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By

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Abstract

We treat two one-dimensional chaotic dynamics derived from both the logistic and the tent transformations from the viewpoint of the theory of stochastic processes. We construct two one-dimensional stochastic processes associated with the above dynamics. Applying the stationarity analysis and the non-linear information analysis in the theory of KM_2O-Langevin equations to them, we investigate the problem of the coexistence of order and chaos from the viewpoint of the fluctuation-dissipation theorem which characterizes the stationarity property of stochastic processes.

Key Words and Phrases: Chaos, Order and Chaos, KM_2O-Langevin equation, Fluctuation-Dissipation Theorem.

1. Introduction

For a given Borel map f from $I = [0, 1]$ into I, we shall consider a discrete dynamical system on the phase space $I = [0, 1]$ defined by

$$x_n = f(x_{n-1}) \qquad (n = 1, 2, 3, \ldots), \tag{1}$$

where x_0 is a given initial point of the phase space I.

By running the initial point x_0 in the phase space I, we can introduce a family $\mathbf{X} = (X(n); n \in \mathbf{N}^*)$ of maps $X(n) : I \to I$ defined by

$$X(n)(x_0) \equiv f^n(x_0) = x_n \qquad (n \in \mathbf{N}^*, x_0 \in I), \tag{2}$$

where $\mathbf{N}^*$ denotes the set $\{0, 1, 2, \ldots\}$.

Moreover, we shall treat the case where the map f has an invariant probability measure μ on the measurable space $(I, \mathcal{B}(I))$, that is, there exists a probability Borel measure μ on the measurable space $(I, \mathcal{B}(I))$ such that

$$\mu(f^{-1}A) = \mu(A) \qquad (\forall A \in \mathcal{B}(I)). \tag{3}$$

Then we can regard the family $\mathbf{X}$ as a one-dimensional stochastic process on the probability space $(I, \mathcal{B}(I), \mu)$. It is to be noted that the stochastic process $\mathbf{X}$ is strictly stationary.

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In this paper, we shall deal with two cases where the map f is a logistic map or a tent map. The discrete dynamical systems derived from these maps have been investigated as typical examples of chaos. The concept of chaos is explained in a sense that the time evolutions of these dynamical systems are deterministic, but it is impossible for us to predict their remote future, that is, there coexist the systematic state, "a state of order" and the random state, " a state of chaos".

The first purpose of this paper is to obtain a mathematical representation for a philosophical concept of the coexistence of order and chaos stated above from the theory of KM_2O-Langevin equations. The second purpose is to find a certain new relation besides the fluctuation-dissipation theorem characterizing the weakly stationarity property of a stochastic process which is represented as the relation among the system of the minimum KM_2O-Langevin matrices associated with the stochastic process.

For that purpose, we shall develop an analysis of chaos based upon the theory of KM_2O-Langevin equations, to be called a chaos analysis. We shall explain its idea. Taking the standardization of the stochastic process $\mathbf{X}$, we define a one-dimensional stochastic process $\mathbf{W} = (W(n); n \in \mathbf{N}^*)$ by

$$W(n) \equiv \frac{1}{\sqrt{V(X(n))}}(X(n) - E(X(n))) \qquad (n \in \mathbf{N}^*). \tag{4}$$

Moreover, for each $n \in \mathbf{N}^*$, we define the non-linear information space $\mathbf{N}_0^n(\mathbf{W})$ and the linear information space $\mathbf{M}_0^n(\mathbf{W})$ as follows:

$$\mathbf{N}_0^n(\mathbf{W}) \equiv L^2(I, \mathcal{B}_0^n(\mathbf{W}), \mu), \tag{5}$$

$$\mathbf{M}_0^n(\mathbf{W}) \equiv \{\sum_{k=0}^{n} c_k W(k); c_k \in \mathbf{R} \ (0 \le k \le n)\}, \tag{6}$$

where $\mathcal{B}_0^n(\mathbf{W})$ stands for the smallest σ-field with respect to which all random variables $W(k)$ $(0 \le k \le n)$ are measurable.

If we project the random variable $W(n+1)$ on the non-linear information space $\mathbf{N}_0^n(\mathbf{W})$, then we have

$$P_{\mathbf{N}_0^n(\mathbf{W})} W(n+1) = W(n+1) \qquad (n \in \mathbf{N}^*). \tag{7}$$

This implies that when we project the random variable $W(n+1)$ on the non-linear information space $\mathbf{N}_0^n(\mathbf{W})$, we can represent a certain existence of order, but we cannot represent any existence of chaos of the stochastic process $\mathbf{W}$.

On the other hand, for the case where the map f is logistic or tent, if we project the random variable $W(n+1)$ on the linear information space $\mathbf{M}_0^n(\mathbf{W})$, then we have

$$P_{\mathbf{M}_0^n(\mathbf{W})} W(n+1) = 0 \qquad (n \in \mathbf{N}^*). \tag{8}$$

This implies that when we project the random variable $W(n+1)$ on the linear information space $\mathbf{M}_0^n(\mathbf{W})$, we can represent a certain existence of chaos, but we cannot represent any existence of order for the stochastic process $\mathbf{W}$.

Therefore, the problem of coexistence of order and chaos is reduced to the following problem: When we project the random variable $W(n+1)$ on what kinds of closed subspace $\mathbf{P}_0^n(\mathbf{W})$ such that

$$\mathbf{M}_0^n(\mathbf{W}) \subset \mathbf{P}_0^n(\mathbf{W}) \subset \mathbf{N}_0^n(\mathbf{W}) \qquad (n \in \mathbf{N}^*), \tag{9}$$

how can we investigate the problem of coexistence of order and chaos for the stochastic process $\mathbf{W}$?

