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## **Some topics related to Hurwitz-Lerch zeta functions**

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# SOME TOPICS RELATED TO HURWITZ-LERCH ZETA FUNCTIONS

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ABSTRACT. In this paper, we consider multiplication formulas and their inversion formulas for Hurwitz-Lerch zeta functions. Inversion formulas give simple proofs of known results, and also show generalizations of those results. Next, we give a generalization of digamma and gamma functions in terms of Hurwitz-Lerch zeta functions, and consider its properties. In all the sections, various kinds of results are always proved by multiplication formulas and inversion formulas.

## 1. INTRODUCTION

**Definition 1.1** ([3, p.27, (1)]). *We define Hurwitz-Lerch zeta functions by*

$$(1.1) \quad \Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}, \quad z \in \mathbb{C}, \quad |z| < 1, \quad a \neq 0, -1, -2, \dots, \quad s \in \mathbb{C}.$$

The function  $\Phi(z, s, a)$  was defined by Erdélyi et al in [3] originally. We put

$$\mathbb{C}_1 := \mathbb{C} \setminus [1, +\infty), \quad \mathbb{C}_2 := \{a; \Re(a) > 0\}, \quad \mathbb{C}_3 := \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

In [4, p.5 Theorem 1], the function  $\Phi(z, s, a)$  is extended to an analytic function in three variables  $z, s, a$  for

$$\begin{aligned} a &\in \mathbb{C}_2, \quad z \in \mathbb{C}_1, \quad \text{and} \\ s &\in \mathbb{C} \quad \text{or} \quad s \in \mathbb{C} \setminus \{1\} \quad \text{according to} \quad z \neq 1 \quad \text{or} \quad z = 1, \end{aligned}$$

by the contour integral representation

$$\Phi(z, s, a) = -\frac{\Gamma(1-s)}{2\pi i} \int_{\infty}^{(+0)} \frac{(-t)^{s-1} e^{-at}}{1 - ze^{-t}} dt, \quad a \in \mathbb{C}_2, \quad |\arg(-t)| \leq \pi.$$

The contour starts at  $\infty$ , encircles the origin once counter-clockwise and returns to its starting point. The initial and final values of  $\arg(-t)$  are  $-\pi$  and  $\pi$  respectively.

In Section 2, we consider a multiplication formula in Theorem 2.1 and an inversion formula in Theorem 2.2. And by that inversion formula, we can give a simple proof of a known result, and also show generalizations of the result. We give a simple proof of

$$(1.2) \quad \zeta(2, k/m) = \frac{\pi^2}{\sin^2 \pi k/m} + 2m \sum_{n=1}^{[(m-1)/2]} \sin(2\pi kn/m) \operatorname{Cl}_2(2\pi n/m),$$

([7, p.358, (16.23)]), where  $\operatorname{Cl}_2(\theta)$  is the Clausen integral defined by (2.9). And we obtain formula (2.8), which is a generalization of (1.2).

In Section 3, by using the inversion formula we can show in Theorem 3.1 that  $\Phi(z, 1, l/m)$  is transcendental if  $m \in \mathbb{N}$ ,  $l = 1, 2, \dots, m$ ,  $z$  is algebraic,  $|z| \leq 1$ ,  $z \neq 1$ , which is a generalization of Uchiyama's result [10]. We consider  $\Phi_r(z, 1, l/m)$ , which is a multiple analogue

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of  $\Phi(z, 1, l/m)$ . In Theorem 3.3, we show that  $\Phi_r(z, 1, l/m)$  is transcendental if  $z$  is algebraic,  $|z| < 1$ ,  $z \neq 1$ .

In Section 4, we treat Hurwitz-Lerch Bernoulli functions. By multiplication formulas, we can give a simple proof of

$$(1.3) \quad m^{1-N} B_N(mx) = \sum_{j=0}^{m-1} B_N(x + j/m)$$

([3, p.37, (11)]), where  $B_N(x)$  is the Bernoulli function defined by (4.6). We can show interesting formulas (4.5) and (4.9) which seem to be new by the inversion formula.

In Section 5 we introduce  $\psi(a, z)$  which is a generalization of digamma functions by using Hurwitz-Lerch zeta functions, and consider its properties. In Theorem 5.4, we can show that if  $z$  is algebraic,  $|z| \leq 1$ ,  $z \neq 1$ , then  $\psi_\Phi(l/m, z)$  is transcendental. By the multiplication formula, we can give a simple proof of

$$(1.4) \quad \psi(ma) = \log m + \frac{1}{m} \sum_{j=0}^{m-1} \psi(a + j/m)$$

([3, p.16, (12)]), where  $\psi(a)$  is the digamma function defined by (5.1). Let  $\gamma := -\psi(1)$  be the Euler constant. Inversion formulas give a simple proof of

$$(1.5) \quad \psi(l/m) = -\gamma - \log m - \frac{\pi}{2} \cot(\pi l/m) + \sum_{n=1}^{m-1} \cos(2\pi n l/m) \log(2 \sin n\pi/m)$$

([3, p.19, (29)]). We have interesting formulas (5.14) and (5.15), which are generalizations of Gauss' first formula (1.5), by inversion formulas. At the end of this section, we show (5.16) which is a generalization of Gauss' second formula (5.17).

In Section 6, we generalize the notion of gamma functions by using Hurwitz-Lerch zeta functions, and consider its properties in Theorem 6.4. At the end of this paper, we evaluate a special value of generalized gamma functions.

In all sections, various kinds of results are always proved by multiplication formulas and inversion formulas.

