LARGE DEVIATION PROBABILITIES FOR MAXIMUM LIKELIHOOD ESTIMATOR AND BAYES ESTIMATOR OF A PARAMETER FOR FRACTIONAL ORNSTEIN-UHLENBECK TYPE PROCESS

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M. N. Mishra* and **B. L. S. Prakasa Rao†**

Abstract

We investigate the probabilities of large deviations of the maximum likelihood estimator and Bayes estimator of the drift parameter for a fractional Ornstein-Uhlenbeck type process.

Key Words and Phrases: Bayes estimator, Fractional Brownian motion, Fractional Ornstein-Uhlenbeck type process Large deviation, Maximum likelihood estimator.

1. Introduction

Statistical inference for diffusion type processes satisfying stochastic differential equations driven by Wiener processes has been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao (1999). There has been a recent interest to study similar problems for stochastic processes driven by a fractional Brownian motion. Le Breton (1998) studied parameter estimation and filtering in a simple linear model driven by a fractional Brownian motion. In a recent paper, Kleptsyna and Le Breton (2002) studied parameter estimation problems for fractional Ornstein-Uhlenbeck type process. This is a fractional analogue of the Ornstein-Uhlenbeck process, that is, a continuous time first order autoregressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a fractional Brownian motion (fBm) $W^H = \{W_t^H, t \geq 0\}$ with Hurst parameter $H \in [1/2, 1)$. Such a process is the unique Gaussian process satisfying the linear integral equation

$$X_t = \theta \int_0^t X_s ds + \sigma W_t^H, \ t \geq 0. \tag{1.1}$$

They investigate the problem of estimation of the parameters θ and σ^2 based on the observation $\{X_s, 0 \leq s \leq T\}$ and prove that the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$.

We now study large deviation probabilities for the maximum likelihood and the Bayes estimators for the drift parameter involved in a fractional Ornstein-Uhlenbeck type process.

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2. Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions. The natural filtration of a stochastic process is understood as the P-completion of the filtration generated by this process.

Let $W^H = \{W_t^H, t \geq 0\}$ be a normalized fractional Brownian motion with Hurst parameter $H \in (0,1)$, that is, a Gaussian process with continuous sample paths such that $W_0^H = 0$, $E(W_t^H) = 0$ and

$$E(W_s^H W_t^H) = \frac{1}{2}[s^{2H} + t^{2H} - |s-t|^{2H}], t \geq 0, \ s \geq 0. \tag{2.1}$$

Let us consider a stochastic process $Y = \{Y_t, t \geq 0\}$ defined by the stochastic integral equation

$$Y_t = \int_0^t C(s)\, ds + \int_0^t B(s)\, dW_s^H, \quad t \geq 0, \tag{2.2}$$

where $C = \{C(t), \ t \geq 0\}$ is an $(\mathcal{F}_t)$-adapted process and $B(t)$ is a non-vanishing non-random function. For convenience we write the above integral equation in the form of a stochastic differential equation

$$dY_t = C(t)\, dt + B(t)\, dW_t^H, \ t \geq 0, Y(0) = 0, \tag{2.3}$$

driven by the fractional Brownian motion W^H. The integral

$$\int_0^t B(s)\, dW_s^H \tag{2.4}$$

is not a stochastic integral in the Ito sense but one can define the integral of a deterministic function with respect to the fBm in a natural sense (cf. Norros et al. (1999)). Even though the process Y is not a semimartingale, one can associate a semimartingale $Z = \{Z_t, t \geq 0\}$ which is called a *fundamental semimartingale* such that the natural filtration $(\mathcal{Z}_t)$ of the process Z coincides with the natural filtration $(\mathcal{Y}_t)$ of the process Y (Kleptsyna et al. (2000)).