We shall state the contents of this paper. In Section 2, Section 3 and Section 4, we shall review the theory of $\mathrm{KM_2O}$-Langevin equations for any d-dimensional degenerate flow $\mathbf{Z} = (Z(n); 0 \leq n \leq N)$ in a real inner product space W with an inner product $(*, \star)$. In Section 2, in particular, we shall recall the theory of weight transformations and show a new theorem (Theorem 2.3) in which the speed of convergence of $\mathrm{KM_2O}$-Langevin dissipation matrices can be estimated.

In Section 3, we shall review the theory of stationarity analysis and give an algorithm (fluctuation-dissipation algorithm) for obtaining all the elements of the set $\mathcal{LM}(R)$ of $\mathrm{KM_2O}$-Langevin matrices from the covariance matrix function R of a degenerate stationary flow.

In Section 4, we shall recall the theory of non-linear information spaces associated with any d-dimensional local stochastic process and construct a generating system of the non-linear information spaces (Theorem 4.1).

In Section 5 and Section 6, we shall treat the stochastic processes associated with the logistic map and the tent map, respectively, and investigate the problem of coexistence of order and chaos by using 18 kinds of two-dimensional stochastic processes taken out from the non-linear transformations of rank 6 introduced in Section 4. Moreover, we shall prove in Theorem 5.5 that the inequalities in Theorem 2.3 are tight.

In Section 7, we shall discuss the results in Sections 5 and 6.

2. $\mathrm{KM_2O}$-Langevin equations

In this section, we shall review the theory of $\mathrm{KM_2O}$-Langevin equations for general flows in a real inner product space (Matsuura and Okabe (2001), Okabe (1999, 2002)). Let $(W, (\star, *))$ be any real inner product space with an inner product $(\star, *)$. By a d-dimensional flow $\mathbf{Z} = (Z(n); \ell \leq n \leq r)$ in W, we mean a function $Z : \{\ell, \ell+1, \ldots, r-1, r\} \to W^d$, where d, ℓ and r $(d \geq 1, \ell \leq r)$ are integers. A d-dimensional flow $\mathbf{Z} = (Z(n); \ell \leq n \leq r)$ is said to be non-degenerate if $\{Z_j(n); 1 \leq j \leq d, \ell \leq n \leq r\}$ is linearly independent in W, where $Z_j(n)$ is the jth component of $Z(n)$. Otherwise, $\mathbf{Z}$ is said to be degenerate. For two integers n_1 and n_2 $(\ell \leq n_1 \leq n_2 \leq r)$, we define a closed subspace $\mathbf{M}_{n_1}^{n_2}(\mathbf{Z})$ of W by

$$\mathbf{M}_{n_1}^{n_2}(\mathbf{Z}) \equiv [\{Z_j(m); 1 \leq j \leq d, n_1 \leq m \leq n_2\}], \tag{10}$$

where for any subset S of W, we denote by $[S]$ the closed subspace of W which is generated by all elements in S.

For a given d-dimensional flow $\mathbf{Z} = (Z(n); 0 \leq n \leq N)$, we define two d-dimensional flows $\mathbf{Z}_+ = (Z_+(n); 0 \leq n \leq N)$ and $\mathbf{Z}_- = (Z_-(\ell); -N \leq \ell \leq 0)$ by

$$Z_+(n) = Z(n) \qquad (0 \leq n \leq N), \tag{11}$$

$$Z_-(\ell) = Z(N+\ell) \qquad (-N \leq \ell \leq 0). \tag{12}$$

We call the pair of flows $[\mathbf{Z}_+, \mathbf{Z}_-]$ the natural pair of flows. Then, we derive a new d-dimensional flow $\nu_+(\mathbf{Z}) = (\nu_+(\mathbf{Z})(n); 0 \leq n \leq N)$ (resp. $\nu_-(\mathbf{Z}) = (\nu_-(\mathbf{Z})(\ell); -N \leq \ell \leq 0)$) by projecting each component of $Z_+(n)$ (resp. $Z_-(\ell)$) onto the subspace $\mathbf{M}_0^{n-1}(\mathbf{Z}_+)$ (resp. $\mathbf{M}_{\ell+1}^{0}(\mathbf{Z}_-)$), that is,

$$\nu_+(\mathbf{Z})(0) \equiv Z_+(0), \tag{13}$$

$$\nu_+(\mathbf{Z})(n) \equiv Z_+(n) - P_{\mathbf{M}_0^{n-1}(\mathbf{Z}_+)} Z_+(n) \qquad (1 \le n \le N), \tag{14}$$

$$\nu_-(\mathbf{Z})(0) \equiv Z_-(0), \tag{15}$$

$$\nu_-(\mathbf{Z})(\ell) \equiv Z_-(\ell) - P_{\mathbf{M}_{\ell+1}^0(\mathbf{Z}_-)} Z_-(\ell) \qquad (-N \le \ell \le -1). \tag{16}$$

We call the flow $\nu_+(\mathbf{Z})$ (resp. $\nu_-(\mathbf{Z})$) the forward (resp. backward) $\mathrm{KM_2O}$-Langevin fluctuation flow associated with the flow $\mathbf{Z}$. The forward (resp. backward) $\mathrm{KM_2O}$-Langevin fluctuation matrix function $V_+(\mathbf{Z}) = (V_+(\mathbf{Z})(n); 0 \le n \le N)$ (resp. $V_-(\mathbf{Z}) = (V_-(\mathbf{Z})(n); 0 \le n \le N)$) associated with the flow $\mathbf{Z}$ is defined by

$$V_\pm(\mathbf{Z})(n) \equiv (\nu_\pm(\mathbf{Z})(\pm n), {}^t\nu_\pm(\mathbf{Z})(\pm n)) \qquad (0 \le n \le N), \tag{17}$$

where $(\star, {}^t*)$ denotes the inner product matrix of order d of the vectors $\star$ and $*$ in W.