## 2. MULTIPLICATION AND INVERSION FORMULAS

Firstly, we quote the multiplication formula for Hurwitz-Lerch zeta functions. If  $a$ ,  $s$  and  $z$  satisfy the conditions

$$(2.1) \quad \begin{aligned} &0 < a, \quad z \in \mathbb{C}_1, \quad -\pi < \arg z \leq \pi, \quad \text{and} \\ &s \in \mathbb{C} \quad \text{or} \quad s \in \mathbb{C} \setminus \{1\} \quad \text{according to} \quad z^m \neq 1 \quad \text{or} \quad z^m = 1, \end{aligned}$$

we write  $(z, s, a) \in D_1$ .

**Theorem 2.1** (The multiplication formula [9, p.339, (15)]). *If  $(z, s, a) \in D_1$ ,  $m \in \mathbb{N}$ , then we have*

$$(2.2) \quad \Phi(z, s, ma) = m^{-s} \sum_{j=0}^{m-1} z^j \Phi(z^m, s, a + j/m).$$

*Proof.* We give a proof for the convenience of readers. It is easy to see that

$$\sum_{n=0}^{\infty} \frac{z^n}{(n+ma)^s} = \sum_{j=0}^{m-1} \sum_{n=0}^{\infty} \frac{z^{mn+j}}{(mn+ma+j)^s}.$$

We can get (2.2) by the above equation.  $\square$

In this paper we prove the following inversion formula. If  $a, s$  and  $z$  satisfy the conditions

$$(2.3) \quad \begin{aligned} &0 < a, \quad z \in \mathbb{C}_1, \quad -\pi/m < \arg z^{1/m} \leq \pi/m, \quad \text{and} \\ &s \in \mathbb{C} \quad \text{or} \quad s \in \mathbb{C} \setminus \{1\} \quad \text{according to} \quad z \neq 1 \quad \text{or} \quad z = 1 \end{aligned}$$

we write  $(z, s, a) \in D_2$ . Let  $i = \sqrt{-1}$ , and

$$\omega_m^j = \exp(2\pi i j / m), \quad j \in \mathbb{N}, \quad 0 \leq j \leq m-1.$$

**Theorem 2.2** (The inversion formula). *If  $(z, s, a) \in D_2$ ,  $m \in \mathbb{N}$ , then we have*

$$(2.4) \quad \Phi(z, s, (a+j)/m) = m^{s-1} z^{-j/m} \sum_{n=0}^{m-1} \omega_m^{-jn} \Phi(\omega_m^n z^{1/m}, s, a).$$

*Proof.* If  $J \in \mathbb{N}$ , we have

$$\sum_{n=0}^{m-1} (\omega_m^j)^n (\omega_m^n)^J = \begin{cases} m & j+J \equiv 0 \pmod{m}, \\ 0 & \text{otherwise.} \end{cases}$$

From this formula, we have

$$m^{-1} \sum_{n=0}^{m-1} \omega_m^{-jn} \sum_{h=0}^{\infty} \frac{\omega_m^{nh} z^{h/m}}{(h+a)^s} = \sum_{h=0}^{\infty} \frac{z^{(mh+j)/m}}{(mh+a+j)^s}.$$

We obtain (2.4) by the above equation.  $\square$

If  $(z, s) \in D_1$ , we define  $\text{Li}_s(z)$  by

$$(2.5) \quad \text{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z, s, 1).$$

We have the next corollary.

**Corollary 2.3** (The fraction formula, the rational number formula).

$$(2.6) \quad \Phi(z, s, a/m) = m^{s-1} \sum_{n=0}^{m-1} \Phi(\omega_m^n z^{1/m}, s, a),$$

$$(2.7) \quad \Phi(z, s, l/m) = m^{s-1} z^{-l/m} \sum_{n=0}^{m-1} \omega_m^{-ln} \text{Li}_s(\omega_m^n z^{1/m}).$$

*Proof.* We have the fraction formula (2.6) by putting  $j = 0$  in (2.4). Taking  $a = 1$ ,  $j+1 = l$ ,  $l = 1, 2, \dots, m$  in (2.4), we have the rational number formula (2.7).  $\square$

By putting  $z^{1/m} = 1$  in (2.7), we have

$$(2.8) \quad \zeta(s, l/m) = m^{s-1} \sum_{n=0}^{m-1} \omega_m^{-ln} \text{Li}_s(\omega_m^n).$$

If  $s = k \in \mathbb{N}$  in (2.5), it is called the  $k$ -th polylogarithm. By the definition of  $\text{Li}_2$ , we have

$$\text{Li}_2(e^{i\theta}) = \sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2} + i \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2}, \quad 0 \leq \theta < 2\pi.$$

Here we recall the Clausen integral defined by

$$(2.9) \quad \text{Cl}_2(\theta) := \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2} = - \int_0^{\theta} \log(2 \sin(\theta/2)) d\theta.$$

In [6, p.105, (4.22)], the formula

$$\text{Cl}_2(\theta) + \text{Cl}_2(2\pi - \theta) = 0$$

is stated. Putting  $s = 2$  in (2.8) and using the formulas

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2} = \frac{\pi^2}{6} - \frac{\theta(2\pi - \theta)}{4}, \quad \zeta(2, x) + \zeta(2, 1-x) = \frac{\pi^2}{\sin^2 \pi x},$$

we obtain (1.2). In [7, pp.357-358], it was proved by using the integral

$$\zeta(2, l/m) = \int_0^1 \frac{m^2 y^{l-1} \log y}{1 - y^m} dy.$$

Therefore, the above proof of (1.2) is apparently new. In order to obtain equations similar to (1.2) by Lewin's method, we have to find some integral representation of  $\zeta(s, l/m)$ , which seems to be difficult. But by using inversion formulas, we can obtain formula (2.8), which is the equation similar to (1.2).

### 3. APPLICATIONS IN THE THEORY OF TRANSCENDENTAL NUMBERS

In this section, we consider the case of  $\Phi(z, s, a)$  at  $s = 1$ . If  $|z| \leq 1$ ,  $z \neq 1$ , we have

$$(3.1) \quad \text{Li}_1(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} = -\log(1-z).$$

If we put  $s = 1$  in (2.7), we have

$$(3.2) \quad \Phi(z, 1, l/m) = -z^{-l/m} \sum_{n=0}^{m-1} \omega_m^{-ln} \log(1 - \omega_m^n z^{1/m}).$$

From this formula, we obtain the following theorem.

