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# TAUBERIAN PROPERTY IN SADDLEPOINT APPROXIMATIONS

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# TAUBERIAN PROPERTY IN SADDLEPOINT APPROXIMATIONS

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**Abstract**

A Tauberian theorem which is related to the saddlepoint in the saddlepoint approximations of $M$-estimates is investigated. It is shown that the asymptotic behavior of the saddlepoint at infinity plays a substantial role in this context. We also show a pointwise convergence of a sequence of the saddlepoints which may be taken as a version of the continuity theorem by Lévy. The saddlepoint has no little amounts of information with respect to underlying distribution function.

*Key Words and Phrases:* Laplace transform, $M$-estimate, Regular variation, Saddlepoint approximations, Tail probability, Tauberian theorem, Weak convergence.

## 1. Introduction

Since from the pioneering paper dues to Daniels (1954), the saddlepoint approximations have been widely studied by many authors, see for example Kolassa (1997), Jensen (1995) or Reid (1988). In the present paper we shall consider the asymptotic behavior of the saddlepoint for a saddlepoint approximations of $M$-estimate of a location parameter.

Let $X_1, \cdots X_n$ be independently and identically distributed random variables whose distribution is $F(x-\theta_0)$, where $F(x)$ is symmetric about the origin and $\theta_0$ is a location parameter. Hereafter we assume that $\theta_0 = 0$ without loss of generality. The $M$-estimate for the location is defined as the solution $\hat{\theta}_n$ to the equation $\sum_{i=1}^{n} \psi(X_i - \theta)|_{\theta=\hat{\theta}_n} = 0$, where $\psi(x)$ is called a score function. Since the saddlepoint approximations to the density for the $M$-estimate is given by Field (1985) and Field and Ronchetti (1990), the details of the method will not be explained here.

In this paper we shall extend those results given by Takeuchi (1999), and show a Tauberian theorem which is related to the saddlepoint. In Section 2, we shall show a uniqueness of the existence of the saddlepoint and its bound. Remark that the class of the score function which is treated in this paper is much larger than that of Takeuchi (1999). In Section 3, we shall give a version of the Lévy-type continuity theorem of which weak convergence of distribution function implies convergence of corresponding saddlepoint. Finally in Section 4, we give a Tauberian theorem related to the saddlepoint. Asymptotic behavior of the saddlepoint at infinity corresponds to tail probability of the underlying distribution function. And it is shown that Tauberian property of the saddlepoint exploits this fact. From these assertion above, we may say that the saddlepoint has no little amounts of information with respect to underlying distribution function.

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We shall use the following assumptions through the paper, and further conditions are to be imposed if necessary.

(A. 1) The distribution function $F(x)$ is continuous with $F(-x) = 1 - F(x)$ for $x \in \mathbf{R}$. And it also satisfies the followings,

$$F(x) \begin{cases} < 1 & \text{for } x < a \\ = 1 & \text{for } x \geq a, \end{cases}$$

where $a > 0$ and if desired $a = \infty$.

(A. 2) The score function $\psi(x)$ is Lebesgue mesurable with $\psi(-x) = -\psi(x)$ for $x \in \mathbf{R}$, $\psi(x) \geq 0$ for $x > 0$, and $\sup_{x>0} \psi(x) \leq M$ for some $M > 0$.

(A. 3) $F(x)$ and $\psi(x)$ satisfy

$$t \int_{-a}^{a} \psi(x - t)\, dF(x) < 0 \quad \text{for} \quad t \neq 0. \tag{1.1}$$

These assumptions of the monotonicity of $F(x)$ in (A. 1) and (A. 3) are natural requirement for *M*-estimation, as they guarantee the uniqueness of the existence of a location parameter. Following notes are illustrative.

NOTE 1.1. Let $F(x)$ be a probability distribution function with density $F'(x) = 1/2 \cdot I_{[-2,-1]}(x) + 1/2 \cdot I_{[1,2]}(x)$ and score function be $\psi(x) = x \cdot I_{\{|x|<k\}} + k \cdot \mathrm{sgn}(x) \cdot I_{\{|x|\geq k\}}$ (*Hubers*), where the function $\mathrm{sgn}(x)$ gives the sign of $x$. If $k$ satisfies $0 < k < 1$, however, there exists a $\theta_0 \neq 0$ such that $\int_{-a}^{a} \psi(x - \theta_0)\, dF(x) = 0$.

