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<https://doi.org/10.5109/12947>

出版情報 : 統計数理研究. 4 (1/2), pp.22-32, 1950-12. Research Association of Statistical Sciences

バージョン :

権利関係 :

ON THE STATISTICAL DECISION FUNCTION¹⁾ I.

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1. Introduction

Let $X = \{X_i\}$ ($i=1, 2, 3, \dots$) be an infinite sequence of chance variables. Any particular observation x on X is given by a sequence $x = \{x_i\}$ ($i=1, 2, 3, \dots$) of real values, where x_i denotes the observed value of X_i . Let the space of all the sample points x be M , the Borel field which contains all the sets $\{x; x_i < a_i, i=1, 2, \dots\}$ be K , where a_i are real numbers or $+\infty$, and the Lebesgue measure on K be m . Suppose that the distribution function $F(x)$ of X is not known, it is, however, known that $F(x)$ has the probability density function $p(F|x)$ and is an element of a given class $\mathcal{Q}$ of distribution functions. There is, furthermore, a space D^* given whose elements d represent the possible decision that can be made in the problem under consideration. The problem is to construct a function $D=D(x)$ called the statistical decision function, which associates with each sample point x an element d of D^* so that the decision $d=D(x)$ is made when x is observed. Let $W(F, d)$ be the loss suffered by the statistician when F is the true distribution of X and the decision d is made. We assume that $W(F, d)$ is a non-negative bounded measurable function of F and d . Let $c(n)$ be the cost of making n observations, i. e., $c(n)$ is the cost of observing the values of $x_1, \dots, x_n$. Thus, when the true distribution function of X is F , and if we decide the element of D^* by the decision function $d_n(x)$ which depends only on the first n coordinates $x_1, x_2, \dots, x_n$ of the sample x , the loss is given by the following sum

$$(1.01) \quad r(F, d_n(x)) = W(F, d_n(x)) + c(n).$$

A sequential statistical decision function $\underline{D}$ is composed of the following two sequences $B = \{B_j\}$ and $D = \{d_j\}$.

(i) $\{B_j\}$ is the sequence of B_0 and disjoint subsets $B_1, B_2, \dots, B_j, \dots$ of M , where B_j depends on the first j coordinates $x_1, \dots, x_j$ of a sample x and B_j indicates that the sampling should stop at the j -th observation when $x \in B_j$ ($j=1, 2, \dots$), B_0 is the event that we do not sample at all, but take some decision immediately and it will have probability either 0 or 1. It should be

$$(1.02) \quad \sum_{j=0}^{\infty} P_r(B_j|F) = 1 \text{ for all } F \in \mathcal{Q}.$$

This sequence $B = \{B_j\}$ ($j=0, 1, 2, \dots$) is called a sequential procedure.

(ii) $\{d_j\}$ is the sequence of d_0 and decision functions $d_1(x), d_2(x), \dots$,

1) Communicated at the Autumn-Meeting of Japanese Math. Soc., at Kôbe Univ., October 20, 1950.

$d_j(x), \dots$, where d_0 is an element of D^* and $d_j(x)$ is the function of the first j coordinates $x_1, \dots, x_j$ of $x \in B_j$ and its value is some decision, i.e., some element of D^* . This sequence $D = \{d_j\}$ is also called a decision function.

The sequential decision function $\underline{D}$ which is determined by two sequence B and D will be denoted by $\underline{D} = (B, D)$. Then the average loss caused by the sequential decision function $\underline{D}$ when F is the true distribution function of X is given by

$$(1.03) \quad r(F, \underline{D}) = \sum_{j=0}^{\infty} \int_{B_j} r(F, d_j(x)) p(F|x) dx.$$

Here, we assume that the series on the right hand member of (1) is always convergent in our problem. Let ξ be an a priori distribution on $\mathcal{Q}$, i. e., ξ is a probability measure defined over a suitably chosen Borel field of subsets of $\mathcal{Q}$.

