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NOTE ON ENUMERATION OF 7 × 7 LATIN SQUARES

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INTRODUCTION. After unifying R. A. FISHER's and S. M. JACOB's results on the number of reduced 6 × 6 Latin squares, ALBERT SADE [4, 5] obtained 16,942,080 as the number of reduced 7 × 7 Latin squares. His tactful method uses successive classification according to the Hall rectangles, and he also explains [6] the discrepancy between his and H. W. NORTON's results. His exhaustive method, however, seems to lack suitable self-checking device, and it appears desirable to improve his method so as to make it self-contained.

In the present paper the author refines SADE's method, extending classification over all Latin squares, and revealing a symmetric relation among the fundamental incidence matrices. This relation, together with the ERDÖS-KAPLANSKY formula [1] for the extensions of Latin rectangles, furnishes a powerful check of the enumeration, making SADE's result absolutely certain (although there are found some insignificant mistakes in his paper [5]).

1. Preliminaries.

1.1. Hall rectangles. An $r \times n$ *Latin rectangle* R is an array of the integers $1, 2, \cdots, n$ in the $r \times n$ matrix such that no coincidences of integers occur in any of its rows or columns. In case $r = n$, it is an $n \times n$ *Latin square.* We sometimes ignore the order of appearances of integers in each column of R, deriving from it an $r \times n$ *Hall rectangle* H. This may be alternately defined as an n-dimensional vector

$$(1) \qquad H = (S_1, \cdots, S_n)$$

with n subsets S_i of $N = \{1, 2, \cdots, n\}$ as components such that

(2) each S_i contains just r integers

and

(3) union of any k of the S_i contains at least k integers.

MARSHALL HALL [2, 3] proved that the condition (3) is also sufficient for a vector (1), (2) to be derived from a suitable Latin rectangle. An $r \times n$ Hall rectangle will be called an *r-rectangle* and the 0-rectangle H_0 is defined to have all components ϕ (the empty subset).

1.2. Equivalence and class of Hall rectangles. An r-rectangle H is transformed into one such by

(i) Permutation α of vector indices:

$$H\alpha = (S_{\alpha(1)}, \cdots, S_{\alpha(n)});$$

(ii) Permutation β of integers:

$$\beta H = (\beta S_1, \cdots, \beta S_n), \quad \beta S = \{\beta(i) : i \in S\};$$

(iii) Interchanging of indices and integers (*conjugation*):

$$H^\dagger = (S^\dagger_1, \cdots, S^\dagger_n), \quad S^\dagger_i = \{j : i \in S_j\}.$$

These transformations constitute a group $\mathfrak{G}$ of operations on the totality of r-rectangles. Its order apparently divides $2(n!)^2$. Equivalence and class are defined with respect to this group. Any of these classes will be called an *r-class*, and for such a class $\mathfrak{G}$ we denote by $\lambda(\mathfrak{G})$ the number of r-rectangles contained in it. Denoting by h_r the number of r-classes, we obviously have

$$\lambda(\mathfrak{G}) \geqq 2(n!)^2. \tag{4}$$

1.3. Complementary rectangle and class. The *complementary rectangle* H^* of H, (1), is defined by

$$H^* = (N - S_1, \cdots, N - S_n).$$

Then equivalence and hence classes of r-rectangles are transferred into $(n-r)$-rectangles by this operation. The *complementary class* of $\mathfrak{G}$ is denoted by $\mathfrak{G}^*$. We have then,

$$\lambda(\mathfrak{G}) = \lambda(\mathfrak{G}^*), \tag{5}$$

$$h_r = h_{n-r}. \tag{5'}$$

1.4. Incidence relations among classes. An r-rectangle H is said to *contain* the s-rectangle K, if each component of H contains the corresponding component of K, and we denote this relation by $H \supset K$ or $K \subset H$. Obviously

$$H \supset K \rightleftarrows H\alpha \supset K\alpha \rightleftarrows \beta H \supset \beta K \rightleftarrows H^\dagger \supset K^\dagger, \tag{6}$$

$$H \supset K \rightleftarrows K^* \supset H^*. \tag{7}$$

The totality $\mathfrak{A}_s(H)$ of s-rectangles incident with an r-rectangle H is classified according to the s-classes $\mathfrak{D}$:

$$\mathfrak{A}_s(H) = \sum \alpha(H, \mathfrak{D})\mathfrak{D}.$$

The incidence number $\alpha(H, \mathfrak{D})$ denotes the number of s-rectangles in $\mathfrak{A}_s(H) \cap \mathfrak{D}$. But from (6) this number depends only on the class $\mathfrak{C}$ of H, and we may write symbolically

$$\mathfrak{A}_s(\mathfrak{G}) = \sum \alpha(\mathfrak{G}, \mathfrak{D})\mathfrak{D}. \tag{8}$$

