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Sato, Tatsuro
Department of Mathematics, Oita National College of Technology

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# ON BEHAVIORS OF CELLULAR AUTOMATA WITH RULE NUMBER 57

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# ON BEHAVIORS OF CELLULAR AUTOMATA WITH RULE NUMBER 57

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**Tatsuro Sato***

**Abstract**

This paper describes the behaviors of one dimensional finite cellular automata with a triplet local transition rule number 57 having the fixed boundary conditions. The behaviors of $CA-57(m)$ were observed by computer experiments, and some formulae about the limit cycles and the transient length were found out. The purpose of this paper is to prove the above formulae theoretically.

*Key Words and Phrases:* Boundary condition, Cellular automaton, Limit cycle, Transient length.

## 1. Introduction

Cellular automata were initially introduced by J. von Neumann as a theoretical model of self-reproducing systems in 1950's. Recently, cellular automata have been realized again as a theoretical model of complex system since 1980's by S. Wolfram et al.. Cellular automata have very simple structure. While cellular automata obey simple transition rules, their behaviors are very complicated. The complication is caused by interaction between cells and is similar to such behaviors of complex systems as fractal and chaotic phenomena. The importance of cellular automata seems to increase in the fields of mathematics, physics, biology, computer science and so on.

Various cellular automata have been analyzed concerning one dimensional and two dimensional cellular automata with cyclic cell array or linear cell array and so on. Wolfram classified one dimensional cellular automata into four complex classes according to patterns generated by the synchronous dynamics. In this paper, we deal with one dimensional finite cellular automata having either state 0 or 1 and works on a linear array of $m$ cells with a triplet local transition rule and fixed boundary conditions. Concerning cellular automata with simple structure, many important results have been reported by Kawahara et al. [Kawahara (1991), Kawahara et al. (1992), Inokuchi and Sato et al. (1996)]. Especially, the author has investigated cellular automata with threshold rules and having limit cycles of period length 3[Sato (1996a), Sato (1996b), Sato (1998), Sato (1999), Inokuchi and Sato (2000)]. As to cellular automata with threshold rules, Shingai reported that period lengths of limit cycles are 4 or less[Shingai (1978)]. And cellular automata having limit cycles of period length 1 or 2 were investigated by Kawahara et al.. So the present authors aimed at cellular automata having limit cycles of period length 3. By computer experiments, we observed

* Department of Mathematics, Oita National College of Technology,1666 Maki Oita 870-0152, Japan.
E-mail : tatsuro@oita-ct.ac.jp

that there were seven cellular automata having limit cycles of period length 3 except cellular automata with threshold rules. They have been investigated but rule numbers 57 and 58.

In this paper, the behavior of a cellular automaton with its rule number 57 having fixed boundary conditions have been reported. Concerning the transient length under the boundary condition 1–0, this cellular automaton has the feature that its transient length is influenced by the cell size $m$ being an odd number or an even number.

## 2. Preliminaries

In this section, we define one dimensional finite cellular automata with triplet local transition rules and introduce notations used in this paper.

When we set $I = \{1,\ 2,\ \cdots,\ m\}$ as a set of cells, a $m-$dimensional vector space $\{0,\ 1\}^m$ is called a configuration space and its element $x = (x_1,\ x_2,\ \cdots,\ x_m) \in \{0,\ 1\}^m$ is called a configuration. The triplet local transition rule $f$ is a function $\{0,\ 1\}^3 \to \{0,\ 1\}$. The rule number $R$ of $f$ is defined as follows :

$$R = 2^7 r_7 + 2^6 r_6 + 2^5 r_5 + 2^4 r_4 + 2^3 r_3 + 2^2 r_2 + 2^1 r_1 + 2^0 r_0$$

where $r_i = f(x,\ y,\ z)$ and $i = 4x + 2y + z$. Then a triplet local transition rule $f$ is illustrated as follows :

$$\begin{array}{cccccccc} 111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ r_7 & r_6 & r_5 & r_4 & r_3 & r_2 & r_1 & r_0 \end{array} .$$

A finite cellular automaton with its rule number $R$ having $m$ cells, the left boundary condition $a(a \in \{0,\ 1\})$ and the right boundary condition $b(b \in \{0,\ 1\})$ is represented as follows :

$$CA - R_{a-b}(m).$$

Definition 2.1. $CA$–$R_{a-b}(m)$ is a pair $(C,\ \delta)$, where $C = \{0,\ 1\}^m$ is a configuration space and $\delta$ is a global transition function from $C$ to $C$ defined as follows :

$$\delta(x) = (f(ax_1x_2),\ f(x_1x_2x_3),\ \cdots,\ f(x_{m-1}x_mb)).$$

Since $CA - R_{a-b}(m)$ has $2^m$ possible configurations, each configuration converges in certain states( that is, a limit cycle) after some transition steps. When there exists a natural number $s \geq 1$ such as $\delta^s(c) = c$ for $c \in C$, we call $c$ a configuration on a limit cycle.