Then the expected value of $r(F, \underline{D})$ is given by

$$(1.04) \quad r(\xi, \underline{D}) = \int_{\mathcal{Q}} r(F, \underline{D}) d\xi$$

$r(\xi, \underline{D})$ is called the risk when ξ is the a priori distribution on $\mathcal{Q}$ and $\underline{D}$ is the decision function adopted. The sequential decision function $\underline{D}^*$ is called a Bayes solution relative to the a priori distribution ξ , if

$$(1.05) \quad r(\xi, \underline{D}^*) = \inf_p r(\xi, \underline{D}).$$

Our object is to determine the necessary and sufficient condition so that a sequential decision function will be a Bayes solution relative to the a priori distribution ξ .

2. Theorem and its Proof

Let D_j^* be the set of all decision functions $d_j(x)$ which depends only on the first j coordinates $x_1, \dots, x_j$ of x ($j=1, 2, \dots$). For $j=0$, d_0 is some element of D^* , i. e., $D_0^* = D^*$. Let $\{d_{j_n}\}$ and d_j^* be a sequence of elements of D_j^* and an element of D_j^* , respectively. If it is valid that

$$W(F, d_{j_n}(x)) \rightarrow W(F, d_j^*(x)),$$

as $n \rightarrow \infty$, for all $F \in \mathcal{Q}$ all $x \in M$, then we say that $\{d_{j_n}\}$ converges to d_j^* . We assume that D_j^* is compact in the sense of the convergence of the above definition.

LEMMA.

For any a priori distribution ξ of F and for any j , there exists a decision function $d_j^*(x)$ such that

$$(2.01) \quad r(\xi, d_j^*(x)) = \inf_{d_j \in D_j^*} r(\xi, d_j(x)) \quad \text{for all } x.$$

where $d_j(x)$ is an element of D_j^* and

$$(2.02) \quad r(\xi, d_j(x)) = \int_{\Omega} r(F, d_j(x)) p(F|x) d\xi.$$

As this lemma will be proved in the same way as done in GIRSHICK'S^b, we do not refer to it here.

We write

$$(2.03) \quad r_j(\xi, x) = r(\xi, d_j(x)) = \inf_{d_j} r(\xi, d_j(x))$$

Let N be any fixed integer, and by the induction backwards we define functions $\alpha_{jN}(\xi, x)$ ($j=0, 1, \dots, N$) which depends only on the first j coordinates of x .

That is

$$(2.04) \quad \alpha_{NN}(\xi, x) = r_N(\xi, x)$$

and, for $j < N$

$$(2.05) \quad \alpha_{jN}(\xi, x) = \min (r_j(\xi, x), E_j\{\alpha_{j+1,N}(\xi, x)\})$$

where E_j is the conditional expectation given $x_1, \dots, x_j$, i. e.,

$$(2.06) \quad E_j\{\alpha_{j+1,N}(\xi, x)\} = \int_{-\infty}^{\infty} \alpha_{j+1,N}(\xi, x) dx_{j+1}.$$

It can be seen easily that when j is fixed, $\alpha_{jN}(\xi, x)$ is non-negative and non-increasing as N increases. Therefore there exists $\lim_{N \rightarrow \infty} \alpha_{jN}(\xi, x)$, and we represent this limit as $\alpha_j(\xi, x)$, i. e.,

$$(2.07) \quad \alpha_j(\xi, x) = \lim_{N \rightarrow \infty} \alpha_{jN}(\xi, x).$$

Then the following relation will hold

$$(2.08) \quad \alpha_j(\xi, x) = \min \{r_j(\xi, x), E_j\{\alpha_{j+1}(\xi, x)\}\},$$

and consequently

$$(2.09) \quad \alpha_j(\xi, x) \leq r_j(\xi, x) \quad (j=0, 1, 2, \dots).$$

By means of these functions $\alpha_j(\xi, x)$ we define subsets S_j of M as follows:

$$(2.10) \quad S_j = \{x; r_i(\xi, x) > \alpha_i(\xi, x) \text{ for } i < j, \text{ and } r_j(\xi, x) = \alpha_j(\xi, x)\}.$$

It is clear that thus defined sequence of subsets $\{S_j\}$ ($j=0, 1, 2, \dots$) forms a sequential procedure.