**Theorem 3.1.** *If  $z$  is algebraic,  $|z| \leq 1$ ,  $z \neq 1$ , then  $\Phi(z, 1, l/m)$  is transcendental.*

*Proof.* If  $z = -1$ , this has been proved by Saburo Uchiyama in [10]. By termwise integration, we can obtain

$$(3.3) \quad \Phi(z, 1, a) = \int_0^1 \sum_{n=0}^{\infty} z^n t^{n+a-1} dt = \int_0^1 \frac{t^{a-1}}{1-zt} dt, \quad 0 < a \leq 1.$$

This can be justified by Abel's theorem. Because of  $\Re(1-zt) > 0$ ,  $t^{a-1} > 0$ , for all  $0 < t < 1$ , we have  $\Phi(z, 1, l/m) \neq 0$ . Since  $(1 - \omega_m^n z^{1/m})$  and  $\omega_m^n$  are algebraic, according to Baker's

theorem [2, p.11, Theorem 2.2] and (3.2), we obtain that  $\Phi(z, 1, l/m)$  is transcendental.  $\square$

If we reform the method introduced in [10], the argument is as follows. Consider

$$\Phi(z, 1, l/m) = \int_0^1 \frac{t^{l/m-1}}{1-zt} dt = m \int_0^1 \frac{u^{l-1}}{1-zu^m} du.$$

Then  $1 - zu^m$  has simple roots  $\alpha_1, \alpha_2, \dots, \alpha_m$ . Therefore the right-hand side becomes

$$\sum_{n=1}^m \frac{\gamma_n}{u - \alpha_n},$$

where  $\alpha_n, \gamma_n$  are algebraic. According to Baker's theorem we obtain the same result. Inversion formulas simplify the argument, because to determine  $\gamma_n$  is not easy.

Next we generalize this result. Let  $r \in \mathbb{N}$ ,  $s \in \mathbb{C}$ ,  $|z| < 1$ ,  $0 < a$  and we define  $\Phi_r(z, s, a)$  by

$$(3.4) \quad \Phi_r(z, s, a) := \sum_{n_1, n_2, \dots, n_r=0}^{\infty} \frac{z^{n_1+n_2+\dots+n_r}}{(n_1+n_2+\dots+n_r+a)^s}.$$

It is easy to see that  $\Phi_r(z, s, a)$  is expressed as

$$\Phi_r(z, s, a) = \sum_{n=0}^{\infty} \binom{n+r-1}{r-1} \frac{z^n}{(n+a)^s}.$$

We show that  $\Phi_r(z, s, a)$  is a sum of  $\Phi(z, s, a)$  and its derivatives.

**Proposition 3.2.** *We have*

$$(3.5) \quad \Phi_r(z, s, a) = \frac{1}{(r-1)!} \frac{\partial^{r-1}}{\partial z^{r-1}} \left( z^{r-1} \Phi(z, s, a) \right).$$

*Proof.* We evaluate the right-hand side of (3.5). We have

$$\begin{aligned} & \frac{1}{(r-1)!} \frac{\partial^{r-1}}{\partial z^{r-1}} \left( z^{r-1} \Phi(z, s, a) \right) \\ &= \frac{1}{(r-1)!} \frac{\partial^{r-1}}{\partial z^{r-1}} \sum_{n=0}^{\infty} \frac{z^{n+r-1}}{(n+a)^s} = \sum_{n=0}^{\infty} \binom{n+r-1}{r-1} \frac{z^n}{(n+a)^s}. \end{aligned}$$

$\square$

By this proposition, we can obtain the following theorem.

**Theorem 3.3.** *If  $z$  is algebraic,  $|z| < 1$ ,  $r \geq 2$ , then  $\Phi_r(z, 1, l/m)$  is transcendental.*

*Proof.* If  $r = 2$ , we have

$$\begin{aligned} & \sum_{n_1, n_2=0}^{\infty} \frac{z^{n_1+n_2}}{n_1+n_2+l/m} = \frac{\partial}{\partial z} \left( z \Phi(z, 1, l/m) \right) \\ &= (l/m - 1) z^{-l/m} \sum_{n=1}^m \omega_m^{-ln} \log(1 - \omega_m^n z^{1/m}) + \frac{z^{(1-l)/m}}{m} \sum_{n=1}^m \frac{\omega_m^{(1-l)n}}{1 - \omega_m^n z^{1/m}} \end{aligned}$$

by (3.2) and Proposition 3.2. Similarly, if  $r \geq 3$ , we can obtain

$$(3.6) \quad \Phi_r(z, 1, l/m) = \frac{l/m - r + 1}{(r-1)!} z^{r-1-l/m} \sum_{n=1}^m \omega_m^{-ln} \log(1 - \omega_m^n z^{1/m}) \\ + (\text{a fractional expression in } \omega_m^n \text{ and } z^{1/m})$$

by (3.2) and Proposition 3.2. We see that the first term on the right-hand side is not equal to 0 by the argument similar to the proof of Theorem 3.1. The “fractional expression” part of the right-hand side is an algebraic number. Therefore we obtain that  $\Phi_r(z, 1, l/m)$  is transcendental by Baker’s theorem.  $\square$

We give another proof of Theorem 3.3. This method can determine the fractional expression on  $\omega_m^n$  and  $z^{1/m}$  on the right-hand side of (3.6). Let  $s(n, r)$  be Stirling numbers of the first kind which are defined by

$$x(x-1) \cdots (x-n+1) = \sum_{r=0}^n s(n, r) x^r.$$

We define  $p_{r,n}(x)$  by

$$p_{r,n}(x) := \frac{1}{(r-1)!} \sum_{k=n}^{r-1} (-1)^{r+1-n} \binom{k}{n} s(r, k+1) x^{k-n}.$$