Definition 2.2. Let $c \in C$ be a configuration on a limit cycle. We define the period $P$ of the limit cycle as follows :

$$P = min\{s \geq 1 : \delta^s(c) = c, c \in C\}.$$

Definition 2.3. Let $h(x)$ be the least number of transition steps needed by a configuration $x \in C$ to converge a limit cycle. We define the transient length $H$ of $CA - R_{a-b}(m)$ as follows :

$$H = max\{h(x) : x \in C\}.$$

Definition 2.4. We define the reflective transition rule $\overline{f}$ of a triplet transition rule $f$ as follows :

$$\overline{f}(x,\ y,\ z) = 1 - f(1 - x,\ 1 - y,\ 1 - z).$$

DEFINITION 2.5. We define the symmetric transition rule $f^T$ of a triplet transition rule $f$ as follows :

$$f^T(x,\ y,\ z) = f(z,\ y,\ x).$$

Taking the reflective rule, the symmetric rule and the symmetric reflective rule into consideration, 256 triplet transition rules can be classified into nonequivalent 88 groups.

In addition to above definitions, the notations in this paper are as follows :

(a) $a^k = \underbrace{aa\cdots a}_{k-times}$ ($a = 0$ or 1).

Let $A$ be a subsequence.

(b) $A^k = \underbrace{AA\cdots A}_{k-times}$.

(c) $[\ A\ )_l^*$ : sequence composed of $l$ bits taken from the left edge when some $A$'s are arranged.

(d) $(\ A\ ]_l^*$ : sequence composed of $l$ bits taken from the right edge when some $A$'s are arranged.

(e) $*$ : an irrelevant bit.

(f) The state of $i-$th cell of a configuration $x$ is denoted by $x_i$.

A finite cellular automaton $CA-57_{a-b}(m)$ treated in this paper has the following triplet local transition rule $f$ by the definition of the rule number :

$$\begin{array}{cccccccc} 111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}$$

because $57 = 2^5 + 2^4 + 2^3 + 2^0$.

## 3. Behaviors of $CA - 57_{0-0}(m)$

In this section, we deal with $CA - 57_{0-0}(m)$.

LEMMA 3.1. *Let $\delta(x) = y$ for any configuration $x$ of $CA$–$57_{0-0}(m)$. Then, the following holds :*

1. *If $x_m = a$, we have $y_m = 1 - a$ where $a = 0$ or 1.*
2. *If $y_i y_{i+1} y_{i+2} = 111$, we have $x_i x_{i+1} x_{i+2} x_{i+3} = 0000$ $(1 \leq i \leq m-2)$.*
3. *If $y_i y_{i+1} y_{i+2} = 000$, we have $x_{i-1} x_i x_{i+1} x_{i+2} = 1111$ $(2 \leq i \leq m-2)$.*
4. $y_1 y_2 y_3 \neq 000$.
5. $y_{m-3} y_{m-2} y_{m-1} y_m \neq 1110$.
6. $y_{m-4} y_{m-3} y_{m-2} y_{m-1} y_m \neq 11101$.
7. *$y$ doesn't contain the subsequence 111011.*

LEMMA 3.2. *For any configuration $x$ of $CA - 57_{0-0}(m)$, we set $\delta^k(x) = y$. Then, we have $y_i y_{i+1} \cdots y_{i+4} \neq 11101$ $(1 \leq i \leq m-6)$ if $k \geq 3$, $y_1 y_2 \neq 11$ if $k \geq 2$ and $y_1 y_2 y_3 y_4 y_5 y_6 \neq 101001$ if $k \geq 7$.*

LEMMA 3.3. *For any configuration $x$ of $CA - 57_{0-0}(m)$, $\delta^k(x)$ doesn't contain the subsequence 000 if $k \geq 2m-5$ and so the subsequence 111 if $k \geq 2m-4$.*

LEMMA 3.4. *For any configuration $x$ of $CA-57_{0-0}(m)$, we set $\delta^k(x)=y$. Then, we have*

