

# Riemann zeta-values, Euler polynomials and the best constant of Sobolev inequality

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## **Riemann zeta-values, Euler polynomials and the best constant of Sobolev inequality**

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# RIEMANN ZETA-VALUES, EULER POLYNOMIALS AND THE BEST CONSTANT OF SOBOLEV INEQUALITY

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ABSTRACT. In this paper, we obtain the value  $\sup_{u \in I_M, u \neq 0} S_M(u)$ , where  $S_M(u)$  is the Sobolev functional and  $I_M$  is a certain pre-Hilbert space. We also show that the functions attaining  $\sup_{u \in I_M, u \neq 0} S_M(u)$  are explicitly written by the Euler polynomials. This result is an analogue of the theorem proved by Kametaka, Yamagishi, Watanabe, Nagai and Takemura. Simultaneously, we obtain a much simpler proof of their theorem.

## 1. INTRODUCTION

Kametaka, Yamagishi, Watanabe, Nagai and Takemura [1] showed the following Theorem A. In order to present their theorem, we define the following function spaces:

$$H_M := \{u(x) : u^{(M)}(x) \in L^2(0, 1), \ u^{(j)}(1) = u^{(j)}(0), \ (0 \leq j \leq M-1), \ \int_0^1 u(x)dx = 0\},$$

where  $M = 1, 2, 3, \dots$  and the Sobolev functional

$$S_M(u) := \left( \sup_{0 \leq y \leq 1} |u(y)| \right)^2 / \int_0^1 |u^{(M)}(x)|^2 dx.$$

**Theorem A** (see [1, Theorem 1.1]). *For  $M = 1, 2, 3, \dots$ , we have the next conclusions.*

- (1)  $\inf_{u \in H_M, u \neq 0} S_M(u) = 0$ .
- (2)  $\sup_{u \in H_M, u \neq 0} S_M(u) = 2\zeta(2M)(2\pi)^{-2M}$ , where  $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ ,  $\Re(s) > 1$ .
- (3) *For any fixed  $y$  satisfying  $0 \leq y \leq 1$ , we have  $S_M(B_{2M}(|x-y|)) = 2\zeta(2M)(2\pi)^{-2M}$ , where  $B_M(x)$  is the Bernoulli polynomials defined by (2.1).*

In this paper, we will show the following theorem which is an analogue of Theorem A. We define the next function spaces:

$$I_M := \{u(x) : u^{(M)}(x) \in L^2(0, 1), \ u^{(j)}(1) = -u^{(j)}(0), \ (0 \leq j \leq M-1)\}.$$

**Theorem 1.1.** *For  $M = 1, 2, 3, \dots$ , we have the following conclusions.*

- (1)  $\inf_{u \in I_M, u \neq 0} S_M(u) = 0$ .
- (2)  $\sup_{u \in I_M, u \neq 0} S_M(u) = K_M$ , where  $K_M$  is defined by (3.2).
- (3) *For any fixed  $y$  satisfying  $0 \leq y \leq 1$ , we have  $S_M(E_{2M-1}(|x-y|)) = K_M$ , where the Euler polynomials  $E_M(x)$  is defined by (2.2).*

**Remark 1.2.** *We obtain a simple proof of Theorem A by changing  $(-1)^M E_{2M-1}(|x-y|)/2(2M-1)!$  into  $(-1)^M B_{2M}(|x-y|)/(2M)!$  and modifying the proof of Theorem 1.1.*

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## 2. EULER POLYNOMIALS

In this section we quote definition and properties of the Euler and Bernoulli polynomials from [2, Section 1.6] in order to prove Theorem 1.1. The Bernoulli polynomials  $B_n(x)$  are defined by the following generating function:

$$(2.1) \quad \frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}, \quad |z| < 2\pi.$$

The Euler polynomials  $E_n(x)$  are an analogue of Bernoulli polynomials  $B_n(x)$ , and are defined by the following generating function:

$$(2.2) \quad \frac{2e^{xz}}{e^z + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!}, \quad |z| < \pi.$$

The Euler polynomials  $E_n(x)$ ,  $n = 0, 1, 2, 3$  are as follows.

$$E_0(x) = 1, \quad E_1(x) = x - \frac{1}{2}, \quad E_2(x) = x^2 - x, \quad E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}.$$

The next formulas are readily derivable from the definition of the Euler polynomials.

- (E1)  $E'_n(x) = nE_{n-1}(x)$ ,  $n \in \mathbb{N}$ .
- (E2)  $E_n(1-x) = (-1)^n E_n(x)$ ,  $n \in \mathbb{N}_0$ .
- (E3)  $(-1)^{n+1} E_n(-x) = E_n(x) - 2x^n$ ,  $n \in \mathbb{N}_0$ .

By using the above properties, we obtain the next lemma.

**Lemma 2.1.** *We define the function  $D_M(x, y) := (-1)^M E_{2M-1}(|x-y|)/2(2M-1)!$ , where  $M = 1, 2, 3, \dots$ . Then we have the following properties.*

- (D1)  $\partial_x^{2M} D_M(x, y) = 0$ ,  $x \neq y$ .
- (D2)  $\partial_x^j D_M(x, y)|_{x=1} = -\partial_x^j D_M(x, y)|_{x=0}$ ,  $0 \leq j \leq 2M-1$ ,  $0 < y < 1$ .
- (D3)  $\partial_x^j D_M(x, y)|_{x=y+0} - \partial_x^j D_M(x, y)|_{x=y-0} = 0$  or  $(-1)^M$ ,  $0 \leq j \leq 2M-2$  or  $j = 2M-1$ , respectively.