Define, for $0 < s < t$,

$$k_H = 2H\Gamma\,(\frac{3}{2} - H)\Gamma(H + \frac{1}{2}), \tag{2.5}$$

$$k_H(t,s) = k_H^{-1} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}, \tag{2.6}$$

$$\lambda_H = \frac{2H\,\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}, \tag{2.7}$$

$$w_t^H = \lambda_H^{-1} t^{2-2H}, \tag{2.8}$$

and

$$M_t^H = \int_0^t k_H(t,s)\, dW_s^H, \quad t \geq 0. \tag{2.9}$$

The process M^H is a Gaussian martingale, called the *fundamental martingale* (cf. Norros et al. (1999)) and its quadratic variation $\langle M_t^H \rangle = w_t^H$. Further more the natural filtration of the martingale M^H coincides with the natural filtration of the fBm W^H. In fact the stochastic integral

$$\int_0^t B(s)\,dW_s^H \tag{2.10}$$

can be represented in terms of the stochastic integral with respect to the martingale M^H. For a measurable function f on $[0,T]$, let

$$K_H^f(t,s) = -2H\frac{d}{ds}\int_s^t f(r)r^{H-\frac{1}{2}}(r-s)^{H-\frac{1}{2}}\,dr, \quad 0 \le s \le t \tag{2.11}$$

when the derivative exists in the sense of absolute continuity with respect to the Lebesgue measure (see Samko et al. (1993) for sufficient conditions). The following result is due to Kleptsyna et al. (2000).

Theorem 2.1. *Let M^H be the fundamental martingale associated with the fBm W^H defined by (2.9). Then*

$$\int_0^t f(s)dW_s^H = \int_0^t K_H^f(t,s)\,dM_s^H,\ t\in[0,T] \tag{2.12}$$

a.s $[P]$ whenever both sides are well defined.

Suppose the sample paths of the process $\{\frac{C(t)}{B(t)}, t \ge 0\}$ are smooth enough (see Samko et al. (1993)) so that

$$Q_H(t) = \frac{d}{dw_t^H}\int_0^t k_H(t,s)\frac{C(s)}{B(s)}\,ds,\ t\in[0,T] \tag{2.13}$$

is well-defined where w^H and k_H are as defined in (2.8) and (2.6) respectively and the derivative is understood in the sense of absolute continuity. The following theorem due to Kleptsyna et al. (2000) associates a *fundamental semimartingale* Z associated with the process Y such that the natural filtration $(\mathcal{Z}_t)$ coincides with the natural filtration $(\mathcal{Y}_t)$ of Y.

Theorem 2.2. *Suppose the sample paths of the process Q_H defined by (2.13) belong P-a.s to $L^2([0,T],dw^H)$ where w^H is as defined by (2.8). Let the process $Z = (Z_t, t\in[0,T])$ be defined by*

$$Z_t = \int_0^t k_H(t,s)B^{-1}(s)\,dY_s, \tag{2.14}$$

where the function $k_H(t,s)$ is as defined in (2.6). Then the following results hold:

(i) *The process Z is an $(\mathcal{F}_t)$-semimartingale with the decomposition*

$$Z_t = \int_0^t Q_H(s)\,dw_s^H + M_t^H, \tag{2.15}$$

where M^H is the fundamental martingale defined by (2.9),

(ii) *the process* Y *admits the representation*

$$Y_t = \int_0^t K_H^B(t,s)\, dZ_s, \tag{2.16}$$

where the function K_H^B *is as defined in (2.11), and*

(iii) *the natural filtration of* $(\mathcal{Z}_t)$ *and* $(\mathcal{Y}_t)$ *coincide.*

Kleptsyna et al. (2000) derived the following Girsanov type formula as a consequence of the Theorem 2.2.

Theorem 2.3. *Suppose the assumptions of Theorem 2.2 hold. Define*

$$\Lambda_H(T) = \exp\left\{ -\int_0^T Q_H(t)\, dM_t^H - \frac{1}{2}\int_0^t Q_H^2(t)\, dw_t^H \right\}. \tag{2.17}$$