1. $y_i y_{i+1}\cdots y_{i+4}\neq 11101$ $(1\le i\le m-6)$ *if* $k\ge 3$.
2. $y_{m-5}y_{m-4}\cdots y_{m-1}\neq 11101$ if $k\ge 5$.
3. $y_i y_{i+1}\cdots y_{i+6}\neq 1110011$ $(1\le i\le m-6)$ *if* $k\ge 5$.
4. $y_i y_{i+1}\cdots y_{i+5}\neq 000101$ $(1\le i\le m-5)$ *if* $k\ge 6$.
5. $y_i y_{i+1}\cdots y_{i+6}\neq 0001001$ $(1\le i\le m-6)$ *if* $k\ge m-3$.
6. $y_{m-3}y_{m-2}y_{m-1}y_m\neq 1011$ *if* $k\ge 2m-2$.
7. $y_3y_4\cdots y_{m-1}$ *doesn't contain the subsequence* 11 *if* $k\ge 3m-6$.
8. $y_1y_2\cdots y_{m-2}$ *doesn't contain the subsequence* 101 *if* $k\ge 4m-10$.

PROPOSITION 3.5. *For any configuration $x$ of $CA-57_{0-0}(m)$, it follows that $\delta^{4m-10}(x)$ $=$ $(010]^*_{m-2}10$, $(001]^*_{m-2}01$, $(100]^*_{m-2}10$, $(010]^*_{m-2}01$, $(001]^*_{m-2}00$ or $(100]^*_{m-2}11$ and these six configurations make a limit cycle.*

THEOREM 3.6. *$CA-57_{0-0}(m)$ has a unique limit cycle of period length 6 and its transient length is $4m-10$.*

*Proof.* From proposition 3.5, it follows that $H\le 4m-10$. Especially, we set $x=0^m$, then, we have $\delta(x)=1^m$, $\delta^2(x)=10^{m-1}$, $\delta^3(x)=01^{m-1}$ and

$$\delta^l(x)=\begin{cases} 01(011)^{k-1}0^{m-3k+1} & (l=6k-2)\\ 00(110)^{k-1}1^{m-3k+1} & (l=6k-1)\\ 101(011)^{k-1}0^{m-3k} & (l=6k)\\ 01(011)^{k}1^{m-3k-2} & (l=6k+1)\\ 00(110)^{k}0^{m-3k-2} & (l=6k+2)\\ 101(011)^{k}1^{m-3k-3} & (l=6k+3)\end{cases}\quad \text{for } k=1,\,2,\,3,\,\cdots.$$

We can prove the statement by induction on $k$.

(i) In the case $m=3k$

Let $l=6k-1$, then, we have $l=2m-1$ and it follows that $\delta^{2m-1}(x)=00(110)^{k-1}1$, $\delta^{2m}(x)=1010(110)^{k-2}10$, $\delta^{2m+1}(x)=010(110)^{k-2}101$ and $\delta^{2m+2}(x)=00(110)^{k-2}1010$. In the same way, we have

$$\delta^{2m-1+3t}(x)=00(110)^{k-1-t}[10)^*_{m-3(k-t)+1}.$$

When we set $t=k-1$, we have $3t=3k-3=m-3$. Therefore, it follows that

$$\delta^{2m-1+m-3}(x)=\delta^{3m-4}(x)=00[10)^*_{m-2}.$$

Furthermore, we have $\delta^{3m-3}(x)=100[10)^*_{m-3}$, $\delta^{3m-2}(x)=0100[10)^*_{m-4}$ and $\delta^{3m-1}(x)=00100[10)^*_{m-5}$. In the same way, we have

$$\delta^{3m-4+3s}(x)=00(100)^s[10)^*_{m-3s-2}.$$

When we set $s=k-2$, we have $3s=3k-6=m-6$. Therefore, it follows that

$$\begin{aligned}\delta^{3m-4+(m-6)}(x)&=\delta^{4m-10}(x)\\&=00(100)^{k-2}[10)^*_4\\&=0(010)^{k-1}10\\&=(010]^*_{m-2}10.\end{aligned}$$

(ii) In the cases $m = 3k + 1$ or $m = 3k + 2$
We have $\delta^{4m-10}(x) = (010]^*_{m-2}10$ with the discussion similar to item (i).

Therefore, we have $h(x) = 4m - 10 = H$. $\square$

## 4. Behaviors of $CA - 57_{1-1}(m)$

The symmetric reflective rule of the rule 57 is equal to 57. Therefore, the behaviors of $CA - 57_{1-1}(m)$ coincide with the behaviors of $CA - 57_{0-0}(m)$.

THEOREM 4.1. *$CA - 57_{1-1}(m)$ has a unique limit cycle of period length 6 and its transient length is $4m - 10$.*

## 5. Behaviors of $CA - 57_{0-1}(m)$

In this section, we deal with $CA - 57_{0-1}(m)$.

