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SOME TOPICS RELATED TO HURWITZ-LERCH ZETA FUNCTIONS

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Abstract. In this paper, we consider multiplication formulas and their inversion formulas for Hurwitz-Lerch zeta functions. Inversion formulas give simple proofs of known results, and also show generalizations of those results. Next, we give a generalization of digamma and gamma functions in terms of Hurwitz-Lerch zeta functions, and consider its properties. In all the sections, various kinds of results are always proved by multiplication formulas and inversion formulas.

1. Introduction

Definition 1.1 ([3, p.27, (1)]). *We define Hurwitz-Lerch zeta functions by*

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}, \qquad z \in \mathbb{C}, \ |z|<1, \ a \neq 0,-1,-2,\dots, \ s \in \mathbb{C}. \tag{1.1}$$

The function $\Phi(z,s,a)$ was defined by Erdélyi et al in [3] originally. We put

$$\mathbb{C}_1 := \mathbb{C} \setminus [1,+\infty), \qquad \mathbb{C}_2 := \{a\,;\, \Re(a)>0\}, \qquad \mathbb{C}_3 := \mathbb{C} \setminus \{0,-1,-2,\dots\}.$$

In [4, p.5 Theorem 1], the function $\Phi(z,s,a)$ is extended to an analytic function in three variables z, s, a for

$$a \in \mathbb{C}_2, \ z \in \mathbb{C}_1, \ \text{and}$$
$$s \in \mathbb{C} \ \text{or} \ s \in \mathbb{C}\setminus\{1\} \ \text{according to} \ z \neq 1 \ \text{or} \ z=1,$$

by the contour integral representation

$$\Phi(z,s,a) = -\frac{\Gamma(1-s)}{2\pi i}\int_{\infty}^{(+0)} \frac{(-t)^{s-1}e^{-at}}{1-ze^{-t}}dt, \qquad a \in \mathbb{C}_2, \ |\arg(-t)| \le \pi.$$

The contour starts at ∞, encircles the origin once counter-clockwise and returns to its starting point. The initial and final values of $\arg(-t)$ are $-\pi$ and π respectively.

In Section 2, we consider a multiplication formula in Theorem 2.1 and an inversion formula in Theorem 2.2. And by that inversion formula, we can give a simple proof of a known result, and also show generalizations of the result. We give a simple proof of

$$\zeta(2,k/m) = \frac{\pi^2}{\sin^2 \pi k/m} + 2m \sum_{n=1}^{[(m-1)/2]} \sin(2\pi kn/m)\,\mathrm{Cl}_2(2\pi n/m), \tag{1.2}$$

([7, p.358, (16.23)]), where $\mathrm{Cl}_2(\theta)$ is the Clausen integral defined by (2.9). And we obtain formula (2.8), which is a generalization of (1.2).

In Section 3, by using the inversion formula we can show in Theorem 3.1 that $\Phi(z,1,l/m)$ is transcendental if $m \in \mathbb{N}$, $l=1,2,\dots m$, z is algebraic, $|z| \le 1$, $z \neq 1$, which is a generalization of Uchiyama's result [10]. We consider $\Phi_r(z,1,l/m)$, which is a multiple analogue

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of $\Phi(z,1,l/m)$. In Theorem 3.3, we show that $\Phi_r(z,1,l/m)$ is transcendental if z is algebraic, $|z|<1$, $z\neq 1$.

In Section 4, we treat Hurwitz-Lerch Bernoulli functions. By multiplication formulas, we can give a simple proof of

$$m^{1-N}B_N(mx)=\sum_{j=0}^{m-1}B_N(x+j/m) \tag{1.3}$$

([3, p.37, (11)]), where $B_N(x)$ is the Bernoulli function defined by (4.6). We can show interesting formulas (4.5) and (4.9) which seem to be new by the inversion formula.

In Section 5 we introduce $\psi(a,z)$ which is a generalization of digamma functions by using Hurwitz-Lerch zeta functions, and consider its properties. In Theorem 5.4, we can show that if z is algebraic, $|z|\leq 1$, $z\neq 1$, then $\psi_\Phi(l/m,z)$ is transcendental. By the multiplication formula, we can give a simple proof of

$$\psi(ma)=\log m+\frac{1}{m}\sum_{j=0}^{m-1}\psi(a+j/m) \tag{1.4}$$

([3, p.16, (12)]), where $\psi(a)$ is the digamma function defined by (5.1). Let $\gamma:=-\psi(1)$ be the Euler constant. Inversion formulas give a simple proof of

$$\psi(l/m)=-\gamma-\log m-\frac{\pi}{2}\cot(\pi l/m)+\sum_{n=1}^{m-1}\cos(2\pi ln/m)\log\left(2\sin n\pi/m\right) \tag{1.5}$$

([3, p.19, (29)]). We have interesting formulas (5.14) and (5.15), which are generalizations of Gauss' first formula (1.5), by inversion formulas. At the end of this section, we show (5.16) which is a generalization of Gauss' second formula (5.17).

In Section 6, we generalize the notion of gamma functions by using Hurwitz-Lerch zeta functions, and consider its properties in Theorem 6.4. At the end of this paper, we evaluate a special value of generalized gamma functions.

In all sections, various kinds of results are always proved by multiplication formulas and inversion formulas.

