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Abstract

The first Painlevé equation is expressed as a vector field on the three dimensional weighted projective space $\mathbb{C}P^3(3,2,4,5)$. With the aid of the dynamical systems theory and the orbifold structure of $\mathbb{C}P^3(3,2,4,5)$, a simple proof of the Painlevé property and the construction of the space of initial conditions are given. In particular, Painlevé's transformation is geometrically derived, which proves to be a Darboux coordinates of a certain algebraic surface with a holomorphic symplectic form.

Keywords: the first Painlevé equation; weighted projective space

1 Introduction

The first Painlevé equation ($\mathrm{P_I}$)

$$\frac{dx}{dz} = 6y^2 + z,\ \frac{dy}{dz} = x, \quad (x, y, z) \in \mathbb{C}^3 \tag{1.1}$$

is invariant under the $\mathbb{Z}_5$ action defined by

$$(x, y, z) \mapsto (\omega^3 x, \omega^2 y, \omega^4 z), \quad \omega := e^{2\pi i/5}. \tag{1.2}$$

Motivated by this symmetry, ($\mathrm{P_I}$) is studied by means of the weighted projective space $\mathbb{C}P^3(3,2,4,5)$. The weighted projective space $\mathbb{C}P^3(3,2,4,5)$ is a three dimensional compact orbifold (toric variety) with singularities, see Sec.2 for the definition. ($\mathrm{P_I}$) is regarded as a 3-dim vector field on $\mathbb{C}P^3(3,2,4,5)$ and the dynamical systems theory will be applied to the vector field. The space $\mathbb{C}P^3(3,2,4,5)$ is decomposed as

$$\mathbb{C}P^3(3,2,4,5) = \mathbb{C}^3/\mathbb{Z}_5 \ \cup\ \mathbb{C}P^2(3,2,4), \quad \text{(disjoint)}. \tag{1.3}$$

This means that $\mathbb{C}P^3(3,2,4,5)$ is a compactification of $\mathbb{C}^3/\mathbb{Z}_5$ obtained by attaching a 2-dim weighted projective space $\mathbb{C}P^2(3,2,4)$ at infinity. ($\mathrm{P_I}$) divided by the $\mathbb{Z}_5$ action is given on $\mathbb{C}^3/\mathbb{Z}_5$, and the 2-dim space $\mathbb{C}P^2(3,2,4)$ describes behavior of ($\mathrm{P_I}$) near infinity (i.e. $x = \infty$ or $y = \infty$ or $z = \infty$). On the "infinity set" $\mathbb{C}P^2(3,2,4)$, there exist two fixed points of the vector field. The one corresponds to movable poles of ($\mathrm{P_I}$), and the other corresponds to the irregular singular point $z = \infty$. Local properties of these fixed points will be investigated by the dynamical systems theory. Our main results include

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- a simple proof of the fact that any solutions of $(\mathrm{P_I})$ are meromorphic on $\mathbb{C}$,
- a simple construction of Okamoto's space of initial conditions,
- a characterization of $(\mathrm{P_I})$ by the geometry of $\mathbb{C}P^3(3,2,4,5)$ and a certain local condition,
- a geometric interpretation of Painlevé's coordinates defined by

$$\left\{\begin{array}{l} x = uw^3 - 2w^{-3} - \frac{1}{2}zw - \frac{1}{2}w^2 \\ y = w^{-2}, \end{array}\right. \tag{1.4}$$

  which was introduced in his original work [17] to prove the Painlevé property of $(\mathrm{P_I})$,
- a geometric interpretation of Boutroux's coordinates/solution introduced in [1] to investigate the irregular singular point of $(\mathrm{P_I})$.

Although these topics have been well studied by many authors, all of our new approaches are based on the geometry of $\mathbb{C}P^3(3,2,4,5)$ and these topics are treated in a unified manner.

If we regard $(\mathrm{P_I})$ as a vector field on $\mathbb{C}P^3(3,2,4,5)$, we will find two fixed points (zeros of the vector field), say Q_1 and Q_2, on the "infinity set" $\mathbb{C}P^2(3,2,4)$ as was mentioned above. In Sec.3, the normal form theory of vector fields is applied to the fixed point Q_1, which corresponds to movable singularities. With the aid of the Poincaré linearization theorem, it is proved that $(\mathrm{P_I})$ is locally transformed into an integrable system $y'' = 6y^2$ near each movable singularity. As a result, two independent first integrals of $(\mathrm{P_I})$ are obtained. By using the implicit function theorem to these integrals, meromorphy of solutions will be immediately proved. Conversely, if a given meromorphic vector field defined on $\mathbb{C}P^3(3,2,4,5)$ satisfies a certain condition on the Jacobi matrix at the point Q_1, then the vector field is equivalent to $(\mathrm{P_I})$ $y'' = 6y^2 + z$ or the integrable system $y'' = 6y^2$. Roughly speaking, $(\mathrm{P_I})$ is characterized by the geometry of $\mathbb{C}P^3(3,2,4,5)$ and the local condition at one point Q_1.

In Sec.4, the space of initial conditions for $(\mathrm{P_I})$ is constructed (see Sec.4 for the definition of the space of initial conditions). The fixed point Q_1 above is a singularity of the foliation; any integral curves of $(\mathrm{P_I})$ pass through Q_1. In order to desingularize the point, the weighted blow-up will be introduced. One point Q_1 is enlarged to a 2-dim weighted projective space $\mathbb{C}P^2(6,4,5)$ by the weighted blow-up. It is remarkable that only one time blow-up is sufficient to obtain the space of initial conditions if we use suitable weights, while Okamoto performed blow-up (without weights) eight times to obtain the space of initial conditions [15, 8]. In particular, our space of initial conditions is covered by only two charts.

By the weighted blow-up of $\mathbb{C}P^3(3,2,4,5)$, we will recover the famous Painlevé's coordinates (1.4) in a purely geometric manner. Painlevé found the coordinates transformation (1.4) in an analytic way to prove the Painlevé property of $(\mathrm{P_I})$ (see [6]). From our approach using the weighted projective space, it turns out that

Painlevé's coordinates is nothing but the Darboux coordinates of the nonsingular algebraic surface $M(z)$ defined by

$$V^2 = UW^4 + 2zW^3 + 4W,$$

which admits a holomorphic symplectic form, where $z \in \mathbb{C}$ is an independent variable of $(\mathrm{P_I})$ and it is a parameter of the surface. Our space of initial conditions is obtained by glueing $\mathbb{C}^2_{(x,y)}$ (the original space for dependent variables) and the surface $M(z)$, which also has a holomorphic symplectic form. Then, $(\mathrm{P_I})$ proves to be a Hamiltonian system with respect to the form.

In Sec.5, a cellular decomposition of the weighted blow-up of $\mathbb{C}P^3(3,2,4,5)$ will be given. We will show that the weighted blow-up of $\mathbb{C}P^3(3,2,4,5)$ is naturally decomposed into the fiber space for $(\mathrm{P_I})$ (a fiber bundle over $\mathbb{C}$ whose fiber is the space of initial conditions), a certain elliptic fibration over the moduli space of complex tori defined by the Weierstrass equation, and the projective curve $\mathbb{C}P^1$. We also show that the extended Dynkin diagram of type $\tilde{E}_8$ is hidden in the weighted blow-up of $\mathbb{C}P^3(3,2,4,5)$.

In Sec.6, the other fixed point Q_2 of the vector field will be investigated. The Jacobi matrix of the vector field at Q_2 has exactly one zero eigenvalue. This means that there exists a one dimensional center manifold. We will show that one of the inhomogeneous coordinates of $\mathbb{C}P^3(3,2,4,5)$ including Q_2 is just Boutroux's coordinates, and Boutroux's tritronquée solution can be defined as the center manifold at Q_2.

An approach using weighted projective spaces is also applicable to the second Painlevé equation to the sixth Painlevé equation, which will appear in a forthcoming paper.

2 Weighted projective space

In this section, a weighted projective space is defined and the first Painlevé equation $(\mathrm{P_I})$ is given as a meromorphic equation on $\mathbb{C}P^3(3,2,4,5)$.

Let $\widetilde{U}$ be a complex manifold and Γ a finite group acting analytically and effectively on $\widetilde{U}$. In general, the quotient space $\widetilde{U}/\Gamma$ is not a smooth manifold if the action has fixed points. Roughly speaking, a (complex) orbifold M is defined by glueing a family of such spaces $\widetilde{U}_\alpha/\Gamma_\alpha$; a Hausdorff space M is called an orbifold if there exist an open covering $\{U_\alpha\}$ of M and homeomorphisms $\varphi_\alpha : U_\alpha \simeq \widetilde{U}_\alpha/\Gamma_\alpha$. See [21] for more details. In this article, we will consider quotient spaces of the form $\mathbb{C}^n/\mathbb{Z}_p$.

A meromorphic differential form on a complex orbifold $M = \bigcup U_\alpha \simeq \bigcup \widetilde{U}_\alpha/\Gamma_\alpha$ is defined to be a family $\widetilde{\omega}_\alpha$ of Γ_α-invariant meromorphic forms on $\widetilde{U}_\alpha$, which is consistent on intersections $U_\alpha \cap U_\beta$. More formally, let $\{\widetilde{\omega}_\alpha\}$ be a family of Γ_α-invariant meromorphic forms on $\widetilde{U}_\alpha$. If there is an open set $U_\gamma \simeq \widetilde{U}_\gamma/\Gamma_\gamma$ such that $U_\gamma \subset U_\alpha \cap U_\beta$, we suppose that there are injections $\lambda_j : \widetilde{U}_\gamma \to \widetilde{U}_j$ satisfying

$\lambda_\alpha^*\widetilde{\omega}_\alpha = \lambda_\beta^*\widetilde{\omega}_\beta$. Then, the family $\{\widetilde{\omega}_\alpha\}$ is called a meromorphic differential form on the orbifold M. We also define a meromorphic ordinary differential equation (ODE) on an orbifold by regarding differential equations as Pfaffian forms.

Consider the weighted $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ given by

$$(x_0, \cdots, x_n) \mapsto (\lambda^{p_0}x_0, \cdots, \lambda^{p_n}x_n), \quad \lambda \in \mathbb{C}^* := \mathbb{C}\backslash\{0\}, \tag{2.1}$$

with the weight $(p_0, \cdots, p_n) \in \mathbb{Z}^n$. The quotient space

$$\mathbb{C}P^n(p_0, \cdots, p_n) := \mathbb{C}^{n+1}/\mathbb{C}^* \tag{2.2}$$

is called the weighted projective space. Note that $\mathbb{C}P^n(1, \cdots, 1)$ is a usual projective space, while otherwise a weighted projective space is not a complex manifold but an orbifold with singularities.

In this article, we only consider the weighted projective space $\mathbb{C}P^3(3, 2, 4, 5)$ for the study of $(\mathrm{P_I})$. Let us confirm that $\mathbb{C}P^3(3, 2, 4, 5)$ is indeed an orbifold.

The space $\mathbb{C}P^3(3, 2, 4, 5)$ is defined by the equivalence relation on $\mathbb{C}^4$

$$(x, y, z, \varepsilon) \sim (\lambda^3x, \lambda^2y, \lambda^4z, \lambda^5\varepsilon).$$

(i) When $x \neq 0$,

$$(x, y, z, \varepsilon) \sim (1,\ x^{-2/3}y,\ x^{-4/3}z,\ x^{-5/3}\varepsilon) =: (1, Y_1, Z_1, \varepsilon_1).$$

However, due to the choice of the branch of $x^{1/3}$, we also obtain

$$(Y_1, Z_1, \varepsilon_1) \sim (e^{\pm 4\pi i/3}Y_1, e^{\pm 8\pi i/3}Z_1, e^{\pm 10\pi i/3}\varepsilon_1),$$

by putting $x \mapsto e^{\pm 2\pi i}x$. This implies that three points $(Y_1, Z_1, \varepsilon_1)$, $(e^{4\pi i/3}Y_1, e^{8\pi i/3}Z_1, e^{10\pi i/3}\varepsilon_1)$ and $(e^{-4\pi i/3}Y_1, e^{-8\pi i/3}Z_1, e^{-10\pi i/3}\varepsilon_1)$ have to be identified. Thus, the subset of $\mathbb{C}P^3(3, 2, 4, 5)$ such that $x \neq 0$ is homeomorphic to $\mathbb{C}^3/\mathbb{Z}_3$.

