2+16n-11)N^3$$
$$- (2n^3-11n^2+40n-39)N^2 + (10n^3-7n^2-32n+45)N$$
$$- 6n(n+1)^2\} \mathrm{I}_u\left(\frac{n+3}{2}, \frac{N-n}{2}\right)$$
$$- \{(n-1)^2 N^4 - (2n^3-7n^2+20n-15)N^3$$
$$- (6n^3-15n^2+64n-71)N^2 + (18n^3+17n^2-76n+105)N$$
$$- 10n(n+1)(n+3)\} \mathrm{I}_u\left(\frac{n+5}{2}, \frac{N-n}{2}\right)\Big]$$

$$(2.15) \quad \mathrm{Q}_{3,3}(u) = \frac{n-1}{24n^2 N^2 (N-n+2)(N-n+4)} \Big[(N-1)\{(3n^2$$
$$- 6n+5)N^3 - (3n^3-18n^2+25n-20)N^2 + (15n^3-49n^2$$
$$+ 11n+15)N - 10n(n+1)(n-3)\} \mathrm{I}_u\left(\frac{n-1}{2}, \frac{N-n}{2}\right)$$
$$- 3(N+1)\{(3n^2-6n+5)N^3 - (3n^3-16n^2+29n-30)N^2$$
$$+ (17n^3-23n^2-15n+45)N - 10n(n^2-1)\} \mathrm{I}_u\left(\frac{n+1}{2}, \frac{N-n}{2}\right)$$
$$+ 3\{(3n^2-6n+5)N^4 - (3n^3-23n^2+51n-55)N^3$$
$$+ 5(2n^3+9n^2-28n+39)N^2 + (47n^3-11n^2-133n+225)N$$
$$- 30n(n+1)^2\} \mathrm{I}_u\left(\frac{n+3}{2}, \frac{N-n}{2}\right)$$
$$- \{(3n^2-6n+5)N^4 - (3n^3-27n^2+67n-75)N^3$$
$$+ (6n^3+89n^2-252n+355)N^2 + 5(19n^3+21n^2-73n+105)N$$
$$- 50n(n+1)(n+3)\} \mathrm{I}_u\left(\frac{n+5}{2}, \frac{N-n}{2}\right)\Big].$$

Corollary. *Under the same hypothesis to Theorem* 2.2, *if we use the same confidence interval for* S^2 *obtained by Theorem* 1.2, *the confidence-coefficient is approximately given by*

$$(2.16) \quad 1 - \alpha - (\beta_2 - 3)\mathrm{Q}_4(u_0) - \frac{1}{10}(\beta_4 - 15\beta_2 + 30)\mathrm{Q}_6(u_0) - \beta_1 \mathrm{Q}_{3,3}(u_0),$$

where we have put $u_0 = \left\{1 + \frac{N-n}{n-1} F_{n-1}^{N-n}(\alpha)\right\}^{-1}$.

The evaluation of the values of $\mathrm{Q}_4(u_0)$, $\mathrm{Q}_6(u_0)$ and $\mathrm{Q}_{3,3}(u_0)$ can also be done. For examples, in the case $\alpha = 0.05$

	$Q_4(u_0)$	$Q_6(u_0)$	$Q_{3,3}(u_0)$
$N = 99,\quad n = 5$	0.0155	−0.0176	0.0115
$N = 100,\quad n = 10$	0.0254	−0.0239	0.0156
$N = 100,\quad n = 16$	0.0306	−0.0220	0.0152
$N = 99,\quad n = 25$	0.0344	−0.0172	0.0131

in the case $\alpha = 0.01$

	$Q_4(u_0)$	$Q_6(u_0)$	$Q_{3,3}(u_0)$
$N = 99,\quad n = 5$	0.0036	−0.0044	0.0029
$N = 100,\quad n = 10$	0.0069	−0.0076	0.0051
$N = 100,\quad n = 16$	0.0090	−0.0080	0.0057
$N = 99,\quad n = 25$	0.0116	−0.0082	0.0067

In these cases, the effects of non-normality due to kurtosis seems to be too large to neglect.