2. Multiplication and inversion formulas

Firstly, we quote the multiplication formula for Hurwitz-Lerch zeta functions. If a, s and z satisfy the conditions

$$\begin{aligned}&0<a,\ \ z\in\mathbb{C}_1,\ \ -\pi<\arg z\leq\pi,\ \text{ and}\\ &s\in\mathbb{C}\ \text{ or }\ s\in\mathbb{C}\setminus\{1\}\ \text{ according to }\ z^m\neq 1\ \text{ or }\ z^m=1,\end{aligned} \tag{2.1}$$

we write $(z,s,a)\in D_1$.

Theorem 2.1 (The multiplication formula [9, p.339, (15)])**.** *If $(z,s,a)\in D_1$, $m\in\mathbb{N}$, then we have*

$$\Phi(z,s,ma)=m^{-s}\sum_{j=0}^{m-1}z^j\Phi\left(z^m,s,a+j/m\right). \tag{2.2}$$

Proof. We give a proof for the convenience of readers. It is easy to see that

$$\sum_{n=0}^{\infty} \frac{z^n}{(n+ma)^s} = \sum_{j=0}^{m-1} \sum_{n=0}^{\infty} \frac{z^{mn+j}}{(mn+ma+j)^s}.$$

We can get (2.2) by the above equation. □

In this paper we prove the following inversion formula. If a, s and z satisfy the conditions

$$\begin{aligned} &0<a, \ \ z \in \mathbb{C}_1, \ \ -\pi/m < \arg z^{1/m} \le \pi/m, \ \text{ and} \\ &s \in \mathbb{C} \ \text{ or } \ s \in \mathbb{C}\setminus\{1\} \ \text{ according to } \ z \neq 1 \ \text{ or } \ z=1 \end{aligned} \tag{2.3}$$

we write $(z,s,a) \in D_2$. Let $i = \sqrt{-1}$, and

$$\omega_m^j = \exp(2\pi i j/m), \qquad j \in \mathbb{N}, \ \ 0 \le j \le m-1.$$

Theorem 2.2 (The inversion formula)**.** *If* $(z,s,a) \in D_2$, $m \in \mathbb{N}$, *then we have*

$$\Phi\big(z,s,(a+j)/m\big) = m^{s-1} z^{-j/m} \sum_{n=0}^{m-1} \omega_m^{-jn} \Phi\left(\omega_m^n z^{1/m}, s, a\right). \tag{2.4}$$

Proof. If $J \in \mathbb{N}$, we have

$$\sum_{n=0}^{m-1} \left(\omega_m^j\right)^n \left(\omega_m^n\right)^J = \begin{cases} m & j+J \equiv 0 \bmod m, \\ 0 & \text{otherwise}. \end{cases}$$

From this formula, we have

$$m^{-1} \sum_{n=0}^{m-1} \omega_m^{-jn} \sum_{h=0}^{\infty} \frac{\omega_m^{nh} z^{h/m}}{(h+a)^s} = \sum_{h=0}^{\infty} \frac{z^{(mh+j)/m}}{(mh+a+j)^s}.$$

We obtain (2.4) by the above equation. □

If $(z,s) \in D_1$, we define $\mathrm{Li}_s(z)$ by

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z,s,1). \tag{2.5}$$

We have the next corollary.

Corollary 2.3 (The fraction formula, the rational number formula)**.**

$$\Phi(z,s,a/m) = m^{s-1} \sum_{n=0}^{m-1} \Phi\left(\omega_m^n z^{1/m}, s, a\right), \tag{2.6}$$

$$\Phi(z,s,l/m) = m^{s-1} z^{-l/m} \sum_{n=0}^{m-1} \omega_m^{-ln} \mathrm{Li}_s\left(\omega_m^n z^{1/m}\right). \tag{2.7}$$

Proof. We have the fraction formula (2.6) by putting $j=0$ in (2.4). Taking $a=1$, $j+1=l$, $l=1,2,\dots m$ in (2.4), we have the rational number formula (2.7). □

By putting $z^{1/m}=1$ in (2.7), we have

$$\zeta(s,l/m)=m^{s-1}\sum_{n=0}^{m-1}\omega_m^{-ln}\mathrm{Li}_s\left(\omega_m^n\right). \tag{2.8}$$

If $s=k\in\mathbb{N}$ in (2.5), it is called the k-th polylogarithm. By the definition of Li_2, we have

$$\mathrm{Li}_2\left(e^{i\theta}\right)=\sum_{n=1}^{\infty}\frac{\cos n\theta}{n^2}+i\sum_{n=1}^{\infty}\frac{\sin n\theta}{n^2}, \qquad 0\le\theta<2\pi.$$

Here we recall the Clausen integral defined by

$$\mathrm{Cl}_2(\theta):=\sum_{n=1}^{\infty}\frac{\sin n\theta}{n^2}=-\int_0^{\theta}\log\bigl(2\sin(\theta/2)\bigr)\,d\theta. \tag{2.9}$$

In [6, p.105, (4.22)], the formula

$$\mathrm{Cl}_2(\theta)+\mathrm{Cl}_2(2\pi-\theta)=0$$

is stated. Putting $s=2$ in (2.8) and using the formulas

$$\sum_{n=1}^{\infty}\frac{\cos n\theta}{n^2}=\frac{\pi^2}{6}-\frac{\theta(2\pi-\theta)}{4}, \qquad \zeta(2,x)+\zeta(2,1-x)=\frac{\pi^2}{\sin^2\pi x},$$

we obtain (1.2). In [7, pp.357-358], it was proved by using the integral

$$\zeta(2,l/m)=\int_0^1\frac{m^2y^{l-1}\log y}{1-y^m}dy.$$

Therefore, the above proof of (1.2) is apparently new. In order to obtain equations similar to (1.2) by Lewin's method, we have to find some integral representation of $\zeta(s,l/m)$, which seems to be difficult. But by using inversion formulas, we can obtain formula (2.8), which is the equation similar to (1.2).