Suppose that $E(\Lambda_H(T)) = 1$. *Then the measure* $P^* = \Lambda_H(T)P$ *is a probability measure and the probability measure of the process* Y *under* P^* *is the same as that of the process* V *defined by*

$$V_t = \int_0^t B(s)\, dW_s^H,\ 0 \le t \le T. \tag{2.18}$$

3. Maximum likelihood estimation and Bayes estimation

Let us consider the stochastic differential equation

$$dX(t) = \theta X(t)dt + dW_t^H, t \ge 0, X(0) = 0, \tag{3.1}$$

where $\theta \in \Theta \subset R, W = \{W_t^H,\ t \ge 0\}$ is a fractional Brownian motion with Hurst parameter H. In other words $X = \{X_t, t \ge 0\}$ is a stochastic process satisfying the stochastic integral equation

$$X(t) = \int_0^t \theta X(s)ds + W_t^H, \quad t \ge 0. \tag{3.2}$$

Let

$$Q_H(t) = \frac{d}{dw_t^H}\int_0^t k_H(t,s)X(s)ds, \quad t \ge 0 \tag{3.3}$$

is well-defined where w_t^H and $k_H(t,s)$ are as defined in (2.8) and (2.6) respectively. Suppose the sample paths of the process $\{Q_H(t), 0 \le t \le T\}$ belong almost surely to $L^2([0,T], dw_t^H)$. Define

$$Z_t = \int_0^t k_H(t,s)dX_s,\ t \ge 0. \tag{3.4}$$

Then the process $Z = \{Z_t,\ t \ge 0\}$ is an $(\mathcal{F}_t)$-semimartingale with the decomposition

$$Z_t = \int_0^t Q_H(s)\, dw_s^H + M_t^H, \tag{3.5}$$

where M^H is the fundamental martingale defined by (2.9) and the process X admits the representation

$$X_t = \int_0^t K_H(t,s)\,dZ_s, \tag{3.6}$$

where the function K_H is as defined by (2.11) with $f \equiv 1$. Let P_θ^T be the measure induced by the process $\{X_t, 0 \le t \le T\}$ when θ is the true parameter. Following Theorem 2.3, we get that the Radon-Nikodym derivative of P_θ^T with respect to P_0^T is given by

$$\frac{dP_\theta^T}{dP_0^T} = \exp\left[\int_0^T Q_H(s)\ dZ_s - \frac{1}{2}\int_0^T Q_H^2(s)dw_s^H\right]. \tag{3.7}$$

Maximum likelihood estimation

The problem of maximum likelihood estimation of the parameter θ is discussed in Kleptsyna and Le Breton (2002) for fractional Ornstein-Uhlenbeck type process and by Prakasa Rao (2003, 2005) for more general processes. Let $L_T(\theta)$ denote the Radon-Nikodym derivative $\frac{dP_\theta^T}{dP_0^T}$. The maximum likelihood estimator (MLE) is defined by the relation

$$L_T(\hat{\theta}_T) = \sup_{\theta\in\Theta} L_T(\theta). \tag{3.8}$$

We assume that there exists a measurable maximum likelihood estimator. Sufficient conditions can be given for the existence of such an estimator (cf. Lemma 3.1.2, Prakasa Rao (1987)). Then

$$\log L_T(\theta) = \int_0^T \theta Q_H(t)dZ_t - \frac{1}{2}\int_0^T \theta^2 Q_H^2(t)dw_t^H \tag{3.9}$$

and the likelihood equation is given by

$$\int_0^T Q_H(t)dZ_t - \int_0^T \theta Q_H^2(t)dw_t^H = 0. \tag{3.10}$$

Hence the MLE $\hat{\theta}_T$ of θ is given by

$$\hat{\theta}_T = \frac{\int_0^T Q_H(t)dZ_t}{\int_0^T Q_H^2(t)\,dw_t^H}. \tag{3.11}$$

Let θ_0 be the true parameter. Using the fact that

$$dZ_t = \theta_0 Q_H(t)\,dw_t^H + dM_t^H, \tag{3.12}$$

it can be shown that

$$\frac{dP_\theta^T}{dP_{\theta_0}^T} = \exp\left[(\theta-\theta_0)\int_0^T Q_H(t)dM_t^H - \frac{1}{2}(\theta-\theta_0)^2\int_0^T Q_H^2(t)dw_t^H\right]. \tag{3.13}$$