(ii) When $y \neq 0$,

$$(x, y, z, \varepsilon) \sim (y^{-3/2}x,\ 1,\ y^{-2}z,\ y^{-5/2}\varepsilon) =: (X_2, 1, Z_2, \varepsilon_2).$$

Because of the choice of the branch of $y^{1/2}$, we obtain

$$(X_2, Z_2, \varepsilon_2) \sim (-X_2, Z_2, -\varepsilon_2). \tag{2.3}$$

Hence, the subset of $\mathbb{C}P^3(3, 2, 4, 5)$ with $y \neq 0$ is homeomorphic to $\mathbb{C}^3/\mathbb{Z}_2$.

(iii) When $z \neq 0$,

$$(x, y, z, \varepsilon) \sim (z^{-3/4}x,\ z^{-1/2}y,\ 1,\ z^{-5/4}\varepsilon) =: (X_3, Y_3, 1, \varepsilon_3).$$

Similarly, the subset $\{z \neq 0\} \subset \mathbb{C}P^3(3, 2, 4, 5)$ is homeomorphic to $\mathbb{C}^3/\mathbb{Z}_4$.

(iv) When $\varepsilon \neq 0$,

$$(x, y, z, \varepsilon) \sim (\varepsilon^{-3/5}x,\ \varepsilon^{-2/5}y,\ \varepsilon^{-4/5}z,\ 1) =: (X_4, Y_4, Z_4, 1).$$

The subset $\{\varepsilon \neq 0\} \subset \mathbb{C}P^3(3,2,4,5)$ is homeomorphic to $\mathbb{C}^3/\mathbb{Z}_5$.

This proves that $\mathbb{C}P^3(3,2,4,5)$ is an orbifold satisfying

$$\mathbb{C}P^3(3,2,4,5) = \mathbb{C}^3/\mathbb{Z}_3 \cup \mathbb{C}^3/\mathbb{Z}_2 \cup \mathbb{C}^3/\mathbb{Z}_4 \cup \mathbb{C}^3/\mathbb{Z}_5.$$

The local charts $(Y_1, Z_1, \varepsilon_1)$, $(X_2, Z_2, \varepsilon_2)$, $(X_3, Y_3, \varepsilon_3)$ and (X_4, Y_4, Z_4) defined above are called inhomogeneous coordinates as the usual projective space. Note that they give coordinates on the lift $\mathbb{C}^3$, not on the quotient $\mathbb{C}^3/\mathbb{Z}_p$. Therefore, any equations written in these inhomogeneous coordinates should be invariant under the corresponding $\mathbb{Z}_p$ actions.

The transformations between inhomogeneous coordinates are give by

$$\left\{\begin{array}{rcrcrcr} X_4 = & & \varepsilon_1^{-3/5} & = & X_2\varepsilon_2^{-3/5} & = & X_3\varepsilon_3^{-3/5} \\ Y_4 = & & Y_1\varepsilon_1^{-2/5} & = & \varepsilon_2^{-2/5} & = & Y_3\varepsilon_3^{-2/5} \\ Z_4 = & & Z_1\varepsilon_1^{-4/5} & = & Z_2\varepsilon_2^{-4/5} & = & \varepsilon_3^{-4/5}. \end{array}\right. \tag{2.4}$$

Now we give $(\mathrm{P_I})$ on the (X_4, Y_4, Z_4)-coordinates as

$$(\mathrm{P_I}) \quad \frac{dX_4}{dZ_4} = 6Y_4^2 + Z_4, \quad \frac{dY_4}{dZ_4} = X_4. \tag{2.5}$$

By (2.4), $(\mathrm{P_I})$ is transformed into the following equations

$$\begin{aligned} \frac{dY_1}{d\varepsilon_1} &= \frac{3 - 12Y_1^3 - 2Y_1Z_1}{\varepsilon_1(-30Y_1^2 - 5Z_1)}, \quad \frac{dZ_1}{d\varepsilon_1} = \frac{3\varepsilon_1 - 24Y_1^2Z_1 - 4Z_1^2}{\varepsilon_1(-30Y_1^2 - 5Z_1)}, \\ \frac{dX_2}{d\varepsilon_2} &= \frac{-12 - 2Z_2 + 3X_2^2}{5X_2\varepsilon_2}, \quad \frac{dZ_2}{d\varepsilon_2} = \frac{-2\varepsilon_2 + 4X_2Z_2}{5X_2\varepsilon_2}, \\ \frac{dX_3}{d\varepsilon_3} &= \frac{-24Y_3^2 - 4 + 3X_3\varepsilon_3}{5\varepsilon_3^2}, \quad \frac{dY_3}{d\varepsilon_3} = \frac{-4X_3 + 2Y_3\varepsilon_3}{5\varepsilon_3^2}, \end{aligned}$$

on the other inhomogeneous coordinates. Although the transformations (2.4) have branches, the above equations are rational due to the symmetry (1.2) of $(\mathrm{P_I})$. It is easy to verify that (2.5) and above equations are invariant under the suitable $\mathbb{Z}_p$ actions. For example, the second one is invariant under the $\mathbb{Z}_2$ action (2.3). Hence, they define a meromorphic ODE on $\mathbb{C}P^3(3,2,4,5)$ in the sense of an orbifold. Indeed, an ODE on $\mathbb{C}P^3(3,2,4,5)$ is meromorphic if and only if the expressions written in the inhomogeneous coordinates are meromorphic.

For our purposes, it is convenient to rewrite above equations as 3-dim vector fields (autonomous ODEs) given by

$$\left\{\begin{array}{l} \dot{Y}_1 = 3 - 12Y_1^3 - 2Y_1Z_1, \\ \dot{Z}_1 = 3\varepsilon_1 - 24Y_1^2Z_1 - 4Z_1^2, \\ \dot{\varepsilon}_1 = \varepsilon_1(-30Y_1^2 - 5Z_1), \end{array}\right. \tag{2.6}$$

$$\left\{\begin{array}{l} \dot{X}_2 = (-12 - 2Z_2 + 3X_2^2)/X_2, \\ \dot{Z}_2 = (-2\varepsilon_2 + 4X_2Z_2)/X_2, \\ \dot{\varepsilon}_2 = 5\varepsilon_2, \end{array}\right. \tag{2.7}$$

$$\begin{cases} \dot{X}_3 = -24Y_3^2 - 4 + 3X_3\varepsilon_3, \\ \dot{Y}_3 = -4X_3 + 2Y_3\varepsilon_3, \\ \dot{\varepsilon}_3 = 5\varepsilon_3^2, \end{cases} \tag{2.8}$$

where $(\dot{\ }) = d/dt$ and t is an additional parameter.

It turns out from (2.4) that if $\varepsilon_i = 0\,(i = 1, 2, 3)$, $X_4 = \infty$ or $Y_4 = \infty$ or $Z_4 = \infty$. Thus, the set $\{\varepsilon_1 = 0\} \cup \{\varepsilon_2 = 0\} \cup \{\varepsilon_3 = 0\}$ is attached at "infinity" of the (X_4, Y_4, Z_4)-coordinates. Further, when $\varepsilon_1 = 0$ or $\varepsilon_2 = 0$ or $\varepsilon_3 = 0$, then (Y_1, Z_1), (X_2, Z_2) and (X_3, Y_3) are related as

$$\begin{cases} Y_1 = X_2^{-2/3} & = Y_3 X_3^{-2/3}, \\ Z_1 = Z_2 X_2^{-4/3} & = X_3^{-4/3}. \end{cases} \tag{2.9}$$

An orbifold obtained by glueing three copies of $\mathbb{C}^2$ by this relation gives the weighted projective space $\mathbb{C}P^2(3, 2, 4)$. This proves the cellular decomposition (1.3). The (X_4, Y_4, Z_4)-coordinates is assigned on the covering space $\mathbb{C}^3$ of $\mathbb{C}^3/\mathbb{Z}_5$, and $(\mathrm{P_I})$ is defined on it. The "infinity set" $\mathbb{C}P^2(3, 2, 4)$ is given by

$$\mathbb{C}P^2(3, 2, 4) = \{\varepsilon_1 = 0\} \cup \{\varepsilon_2 = 0\} \cup \{\varepsilon_3 = 0\}. \tag{2.10}$$

According to Eqs.(2.6),(2.7),(2.8), the set $\mathbb{C}P^2(3, 2, 4)$ is an invariant manifold of the vector fields; that is, $\varepsilon_i(t) \equiv 0$ when $\varepsilon_i(0) = 0$ at an initial time. The dynamics on the invariant manifold describes behavior of $(\mathrm{P_I})$ near infinity. In particular, fixed points of the vector fields play an important role. Vector fields (2.6),(2.7),(2.8) have exactly two fixed points on $\mathbb{C}P^2(3, 2, 4)$;

(i). $(X_2, Z_2, \varepsilon_2) = (\pm 2, 0, 0)$.

Due to the $\mathbb{Z}_2$ action (2.3) on the $(X_2, Z_2, \varepsilon_2)$-coordinates, two points $(2, 0, 0)$ and $(-2, 0, 0)$ represent the same point on $\mathbb{C}P^3(3, 2, 4, 5)$ and it is sufficient to consider one of them. In the next section, we will show that this fixed point corresponds to movable singularities of $(\mathrm{P_I})$. By applying the normal form theory of vector fields, we will prove that any solutions of $(\mathrm{P_I})$ are meromorphic; in particular, $(\mathrm{P_I})$ has the Painlevé property. In Sec.4, the space of initial conditions for $(\mathrm{P_I})$ is constructed by applying the weighted blow-up at this point.

(ii). $(X_3, Y_3, \varepsilon_3) = (0, (-6)^{-1/2}, 0)$

Again, two points should be identified due to the $\mathbb{Z}_4$ action on $(X_3, Y_3, \varepsilon_3)$. This fixed point corresponds to the irregular singular point of $(\mathrm{P_I})$, which will be considered in Sec.6.

Note that fixed points obtained from the $(Y_1, Z_1, \varepsilon_1)$-coordinates are the same as one of the above. For example, the fixed point $(Y_1, Z_1, \varepsilon_1) = ((1/4)^{1/3}, 0, 0)$ is the same as $(X_2, Z_2, \varepsilon_2) = (\pm 2, 0, 0)$ due to (2.9).

3 Movable singularities

Recall that a singularity $Z_4 = z_*$ of a solution of $(\mathrm{P_I})$, Eq.(2.5), is called movable if the position of z_* depends on the choice of an initial condition. In this section, we give local analysis near such movable singularities based on the normal form theory. Though it is known that all movable singularities of $(\mathrm{P_I})$ are poles (Painlevé property), we do not use this fact to give a new proof of Thm.3.2 below.