Furthermore, there exist two matrix functions $\gamma_+ = (\gamma_+(n,k); 0 \le k < n \le N)$ and $\gamma_- = (\gamma_-(n,k); 0 \le k < n \le N)$ such that

$$P_{\mathbf{M}_0^{n-1}(\mathbf{Z}_+)} Z_+(n) = -\sum_{k=0}^{n-1} \gamma_+(n,k) Z_+(k) \qquad (1 \le n \le N), \tag{18}$$

$$P_{\mathbf{M}_{-n+1}^0(\mathbf{Z}_-)} Z_-(-n) = -\sum_{k=0}^{n-1} \gamma_-(n,k) Z_-(-k) \qquad (1 \le n \le N). \tag{19}$$

In general, these matrix functions are not uniquely determined. We denote by $\mathcal{LMD}_+(\mathbf{Z})$ the set of all matrix functions γ_+ for which (18) holds and by $\mathcal{LMD}_-(\mathbf{Z})$ the set of all matrix functions γ_- for which (19) holds. Any element of $\mathcal{LMD}_+(\mathbf{Z})$ (resp. $\mathcal{LMD}_-(\mathbf{Z})$) is called a forward (resp. backward) $\mathrm{KM_2O}$-Langevin dissipation matrix function associated with the flow $\mathbf{Z}$.

We note that if the flow $\mathbf{Z}$ is non-degenerate, the matrix function γ_+ (resp. γ_-) is uniquely determined only through relation (18) (resp. (19)). To find a constructive and efficient way to obtain appropriate $\mathrm{KM_2O}$-Langevin dissipation matrix functions for degenerate flows, we have analyzed weight transformation in Matsuura and Okabe (2001). Let $\boldsymbol{\xi} = (\xi(n); 0 \le n \le N)$ be any non-degenerate d-dimensional flow in W such that

$$(Z(m), {}^t\xi(n)) = 0 \quad \text{and} \quad (\xi(m), {}^t\xi(n)) = \delta_{mn} I_d \qquad (0 \le m, n \le N), \tag{20}$$

where I_d denotes the identity matrix of order d.

For each $w > 0$, we define a d-dimensional flow $\mathbf{Z}^w = (Z^w(n); 0 \le n \le N)$ in W by

$$Z^w(n) \equiv Z(n) + w\,\xi(n). \tag{21}$$

This transformation from $\mathbf{Z}$ into $\mathbf{Z}^w$ is called a weight transformation with the weight w, and $\boldsymbol{\xi}$ is called an additive white noise flow for the flow $\mathbf{Z}$. Furthermore, we define a norm $||\gamma_+||$ (resp. $||\gamma_-||$) on the set $\mathcal{LMD}_+(\mathbf{Z})$ (resp. $\mathcal{LMD}_-(\mathbf{Z})$) by

$$||\gamma_\pm|| \equiv (\sum_{n=1}^{N} \sum_{k=0}^{n-1} \sum_{j=1}^{d} \sum_{\ell=1}^{d} \gamma_{\pm j\ell}(n,k)^2)^{1/2},$$

where $\gamma_{\pm j\ell}(n,k)$ denotes the (j,ℓ)th component of $\gamma_\pm(n,k)$.

The following theorem has been proved in Matsuura and Okabe (2001).

Theorem 2.1 (Matsuura and Okabe, 2001) *There exist matrix functions* $\gamma_+^0(\mathbf{Z})$ *and* $\gamma_-^0(\mathbf{Z})$ *that satisfy*

(i) $\gamma_+^0(\mathbf{Z}) \in \mathcal{LMD}_+(\mathbf{Z})$ *and* $\gamma_-^0(\mathbf{Z}) \in \mathcal{LMD}_-(\mathbf{Z})$;

(ii) *for any elements* γ_+ *of* $\mathcal{LMD}_+(\mathbf{Z})$ *and* γ_- *of* $\mathcal{LMD}_-(\mathbf{Z})$ *such that* $\gamma_+ \neq \gamma_+^0(\mathbf{Z})$ *and* $\gamma_- \neq \gamma_-^0(\mathbf{Z})$,

$$||\gamma_+^0(\mathbf{Z})|| < ||\gamma_+|| \quad \textit{and} \quad ||\gamma_-^0(\mathbf{Z})|| < ||\gamma_-||.$$

The following lemma and theorem have been proved in Matsuura and Okabe (2001).

Lemma 2.1 (Matsuura and Okabe, 2001) *For any* $w > 0$, *the flow* $\mathbf{Z}^w$ *is non-degenerate.*

It follows immediately from Lemma 2.1 that for each $w > 0$, there exist uniquely two matrix functions $\gamma_\pm(\mathbf{Z}^w) = (\gamma_\pm(\mathbf{Z}^w)(n,k); 0 \le k < n \le N)$ such that

$$\mathcal{LMD}_+(\mathbf{Z}^w) = \{\gamma_+(\mathbf{Z}^w)\} \quad \text{and} \quad \mathcal{LMD}_-(\mathbf{Z}^w) = \{\gamma_-(\mathbf{Z}^w)\}. \tag{22}$$

Theorem 2.2 (Matsuura and Okabe, 2001) *For each* m, n, k $(0 \le m \le N, 0 \le k < n \le N)$,

(i) $\lim_{w\to 0} \nu_+(\mathbf{Z}^w)(m) = \nu_+(\mathbf{Z})(m)$ *and* $\lim_{w\to 0} \nu_-(\mathbf{Z}^w)(-m) = \nu_-(\mathbf{Z})(-m)$,

(ii) $\lim_{w\to 0} V_+(\mathbf{Z}^w)(m) = V_+(\mathbf{Z})(m)$ *and* $\lim_{w\to 0} V_-(\mathbf{Z}^w)(m) = V_-(\mathbf{Z})(m)$,

(iii) $\lim_{w\to 0} \gamma_+(\mathbf{Z}^w)(n,k) = \gamma_+^0(\mathbf{Z})(n,k)$ *and* $\lim_{w\to 0} \gamma_-(\mathbf{Z}^w)(n,k) = \gamma_-^0(\mathbf{Z})(n,k)$.