NOTE 1.2. If $F(x) = 1/2a \cdot I_{[-a,a]}(x)$ and $\psi(x) = \sin(bx) \cdot I_{\{|x|<\pi/b\}}$ (*a smoothed Hampels*) then the same matter as in Note 1.1 occurs when $ab > \pi$.

The class of score functions that satisfy (A. 2) includes many important ones, and for practical form of the score functions, see for example, Serfling (1980).

The saddlepoint is denoted by $\alpha_t$ and it is defined as a solution to the saddlepoint equation $g(t, \alpha) = 0$ with respect to $\alpha$, for fixed $t$, where

$$g(t, \alpha) = \int_{-a}^{a} \psi(x - t) \exp[\alpha \psi(x - t)]\, dF(x). \tag{1.2}$$

Since from (1.2), the saddlepoint itself does not depend on the sample size. And it is obvious that $g(-t, -\alpha) = -g(t, \alpha)$, we call this as an odd property of the saddlepoint equation. In this paper $F$ also denotes Lebesgue-Stieltjes measure induced by the distribution function. Note that the score function is Lebesgue-Stieltjes measurable with respect to $F$.

It should be noted that there are two ways to get the saddlepoint approximations to the probability density function of the *M*-estimate. The formula (1.2) appears when the conjugate density with the Edgeworth expansions approach is employed. And the other is by means of the method of steepest descent, see for example deBruijn (1981), however there is no difference to the value of the saddlepoint (Field and Ronchetti (1990)).

## 2. Existence of the Saddlepoint

For the existence of the saddlepoint we have the following theorem for the *M*-estimator of location. We further assume the following condition through this section.

(A. 2') The score function $\psi(x)$ is continuous and strictly increasing in a neighborhood of $x = 0$.

Note that we do not use implicit function theorem here, since it needs much stronger conditions than our ones.

THEOREM 2.1. *For any fixed* $t \in (-a, a)$*, the saddlepoint equation* $g(t, \alpha) = 0$ *has at least one root* $\alpha_t$*. If the distribution function* $F(x)$ *is strictly increasing in a neighborhood of* $x = t$ *then the root is unique.*

PROOF. By the boundedness of the score function

$$|g(t, \alpha + h) - g(t, \alpha)| \leq Me^{|\alpha|M} \int_{-a}^{a} \bigl|\exp[h\psi(x - t)] - 1\bigr| \, dF(x).$$

Hence from this, $g(t, \alpha)$ is continuous with respect to $\alpha$ uniformly for $t$. By (A. 2') there exists a $\delta > 0$ such that $\psi(x)$ is strictly increasing on $(-\delta, \delta)$, and we decompose the right hand side of (1.2) into four parts;

$$g(t, \alpha) = \int_{-a}^{t} + \int_{t}^{t+\delta_1} + \int_{t+\delta_1}^{t+\delta_2} + \int_{t+\delta_2}^{a} \psi(x - t) \exp[\alpha\psi(x - t)] \, dF(x), \tag{2.1}$$

where $\delta_1$ and $\delta_2$ satisfies $0 < \delta_1 < \delta_2 \leq \delta$. We have then $\inf_{\delta_1 < x < \delta_2} \psi(x) = \psi(\delta_1) > 0$ by (A. 2). Notice that for each $t$ we can select $\delta_2$ such that $t + \delta_2 < a$, so (2.1) is well-defined. The first part of those of integrals in (2.1) tends to 0 by the dominated convergence theorem, and the both of second and last parts are nonnegative. Assume that $F(x)$ is strictly increasing in a neighborhood of $x = t$ then the third part tends to infinity by what we have mentioned above, as

$$\int_{t+\delta_1}^{t+\delta_2} \psi(x - t) \exp[\alpha\psi(x - t)] \, dF(x) \geq \psi(\delta_1) \exp\bigl[\alpha\psi(\delta_1)\bigr] (F(t + \delta_2) - F(t + \delta_1)) \\ \to \infty, \quad \text{as} \quad \alpha \to \infty.$$