Denote by $M_{r,s}$ the $h_r \times h_s$ *incidence matrix* $\|\alpha(\mathfrak{C}, \mathfrak{D})\|$, and by Δ_r the $h_r \times h_r$ diagonal matrix

$$\Delta_r = \text{diag.}\{\lambda(\mathfrak{G})\}.$$

Then we have the symmetric relation;

LEMMA 1. $\Delta_r M_{r,s} = M^T_{s,r} \Delta_s = (\Delta_s M_{s,r})^T$,

T indicating matrix transposition.

In fact, the two incidence matrices among the totalities of r- and s-rectangles (instead of their classes) are transpose of each other. We obtain $M_{r,s}$ and $M_{s,r}$ by selecting suitable number of rows and then pooling columns. Thus if we pool the rows of the first two matrices in the same fashion as the columns, the resulting matrices are again transpose of each other; but these are represented by $\Delta_r M_{r,s}$ and $\Delta_s M_{s,r}$.

Finally if we number for $r \neq n/2$, the r- and $(n-r)$-classes in such a way that the $\mathfrak{G}$'s appear in the same order as the corresponding $\mathfrak{G}^*$'s, we have by (7)

(9) $$M_{n-r,\, n-s} = M_{r,\, s},$$

and combining this with (5) and with $\Delta_{n-r} = \Delta_r$,

(10) $$\Delta_{n-s} M_{n-s,\, n-r} = (\Delta_r M_{r,\, s})^T.$$

REMARK. It can be shown by examples that, in case $n \geqq 8$, the $(n/2)$-classes are not always self-complementary, contrary to SADE's assertion [5, p. 2] that the invariant uniquely determines the class.

2. Enumeration of Latin squares.

2.1. Enumeration of Latin rectangles.

LEMMA 2. *An ascending chain*

$$H_0 \subset H_1 \subset \cdots \subset H_r$$

of Hall rectangles H_i, H_i being an i-rectangle, corresponds biuniquely to an $r \times n$ Latin rectangle.

Thus the matric product

(11) $$M_{0,1} M_{1,2} \cdots M_{r-1,\, r} = G_r$$

is a $1 \times h_r$ matrix whose $\mathfrak{G}$-component gives the number of $r \times n$ Latin rectangles, the derived Hall rectangle of which belongs to $\mathfrak{G}$. And

(12) $$L(r, n) = G_r E_r$$

gives the number of $r \times n$ Latin rectangles, where E_r is the $h_r \times 1$ matrix with all components equal to 1. In particular,

(13) $$L_n = G_n = M_{0,1} M_{1,2} \cdots M_{n-1,\, n}.$$

Thus the task of finding the number L_n of $n \times n$ Latin squares is reduced to that of finding the fundamental incidence matrices $M_{r,\, r+1}$ $(r = 0, 1, \cdots, n-1)$. These matrices must satisfy the symmetric relation (10); or if we write $F_r = \Delta_r M_{r,\, r+1}$,

(14) $$F_{n-1-r} = F_r^T.$$

2.2. Extensions of Latin rectangles. Let R be an $r \times n$ Latin rectangle. ERDÖS and KAPLANSKY [1] have given a formula for the number of ways of extending R by one new row. This number, $\nu(R)$, however, depends only on the structure of its Hall rectangle H; in fact only on the class $\mathfrak{G}$ of this r-rectangle H. This number $\nu(R) = \nu(H) = \nu(\mathfrak{G})$ is the number of ways of selecting a so-called ***distinct representative*** (MARSHALL HALL [2, 3]) of the n subsets in the complementary Hall rectangle H^*. This function of the class $\mathfrak{C}$ is given as the $\mathfrak{G}$-component of the $h_r \times 1$ matrix

$$M_{r,\,r+1} E_{r+1} = \| \nu(\mathfrak{C}) \|, \tag{15}$$

i.e., as the sum of numbers in the $\mathfrak{G}$-th row of $M_{r,\,r+1}$. In the same manner as in Lemma 2 and (15), the $h_r \times 1$ matrix

$$M_{r,\,r+1} M_{r+1,\,r+2} \cdots M_{n-1,\,n} = \bar{G}_{n-r} \tag{16}$$

gives the number of extensions of a given $r \times n$ Latin rectangle to a whole Latin square. We have already observed that

$$L_n = G_r \bar{G}_{n-r} \qquad (r = 0, 1, \cdots, n-1). \tag{17}$$

But we find on the other hand

LEMMA 3. $\qquad G_r = (\varDelta_r \bar{G}_r)^T = \bar{G}_r^T \varDelta_r$.