LEMMA 5.1. *Let $\delta(x) = y$ for any configuration $x$ of $CA - 57_{0-1}(m)$. Then, we have $y_{m-2}y_{m-1}y_m \neq 111$, $y_iy_{i+1}\cdots y_{i+5} \neq 111000$ and $y_iy_{i+1}\cdots y_{i+5} \neq 001000$.*

LEMMA 5.2. *For any configuration $x$ of $CA - 57_{0-1}(m)$, we set $\delta^k(x) = y$. Then, the following holds :*

1. $y_{m-1}y_m \neq 00$ *if* $k \geq 2$.
2. $y_iy_{i+1}\cdots y_{i+5} \neq 000111$ *if* $k \geq 2$.
3. $y_iy_{i+1}\cdots y_{i+5} \neq 100111$ *if* $k \geq 2$.
4. $y_1y_2y_3 \neq 011$ *if* $k \geq 4$.
5. $y_{m-2}y_{m-1}y_m \neq 001$ *if* $k \geq 4$.
6. $y_1y_2y_3y_4 \neq 1011$ *if* $k \geq 3$.
7. $y_{m-3}y_{m-2}y_{m-1}y_m \neq 0010$ *if* $k \geq 3$.
8. $y_{m-4}y_{m-3}y_{m-2}y_{m-1}y_m \neq 11010$ *if* $k \geq 7$.

LEMMA 5.3. *For any configuration $x$ of $CA - 57_{0-1}(m)$, $\delta^k(x)$ doesn't contain subsequences 000 and 111 if $k \geq m - 2$.*

LEMMA 5.4. *For any configuration $x$ of $CA - 57_{0-1}(m)$, we set $\delta^k(x) = y$. Then, we have $y_iy_{i+1}y_{i+2}y_{i+3} \neq 1100$ $(1 \leq i \leq m-3)$ if $k \geq m - 1$ and $y_iy_{i+1}\cdots y_{i+6} \neq 1101001$ $(1 \leq i \leq m-7)$ if $k \geq m$.*

LEMMA 5.5. *For any configuration $x$ of $CA - 57_{0-1}(m)$, we set $\delta^k(x) = y$. Then, we have $y_iy_{i+1}\cdots y_{i+5} \neq 101001$ $(2 \leq i \leq m-6)$ and $y_iy_{i+1}\cdots y_{i+4} \neq 11010$ $(3 \leq i \leq m-5)$ where $k \geq 3n - 4$ if $m = 2n$ and $k \geq 3n - 6$ if $m = 2n - 1$.*

We set $\delta^k(x) = y$ where $k \geq 3n - 4$ if $m = 2n$ and $k \geq 3n - 6$ if $m = 2n - 1$, then, we have
$$y = (100]^*_p(10)^q[110)^*_{m-p-2q} \ (p \geq 0,\ q \geq 0)$$
by lemma 5.3, lemma 5.4 and lemma 5.5.

LEMMA 5.6. *For* $y = (100]^*_p(10)^q[110)^*_{m-p-2q}$, *we have*
$$\delta^i(y) = (001]^*_{p+i}(10)^{q-i}[110)^*_{m-p-2q+i}.$$

About the configuration $y = (100]^*_p(10)^q[110)^*_{m-p-2q}$, the following holds :

(i) In the case $q = 0$
We have $y = (100]^*_p[110)^*_{m-p}$, $p \geq 2$ and $m - p \geq 2$ by lemma 3.2 item 2, lemma 5.2 item 1, item 4 and item 5. As $2 \leq p \leq m - 2$, we can construct $m - 3$ $y$'s. Then, we have
$\delta(y) = (010]^*_p[101)^*_{m-p}$, $\delta^2(y) = (001]^*_p[011)^*_{m-p}$ and $\delta^3(y) = (100]^*_p[110)^*_{m-p} = y$.

(ii) In the case $q = 1$
We have $y = (100]^*_p(10)[110)^*_{m-p-2}$, $p \geq 1$ and $m - p - 2 \geq 1$ by lemma 5.2 item 6 and item 7. As $1 \leq p \leq m - 3$, we can construct $m - 3$ $y$'s. Then, we have
$\delta(y) = (010]^*_p(01)[101)^*_{m-p-2}$, $\delta^2(y) = (001]^*_p(01)[011)^*_{m-p-2}$ and
$\delta^3(y) = (100]^*_p(10)[110)^*_{m-p-2} = y$.

(iii) In the case $q = 2$
We have $y = (100]^*_p(10)^2[110)^*_{m-p-4}$, $p \geq 0$ and $m - p - 4 \geq 0$. As $0 \leq p \leq m - 4$, we can construct $m - 3$ $y$'s. Then, we have
$\delta(y) = (010]^*_p(0101)[101)^*_{m-p-4}$, $\delta^2(y) = (001]^*_p(0011)[011)^*_{m-p-4}$ and
$\delta^3(y) = (100]^*_p(1010)[110)^*_{m-p-4} = y$.