3.1 Meromorphy of solutions

At first, let us show that $(\mathrm{P_I})$ is locally transformed into a linear system. By putting $\hat{X}_2 = X_2 - 2$, Eq.(2.7) is rewritten as

$$\begin{cases} \dot{\hat{X}}_2 = 6\hat{X}_2 - Z_2 + \dfrac{-3\hat{X}_2^2 + \hat{X}_2 Z_2}{2+\hat{X}_2}, \\ \dot{Z}_2 = 4Z_2 - \varepsilon_2 + \dfrac{\hat{X}_2\varepsilon_2}{2+\hat{X}_2}, \\ \dot{\varepsilon}_2 = 5\varepsilon_2. \end{cases} \tag{3.1}$$

The origin is a fixed point with the Jacobi matrix

$$J = \begin{pmatrix} 6 & -1 & 0 \\ 0 & 4 & -1 \\ 0 & 0 & 5 \end{pmatrix}. \tag{3.2}$$

The eigenvalues $\lambda = 6, 4, 5$ satisfy the condition on the Poincaré linearization theorem (for the convenience of the reader, we give the statement of this theorem in the end of this subsection). Hence, there exists a neighborhood U of the origin and a local analytic transformation defined on U of the form

$$\begin{pmatrix} \hat{X}_2 \\ Z_2 \end{pmatrix} \mapsto \begin{pmatrix} \hat{u}_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} \hat{X}_2 + \varphi_1(\hat{X}_2, Z_2, \varepsilon_2) \\ Z_2 + \hat{\varphi}_2(\hat{X}_2, Z_2, \varepsilon_2) \end{pmatrix},$$

such that Eq.(3.1) is linearized as

$$\begin{cases} \dot{\hat{u}}_2 = 6\hat{u}_2 - v_2, \\ \dot{v}_2 = 4v_2 - \varepsilon_2, \\ \dot{\varepsilon}_2 = 5\varepsilon_2, \end{cases} \tag{3.3}$$

where local analytic functions φ_1 and $\hat{\varphi}_2$ satisfy $\varphi_1, \hat{\varphi}_2 \sim O(||\boldsymbol{X}||^2)$, $\boldsymbol{X} := (\hat{X}_2, Z_2, \varepsilon_2)$. Note that we need not change ε_2 because the equation of ε_2 is already linear. Furthermore, we have $\hat{\varphi}_2(\hat{X}_2, Z_2, 0) = 0$ because the equation of Z_2 is linear when $\varepsilon_2 = 0$. Thus, we can set $\hat{\varphi}_2 = \varepsilon_2\varphi_2$ and the above transformation takes the form

$$\begin{pmatrix} \hat{X}_2 \\ Z_2 \end{pmatrix} \mapsto \begin{pmatrix} \hat{u}_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} \hat{X}_2 + \varphi_1(\hat{X}_2, Z_2, \varepsilon_2) \\ Z_2 + \varepsilon_2\varphi_2(\hat{X}_2, Z_2, \varepsilon_2) \end{pmatrix}, \tag{3.4}$$

with $\varphi_1 \sim O(||\boldsymbol{X}||^2)$ and $\varphi_2 \sim O(X_2, Z_2, \varepsilon_2)$.

Remark 3.1. For the linear system Eq.(3.3), let us change coordinates as $\hat{u}_2 =$

$u_2 - 2$, and $x = u_2\varepsilon_2^{-3/5}, y = \varepsilon_2^{-2/5}, z = v_2\varepsilon_2^{-4/5}$; that is, we move to the original chart for (P_I), see Eq.(2.4). Then, y satisfies the equation $d^2y/dz^2 = 6y^2$, which is exactly solved as $y = (z - C)^{-2}$. This proves that there is a local analytic transformation defined near $(\hat{X}_2, Z_2, \varepsilon_2) = (0, 0, 0)$ such that (P_I) $Y_4'' = 6Y_4^2 + Z_4$ is transformed into $y'' = 6y^2$. This fact was first obtained by [4] for (P_I). Our proof using weighted projective spaces and the Poincaré theorem is also applicable to the second Painlevé to sixth Painlevé equations to prove that they are locally transformed to solvable systems. This result does not prove the meromorphy of solutions of (P_I) because the independent variable is also changed $Z_4 \mapsto z$.

Since Eq.(3.3) is linear, we can construct two integrals explicitly as

$$C_1 = \varepsilon_2^{-4/5}v_2 + \varepsilon_2^{1/5}, \quad C_2 = \frac{1}{2}\varepsilon_2^{-1/5} + \varepsilon_2^{-6/5}\hat{u}_2 - \frac{1}{2}\varepsilon_2^{-6/5}v_2.$$

By applying the transformations (3.4), $\hat{X}_2 = X_2 - 2$ and (2.4), we obtain the local integrals of (P_I) of the form

$$\left\{\begin{array}{l} C_1 = Z + Y^{-1/2} + Y^{-1/2}\varphi_2(XY^{-3/2} - 2, ZY^{-2}, Y^{-5/2}), \\ C_2 = \dfrac{1}{2}Y^{1/2} - 2Y^3 + XY^{3/2} - \dfrac{1}{2}YZ + Y^3\varphi_1(\cdots) - \dfrac{1}{2}Y^{1/2}\varphi_2(\cdots), \end{array}\right. \tag{3.5}$$

where the subscripts for X_4, Y_4, Z_4 are omitted. Arguments of φ_1 and φ_2 in the second line are the same as that of the first line. Now we give a new proof of the well known theorem:

Theorem.3.2. Any solutions of (P_I) are meromorphic on $\mathbb{C}$.

A well known proof of this result is essentially based on Painlevé's argument modified by Hukuhara ([16], [7], see also [6]). Here, we will prove the theorem by applying the implicit function theorem to the above integrals.

Proof. Fix a solution $(X_4(Z_4), Y_4(Z_4))$ of (P_I) with an initial condition $(X_4(z_0), Y_4(z_0)) = (x_0, y_0)$. The existence theorem on ODE shows that the solution is holomorphic near z_0. Let $B(z_0, R)$ be the largest disk of radius R centered at z_0 such that the solution is holomorphic inside the disk. Let z_* ($\neq \infty$) be a singularity on the boundary of the disk (if $R = \infty$, there remains nothing to prove). The next lemma implies that the fixed point $(X_2, Z_2, \varepsilon_2) = (\pm 2, 0, 0)$ corresponds to the singularity z_*.

Lemma.3.3. $(X_2, Z_2, \varepsilon_2) \to (\pm 2, 0, 0)$ as $Z_4 \to z_*$ along a curve Γ inside the disk $B(z_0, R)$.

Proof. Suppose that there exists a sequence $\{z_n\}_{n=1}^{\infty}$ converging to z_* on the curve Γ such that both of $X_4(z_n)$ and $Y_4(z_n)$ are bounded as $n \to \infty$. Taking a subsequence if necessary, we can assume that (X_4, Y_4) converges to some point (x_*, y_*). Because of the existence theorem on ODE, a solution of (P_I) satisfying the initial condition (x_*, y_*, z_*) is holomorphic near this point, which contradicts with the definition of z_*. Hence, either X_4 or Y_4 diverges as $Z_4 \to z_*$.

(i) Suppose that $Y_4 \to \infty$ as $Z_4 \to z_*$. We move to the $(X_2, Z_2, \varepsilon_2)$-coordinates. Eq.(2.4) provides

$$X_2 = X_4 Y_4^{-3/2}, \quad Z_2 = Z_4 Y_4^{-2}, \quad \varepsilon_2 = Y_4^{-5/2}. \tag{3.6}$$

This immediately yields $Z_2 \to 0$, $\varepsilon_2 \to 0$ as $Z_4 \to z_*$. Let us show $X_2 \to \pm 2$. $(\mathrm{P_I})$ is a Hamiltonian system with the Hamiltonian function $H = X_4^2/2 - 2Y_4^3 - Z_4 Y_4$. Thus, the equality $H = -\int Y_4(Z_4) dZ_4$ holds along a solution. In the $(X_2, Z_2, \varepsilon_2)$-coordinates, this is written as

$$X_2(Z_2)^2 = 4 + \frac{2}{3} Z_2 - \frac{2}{3}\varepsilon_2(Z_2)^{6/5} \int_\xi^{Z_2} \varepsilon_2(z)^{-6/5} dz,$$

where $(X_2(Z_2), \varepsilon_2(Z_2))$ is a solution of the ODE solved as a function of Z_2, and ξ is a certain nonzero number determined by the initial condition. Since $Z_2 \to 0$, $\varepsilon_2 \to 0$ as $Z_4 \to z_*$, we obtain $X_2^2 \to 4$.

(ii) Suppose that $X_4 \to \infty$ as $Z_4 \to z_*$. In this case, we use the $(Y_1, Z_1, \varepsilon_1)$-coordinates given by

$$Y_1 = Y_4 X_4^{-2/3}, \quad Z_1 = Z_4 X_4^{-4/3}, \quad \varepsilon_1 = X_4^{-5/3}. \tag{3.7}$$

By the assumption, we have $Z_1 \to 0$, $\varepsilon_1 \to 0$ as $Z_4 \to z_*$. Then, we can show that $Y_1^3 \to 1/4$ as $Z_4 \to z_*$ by the same way as above. This means that $(Y_1, Z_1, \varepsilon_1)$ converges to the fixed point $((1/4)^{1/3}, 0, 0)$ of the vector field (2.6). It is easy to verify that this fixed point is the same point as $(X_2, Z_2, \varepsilon_2) = (\pm 2, 0, 0)$ if written in the $(X_2, Z_2, \varepsilon_2)$-coordinates. □

The sign of $X_2 = X_4 Y_4^{-3/2}$ depends on the choice of the branch of $Y_4^{1/2}$ and two points $(2, 0, 0)$ and $(-2, 0, 0)$ are the essentially the same. In what follows, we assume that $(X_2, Z_2, \varepsilon_2) \to (2, 0, 0)$ as $Z_4 \to z_*$. Due to the above lemma, when Z_4 is sufficiently close to z_*, the solution is included the neighborhood U, on which local holomorphic functions φ_1 and φ_2 are well defined. Then, the integrals (3.5) are available. To apply the implicit function theorem, put

$$\begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} Y^{-1/2} \\ \frac{1}{2} Y^{1/2} - 2Y^3 + XY^{3/2} - \frac{1}{2} YZ \end{pmatrix}, \tag{3.8}$$

(again, the subscripts for X_4, Y_4, Z_4 are omitted). Then, (3.5) takes the form

$$\begin{cases} C_1 = Z + w + w\varphi_2(uw^6 + \frac{1}{2} Zw^4 - \frac{1}{2} w^5, Zw^4, w^5) \\ C_2 = u + w^{-6}\varphi_1(\cdots) - \frac{1}{2} w^{-1} \varphi_2(\cdots). \end{cases}$$

Note that $C_1 = Z + w + O(w^5)$ and $C_2 = u + O(w^2)$ as $w \to 0$. Since $w \to 0$ as $Z_4 \to z_*$, the constant $C_1 = z_*$ is just the position of the singularity. If we set

$$\begin{aligned} f_1(w, u, Z) &= Z + w + w\varphi_2(\cdots) - z_* \\ f_2(w, u, Z) &= u + w^{-6}\varphi_1(\cdots) - \frac{1}{2} w^{-1} \varphi_2(\cdots) - C_2, \end{aligned}$$

then $f_i(0, C_2, z_*) = 0$. The Jacobi matrix of (f_1, f_2) with respect to (w, u) at $(w, u, Z) = (0, C_2, z_*)$ is the identity matrix. Hence, the implicit function theorem proves that there exists a local holomorphic function $g(Z)$ such that $f_1 = f_2 = 0$ is solved as $w = g(Z) \sim O(Z - z_*)$. Since $Y_4 = w^{-2}$, $Z_4 = z_*$ is a pole of second order of Y_4. This completes the proof of Thm.3.2. □

The Poincaré linearization theorem used in this subsection is stated as follows: Let $Ax + f(x)$ be a holomorphic vector field on $\mathbb{C}^n$ with a fixed point $x = 0$, where A is an $n \times n$ constant matrix and $f(x) \sim O(|x|^2)$ is a nonlinearity. Let $\lambda_1, \cdots, \lambda_n$ be eigenvalues of A. We consider the following two conditions:

(Nonresonance) There are no $j \in \{1, \cdots, n\}$ and non-negative integers $m_1, \cdots, m_n$ satisfying the resonant condition

$$m_1\lambda_1 + \cdots + m_n\lambda_n = \lambda_j, \quad (m_1 + \cdots + m_n \geq 2). \tag{3.9}$$

(Poincaré domain) The convex hull of $\{\lambda_1, \cdots, \lambda_n\}$ in $\mathbb{C}$ does not include the origin.

Suppose that A is diagonal and eigenvalues satisfy the above two conditions. Then, there exists a local analytic transformation $y = x + \varphi(x)$, $\varphi(x) \sim O(|x|^2)$ defined near the origin such that the equation $dx/dt = Ax + f(x)$ is transformed into the linear system $dy/dt = Ay$. See [3] for the proof.