Let us denote the sequential procedure which consists of the sequence $\{S_j\}$ by S_ξ and the decision function which consists of the sequence $\{d_j\}$ by D° . Let D_ξ be the sequential decision function which is determined by S_ξ and D° , that is,

$$(2.11) \quad D_\xi = (S_\xi, D^\circ)$$

We assume that if for any fixed element x of some subset $A \subset M$,

$$(2.12) \quad r(\xi, d_j(x)) = \int_{\Omega} r(F, d_j(x)) p(F|x) d\xi,$$

has the minimum value with respect to d_j , for $d_j = d_j^\circ$ and $d_j = d_j^*$, then it holds $d_j^\circ(x) = d_j^*(x)$ almost everywhere on A .

Then the following theorem holds.

THEOREM .

There exists a Bayes solution relative to any a priori distribution ξ . The necessary and sufficient condition for that the sequential decision function $\underline{D} = (T, D)$, where $T = \{B_j\}$, $D = \{d_j\}$, will be a Bayes solution relative to the a priori distribution ξ is that

(i) $d_j(x) = d_j^\circ(x)$, almost everywhere on B_j , ($j=0, 1, 2, \dots$)

(ii) the following relation holds except for a set of measure 0

$$(2.13) \quad B_0 = S_0, B_1 \subset S_1, B_2 \subset S_1 + S_2, \dots, B_j \subset S_1 + S_2 + \dots + S_j, \dots$$

Consequently, if we write

$$(2.14) \quad B_i \cap S_l = D_i, S_l - D_i = S'_i, B_l \cap S'_k = B_l^k \quad (l=2, 3, \dots; k=1, 2, \dots, l-1)$$

then it holds

$$(2.15) \quad B_1 = D_1, B_2 = D_2 + B_2^1, B_3 = D_3 + B_3^1 + B_3^2, \dots, B_j = D_j + B_j^1 + B_j^2 + \dots + B_j^{j-1}, \dots$$

and

$$(2.16) \quad S'_i = B_{i+1}^i + B_{i+2}^i + B_{i+3}^i + \dots, \quad (i=1, 2, 3, \dots).$$

(iii) if there exists some k such that $m(S'_k) > 0$, then except for the set of measure 0 it holds

$$(2.17) \quad r_{k+1}(\xi, x) = E_{k+1} \{ r_{k+2}(\xi, x) \},$$

on the complement C B_{k+1}^k of B_{k+1}^k with respect to S'_k

$$(2.18) \quad r_{k+2}(\xi, x) = E_{k+2} \{ r_{k+3}(\xi, x) \},$$

on the complement $C[B_{k+1}^k + B_{k+2}^k]$ of $[B_{k+1}^k + B_{k+2}^k]$ with respect to S'_k , and so on.

PROOF: It can be shown that the above defined sequential decision function $\underline{D}_\xi = (S_\xi, D^\circ)$ is a Bayes solution relative to the a priori distribution ξ in a analogous way as done in Girshick's¹⁾, so we will omit it here. Let $\underline{D} = (T, D)$, $T = \{B_j\}$, $D = \{d_j\}$ be a Bayes solution relative to the a priori distribution ξ .

At first we will prove that the condition (i) is necessary. It follows from the lemma that

$$(2.19) \quad r(\xi, d_j(x)) \geq r(\xi, d_j^\circ(x)).$$

Now let us put

$$(2.20) \quad R_j = \{x; x \in B_j \text{ and } r(\xi, d_j(x)) > r(\xi, d_j^\circ(x))\},$$

and suppose that $m(R_j) > 0$.

Then, we can choose $\delta > 0$ and $R_j' \subset R_j$ such that $m(R_j') > 0$, and the following relation holds for $x \in R_j'$

$$(2.21) \quad r(\xi, d_j(x)) > r(\xi, d_j^\circ(x)) + \delta.$$

Consequently it follows

$$\begin{aligned} (2.22) \quad r(\xi; B_j, d_j) &= \int_{B_j} \int_{\Omega} r(F, d_j(x)) p(F|x) d\xi dx = \int_{B_j} r(\xi, d_j(x)) dx \\ &> \int_{R_j'} r(\xi, d_j^\circ(x)) dx + \delta \int_{R_j'} dx + \int_{CR_j'} r(\xi, d_j^\circ(x)) dx \\ &= \int_{B_j} r(\xi, d_j^\circ(x)) p x + \delta m(R_j') \\ &> \int_{B_j} r(\xi, d_j^\circ(x)) dx = r(\xi; B_j, d_j^\circ). \end{aligned}$$