Thus we have $\lim_{\alpha\to\infty} g(t, \alpha) = \infty$ for any fixed $t \in (-a, a)$. Since from the odd property of the saddlepoint equation, we also get $\lim_{\alpha\to-\infty} g(t, \alpha) = -\infty$. Finally $g(t, \alpha)$ is strictly increasing with respect to $\alpha$ for any fixed $t \in \mathbf{R}$, as, for any $h > 0$

$$g(t, \alpha + h) - g(t, \alpha) \geq h \int_{-a}^{a} \{\psi(x - t)\}^2 \exp[\alpha\psi(x - t)] \, dF(x)$$
$$\geq \begin{cases} h\,\psi(\delta_1)^2 \exp\bigl[\alpha\psi(\delta_1)\bigr] (F(t + \delta_2) - F(t + \delta_1)) > 0 & \text{if } \alpha \geq 0 \\ h\,\psi(\delta_1)^2 \exp\bigl[\alpha\psi(\delta_2)\bigr] (F(t + \delta_2) - F(t + \delta_1)) > 0 & \text{if } \alpha < 0, \end{cases}$$

with using same $\delta_1$ and $\delta_2$ as above.

If $F(x)$ is not strictly increasing in a neighborhood of $x = t$, by the assertion above, we have

$$\lim_{\alpha\to\infty} g(t, \alpha) \geq 0, \lim_{\alpha\to-\infty} g(t, \alpha) \leq 0$$

and $g(t,\alpha)$ is a monotone increasing function with respect to $\alpha$. Hence we get the conclusion. $\square$

NOTE 2.2. As in the proof of Theorem 2.1, strict increase of the distribution function yields not only uniqueness of the saddlepoint but also strict increase of $g(t,\alpha)$ with respect to $\alpha$. This fact will be often used in the rest of this paper.

As Theorem 2.1 states a sufficient condition for uniqueness of the saddlepoint, hereafter we suppose that the saddlepoint is uniquely determined by $t \in (-a,a)$ and denote it as $\alpha(t)$. In other words $\alpha = \alpha(t)$ is the only solution that satisfies the saddlepoint equation $g(t,\alpha) = 0$.

The odd property of the saddlepoint equation implies that the saddlepoint itself is an odd function, i.e. $\alpha(-t) = -\alpha(t)$. Since we have $\alpha(0) = 0$, so, we may confine ourselves to examine the behavior of the saddlepoint for $t > 0$.

To construct a bound for the saddlepoint, we need the following lemmata.

LEMMA 2.3. *The saddlepoint satisfies* $\alpha(t) > 0$ *for* $0 < t < a$.

PROOF. By Proposition 2.2 in Takeuchi (1999), which is also valid in this case, it is sufficient to show that $\int_{-a}^{a} \psi(x-t)\,dF(x) < 0$ for $t > 0$. But this is clear from (1.1). $\square$

By the lemma above we have that $\alpha(t) = 0$ if and only if $t = 0$.

LEMMA 2.4. *The saddlepoint* $\alpha(t)$ *is continuous on* $0 < t < a$.

PROOF. We omit the proof because it is completely same as one in Lemma 2.5 in Takeuchi (1999). $\square$

PROPOSITION 2.5. *Adding to (A. 2) that if there exists a* $\gamma > 0$ *such that* $\inf_{\gamma<x<a} \psi(x) > 0$, *then there exists a lower bound for* $\alpha(t)$ *such that*

$$\alpha(t) \geq \frac{1}{M + c(\gamma)} \log \frac{c(\gamma)F(t-\gamma)}{M(1-F(t))} \tag{2.2}$$

*for* $t \in (-a,a)$, *where* $c(\gamma) = \inf_{\gamma<x<a} \psi(x)$.