(iv) In the case $q \geq 3$
We have $\delta^{q-2}(y) = (100]^*_{p+q-2}(10)^2[110)^*_{m-p-q-2}$ by lemma 5.6. So, this case results in the case (iii).

Moreover, for $y = (100]^*_p[110)^*_{m-p}$, we have the following :
$$\delta(y) = (010]^*_p[101)^*_{m-p} = (100]^*_{p-2}(10)^2[110)^*_{m-p-2},$$
$$\delta^2(y) = (010]^*_{p-1}(0101)[101)^*_{m-p-2} = (100]^*_{p-1}(10)[110)^*_{m-p-1} \text{ and}$$
$$\delta^3(y) = (010]^*_{p-1}(01)[101)^*_{m-p-1} = (100]^*_p[110)^*_{m-p} = y.$$
As $0 \leq p - 2 \leq m - 4$ and $1 \leq p - 1 \leq m - 3$ where $2 \leq p \leq m - 2$, the constructed configurations $y$'s where $q = 0$, $q = 1$ or $q = 2$ make $m - 3$ limit cycles of period length 3.

LEMMA 5.7. *For any configuration* $x$ *of* $CA - 57_{0-1}(m)$, *we have that* $\delta^{3n-5}(x)$ *doesn't contain the subsequence* 11010 *at the left side of the subsequence* 101001 *simultaneously and* $\delta^{3n-4}(x) \neq (10)^n$ *if* $m = 2n$.

For any configuration $x$ of $CA - 57_{0-1}(m)$, $\delta^k(x) = (100]^*_p(10)^q[110)^*_{m-p-2q}$ ($p \geq 0$, $q \geq 0$) where $k \geq 3n - 4$ if $m = 2n$ and $k \geq 3n - 6$ if $m = 2n - 1$. Considering lemma 5.7, for $m = 2n$ and $y = (10)^{n-1}11$, we have $\delta^{n-3}(y) = (001]^*_{n-3}(10)^2[110)^*_{n-1}$ which is a configuration on a limit cycle by lemma 5.5 and $h(x) \leq (3n - 4) + (n - 3) = 4n - 7 = 2m - 7$. For $m = 2n - 1$ and $y = 0(10)^{n-1}$, we have $\delta^{n-3}(y) = (001]^*_{n-2}(10)^2[110)^*_{n-3}$ which is a configuration on a limit cycle by lemma 5.6 and $h(x) \leq (3n - 6) + (n - 3) = 4n - 9 = 2m - 7$.

Therefore, we have that $CA - 57_{0-1}(m)$ has $m - 3$ limit cycles of period length 3 and its transient length is $2m - 7$ or less. Especially, we set $x = 1^m$, then, we have
$$\delta(x) = 10^{m-1},\ \delta^2(x) = 01^{m-2}0,\ \delta^3(x) = 010^{m-3}1$$
and for $i \geq 1$ where $k \geq 4$, we have

$$\delta^k(x) = y = \begin{cases} 00(110)^{i-1}1^{m-6i+2}(001)^{i-1}01 & = y_1^i \ (k = 6i-2) \\ 101(011)^{i-1}0^{m-6i+1}(100)^{i-1}11 & = y_2^i \ (k = 6i-1) \\ 010(110)^{i-1}1^{m-6i}(001)^{i-1}010 & = y_3^i \ (k = 6i) \\ 0(011)^i0^{m-6i-1}(100)^{i-1}101 & = y_4^i \ (k = 6i+1) \\ 1(010)^i1^{m-6i-2}(001)^i1 & = y_5^i \ (k = 6i+2) \\ 01(011)^i0^{m-6i-3}(100)^{i-1}1010 & = y_6^i \ (k = 6i+3). \end{cases}$$

By direct calculations, it holds that $\delta^4(x) = y_1^1$, $\delta(y_1^i) = y_2^i$, $\delta(y_2^i) = y_3^i$, $\delta(y_3^i) = y_4^i$, $\delta(y_4^i) = y_5^i$, $\delta(y_5^i) = y_6^i$ and $\delta(y_6^i) = y_1^{i+1}$.

When $m$ is an even number, we discuss about the following cases :

(i) $m = 2n$ and $n = 3l-2$, (ii) $m = 2n$ and $n = 3l-1$ or (iii) $m = 2n$ and $n = 3l$

First, we discuss about the case (i).