3. Applications in the theory of transcendental numbers

In this section, we consider the case of $\Phi(z,s,a)$ at $s=1$. If $|z|\le 1$, $z\neq 1$, we have

$$\mathrm{Li}_1(z)=\sum_{n=1}^{\infty}\frac{z^n}{n}=-\log(1-z). \tag{3.1}$$

If we put $s=1$ in (2.7), we have

$$\Phi(z,1,l/m)=-z^{-l/m}\sum_{n=0}^{m-1}\omega_m^{-ln}\log\left(1-\omega_m^n z^{1/m}\right). \tag{3.2}$$

From this formula, we obtain the following theorem.

Theorem 3.1. *If z is algebraic, $|z|\le 1$, $z\neq 1$, then $\Phi(z,1,l/m)$ is transcendental.*

Proof. If $z=-1$, this has been proved by Saburo Uchiyama in [10]. By termwise integration, we can obtain

$$\Phi(z,1,a)=\int_0^1\sum_{n=0}^{\infty}z^nt^{n+a-1}dt=\int_0^1\frac{t^{a-1}}{1-zt}dt, \qquad 0<a\le 1. \tag{3.3}$$

This can be justified by Abel's theorem. Because of $\Re(1-zt)>0$, $t^{a-1}>0$, for all $0<t<1$, we have $\Phi(z,1,l/m)\neq 0$. Since $(1-\omega_m^n z^{1/m})$ and ω_m^n are algebraic, according to Baker's theorem [2, p.11, Theorem 2.2] and (3.2), we obtain that $\Phi(z,1,l/m)$ is transcendental. □

If we reform the method introduced in [10], the argument is as follows. Consider

$$\Phi(z,1,l/m)=\int_0^1\frac{t^{l/m-1}}{1-zt}dt=m\int_0^1\frac{u^{l-1}}{1-zu^m}du.$$

Then $1-zu^m$ has simple roots $\alpha_1,\alpha_2,\dots,\alpha_m$. Therefore the right-hand side becomes

$$\sum_{n=1}^{m}\frac{\gamma_n}{u-\alpha_n},$$

where α_n, γ_n are algebraic. According to Baker's theorem we obtain the same result. Inversion formulas simplify the argument, because to determine γ_n is not easy.

Next we generalize this result. Let $r\in\mathbb{N}$, $s\in\mathbb{C}$, $|z|<1$, $0<a$ and we define $\Phi_r(z,s,a)$ by

$$\Phi_r(z,s,a):=\sum_{n_1,n_2,\dots,n_r=0}^{\infty}\frac{z^{n_1+n_2+\cdots+n_r}}{(n_1+n_2+\cdots+n_r+a)^s}. \tag{3.4}$$

It is easy to see that $\Phi_r(z,s,a)$ is expressed as

$$\Phi_r(z,s,a)=\sum_{n=0}^{\infty}\binom{n+r-1}{r-1}\frac{z^n}{(n+a)^s}.$$

We show that $\Phi_r(z,s,a)$ is a sum of $\Phi(z,s,a)$ and its derivatives.

Proposition 3.2. *We have*

$$\Phi_r(z,s,a)=\frac{1}{(r-1)!}\frac{\partial^{r-1}}{\partial z^{r-1}}\Big(z^{r-1}\Phi(z,s,a)\Big). \tag{3.5}$$

Proof. We evaluate the right-hand side of (3.5). We have

$$\begin{aligned}&\frac{1}{(r-1)!}\frac{\partial^{r-1}}{\partial z^{r-1}}\Big(z^{r-1}\Phi(z,s,a)\Big)\\&=\frac{1}{(r-1)!}\frac{\partial^{r-1}}{\partial z^{r-1}}\sum_{n=0}^{\infty}\frac{z^{n+r-1}}{(n+a)^s}=\sum_{n=0}^{\infty}\binom{n+r-1}{r-1}\frac{z^n}{(n+a)^s}.\end{aligned}$$

□

By this proposition, we can obtain the following theorem.

Theorem 3.3. *If z is algebraic, $|z|<1$, $r\geq 2$, then $\Phi_r(z,1,l/m)$ is transcendental.*

Proof. If $r=2$, we have

$$\begin{aligned}&\sum_{n_1,n_2=0}^{\infty}\frac{z^{n_1+n_2}}{n_1+n_2+l/m}=\frac{\partial}{\partial z}\Big(z\Phi(z,1,l/m)\Big)\\&=(l/m-1)z^{-l/m}\sum_{n=1}^{m}\omega_m^{-ln}\log\left(1-\omega_m^n z^{1/m}\right)+\frac{z^{(1-l)/m}}{m}\sum_{n=1}^{m}\frac{\omega_m^{(1-l)n}}{1-\omega_m^n z^{1/m}}\end{aligned}$$

by (3.2) and Proposition 3.2. Similarly, if $r \geq 3$, we can obtain

$$\Phi_r(z,1,l/m) = \frac{l/m-r+1}{(r-1)!} z^{r-1-l/m} \sum_{n=1}^{m} \omega_m^{-ln} \log\left(1-\omega_m^n z^{1/m}\right) + \left(\text{a fractional expression in } \omega_m^n \text{ and } z^{1/m}\right) \tag{3.6}$$

by (3.2) and Proposition 3.2. We see that the first term on the right-hand side is not equal to 0 by the argument similar to the proof of Theorem 3.1. The "fractional expression" part of the right-hand side is an algebraic number. Therefore we obtain that $\Phi_r(z,1,l/m)$ is transcendental by Baker's theorem. □