(where CR_j' denotes the complement of R_j' with respect to B_j)

And generally it holds

$$(2.23) \quad r(\xi; B_i, d_i) \geq r(\xi; B_i, d_i^\circ).$$

Consequently, it holds

$$(2.24) \quad r(\xi, \underline{D}) = \sum_{j=0}^{\infty} r(\xi, B_j, d_j) > \sum_{j=0}^{\infty} r(\xi; B_j, d_j^\circ) = r(\xi; T, D^\circ)$$

Thus we have a sequential decision function $\underline{D}' = (T, D^\circ)$ such that $r(\xi, \underline{D}) > r(\xi, \underline{D}')$. This contradicts with the assumption that the sequential decision function $\underline{D} = (T, D)$ is a Bayes solution relative to ξ .

Therefore, it holds on B_j

$$(2.26) \quad r(\xi, d_j(x)) = r(\xi, d_j^\circ(x)) \quad (j=0, 1, 2, \dots)$$

except for a set of measure 0. Consequently it follows from our assumption that $d_j(x) = d_j^\circ(x)$ almost everywhere on B_j ($j=0, 1, 2, \dots$).

Next we will prove that the condition (ii) is necessary.

We put

$$(2.27) \quad B_i' = (B_i - D_i) \cap C[S_1 + S_2 + \dots + S_i],$$

where $C[S_1 + \dots + S_i]$ is the complement of $[S_1 + \dots + S_i]$ with respect to M ,

$$(2.28) \quad B_i'' = (B_i - D_i) \cap [S_1 + S_2 + \dots + S_i]$$

$$S_i^l = B_i' \cap S_i \quad (i = l+1, l+2, \dots; l = 1, 2, \dots)$$

Let $S_i^l(1, 2, \dots, l)$ be the intersection of S_i^l and the subset $C(x_1, \dots, x_l)$ of M defined by $x_1 = \text{const.}, \dots, x_l = \text{const.}$ and $CS_i^l(1, 2, \dots, l)$ be the complement of $S_i^l(1, 2, \dots, l)$ with respect to $C(x_1, \dots, x_l)$.

To prove that $m(B_i') = 0$ ($i = 1, 2, \dots$), we assume that $m(B_i') > 0$ for some l .

Now, by the definition of S_j , it follows that if $x \in B_i'$

$$(2.29) \quad r_i(\xi, x) > \alpha_i(\xi, x) = E_i \{ \alpha_{i+1}(\xi, x) \} = \int_{-\infty}^{\infty} \alpha_{i+1}(\xi, x) dx_{i+1} \\ = \int_{S_{i+1}^l(1, 2, \dots, l)} \alpha_{i+1}(\xi, x) dx_{i+1} + \int_{CS_{i+1}^l(1, 2, \dots, l)} \alpha_{i+1}(\xi, x) dx_{i+1}.$$

And if $x \in S_{i+1}^l(1, 2, \dots, l)$, then we have

$$(2.30) \quad \alpha_{i+1}(\xi, x) = r_{i+1}(\xi, x).$$

Therefore it follows

$$(2.31) \quad r_i(\xi, x) > \int_{S_{i+1}^l(1, 2, \dots, l)} r_{i+1}(\xi, x) dx_{i+1} + \int_{CS_{i+1}^l(1, 2, \dots, l)} \alpha_{i+1}(\xi, x) dx_{i+1}.$$