By reforming the proof of [9, p.86, (21)] we have

$$(3.7) \quad \Phi_r(z, s, a) = \frac{1}{(r-1)!} \sum_{n=0}^{r-1} p_{r,n}(a) \Phi(z, s-n, a).$$

Let  $S(n, r)$  be Stirling numbers of the second kind, which are defined by

$$x^n = \sum_{r=0}^n S(n, r) x(x-1) \cdots (x-r+1).$$

The following formula has been showed in [4, p.14, Theorem 6]

$$(3.8) \quad \Phi(z, -N, a) = \sum_{r=0}^N \sum_{n=r}^N \binom{N}{n} \frac{r! z^r a^{N-n} S(n, r)}{(1-z)^{r+1}}, \quad N = 0, 1, 2, \dots$$

Taking  $s = 1$  in (3.7) and using (3.8), we can determine the fractional expression in  $\omega_m^n$  and  $z^{1/m}$  on the right-hand side of (3.6).

#### 4. HURWITZ-LERCH BERNOULLI FUNCTIONS

In this section, we study Hurwitz-Lerch Bernoulli functions  $\mathcal{B}_N(a, z)$ . By (4.3) below, it is known that the right-hand side of (3.8) is related to Hurwitz-Lerch Bernoulli functions. They are already included in [1], [4] and [9], hence we may call them Apostol-Bernoulli functions.

**Definition 4.1** ([9, p.126, (41)]). *If  $|z| \leq 1$ ,  $z \neq 1$ ,  $N = 0, 1, 2, \dots$ ,  $0 \leq a$ , we define Hurwitz-Lerch Bernoulli functions by*

$$(4.1) \quad \frac{te^{at}}{ze^t - 1} = \sum_{N=0}^{\infty} \mathcal{B}_N(a, z) \frac{t^N}{N!}, \quad |t + \log z| < 2\pi.$$



We define Hurwitz-Lerch Bernoulli numbers by

$$(4.2) \quad \mathcal{B}_N(z) := \mathcal{B}_N(0, z).$$

In [9, p.126, (40)] there is the formula

$$(4.3) \quad \mathcal{B}_{N+1}(a, z) = -(N+1)\Phi(z, -N, a), \quad 0 < a.$$

The following multiplication formula and the inversion formula for  $\mathcal{B}_N(a, z)$  are direct consequences of Theorem 2.1 and Theorem 2.2.

**Theorem 4.2.** *If  $z^m \neq 1$ , we have the multiplication formula*

$$(4.4) \quad \mathcal{B}_N(ma, z) = m^{1-N} \sum_{j=0}^{m-1} z^j \mathcal{B}_N(a + j/m, z^m).$$

*If  $z \neq 1$ , we have the inversion formula*

$$(4.5) \quad \mathcal{B}_N((a+j)/m, z) = m^{-N} z^{-j/m} \sum_{j=0}^{m-1} \omega_m^{-jn} \mathcal{B}_N(a + j/m, \omega_m^n z^{1/m}).$$

*Proof.* We get (4.4), (4.5) by (3.8) and putting  $s = 1 - N$  in (2.2) and (2.4).  $\square$

We recall Bernoulli functions defined by

$$(4.6) \quad \frac{te^{at}}{e^t - 1} = \sum_{N=0}^{\infty} B_N(a) \frac{t^N}{N!}, \quad |t| < 2\pi.$$

The number  $B_N := B_N(0)$  are called Bernoulli numbers. The formula

$$(4.7) \quad B_{N+1}(a) = -(N+1)\zeta(-N, a), \quad 0 < a$$

is stated in [9, p.85, (17)]. We get (1.3) by this definition and (2.2). If  $k = 1, 2, \dots, m-1$ , we have the next theorem.

**Theorem 4.3** (Relations between Hurwitz-Lerch-Bernoulli functions and Bernoulli functions).

$$(4.8) \quad \mathcal{B}_N(a, \omega_m^k) = m^{1-N} \sum_{j=0}^{m-1} \omega_m^{kj} B_N(a + j/m),$$

$$(4.9) \quad B_N((a+j)/m) - m^{-N} B_N(a) = m^{-N} \sum_{n=1}^{m-1} \omega_m^{-jn} \mathcal{B}_N(a, \omega_m^n).$$

*Proof.* We have (4.8) by putting  $s = 1 - N$ ,  $z = \omega_m^k$  in (2.2). We have (4.9) by putting  $s = 1 - N$ ,  $z = 1$  in (2.4).  $\square$

Using (3.8) and taking  $a = 1$  in (4.9), we have the rational number formula

$$(4.10) \quad B_N(l/m) = m^{-N} B_N(1) - \frac{N}{m^N} \sum_{h=1}^{m-1} \sum_{r=0}^{N-1} \sum_{n=r}^{N-1} \binom{N-1}{n} \frac{r! \omega_m^{(r-j)h} S(n, r)}{(1 - \omega_m^h)^{r+1}}.$$

Formula (4.10) should be compared with the formulas

$$B_{2N-1}(l/m) = (-1)^N \frac{2(2N-1)!}{(2m\pi)^{2N-1}} \sum_{n=1}^N \zeta(2N-1, n/m) \sin(2\pi ln/m), \quad N \neq 1,$$

$$B_{2N}(l/m) = (-1)^{N-1} \frac{2(2N)!}{(2m\pi)^{2N}} \sum_{n=1}^N \zeta(2N, n/m) \cos(2\pi n l/m), \quad N \neq 0$$

([9, p.336 Theorem 6.2]). They are proved by the known formula

$$B_N(x) = \frac{2 \cdot N!}{(2\pi)^N} \sum_{n=1}^{\infty} \frac{1}{n^N} \cos(2\pi n x - \pi N/2), \quad N \neq 0, 1, \quad 0 \leq x \leq 1,$$

and the multiplication formula (2.2) in [9, p.337, (8)].