Following KITAGAWA's Theorem 1.3 which gives the confidence interval for the difference of two means $\tilde{x}_1$ and $\tilde{x}_2$ that are recognized the population means of two finite populations, by means of sample means $\bar{x}_1$, $\bar{x}_2$ and sample variances s_1^2 and s_2^2, we can derive the following

Theorem 2.3. *Under the Assumptions* I, II *and* III', *when we have* $\sigma_1^2 = \sigma_2^2 = \sigma^2$, *neglecting the higher powers of* $a_3^{(i)}$, $a_4^{(i)}$, $a_5^{(i)}$ *and* $a_6^{(i)}$, $(i = 1, 2)$, *the distribution of*

$$\tau = \frac{\bar{x}_2 - \bar{x}_1 - (\tilde{x}_2 - \tilde{x}_1)}{s} \Bigg/ \sqrt{\frac{n_1+n_2}{n_1}\cdot\frac{N_1-n_1}{N_1} + \frac{n_1+n_2}{n_2}\cdot\frac{N_2-n_2}{N_2}}$$

is given as follows:

for any assigned $\tau_0 (\geq 0)$, *putting* $u = (1+\tau_0^2)^{-1}$

$$\text{(2.17)}\quad \Pr.\{|\tau| > \tau_0\} = I_u\left(\frac{n_1+n_2-2}{2}, \frac{1}{2}\right) + \frac{\mu_3^{(1)}\mu_3^{(2)}}{\sigma^6} Q_{3,3}^{(0)}(u)$$
$$+ \sum_{i=1}^{2} (\beta_2^{(i)} - 3) P_4^{(i)}(u) + \sum_{i=1}^{2} \frac{1}{10} (\beta_4^{(i)} - 15\beta_2^{(i)} + 30) P_6^{(i)}(u)$$
$$+ \sum_{i=1}^{2} \beta_1^{(i)} P_{3,3}^{(i)}(u),$$

where we have put $s^2 = (n_1 s_1^2 + n_2 s_2^2)/(n_1+n_2)$

$$\text{(2.18)}\quad P_4^{(i)}(u) = \frac{1}{24}\Bigg[A_4^{(i)}\Bigg\{I_u\left(\frac{n_1+n_2-2}{2}, \frac{1}{2}\right)$$
$$- 2I_u\left(\frac{n_1+n_2}{2}, \frac{1}{2}\right) + I_u\left(\frac{n_1+n_2+2}{2}, \frac{1}{2}\right)\Bigg\}$$

$$- B_4^{(i)} \Big\{(n_1 + n_2 - 2)\, \mathrm{I}_u\Big(\frac{n_1 + n_2 - 2}{2}, \frac{1}{2}\Big) - 2(n_1 + n_2 - 1)\, \mathrm{I}_u\Big(\frac{n_1 + n_2}{2}, \frac{1}{2}\Big) + (n_1 + n_2)\, \mathrm{I}_u\Big(\frac{n_1 + n_2 + 2}{2}, \frac{1}{2}\Big)\Big\}$$

$$+ C_4^{(i)} \Big\{(n_1 + n_2 - 4)\, \mathrm{I}_u\Big(\frac{n_1 + n_2 - 2}{2}, \frac{1}{2}\Big) - 2(n_1 + n_2 - 2)\, \mathrm{I}_u\Big(\frac{n_1 + n_2}{2}, \frac{1}{2}\Big) + (n_1 + n_2)\, \mathrm{I}_u\Big(\frac{n_1 + n_2 + 2}{2}, \frac{1}{2}\Big)\Big\}\Big]$$

$$(2.19)\quad \mathrm{P}_6^{(i)}(u) = \frac{-1}{72}\,[A_6^{(i)} E - B_6^{(i)} F + C_6^{(i)} G - D_6^{(i)} H]$$

$$(2.20)\quad \mathrm{P}_{3,3}^{(i)}(u) = \frac{1}{72}\,[A_{3,3}^{(i)} E + B_{3,3}^{(i)} F + C_{3,3}^{(i)} G + D_{3,3}^{(i)} H]$$