We call the matrix function $\gamma_+^0(\mathbf{Z}) = (\gamma_+^0(\mathbf{Z})(n,k); 0 \le k < n \le N)$ (resp. $\gamma_-^0(\mathbf{Z}) = (\gamma_-^0(\mathbf{Z})(n,k); 0 \le k < n \le N)$) the minimum forward (resp. backward) KM$_2$O-Langevin dissipation matrix function associated with the flow $\mathbf{Z}$.

After the above preparations, we can introduce a system $\mathcal{LM}(\mathbf{Z})$ of matrices of order d by

$$\mathcal{LM}(\mathbf{Z}) \equiv \{\gamma_+^0(\mathbf{Z})(n,k), \gamma_-^0(\mathbf{Z})(n,k), V_+(\mathbf{Z})(m), V_-(\mathbf{Z})(m); \\ 0 \le k < n \le N, 0 \le m \le N\} \tag{23}$$

and call it the system of the minimum KM$_2$O-Langevin matrices associated with the flow $\mathbf{Z}$.

From the theorems stated above, we can derive the following forward (resp. backward) KM$_2$O-Langevin equation (25) (resp. (27)) with (24) (resp. (26)) associated with the flow $\mathbf{Z}$:

$$Z(0) = \nu_+(\mathbf{Z})(0), \tag{24}$$

$$Z(n) = -\sum_{k=0}^{n-1} \gamma_+^0(\mathbf{Z})(n,k)Z(k) + \nu_+(\mathbf{Z})(n) \qquad (1 \le n \le N), \tag{25}$$

$$Z(N) = \nu_-(\mathbf{Z})(0), \tag{26}$$

$$Z(N-n) = -\sum_{k=0}^{n-1} \gamma_-^0(\mathbf{Z})(n,k)Z(N-k) + \nu_-(\mathbf{Z})(-n) \qquad (1 \le n \le N). \tag{27}$$

Finally, in order to prove Theorem 2.3 concerning the speed of convergence of KM$_2$O-Langevin dissipation matrices in Theorem 2.2, we shall introduce some notations. Let $R_+(\mathbf{Z}) = (R_+(\mathbf{Z})(m,n); 0 \le m,n \le N)$ and $R_-(\mathbf{Z}) = (R_-(\mathbf{Z})(\ell,k); -N \le \ell,k \le 0)$ be the covariance matrix functions for the d-dimensional flows $\mathbf{Z}_+$ and $\mathbf{Z}_-$ in W:

$$R_\pm(\mathbf{Z})(\pm m, \pm n) \equiv (Z_\pm(\pm m),\ {}^t Z_\pm(\pm n)) \qquad (0 \le m,n \le N). \tag{28}$$

Next, for any n $(1 \le n \le N)$, we define four matrices $\Gamma_+(\mathbf{Z})(n), \Gamma_-(\mathbf{Z})(n), S_+(\mathbf{Z})(n)$ and $S_-(\mathbf{Z})(n)$ of (nd,d)-type by

$$\begin{aligned} \Gamma_\pm(\mathbf{Z})(n) &\equiv {}^t(\gamma^0_\pm(\mathbf{Z})(n,0), \gamma^0_\pm(\mathbf{Z})(n,1), \ldots, \gamma^0_\pm(\mathbf{Z})(n,n-1)), & (29)\\ S_\pm(\mathbf{Z})(n) &\equiv {}^t(R_\pm(\mathbf{Z})(\pm n,0), R_\pm(\mathbf{Z})(\pm n,\pm 1), \ldots, R_\pm(\mathbf{Z})(\pm n, \pm(n-1))). & (30) \end{aligned}$$

Moreover, we define two matrices $T_+(\mathbf{Z})(n)$ and $T_-(\mathbf{Z})(n)$ of order nd by

$$\begin{aligned} &T_\pm(\mathbf{Z})(n) \\ &= \begin{pmatrix} R_\pm(\mathbf{Z})(0,0) & R_\pm(\mathbf{Z})(0,\pm 1) & \ldots & R_\pm(\mathbf{Z})(0,\pm(n-1)) \\ R_\pm(\mathbf{Z})(\pm 1,0) & R_\pm(\mathbf{Z})(\pm 1,\pm 1) & \ldots & R_\pm(\mathbf{Z})(\pm 1,\pm(n-1)) \\ \vdots & \vdots & \ddots & \vdots \\ R_\pm(\mathbf{Z})(\pm(n-1),0) & R_\pm(\mathbf{Z})(\pm(n-1),\pm 1) & \cdots & R_\pm(\mathbf{Z})(\pm(n-1),\pm(n-1)) \end{pmatrix}. \end{aligned} \tag{31}$$

Similarly, for each $w > 0$, we can introduce four kinds of matrix functions $R_\pm(\mathbf{Z}^w) = (R_\pm(\mathbf{Z}^w)(\pm m, \pm n); 0 \le m,n \le N), \Gamma_\pm(\mathbf{Z}^w) = (\Gamma_\pm(\mathbf{Z}^w)(n); 1 \le n \le N), S_\pm(\mathbf{Z}^w) = (S_\pm(\mathbf{Z}^w)(n); 1 \le n \le N)$ and $T_\pm(\mathbf{Z}^w) = (T_\pm(\mathbf{Z}^w)(n); 1 \le n \le N)$.

Immediately, we have

Lemma 2.2 *For each $w > 0$,*

(i) $R_\pm(\mathbf{Z}^w)(\pm m, \pm n) = R_\pm(\mathbf{Z})(\pm m, \pm n) + w^2 \delta_{mn} I_d \quad (0 \le m,n \le N)$,

(ii) $S_\pm(\mathbf{Z}^w)(n) = S_\pm(\mathbf{Z})(n) \quad (1 \le n \le N)$,

(iii) $T_\pm(\mathbf{Z}^w)(n) = T_\pm(\mathbf{Z})(n) + w^2 I_{nd} \quad (1 \le n \le N)$.