## 5. HURWITZ-LERCH DIGAMMA FUNCTIONS

The values of  $\Phi(z, s, a)$  at  $s = -N$  are considered as Hurwitz-Lerch-Bernoulli functions in the preceding section. In this section we consider the case of  $s = 1$ . The function  $\zeta(s, a)$  has a simple pole at  $s = 1$ . But  $\Phi(z, s, a)$  does not have a pole at  $s = 1$ , if  $|z| \leq 1$ ,  $z \neq 1$ . Therefore it is easier to treat. We put  $S = \{z; |z| \leq 1\}$ . Using  $\Phi(z, 1, a)$ , we define the following generalization of digamma functions  $\psi(a)$  which are defined by

$$(5.1) \quad \psi(a) := \lim_{N \rightarrow \infty} \left( \log(N+1) - \sum_{n=0}^N \frac{1}{n+a} \right).$$

**Definition 5.1.** If  $0 < a$ ,  $|z| \leq 1$ , we define Hurwitz-Lerch digamma functions by

$$(5.2) \quad \psi_{\Phi}(a, z) := \begin{cases} -\log(1-z) - \Phi(z, 1, a) & z \in S \setminus [0, 1], \\ \lim_{N \rightarrow \infty} \left( \log \left( \sum_{n=0}^N z^n \right) - \sum_{n=0}^N \frac{z^n}{n+a} \right) & 0 \leq z \leq 1. \end{cases}$$

If  $0 < z < 1$ , we have

$$\lim_{N \rightarrow \infty} \left( \log \left( \sum_{n=0}^N z^n \right) - \sum_{n=0}^N \frac{z^n}{n+a} \right) = -\log(1-z) - \Phi(z, 1, a).$$

By Abel's theorem, when  $z \uparrow 1$  on the real axis, we have

$$\lim_{z \uparrow 1} \lim_{N \rightarrow \infty} \left( \log \left( \sum_{n=0}^N z^n \right) - \sum_{n=0}^N \frac{z^n}{n+a} \right) = \lim_{N \rightarrow \infty} \left( \log(N+1) - \sum_{n=0}^N \frac{1}{n+a} \right) = \psi(a).$$

Therefore we can write

$$(5.3) \quad \psi_{\Phi}(a, z) = \begin{cases} -\log(1-z) - \Phi(z, 1, a) & z \in S \setminus \{1\}, \\ \psi(a) & z = 1. \end{cases}$$

Here we show other representations of Hurwitz-Lerch digamma functions  $\psi_{\Phi}(a, z)$  in the case of  $z \neq 1$ . By (2.5), (3.1) and (5.3), we have

$$(5.4) \quad \psi_{\Phi}(a, z) = z\Phi(z, 1, 1) - \Phi(z, 1, a).$$

By (3.3) and (5.4), we obtain

$$(5.5) \quad \psi_{\Phi}(a, z) = \int_0^1 \frac{z - t^{a-1}}{1 - zt} dt.$$

Recall Pochhammer's symbol

$$(\lambda)_0 := 1, \quad (\lambda)_n := \lambda(\lambda+1) \cdots (\lambda+n) \quad n = 1, 2, 3, \dots$$

for  $\lambda \in \mathbb{C}$ , and Gauss' hypergeometric series

$$F(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

for  $a, b \in \mathbb{C}$ ,  $c \in \mathbb{C}_3$ . Using these symbols, Hurwitz-Lerch digamma functions  $\psi_{\Phi}(a, z)$  are written by

$$\begin{aligned} \psi_{\Phi}(a, z) &= -\log(1-z) - a^{-1}F(1, a; a+1; z) \\ &= zF(1, 2; 2; z) - a^{-1}F(1, a; a+1; z). \end{aligned}$$

Now we consider properties of Hurwitz-Lerch digamma functions.

**Theorem 5.2** (The integral representation, the asymptotic expansion). *If  $|z| \leq 1$ ,  $z \neq 1$ ,  $0 < a$ , we have*

$$(5.6) \quad \psi_{\Phi}(a, z) = \int_0^{\infty} \frac{e^{(z-1)t} - e^{-t}}{t} dt - \int_0^{\infty} \frac{e^{-at}}{1 - ze^{-t}} dt.$$

$$(5.7) \quad \psi_{\Phi}(a, z) = -\log(1-z) - \sum_{n=1}^M \frac{\mathcal{B}_n(z)}{n} (-a)^{-n} + O(a^{-M-1}).$$

*Proof.* Using

$$-\log(1-z) = -\int_0^{\infty} \frac{e^{-t} - e^{-(1-z)t}}{t} dt$$

and putting  $x = e^{-t}$  in (3.3), we have (5.6). By

$$-\int_0^{\infty} \frac{e^{-at}}{1 - ze^{-t}} dt = \int_0^{\infty} \frac{-t}{ze^{-t} - 1} \frac{e^{-at}}{-t} dt, \quad n! = \int_0^{\infty} t^n e^{-t} dt,$$

(4.1), (5.6) and  $\mathcal{B}_0(z) = 0$ , which is in [9, p.127, (46)], we have

$$(5.8) \quad \psi_{\Phi}(a, z) \sim -\log(1-z) - \sum_{n=1}^{\infty} \frac{\mathcal{B}_n(z)}{n} (-a)^{-n}.$$

This formula implies (5.7). □

**Theorem 5.3.** *If  $|z| \leq 1$ ,  $z^m \neq 1$ , we have the multiplication formula*

$$(5.9) \quad \psi_{\Phi}(ma, z) + \log(1-z) = \frac{1}{m} \sum_{j=0}^{m-1} z^j (\psi_{\Phi}(a + j/m, z^m) + \log(1 - z^m)).$$