We give another proof of Theorem 3.3. This method can determine the fractional expression on ω_m^n and $z^{1/m}$ on the right-hand side of (3.6). Let $s(n,r)$ be Stirling numbers of the first kind which are defined by

$$x(x-1)\cdots(x-n+1) = \sum_{r=0}^{n} s(n,r)x^r.$$

We define $p_{r,n}(x)$ by

$$p_{r,n}(x) := \frac{1}{(r-1)!} \sum_{k=n}^{r-1} (-1)^{r+1-n} \binom{k}{n} s(r,k+1) x^{k-n}.$$

By reforming the proof of [9, p.86, (21)] we have

$$\Phi_r(z,s,a) = \frac{1}{(r-1)!} \sum_{n=0}^{r-1} p_{r,n}(a)\, \Phi(z,s-n,a). \tag{3.7}$$

Let $S(n,r)$ be Stirling numbers of the second kind, which are defined by

$$x^n = \sum_{r=0}^{n} S(n,r)x(x-1)\cdots(x-r+1).$$

The following formula has been showed in [4, p.14, Theorem 6]

$$\Phi(z,-N,a) = \sum_{r=0}^{N} \sum_{n=r}^{N} \binom{N}{n} \frac{r!\, z^r a^{N-n} S(n,r)}{(1-z)^{r+1}}, \qquad N = 0,1,2\dots . \tag{3.8}$$

Taking $s=1$ in (3.7) and using (3.8), we can determine the fractional expression in ω_m^n and $z^{1/m}$ on the right-hand side of (3.6).

4. Hurwitz-Lerch Bernoulli functions

In this section, we study Hurwitz-Lerch Bernoulli functions $\mathcal{B}_N(a,z)$. By (4.3) below, it is known that the right-hand side of (3.8) is related to Hurwitz-Lerch Bernoulli functions. They are already included in [1], [4] and [9], hence we may call them Apostol-Bernoulli functions.

Definition 4.1 ([9, p.126, (41)]). *If $|z| \leq 1$, $z \neq 1$, $N = 0,1,2,\dots$, $0 \leq a$, we define Hurwitz-Lerch Bernoulli functions by*

$$\frac{te^{at}}{ze^t-1} = \sum_{N=0}^{\infty} \mathcal{B}_N(a,z) \frac{t^N}{N!}, \qquad |t+\log z| < 2\pi. \tag{4.1}$$

We define Hurwitz-Lerch Bernoulli numbers by

$$\mathcal{B}_N(z) := \mathcal{B}_N(0, z). \tag{4.2}$$

In [9, p.126, (40)] there is the formula

$$\mathcal{B}_{N+1}(a, z) = -(N+1)\Phi(z, -N, a), \qquad 0 < a. \tag{4.3}$$

The following multiplication formula and the inversion formula for $\mathcal{B}_N(a, z)$ are direct consequences of Theorem 2.1 and Theorem 2.2.

Theorem 4.2. *If $z^m \neq 1$, we have the multiplication formula*

$$\mathcal{B}_N(ma, z) = m^{1-N} \sum_{j=0}^{m-1} z^j \mathcal{B}_N(a + j/m, z^m). \tag{4.4}$$

If $z \neq 1$, we have the inversion formula

$$\mathcal{B}_N\big((a+j)/m, z\big) = m^{-N} z^{-j/m} \sum_{j=0}^{m-1} \omega_m^{-jn} \mathcal{B}_N\left(a + j/m, \omega_m^n z^{1/m}\right). \tag{4.5}$$

Proof. We get (4.4), (4.5) by (3.8) and putting $s = 1 - N$ in (2.2) and (2.4). □

We recall Bernoulli functions defined by

$$\frac{te^{at}}{e^t - 1} = \sum_{N=0}^{\infty} B_N(a) \frac{t^N}{N!}, \qquad |t| < 2\pi. \tag{4.6}$$

The number $B_N := B_N(0)$ are called Bernoulli numbers. The formula

$$B_{N+1}(a) = -(N+1)\zeta(-N, a), \qquad 0 < a \tag{4.7}$$

is stated in [9, p.85, (17)]. We get (1.3) by this difinition and (2.2). If $k = 1, 2 \dots m-1$, we have the next theorem.

Theorem 4.3 (Relations between Hurwitz-Lerch-Bernoulli functions and Bernoulli functions)**.**

$$\mathcal{B}_N(a, \omega_m^k) = m^{1-N} \sum_{j=0}^{m-1} \omega_m^{kj} B_N(a + j/m), \tag{4.8}$$

$$B_N\big((a+j)/m\big) - m^{-N} B_N(a) = m^{-N} \sum_{n=1}^{m-1} \omega_m^{-jn} \mathcal{B}_N(a, \omega_m^n). \tag{4.9}$$

Proof. We have (4.8) by putting $s = 1 - N$, $z = \omega_m^k$ in (2.2). We have (4.9) by putting $s = 1 - N$, $z = 1$ in (2.4). □

Using (3.8) and taking $a = 1$ in (4.9), we have the rational number formula

$$B_N(l/m) = m^{-N} B_N(1) - \frac{N}{m^N} \sum_{h=1}^{m-1} \sum_{r=0}^{N-1} \sum_{n=r}^{N-1} \binom{N-1}{n} \frac{r!\, \omega_m^{(r-j)h} S(n, r)}{(1 - \omega_m^h)^{r+1}}. \tag{4.10}$$