Further, we shall prove the following.

Lemma 2.3 *For each n $(1 \le n \le N)$, $T_\pm(\mathbf{Z})(n)\Gamma_\pm(\mathbf{Z})(n) = S_\pm(\mathbf{Z})(n)$.*

Proof. For each ℓ $(0 \le \ell \le n-1)$, taking the inner product of the both-hand sides of the forward KM$_2$O-Langevin equation (25) by the vector $Z_+(\ell)$, we have

$$R_+(\mathbf{Z})(n,\ell) = -\sum_{k=0}^{n-1} \gamma^0_+(\mathbf{Z})(n,k) R_+(\mathbf{Z})(k,\ell). \tag{32}$$

Using three matrices $\Gamma_+(\mathbf{Z})(n), S_+(\mathbf{Z})(n)$ and $T_+(\mathbf{Z})(n)$ in (29), (30) and (31), we find that the plus part comes from (32). Similarly, the minus part is proved. (Q.E.D.)

Theorem 2.3 *For any $w > 0$ and any n $(1 \le n \le N)$, the following hold*:

(i) $\dfrac{\|\Gamma_+(\mathbf{Z}^w)(n) - \Gamma_+(\mathbf{Z})(n)\|}{\|\Gamma_+(\mathbf{Z})(n)\|} \le \dfrac{w^2}{w^2 + \lambda_+(\mathbf{Z})(n)} \quad (\|\Gamma_+(\mathbf{Z})(n)\| \ne 0)$,

(ii) $\dfrac{\|\Gamma_-(\mathbf{Z}^w)(n) - \Gamma_-(\mathbf{Z})(n)\|}{\|\Gamma_-(\mathbf{Z})(n)\|} \le \dfrac{w^2}{w^2 + \lambda_-(\mathbf{Z})(n)} \quad (\|\Gamma_-(\mathbf{Z})(n)\| \neq 0),$

where the symbol $\|A\|$ stands for the Euclidean norm of the matrix A, and $\lambda_+(\mathbf{Z})(n)$ and $\lambda_-(\mathbf{Z})(n)$ are the minimums of positive eigenvalues of the matrices $T_+(\mathbf{Z})(n)$ and $T_-(\mathbf{Z})(n)$, respectively.

Proof. We prove only (i). It follows from Lemmas 2.2(ii) and 2.3 that

$$\Gamma_+(\mathbf{Z}^w)(n) - \Gamma_+(\mathbf{Z})(n) = (T_+(\mathbf{Z}^w)(n)^{-1}T_+(\mathbf{Z})(n) - I)\Gamma_+(\mathbf{Z})(n). \tag{33}$$

Since the matrix $T_+(\mathbf{Z})(n)$ is symmetric and non-negative definite, there exists an orthogonal matrix U such that

$$T_+(\mathbf{Z})(n) = U\mathrm{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_r, 0, \ldots, 0)U^{-1},$$

where λ_k $(1 \le k \le r)$ are positive eigenvalues of the matrix $T_+(\mathbf{Z})(n)$. It follows from Lemma 2.2(iii) that for any $w > 0$,

$$T_+(\mathbf{Z}^w)(n)^{-1} = U\mathrm{Diag}(\frac{1}{w^2+\lambda_1}, \frac{1}{w^2+\lambda_2}, \ldots, \frac{1}{w^2+\lambda_r}, \frac{1}{w^2}, \ldots, \frac{1}{w^2})U^{-1}.$$

Substituting this into (33), we have

$$\Gamma_+(\mathbf{Z}^w)(n) - \Gamma_+(\mathbf{Z})(n) = -U\mathrm{Diag}(\frac{w^2}{w^2+\lambda_1}, \frac{w^2}{w^2+\lambda_2}, \ldots, \frac{w^2}{w^2+\lambda_r}, 1, \ldots, 1)U^{-1}\Gamma_+(\mathbf{Z})(n).$$

On the other hand, noting that the matrix $\Gamma_+(\mathbf{Z})(n)$ can be determined from the minimum norm property, we can find that all rows after the rth row of the matrix $U^{-1}\Gamma_+(\mathbf{Z})(n)$ are 0 ((4.39) in Matsuura and Okabe (2001)). Therefore, we have

$$\Gamma_+(\mathbf{Z}^w)(n) - \Gamma_+(\mathbf{Z})(n) = -U\mathrm{Diag}(\frac{w^2}{w^2+\lambda_1}, \frac{w^2}{w^2+\lambda_2}, \ldots, \frac{w^2}{w^2+\lambda_r}, 0, \ldots, 0)U^{-1}\Gamma_+(\mathbf{Z})(n)$$

and so

$$\frac{\|\Gamma_+(\mathbf{Z}^w)(n) - \Gamma_+(\mathbf{Z})(n)\|}{\|\Gamma_+(\mathbf{Z})(n)\|} \le \frac{w^2}{w^2 + \lambda_+(\mathbf{Z})(n)} \qquad (\|\Gamma_+(\mathbf{Z})(n)\| \neq 0),$$

which proves (i). (Q.E.D.)

We shall find in Section 5 that the inequalities in Theorem 2.3 are tight.

3. Stationarity property and Fluctuation-Dissipation Theorem

In this section, we shall recall the results of stationarity analysis in the theory of KM$_2$O-Langevin equations.

[3.1] We first recall the definition of stationarity property for flows in a real inner product space W (Okabe (1999)). Let $\mathbf{Z} = (Z(n); 0 \le n \le N)$ be any d-dimensional flow in W. We say that the flow $\mathbf{Z}$ has stationarity property if there exists a covariance matrix function $R = (R(n); |n| \le N)$ of $\mathbf{Z}$ such that

$$R(\mathbf{Z})(m, n) = R(m - n) \qquad (0 \le m, n \le N). \tag{34}$$

We are now going to state the fluctuation-dissipation theorem for stationary flows.