3.2 Characterization of $(\mathrm{P_I})$

In order to apply the Poincaré linearization theorem to Eq.(3.1), eigenvalues of the Jacobi matrix (3.2) have to satisfy certain conditions and the other components of the matrix are not important. However, to prove the meromorphy of solutions, the $(2, 3)$-component of the Jacobi matrix also plays an important role. If the $(2, 3)$-component of the Jacobi matrix were zero, the function $f_1(w, u, Z)$ defined in the proof of Thm.3.2 becomes $f_1 = Z + w\varphi_2(\cdots) - z_*$ (i.e. the term w does not appear). As a result, the implicit function theorem is not applicable and we can not prove Thm.3.2. To see the geometric role of the $(2, 3)$-component, let us consider the dynamical system

$$\begin{cases} \dot{X}_4 = 6Y_4^2 + Z_4, \\ \dot{Y}_4 = X_4, \\ \dot{Z}_4 = \beta, \end{cases} \tag{3.10}$$

where $\beta \in \mathbb{C}$ is a constant. When $\beta \neq 0$, this is reduced to $(\mathrm{P_I})$ by a suitable scaling. This system defines a family of integral curves on $\mathbb{C}^3$. We regard $\mathbb{C}^3$ as a vector bundle; Z_4-space is a base and (X_4, Y_4)-space is a fiber. As long as $\beta \neq 0$, each integral curve is a local section of the bundle, while if $\beta = 0$, integral curves are tangent to a fiber and we can not solve the system as a function of Z_4. Now we change the coordinates by (2.4) and $\hat{X}_2 = X_2 - 2$. Then, Eq.(3.10) is brought into the system

$$
\begin{cases}
\dot{\hat{X}}_2 = 6\hat{X}_2 - Z_2 + \dfrac{-3\hat{X}_2^2 + \hat{X}_2 Z_2}{2+\hat{X}_2}, \\
\dot{Z}_2 = 4Z_2 - \beta\varepsilon_2 + \dfrac{\beta \hat{X}_2 \varepsilon_2}{2+\hat{X}_2}, \\
\dot{\varepsilon}_2 = 5\varepsilon_2.
\end{cases} \tag{3.11}
$$

Hence, integral curves give local sections if and only if the $(2,3)$-component of the Jacobi matrix of the above system is not zero.

This suggests that the $(2,3)$-component is closely related with the Painlevé property. On the (X_4, Y_4, Z_4)-coordinates of $\mathbb{C}P^3(3,2,4,5)$, give the ODE

$$
\frac{dX_4}{dZ_4} = f(X_4, Y_4, Z_4), \quad \frac{dY_4}{dZ_4} = g(X_4, Y_4, Z_4), \tag{3.12}
$$

where f and g are holomorphic in X_4, Y_4 and meromorphic in Z_4. We suppose that this equation defines a meromorphic ODE on $\mathbb{C}P^3(3,2,4,5)$. This means that the equations expressed in the other inhomogeneous coordinates are also meromorphic. We will show later that these equations are rational (recall that a meromorphic function on a projective space is rational). Thus, there are relatively prime polynomials h_1, h_2, h_3 such that the equation written in the $(X_2, Z_2, \varepsilon_2)$-coordinates is given by $dX_2/d\varepsilon_2 = h_1(X_2, Z_2, \varepsilon_2)/h_3(X_2, Z_2, \varepsilon_2)$ and $dZ_2/d\varepsilon_2 = h_2(X_2, Z_2, \varepsilon_2)/h_3(X_2, Z_2, \varepsilon_2)$. As before, we introduce a vector field

$$
\begin{cases}
\dot{X}_2 = h_1(X_2, Z_2, \varepsilon_2), \\
\dot{Z}_2 = h_2(X_2, Z_2, \varepsilon_2), \\
\dot{\varepsilon}_2 = h_3(X_2, Z_2, \varepsilon_2).
\end{cases}
$$

We call it the associated vector field with $dX_2/d\varepsilon_2 = h_1/h_3, dZ_2/d\varepsilon_2 = h_2/h_3$. The next theorem shows that $(\mathrm{P_I})$ is characterized by the (i) geometry of $\mathbb{C}P^3(3,2,4,5)$ and (ii) a local condition at a fixed point. Note that there are infinitely many equations satisfying only the condition (i) below. It is remarkable that the condition (ii) seems to be very weak, however, it completely determines an equation.

Theorem.3.4. Consider the ODE (3.12), where f and g are holomorphic in X_4, Y_4 and meromorphic in Z_4. Suppose the following two conditions:

(i) Eq.(3.12) defines a meromorphic ODE on $\mathbb{C}P^3(3,2,4,5)$.
(ii) The associated polynomial vector field in the $(X_2, Z_2, \varepsilon_2)$-coordinates has a fixed point of the form $(X_2, Z_2, \varepsilon_2) = (X_*, 0, 0)$. Eigenvalues and the $(2,3)$-component of the Jacobi matrix at this point are not zero.

Then, Eq.(3.12) is of the form

$$
\frac{dX_4}{dZ_4} = aY_4^2 + bZ_4, \quad \frac{dY_4}{dZ_4} = cX_4, \tag{3.13}
$$

where $a \neq 0, c \neq 0$ and b are constants. When $b \neq 0$, this is equivalent to $(\mathrm{P_I})$, and when $b = 0$, this is equivalent to the integrable equation $y'' = 6y^2$.

Proof. At first, we show that f and g are polynomial in X_4, Y_4 and rational in Z_4. In the $(X_2, Y_2, \varepsilon_2)$-coordinates, Eq.(3.12) is written by

$$\begin{cases} \dfrac{dX_2}{d\varepsilon_2} = \dfrac{1}{5\varepsilon_2}\left(3X_2 - 2\varepsilon_2^{1/5}\dfrac{f(X_2\varepsilon_2^{-3/5}, \varepsilon_2^{-2/5}, Z_2\varepsilon_2^{-4/5})}{g(X_2\varepsilon_2^{-3/5}, \varepsilon_2^{-2/5}, Z_2\varepsilon_2^{-4/5})}\right), \\ \dfrac{dZ_2}{d\varepsilon_2} = \dfrac{1}{5\varepsilon_2}\left(4Z_2 - 2\varepsilon_2^{2/5}\dfrac{1}{g(X_2\varepsilon_2^{-3/5}, \varepsilon_2^{-2/5}, Z_2\varepsilon_2^{-4/5})}\right). \end{cases}$$

By the condition (i), the right hand sides are meromorphic in ε_2. In particular, $f(X_2\varepsilon_2^{-3/5}, \varepsilon_2^{-2/5}, Z_2\varepsilon_2^{-4/5})$ and $g(X_2\varepsilon_2^{-3/5}, \varepsilon_2^{-2/5}, Z_2\varepsilon_2^{-4/5})$ are meromorphic as a function of $\varepsilon_2^{1/5}$. On the other hand, they are obviously meromorphic in $\varepsilon_2^{-1/5}$. Thus, they are rational in $\varepsilon_2^{1/5}$. Therefore, the right hand sides above are rational in ε_2. This implies that $\varepsilon_2^{-2/5}g$ and $\varepsilon_2^{-1/5}f$ are rational in ε_2 and we can set

$$\varepsilon_2^{-1/5}f(X_2\varepsilon_2^{-3/5}, \varepsilon_2^{-2/5}, Z_2\varepsilon_2^{-4/5}) = \frac{\sum h_i^1(X_2, Z_2)\varepsilon_2^i}{\sum h_i^2(X_2, Z_2)\varepsilon_2^i},$$
$$\varepsilon_2^{-2/5}g(X_2\varepsilon_2^{-3/5}, \varepsilon_2^{-2/5}, Z_2\varepsilon_2^{-4/5}) = \frac{\sum h_i^3(X_2, Z_2)\varepsilon_2^i}{\sum h_i^4(X_2, Z_2)\varepsilon_2^i},$$

where all $\sum$ are finite sums, and $h_i^1, h_i^2, h_i^3, h_i^4$ are meromorphic in X_2, Z_2. In the original chart, f and g are written as

$$f(X_4, Y_4, Z_4) = \frac{\sum h_i^1(X_4Y_4^{-3/2}, Z_4Y_4^{-2})Y_4^{-5i/2-1/2}}{\sum h_i^2(X_4Y_4^{-3/2}, Z_4Y_4^{-2})Y_4^{-5i/2}},$$
$$g(X_4, Y_4, Z_4) = \frac{\sum h_i^3(X_4Y_4^{-3/2}, Z_4Y_4^{-2})Y_4^{-5i/2-1}}{\sum h_i^4(X_4Y_4^{-3/2}, Z_4Y_4^{-2})Y_4^{-5i/2}}. \tag{3.14}$$

Next, we move to the $(X_3, Y_3, \varepsilon_3)$-coordinates. Eq.(3.12) is written as

$$\begin{cases} \dfrac{dX_3}{d\varepsilon_3} = \dfrac{1}{5\varepsilon_2^2}\left(3X_3\varepsilon_3 - 4\varepsilon_3^{4/5}f(X_3\varepsilon_3^{-3/5}, Y_3\varepsilon_3^{-2/5}, \varepsilon_3^{-4/5})\right), \\ \dfrac{dY_3}{d\varepsilon_3} = \dfrac{1}{5\varepsilon_3^2}\left(2Y_3\varepsilon_3 - 4\varepsilon_3^{3/5}g(X_3\varepsilon_3^{-3/5}, Y_3\varepsilon_3^{-2/5}, \varepsilon_3^{-4/5})\right). \end{cases} \tag{3.15}$$

Substituting (3.14) yields

$$\begin{cases} \dfrac{dX_3}{d\varepsilon_3} = \dfrac{1}{5\varepsilon_2^2}\left(3X_3\varepsilon_3 - 4\dfrac{\sum h_i^1(X_3Y_3^{-3/2}, Y_3^{-2})Y_3^{-5i/2-1/2}\varepsilon_3^{i+1}}{\sum h_i^2(X_3Y_3^{-3/2}, Y_3^{-2})Y_3^{-5i/2}\varepsilon_3^i}\right), \\ \dfrac{dY_3}{d\varepsilon_3} = \dfrac{1}{5\varepsilon_3^2}\left(2Y_3\varepsilon_3 - 4\dfrac{\sum h_i^3(X_3Y_3^{-3/2}, Y_3^{-2})Y_3^{-5i/2-1}\varepsilon_3^{i+1}}{\sum h_i^4(X_3Y_3^{-3/2}, Y_3^{-2})Y_3^{-5i/2}\varepsilon_3^i}\right). \end{cases} \tag{3.16}$$

The right hand sides are obviously rational in ε_3. By the same argument as before, they are also rational in Y_3. Hence, the right hand sides of Eq.(3.15) are rational in

Y_3, ε_3, and we can set

$$
\begin{aligned}
\varepsilon_3^{4/5} f(X_3\varepsilon_3^{-3/5}, Y_3\varepsilon_3^{-2/5}, \varepsilon_3^{-4/5}) &= \frac{\sum h_{ij}^5(X_3)Y_3^i\varepsilon_3^j}{\sum h_{ij}^6(X_3)Y_3^i\varepsilon_3^j}, \\
\varepsilon_3^{3/5} g(X_3\varepsilon_3^{-3/5}, Y_3\varepsilon_3^{-2/5}, \varepsilon_3^{-4/5}) &= \frac{\sum h_{ij}^7(X_3)Y_3^i\varepsilon_3^j}{\sum h_{ij}^8(X_3)Y_3^i\varepsilon_3^j},
\end{aligned}
$$

where $\sum$ are finite sums and $h_{ij}^5, h_{ij}^6, h_{ij}^7, h_{ij}^8$ are meromorphic in X_3.

Finally, we move to the $(Y_1, Z_1, \varepsilon_1)$-coordinates. Repeating the same procedure, we can verify that $\varepsilon_1^{-1/5} f(\varepsilon_1^{-3/5}, Y_1\varepsilon_1^{-2/5}, Z_1\varepsilon_1^{-4/5})$ and $\varepsilon_1^{-2/5} g(\varepsilon_1^{-3/5}, Y_1\varepsilon_1^{-2/5}, Z_1\varepsilon_1^{-4/5})$ are rational in Y_1, Z_1, ε_1. This proves that $f(X_4, Y_4, Z_4)$ and $g(X_4, Y_4, Z_4)$ are rational functions. By the assumption, they are polynomial in X_4 and Y_4.