PROOF. The proof is almost same as Proposition 2.3 in Takeuchi (1999). To find a continuous function $g^+(t,\alpha)$ with respect to $\alpha$ such that $\lim_{\alpha\to\pm\infty} g^+(t,\alpha) = \pm\infty$, $g^+(t,\alpha) \geq g(t,\alpha)$ and has a unique solution to $g^+(t,\alpha) = 0$ for each $t \in (-a,a)$, we may evaluate the integral in (1.2) for $t \geq 0$ as;

$$\begin{aligned} g(t,\alpha) &\leq \int_{-a}^{t-\gamma} + \int_{t}^{a} \psi(x-t)\exp\big[\alpha\psi(x-t)\big]\,dF(x) \\ &\leq -c(\gamma)\exp[-\alpha c(\gamma)]F(t-\gamma) + M\exp[\alpha M](1-F(t)). \end{aligned} \tag{2.3}$$

Define the right hand side of (2.3) as $g^+(t,\alpha)$ then it obviously satisfies all of required conditions as described above. The lower bound, $\alpha_t^+$ say, is given as a solution to $g^+(t,\alpha_t^+) = 0$. $\square$

The procedure in the proof above will be used again in the rest of this paper. And from this proposition we find that the saddlepoint tends to infinity as $t \to \infty$.

Under the same conditions as in Proposition 2.5, we have an upper bound for some restricted $t$.

COROLLARY 2.6. *If $t \in \{t : \alpha(t) \le 1/M\}$ then we have an upper bound such that*

$$\alpha(t) \le \frac{1}{M + c(\gamma)} \log \frac{MF(t)}{c(\gamma)(1 - F(t+\gamma))}. \tag{2.4}$$

PROOF. We can evaluate (1.2) from below by

$$g^{-}(t, \alpha) = -M \exp[-\alpha M] F(t) + c(\gamma) \exp[\alpha c(\gamma)](1 - F(t+\gamma)).$$

□

The bounds (2.2) and (2.4) are an extended result of Proposition 2.3 in Takeuchi (1999), because if the score function is *Hubers* (see Note 1.1) then by setting $\gamma = c(\gamma) = M$ as $k$ in those of (2.2) and (2.4), we have the same bounds.

## 3. Continuity Theorem

In this section we give an Lévy-type continuity theorem with respect to probability distribution functions and corresponding saddlepoints. We assume the following condition through this section.

(A. 2") There exists a $\gamma > 0$ such that $c(\gamma) = \inf_{\gamma < x < a} \psi(x) > 0$.

THEOREM 3.1. *Let $\{F_n\}$ be a sequence of probability distribution functions and $\{\alpha_n\}$ be the corresponding sequence of saddlepoints. If $F_n \xrightarrow{w} F$ and $F(x)$ is continuous on $\mathbf{R}$ and strictly increasing at $x = t$ then we have*

$$\alpha_n(t) \to \alpha(t) \quad as \quad n \to \infty,$$

*where $\alpha(t)$ is the saddlepoint of $F$.*

PROOF. For each fixed $t \in \mathbf{R}$, we have from Theorem 2.1, there exists a saddlepoint $\alpha_n(t)$ such that

$$\begin{aligned} 0 &= \int_{-a}^{a} \psi(x-t) \exp[\alpha_n(t)\psi(x-t)]\, dF_n(x) \\ &= \int_{-a}^{a} \psi(x-t) \exp[\alpha_n(t)\psi(x-t)]\, d(F_n(x) - F(x)) \\ &\qquad + \int_{-a}^{a} \psi(x-t) \exp[\alpha_n(t)\psi(x-t)]\, dF(x) \end{aligned} \tag{3.1}$$

for each $F_n$, $n \ge 1$. Now assume that the sequence $\{\alpha_n(t)\}$ fails to converge to $\alpha(t)$, where $\alpha(t)$ is defined as the saddlepoint of $F$ in the sense that they satisfies the saddlepoint equation. Let $\sup_{n \ge 1} |\alpha_n(t)| < \infty$ then by the weak convergence assumption the first term of the right hand side of (3.1) converges to 0, but the second term of (3.1) is shown to not converge to 0 as follows. If $\limsup_{n\to\infty} \alpha_n(t) < \alpha(t)$ then the second term takes negative value for sufficiently large $n$, since we have

$$\limsup_{n\to\infty} \int_{-a}^{a} \psi(x-t)\exp[\alpha_n(t)\psi(x-t)]\,dF(x) < 0$$