Theorem 3.1 (Matsuura and Okabe, 2001) *The flow* $\mathbf{Z}$ *is stationary if and only if the followings hold*:

$$\text{(i)} \begin{cases} \gamma_+^0(\mathbf{Z})(n,k) = \gamma_+^0(\mathbf{Z})(n-1,k-1) + \delta_+^0(\mathbf{Z})(n)\gamma_-^0(\mathbf{Z})(n-1,n-k-1) \\ \hfill (1 \le k < n \le N), \\ \gamma_-^0(\mathbf{Z})(n,k) = \gamma_-^0(\mathbf{Z})(n-1,k-1) + \delta_-^0(\mathbf{Z})(n)\gamma_+^0(\mathbf{Z})(n-1,n-k-1) \\ \hfill (1 \le k < n \le N), \end{cases}$$

$$\text{(ii)} \begin{cases} V_+(\mathbf{Z})(0) = V_-(\mathbf{Z})(0), \\ V_+(\mathbf{Z})(n) = (I_d - \delta_+^0(\mathbf{Z})(n)\delta_-^0(\mathbf{Z})(n))V_+(\mathbf{Z})(n-1) \qquad (1 \le n \le N), \\ V_-(\mathbf{Z})(n) = (I_d - \delta_-^0(\mathbf{Z})(n)\delta_+^0(\mathbf{Z})(n))V_-(\mathbf{Z})(n-1) \qquad (1 \le n \le N), \end{cases}$$

$$\text{(iii)} \ \delta_+^0(\mathbf{Z})(n)V_-(\mathbf{Z})(n-1) = V_+(\mathbf{Z})(n-1)\, {}^t\delta_-^0(\mathbf{Z})(n) \qquad (1 \le n \le N),$$

where $\delta_+^0(\mathbf{Z})(n) \equiv \gamma_+^0(\mathbf{Z})(n,0)$ *and* $\delta_-^0(\mathbf{Z})(n) \equiv \gamma_-^0(\mathbf{Z})(n,0)$.

We call the matrix function $\delta_+^0(\mathbf{Z}) = (\delta_+^0(\mathbf{Z})(n); 1 \le n \le N)$ (resp. $\delta_-^0(\mathbf{Z}) = (\delta_-^0(\mathbf{Z})(n); 1 \le n \le N)$) the minimum forward (resp. backward) $\mathrm{KM_2O}$-Langevin partial autocorrelation matrix function associated with the flow $\mathbf{Z}$. The algorithms in Theorems 3.1(i) and (ii),(iii) are said to be (DDT) and (FDT), respectively (Okabe (1999)).

[3.2] We shall give the algorithm for calculating the minimum $\mathrm{KM_2O}$-Langevin matrices associated with the flow $\mathbf{Z}$ from the covariance matrix function R.

For that purpose, we define a d-dimensional flow $\mathbf{Z}^w$ in (21). Then, we define for any $w > 0$ and any n $(0 \le n \le N)$ a subsystem $\mathcal{LM}(\mathbf{Z}^w; n)$ of the $\mathrm{KM_2O}$-Langevin matrix $\mathcal{LM}(\mathbf{Z}^w)$ by

$$\mathcal{LM}(\mathbf{Z}^w; n) \equiv \{\gamma_+(\mathbf{Z}^w)(m,k), \gamma_-(\mathbf{Z}^w)(m,k), V_+(\mathbf{Z}^w)(\ell), V_-(\mathbf{Z}^w)(\ell); \\ 0 \le k < m \le n, 0 \le \ell \le n\}. \tag{35}$$

We note that

$$R(\mathbf{Z}^w)(n) = R(n) + w^2\delta_{n0}I_d \qquad (0 \le n \le N), \tag{36}$$

where $R(\mathbf{Z}^w)$ is the covariance matrix function of the flow $\mathbf{Z}^w$.

It follows from Theorem 3.1(i) and (ii) for the flow $\mathbf{Z}^w$ that for any $w > 0$ and any n $(1 \le n \le N)$, the matrices $\gamma_+(\mathbf{Z}^w)(n,k), \gamma_-(\mathbf{Z}^w)(n,k), V_+(\mathbf{Z}^w)(n)$ and $V_-(\mathbf{Z}^w)(n)$ can be calculated from the matrices $\delta_+(\mathbf{Z}^w)(n), \delta_-(\mathbf{Z}^w)(n)$ and the system $\mathcal{LM}(\mathbf{Z}^w; n-1)$ with $V_\pm(\mathbf{Z}^w)(0) = R(0) + w^2 I_d$ $(0 \le k < n)$, where $\delta_\pm(\mathbf{Z}^w)(n) = \gamma_\pm(\mathbf{Z}^w)(n,0)$. Therefore, we have only to obtain an algorithm by which the matrices $\delta_+(\mathbf{Z}^w)(n)$ and $\delta_-(\mathbf{Z}^w)(n)$ can be calculated from the system $\mathcal{LM}(\mathbf{Z}^w; n-1)$ and the matrices $R(m)$ $(0 \le m \le n)$. Applying Theorem 6.1 of Okabe (1993a), to be called (PAC) in Okabe (1993b), to the non-degenerate flow $\mathbf{Z}^w$, we see that for any positive weight w and any n $(1 \le n \le N)$,

$$\delta_\pm(\mathbf{Z}^w)(n) = -\{R(\pm n) + \sum_{k=0}^{n-2} \gamma_\pm(\mathbf{Z}^w)(n-1,k)R(\pm(k+1))\}V_\mp(\mathbf{Z}^w)(n-1)^{-1}. \tag{37}$$

On the other hand, it follows from Theorem 2.2 that

$$\lim_{w\to 0} \delta_+(\mathbf{Z}^w)(n) = \delta_+^0(\mathbf{Z})(n) \quad \text{and} \quad \lim_{w\to 0} \delta_-(\mathbf{Z}^w)(n) = \delta_-^0(\mathbf{Z})(n) \quad (1 \le n \le N). \tag{38}$$

Thus we have obtained the algorithm for calculating the minimum $\mathrm{KM_2O}$-Langevin matrix from the covariance matrix function. We shall call it the fluctuation-dissipation algorithm.