If $x \in CS_{i+1}^l(1, 2, \dots, l)$, it holds by the definition of S_j

$$(2.32) \quad r_{i+1}(\xi, x) > \alpha_{i+1}(\xi, x)$$

hence

$$\alpha_{i+1}(\xi, x) = E_{i+1} \{ \alpha_{i+2}(\xi, x) \}.$$

Consequently it follows

$$(2.33) \quad \int_{CS_{i+1}^l(1, 2, \dots, l)} \alpha_{i+1}(\xi, x) dx_{i+1} = \int_{CS_{i+1}^l(1, 2, \dots, l)} \int_{-\infty}^{\infty} \alpha_{i+2}(\xi, x) dx_{i+2} dx_{i+1} \\ = \int_{S_{i+2}^l(1, 2, \dots, l)} \int_{-\infty}^{\infty} \alpha_{i+2}(\xi, x) dx_{i+2} dx_{i+1} + \int_{C[S_{i+1}^l(1, 2, \dots, l) + S_{i+2}^l(1, 2, \dots, l)]} \int_{-\infty}^{\infty} \alpha_{i+2}(\xi, x) dx_{i+2} dx_{i+1} \\ = \int_{S_{i+2}^l(1, 2, \dots, l)} \int_{S_{i+2}^l(1, 2, \dots, l, l+1)} \alpha_{i+2}(\xi, x) dx_{i+2} dx_{i+1} \\ + \int_{S_{i+2}^l(1, \dots, l)} \int_{CS_{i+2}^l(1, 2, \dots, l, l+1)} \alpha_{i+2}(\xi, x) dx_{i+2} dx_{i+1} \\ + \int_{C[S_{i+1}^l(1, \dots, l) + S_{i+2}^l(1, \dots, l)]} \int_{-\infty}^{\infty} \alpha_{i+2}(\xi, x) dx_{i+2} dx_{i+1} \\ = \int_{S_{i+2}^l(1, \dots, l)} r_{i+2}(\xi, x) dx_{i+2} dx_{i+1} + \int_{C[S_{i+1}^l(1, \dots, l) + S_{i+2}^l(1, \dots, l)]} \alpha_{i+2}(\xi, x) dx_{i+2} dx_{i+1}.$$

Since we have

$$(2.33) \quad \alpha_{l+2}(\xi, x) = r_{l+2}(\xi, x) \text{ on } S_{l+2}^l(1, \dots, l, l+1).$$

From (2.31) and (2.33) it follows

$$(2.34) \quad r_l(\xi, x) > \int_{S_{l+1}^l(1, \dots, l)} r_{l+1}(\xi, x) dx_{l+1} + \int_{S_{l+2}^l(1, \dots, l)} r_{l+2}(\xi, x) dx_{l+2} dx_{l+1} \\ + \int_{C[S_{l+1}^l(1, \dots, l) + S_{l+2}^l(1, \dots, l)]} \alpha_{l+2}(\xi, x) dx_{l+2} dx_{l+1}.$$

If $x \in B_l'$, using the analogous method as above, we can conclude the following relation,

$$(2.35) \quad r_l(\xi, x) > \int_{S_{l+1}^l(1, \dots, l)} r_{l+1}(\xi, x) dx_{l+1} + \int_{S_{l+2}^l(1, \dots, l)} r_{l+2}(\xi, x) dx_{l+2} dx_{l+1} \\ + \int_{S_{l+3}^l(1, \dots, l)} r_{l+3}(\xi, x) dx_{l+3} dx_{l+2} dx_{l+1} + \dots$$

Therefore, if $m(B_l') > 0$, we have

$$(2.36) \quad \int_{B_l'} r_l(\xi, x) dx > \int_{S_{l+1}^l} r_{l+1}(\xi, x) dx + \int_{S_{l+2}^l} r_{l+2}(\xi, x) dx + \int_{S_{l+3}^l} r_{l+3}(\xi, x) dx + \dots$$

Now, we construct the sequential procedure $T^* = \{C_j\}$ as follows:

$$C_0 = B_0, C_1 = B_1, \dots, C_{l-1} = B_{l-1}, C_l = D_l + B_l'', C_{l+1} = B_{l+1} + S_{l+1}^l, C_{l+2} = B_{l+2} + S_{l+2}^l, \dots$$

Then, it is clear that any two sets C_i and C_j ($i \neq j$) are disjoint, and

$$(2.37) \quad C_1 + C_2 + C_3 + \dots \\ = B_1 + \dots + B_{l-1} + (D_l + B_l'' + S_{l+1}^l + S_{l+2}^l + \dots) + B_{l+1} + B_{l+2} + \dots \\ = M.$$