Now we can write g and f/g as quotients of polynomials as

$$
g(X,Y,Z) = \frac{\sum a_{ijk}X^iY^jZ^k}{\sum b_k Z^k}, \quad \frac{f(X,Y,Z)}{g(X,Y,Z)} = \frac{\sum p_{ijk}X^iY^jZ^k}{\sum q_{ijk}X^iY^jZ^k}. \tag{3.17}
$$

Our purpose is to determine coefficients a_{ijk}, b_k, p_{ijk} and q_{ijk} by the conditions (i) and (ii). In the (X_2, Y_2, Z_2)-coordinates, Eq.(3.12) with (3.17) is given by

$$
\left\{
\begin{aligned}
\frac{dX_2}{d\varepsilon_2} &= \frac{1}{5\varepsilon_2}\left(3X_2 - 2\varepsilon_2^{1/5}\frac{\sum p_{ijk}X_2^iZ_2^k\varepsilon_2^{-(3i+2j+4k)/5}}{\sum q_{ijk}X_2^iZ_2^k\varepsilon_2^{-(3i+2j+4k)/5}}\right), \\
\frac{dZ_2}{d\varepsilon_2} &= \frac{1}{5\varepsilon_2}\left(4Z_2 - 2\varepsilon_2^{2/5}\frac{\sum b_kZ_2^k\varepsilon_2^{-4k/5}}{\sum a_{ijk}X_2^iZ_2^k\varepsilon_2^{-(3i+2j+4k)/5}}\right).
\end{aligned}
\right. \tag{3.18}
$$

Due to the condition (i), the right hand sides are rational in ε_2. This yields the conditions for coefficients as

$$
\left\{
\begin{array}{lll}
p_{ijk} \neq 0 & \text{only when} & 3i+2j+4k-1 = 5m+\delta, \quad (m = -1, 0, \cdots, M), \\
q_{ijk} \neq 0 & \text{only when} & 3i+2j+4k = 5m'+\delta, \quad (m' = 0, 1, \cdots, M'), \\
b_k \neq 0 & \text{only when} & 4k-2 = 5n+\delta', \quad (n = -1, 0, \cdots, N), \\
a_{ijk} \neq 0 & \text{only when} & 3i+2j+4k = 5n'+\delta', \quad (n' = 0, 1, \cdots, N'),
\end{array}
\right. \tag{3.19}
$$

where $\delta, \delta' \in \{0,1,2,3,4\}$. More precisely, the first line means that $p_{ijk} \neq 0$ only when there are integers m and δ such that (i,j,k) satisfies $3i+2j+4k-1 = 5m+\delta$, where $\delta \in \{0,1,2,3,4\}$ is independent of (i,j,k). Since Eq.(3.18) is rational, there is the largest integer m satisfying $3i+2j+4k-1 = 5m+\delta$ and $p_{ijk} \neq 0$. In (3.19), the largest integer is denoted by M. Integers M', N and N' play a similar role. Then, Eq.(3.18) is rewritten as

$$
\left\{
\begin{aligned}
\frac{dX_2}{d\varepsilon_2} &= \frac{1}{5\varepsilon_2}\left(3X_2 - 2\frac{\sum p_{ijk}X_2^iZ_2^k\varepsilon_2^{-m}}{\sum q_{ijk}X_2^iZ_2^k\varepsilon_2^{-m'}}\right), \\
\frac{dZ_2}{d\varepsilon_2} &= \frac{1}{5\varepsilon_2}\left(4Z_2 - 2\frac{\sum b_kZ_2^k\varepsilon_2^{-n}}{\sum a_{ijk}X_2^iZ_2^k\varepsilon_2^{-n'}}\right).
\end{aligned}
\right. \tag{3.20}
$$

In order to confirm the condition (ii), we shall rewrite it as a polynomial vector field.

(I). When $M > M'$ and $N > N'$, the associated polynomial vector field is of the form

$$\begin{cases} \dot{X}_2 = 3X_2(\sum a_{ijk}X_2^iZ_2^k\varepsilon_2^{N-n'})(\sum q_{ijk}X_2^iZ_2^k\varepsilon_2^{M-m'}) \\ \qquad\qquad -2(\sum p_{ijk}X_2^iZ_2^k\varepsilon_2^{M-m})(\sum a_{ijk}X_2^iZ_2^k\varepsilon_2^{N-n'}), \\ \dot{Z}_2 = 4Z_2(\sum a_{ijk}X_2^iZ_2^k\varepsilon_2^{N-n'})(\sum q_{ijk}X_2^iZ_2^k\varepsilon_2^{M-m'}) \\ \qquad\qquad -2(\sum b_kZ_2^k\varepsilon_2^{N-n})(\sum q_{ijk}X_2^iZ_2^k\varepsilon_2^{M-m'}), \\ \dot{\varepsilon}_2 = 5\varepsilon_2(\sum a_{ijk}X_2^iZ_2^k\varepsilon_2^{N-n'})(\sum q_{ijk}X_2^iZ_2^k\varepsilon_2^{M-m'}). \end{cases} \tag{3.21}$$

Because of the condition (ii), we seek a fixed point of the form $(X_*, 0, 0)$ with nonzero eigenvalues. Since $N - n' > 0$ and $M - m' > 0$, the right hand side of the equation of ε_2 is of order $O(\varepsilon_2^3)$. Hence, the Jacobi matrix at the point $(X_*, 0, 0)$ has a zero eigenvalue. In a similar manner, we can verify that two cases $M > M', N \leq N'$ and $M \leq M', N > N'$ are excluded. In these cases, the right hand side of the equation of ε_2 is of order $O(\varepsilon_2^2)$ and the Jacobi matrix has a zero eigenvalue.

(II). When $M \leq M'$ and $N \leq N'$, the associated polynomial vector field is of the form

$$\begin{cases} \dot{X}_2 = 3X_2(\sum a_{ijk}X_2^iZ_2^k\varepsilon_2^{N'-n'})(\sum q_{ijk}X_2^iZ_2^k\varepsilon_2^{M'-m'}) \\ \qquad\qquad -2(\sum p_{ijk}X_2^iZ_2^k\varepsilon_2^{M'-m})(\sum a_{ijk}X_2^iZ_2^k\varepsilon_2^{N'-n'}), \\ \dot{Z}_2 = 4Z_2(\sum a_{ijk}X_2^iZ_2^k\varepsilon_2^{N'-n'})(\sum q_{ijk}X_2^iZ_2^k\varepsilon_2^{M'-m'}) \\ \qquad\qquad -2(\sum b_kZ_2^k\varepsilon_2^{N'-n})(\sum q_{ijk}X_2^iZ_2^k\varepsilon_2^{M'-m'}), \\ \dot{\varepsilon}_2 = 5\varepsilon_2(\sum a_{ijk}X_2^iZ_2^k\varepsilon_2^{N'-n'})(\sum q_{ijk}X_2^iZ_2^k\varepsilon_2^{M'-m'}). \end{cases} \tag{3.22}$$

Suppose that this has a fixed point of the form $(X_*, 0, 0)$. The $(2, 3)$-component of the Jacobi matrix at this point is given by

$$-2 \cdot \frac{\partial}{\partial \varepsilon_2}\Big|_{(X_*,0,0)} (\sum b_kZ_2^k\varepsilon_2^{N'-n})(\sum q_{ijk}X_2^iZ_2^k\varepsilon_2^{M'-m'}).$$

We require that this quantity is not zero.

(II-a). Suppose that the polynomial $\sum b_kZ_2^k\varepsilon_2^{N'-n}$ includes a constant term. This means that $b_k \neq 0$ when $k = 0$ and $n = N'$. Substituting it to the third condition of (3.19) provides $-2 = 5N' + \delta'$. This proves that $N' = -1$ and $\delta' = 3$. Then, the fourth condition of (3.19) yields $3i + 2j + 4k = -2$. Since there are no nonnegative integers i, j, k satisfying this relation, $a_{ijk} = 0$ for any i, j, k. In this case, we obtain $\dot{\varepsilon}_2 = 0$ and the Jacobi matrix has a zero eigenvalue.

(II-b). When $\sum b_kZ_2^k\varepsilon_2^{N'-n}$ does not have a constant term, it has to include a monomial ε_2. Otherwise, the $(2, 3)$-component of the Jacobi matrix becomes zero. This means that $b_k \neq 0$ when $k = 0$ and $n = N' - 1$. The third condition of (3.19) provides $-2 = 5(N'-1)+\delta'$, which proves $N'=0$ and $\delta'=3$. Since $N \le N'$, we have $N=-1$ or $N=0$. In this case, (3.19) becomes

$$\begin{cases} b_k \neq 0 \quad \text{only when} \quad 4k-2=-2 \quad \text{or} \quad 3, \\ a_{ijk} \neq 0 \quad \text{only when} \quad 3i+2j+4k=3. \end{cases} \tag{3.23}$$

Therefore, nonzero numbers among these coefficients are only b_0 and a_{100}. This proves that $g(X,Y,Z)$ is given by $a_{100}X/b_0$. In what follows, we put $a_{100}/b_0 = c$.

Since $g = cX$ and f is polynomial in X, Y, f/g can be written as

$$\frac{f(X,Y,Z)}{g(X,Y,Z)} = \frac{\sum p_{ijk}X^iY^jZ^k}{\sum q_{10k}XZ^k}.$$

In this case, the equation of ε_2 in (3.22) is given by

$$\dot{\varepsilon}_2 = 5a_{100}X_2^2\varepsilon_2 \cdot \left(\sum q_{10k}Z_2^k\varepsilon_2^{M'-m'}\right). \tag{3.24}$$

The polynomial $\sum q_{10k}Z_2^k\varepsilon_2^{M'-m'}$ has to include a constant term so that the Jacobi matrix at $(X_*,0,0)$ does not have a zero eigenvalue. This means that $q_{10k}\neq 0$ when $k=0$ and $m'=M'$. Thus (3.19) provides $3 = 5M'+\delta$. This shows $M'=0$ and $\delta=3$. Then, (3.19) becomes

$$\begin{cases} p_{ijk} \neq 0 \quad \text{only when} \quad 3i+2j+4k-1=3, \\ q_{10k} \neq 0 \quad \text{only when} \quad 3+4k=3. \end{cases} \tag{3.25}$$

Therefore, nonzero numbers among these coefficients are only p_{020}, p_{001} and q_{100}. Putting $a := cp_{020}/q_{100}$ and $b := cp_{001}/q_{100}$, we obtain the ODE (3.13) as a necessary condition for (i) and (ii). It is straightforward to confirm that (3.13) actually satisfies the conditions (i) and (ii) when $a \neq 0, c \neq 0$. This completes the proof. □

4 The space of initial conditions

4.1 The space of initial conditions for $(\mathrm{P_I})$

In this section, we construct the space of initial conditions for $(\mathrm{P_I})$ by the weighted blow-up of $\mathbb{C}P^3(3,2,4,5)$. Let us recall the definition of a space of initial conditions of a differential equation. Let $\mathcal{P} = (E,\pi,B)$ be a fiber bundle and $\mathcal{F}$ a foliation on $\mathcal{P}$ defined by a given differential equation. Roughly speaking, the base space B is a space for an independent variable, and a fiber is a space for dependent variables. If they satisfy the following conditions, a fiber of $\mathcal{P}$ is called a space of initial conditions.

(a) Each leaf of $\mathcal{F}$ transversely intersects with fibers.
(b) Each path γ on B is lifted to a leaf $\tilde{\gamma}_p$ which passes through a given point $p \in \pi^{-1}(\gamma(0))$.
(c) $\pi|_{\tilde{\gamma}_p} : \tilde{\gamma}_p \to B$ is surjective and $\tilde{\gamma}_p$ is a covering space of B.

For ($\mathrm{P_I}$), $B = \mathbb{C}$ is a Z_4-space and the fiber (the space of initial conditions) is a certain rational surface. It was first constructed by Okamoto [15] by a blow-up of a Hirzebruch surface eight times and by removing a certain divisor called vertical leaves. Different approaches are proposed by Duistermaat and Joshi [5] and Iwasaki and Okada [8]. They also performed a blow-up many times.

Here, we will obtain the space of initial conditions by a weighted blow-up only one time. We will recover Painlevé's coordinates in a purely geometric manner. We find a symplectic structure of the space of initial conditions and show that Painlevé's coordinates is a Darboux coordinates of the symplectic structure.