Following this representation of the Radon-Nikodym derivative, we obtain that

$$\hat{\theta}_T - \theta_0 = \frac{\int_0^T Q_H(t)dM_t^H}{\int_0^T Q_H^2(t)\,dw_t^H}. \tag{3.14}$$

Note that the quadratic variation $\langle Z \rangle$ of the process Z is the same as the quadratic variation $\langle M^H \rangle$ of the martingale M^H which in turn is equal to w^H. This follows from the relations (2.15) and (2.9). Hence we obtain that

$$[w_T^H]^{-1} \lim_n \Sigma[Z_{t_{i+1}^{(n)}} - Z_{t_i^{(n)}}]^2 = 1 \text{ a.s } [P_{\theta_0}],$$

where $(t_i^{(n)})$ is a partition of the interval $[0,T]$ such that $\sup |t_{i+1}^{(n)} - t_i^{(n)}|$ tends to zero as $n \to \infty$.

Bayes estimation

Suppose that the parameter space Θ is open and Λ is a prior probability measure on the parameter space $\Theta \subset R$. Further suppose that the probability measure Λ has a density $\lambda(.)$ with respect to the Lebesgue measure and the density function is continuous and positive in an open neighborhood of θ_0, the true parameter.

The posterior density of θ given the observation $X^T \equiv \{X_s, 0 \le s \le T\}$ is given by

$$p(\theta|X^T) = \frac{\frac{dP_\theta^T}{dP_{\theta_0}^T}\lambda(\theta)}{\int_\Theta \frac{dP_\theta^T}{dP_{\theta_0}^T}\lambda(\theta)\, d\theta}. \tag{3.15}$$

We define the Bayes estimate (BE) $\tilde{\theta}_T$ of the parameter θ based on the path X^T and the prior density $\lambda(\theta)$, to be the minimizer of the function

$$B_T(\phi) = \int_\Theta L(\theta, \phi)\, p(\theta|X^T)\, d\theta,\ \phi \in \Theta$$

where $L(\theta, \phi)$ is a given loss function defined on $\Theta \times \Theta$. In particular, for the quadratic loss function $|\theta - \phi|^2$, the Bayes estimator is the posterior mean given by

$$\tilde{\theta}_T = \frac{\int_\Theta u p^T(u|X^T) du}{\int_\Theta p^T(v|X^T) dv}.$$

Suppose the loss function $L(\theta, \phi) : \Theta \times \Theta \subset R$ satisfies the following conditions:

D(i) $L(\theta, \phi) = L(|\theta - \phi|)$;

D(ii) $L(\theta)$ is non-negative and continuous on R;

D(iii) $L(.)$ is symmetric;

D(iv) the sets $\{\theta : L(\theta) < c\}$ are convex sets and are bounded for all $c > 0$; and

D(v) there exists numbers $\gamma > 0$, $H_0 \ge 0$ such that for $H \ge H_0$,

$$\sup \{L(\theta) : |\theta| \le H^\gamma\} \le \inf \{L(\theta) : |\theta| \ge H\}.$$

Obviously, the loss function of the form $L(\theta, \phi) = |\theta - \phi|^2$ satisfies the conditions D(i) - D(v).

4. Probabilities of large deviations

Let

$$\psi_T^H(\theta; a) = E_\theta[\exp(-a \int_0^T Q_H^2(t) dw^H(t))]$$

for $a > 0$. Kleptsyna and Le Breton (2002) proved that

$$\psi_T^H(\theta; a) = (\frac{4(sin\pi H)\sqrt{\theta^2 + 2a}e^{-\theta T}}{\pi T D_T^H(\theta; \sqrt{\theta^2 + 2a})})^{1/2},$$

where

$$\begin{aligned} D_T^H(\theta; \beta) &= [\beta\, cosh\frac{\beta}{2}T - \theta\, sinh\frac{\beta}{2}T]^2 I_{-H}(\frac{\beta}{2}T) I_{H-1}(\frac{\beta}{2}T) \\ &\quad -[\beta\, sinh\frac{\beta}{2}T - \theta\, cosh\frac{\beta}{2}T]^2 I_{1-H}(\frac{\beta}{2}T) I_H(\frac{\beta}{2}T) \end{aligned}$$

where I_ν is the Bessel function of the first kind and order ν (cf. Watson (1995)). It was also proved in Kleptsyna and Le Breton (2002) that

$$\lim_{T\to\infty} \psi_T^H(\theta; a) = 0,$$

and hence

$$\lim_{T\to\infty} \int_0^T Q_H^2(t) dw^H(t) = +\infty \text{ a.s. } P_\theta.$$

In view of these observations, we make the following assumption.