Therefore $T^* = \{C_j\}$ is certainly a sequential procedure. Then, for the sequential decision function $\underline{D}^*$, which is determined by T^* and D , we have

$$(2.38) \quad r(\xi, D^*) = r_0 P_r(C_0) + \int_{C_1} r_1(\xi, x) dx + \dots + \int_{C_{l+1}} r_{l+1}(\xi, x) dx \\ + \int_{C_l} r_l(\xi, x) dx + \int_{C_{l+1}} r_{l+1}(\xi, x) dx + \dots \\ = r_0 P_r(B_0) + \int_{B_1} r_1(\xi, x) dx + \dots + \int_{B_{l-1}} r_{l-1}(\xi, x) dx + \int_{D_l + B_l''} r_l(\xi, x) dx \\ + \int_{B_{l+1} + S_{l+1}^l} r_{l+1}(\xi, x) dx + \int_{B_{l+2} + S_{l+2}^l} r_{l+2}(\xi, x) dx + \dots$$

$$\begin{aligned}
&= r_0 P_r(B_0) + \int_{B_1} r_1(\xi, x) dx + \dots + \int_{B_{l-1}} r_{l-1}(\xi, x) dx \\
&\quad + \left\{ \int_{D_l + B_{l'}''} r_l(\xi, x) dx + \int_{S_{l+1}^l} r_{l+1}(\xi, x) dx + \int_{S_{l+2}^l} r_{l+2}(\xi, x) dx + \dots \right\} \\
&\quad + \int_{B_{l+1}} r_{l+1}(\xi, x) dx + \int_{B_{l+2}} r_{l+2}(\xi, x) dx + \dots
\end{aligned}$$

So that, from (2.35) and (2.38), it follows

$$\begin{aligned}
(2.39) \quad r(\xi, D^*) &< r_0 P_r(B_0) + \int_{B_1} r_1(\xi, x) dx + \dots + \int_{B_{l-1}} r_{l-1}(\xi, x) dx \\
&\quad + \left\{ \int_{D_l + B_{l'}''} r_l(\xi, x) dx + \int_{B_{l'}'} r_l(\xi, x) dx \right\} \\
&\quad + \int_{B_{l+1}} r_{l+1}(\xi, x) dx + \int_{B_{l+2}} r_{l+2}(\xi, x) dx + \dots \\
&= r(\xi; T, D^0).
\end{aligned}$$

That is $r(\xi, D^*) < r(\xi; T, D^0)$. This contradicts with the assumption that the sequential decision function (T, D^0) is a Bayes solution relative to the a priori distribution ξ . Consequently we may conclude $m(B_l') = 0$ ($l=1, 2, \dots$). This shows that the condition (ii) is necessary.

Lastly we will prove that the condition (iii) is necessary.

Let us assume that for some k , say $k=1$, $m(S_1') > 0$. If $n \geq N$, it can be shown easily that $r_N(\xi, x) \geq E_N\{r_n(\xi, x)\}$, so that

$$(2.40) \quad r_1(\xi, x) \geq E_1\{r_2(\xi, x)\}.$$

On the other hand, if $x \in S_1'$, we have $r_1(\xi, x) = \alpha_1(\xi, x)$, so that

$$(2.41) \quad r_1(\xi, x) \leq E_1\{\alpha_2(\xi, x)\} \leq E_1\{r_2(\xi, x)\}.$$

From (2.40) and (2.41), it follows

$$(2.42) \quad r_1(\xi, x) = E_1\{r_2(\xi, x)\}, \text{ if } x \in S_1'.$$

Accordingly we have

$$(2.43) \quad \int_{S_1'} r_1(\xi, x) dx_1 = \int_{S_1'} \int_{-\infty}^{\infty} r_2(\xi, x) dx_2 dx_1.$$

Let $B_l'(1, \dots, k)$ be the intersection of B_l' and $C(x_1, \dots, x_k)$, and let $\bar{C}B_l'(1, 2, \dots, k)$ be the complement of $B_l'(1, \dots, k)$ with respect to $C(x_1, \dots, x_k)$, where $C(x_1, \dots, x_k)$ is the subset of M defined by $x_1 = \text{const.}, \dots, x_k = \text{const.}$. Then

$$(2.44) \quad \int_{-\infty}^{\infty} r_2(\xi, x) dx_2 = \int_{B'_2(1)} r_2(\xi, x) dx_2 + \int_{C B'_2(1)} r_2(\xi, x) dx_2.$$

So that (2.43) can be written as follows,

$$(2.44) \quad \int_{S'_1} r_1(\xi, x) dx_1 = \int_{B'_2} r_2(\xi, x) dx_2 dx_1 + \int_{C B'_2} r_2(\xi, x) dx_2 dx_1,$$

where $C B'_2$ is the complement of B'_2 with respect to S'_1 .