*If  $|z| \leq 1$ ,  $z \neq 1$ , we have the inversion formula*

$$(5.10) \quad \begin{aligned} &\psi_{\Phi}((a+j)/m, z) + \log(1-z) \\ &= z^{-j/m} \sum_{n=0}^{m-1} \omega_m^{-jn} \left( \psi_{\Phi}(a, \omega_m^n z^{1/m}) + \log(1 - \omega_m^n z^{1/m}) \right). \end{aligned}$$

*Proof.* We have (5.9) by using the definition of  $\psi_{\Phi}(a, z)$  and putting  $s = 1$  in (2.2). We have (5.10) by using the definition of  $\psi_{\Phi}(a, z)$  and putting  $s = 1$  in (2.4). □

By putting  $j = 0$  in (5.10) we have the fraction formula

$$\psi_{\Phi}(a/m, z) + \log(1 - z) = z^{-j/m} \sum_{n=0}^{m-1} \left( \psi_{\Phi}(a, \omega_m^n z^{1/m}) + \log(1 - \omega_m^n z^{1/m}) \right).$$

Using the definition of  $\psi_{\Phi}(a, z)$  and putting  $a = 1$ ,  $j + 1 = l$  in (5.10), we have the rational number formula

$$(5.11) \quad \psi_{\Phi}(l/m, z) = -\log(1 - z) + z^{-l/m} \sum_{n=0}^{m-1} \omega_m^{-ln} \log(1 - \omega_m^n z^{1/m}).$$

In [5, p.937] the  $q$ -Euler constant is defined by

$$\gamma(q) := \frac{(q-1)\log(q-1)}{\log q} + (q-1) \sum_{n=1}^{\infty} \frac{1}{q^n - 1} - \frac{q-1}{2},$$

and it is shown in [5, p.938, Theorem 2.4] that if  $q \geq 2$  is an integer, then

$$\gamma(q) - \frac{(q-1)\log(q-1)}{\log q}$$

is an irrational number. Therefore here we consider  $\psi_{\Phi}(l/m, z)$ .

**Theorem 5.4.** *If  $z$  is algebraic,  $|z| \leq 1$ ,  $z \neq 1$ , then  $\psi_{\Phi}(l/m, z)$  is transcendental. Especially, the Hurwitz-Lerch Euler constant  $-\psi_{\Phi}(1, z)$  is transcendental if  $z$  is algebraic,  $|z| \leq 1$ ,  $z \neq 1$ .*

*Proof.* We obtain  $\psi_{\Phi}(l/m, z) \neq 0$  by (5.5) and inequalities

$$\frac{z - t^{a-1}}{1 - zt} = \frac{(z - t^{a-1})(1 - \bar{z}t)}{|1 - zt|^2},$$

$$\Re((z - t^{a-1})(1 - \bar{z}t)) < 0, \quad 0 < a \leq 1, \quad 0 < t < 1.$$

According to Baker's theorem and (5.11),  $\psi_{\Phi}(l/m, z)$  is transcendental. □

When  $k = 1, 2, \dots, m-1$ , we have the next theorem.

**Theorem 5.5** (Relations between Hurwitz-Lerch digamma functions and digamma functions).

$$(5.12) \quad \psi_{\Phi}(ma, \omega_m^k) = \frac{1}{m} \sum_{j=0}^{m-1} \omega_m^{kj} \psi(a + j/m) - \log(1 - \omega_m^k),$$

$$(5.13) \quad \psi((a+j)/m) - \psi(a) + \log m = \sum_{n=1}^{m-1} \omega_m^{-jn} \left( \psi_{\Phi}(a, \omega_m^n) + \log(1 - \omega_m^n) \right).$$

*Proof.* We have (5.12) by putting  $z = \lambda \omega_m^k$  ( $0 < \lambda < 1$ ) and taking  $\lambda \uparrow 1$  in (5.9), and

$$\lim_{z \rightarrow \omega_m^k} \sum_{j=0}^{m-1} z^j \log(1 - z^m) = \lim_{z \rightarrow \omega_m^k} \frac{z^m - 1}{z - 1} \log(1 - z^m) = 0.$$

We have (5.13) by letting  $z \uparrow 1$  on the real axis in (5.10) and

$$\lim_{z \uparrow 1} (-\log(1 - z) + \log(1 - z^{1/m})) = -\log m.$$

□

**Corollary 5.6** (The inversion formula, the fraction formula).

$$(5.14) \quad \psi((a+j)/m) = -\log m + \psi(a) - \sum_{n=1}^{m-1} \omega_m^{-jn} \Phi(\omega_m^n 1, a),$$

$$(5.15) \quad \psi(a/m) = \sum_{n=0}^{m-1} \psi_{\Phi}(a, \omega_m^n) = -\log m + \psi(a) - \sum_{n=1}^{m-1} \Phi(\omega_m^n 1, a).$$

*Proof.* We have the inversion formula (5.14) by the definition of  $\psi_{\Phi}(\omega_m^n, a)$  and (5.13). The first equality of the fraction formula (5.15) is proved by putting  $j = 0$  in (5.13), and

$$m = \lim_{x \rightarrow 1} \frac{1-x^m}{1-x} = \lim_{x \rightarrow 1} \prod_{n=1}^{m-1} (1-x\omega_m^n) = \prod_{n=1}^{m-1} (1-\omega_m^n).$$