Formula (4.10) should be compared with the formulas

$$B_{2N-1}(l/m) = (-1)^N \frac{2(2N-1)!}{(2m\pi)^{2N-1}} \sum_{n=1}^{N} \zeta(2N-1, n/m) \sin(2\pi l n/m), \qquad N \neq 1,$$

$$B_{2N}(l/m) = (-1)^{N-1} \frac{2(2N)!}{(2m\pi)^{2N}} \sum_{n=1}^{N} \zeta(2N, n/m) \cos(2\pi ln/m), \qquad N \neq 0$$

([9, p.336 Theorem 6.2]). They are proved by the known formula

$$B_N(x) = \frac{2 \cdot N!}{(2\pi)^N} \sum_{n=1}^{\infty} \frac{1}{n^N} \cos(2\pi nx - \pi N/2), \qquad N \neq 0, 1, \quad 0 \leq x \leq 1,$$

and the multiplication formula (2.2) in [9, p.337, (8)].

5. Hurwitz-Lerch digamma functions

The values of $\Phi(z, s, a)$ at $s = -N$ are considered as Hurwitz-Lerch-Bernoulli functions in the preceding section. In this section we consider the case of $s = 1$. The function $\zeta(s, a)$ has a simple pole at $s = 1$. But $\Phi(z, s, a)$ does not have a pole at $s = 1$, if $|z| \leq 1$, $z \neq 1$. Therefore it is easier to treat. We put $S = \{z\,;\, |z| \leq 1\}$. Using $\Phi(z, 1, a)$, we define the following generalization of digamma functions $\psi(a)$ which are defined by

$$\psi(a) := \lim_{N\to\infty} \left(\log(N+1) - \sum_{n=0}^{N} \frac{1}{n+a} \right). \tag{5.1}$$

Definition 5.1. *If* $0 < a$, $|z| \leq 1$, *we define Hurwitz-Lerch digamma functions by*

$$\psi_\Phi(a, z) := \begin{cases} -\log(1-z) - \Phi(z, 1, a) & z \in S \setminus [0, 1], \\ \displaystyle\lim_{N\to\infty} \left(\log\left(\sum_{n=0}^{N} z^n \right) - \sum_{n=0}^{N} \frac{z^n}{n+a} \right) & 0 \leq z \leq 1. \end{cases} \tag{5.2}$$

If $0 < z < 1$, we have

$$\lim_{N\to\infty} \left(\log\left(\sum_{n=0}^{N} z^n \right) - \sum_{n=0}^{N} \frac{z^n}{n+a} \right) = -\log(1-z) - \Phi(z, 1, a).$$

By Abel's theorem, when $z \uparrow 1$ on the real axis, we have

$$\lim_{z\uparrow 1} \lim_{N\to\infty} \left(\log\left(\sum_{n=0}^{N} z^n \right) - \sum_{n=0}^{N} \frac{z^n}{n+a} \right) = \lim_{N\to\infty} \left(\log(N+1) - \sum_{n=0}^{N} \frac{1}{n+a} \right) = \psi(a).$$

Therefore we can write

$$\psi_\Phi(a, z) = \begin{cases} -\log(1-z) - \Phi(z, 1, a) & z \in S \setminus \{1\}, \\ \psi(a) & z = 1. \end{cases} \tag{5.3}$$

Here we show other representations of Hurwitz-Lerch digamma functions $\psi_\Phi(a, z)$ in the case of $z \neq 1$. By (2.5), (3.1) and (5.3), we have

$$\psi_\Phi(a, z) = z\Phi(z, 1, 1) - \Phi(z, 1, a). \tag{5.4}$$

By (3.3) and (5.4), we obtain

$$\psi_\Phi(a, z) = \int_0^1 \frac{z - t^{a-1}}{1 - zt} dt. \tag{5.5}$$

Recall Pochhammer's symbol

$$(\lambda)_0 := 1\,, \qquad (\lambda)_n := \lambda(\lambda+1)\cdots(\lambda+n) \qquad n = 1, 2, 3, \ldots$$

for $\lambda \in \mathbb{C}$, and Gauss' hypergeometric series

$$F(a,b\,;\,c:z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

for $a, b \in \mathbb{C}$, $c \in \mathbb{C}_3$. Using these symbols, Hurwitz-Lerch digamma functions $\psi_\Phi(a,z)$ are written by

$$\begin{aligned}\psi_\Phi(a,z) &= -\log(1-z) - a^{-1}F(1,a\,;\,a+1:z)\\ &= zF(1,2\,;\,2:z) - a^{-1}F(1,a\,;\,a+1:z).\end{aligned}$$

Now we consider properties of Hurwitz-Lerch digamma functions.

Theorem 5.2 (The integral representation, the asymptotic expansion)**.** *If* $|z| \le 1$, $z \neq 1$, $0 < a$, *we have*

$$\psi_\Phi(a,z) = \int_0^\infty \frac{e^{(z-1)t} - e^{-t}}{t} dt - \int_0^\infty \frac{e^{-at}}{1 - ze^{-t}} dt. \tag{5.6}$$

$$\psi_\Phi(a,z) = -\log(1-z) - \sum_{n=1}^{M} \frac{\mathcal{B}_n(z)}{n} (-a)^{-n} + O(a^{-M-1}). \tag{5.7}$$

Proof. Using

$$-\log(1-z) = -\int_0^\infty \frac{e^{-t} - e^{-(1-z)t}}{t} dt$$

and putting $x = e^{-t}$ in (3.3), we have (5.6). By

$$-\int_0^\infty \frac{e^{-at}}{1-ze^{-t}} dt = \int_0^\infty \frac{-t}{ze^{-t}-1} \frac{e^{-at}}{-t} dt, \qquad n! = \int_0^\infty t^n e^{-t} dt,$$