Generally it holds that

$$(2.45) \quad r_2(\xi, x) \geq E_2\{r_3(\xi, x)\},$$

but now we assume that the condition (iii) does not hold and on some subset of positive measure of $C B'_2$ it holds

$$(2.46) \quad r_2(\xi, x) > E_2\{r_3(\xi, x)\}.$$

Then, it follows

$$(2.47) \quad \int_{C B'_2} r_2(\xi, x) dx_2 dx_1 > \int_{C B'_2} \int_{-\infty}^{\infty} r_3(\xi, x) dx_3 dx_2 dx_1.$$

Now

$$(2.48) \quad \begin{aligned} \int_{C B'_2(1)} \int_{-\infty}^{\infty} r_3(\xi, x) dx_3 dx_2 &= \int_{B'_3(1)} \int_{-\infty}^{\infty} r_3(\xi, x) dx_3 dx_2 \\ &\quad + \int_{C[B'_2(1)+B'_3(1)]} \int_{-\infty}^{\infty} r_3(\xi, x) dx_3 dx_2 \\ &= \int_{B'_3(1)} \int_{B'_3(1,2)} r_3(\xi, x) dx_3 dx_2 + \int_{B'_3(1)} \int_{C B'_3(1,2)} r_3(\xi, x) dx_3 dx_2 \\ &\quad + \int_{C[B'_2(1)+B'_3(1)]} \int_{-\infty}^{\infty} r_3(\xi, x) dx_3 dx_2 \\ &= \int_{B'_3(1)} r_3(\xi, x) dx_3 dx_2 + \int_{C[B'_2(1)+B'_3(1)]} r_3(\xi, x) dx_3 dx_2. \end{aligned}$$

Therefore it follows from (2.44), (2.47) and (2.48)

$$(2.49) \quad \int_{S'_1} r_1(\xi, x) dx_1 > \int_{B'_2} r_2(\xi, x) dx + \int_{B'_3} r_3(\xi, x) dx + \int_{C[B'_2+B'_3]} r_3(\xi, x) dx,$$

where $C[B'_2+B'_3]$ is the complement of $B'_2+B'_3$ with respect to S'_1 .

Generally it holds that

$$(2.50) \quad r_3(\xi, x) \geq E_3\{r_4(\xi, x)\},$$

but now we assume that on some subset of positive measure of $C[B_2' + B_3']$ it holds

$$(2.51) \quad r_3(\xi, x) > E_3\{r_4(\xi, x)\}.$$

Then it follows

$$(2.52) \quad \int_{C[B_2'+B_3']} r_3(\xi, x) dx > \int_{C[B_2'+B_3']} \int_{-\infty}^{\infty} r_4(\xi, x) dx_4 dx_3 dx_2 dx_1.$$

Let $C[B_2' + B_3'] (1, 2)$ be the intersection of $C[B_2' + B_3']$ and $C(x_1, x_2)$, etc. Then we have

$$\begin{aligned} (2.53) \quad & \int_{C[B_2'+B_3'](1,2)} \int_{-\infty}^{\infty} r_4(\xi, x) dx_4 dx_3 \\ &= \int_{B_4'(1,2)} \int_{-\infty}^{\infty} r_4(\xi, x) dx_4 dx_3 + \int_{C[B_2'+B_3'+B_4'](1,2)} \int_{-\infty}^{\infty} r_4(\xi, x) dx_4 dx_3 \\ &= \int_{B_4'(1,2)} \int_{B_4'(1,2,3)} r_4(\xi, x) dx_4 dx_3 + \int_{B_4'(1,2)} \int_{CB_4'(1,2,3)} r_4(\xi, x) dx_4 dx_3 \\ & \quad + \int_{C[B_2'+B_3'+B_4'](1,2)} \int_{-\infty}^{\infty} r_4(\xi, x) dx_4 dx_3 \\ &= \int_{B_4'(1,2)} r_4(\xi, x) dx_4 dx_3 + \int_{C[B_2'+B_3'+B_4'](1,2)} r_4(\xi, x) dx_4 dx_3. \end{aligned}$$