*Proof.* By using (E1), we obtain the following formula:

$$(2.3) \quad \partial_x^j D_M(x, y) = (-1)^M (\operatorname{sgn}(x-y))^j \frac{E_{2M-j-1}(|x-y|)}{2(2M-j-1)!}, \quad \begin{cases} 0 \leq j \leq 2M-2, & 0 < x, y < 1, \\ 0 \leq j \leq 2M-1, & 0 < x \neq y < 1. \end{cases}$$

We have (D1) by (2.3) and differentiating  $E_1(|x-y|)$  twice. We obtain (D2) by using (2.3) and (E2). When  $0 \leq j \leq 2M-2$ , we have (D3) by (2.3) and (E3). When  $j = 2M-1$ , we obtain (D3) by using (2.3) and differentiating  $E_1(|x-y|)$ .  $\square$

## 3. PROOF OF THEOREM 1.1

**Proposition 3.1.** *For the function space  $I_M$ , the sesquilinear form*

$$\langle u, v \rangle_M := \int_0^1 u^{(M)}(x) v^{(M)}(x) dx$$

*is an inner product. Moreover when  $u(x) \in I_M$ , we have the following reproducing relation,*

$$(3.1) \quad \int_0^1 u^{(M)}(x) \partial_x^M D_M(x, y) dx = u(y), \quad 0 \leq y \leq 1.$$

*Proof.* By integrating with respect to  $x$  on the intervals  $0 < x < y$  and  $y < x < 1$  with arbitrarily fixed  $0 < y < 1$  and using (D1), (D2), (D3) and the condition  $u^{(j)}(1) = -u^{(j)}(0)$ ,  $0 \leq j \leq M-1$ , we have

$$\begin{aligned} \langle u, D_M \rangle_M &= \left[ u^{(M-1)}(x) \partial_x^M D_M(x, y) \right]_{x=0}^{x=1} - \int_0^1 u^{(M-1)}(x) \partial_x^{M+1} D_M(x, y) dx \\ &= \cdots = (-1)^{M-1} \int_0^1 u^{(1)}(x) \partial_x^{2M-1} D_M(x, y) dx \\ &= (-1)^{M-1} \left[ u(x) \partial_x^{2M-1} D_M(x, y) \right] \left\{ \left|_0^y + \left|_y^1 \right\} + (-1)^M \int_0^1 u(x) \partial_x^{2M} D_M(x, y) dx = u(y). \right. \end{aligned}$$

Next we will compute the value  $K_M := D_M(y, y) = (-1)^M E_{2M-1}(0)/2(2M-1)!$ . By using the Fourier series expansions of  $E_n(x)$ , we have

$$(3.2) \quad K_M := \frac{(-1)^M E_{2M-1}(0)}{2(2M-1)!} = \frac{2}{\pi^{2M}} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^{2M}} = \frac{(2^{2M+1} - 2)\zeta(2M)}{(2\pi)^{2M}}.$$

Putting  $u(x) = D_M(x, y) \in I_M$  (this is proved by Lemma 2.1) in (3.1), we obtain

$$(3.3) \quad \int_0^1 |\partial_x^M D_M(x, y)|^2 dx = D_M(y, y) = K_M.$$

Hence applying the Schwarz inequality to (3.1), we have

$$(3.4) \quad |u(y)|^2 \leq \int_0^1 |u^{(M)}(x)|^2 dx \int_0^1 |\partial_x^M D_M(x, y)|^2 dx = K_M \int_0^1 |u^{(M)}(x)|^2 dx.$$

This inequality implies that  $\langle u, u \rangle_M = \int_0^1 |u^{(M)}(x)|^2 dx = 0$  holds if and only if  $u(x) \equiv 0$ , which shows that the form  $\langle \cdot, \cdot \rangle_M$  is an inner product of  $I_M$ .  $\square$

*Proof of Theorem 1.1.* Applying  $u(x) = D(x, y_0)$  to (3.4) and using (3.3), we have

$$\sup_{0 \leq y \leq 1} |D_M(y, y_0)|^2 \leq K_M \int_0^1 |\partial_x^M D_M(x, y)|^2 dx = K_M^2.$$

Obviously, we have  $K_M := D_M(y_0, y_0) \leq \sup_{0 \leq y \leq 1} |D_M(y, y_0)|$ . Hence we obtain (2) and (3) of Theorem 1.1. We can prove (1) of Theorem 1.1 by taking  $u(x) = \sin(2n+1)\pi x$  or  $\cos(2n+1)\pi x$ ,  $n = 1, 2, \dots$ .  $\square$

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## REFERENCES

- [1] Y. Kametaka, H. Yamagishi, K. Watanabe A. Nagai and K. Takemura, Riemann zeta function, Bernoulli polynomials and the best constant of Sobolev inequality, *Scientiae Mathematicae Japonicae in Editione Electronica*, 65 (2007) 333-359.
- [2] H. M. Srivastava and Junesang Choi, Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, 2001.

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