Assumption (A) Fix $\theta \in \Theta$. Suppose that there exists a function α_T tending to zero as $T \to \infty$ such that,

$$\alpha_T \int_0^T Q_H^2(t) dw^H(t) \to c > 0 \text{ a.s}[P_\theta] \text{ as } T \to \infty,$$

and there exists a neighborhood N_θ of θ such that

$$\sup_{\phi\in N_\theta} E_\phi[\alpha_T \int_0^T Q_H^2(t) dw^H(t)] = O(1)$$

as $T \to \infty$. We now prove the following theorems giving the large deviation probabilities for the MLE and BE discussed in Section 3.

Theorem 4.1. *Under the conditions stated above, there exists positive constants C_1 and C_2, depending on θ and T, such that for every $\gamma > 0$,*

$$P_\theta^T\{|\alpha_T^{-1/2}(\hat{\theta}_T - \theta)| > \gamma\} \le C_1 e^{-C_2\gamma^2}$$

where $\hat{\theta}_T$ is the MLE of the parameter θ.

Theorem 4.2. *Under the conditions stated above, there exists positive constants C_3 and C_4, depending on θ and T, such that for every $\gamma > 0$,*

$$P_\theta^T\{|\alpha_T^{-1/2}(\tilde{\theta_T} - \theta)| > \gamma\} \le C_3 e^{-C_4\gamma^2},$$

where $\tilde{\theta}_T$ is the BE of the parameter θ with respect to the prior $\lambda(.)$ and the loss function $L(.,.)$ satisfying the conditions D(i)-D(v).

Let E_θ^T denote the expectation with respect to the probability measure P_θ^T. Fix $\theta \in \Theta$. For proofs of theorems stated above, we need the following lemmas. Define

$$Z_T(u) = \frac{dP^T_{\theta+u\alpha_T{}^{1/2}}}{dP^T_\theta}.$$

Lemma 4.3. *Under the conditions stated above, there exists positive constants c_1 and d_1 such that*

$$E_\theta^T[Z_T^{\frac{1}{2}}(u)] \leq d_1 e^{-c_1 u^2}$$

for $-\infty < u < \infty$.

Lemma 4.4. *Under the conditions stated above, there exists a positive constant c_2 such that*

$$E_\theta^T \left\{ Z_T^{\frac{1}{2}}(u_1) - Z_T^{\frac{1}{2}}(u_2) \right\}^2 \leq c_2(u_1 - u_2)^2$$

for $-\infty < u_1, u_2 < \infty$.

Lemma 4.5. *Let $\xi(x)$ be a real valued random function defined on a closed subset F of the Euclidean space R^k. Assume that random process $\xi(x)$ is measurable and separable. Assume that the following conditions are fulfilled: there exists numbers $m \geq r > k$ and a positive continuous function on $G(x) : R^k \to R$ bounded on the compact sets such that for all $x, h \in F, x+h \in F$,*

$$E|\xi(x)|^m \leq G(x), \;\; E|\xi(x+h) - \xi(x)|^m \leq G(x)\|h\|^r.$$

Then, with probability one, the realizations of $\xi(t)$ are continuous functions on F. Moreover let

$$\omega(\delta, \xi, L) = \sup |\xi(x) - \xi(y)|,$$

where the upper bound is taken over $x, y \in F$ with $\|x-y\| \leq h, \;\; \|x\| \leq L, \;\; \|y\| \leq L$; then

$$E(\omega(h, \xi, L)) \leq B_0 \left(\sup_{\|x\| \leq L} G(x) \right)^{\frac{1}{m}} L^{k/m} h^{\frac{r-k}{m}} \log(h^{-1}),$$

where the constant B_0 depends on $m, r,$ and k.