by Fatou's lemma and strict increase of $g(t,\alpha)$ with respect to $\alpha$. If $\limsup_{n\to\infty} \alpha_n(t) > \alpha(t)$ then there exists a subsequence $\{\alpha_{n'}(t)\}$ that converges to $\limsup_{n\to\infty} \alpha_n(t)$ and the second term of (3.1) converges to a positive value with this subsequence. In the similar way, $\limsup_{n\to\infty} \alpha_n(t) = \alpha(t)$ implies $\liminf_{n\to\infty} \alpha_n(t) < \alpha(t)$, there exists another subsequence $\{\alpha_{n'}(t)\}$ such that $\alpha_{n'}(t)$ converges to $\liminf_{n\to\infty} \alpha_n(t)$ and in this case, the same term converges to negative. All of these results contradicts to (3.1).

Finally assume that $\sup_{n\geq 1} |\alpha_n(t)|$ is infinite. For each $x \in (-a,a)$ and sufficiently small $\varepsilon > 0$, there exists an $N_0 \geq 1$ such that for all $n \geq N_0$ we have

$$1 - F_n(x) > 1 - F(x) - \varepsilon > 0 \tag{3.2}$$

by the assumption. Since from the odd property of $g(t,\alpha)$, without loss of generality, we may only consider when $\limsup_{n\to\infty} \alpha_n(t)$ is infinite. There exists a subsequence that $\alpha_{n'}(t)$ diverges to plus infinity. Take $\delta > \gamma$ such that $F(t+\delta) < 1$ then for sufficiently large $n'$ we have

$$\begin{aligned} 0 &= \int_{-a}^{a} \psi(x-t)\exp[\alpha_{n'}(t)\psi(x-t)]\,dF_{n'}(x) \\ &\geq -M\int_{-a}^{t} dF_{n'}(x) + \int_{t+\delta}^{a} \psi(x-t)\exp[\alpha_{n'}(t)\psi(x-t)]dF_{n'}(x) \\ &\geq -M + c(\delta)\exp[\alpha_{n'}(t)c(\delta)]\{1 - F(t+\delta) - \varepsilon\} \end{aligned}$$

Since from (3.2) the last term tends to infinity as $n' \to \infty$, and this is also a contradiction. Hence we get the conclusion. □

## 4. Tauberian Property of the Saddlepoint

In this section, we shall investigate an asymptotic behavior of the saddlepoint at infinity by evaluating the tail probability of underlying distribution function. And this makes possible to get an asymptotically equivalent form of the saddlepoint by means of Tauberian theorem. The relationship between Laplace transform of the distribution function and the corresponding saddlepoint is clarified, and in this respect, the following theorem plays a key role in this section.

Theorem 4.1. *We assume that the support of $F$ is unbounded, i.e. $a = \infty$ in (A. 1). Then for a bounded score function $\psi$, there exists a saddlepoint $\alpha(t)$ such that*

$$\alpha(t) \sim -\frac{1}{2M}\log(1 - F(t)), \quad as \quad t \to \infty, \tag{4.1}$$

*where $M = \sup_{x>0} \psi(x)$.*

Proof. First of all we define the score function as follows. It is continuous and strictly increasing on $\mathbf{R}$ with $\lim_{x\to\infty} \psi(x) = M$. And for sufficiently large $m > 0$,

$\psi(x) \geq \frac{\psi(m)}{m} x$ for $x \in (0,\ m]$. Let $m(\varepsilon)$ be a positive number such that $\psi(m(\varepsilon)) = M - \varepsilon$, where $\varepsilon > 0$ is sufficiently small. We may evaluate $g(t, \alpha)$ by $g^+$ and $g^-$ such that

$$g^+(t,\alpha) = -M\left\{\exp[-\alpha M]F(t-m(\varepsilon)) - \exp[\alpha M](1-F(t))\right\}, \tag{4.2}$$

$$\begin{aligned} g^-(t,\alpha) = -(M-\varepsilon)\{&\exp[-\alpha(M-\varepsilon)]F(t-m(\varepsilon)) \\ &+\exp[\alpha(M-\varepsilon)](F(t)-F(t-m(\varepsilon))) \\ &-\exp[\alpha(M-\varepsilon)](1-F(t+m(\varepsilon))\}. \end{aligned} \tag{4.3}$$

Note that to evaluate $g(t, \alpha)$ from below, (1.2) may be decomposed as follows.