We have the second equality of (5.15) by putting  $j = 0$  in (5.14).  $\square$

Now we prove (1.4) and (1.5). By letting  $z \uparrow 1$  on the real axis and

$$\lim_{z \uparrow 1} \left( -\log(1-z) + \frac{1}{m} \sum_{j=0}^{m-1} z^j \log(1-z^m) \right) = \log m,$$

we have (1.4). If  $k = 1, 2, \dots, m-1$ , we have

$$\log(1-\omega_m^k) = \log(2 \sin \pi/m) + i(\pi k/m - \pi/2),$$

$$i\pi \left( \frac{1}{2} + \frac{1}{\omega_m^k - 1} \right) = \frac{\pi}{2} \cot(\pi k/m), \quad \sum_{n=1}^{m-1} \omega_m^{kn} = -1, \quad \sum_{n=1}^{m-1} n\omega_m^{kn} = \frac{m}{\omega_m^{kn} - 1}.$$

By using  $\gamma = -\psi(1)$ , (3.1) and putting  $a = 1$ ,  $j+1 = l$ ,  $l = 1, 2, \dots, m$  in (5.14), we have Gauss' formula (1.5). Therefore (5.14) is a generalization of (1.5). Using the definition of  $\psi_{\Phi}(ma, \omega_m^k)$  and replacing  $a$  by  $a/m$  in (5.12), we have

$$(5.16) \quad \Phi(\omega_m^k, 1, a) = -\frac{1}{m} \sum_{j=0}^{m-1} \omega_m^{kj} \psi((a+j)/m).$$

Putting  $a = 1$  in (5.16), we have Gauss' second formula [9, pp.19, (49)]

$$(5.17) \quad \sum_{n=1}^m \omega_m^{kn} \psi(n/m) = m \log(1-\omega_m^k).$$

## 6. HURWITZ-LERCH GAMMA FUNCTIONS

In the preceding section we consider Hurwitz-Lerch digamma functions. It is natural that we define the following generalization of gamma functions by using Hurwitz-Lerch digamma functions. In many books, for example [3], [9], and [11], the gamma function is defined before the definition of the digamma function. In this paper, the order is reversed.

**Definition 6.1.** If  $|z| \leq 1$ ,  $0 < a$ , we define Hurwitz-Lerch gamma functions by

$$(6.1) \quad \log \Gamma_{\Phi}(a, z) := \int_1^a \psi_{\Phi}(x, z) dx.$$

Letting  $z \uparrow 1$  on the real axis we have

$$\lim_{z \uparrow 1} \int_1^a \psi_\Phi(x, z) dx = \int_1^a \psi(x) dx = \log \Gamma(a)$$

by Abel's theorem. Therefore Hurwitz-Lerch gamma functions are generalizations of the gamma function.

**Theorem 6.2** (The infinite product, Lerch's formula). *If  $|z| \leq 1$ ,  $z \neq 1$ ,  $0 < a$ , we have*

$$(6.2) \quad \Gamma_\Phi(a, z) = (1 - z)^{1-a} \prod_{n=0}^{\infty} \left( \frac{n+1}{n+a} \right)^{z^n},$$

$$(6.3) \quad \log \Gamma_\Phi(a, z) = \frac{\partial}{\partial s} \Phi(z, 0, a) - \frac{\partial}{\partial s} \Phi(z, 0, 1) - (a-1) \log(1-z).$$

*Proof.* By the uniformity of the convergence, we have

$$(6.4) \quad \log \Gamma_\Phi(a, z) = \int_1^a \psi_\Phi(x, z) dx = (1-a) \log(1-z) - \sum_{n=0}^{\infty} z^n \log \frac{n+a}{n+1}.$$

This formula implies (6.2). If  $|z| < 1$ , we have (6.3) by termwise differentiation of  $\Phi(z, s, a)$  and (6.4). If  $|z| \leq 1$ ,  $z \neq 1$ , we have

$$\frac{\partial}{\partial a} \Phi(z, s, a) = -s \Phi(z, s, a) = -s \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^{s+1}} \quad \Re(s) > -1.$$

Therefore we have

$$\frac{\partial^2}{\partial s \partial a} [\Phi(z, s, a)]_{s=0} = -\Phi(z, 1, a).$$

We put

$$(6.5) \quad f(a, z) = \log \Gamma_\Phi(a, z) - \frac{\partial}{\partial s} \Phi(z, 0, a).$$

By the definition of  $\log \Gamma_\Phi(a, z)$  we have

$$\frac{\partial}{\partial a} f(a, z) = -\log(1-z).$$

Hence we have

$$f(a, z) = -a \log(1-z) + g(z).$$

Next we determine  $g(z)$ . By the definition of  $\log \Gamma_\Phi(a, z)$ , we have  $\log \Gamma_\Phi(1, z) = 0$ . Therefore by (6.5), we have

$$f(1, z) = -\frac{\partial}{\partial s} \Phi(z, 0, 1).$$

Therefore we have

$$g(z) = \log(1-z) - \frac{\partial}{\partial s} \Phi(z, 0, 1).$$

This formula implies (6.3). □

**Theorem 6.3** (The integral representation, the asymptotic expansion). *If  $|z| \leq 1$ ,  $z \neq 1$ ,  $0 < a$ , we have*

$$(6.6) \quad \log \Gamma_\Phi(a, z) = \int_0^\infty \left( (a-1) (e^{(z-1)t} - e^{-t}) - \frac{e^{-t} - e^{-at}}{1 - ze^{-t}} \right) \frac{dt}{t},$$

$$(6.7) \quad \log \Gamma_{\Phi}(a, z) = (1-a) \log(1-z) + \mathcal{B}_1(z) \log a - \frac{\partial}{\partial s} \Phi(z, 0, 1) - \sum_{n=2}^M \frac{\mathcal{B}_n(z)}{n(n-1)} (-a)^{-n+1} + O(a^{-M}).$$

*Proof.* Formula (6.6) is a generalization of Malmstén's formula [3, p.21, (1)]. By integrating (5.6) from 1 to  $a$  and changing the order of integration, we have (6.6). Formula (6.7) is a generalization of Stirling's formula [3, p.47, (1)]. By putting

$$f(t) = \frac{te^{-at}}{1 - ze^{-t}}$$

and using Ruijsenaars' method introduced in [8, p.118, (3.13)], we have (6.7).  $\square$