We will use this lemma with $\xi(u) = Z_T^{1/2}(u), m = 2, r = 2, k = 1, G(x) = e^{-cx^2}$ and $L = H + r + 1$. For the proof of this lemma, see Ibragimov and Khasminskii (1981) (Correction, cf. Kallianpur and Selukar (1993)).

Proof of Lemma 4.3. We know that

$$\begin{aligned}
E_\theta^T(Z_T^{1/2}(u)) &= E_\theta^T(\frac{dP^T_{\theta+u\sqrt{\alpha_T}}}{dP_\theta^T})^{1/2} \\
&= E_\theta^T[\exp\{\frac{u\sqrt{\alpha_T}}{2}\int_0^T Q_H(t)dM^H(t) - \frac{1}{4}u^2\alpha_T\int_0^T Q_H^2(t)dw^H(t)\}] \\
&= E_\theta^T[\exp\{\frac{u\sqrt{\alpha_T}}{2}\int_0^T Q_H(t)dM^H(t) - \frac{u^2\alpha_T}{6}\int_0^T Q_H^2(t)dw^H(t)\} \\
&\quad \times\exp\{-\frac{u^2\alpha_T}{12}\int_0^T Q_H^2(t)dw^H(t)\}] \\
&\leq \Bigg[E_\theta^T\Bigg\{\exp\left(\frac{1}{2}u\sqrt{\alpha_T}\int_0^T Q_H(t)dM^H(t)\right. \\
&\quad \left.-(1/6)u^2\alpha_T\int_0^T Q_H^2(t)dw^H(t)\right)\Bigg\}^{4/3}\Bigg]^{3/4} \\
&\quad \times\left[E_\theta^T\left\{\exp\left(-(1/12)u^2\alpha_T\int_0^T Q_H^2(t)dw^H(t)\right)\right\}^4\right]^{1/4} \\
&\quad \text{(by Holder's inequality)} \\
&= \Bigg\{E_\theta^T\exp\left(\frac{2}{3}u\sqrt{\alpha_T}\int_0^T Q_H(t)dM^H(t)\right. \\
&\quad \left.-(2/9)u^2\alpha_T\int_0^T Q_H^2(t)dw^H(t)\right)\Bigg\}^{3/4} \\
&\quad \times\{E_\theta^T\exp(-(1/3)u^2\alpha_T Q_H^2(t)dt)\}^{1/4} \\
&\leq \left[E_\theta^T\left\{\exp\left(-\frac{1}{3}u^2\alpha_T\int_0^T Q_H^2(t)dw^H(t)\right)\right\}\right]^{1/4}
\end{aligned}$$

(since the first term is less than or equal to one (cf. Gikhman and Skorokhod (1972))). The last term is bounded by $e^{-c_1u^2}$ for some positive constant c_1 depending on θ and T by assumption (A) which completes the proof of Lemma 4.3.

We now prove Lemma 4.4.

Proof of Lemma 4.4. Note that

$$\begin{aligned}
&E_\theta^T\left\{Z_T^{\frac{1}{2}}(u_1) - Z_T^{\frac{1}{2}}(u_2)\right\}^2 \\
&= E_\theta^T\{Z_T(u_1)+Z_T(u_2)\} - 2E_\theta^T\left\{Z_T^{\frac{1}{2}}(u_1)Z_T^{\frac{1}{2}}(u_2)\right\} \\
&= 2\left[1 - E_\theta^T\left\{Z_T^{\frac{1}{2}}(u_1)Z_T^{\frac{1}{2}}(u_2)\right\}\right] \\
&\quad (\text{since } E_\theta^T Z_T(u) = E_\theta^T[\exp\{u\sqrt{\alpha_T}\int_0^T Q_H(t)dM^H(t) \\
&\quad -\frac{1}{2}u^2\alpha_T\int_0^T Q_H^2(t)dw^H(t)\}] = 1.)
\end{aligned}$$