$$g(t,\alpha) \geq \int_{-a-t}^{-m(\varepsilon)} + \int_{-m(\varepsilon)}^{0} + \int_{m(\varepsilon)}^{a-t} \psi(x)\exp[\alpha\psi(x)]\, dF(x+t) \tag{4.4}$$

By the definition of the $\psi$, we have for $-m(\varepsilon) \leq x \leq 0$

$$e^{\alpha\psi(x)} \leq \frac{1}{2m(\varepsilon)}(e^{\alpha\psi(m(\varepsilon))} - e^{-\alpha\psi(m(\varepsilon))})x + \frac{1}{2}(e^{\alpha\psi(m(\varepsilon))} + e^{-\alpha\psi(m(\varepsilon))}),$$

and apply this inequality to the second term of the right hand side of (4.4) to get (4.3). Together with (4.2) and (4.3), we have a bound for the saddlepoint at $t$ $(0 < t < a)$ such that

$$\frac{1}{2M}\log\frac{F(t-m(\varepsilon))}{1-F(t)} \leq \alpha(t) \leq \frac{1}{2(M-\varepsilon)}\log\frac{F(t-m(\varepsilon))}{1-F(t)-F(t+m(\varepsilon))+F(t-m(\varepsilon))}. \tag{4.5}$$

For any $\varepsilon' > 0$ there exists a $T > 0$ such that

$$-\frac{1}{2M}\left\{\log(1-F(t))+\varepsilon'\right\} \leq \alpha(t) \leq -\frac{1}{2(M-\varepsilon)}\left\{\log(1-F(t))-\varepsilon'\right\}.$$

for all $t > T$. Since $\alpha(t)$ tends to infinity as $t \to \infty$, we have

$$\frac{1}{2M}\limsup_{t\to\infty}\frac{-1}{\alpha(t)}\log(1-F(t)) \leq 1$$

$$\frac{1}{2(M-\varepsilon)}\liminf_{t\to\infty}\frac{-1}{\alpha(t)}\log(1-F(t)) \geq 1,$$

and letting $\varepsilon \to 0$ completes the proof of the theorem. □

Theorem 4.1 not only shows that a relationship between the distribution function and the corresponding saddlepoint, but also reveals a statistical meaning of the saddlepoint. Note that (4.5) is useful for determining an initial value for the saddlepoint in terms of computational effort.

The rapidity of increase of the saddlepoint may be evaluated as following Notes.

NOTE 4.2. If $F$ is the standard normal then we have $\alpha(t) \sim t^2/4M$ as $t \to \infty$, since from the relation;

$$\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{u^2}{2}}\, du \sim \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}} \quad \text{as} \quad x \to \infty,$$

and from (4.1). It is interesting that in the saddlepoint approximation of the sample mean, i.e. $\psi(x) = x$, we have $\alpha(t) = t$ for $-\infty < t < \infty$ under the same distribution.

Note 4.3. If $F$ has a tail probability such that $1-F(x) \sim C\exp[-x^\gamma]$ as $x \to \infty$, where $C > 0$ and $\gamma > 0$, then from (4.1) we have $\alpha(t) \sim t^\gamma/2M$ as $t \to \infty$.

Note 4.4. A computation was done to obtain Table 1 which exhibits the asymptotic evaluation (4.1). The underlying p.d.f. is $F'(x) = 0.5\exp(-|x|)$, $(|x| < \infty)$ with Huber type score function $k = 1.5$, see Note 1.1. The "ratio" denotes $(-2k\alpha(t))^{-1}\log(1-F(t))$, and it seems to converge to unity. It is also confirmed that $\alpha(t) \sim t/2k$ as in Note 4.3.

Table 1.

| $t$ | 20 | 40 | 60 | 80 | 100 | 200 | 300 |
|---|---|---|---|---|---|---|---|
| $\alpha(t)$ | 7.3538 | 14.041 | 20.715 | 27.386 | 34.055 | 67.393 | 100.73 |
| ratio | 0.93798 | 0.96606 | 0.97663 | 0.98218 | 0.98560 | 0.99265 | 0.99507 |

It is of interest whether there exists some correspondence between probability measure $P$ and an induced measure by the saddlepoint, especially in the sense that

$$Q(B) = -\frac{1}{2M}\log(1-P(B)) \quad \text{for some set } B \in \mathscr{F},$$

where $Q$ denotes the induced measure and $\mathscr{F}$ a $\sigma$-field. Although we shall not consider this subject here, certain conditions are needed to which the saddlepoint induces a measure.