**Theorem 6.4.** *If  $|z| \leq 1$ ,  $z^m \neq 1$  we have the multiplication formula*

$$(6.8) \quad \frac{\Gamma_{\Phi}(ma, z)}{\Gamma_{\Phi}(m, z)} = \prod_{j=0}^{m-1} \frac{(1-z^m)^{(a-1)z^j}}{(1-z)^{(a-1)}} \left( \frac{\Gamma_{\Phi}(a + j/m, z^m)}{\Gamma_{\Phi}(1 + j/m, z^m)} \right)^{z^j}.$$

*If  $|z| \leq 1$ ,  $z \neq 1$  we have the inversion formula*

$$(6.9) \quad \frac{(1-z)^{(a-1)/m} \Gamma_{\Phi}((a+j)/m, z)}{\Gamma_{\Phi}((1+j)/m, z)} = \prod_{n=0}^{m-1} \left( (1 - \omega_m^n z^{1/m})^{(a-1)} \Gamma_{\Phi}(a, \omega_m^n z^{1/m}) \right)^{\omega_m^{-jn} z^{-j/m/m}}.$$

*Proof.* By (5.9) and

$$\int_1^a \psi_{\Phi}(mx, z) dx = \frac{1}{m} \log \frac{\Gamma_{\Phi}(ma, z)}{\Gamma_{\Phi}(m, z)}, \quad \int_1^a \psi_{\Phi}(x + j/m, z) dx = \log \frac{\Gamma_{\Phi}(a + j/m, z)}{\Gamma_{\Phi}(1 + j/m, z)},$$

we have (6.8). By (5.10) and

$$\int_1^a \psi_{\Phi}((x+j)/m, z^m) dx = m \log \frac{\Gamma_{\Phi}((a+j)/m, z^m)}{\Gamma_{\Phi}((1+j)/m, z^m)},$$

we have (6.9).  $\square$

By putting  $j = 0$  in (6.9), we have the fraction formula

$$\frac{(1-z)^{(a-1)/m} \Gamma_{\Phi}(a/m, z)}{\Gamma_{\Phi}(1/m, z)} = \prod_{n=0}^{m-1} \left( (1 - \omega_m^n z^{1/m})^{(a-1)} \Gamma_{\Phi}(a, \omega_m^n z^{1/m}) \right)^{1/m}.$$

We have the following theorem by integrating (5.12) and (5.13) from 1 to  $a$ .

**Theorem 6.5** (Relations between Hurwitz-Lerch gamma functions and the gamma function).

$$(6.10) \quad \frac{\Gamma_{\Phi}(ma, \omega_m^k)}{\Gamma_{\Phi}(m, \omega_m^k)} = (1 - \omega_m^k)^{m(1-a)} \prod_{j=0}^{m-1} \frac{\Gamma(a + j/m)}{\Gamma(1 + j/m)},$$

$$(6.11) \quad \frac{(m^{1-a} \Gamma(a))^{-1/m} \Gamma((a+j)/m)}{\Gamma((1+j)/m)} = \prod_{n=1}^{m-1} \left( (1 - \omega_m^n)^{(a-1)} \Gamma_{\Phi}(a, \omega_m^n) \right)^{\omega_m^{-jn}/m}.$$

Puttig  $j = 0$ , we have the fraction formula

$$(6.12) \quad \Gamma(a/m) = \Gamma(1/m) \prod_{n=0}^{m-1} \left( \Gamma_{\Phi}(a, \omega_m^n) \right)^{1/m}.$$

Finally we calculate the value of  $\Gamma_{\Phi}(a, \omega_m^k)$ . By (6.1), we have  $\Gamma_{\Phi}(1, z) = 1$ . Therefore we have

$$\Gamma_{\Phi}(m, \omega_m^k) = (1 - \omega_m^k)^{1-m} \prod_{j=0}^{m-1} \frac{\Gamma(1 + j/m)}{\Gamma((1 + j)/m)}$$

by putting  $a = 1/m$  in (6.10). Replacing  $a$  by  $a/m$  in (6.10), we have

$$\Gamma_{\Phi}(a, \omega_m^k) = (1 - \omega_m^k)^{1-a} \prod_{j=0}^{m-1} \frac{\Gamma((a + j)/m)}{\Gamma((1 + j)/m)}.$$

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#### REFERENCES

- [1] T. M. Apostol, "On the Lerch zeta-function," *Pacific J. Math.* **1** (1951), 161-167.
- [2] A. Baker, *Transcendental number theory*, Cambridge Univ. Press, 1975.
- [3] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental functions Vol 1*, McGraw-Hill, New York, 1953.
- [4] S. Kanemitsu, M. Katsurada and M. Yoshimoto, "On the Hurwitz-Lerch zeta-function," *Aequationes Math.* **59** (2000), 1-19.
- [5] N. Kurokawa and S. Koyama, " $q$ -analogue of the Euler constant; Lerch's limit formula," *Proc. Amer. Math. Soc.* **132** (2003), 935-943.
- [6] L. Lewin, *Polylogarithms and associated functions*, North Holland, New York and Oxford, 1981.
- [7] L. Lewin, "Some miscellaneous results," in *Structural properties of polylogarithms* (L. Lewin et al., eds.), American Mathematical Society 1991, pp. 355-375.
- [8] S. N. M. Ruijsenaars, "On Barnes' multiple zeta and gamma functions," *Adv. in Math.* **156** (2000), 107-132.
- [9] H. M. Srivastava and Junesang Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston, and London, 2001.
- [10] S. Uchiyama, "An application of A. Baker's fundamental theorem (in Japanese)," in *Proceedings of a Meeting on Transcendental Number Theory*, (Takayoshi, Mitsui ed.), Gakushuin University, 1983, pp. 91-94.
- [11] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, Cambridge Univ. Press, 1945.

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