Recall that ($\mathrm{P_I}$) written in the $(X_2, Z_2, \varepsilon_2)$-coordinates has two fixed points $(\pm 2, 0, 0)$. Putting $\hat{X}_2 = X_2 \pm 2$, we obtain

$$\begin{cases} \dot{\hat{X}}_2 = 6\hat{X}_2 \pm Z_2 + \dfrac{\pm 3\hat{X}_2^2 + \hat{X}_2 Z_2}{2 \mp \hat{X}_2}, \\ \dot{Z}_2 = 4Z_2 \pm \varepsilon_2 + \dfrac{\hat{X}_2 \varepsilon_2}{2 \mp \hat{X}_2}, \\ \dot{\varepsilon}_2 = 5\varepsilon_2. \end{cases} \tag{4.1}$$

(In Sec.3, we used only $\hat{X}_2 = X_2 - 2$). The origin is a fixed point of the vector field and it is a singularity of the foliation defined by integral curves. We apply a blow-up to this point. At first, we change the coordinates by the linear transformation

$$\begin{cases} \hat{X}_2 = u \mp \frac{1}{2}v - \frac{1}{2}w, \\ Z_2 = v, \\ \varepsilon_2 = w. \end{cases}$$

Then, we obtain

$$\begin{cases} \dot{u} = 6u + f_1(u, v, w), \\ \dot{v} = 4v \pm w + f_2(u, v, w), \\ \dot{w} = 5w, \end{cases} \tag{4.2}$$

where f_1 and f_2 denote nonlinear terms. Note that the linear part is *not* diagonalized; since we know that the $(2,3)$-component is important (Thm.3.5), we remove only the $(1,2)$-component of the linear part of (4.1). Now we introduce the weighted blow-up with weights $6, 4, 5$, which are taken from eigenvalues of the Jacobi matrix, defined by the following transformations

$$\begin{cases} u &= u_1^6 &= v_2^6 u_2 &= w_3^6 u_3, \\ v &= u_1^4 v_1 &= v_2^4 &= w_3^4 v_3, \\ w &= u_1^5 w_1 &= v_2^5 w_2 &= w_3^5. \end{cases} \tag{4.3}$$

The exceptional divisor $\{u_1 = 0\} \cup \{v_2 = 0\} \cup \{w_3 = 0\}$ is a 2-dim weighted projective space $\mathbb{C}P^2(6, 4, 5)$ and the blow-up of (u, v, w)-space is a (singular) line bundle over $\mathbb{C}P^2(6,4,5)$. We mainly use the (u_3,v_3,w_3)-coordinates. In the (u_3,v_3,w_3)-coordinates, Eq.(4.2) is written as

$$\left\{\begin{aligned} \frac{du}{dv} &= \frac{1}{8}\left(v^2w \pm 3vw^2 + 2w^3 \mp 8uvw^3 - 10uw^4 + 12u^2w^5\right), \\ \frac{dw}{dv} &= \frac{1}{4}\left(\pm 4 \pm vw^4 + w^5 - 2uw^6\right), \end{aligned}\right. \tag{4.4}$$

where the subscripts for u_3, v_3, w_3 are omitted for simplicity. The relation between the original chart (X_4,Y_4,Z_4) and (u_3,v_3,w_3) is

$$\left\{\begin{array}{l} X_4 = u_3w_3^3 \mp 2w_3^{-3} \mp \frac{1}{2}v_3w_3 - \frac{1}{2}w_3^2, \\ Y_4 = w_3^{-2}, \\ Z_4 = v_3, \end{array}\right. \tag{4.5}$$

or

$$\left\{\begin{array}{l} u_3 = X_4Y_4^{3/2} \pm 2Y_4^3 \pm \frac{1}{2}Z_4Y_4 + \frac{1}{2}Y_4^{1/2}, \\ w_3 = Y_4^{-1/2}, \\ v_3 = Z_4. \end{array}\right. \tag{4.6}$$

It is remarkable that the independent variable Z_4 is not changed despite the fact that Z_4 is changed in each step of transformations. Now we have recovered Painlevé's coordinates (4.5) which was introduced in his paper to prove the Painlevé property of ($\mathrm{P_I}$). He found this transformation by observing a Laurent series of a solution, see also Gromak, Laine and Shimomura [6]. At a first glance, $\mathbb{C}^2_{(X_4,Y_4)} \cup \mathbb{C}^2_{(u_3,w_3)}$ does not define a manifold glued by (4.5) because (4.5) is not one-to-one but one-to-two. Nevertheless, we can show that (4.5) defines a certain algebraic surface. Recall that the (X_2,Z_2,ε_2)-space should be divided by the $\mathbb{Z}_2$ action $(X_2,Z_2,\varepsilon_2) \mapsto (-X_2,Z_2,-\varepsilon_2)$ due to the orbifold structure of $\mathbb{C}P^3(3,2,4,5)$. This action induces a certain $\mathbb{Z}_2$ action on the (u_3,v_3,w_3)-space. If we divide the (u_3,v_3,w_3)-space by the $\mathbb{Z}_2$ action, we can prove that (4.5) becomes a one-to-one mapping. Then, $\mathbb{C}^2_{(X_4,Y_4)}$ and $\mathbb{C}^2_{(u_3,w_3)}/\mathbb{Z}_2$ are glued by (4.5) to define an algebraic surface, which gives the space of initial conditions for ($\mathrm{P_I}$). The space $\mathbb{C}^2_{(u_3,w_3)}/\mathbb{Z}_2$ is realized as a nonsingular algebraic surface defined by

$$M(Z_4) : V^2 = UW^4 + 2Z_4W^3 + 4W \tag{4.7}$$

with the parameter Z_4 (the proof is given below). By using (U,V,W), the relation (4.5),(4.6) is rewritten as

$$\left\{\begin{array}{l} X_4 = VW^{-2} - \frac{1}{2}W, \\ Y_4 = W^{-1}. \end{array}\right. , \quad \left\{\begin{array}{l} V = X_4Y_4^{-2} + \frac{1}{2}Y_4^{-3}, \\ W = Y_4^{-1}. \end{array}\right. \tag{4.8}$$

Hence, $\mathbb{C}^2_{(X_4,Y_4)}$ and $M(Z_4)$ glued by this relation defines a nonsingular algebraic surface denoted by $E(Z_4)$. The surface $M(Z_4)$ admits a holomorphic symplectic form

$$-\frac{1}{W^4}dV \wedge dW = \frac{1}{4UW^3 + 6Z_4W^2 + 4}dV \wedge dU. \tag{4.9}$$

We can verify that

$$dX_4 \wedge dY_4 = -\frac{1}{W^4} dV \wedge dW. \tag{4.10}$$

Thus, $E(Z_4)$ also has a holomorphic symplectic form. ($\mathrm{P_I}$) written by (V, W) is

$$\left\{ \begin{array}{l} \dfrac{dV}{dZ_4} = 6 + Z_4 W^2 + \dfrac{1}{4W}(W^3 + 4V)(W^3 - 2V) = W^4 \dfrac{\partial H}{\partial W} \\ \dfrac{dW}{dZ_4} = \dfrac{1}{2}W^3 - V = -W^4 \dfrac{\partial H}{\partial V}, \end{array} \right. \tag{4.11}$$

where H is given by

$$\begin{aligned} H &= \frac{1}{2}X_4^2 - 2Y_4^3 - Z_4 Y_4 \\ &= \frac{V^2}{2W^4} - \frac{V}{2W} + \frac{1}{8}W^2 - \frac{2}{W^3} - \frac{Z_4}{W}. \end{aligned} \tag{4.12}$$

Hence, Eq.(4.11) is a Hamiltonian system with respect to the symplectic form $-W^{-4} dV \wedge dW$ as well as the original ($\mathrm{P_I}$) written by (X_4, Y_4). Let z_* be a pole of a solution of ($\mathrm{P_I}$). As $Z_4 \to z_*$, $X_4, Y_4 \to \infty$ and $V, W \to 0$. By using (4.7), it is easy to verify that Eq.(4.11) is holomorphic even at $W = 0$. This proves that $E(Z_4) = \mathbb{C}^2_{(X_4, Y_4)} \cup M(Z_4)$ is the desired space of initial conditions. Note that the system (4.4) is already a Hamiltonian system with the Hamiltonian

$$H = \frac{1}{8}\left(\pm 8u \pm 2uvw^4 + 2uw^5 - 2u^2 w^6 - \frac{1}{2}v^2 w^2 \mp vw^3 - \frac{1}{2}w^4 \right). \tag{4.13}$$

The transformation (4.5) yields

$$dX_4 \wedge dY_4 = -2du_3 \wedge dw_3. \tag{4.14}$$

This means that (u_3, v_3)-coordinates is a Darboux coordinates for the form $-W^{-4}dV \wedge dW$ (if we remove the factor -2 by a suitable scaling). These results are summarized as follows.

Theorem.4.1.
(i) The space $\mathbb{C}^2_{(u_3, w_3)}$ divided by the $\mathbb{Z}_2$ action induced from the orbifold structure of $\mathbb{C}P^3(3, 2, 4, 5)$ gives the algebraic surface $M(Z_4)$. The space of initial conditions $E(Z_4)$ for ($\mathrm{P_I}$) is given by $\mathbb{C}^2_{(X_4, X_4)} \cup M(Z_4)$ glued by (4.8).
(ii) $M(Z_4)$ and $E(Z_4)$ have holomorphic symplectic forms and ($\mathrm{P_I}$) is a Hamiltonian system with respect to the form. Painlevé's coordinates defined by (4.5) is the Darboux coordinates of the symplectic form on $M(Z_4)$.
(iii) Consider an ODE (3.12) defined on the (X_4, Y_4, Z_4)-coordinates, where f and g are polynomials in X_4 and Y_4. If it is also expressed as a polynomial ODE in the Painlevé's coordinates, then (3.12) is ($\mathrm{P_I}$).

Recently, a similar result is obtained by Iwasaki and Okada [8] by a different approach. A symplectic atlas for the Painlevé equation is found by Takano et al. [20, 13, 14] for the second Painlevé to sixth Painlevé equations, while left open for ($\mathrm{P_I}$).

Proof. Due to the orbifold structure of $\mathbb{C}P^3(3,2,4,5)$, the $(X_2, Z_2, \varepsilon_2)$-space should be divided by the $\mathbb{Z}_2$ action $(X_2, Z_2, \varepsilon_2) \mapsto (-X_2, Z_2, -\varepsilon_2)$. It is straightforward to show that in the (u_3, v_3, w_3)-coordinates, this action is written by

$$\begin{pmatrix} u_3 \\ v_3 \\ w_3 \end{pmatrix} \mapsto \begin{pmatrix} -u_3 + v_3 w_3^{-2} + 4w_3^{-6} \\ v_3 \\ -w_3 \end{pmatrix}. \tag{4.15}$$

Since v_3 is fixed, we consider $\mathbb{C}^2_{(u_3,w_3)}/\mathbb{Z}_2$. The invariants of this action are given by

$$\begin{cases} U = u_3(u_3 w_3^6 - v_3 w_3^4 - 4) + \dfrac{1}{4} w_3^2 v_3^2, \\ V = w_3^7 (u_3 - \dfrac{1}{2} v_3 w_3^{-2} - 2w_3^{-6}), \\ W = w_3^2. \end{cases} \tag{4.16}$$

They satisfy the equation (4.7), which proves $\mathbb{C}^2_{(u_3,w_3)}/\mathbb{Z}_2 = M(Z_4)$. The rest of (i) and (ii) have already been shown. Let us prove (iii). In what follows, we omit the subscripts for u_3, v_3, w_3. By (4.5) with the upper sign, Eq.(3.12) is written in Painlevé's coordinates as

$$\begin{aligned} \frac{d}{dZ_4}\begin{pmatrix} u \\ w \end{pmatrix} &= \begin{pmatrix} w^{-3}\cdot f + \dfrac{3}{2}uw^2 \cdot g - \dfrac{1}{4} Z_4 \cdot g + 3w^{-4}\cdot g - \dfrac{1}{2} w \cdot g + \dfrac{1}{2} w^{-2} \\ -\dfrac{1}{2} w^3 \cdot g \end{pmatrix} \\ &=: \; T(f,g) + \begin{pmatrix} \frac{1}{2} w^{-2} \\ 0 \end{pmatrix} =: \hat{T}(f,g). \end{aligned} \tag{4.17}$$