Proposition 4.5. *Assume that the distribution function $F$ is strictly increasing, and for any fixed $t > 0$, $g(u, \alpha(t))$ is monotone decreasing with respect to $u$ in a neighborhood of $t$. Then there exists an induced Lebesgue-Stieltjes measure on $[0, \infty)$ by the saddlepoint.*

Proof. Assume that for some $t_0 > 0$ there exists sufficiently small $h_0 > 0$ such that $\alpha(t_0 + h_0) \le \alpha(t_0)$. Then by the proof of Theorem 2.1, $g(t, \alpha)$ is strictly increasing with respect to $\alpha$, we have $g(t_0, \alpha(t_0 + h_0)) < g(t_0, \alpha(t_0)) = 0$. This contradicts to the assumption of this proposition. Hence $\alpha(t)$ is a continuous and strictly increasing function on $0 < t < a$, and by defining $Q(a, b] = \alpha(b) - \alpha(a)$ for $0 \le a < b$, we have the required measure on $[0, \infty)$. □

If the saddlepoint has a derivative which is Lebesgue measurable and integrable, then for any fixed $T > 0$, $Q$ is absolutely continuous on $[0, T]$.

From Theorem 4.1, tail probability of a distribution function $F$ can be asymptotically evaluated by the corresponding saddlepoint. By using this fact and Tauberian theorem in Feller (1971), we shall show an asymptotic behavior of the saddlepoint in terms of slowly varying function in what follows. Let $\varphi_G$ be Laplace transform of a probability distribution function $G$, which is concentrated on $[0, \infty)$, such that

$$\varphi_G(\lambda) = \int_0^\infty e^{-\lambda x}\, dG(x), \quad \lambda > 0.$$

Each of slowly varying and regularly varying function is defined as follows.

Definition 4.6. (Bingham et al. (1987)) A measurable function $l > 0$ satisfying (4.6) is called regularly varying at infinity with index $\rho$, and we write $l \in R_\rho$.

$$l(\lambda x)/l(x) \to \lambda^\rho \quad \text{as } x \to \infty,\ \forall \lambda > 0. \tag{4.6}$$

If $\rho = 0$ then $l$ is said to be slowly varying at infinity and we write $l \in R_0$.

Definition 4.7. (Bingham et al. (1987)) A measurable and positive function $l$ satisfying (4.7) is said to be regularly varying at the origin with index $\rho$, and we write $l \in R_\rho(0+)$.

$$l(\lambda x)/l(x) \to \lambda^\rho \quad \text{as } x \downarrow 0,\ \forall \lambda > 0. \tag{4.7}$$

Note 4.8. It is obvious from definition that $l(x) \in R_\rho(0+)$ if and only if $l(1/x) \in R_{-\rho}$.

We need the following theorem due to Feller (1971) p.447.

Theorem 4.9. *(Feller) Let $\varphi_G(\lambda)$ exist for $\lambda > 0$. If $l$ is slowly varying, where $\rho > 0$, then we have*

$$1 - \varphi_G(\lambda) \sim \frac{1}{\lambda^{\rho-1}} l\left(\frac{1}{\lambda}\right) \quad \textit{as} \quad \lambda \downarrow 0 \tag{4.8}$$

*if and only if*

$$1 - G(x) \sim \frac{1}{\Gamma(\rho)} x^{\rho-1} l(x) \quad \textit{as} \quad x \to \infty. \tag{4.9}$$

In the rest of this section we assume that the score function satisfies those of conditions in the proof of Theorem 4.1.