4. Non-linear information spaces

In this section, we shall recall the results of the non-linear information spaces for one-dimensional stochastic processes by rearranging the results in Matsuura and Okabe (2001) and Okabe and Kaneko (2000).

Let $\mathbf{Z} = (Z(n); n \in \mathbf{N}^*)$ be a one-dimensional stochastic process defined on a probability space $(\Omega, \mathcal{B}, P)$ satisfying the following condition (E):

(E) For any $n \in \mathbf{N}^*$, there exists $\lambda_0 > 0$ such that for any $\lambda \in \mathbf{R}$ $(|\lambda| \leq \lambda_0)$,

$$E(\exp\{\lambda Z(n)\}) < \infty.$$

[4.1] For any n_1, n_2 $(0 \leq n_1 \leq n_2 < \infty)$, we define two closed subspaces $\mathbf{M}_{n_1}^{n_2}(\mathbf{Z})$ and $\mathbf{N}_{n_1}^{n_2}(\mathbf{Z})$ of $L^2(\Omega, \mathcal{B}, P)$ by

$$\mathbf{M}_{n_1}^{n_2}(\mathbf{Z}) \equiv [\{Z(m); n_1 \leq m \leq n_2\}], \tag{39}$$

$$\mathbf{N}_{n_1}^{n_2}(\mathbf{Z}) \equiv \{Y \in L^2(\Omega, \mathcal{B}, P); Y \text{ is } \mathcal{B}_{n_1}^{n_2}(\mathbf{Z})\text{-measurable}\}, \tag{40}$$

where for any subset S of $L^2(\Omega, \mathcal{B}, P)$, we denote by $[S]$ the closed subspace of $L^2(\Omega, \mathcal{B}, P)$ which is generated by all elements in S and by $\mathcal{B}_{n_1}^{n_2}(\mathbf{Z})$ the smallest σ-field with respect to which all random variables $Z(m)$ $(n_1 \leq m \leq n_2)$ are measurable. We call $\mathbf{M}_{n_1}^{n_2}(\mathbf{Z})$ (resp. $\mathbf{N}_{n_1}^{n_2}(\mathbf{Z})$) linear (resp. non-linear) information spaces associated with the stochastic process $\mathbf{Z}$.

As noted in Dobrushin and Minlos (1977), it follows from condition (E) that

Lemma 4.1 (i) *For any integer $k \in \mathbf{N}^*$, $Z(k) \in \bigcap_{1 \leq p < \infty} L^p(\Omega, \mathcal{B}, P)$.*

(ii) *For any integers $n, p_k \in \mathbf{N}^*$ $(0 \leq k \leq n)$, $\prod_{k=0}^{n} Z(k)^{p_k} \in \mathbf{N}_0^n(\mathbf{Z})$.*

[4.2] (Generator of non-linear information spaces) We shall construct a generator of the non-linear information spaces by introducing a system of multi-dimensional stochastic processes.

[a] (Parameter space Λ and lexicographical order) We define a subset Λ of the product space ${\mathbf{N}^*}^{\mathbf{N}^*}$ by

$$\Lambda \equiv \{\mathbf{p} = (p_0, p_1, \ldots) \in {\mathbf{N}^*}^{\mathbf{N}^*}; p_0 \geq 1, \exists m \in \mathbf{N}, p_k = 0 \ (\forall k \geq m)\}. \tag{41}$$

Furthermore, for any $q \in \mathbf{N}$, we define a subset $\Lambda(q)$ of the set Λ by

$$\Lambda(q) \equiv \{\mathbf{p} \in \Lambda; \sum_{k=0}^{\infty}(k+1)p_k = q\}. \tag{42}$$

Then we have an orthogonal decomposition of the set Λ:

$$\Lambda = \bigcup_{q \in \mathbf{N}} \Lambda(q). \tag{43}$$

Next, we introduce a lexicographical order in the set Λ. Let $\mathbf{p}, \mathbf{p}'$ be any fixed elements of Λ. There exist $q, q' \in \mathbf{N}$ such that $\mathbf{p} \in \Lambda(q), \mathbf{p}' \in \Lambda(q')$. We say that $\mathbf{p}$

precedes $\mathbf{p}'$ if $q < q'$ or if $q = q'$ and $p_{k_0} > p'_{k_0}$, where k_0 is given by

$$k_0 \equiv \min \left\{ 0 \le k < \infty; p_k \ne p'_k \right\}. \tag{44}$$

[b] ($G(\mathbf{Z})$ and $G(\mathbf{Z})(q)$ $(q \in \mathbf{N})$) For each element $\mathbf{p}$ of Λ, we define an integer $\tau(\mathbf{p}) \in \mathbf{N}^*$ by

$$\tau(\mathbf{p}) \equiv \max\{k \in \mathbf{N}^*; p_k > 0\} \tag{45}$$

and define a one-dimensional stochastic process $\boldsymbol{\varphi}_{\mathbf{p}}(\mathbf{Z}) = (\varphi_{\mathbf{p}}(\mathbf{Z})(n); \tau(\mathbf{p}) \le n < \infty)$ by

$$\varphi_{\mathbf{p}}(\mathbf{Z})(n) \equiv \prod_{k=0}^{\tau(\mathbf{p})} Z(n-k)^{p_k}. \tag{46}$$