For $y_2^i$, we set $m - 6i + 1 = 1$ i.e. $i = l$. Then, we have $y_2^l = 101(011)^{l-1}0(100)^{l-1}010$, $k = 6l - 1$ and

$$\delta^j(y_2^l) = z = \begin{cases} 010(110)^{l-s}(10)^{3s-2}(100)^{l-1-s}1010 & = z_1^s \ (j = 3s-2) \\ 00(110)^{l-s}(10)^{3s-1}(100)^{l-1-s}101 & = z_2^s \ (j = 3s-1) \\ 1010(110)^{l-1-s}(10)^{3s}(100)^{l-1-s}11 & = z_3^s \ (j = 3s). \end{cases}$$

By direct calculations, it holds that $\delta(y_2^l) = z_1^s, \delta(z_1^s) = z_2^s, \delta(z_2^s) = z_3^s$ and $\delta(z_3^s) = z_1^{s+1}$.

For $z_3^s$, we set $l - 1 - s = 0$ i.e. $s = l - 1$. Then, we have $j = 3s = 3l - 3$ and

$$z_3^s = 1010(10)^{3s}11 = (10)^{3l-1}11 = (10)^{n-1}11$$

where $(6l-1) + (3l-3) = 9l - 4 = 3n - 4$. Therefore, we have $\delta^{3n-4}(x) = (10)^{n-1}11$. Furthermore, we have $\delta^{3n-4+(n-3)}(x) = \delta^{4n-7}(x) = \delta^{2m-7}(x) = (001]^*_{n-3}(10)^2[110)^*_{n-1}$. Both the cases (ii) and (iii) are similar to the case (i). So we have $h(x) = 2m - 7$. When $m$ is an odd number, we have $h(x) = 2m - 7$ similarly.

Finally, we have the following theorem as to $CA - 57_{0-1}(m)$.

THEOREM 5.8. *$CA - 57_{0-1}(m)$ has $m - 3$ limit cycles of period length 3 and its transient length is $2m - 7$.*

## 6. Behaviors of $CA - 57_{1-0}(m)$

In this section, we deal with $CA$–$57_{1-0}(m)$.

PROPOSITION 6.1. *From the rule 57, we have that two configurations $0^m$ and $1^m$ make a limit cycle of period length 2, and that two configurations $[01)^*_m$ and $[10)^*_m$ make a limit cycle of period length 2.*

LEMMA 6.2. *For any configuration $x$ of $CA$–$57_{1-0}(m)$ except $0^m$ and $1^m$, we set $\delta(x) = y$. Then, the following holds :*

1. *If $x_m = a$, $y_m = 1 - a$ ($a = 0$ or 1).*

2. $y_1y_2y_3y_4 \neq 1000$.

LEMMA 6.3. *For $k \geq 2m - 6$, $\delta^k(x)$ doesn't contain subsequences 000 and 111 if $\delta^l(x) \neq 0^m$ and $1^m$ where $0 \leq l \leq k$.*

LEMMA 6.4. *For any configuration $x$ of $CA - 57_{1-0}(m)$ except $0^m$ and $1^m$, $\delta^k(x)$ doesn't contain the subsequence 1100 where $k \geq 2m - 5$.*

LEMMA 6.5. *For any configuration $x$ of $CA$–$57_{1-0}(m)$ except $0^m$ and $1^m$, we set $\delta^k(x) = y$ and $\delta^{k-1}(x) = z$ where $k \geq 2m - 5$. Then, the following holds :*

1. *If* $y_iy_{i+1} = 00$, *we have* $z_{i-1}z_i = 00$ $(2 \le i \le m-1)$.

2. *If* $y_iy_{i+1} = 11$, *we have* $z_{i+1}z_{i+2} = 11$ $(1 \le i \le m-2)$.

LEMMA 6.6. *For any configuration* $x$ *of* $CA - 57_{1-0}(m)$ *except* $0^m$ *and* $1^m$, $\delta^k(x)$ *doesn't contain two subsequences* 00 *and* 00 *where* $k \ge 3m - 11$ *if* $m$ *is an odd number and* $k \ge 3m - 9$ *if* $m$ *is an even number.*

*Proof.* We set $\delta^k(x) = y$. Then, $y$ doesn't contain subsequences 000 and 111. Let $y_iy_{i+1} = 00$ and $y_jy_{j+1} = 00$ $(1 \le i \le m-4,\ i+3 \le j \le m-1)$. Here, we set $\delta^{k-(i-1)}(x) = z$. Then, we have $z_1z_2 = 00$, $z_{j-(i-1)}z_{j+1-(i-1)} = 00$ by using lemma 6.5 item 1 repeatedly. Furthermore, we set $\delta^{k-(i-1)-1}(x) = \delta^{k-i}(x) = w$. Then, we have $w_1w_2 = 11$ and $w_{j-i}w_{j-i+1} = 00$. Here, we set $j - i - 3 = s$.