(4.1), (5.6) and $\mathcal{B}_0(z) = 0$, which is in [9, p.127, (46)], we have

$$\psi_\Phi(a,z) \sim -\log(1-z) - \sum_{n=1}^{\infty} \frac{\mathcal{B}_n(z)}{n} (-a)^{-n}. \tag{5.8}$$

This formula implies (5.7). □

Theorem 5.3. *If* $|z| \le 1$, $z^m \neq 1$, *we have the multiplication formula*

$$\psi_\Phi(ma,z) + \log(1-z) = \frac{1}{m} \sum_{j=0}^{m-1} z^j \big(\psi_\Phi(a + j/m, z^m) + \log(1-z^m) \big). \tag{5.9}$$

If $|z| \le 1$, $z \neq 1$, *we have the inversion formula*

$$\begin{aligned}&\psi_\Phi\big((a+j)/m, z\big) + \log(1-z)\\ &\quad = z^{-j/m} \sum_{n=0}^{m-1} \omega_m^{-jn} \Big(\psi_\Phi\left(a, \omega_m^n z^{1/m}\right) + \log\left(1 - \omega_m^n z^{1/m}\right) \Big).\end{aligned} \tag{5.10}$$

Proof. We have (5.9) by using the definition of $\psi_\Phi(a,z)$ and putting $s = 1$ in (2.2). We have (5.10) by using the definition of $\psi_\Phi(a,z)$ and putting $s = 1$ in (2.4). □

By putting $j=0$ in (5.10) we have the fraction formula

$$\psi_\Phi(a/m,z)+\log(1-z)=z^{-j/m}\sum_{n=0}^{m-1}\Big(\psi_\Phi\left(a,\omega_m^n z^{1/m}\right)+\log\left(1-\omega_m^n z^{1/m}\right)\Big).$$

Using the definition of $\psi_\Phi(a,z)$ and putting $a=1$, $j+1=l$ in (5.10), we have the rational number formula

$$\psi_\Phi(l/m,z)=-\log(1-z)+z^{-l/m}\sum_{n=0}^{m-1}\omega_m^{-ln}\log\left(1-\omega_m^n z^{1/m}\right). \tag{5.11}$$

In [5, p.937] the q-Euler constant is defined by

$$\gamma(q):=\frac{(q-1)\log(q-1)}{\log q}+(q-1)\sum_{n=1}^{\infty}\frac{1}{q^n-1}-\frac{q-1}{2},$$

and it is shown in [5, p.938, Theorem 2.4] that if $q\ge 2$ is an integer, then

$$\gamma(q)-\frac{(q-1)\log(q-1)}{\log q}$$

is an irrational number. Therefore here we consider $\psi_\Phi(l/m,z)$.

Theorem 5.4. *If z is algebraic, $|z|\le 1$, $z\ne 1$, then $\psi_\Phi(l/m,z)$ is transcendental. Especially, the Hurwitz-Lerch Euler constant $-\psi_\Phi(1,z)$ is transcendental if z is algebraic, $|z|\le 1$, $z\ne 1$.*

Proof. We obtain $\psi_\Phi(l/m,z)\ne 0$ by (5.5) and inequalities

$$\frac{z-t^{a-1}}{1-zt}=\frac{(z-t^{a-1})(1-\bar{z}t)}{|1-zt|^2},$$

$$\Re\Big((z-t^{a-1})(1-zt)\Big)<0,\qquad 0<a\le 1,\quad 0<t<1.$$

According to Baker's theorem and (5.11), $\psi_\Phi(l/m,z)$ is transcendental. □

When $k=1,2\dots m-1$, we have the next theorem.

Theorem 5.5 (Relations between Hurwitz-Lerch digamma functions and digamma functions)**.**

$$\psi_\Phi\left(ma,\omega_m^k\right)=\frac{1}{m}\sum_{j=0}^{m-1}\omega_m^{kj}\psi(a+j/m)-\log\left(1-\omega_m^k\right), \tag{5.12}$$

$$\psi\big((a+j)/m\big)-\psi(a)+\log m=\sum_{n=1}^{m-1}\omega_m^{-jn}\Big(\psi_\Phi\left(a,\omega_m^n\right)+\log\left(1-\omega_m^n\right)\Big). \tag{5.13}$$

Proof. We have (5.12) by putting $z=\lambda\omega_m^k$ $(0<\lambda<1)$ and taking $\lambda\uparrow 1$ in (5.9), and

$$\lim_{z\to\omega_m^k}\sum_{j=0}^{m-1}z^j\log\left(1-z^m\right)=\lim_{z\to\omega_m^k}\frac{z^m-1}{z-1}\log\left(1-z^m\right)=0.$$

We have (5.13) by letting $z\uparrow 1$ on the real axis in (5.10) and

$$\lim_{z\uparrow 1}\Big(-\log(1-z)+\log(1-z^{1/m})\Big)=-\log m.$$

□

Corollary 5.6 (The inversion formula, the fraction formula)**.**

$$\psi\big((a+j)/m\big) = -\log m + \psi(a) - \sum_{n=1}^{m-1} \omega_m^{-jn} \Phi\left(\omega_m^n 1, a\right), \tag{5.14}$$

$$\psi(a/m) = \sum_{n=0}^{m-1} \psi_\Phi\left(a, \omega_m^n\right) = -\log m + \psi(a) - \sum_{n=1}^{m-1} \Phi(\omega_m^n 1, a). \tag{5.15}$$