Denote

$$\begin{aligned} V_T &= (\frac{dP^T_{\theta_2}}{dP^T_{\theta_1}})^{1/2} \quad \text{where } \theta_1 = \theta + u_1\sqrt{\alpha_T} \text{ and } \theta_2 = \theta + u_2\sqrt{\alpha_T} \\ &= \exp\{\frac{1}{2}(u_2 - u_1)\sqrt{\alpha_T}\int_0^T Q_H(t)dM^H(t) \\ &\quad -\frac{1}{4}(u_2-u_1)^2\alpha_T\int_0^T Q_H^2(t)dw^H(t)\}. \end{aligned}$$

Now

$$\begin{aligned} &E_\theta^T\{Z_T^{\frac{1}{2}}(u_1)Z_T^{\frac{1}{2}}(u_2)\} \\ &= E_\theta^T\{(\frac{dP^T_{\theta+u_1\sqrt{\alpha_T}}}{dP^T_\theta})^{1/2}(\frac{dP^T_{\theta+u_2\sqrt{\alpha_T}}}{dP^T_\theta})^{1/2}\} \\ &= \int(\frac{dP^T_{\theta_1}}{dP^T_\theta})^{1/2}(\frac{dP^T_{\theta_2}}{dP^T_\theta})^{1/2}dP^T_\theta \\ &= \int(\frac{dP^T_{\theta_2}}{dP^T_{\theta_1}})^{1/2}dP^T_{\theta_1} = E^T_{\theta_1}(V_T) \\ &= E^T_{\theta_1}[\exp\{\frac{1}{2}(u_2-u_1)\sqrt{\alpha_T}\int_0^T Q_H(t)dM^H(t) \\ &\quad -\frac{1}{4}(u_2-u_1)^2\alpha_T\int_0^T Q_H^2(t)dw^H(t)\}]. \end{aligned}$$

Thus

$$\begin{aligned} &2\{1 - E_\theta^T(Z_T^{\frac{1}{2}}(u_1)Z_T^{\frac{1}{2}}(u_2))\} \\ &= 2[1 - E^T_{\theta_1}(\exp\{\frac{1}{2}(u_2-u_1)\sqrt{\alpha_T}\int_0^T Q_H(t)dM^H(t) \\ &\quad -\frac{1}{4}(u_2-u_1)^2\alpha_T\int_0^T Q_H^2(t)dw^H(t)\})] \\ &\leq 2[1 - \exp E^T_{\theta_1}\{\frac{1}{2}(u_2-u_1)\sqrt{\alpha_T}\int_0^T Q_H(t)dM^H(t) \\ &\quad -\frac{1}{4}(u_2-u_1)^2\alpha_T\int_0^T Q_H^2(t)dw^H(t)\}] \text{ (by Jensen's inequality)} \\ &= 2[1 - \exp\{-\frac{(u_2-u_1)^2}{4}\alpha_T E^T_{\theta_1}\int_0^T Q_H^2(t)dw^H(t)\}] \\ &\leq 2[\frac{(u_2-u_1)^2}{4}\alpha_T E^T_{\theta_1}\int_0^T Q_H^2(t)dw^H(t)] \text{ (since } 1 - e^{-x} \leq x) \\ &= c_2(u_2-u_1)^2 \end{aligned}$$

for some positive constant c_2 depending on θ and T.

Proof of Theorem 4.1. Denote $U = \{u : \theta + u \in \Theta\}$. Let Γ_r be the interval $L + r \leq$

$|u| \leq l + r + 1$. We use the following inequality to prove the theorem:

$$P_\theta^T\{\sup_{\Gamma_r} Z_T(u) \geq 1\} \leq c_3(1+L+r)^{\frac{1}{2}} e^{-\frac{1}{4}(L+r)^2} \tag{4.1}$$

for some positive constant c_3. Observe that

$$\begin{aligned}
P_\theta^T\{|\alpha_T^{-1/2}(\hat{\theta}_T - \theta)| > L\} &\leq P_\theta^T\{\sup_{|u|>l, u\in U} Z_T(u) \geq Z_T(0)\} \\
&\leq \sum_{r=0}^{\infty} P_\theta^T\{\sup_{\Gamma_r} Z_T(u) \geq 1\} \\
&\leq c_4 \sum_{r=0}^{\infty} e^{-c_5(L+r)^2} \\
&\leq c_6 e^{-c_7 L^2}.
\end{aligned}$$