$$(2.21)\quad \mathrm{Q}_{3,3}^{(0)}(u) = \frac{-1}{36}\,[B_3 F - C_3 G + D_3 H]\,,$$

where, putting $K_i = n_j N_j (N_i - n_i)$ *for* $i, j = 1, 2, (i \neq j)$

$$\begin{aligned}
A_4^{(i)} &= 3(n_i - 1)^2/n_i \\
B_4^{(i)} &= 6K_i (N_i - n_i)(n_i - 1)/[n_i N_i (K_1 + K_2)] \\
C_4^{(i)} &= K_i^2 (n_1 + n_2 - 2)\{n_i^2 - n_i(N_i - n_i) \\
&\qquad + (N_i - n_i)^2\}/[n_i N_i (N_i - n_i)(K_1 + K_2)^2] \\
A_6^{(i)} &= 15(n_i - 1)^3/n_i^2 \\
B_6^{(i)} &= 45K_i (N_i - n_i)(n_i - 1)^2/[n_i^2 N_i (K_1 + K_2)] \\
C_6^{(i)} &= 15K_i^2 (N_i - n_i)^2 (n_i - 1)/[n_i^2 N_i^2 (K_1 + K_2)^2] \\
D_6^{(i)} &= K_i^3 (n_1 + n_2 - 2)\{n_i^5 + (N_i - n_i)^5\}/[n_i^2 (N_i - n_i)^2 N_i^3 (K_1 + K_2)^3] \\
A_{3,3}^{(i)} &= 3(n_i - 1)(3n_i^2 - 6n_i + 5)/n_i^2 \\
B_{3,3}^{(i)} &= 9K_i (N_i - n_i)(n_i - 1)(n_i^2 - 4n_i + 5)/[n_i^2 N_i (K_1 + K_2)] \\
C_{3,3}^{(i)} &= 3K_i^2 (n_i - 1)\{2n_i^3 - (2n_i - 5)(N_i - n_i)^2\}/[n_i^2 N_i^2 (K_1 + K_2)^2] \\
D_{3,3}^{(i)} &= K_i^3 (n_1 + n_2 - 2)[n_i (N_i - n_i)\{(N_i - n_i)^2 - n_i^2\}^2 \\
&\qquad - \{n_i^5 + (N_i - n_i)^5\}]/[n_i^2 (N_i - n_i)^2 N_i^3 (K_1 + K_2)^3] \\
B_3 &= 9(N_1 - n_1)(N_2 - n_2)(n_1 - 1)(n_2 - 1)/(K_1 + K_2) \\
C_3 &= 3\{K_1 (N_1 - 2n_1)(N_2 - n_2)(n_2 - 1) \\
&\qquad + K_2 (N_2 - 2n_2)(N_1 - n_1)(n_1 - 1)\}/(K_1 + K_2)^2 \\
D_3 &= K_1 K_2 (N_1 - 2n_1)(N_2 - 2n_2)(n_1 + n_2 - 2)/(K_1 + K_2)^3
\end{aligned}$$

$$E = \mathrm{I}_u\Big(\frac{n_1 + n_2 - 2}{2}, \frac{1}{2}\Big) - 3\mathrm{I}_u\Big(\frac{n_1 + n_2}{2}, \frac{1}{2}\Big) + 3\mathrm{I}_u\Big(\frac{n_1 + n_2 + 2}{2}, \frac{1}{2}\Big) - \mathrm{I}_u\Big(\frac{n_1 + n_2 + 4}{2}, \frac{1}{2}\Big)$$