Proof. We have the inversion formula (5.14) by the definiton of $\psi_\Phi\left(\omega_m^n, a\right)$ and (5.13). The first equality of the fraction formula (5.15) is proved by putting $j=0$ in (5.13), and

$$m = \lim_{x\to 1} \frac{1-x^m}{1-x} = \lim_{x \to 1} \prod_{n=1}^{m-1} (1 - x\omega_m^n) = \prod_{n=1}^{m-1} (1-\omega_m^n).$$

We have the second equality of (5.15) by putting $j=0$ in (5.14). □

Now we prove (1.4) and (1.5). By letting $z \uparrow 1$ on the real axis and

$$\lim_{z \uparrow 1} \left(-\log(1-z) + \frac{1}{m} \sum_{j=0}^{m-1} z^j \log(1-z^m) \right) = \log m,$$

we have (1.4). If $k = 1, 2, \dots m-1$, we have

$$\log(1-\omega_m^k) = \log(2\sin\pi/m) + i\left(\pi k/m - \pi/2\right),$$

$$i\pi\left(\frac{1}{2} + \frac{1}{\omega_m^k - 1}\right) = \frac{\pi}{2}\cot(\pi k/m), \qquad \sum_{n=1}^{m-1} \omega_m^{kn} = -1, \qquad \sum_{n=1}^{m-1} n\omega_m^{kn} = \frac{m}{\omega_m^{kn} - 1}.$$

By using $\gamma = -\psi(1)$, (3.1) and putting $a=1$, $j+1=l$, $l = 1, 2, \dots m$ in (5.14), we have Gauss' formula (1.5). Therefore (5.14) is a generalization of (1.5). Using the definition of $\psi_\Phi(ma, \omega_m^k)$ and replacing a by a/m in (5.12), we have

$$\Phi\left(\omega_m^k, 1, a\right) = -\frac{1}{m} \sum_{j=0}^{m-1} \omega_m^{kj} \psi\big((a+j)/m\big). \tag{5.16}$$

Putting $a=1$ in (5.16), we have Gauss' second formula [9, pp.19, (49)]

$$\sum_{n=1}^{m} \omega_m^{kn} \psi(n/m) = m \log\left(1 - \omega_m^k\right). \tag{5.17}$$

6. Hurwitz-Lerch gamma functions

In the preceding section we consider Hurwitz-Lerch digamma functions. It is natural that we define the following generalization of gamma functions by using Hurwitz-Lerch digamma functions. In many books, for example [3], [9], and [11], the gamma function is defined before the definition of the digamma function. In this paper, the order is reversed.

Definition 6.1. *If $|z| \le 1$, $0 < a$, we define Hurwitz-Lerch gamma functions by*

$$\log \Gamma_\Phi(a, z) := \int_1^a \psi_\Phi(x, z)\, dx. \tag{6.1}$$

Letting $z \uparrow 1$ on the real axis we have

$$\lim_{z \uparrow 1} \int_1^a \psi_\Phi(x,z)\,dx = \int_1^a \psi(x)\,dx = \log\Gamma(a)$$

by Abel's theorem. Therefore Hurwitz-Lerch gamma functions are generalizations of the gamma function.

Theorem 6.2 (The infinite product, Lerch's formula)**.** *If $|z| \le 1$, $z \neq 1$, $0 < a$, we have*

$$\Gamma_\Phi(a,z) = (1-z)^{1-a} \prod_{n=0}^{\infty} \left(\frac{n+1}{n+a} \right)^{z^n}, \tag{6.2}$$

$$\log\Gamma_\Phi(a,z) = \frac{\partial}{\partial s}\Phi(z,0,a) - \frac{\partial}{\partial s}\Phi(z,0,1) - (a-1)\log(1-z). \tag{6.3}$$

Proof. By the uniformity of the convergence, we have

$$\log\Gamma_\Phi(a,z) = \int_1^a \psi_\Phi(x,z)\,dx = (1-a)\log(1-z) - \sum_{n=0}^{\infty} z^n \log\frac{n+a}{n+1}. \tag{6.4}$$

This formula implies (6.2). If $|z| < 1$, we have (6.3) by termwise differentiation of $\Phi(z,s,a)$ and (6.4). If $|z| \le 1$, $z \neq 1$, we have

$$\frac{\partial}{\partial a}\Phi(z,s,a) = -s\Phi(z,s,a) = -s\sum_{n=0}^{\infty} \frac{z^n}{(n+a)^{s+1}} \qquad \Re(s) > -1.$$

Therefore we have

$$\frac{\partial^2}{\partial s \partial a}\left[\Phi(z,s,a)\right]_{s=0} = -\Phi(z,1,a).$$

We put

$$f(a,z) = \log\Gamma_\Phi(a,z) - \frac{\partial}{\partial s}\Phi(z,0,a). \tag{6.5}$$