(i) In the case $s = 2t$ $(t \ge 0)$
We set $\delta^{k-i-t}(x) = u$. By using lemma 6.5, we have $u_{1+t}u_{2+t}u_{3+t}u_{4+t} = 1100$. As $i+t = i + \dfrac{j-i-3}{2} = \dfrac{i+j-3}{2}$ and $5 \le i+j \le 2m-5$, we have $1 \le i+t \le m-4$.

(i-a) When $m$ is an even number
As $k-i-t \ge 3m-9-(m-4) = 2m-5$ from $1 \le i+t \le m-4$, $u_{1+t}u_{2+t}u_{3+t}u_{4+t} = 1100$ contradicts lemma 6.4.

(i-b) When $m$ is an odd number
We have the maximum value $m-4$ of $i+t$ when $i = m-4$ and $j = m-1$. Then, we have $t = 0$, $y_{m-4}y_{m-3}\cdots y_m = 00100$, $u_1u_2u_3u_4 = 1100$ and $u_m = 1$. We set $\delta^{k-\{(m-4)+1\}}(x) = v$, we have $v_1v_2\cdots v_5 = 10000$ or 00010 and $v_m = 0$. But, $v_1v_2\cdots v_5 = 10000$ contradicts lemma 6.2 item 2. So, in the case $v_1v_2\cdots v_5 = 00010$, we have $\delta^{k-\{(m-4)+1+2(m-5)\}}(x) = 0^{m-2}10$ and $\delta^{k-\{(m-4)+1+2(m-5)+1\}}(x) = 1^{m-2}01$ by using lemma 3.1 item 2, item 3 and lemma 6.2 item 1 repeatedly. But this contradicts lemma 3.1 item 6 as $k - \{(m-4) + 1 + 2(m-5) + 1\} = k - (3m-12) \ge 3m - 11 - (3m-12) = 1$.

(ii) In the case $s = 2t - 1$ $(t \ge 1)$
We set $\delta^{k-i-(t-1)}(x) = u$. By using lemma 6.5, we have $u_tu_{1+t}u_{2+t}u_{3+t}u_{4+t} = 11*00$. As $i+t-1 = i + \dfrac{j-i-2}{2} - 1 = \dfrac{i+j-4}{2}$ and $5 \le i+j \le 2m-5$, we have $\dfrac{1}{2} \le i+t \le m - \dfrac{9}{2}$. So, we have $1 \le i+t \le m-5$. Therefore, we have $k-(i+t-1) \ge 3m-11-(m-5) = 2m-6$ where $m$ is an odd number and $k-(i+t-1) \ge 3m-9-(m-5) = 2m-4 \ge 2m-6$ where $m$ is an even number. But, these cases contradict lemma 6.3. □

COROLLARY 6.7. *For any configuration* $x$ *of* $CA - 57_{1-0}(m)$ *except* $0^m$ *and* $1^m$, $\delta^k(x)$ *doesn't contain the subsequence* $aa$ $(a \in \{0,\ 1\})$ *at the left side of the subsequence* $(1-a)(1-a)$ *simultaneously where* $k \ge 3m-10$ *if* $m$ *is an odd number and* $k \ge 3m-8$ *if* $m$ *is an even number.*

LEMMA 6.8. $CA - 57_{1-0}(m)$ *has one limit cycle of period length* $2m-2$ *and two limit cycles of period length* 2.

*Proof.*

(i) For any configuration $x$ of $CA-57_{1-0}(m)$ except $0^m$ and $1^m$, we set $\delta^{3m-10}(x) = y$, where $m$ is an odd number and $\delta^{3m-8}(x) = y$ where $m$ is an even number. By lemma 6.6 and corollary 6.7, we can construct the configuration $y$ as follows :

(i-a) When $y$ contains the subsequence either 00 or 11.
We have $y = (01]^*_k 00[10)^*_{m-k-2}$ or $(10]^*_k 11[01)^*_{m-k-2}$ ($k = 0,\ 1,\ \cdots,\ m-2$). By direct calculations, we have the follows :
$\delta((01]^*_k 00[10)^*_{m-k-2}) = (01]^*_{k+1} 00[10)^*_{m-k-3}$ ($k = 0,\ 1,\ \cdots,\ m-2$), $\delta((01]^*_{m-2} 00[10)^*_0) = (10]^*_{m-3} 11[01)^*_1$, $\delta((10]^*_{m-l-2} 11[01)^*_l = (10]^*_{m-l-3} 11[01)^*_{l+1}$ ($l = 0,\ 1,\ \cdots,\ m-2$) and $\delta((10]^*_0 11[01)^*_{m-2} = (01]^*_0 00[10)^*_{m-2}$. This means that these $2m-2$ configurations make a limit cycle of period length $2m-2$.