Our purpose is to show that if $\hat{T}(f,g)$ is a polynomial, then Eq.(3.12) is ($\mathrm{P_I}$). Since we know that $\hat{T}(f,g)$ is a polynomial for ($\mathrm{P_I}$), it is sufficient to show the uniqueness. We define operators T and $\hat{T}$ as above. The operator T is a linear mapping from the space of polynomials $\mathbb{C}[X_4, Y_4] \times \mathbb{C}[X_4, Y_4]$ into the space of Laurent polynomials $\mathbb{C}[u, w, w^{-1}] \times \mathbb{C}[u, w, w^{-1}]$ for each Z_4. Let

$$\Pi : \mathbb{C}[u, w, w^{-1}] \times \mathbb{C}[u, w, w^{-1}] \to \mathbb{C}[u, w^{-1}] \times \mathbb{C}[u, w^{-1}] \tag{4.18}$$

be the natural projection to the principle part. If there are two pairs of polynomials (f_1, g_1) and (f_2, g_2) such that $\hat{T}(f_i, g_i)$, $(i = 1, 2)$ are polynomials, then

$$\hat{T}(f_1, g_1) - \hat{T}(f_2, g_2) = T(f_1 - f_2, g_1 - g_2)$$

is also a polynomial and $\Pi \circ T(f_1 - f_2, g_1 - g_2) = 0$. Hence, it is sufficient to prove $\mathrm{Ker}\, \Pi \circ T = \{0\}$. For this purpose, we show that images of monomials of the form $(X^mY^n, 0), (0, X^mY^n)$, $(m, n = 0, 1, \cdots)$ are linearly independent. They are calculated as

$$\begin{aligned} T(X^mY^n, 0) &= \begin{pmatrix} w^{-2n-3}(uw^3 - 2w^{-3} - \frac{1}{2}Z_4 w - \frac{1}{2}w^2)^m \\ 0 \end{pmatrix}, \\ T(0, X^mY^n) &= \begin{pmatrix} * \\ -\frac{1}{2}w^{-2n+3}(uw^3 - 2w^{-3} - \frac{1}{2}Z_4 w - \frac{1}{2}w^2)^m \end{pmatrix}. \end{aligned}$$

It is easy to verify that the principle parts of them are linearly independent. □

Remark.4.2. We have constructed the space of initial conditions by the weighted blow-up at the fixed point $(X_2, Z_2, \varepsilon_2) = (2, 0, 0)$. We can also construct it by using the fixed point $(Y_1, Z_1, \varepsilon_1) = ((1/4)^{1/3}, 0, 0)$, which represents the same point as $(X_2, Z_2, \varepsilon_2) = (2, 0, 0)$. By the same procedure as before (an affine transformation and the weighted blow-up in the $(Y_1, Z_1, \varepsilon_1)$-coordinates), we obtain the one-to-three transformation

$$\begin{cases} X_4 = w_3^{-3}, \\ Y_4 = u_3 w_3^4 - \dfrac{2^{-1/3}}{3} v_3 w_3^2 + 2^{-2/3} w_3^{-2}, \\ Z_4 = v_3. \end{cases} \tag{4.19}$$

Due to the orbifold structure, the $(Y_1, Z_1, \varepsilon_1)$-space should be divided by the $\mathbb{Z}_3$ action given in Sec.2 (see also (5.3)). This induces the $\mathbb{Z}_3$ action in the (u_3, v_3, w_3)-space and we can show that $\mathbb{C}^2_{(X_4, Y_4)}$ and $\mathbb{C}^2_{(u_3, w_3)}/\mathbb{Z}_3$ glued by (4.19) gives the same algebraic surface $E(Z_4)$ as before.

4.2 The space of initial conditions in the Boutroux coordinates

We have proved that (X_4, Y_4)-space and (u_3, w_3)-space are glued to give the space of initial conditions. In this subsection, it is shown that (X_3, Y_3)-space and (u_2, v_2)-space give the space of initial conditions for $(\mathrm{P_I})$ written in the Boutroux coordinates. Recall that on the $(X_3, Y_3, \varepsilon_3)$-coordinates, $(\mathrm{P_I})$ is given as

$$\frac{dX_3}{d\varepsilon_3} = \frac{1}{5\varepsilon_3^2}\left(-24Y_3^2 - 4 + 3X_3\varepsilon_3\right), \quad \frac{dY_3}{d\varepsilon_3} = \frac{1}{5\varepsilon_3^2}\left(-4X_3 + 2Y_3\varepsilon_3\right). \tag{4.20}$$

It has an irregular singular point $\varepsilon_3 = 0$. Since $Z_4 = \varepsilon_3^{-4/5}$, this equation also has the Painlevé property with a possible branch point $\varepsilon_3 = 0$. The coordinates $(X_3, Y_3, \varepsilon_3)$ is equivalent to the Boutroux coordinates introduced by Boutroux [1] to investigate solutions of $(\mathrm{P_I})$ near $Z_4 = \infty$. Geometrically, his coordinates is nothing but the inhomogeneous coordinates of $\mathbb{C}P^3(3, 2, 4, 5)$.

Recall that the (u_2, v_2, w_2)-coordinates is defined by (4.3). In this coordinates,

$(\mathrm{P_I})$ is written as

$$\left\{\begin{aligned} \frac{du_2}{dw_2} &= \frac{1}{5w_2^2}\left(-\frac{1}{2}v_2 - 6u_2^2v_2^5 + 6u_2w_2 \mp \frac{3}{2}v_2^2w_2 + 5u_2v_2^4w_2 \pm 4u_2v_2^3 - v_2^3w_2^2\right), \\ \frac{dv_2}{dw_2} &= \frac{1}{5w_2^2}\left(\mp 4 \mp v_2^4 + 2u_2v_2^6 - v_2w_2 - v_2^5w_2\right). \end{aligned}\right. \tag{4.21}$$

Again, we obtained a polynomial ODE with an irregular singular point $w_2 = 0$. The relation between two charts are

$$\left\{\begin{array}{l} X_3 = v_2^{-3}(v_2^6u_2 \mp \frac{1}{2}v_2^4 - \frac{1}{2}v_2^5w_2 \mp 2), \\ Y_3 = v_2^{-2}, \\ \varepsilon_3 = w_2, \end{array}\right. \tag{4.22}$$

or

$$\left\{\begin{array}{l} u_2 = X_3Y_3^{3/2} + \frac{1}{2}Y_3^{1/2}\varepsilon_3 \pm \frac{1}{2}Y_3 \pm 2Y_3^3, \\ v_2 = Y_3^{-1/2}, \\ w_2 = \varepsilon_3. \end{array}\right. \tag{4.23}$$

We should divide the (u_2, v_2, w_2)-space by the $\mathbb{Z}_2$ action as before. The $\mathbb{Z}_2$ action induced from the orbifold structure is given by

$$\begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix} \mapsto \begin{pmatrix} -u_2 + v_2^{-2} + 4v_2^{-6} \\ -v_2 \\ w_2 \end{pmatrix}. \tag{4.24}$$

We define invariants of this action by

$$\left\{\begin{array}{l} U = u_2(u_2v_2^6 - v_2^4 - 4) + \frac{1}{4}v_2^2 \\ V = v_2^7(u_2 - \frac{1}{2}v_2^{-2} - 2v_2^{-6}) \\ W = v_2^2. \end{array}\right. \tag{4.25}$$

This defines a nonsingular algebraic surface $M = \mathbb{C}^2_{(u_2,v_2)}/\mathbb{Z}_2$

$$M : V^2 = UW^4 + 2W^3 + 4W. \tag{4.26}$$

Note that it is independent of a parameter. The relation (4.22) and Eq.(4.21) are rewritten as

$$\left\{\begin{array}{l} X_3 = W^{-2}V - \frac{1}{2}W\varepsilon_3 \\ Y_3 = W^{-1}, \end{array}\right. \qquad \left\{\begin{array}{l} V = X_3Y_3^{-2} + \frac{1}{2}Y_3^{-3}\varepsilon_3 \\ W = Y_3^{-1}, \end{array}\right. \tag{4.27}$$

and

$$\left\{\begin{aligned} \frac{dV}{d\varepsilon_3} &= \frac{1}{5\varepsilon_3^2}\left(8\frac{V^2}{W} - \varepsilon_3V - 2\varepsilon_3VW^2 - 4W^2 - \varepsilon_2^2W^5 - 24\right), \\ \frac{dW}{d\varepsilon_3} &= \frac{1}{5\varepsilon_3^2}\left(-2W\varepsilon_3 + 4V - 2W^3\varepsilon_3\right), \end{aligned}\right. \tag{4.28}$$

respectively. Hence, $\mathbb{C}^2_{(X_3,Y_3)} \cup M$ gives the space of initial conditions for the Boutroux coordinates. Note that Eqs.(4.19) and (4.28) are not Hamiltonian systems.

5 Cellular decomposition

In this section, the weighted blow-up of $\mathbb{C}P^3(3,2,4,5)$ with weights $6,4,5$ defined in Sec.4 is denoted by $\mathbb{C}P^3(3,2,4,5;6,4,5)$. We calculate the cellular decomposition of it. It will be shown that $\mathbb{C}P^3(3,2,4,5;6,4,5)$ is decomposed into the fiber space $\mathcal{P} = (E, \pi, B)$ for $(\mathrm{P_I})$, an elliptic fibration over the moduli space of complex tori defined by the Weierstrass equation and a projective line. We also shows that the extended Dynkin diagram of type $\tilde{E}_8$ is hidden in the space $\mathbb{C}P^3(3,2,4,5;6,4,5)$.

5.1 The elliptic fibration

Let us calculate a cellular decomposition of $\mathbb{C}P^3(3,2,4,5;6,4,5)$. $\mathbb{C}P^3(3,2,4,5)$ is decomposed as (1.3). Furthermore, $\mathbb{C}P^2(3,2,4)$ is decomposed as $\mathbb{C}P^2(3,2,4) = \mathbb{C}^2/\mathbb{Z}_4 \cup \mathbb{C}P^1(3,2)$. Since $\mathbb{C}P^1(3,2)$ is isomorphic to the Riemann sphere, we obtain

$$\mathbb{C}P^3(3,2,4,5) \;=\; \mathbb{C}^3/\mathbb{Z}_5 \,\cup\, \mathbb{C}^2/\mathbb{Z}_4 \,\cup\, \mathbb{C} \,\cup\, \{p\},$$

where $\{p\}$ denotes the point $(X_2, Z_2, \varepsilon_2) = (2,0,0)$, at which the weighted blow-up was performed. In local coordinates, $\mathbb{C}^3/\mathbb{Z}_5$ is the (X_4, Y_4, Z_4)-space divided by the $\mathbb{Z}_5$ action, and $\mathbb{C}^2/\mathbb{Z}_4$ is $\{(X_3, Y_3, \varepsilon_3) \,|\, \varepsilon_3 = 0\}$ divided by the $\mathbb{Z}_4$ action. On $\{(X_3, Y_3, \varepsilon_3)\}$, the vector field (2.8) is defined. When $\varepsilon_3 = 0$, it is reduced to the Hamiltonian system

$$\left\{ \begin{array}{l} \dot{X}_3 = -24Y_3^2 - 4, \\ \dot{Y}_3 = -4X_2, \end{array} \right. \tag{5.1}$$

with the Hamiltonian function $H = 2X^2 - 8Y^3 - 4Y$. This equation is actually invariant under the $\mathbb{Z}_4$ action given by $(X_3, Y_3, t) \mapsto (iX_3, -Y_3, it)$.