$$F = (n_1+n_2-2)\mathrm{I}_u\left(\frac{n_1+n_2-2}{2}, \frac{1}{2}\right) - 3\left(n_1+n_2-\frac{2}{3}\right)\mathrm{I}_u\left(\frac{n_1+n_2}{2}, \frac{1}{2}\right)$$
$$+3\left(n_1+n_2+\frac{2}{3}\right)\mathrm{I}_u\left(\frac{n_1+n_2+2}{2}, \frac{1}{2}\right) - (n_1+n_2+2)\mathrm{I}_u\left(\frac{n_1+n_2+4}{2}, \frac{1}{2}\right)$$
$$G = (n_1+n_2-2)(n_1+n_2-4)\mathrm{I}_u\left(\frac{n_1+n_2-2}{2}, \frac{1}{2}\right)$$
$$-3(n_1+n_2-2)\left(n_1+n_2-\frac{4}{3}\right)\mathrm{I}_u\left(\frac{n_1+n_2}{2}, \frac{1}{2}\right)$$
$$+3(n_1+n_2)\left(n_1+n_2-\frac{2}{3}\right)\mathrm{I}_u\left(\frac{n_1+n_2+2}{2}, \frac{1}{2}\right)$$
$$-(n_1+n_2)(n_1+n_2+2)\mathrm{I}_u\left(\frac{n_1+n_2+4}{2}, \frac{1}{2}\right)$$
$$H = (n_1+n_2-4)(n_1+n_2-6)\mathrm{I}_u\left(\frac{n_1+n_2-2}{2}, \frac{1}{2}\right)$$
$$-3(n_1+n_2-2)(n_1+n_2-4)\mathrm{I}_u\left(\frac{n_1+n_2}{2}, \frac{1}{2}\right)$$
$$+3(n_1+n_2-2)(n_1+n_2)\mathrm{I}_u\left(\frac{n_1+n_2+2}{2}, \frac{1}{2}\right)$$
$$-(n_1+n_2)(n_1+n_2+2)\mathrm{I}_u\left(\frac{n_1+n_2+4}{2}, \frac{1}{2}\right).$$

Corollary. *Under the same hypothesis to Theorem* 2.3, *if we use the same confidence interval for* $\bar{x}_2-\bar{x}_1$ *obtained by Theorem* 1.3, *the confidence-coefficient is approximately given by*

$$(2.22)\quad 1-\alpha-\frac{\mu_3^{(1)}\mu_3^{(2)}}{\sigma^6}\mathrm{Q}_{3,3}^{(0)}(u_0)-\sum_{i=1}^{2}(\beta_2^{(i)}-3)\mathrm{P}_4^{(i)}(u_0)$$
$$-\sum_{i=1}^{2}\frac{1}{10}(\beta_4^{(i)}-15\beta_2^{(i)}+30)P_6^{(i)}(u_0)-\sum_{i=1}^{2}\beta_1^{(i)}P_{3,3}^{(i)}(u_0)$$

where we have put $u_0=\left\{1+\dfrac{t^2_{n_1+n_2-2}(\alpha)}{n_1+n_2-2}\right\}^{-1}$.

The evaluation of the values of $\mathrm{Q}_{3,3}^{(0)}(u_0)$, $\mathrm{P}_4^{(i)}(u_0)$, $\mathrm{P}_6^{(i)}(u_0)$ and $\mathrm{P}_{3,3}^{(i)}(u_0)$ can easily be done by the use of the PEASON's Tables of the Incomplete Beta Functions. For examples:

in the case $N_1=N_2=100$, $n_1=n_2=10$,

	$\alpha=0.05$	$\alpha=0.01$
$\mathrm{P}_4^{(1)}(u_0)=\mathrm{P}_4^{(2)}(u_0)$	-0.0003	0.0004
$\mathrm{P}_6^{(1)}(u_0)=\mathrm{P}_6^{(2)}(u_0)$	0.00004	0.0003
$\mathrm{P}_{3,3}^{(1)}(u_0)=\mathrm{P}_{3,3}^{(2)}(u_0)$	0.0023	0.0016
$\mathrm{Q}_{3,3}^{(0)}(u_0)$	-0.0047	-0.0034

in the case $N_1 = N_2 = 100,\ n_1 = 5,\ n_2 = 15$,

	$\alpha = 0.05$	$\alpha = 0.01$
$P_4^{(1)}(u_0)$	-0.0028	-0.0012
$P_4^{(2)}(u_0)$	0.0030	0.0012
$P_6^{(1)}(u_0)$	0.0013	0.0006
$P_6^{(2)}(u_0)$	-0.0013	-0.0006
$P_{3,3}^{(1)}(u_0)$	0.0004	0.0001
$P_{3,3}^{(2)}(u_0)$	0.0022	0.0018
$Q_{3,3}^{(0)}(u_0)$	-0.0040	-0.0018

The effects of non-normality seems to be negligible small in these cases.