(i-b) When $y$ contains the subsequence neither 00 nor 11.
We have $y = [01)^*_m$ or $[10)^*_m$ and these two configurations make a limit cycle of period length 2. (cf. proposition 6.1)

(ii) In the case $x = 0^m$ or $x = 1^m$
These two configurations make a limit cycle of period length 2. (cf. proposition 6.1) □

LEMMA 6.9. *The transient length of $CA-57_{1-0}(m)$ is $3m-10$ where $m$ is an odd number and $3m-8$ where $m$ is an even number.*

*Proof.* When $m$ is an odd number.
By lemma 6.8, we have $H \le 3m-10$. Here, we set $x = 1^{m-2}01$. Then, we have

$$\delta^k(x) = y = \begin{cases} 1^{m-3i-2}(001)^i 01 & = y_1^i \quad (k = 6i) \\ 0^{m-3i-2}(100)^i 10 & = y_2^i \quad (k = 6i+1) \\ 1^{m-3i-3}(001)^i 001 & = y_3^i \quad (k = 6i+2) \\ 0^{m-3i-3}(100)^i 100 & = y_4^i \quad (k = 6i+3) \\ 1^{m-3i-4}(001)^i 0011 & = y_5^i \quad (k = 6i+4) \\ 0^{m-3i-4}(100)^i 1010 & = y_6^i \quad (k = 6i+5) \end{cases} \quad for\ i \ge 0$$

by direct calculations. Moreover, it holds that $\delta(y_1^i) = y_2^i$, $\delta(y_2^i) = y_3^i$, $\delta(y_3^i) = y_4^i$, $\delta(y_4^i) = y_5^i$, $\delta(y_5^i) = y_6^i$ and $\delta(y_6^i) = y_1^{i+1}$. Here, we discuss following cases :
(i) $m = 2n-1$ and $n = 3l-2$, (ii) $m = 2n-1$ and $n = 3l-1$, (iii) $m = 2n-1$ and $n = 3l$.
First, we discuss the case (i-a) i.e. $m = 6l-5$
For $y_2^i$, we set $m-3i-2 = 6l-5-3i-2 = 2$. Then, we have $i = 2l-3$ and $k = 6(2l-3)+1 = 2(6l-5)-7 = 2m-7$. Therefore, we have $\delta^{2m-7}(x) = 00(100)^{2l-3}10$. And, we set $00(100)^{2l-3}10 = z$. Then, we have $\delta^{m-5}(z) = (01]^*_{m-5}00100$, $\delta^{m-4}(z) = (01]^*_{m-4}0011$ and $\delta^{m-3}(z) = (10]^*_{m-4}1010 = [01)^*_m$. So, we have $\delta^{(2m-7)+(m-3)}(x) = \delta^{3m-10}(x) = [01)^*_m$, which is a configuration of the limit cycle of period length 2. (cf. proposition 6.1 item 2)
Both the cases (ii) and (iii) are similar to the case (i). Therefore, we have $h(x) = H = 3m-10$. When $m$ is an even number, we have $H = 3m-8$ similarly. □
Finally, we have the following theorem as to $CA-57_{1-0}(m)$.

THEOREM 6.10. *$CA-57_{1-0}(m)$ has one limit cycle of period length $2m-2$ and two limit cycles of period length 2, and its transient length is $3m-10$, where $m$ is an odd number and $3m-8$, where $m$ is an even number.*

## 7. Conclusions

As conclusive remarks, the following can result : (1) Both $CA - 57_{0-0}(m)$ and $CA - 57_{1-1}(m)$ have a unique limit cycle of period length 6 and their transient length is $4m - 10$, (2) $CA - 57_{0-1}(m)$ has $m - 3$ limit cycles of period length 3 and its transient length is $2m - 7$, (3) $CA - 57_{1-0}(m)$ has one limit cycle of period length $2m - 2$ and two limit cycles of period length 2 and its transient length is $3m - 10$, where $m$ is an odd number and $3m - 8$, where $m$ is an even number.

There are some future tasks. The first is to investigate $CA - 58(m)$. The second is to investigate 1–D CA with threshold rules under free boundary conditions[ Sato and Nishi (2001)]. The last is to investigate 1–D CA with reversible transition functions considered as a special type of quantum cellular automata[Inokuchi et al. (2005)] and 2–D CA with reversible transition functions using von Neumann neighbourhood.

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