By the definition of $\log\Gamma_\Phi(a,z)$ we have

$$\frac{\partial}{\partial a} f(a,z) = -\log(1-z).$$

Hence we have

$$f(a,z) = -a\log(1-z) + g(z).$$

Next we determine $g(z)$. By the definition of $\log\Gamma_\Phi(a,z)$, we have $\log\Gamma_\Phi(1,z) = 0$. Therefore by (6.5), we have

$$f(1,z) = -\frac{\partial}{\partial s}\Phi(z,0,1).$$

Therefore we have

$$g(z) = \log(1-z) - \frac{\partial}{\partial s}\Phi(z,0,1).$$

This formula implies (6.3). □

Theorem 6.3 (The integral representation, the asympotic expansion)**.** *If $|z| \le 1$, $z \neq 1$, $0 < a$, we have*

$$\log\Gamma_\Phi(a,z) = \int_0^\infty \left((a-1)\left(e^{(z-1)t} - e^{-t}\right) - \frac{e^{-t} - e^{-at}}{1 - ze^{-t}} \right) \frac{dt}{t}, \tag{6.6}$$

$$\begin{aligned}\log\Gamma_\Phi(a,z) &= (1-a)\log(1-z)+\mathcal{B}_1(z)\log a-\frac{\partial}{\partial s}\Phi(z,0,1)\\ &\quad -\sum_{n=2}^{M}\frac{\mathcal{B}_n(z)}{n(n-1)}(-a)^{-n+1}+O(a^{-M}).\end{aligned}\tag{6.7}$$

Proof. Formula (6.6) is a generalization of Malmstén's formula [3, p.21, (1)]. By integrating (5.6) from 1 to a and changing the order of integration, we have (6.6) . Formula (6.7) is a generalization of Stirling's formula [3, p.47, (1)]. By putting

$$f(t)=\frac{te^{-at}}{1-ze^{-t}}$$

and using Ruijsenaars' method introduced in [8, p.118, (3.13)], we have (6.7). □

Theorem 6.4. *If $|z|\le 1$, $z^m\ne 1$ we have the multiplication formula*

$$\frac{\Gamma_\Phi(ma,z)}{\Gamma_\Phi(m,z)}=\prod_{j=0}^{m-1}\frac{(1-z^m)^{(a-1)z^j}}{(1-z)^{(a-1)}}\left(\frac{\Gamma_\Phi(a+j/m,z^m)}{\Gamma_\Phi(1+j/m,z^m)}\right)^{z^j}.\tag{6.8}$$

If $|z|\le 1$, $z\ne 1$ we have the inversion formula

$$\begin{aligned}&\frac{(1-z)^{(a-1)/m}\Gamma_\Phi\big((a+j)/m,z\big)}{\Gamma_\Phi\big((1+j)/m,z\big)}\\ &=\prod_{n=0}^{m-1}\left(\left(1-\omega_m^n z^{1/m}\right)^{(a-1)}\Gamma_\Phi\left(a,\omega_m^n z^{1/m}\right)\right)^{\omega_m^{-jn}z^{-j/m}/m}.\end{aligned}\tag{6.9}$$

Proof. By (5.9) and

$$\int_1^a\psi_\Phi(mx,z)\,dx=\frac{1}{m}\log\frac{\Gamma_\Phi(ma,z)}{\Gamma_\Phi(m,z)},\quad \int_1^a\psi_\Phi(x+j/m,z)\,dx=\log\frac{\Gamma_\Phi(a+j/m,z)}{\Gamma_\Phi(1+j/m,z)},$$

we have (6.8). By (5.10) and

$$\int_1^a\psi_\Phi\big((x+j)/m,z^m\big)\,dx=m\log\frac{\Gamma_\Phi\big((a+j)/m,z^m\big)}{\Gamma_\Phi\big((1+j)/m,z^m\big)},$$

we have (6.9). □

By putting $j=0$ in (6.9), we have the fraction formula

$$\frac{(1-z)^{(a-1)/m}\Gamma_\Phi(a/m,z)}{\Gamma_\Phi(1/m,z)}=\prod_{n=0}^{m-1}\left(\left(1-\omega_m^n z^{1/m}\right)^{(a-1)}\Gamma_\Phi\left(a,\omega_m^n z^{1/m}\right)\right)^{1/m}.$$

We have the following theorem by integrating (5.12) and (5.13) from 1 to a.

Theorem 6.5 (Relations between Hurwitz-Lerch gamma functions and the gamma function)**.**

$$\frac{\Gamma_\Phi\left(ma,\omega_m^k\right)}{\Gamma_\Phi\left(m,\omega_m^k\right)}=(1-\omega_m^k)^{m(1-a)}\prod_{j=0}^{m-1}\frac{\Gamma(a+j/m)}{\Gamma(1+j/m)},\tag{6.10}$$

$$\frac{\left(m^{1-a}\Gamma(a)\right)^{-1/m}\Gamma\big((a+j)/m\big)}{\Gamma\big((1+j)/m\big)}=\prod_{n=1}^{m-1}\left((1-\omega_m^n)^{(a-1)}\Gamma_\Phi\left(a,\omega_m^n\right)\right)^{\omega_m^{-jn}/m}.\tag{6.11}$$

Puttig $j = 0$, we have the fraction formula

$$\Gamma(a/m) = \Gamma(1/m) \prod_{n=0}^{m-1} \Big(\Gamma_\Phi\left(a, \omega_m^n\right)\Big)^{1/m}. \tag{6.12}$$

Finally we calculate the value of $\Gamma_\Phi\left(a, \omega_m^k\right)$. By (6.1), we have $\Gamma_\Phi(1, z) = 1$. Therefore we have

$$\Gamma_\Phi\left(m, \omega_m^k\right) = (1 - \omega_m^k)^{1-m} \prod_{j=0}^{m-1} \frac{\Gamma(1 + j/m)}{\Gamma\big((1 + j)/m\big)}$$

by putting $a = 1/m$ in (6.10). Replacing a by a/m in (6.10), we have

$$\Gamma_\Phi\left(a, \omega_m^k\right) = (1 - \omega_m^k)^{1-a} \prod_{j=0}^{m-1} \frac{\Gamma\big((a + j)/m\big)}{\Gamma\big((1 + j)/m\big)}.$$

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