In fact, the $\mathfrak{G}$-component of G_r gives the number of $r \times n$ Latin rectangles corresponding to a fixed Hall rectangle H in $\mathfrak{G}$, while that of G_r gives the number of $r \times n$ Latin rectangles corresponding to an arbitrary Hall rectangle H in $\mathfrak{G}$.

This Lemma in turn enables us to compute the numbers $\lambda(\mathfrak{G})$ in terms of the fundamental incidence matrices $M_{r,\,r+1}$.

3. Application to the case $n = 7$.

3.1. Notations. In the case $n = 7$, we have $h_0 = h_1 = h_6 = h_7 = 1$, $h_2 = h_5 = 4$, $h_3 = h_4 = 14$. In this section we follow after SADE's notations [5] of 2- and 3-classes, with modifications (i)—(v):

(i) His class M is absorbed into the class K, and S in Q. These are consequences of the conjugation.

(ii) There appears a new class, denoted by I, and is represented by the so-called YOUDEN square

$$R_I = \begin{matrix} 1\ 2\ 3\ 4\ 5\ 6\ 7 \\ 2\ 3\ 4\ 5\ 6\ 7\ 1 \\ 4\ 5\ 6\ 7\ 1\ 2\ 3 \end{matrix}$$

or by a regular heptagon with all its 14 diagonals when we connect indexed points in the same column of R. This is a result of taking away the reduced restriction.

(iii) Classes are arranged in the order of magnitudes of automorph groups.

(iv) The supplemented columns represent the numbers $\lambda(\mathfrak{C})/7!$ and $\nu(\mathfrak{G})$. (See 1.3. and 2.2.)

(v) The table for $M_{r,\,r+1}$ was prepared by counting out all possibilities at each step, without restricting to reduced rectangles.

3.2. Verifications.

(i) The Δ_r are computed in two ways, Δ_r and Δ_{7-r}, from Lemma 3 and verified to coincide and satisfy (4) and (5).

(ii) The $M_{r,\,r+1}$ are multiplied from left by Δ_r to reveal the fundamental symmetric relations, Lemma 1, or (14).

(iii) In order to provide a verification from another source, the function $\nu(\mathfrak{G})$ was computed.

First of all the ν are, the *rencontre* number $\Delta^7 0! = 1854$ for $r = 1$; the generalized *ménages* or TOUCHARD numbers u for $r = 2$ (Cf. [5], [7]). It is also evident that $\nu = 2^m$ for $r = 5$, where m is the number of cycles of the permutation which is formed from the 2×7 rectangle in the complementary class; and $\nu = 1$ for $r = 6$.

For the cases $r = 3$ and $r = 4$, we must make use of the ERDÖS-KAPLANSKY formula cited in 2.2. We may write this in the form for $r = 3$:

$$\nu(\mathfrak{G}) = \nu = 32 + 16(\xi - \eta) + 4\zeta - 4\rho,$$

the same for $r = 4$ is ([1, p. 231])

$$\nu(\mathfrak{G}^*) = \nu^*(\mathfrak{G}) = \nu^* = 108 - 12(\xi - \eta) - 3\zeta + 4\rho,$$

where ξ, η, ζ and ρ are the intercalate functions of $\mathfrak{G}$, counting the instances for an H in $\mathfrak{G}$, in which

ξ: 2 integers are contained together in 2 subsets;

η: 2 integers are contained together in 3 subsets, or dually, 3 integers are contained together in 2 subsets;

ζ: triangular relation "$i, j \in S_u$; $j, k \in S_v$; $k, i \in S_w$" occurs (i, j, k all $\neq$, and u, v, w all $\neq$);

ρ: the same as ζ, but plus the conditon "$k \in S_u$".

The procedure (ii) verifies the whole incidence numbers, except on the main diagonal of $M_{3,\,4}$; while (iii) verifies the sum of each row of $M_{r,\,r+1}$, hence together with (ii) those numbers just excluded.

(iv) The elements of $\bar{G}_r$ are compared with SADE's enumeration to show the complete coincidence; especially $\bar{G}_7 = L_7 = 7! \cdot 12{,}198{,}297{,}600 = 7!\,6! \cdot 16{,}942{,}080$.

REMARK. SADE's paper contains mistakes in tables [5, p. 6]. In the first of his tables, the eight z's in K-column should be corrected to s's, and hence the (K, z)-entry 8 in the second table should be transferred to (K, s)-entry, improving the latter to the value 12.