Corollary 4.10. *Suppose that (4.1) is valid. Let $\varphi_F(\lambda)$ exist for $\lambda > 0$. If $l$ is slowly varying, where $\rho > 0$, then we have*

$$1 - 2\varphi_F(\lambda) \sim \frac{1}{\lambda^{\rho-1}} l\left(\frac{1}{\lambda}\right) \quad \textit{as} \quad \lambda \downarrow 0 \tag{4.10}$$

*if and only if*

$$\alpha(t) \sim -\frac{1}{2M} \log\left\{\frac{1}{2\Gamma(\rho)} t^{\rho-1} l(t)\right\} \quad \textit{as} \quad t \to \infty, \tag{4.11}$$

*where $\alpha(t)$ is the corresponding saddlepoint of $F$.*

Proof. To apply Theorem 4.9, we set $G(x) = 2F(x) - 1$ for $x \geq 0$. Then $\varphi_G(\lambda)$ exists for $\lambda > 0$ by the assumption, and the equivalence of (4.8) and (4.10) is obvious. We have from (4.1) that

$$1 - G(t) \sim 2\exp[-2M\alpha(t)] \quad \text{as} \quad t \to \infty$$

and this clarify the relation between (4.9) and (4.11). □

Because of Field (1985) and Takeuchi (1999), it is interesting to evaluate behavior of the saddlepoint at the origin. The following result is a corollary to representation theorem 1.3.1 in Bingham et al. (1987).

COROLLARY 4.11. *We further assume that* $\sup_{x>0}|\psi^{(i)}(x)| < M$ $(i = 1, 2)$ *for some* $M > 0$. *The saddlepoint may be written in the form*

$$\alpha\big(t^{-1}\big) = t^{-1}c(t)\exp\left[\int_b^t \varepsilon(u)/u\,du\right] \quad t \ge b \tag{4.12}$$

*for some* $b > 0$, *where* $c(\cdot)$ *is measurable and* $c(t) \to c \in (0,\infty)$, $\varepsilon(t) \to 0$ *as* $t \to \infty$.

PROOF. We have from Theorem 2.1 and (1.2) that

$$\int \psi(x-t)\,dF(x)+\alpha(t)\int \psi^2(x-t)\,dF(x)+\frac{\alpha^2(t)}{2}\int \exp[\theta\alpha(t)\psi(x-t)]\psi^3(x-t)\,dF(x) = 0$$

for some $0 < \theta < 1$. Since $\lim_{t\to 0}\alpha(t) = 0$, the third term is $O(\alpha^2(t))$ as $t \to 0$. Thus we have

$$\alpha(t) \sim -\int \psi(x-t)\,dF(x)\Big/\int \psi^2(x-t)\,dF(x) \quad t \to 0. \tag{4.13}$$

Note that (4.13) is well defined, since the denominator is positive for sufficiently small $t > 0$. It is sufficient to show that $\alpha(t)$ is regularly varying at the origin with index $\rho = -1$. The numerator is shown to be regularly varying at the origin with index $\rho = 1$, as follows.

$$\begin{aligned}&\int \psi(x-\lambda t)\,dF(x)\Big/\int \psi(x-t)\,dF(x)\\ &= \left\{-\lambda t\int \psi'(x)\,dF(x) + \lambda^2 O(t^2)\right\}\Big/\left\{-t\int \psi'(x)\,dF(x) + O(t^2)\right\}\\ &\to \lambda \quad \text{as} \quad t \to 0\end{aligned}$$

In the same way we can show that the denominator is slowly varying at the origin. Hence from this the saddlepoint $\alpha(t)$ is regularly varying at the origin with index $\rho = 1$, and this is equivalent to that $\alpha(t^{-1})$ is regularly varying at infinity with index $\rho = -1$ by Note 4.8. Then by the Characterization Theorem 1.4.1 in Bingham et al. (1987) p.17, there exists a slowly varying function $l$ such that $\alpha(t^{-1}) = t^{-1}l(t)$. Hence we have the conclusions by the Representation Theorem 1.3.1 in Bingham et al. (1987) p.12. □

NOTE 4.12. While (4.12) is an exact representation of the saddlepoint in a neighborhood of the origin, (4.13) is an asymptotically equivalent evaluation that is an extension of Corollary 2.10 in Takeuchi (1999).

It seems to be interesting that from (4.13), the saddlepoint is depending on the score function for $t \to 0$. But from (4.1) it is not depending on the one at infinity, except for the parameter $M = \lim_{x\to\infty}\psi(x)$.

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