This proves the Theorem 4.1. We now prove the inequality (4.1). We divide the interval Γ_r into N sub-intervals $\{\Gamma_r^{(j)}, 1 \leq j \leq N\}$ each with length at most h. The number of such sub-intervals $N \leq [\frac{1}{h}] + 1$. Choose $u_j \in \Gamma_r^{(j)}$, $1 \leq j \leq N$. Then

$$\begin{aligned}
P_\theta^T\{\sup_{u\in\Gamma_r} Z_T(u) \geq 1\} &\leq \sum_{j=1}^{N} P_\theta^T\{Z_T(u_j) \geq \frac{1}{2}\} \\
&\quad + P_\theta^T\{\sup_{|u-v|\leq h, |u|,|v|\leq L+r+1} |Z_T^{\frac{1}{2}}(u) - Z_T^{\frac{1}{2}}(v)| \geq \frac{1}{2}\}.
\end{aligned} \tag{4.2}$$

From the Chebyshev's inequality and in view of Lemma 4.3 , it follows that

$$P_\theta^T\{Z_T^{\frac{1}{2}}(u_j) \geq \frac{1}{2}\} \leq c_8 e^{-(L+r)^2}, 1 \leq j \leq N$$

for some positive constant c_8. Applying Lemma 4.5 with $\xi(u) = Z_T^{1/2}(u)$, and using Lemma 4.4, we obtain that

$$E_\theta^T[\sup_{\substack{|u-v|\leq h \\ |u|,|v|\leq(L+r+1)}} |Z_T^{1/2}(u) - Z_T^{1/2}(v)|] \leq c_9(L+r+1)^{\frac{1}{2}} h^{1/2} \log(h^{-1})$$

for some positive constant c_9. Hence

$$P_\theta^T\{\sup_{u\in\Gamma_r} Z_T(u) \geq 1\} \leq c_{10}\{\frac{1}{h} e^{-(L+r)^2} + (L+r+1)^{\frac{1}{2}} h^{1/2} \log(h^{-1})\}$$

for some positive constant c_{10} depending on θ and T by using (4.2). Choosing $h = e^{-(L+r)^2/2}$, we prove the inequality in Theorem 4.1.

Proof of Theorem 4.2. Observe that the conditions (1) and (2) in Theorem 5.2 of Ibragimov and Khasminskii (1981) are satisfied by Lemmas 4.3 and 4.4. In view of the conditions on the loss function mentioned in Section 3, we can prove Theorem 4.2 by using Theorem 5.2 of Ibragimov and Khasminskii (1981) with $\alpha = 2$ and $g(u) = u^2$. We omit the details.

Remarks 4.6. Bahadur (1960) suggested measuring the asymptotic efficiency of an estimator δ_T of a parameter θ by the magnitude of concentration of the estimator the interval of a fixed length (independent of T), that is by the magnitude of the probability $P_\theta\{|\delta_T - \theta| < \gamma\}$. From the result obtained in Theorem 4.1 proved above, we note that the probability $P_\theta\{|\hat{\theta}_T - \theta| < \gamma\}$ is bounded above by $C_1 e^{-C_2\gamma^2\alpha_T^{-1}}, C_1 > 0, C_2 > 0$ for the maximum likelihood estimator $\hat{\theta}_T$. This bound in turn decreases exponentially to zero as $T \to \infty$ for any fixed $\gamma > 0$. Following the techniques in Theorem 9.3 in Ibragimov and Khasminskii (1981), it can be shown that the the MLE is Bahadur efficient under some additional conditions. Similar result follows for the Bayes estimator $\tilde{\theta}_T$ following Theorem 4.2. The norming factor α_T can be chosen to be T^{-1} if $\theta < 0$, T^{-2} if $\theta = 0$ and $e^{-\theta T}T^{1/2}$ in case $\theta > 0$. This can be seen from Proposition 2.3 of Kleptsyna and Le Breton (2002). Observe that the norming factor α_T tends to zero as $T \to \infty$.

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