3.3. Tables.

(i) Incidence matrices $M_{r,\,r+1}$ ($r = 1, 2, 3, 4, 5$).

$\mathfrak{G}$	$\lambda/7!$	A	C	B	D	ν
E	1	210	420	504	720	1,854

$\mathfrak{G}$	$\lambda/7!$	X	I	J	K	P	U	N	Y	W	Z	R	Q	V	T	ν
A	105/4	4					96		48	96	48			192	96	580
C	105	2		24			24	48	24		48	48	96	144	120	578
B	126						20	20	40	80	40	40	120	80	140	580
D	360		2	7	14	21	10	21	28	28	56	84	112	98	98	579

$\mathfrak{G}$	$\lambda/7!$	X^*	I^*	J^*	K^*	P^*	U^*	N^*	Y^*	W^*	Z^*	R^*	Q^*	V^*	T^*	ν
X	35/6									144						144
I	30		8				24			56		56				144
J	315/2								16	32	32			32	32	144
K	210												48	48	48	144
P	315					8	16				24	16	32	16	32	144
U	360		2			14	2	14	14	14	14	14	28	28		144
N	630						8	16	8	16	8	16	16	24	32	144
Y	630			4			8	8	16	12	4	24	16	24	32	148
W	840	1	2	6			6	12	9	6	12	10	24	36	24	148
Z	1,260			4		6	4	4	2	8	16	24	24	24	28	144
R	1,680		1			3	3	6	9	5	18	15	24	30	30	144
Q	2,520				4	4	4	4	4	8	12	16	36	24	28	144
V	2,520			2	4	2	4	6	6	12	12	20	24	32	20	144
T	2,520			2	4	4		8	8	8	14	20	28	20	28	144

$\mathfrak{G}$	$\lambda/7!$	A^*	C^*	B^*	D^*	ν
X^*	35/6	18	36			54
I^*	30				24	24
J^*	315/2		16		16	32
K^*	210				24	24
P^*	315				24	24
U^*	360	7	7	7	10	31
N^*	630		8	4	12	24
Y^*	630	2	4	8	16	30
W^*	840	3		12	12	27
Z^*	1,260	1	4	4	16	25
R^*	1,680		3	3	18	24
Q^*	2,520		4	6	16	26
V^*	2,520	2	6	4	14	26
T^*	2,520	1	5	7	14	27

$\mathfrak{G}$	$\lambda/7!$	E^*	ν
A^*	105/4	8	8
C^*	105	4	4
B^*	126	4	4
D^*	360	2	2

(ii) **The intercalate functions.**

𝔊	ξ	η	ζ	ρ	ν	ν^*
X	15	6	22	30	144	54
I			28		144	24
J	10	4	12	8	144	32
K	6	2	12		144	24
P	7	2	8		144	24
U	7		7	7	144	31
N	2		20		144	24
Y	9	2	10	9	148	30
W	6		11	6	148	27
Z	4		13	1	144	25
R	3		16		144	24
Q	6	1	10	2	144	26
V	4		14	2	144	26
T	5		11	3	144	27

(iii) **The matrices** $G^T{}_r$, $\bar{G}_r$ $(r = 1, 2, 3, 4, 5, 6)$.

𝔊	$G^T{}_r/7!$	$\bar{G}_{7-r}$	$\lambda/7!$
E	1	12,198,297,600	1
A	210	6,635,520	105/4
C	420	6,543,360	105
B	504	6,604,800	126
D	720	6,566,400	360
X	1,680	13,824	35/6
I	1,440	12,288	30
J	15,120	12,288	315/2
K	10,080	11,520	210
P	15,120	11,520	315
U	47,520	10,944	360
N	45,360	11,904	630
Y	60,480	12,288	630
W	80,640	12,288	840
Z	90,720	11,136	1,260
R	100,800	11,328	1,680
Q	181,440	11,136	2,520
V	211,680	11,328	2,520
T	211,680	10,944	2,520
X^*	80,640	288	35/6
I^*	368,640	48	30
J^*	1,935,360	96	315/2

K^*	2,419,200	48	210
P^*	3,628,800	48	315
U^*	3,939,840	132	360
N^*	7,499,520	72	630
Y^*	7,741,440	96	630
W^*	10,321,920	96	840
Z^*	14,031,360	72	1,260
R^*	19,031,040	60	1,680
Q^*	28,062,720	72	2,520
V^*	28,546,560	84	2,520
T^*	27,578,880	84	2,520
A^*	174,182,400	8	105/4
C^*	687,052,800	4	105
B^*	832,204,800	4	126
D^*	2,363,904,000	2	360
E^*	12,198,297,600	1	1

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