Theorem 2.4 *Under the Assumptions* I, II *and* III', *and neglecting the higher powers of* $a_3^{(i)}$, $a_4^{(i)}$, $a_5^{(i)}$ *and* $a_6^{(i)}$, *the simultaneous probability density of*

$$F_i = \frac{N_i S_i^2 - n_i s_i^2}{n_i s_i^2} \cdot \frac{n_i - 1}{N_i - n_i}, \quad (i = 1, 2)$$

is given by

$$(2.23)\quad p(F_1, F_2) = h_{N_1-n_1,\, n_1-1}(F_1)\ h_{N_2-n_2,\, n_2-1}(F_2)$$
$$\times \left\{ 1 + \sum_{i=1}^{2} (\beta_2^{(i)} - 3)\, p_4^{(i)} + \sum_{i=1}^{2} \frac{1}{10} (\beta_4^{(i)} - 15\beta_2^{(i)} + 30)\, p_6^{(i)} + \sum_{i=1}^{2} \beta_1^{(i)}\, p_{3,3}^{(i)} \right\},$$

where, putting $G_i = \left(1 + \dfrac{N_i - n_i}{n_i - 1} F_i\right)^{-1}$ *and omitting the suffix* i *in both sides of the following equations*

$$(2.24)\quad p_4 = \frac{N-1}{8n N (N - n + 2)} \Big[(n-1)\{(n-1)N - (n+1)\} - 2(N+1) G \{(n-1)N - (n+1)\} + \frac{N+1}{n+1} G^2 \{(n-1)N^2 + 2(n-2)N - 3(n+1)\} \Big]$$

$$(2.25)\quad p_6 = \frac{-5(N-1)}{24n^2N^2(N-n+2)(N-n+4)} \Big[(n-1)\{(n-1)^2 N^3 - 2(n-1)^2(n-2)N^2 + (4n^3 - 13n^2 - 2n + 3)N - 2n(n+1)(n-3)\}$$
$$- 3(n-1)(N+1)G\{(n-1)N^3 - 2(n^2-2n+3)N^2 + (4n^2-3n-9)N - 2n(n+1)\} + \frac{3(N+1)}{n+1} G^2\{(n-1)^2N^4 - (2n^3-7n^2+16n-11)N^3$$
$$- (2n^3 - 11n^2 + 40n - 39)N^2 + (10n^3 - 7n^2 - 32n + 45)N - 6n(n+1)^2\}$$

$$-\frac{(N+1)(N+3)}{(n+1)(n+3)}G^3\{(n-1)^2N^4-(2n^3-7n^2+20n-15)N^3$$
$$-(6n^3-15n^2+64n-71)N^2+(18n^3+17n^2-76n+105)N$$
$$-10n(n+1)(n+3)\}\Big]$$

$$(2.26)\quad p_{3,3}=\frac{N-1}{24n^2N^2(N-n+2)(N-n+4)}\Big[(n-1)\{(3n^2-6n+5)N^3$$
$$-(3n^3-18n^2+25n-20)N^2+(15n^3-49n^2+11n+15)N$$
$$-10n(n+1)(n-3)\}$$
$$-3(N+1)G\{(3n^2-6n+5)N^3-(3n^3-16n^2+29n-30)N^2$$
$$+(17n^3-23n^2-15n+45)N-10n(n^2-1)\}$$
$$+\frac{3(N+1)}{n+1}G^2\{(3n^2-6n+5)N^4-(3n^3-23n^2+51n-55)N^3$$
$$+(10n^3+45n^2-140n+195)N^2+(47n^3-11n^2-133n+225)N$$
$$-30n(n+1)^2\}$$
$$-\frac{(N+1)(N+3)}{(n+1)(n+3)}G^3\{(3n^2-6n+5)N^4-(3n^3-27n^2+67n-75)N^3$$
$$+(6n^3+89n^2-252n+355)N^2+5(19n^3+21n^2-73n+105)N$$
$$-50n(n+1)(n+3)\}\Big].$$

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