Next, due to the definition of the weighted blow-up, we have

$$\mathbb{C}P^3(3,2,4,5;6,4,5) = \mathbb{C}^3/\mathbb{Z}_5 \,\cup \mathbb{C}^2/\mathbb{Z}_4 \,\cup\, \mathbb{C} \,\cup\, \mathbb{C}P^2(6,4,5).$$

Since $\mathbb{C}P^2(6,4,5) = \mathbb{C}^2/\mathbb{Z}_5 \cup \mathbb{C}P^1(6,4)$, we obtain

$$\begin{array}{rccccc} = & \mathbb{C}^3/\mathbb{Z}_5 & \cup & \mathbb{C}^2/\mathbb{Z}_4 & \cup & \mathbb{C} \\ \cup & \mathbb{C}^2/\mathbb{Z}_5 & \cup & \mathbb{C}P^1(6,4)\backslash\{q\} & \cup & \{q\}, \end{array} \tag{5.2}$$

where $\{q\}$ denotes the point $(u_1, v_1, w_1) = (0,0,0)$. In local coordinates, $\mathbb{C}^2/\mathbb{Z}_5$ is given as $\{(u_3, v_3, w_3) \,|\, w_3 = 0\}$ divided by $\mathbb{Z}_5$. This implies that the first column $\mathbb{C}^3/\mathbb{Z}_5 \cup \mathbb{C}^2/\mathbb{Z}_5$ is just the fiber space $\mathcal{P} = (E, \pi, B)$ for $(\mathrm{P_I})$ divided by the $\mathbb{Z}_5$ action, the fiber space over $\mathbb{C}$ whose fiber is the space of initial conditions. The last column $\mathbb{C} \cup \{q\}$ is the Riemann sphere.

Let us investigate the second column $\mathbb{C}^2/\mathbb{Z}_4 \cup \mathbb{C}P^1(6,4)\backslash\{q\}$. On the space $\mathbb{C}^2/\mathbb{Z}_4$, the equation (5.1) divided by the $\mathbb{Z}_4$ action is defined, and $\mathbb{C}P^1(6,4)$ is attached at "infinity". Note that each integral curve $X^2 = 4Y^3 + 2Y - g_3$ of (5.1) defines an elliptic curve, where g_3 is an integral constant; compare with the Weierstrass normal form $X^2 = 4Y^3 - g_2Y - g_3$.

Recall that the Weierstrass normal form defines a complex torus when $g_2^3 - 27g_3^2 \neq 0$. Two complex tori defined by (g_2, g_3) and (g_2', g_3') are isomorphic to one another if there is $\lambda \neq 0$ such that $(g_2, g_3) = (\lambda^4 g_2', \lambda^6 g_3')$. Hence, $\mathbb{C}P^1(6,4)\backslash\{\text{one point}\}$ is a moduli space of complex tori.

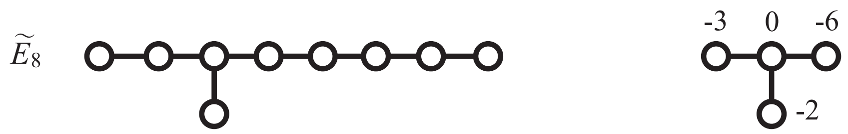
According to Eq.(5.1), the (X_3, Y_3)-space is foliated by a family of elliptic curves (including two singular curves $g_3 = \pm(8/27)^{1/2}$) defined by the Weierstrass normal form $X^2 = 4Y^3 + 2Y - g_3$. By the $\mathbb{Z}_4$ action $(X_3, Y_3) \mapsto (iX_3, -Y_3)$, the normal form is mapped to $X^2 = 4Y^3 + 2Y + g_3$. This means that by the $\mathbb{Z}_4$ action induced from the orbifold structure, two elliptic curves having parameters $(-2, g_3)$ and $(-2, -g_3)$ are identified. However, the equality $(-2, g_3) = (-2\lambda^4, g_3'\lambda^6)$ holds for some λ if and only if $g_3' = -g_3$. This proves that two elliptic curves identified by the $\mathbb{Z}_4$ action are isomorphic with one another, and $\mathbb{C}^2/\mathbb{Z}_4$ is foliated by isomorphism classes of elliptic curves (including a singular one, while the case $g_2 = 0$ is excluded). The set $\mathbb{C}P^1(6,4)\backslash\{q\}$ is expressed as $\{(u_2, v_2, w_2)\,|\,v_2 = w_2 = 0\}$ divided by the $\mathbb{Z}_2$ action $u_2 \mapsto -u_2$. We can show that each isomorphism class of an elliptic curve $X^2 = 4Y^3 + 2Y \mp g_3$ intersects with the moduli space $\mathbb{C}P^1(6,4)\backslash\{q\}$ at the point $(u_2, v_2, w_2) = (g_3/4, 0, 0) \sim (-g_3/4, 0, 0)$. This proves that the second column of (5.2) gives an elliptic fibration whose base space is the moduli space $\mathbb{C}P^1(6,4)\backslash\{q\}$ and fibers are isomorphism classes of elliptic curves including the singular curve, but excluding the curve of $g_2 = 0$.

Theorem.5.1. The space $\mathbb{C}P^3(3,2,4,5;6,4,5)$ is decomposed into the disjoint union of the fiber space for $(\mathrm{P_I})$ divided by $\mathbb{Z}_5$, an elliptic fibration obtained from the Weierstrass normal form as above, and $\mathbb{C}P^1$.

5.2 The extended Dynkin diagram $\tilde{E}_8$

It is known that an extended Dynkin diagram is associated with each Painlevé equation (Sakai [18]). For example, the diagram of type $\tilde{E}_8$ is associated with $(\mathrm{P_I})$. Okamoto obtained $\tilde{E}_8$ as follows: In order to construct the space of initial conditions of $(\mathrm{P_I})$, he performed a blow-up eight times to a Hirzebruch surface. After that, vertical leaves, which are the pole divisor of the symplectic form, are removed. The configuration of irreducible components of the vertical leaves is described by the Dynkin diagram of type $\tilde{E}_8$, see Fig.1(a). Our purpose is to find the diagram $\tilde{E}_8$ hidden in $\mathbb{C}P^3(3,2,4,5;6,4,5)$.

Recall that $\mathbb{C}P^3(3,2,4,5;6,4,5)$ is covered by seven local coordinates; the inhomogeneous coordinates (2.4) of $\mathbb{C}P^3(3,2,4,5)$ and (u_i, v_i, w_i) defined by (4.3). These local coordinates should be divided by the suitable actions due to the orbifold structure. For example, the actions on $(Y_1, Z_1, \varepsilon_1)$, $(X_2, Z_2, \varepsilon_2)$ and (u_1, v_1, w_1) are given



(a) (b)

Fig. 1: (a) The usual extended Dynkin diagram of type $\tilde{E}_8$. Each vertex denotes $\mathbb{C}P^1$, and two $\mathbb{C}P^1$ connected by an edge intersect with each other. All self-intersection numbers are -2. (b) The diagram obtained from $\mathbb{C}P^3(3,2,4,5;6,4,5)$. Each number denotes the self-intersection number.

by

$$(Y_1, Z_1, \varepsilon_1) \mapsto (\omega Y_1, \omega^2 Z_1, \omega\varepsilon_1), \quad \omega = e^{2\pi i/3}, \tag{5.3}$$
$$(X_2, Z_2, \varepsilon_2) \mapsto (-X_2, Z_2, -\varepsilon_2), \tag{5.4}$$
$$(u_1, v_1, w_1) \mapsto (\zeta u_1, \zeta^2 v_1, \zeta w_1), \quad \zeta = e^{2\pi i/6}. \tag{5.5}$$

Each fiber (the space of initial conditions) for ($\mathrm{P_I}$) is not invariant under the $\mathbb{Z}_5$ action (1.2) except for the fiber on $Z_4 = 0$. Hence, we consider the closure of the fiber on $Z_4 = 0$ in $\mathbb{C}P^3(3,2,4,5;6,4,5)$. The closure is a 2-dim orbifold expressed as

$$N := \{(X_4, Y_4, 0)\} \cup \{(Y_1, 0, \varepsilon_1)\} \cup \{(X_2, 0, \varepsilon_2)\} \cup \{(u_3, 0, w_3)\} \cup \{(u_1, 0, w_1)\}. \tag{5.6}$$

N is a compactification of the space of initial conditions $E(0) = \{(X_4, Y_4, 0)\} \cup \{(u_3, 0, w_3)\}$ obtained by attaching a 1-dim space

$$D := \{(Y_1, 0, 0)\} \cup \{(X_2, 0, 0) \,|\, X_2 \neq \pm 2\} \cup \{(u_1, 0, 0)\}, \quad X_2 = Y_1^{-3/2} = u_1^6 \mp 2. \tag{5.7}$$

N has three orbifold singularities on D given by

$$(Y_1, \varepsilon_1) = (0,0), \quad (X_2, \varepsilon_2) = (0,0), \quad (u_1, w_1) = (0,0). \tag{5.8}$$

Let us calculate the minimal resolution of these singularities. For example, the singularity $(X_2, \varepsilon_2) = (0,0)$ is defined by the $\mathbb{Z}_2$ action $(X_2, \varepsilon_2) \mapsto (-X_2, -\varepsilon_2)$; i.e. this is a A_1 singularity, and it is resolved by the standard one time blow-up. The self-intersection number of the exceptional divisor is -2. Similarly, singularities $(Y_1, \varepsilon_1) = (0,0)$ and $(u_1, w_1) = (0,0)$ are resolved by one time blow-up, whose self-intersection numbers of the exceptional divisors are -3 and -6, respectively. From the minimal resolution of singularities of N, we remove the space of initial conditions $E(0)$. Then, we obtain the union of four projective lines, whose configuration is described as Fig.1(b), see also Fig.2.

Although the diagram Fig.1(b) is different from $\tilde{E}_8$, it is remarkable that the self-intersection numbers $-2, -3, -6$ are the same as the lengths of arms from the center of $\tilde{E}_8$. The same facts also hold for the other Painlevé equations, which will be reported in a forthcoming paper.

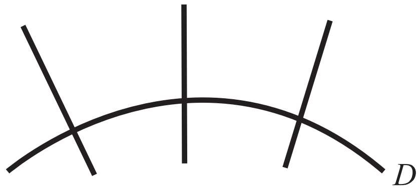


Fig. 2: The minimal resolution of singularities of the closure of $E(0)$.

6 Irregular singular point

In his paper [1], Boutroux introduced the following coordinates

$$X_4 = xz^{3/5}, \quad Y_4 = yz^{2/5}, \quad Z_4 = z^{4/5}. \tag{6.1}$$

With this coordinates, $(\mathrm{P_I})$ is rewritten as

$$\frac{dx}{dz} = \frac{1}{5}\left(24y^2 + 4 - \frac{3x}{z}\right), \quad \frac{dy}{dz} = \frac{1}{5}\left(4x + \frac{2y}{z}\right). \tag{6.2}$$

Since this system is reduced to the solvable Hamiltonian system as $z \to \infty$, it is suitable for the study of the irregular singular point $Z_4 = \infty$, see [5, 9, 10].

Although this transformation can be derived by observing the magnitude of solutions [10], it has a purely geometric meaning: putting $z = \varepsilon_3^{-1}$, we recover the inhomogeneous coordinates $(X_3, Y_3, \varepsilon_3)$ of $\mathbb{C}P^3(3, 2, 4, 5)$. It is convenient to use ε_3 instead of z when we apply the dynamical systems theory. The vector field (2.8) has a fixed point $(X_3, Y_3, \varepsilon_3) = (0, (-6)^{-1/2}, 0)$. The Jacobi matrix at this point has exactly one zero eigenvalue. Thus, the vector field has a one dimensional center manifold at this point. With the aid of the center manifold theory [19, 3], we can prove the following well known result (in the statement, the subscripts for X_4, Y_4, Z_4 are omitted).

Theorem.6.1. There exist five solutions $Y_n(Z)$ of $(\mathrm{P_I})$ such that

(i) they are analytic on each sector

$$-\frac{\pi}{5} - \frac{4\pi}{5}n < \arg(Z) < \frac{7\pi}{5} - \frac{4\pi}{5}n, \quad n = 0, \pm 1, \pm 2, \cdots \tag{6.3}$$

when $|Z| >> 1$. In particular, they do not have poles in the respective sectors when $|Z| >> 1$.

(ii) they are exponentially close to each other on the common region of sectors.

Now these solutions $Y_n(Z)$ are called Boutroux's tritronquée solutions. For the classical proof, see Joshi and Kitaev [9]. For the proof using the center manifold theory, see Chiba [2], which provides the coordinates-free definition of the tritronquée solution: Boutroux's tritronquée solution is the center manifold of the fixed point of the vector field (2.8). If we employ the Riemann-Hilbert approach, more refined results such as the connection formulae can be obtained, see Kapaev et al. [12, 11].

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