Consequently

$$(2.54) \quad \int_{C[B_2'+B_3']} \int_{-\infty}^{\infty} r_4(\xi, x) dx_4 dx_3 dx_2 dx_1 = \int_{B_4'} r_4(\xi, x) dx + \int_{C[B_2'+B_3'+B_4']} r_4(\xi, x) dx.$$

From (2.49), (2.52) and (2.54), it follows

$$\begin{aligned} (2.55) \quad & \int_{S_1'} r_1(\xi, x) dx > \int_{B_2'} r_2(\xi, x) dx + \int_{B_3'} r_3(\xi, x) dx + \int_{B_4'} r_4(\xi, x) dx \\ & \quad + \int_{C[B_2'+B_3'+B_4']} r_4(\xi, x) dx. \end{aligned}$$

Thus proceeding as above, if the condition (iii) is not satisfied for $k=1$, then we have

$$(2.56) \quad \int_{S_1'} r_1(\xi, x) dx > \int_{B_2'} r_2(\xi, x) dx + \int_{B_3'} r_3(\xi, x) dx + \dots + \int_{B_i'} r_i(\xi, x) dx + \dots$$

If the condition (iii) is satisfied here, the left and the right hand members of (2.56) are equal.

As when $k=1$, we have generally

$$(2.57) \quad \int_{S_k^c} r_k(\xi, x) dx \geq \int_{B_{k+1}^k} r_{k+1}(\xi, x) dx + \int_{B_{k+2}^k} r_{k+2}(\xi, x) dx + \dots$$

On the otherhand, we have

$$(2.58) \quad D_i + S_i' = S_i \quad (i=1, 2, \dots)$$

$$D_1 = B_1, \quad D_i + B_i^1 + B_i^2 + \dots + B_i^{i-1} = B_i \quad (i=2, 3, \dots).$$

Therefore if the condition (iii) is not satisfied for some k , say $k=1$, it follows from (2.56), (2.57) and (2.58)

$$(2.59) \quad r(\xi; S_\xi, D^\circ) = r_0 P_r(S_0) + \left[\int_{D_1} r_1(\xi, x) dx + \int_{S_1'} r_1(\xi, x) dx \right]$$

$$+ \left[\int_{D_2} r_2(\xi, x) dx + \int_{S_2'} r_2(\xi, x) dx \right] + \dots$$

$$> r_0 P_r(S_0) + \int_{D_1} r_1(\xi, x) dx + \left[\int_{D_2} r_2(\xi, x) dx + \int_{B_2^1} r_2(\xi, x) dx \right]$$

$$+ \left[\int_{D_3} r_3(\xi, x) dx + \int_{B_3^1} r_3(\xi, x) dx + \int_{B_3^2} r_3(\xi, x) dx \right] + \dots$$

$$= r_0 P_r(S_0) + \int_{B_1} r_1(\xi, x) dx + \int_{B_2} r_2(\xi, x) dx + \int_{B_3} r_3(\xi, x) dx + \dots$$

$$= r(\xi; T, D^\circ).$$

That is

$$(2.60) \quad r(\xi; S_\xi, D^\circ) > r(\xi; T, D^\circ).$$

This contradicts with the fact that the sequential decision function $D_\xi = (S_\xi, D^\circ)$ is a Bayes solution relative to the a priori distribution ξ . Therefore the condition (iii) is necessary.

On the otherhand if the conditions (i), (ii) and (iii) are satisfied for the sequential decision function $\underline{D} = (T, D)$, then we known from the above proof that $r(\xi, \underline{D}) = r(\xi, D_\xi)$. Therefore in this case the sequential decision function $\underline{D} = (T, D)$ is a Bayes solution relative to ξ .

Consequently, the conditions (i), (ii) and (iii) are the necessary and sufficient condition, so that a sequential decision function may be a Bayes solution relative to the a priori distribution ξ .

References

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