We denote by $G(\mathbf{Z})$ all of these stochastic processes:

$$G(\mathbf{Z}) \equiv \{\boldsymbol{\varphi}_{\mathbf{p}}(\mathbf{Z}); \mathbf{p} \in \Lambda\}. \tag{47}$$

Moreover, for any $q \in \mathbf{N}$, we define a subset $G(\mathbf{Z})(q)$ of the set $G(\mathbf{Z})$ by

$$G(\mathbf{Z})(q) \equiv \{\boldsymbol{\varphi}_{\mathbf{p}}(\mathbf{Z}); \mathbf{p} \in \Lambda(q)\}. \tag{48}$$

We note that the set $G(\mathbf{Z})$ can be decomposed into the following orthogonal sums:

$$G(\mathbf{Z}) = \bigcup_{q \in \mathbf{N}} G(\mathbf{Z})(q). \tag{49}$$

[c] (Lexicographical order in $G(\mathbf{Z})$) Using the lexicographical order in the set Λ introduced in [a] and noting that there exists a one-to-one correspondence between $G(\mathbf{Z})$ and Λ, we can introduce an order into $G(\mathbf{Z})$ according to the lexicographical order in Λ and parameterize the set $G(\mathbf{Z})$ as follows:

$$G(\mathbf{Z}) = \{\boldsymbol{\varphi}_j(\mathbf{Z}); j \in \mathbf{N}^*\}. \tag{50}$$

Since there exists for each $j \in \mathbf{N}^*$ a unique element $\mathbf{p}_j$ of the set Λ such that $\boldsymbol{\varphi}_j(\mathbf{Z}) = \boldsymbol{\varphi}_{\mathbf{p}_j}(\mathbf{Z})$, we can define an integer $\tau(j) \equiv \tau(\mathbf{p}_j)$ and represent the stochastic processes $\boldsymbol{\varphi}_j(\mathbf{Z}) = (\varphi_j(\mathbf{Z})(n); \tau(j) \le n < \infty)$ as

$$\varphi_j(\mathbf{Z})(n) \equiv \varphi_{\mathbf{p}_j}(\mathbf{Z})(n) \qquad (\sigma(j) \le n < \infty). \tag{51}$$

[d] (System of stochastic processes of rank q) Let us fix any $q \in \mathbf{N}$. We define a natural number d_q by

$$d_q \equiv (\text{the number of elements in } \bigcup_{s=1}^{q} \Lambda(s)) - 1. \tag{52}$$

Then we can see that

$$G(\mathbf{Z})(q) = \{\boldsymbol{\varphi}_{d_{q-1}+1}(\mathbf{Z}), \boldsymbol{\varphi}_{d_{q-1}+2}(\mathbf{Z}), \dots, \boldsymbol{\varphi}_{d_q}(\mathbf{Z})\}. \tag{53}$$

Then, we call the system of stochastic processes $\boldsymbol{\varphi}_j(\mathbf{Z})$ $(0 \le j \le d_q)$ the system of stochastic processes of rank q.

We note that the numbers d_q $(1 \le q \le 6)$ are given by

$$(d_1, d_2, d_3, d_4, d_5, d_6) = (0, 1, 3, 6, 11, 18). \tag{54}$$

Further, the system of stochastic processes of rank 6 consists of the following 19 one-dimensional stochastic processes $\boldsymbol{\varphi}_j(\mathbf{Z})$ $(0 \le j \le 18)$:

$$\begin{cases} \boldsymbol{\varphi}_0(\mathbf{Z}) = (Z(n); 0 \le n < \infty), \\ \boldsymbol{\varphi}_1(\mathbf{Z}) = (Z(n)^2; 0 \le n < \infty), \\ \boldsymbol{\varphi}_2(\mathbf{Z}) = (Z(n)^3; 0 \le n < \infty), \\ \boldsymbol{\varphi}_3(\mathbf{Z}) = (Z(n)Z(n-1); 1 \le n < \infty), \\ \boldsymbol{\varphi}_4(\mathbf{Z}) = (Z(n)^4; 0 \le n < \infty), \\ \boldsymbol{\varphi}_5(\mathbf{Z}) = (Z(n)^2Z(n-1); 1 \le n < \infty), \\ \boldsymbol{\varphi}_6(\mathbf{Z}) = (Z(n)Z(n-2); 2 \le n < \infty), \\ \boldsymbol{\varphi}_7(\mathbf{Z}) = (Z(n)^5; 0 \le n < \infty), \\ \boldsymbol{\varphi}_8(\mathbf{Z}) = (Z(n)^3Z(n-1); 1 \le n < \infty), \\ \boldsymbol{\varphi}_9(\mathbf{Z}) = (Z(n)^2Z(n-2); 2 \le n < \infty), \\ \boldsymbol{\varphi}_{10}(\mathbf{Z}) = (Z(n)Z(n-1)^2; 1 \le n < \infty), \\ \boldsymbol{\varphi}_{11}(\mathbf{Z}) = (Z(n)Z(n-3); 3 \le n < \infty), \\ \boldsymbol{\varphi}_{12}(\mathbf{Z}) = (Z(n)^6; 0 \le n < \infty), \\ \boldsymbol{\varphi}_{13}(\mathbf{Z}) = (Z(n)^4Z(n-1); 1 \le n < \infty), \\ \boldsymbol{\varphi}_{14}(\mathbf{Z}) = (Z(n)^3Z(n-2); 2 \le n < \infty), \\ \boldsymbol{\varphi}_{15}(\mathbf{Z}) = (Z(n)^2Z(n-1)^2; 1 \le n < \infty), \\ \boldsymbol{\varphi}_{16}(\mathbf{Z}) = (Z(n)^2Z(n-3); 3 \le n < \infty), \\ \boldsymbol{\varphi}_{17}(\mathbf{Z}) = (Z(n)Z(n-1)Z(n-2); 2 \le n < \infty), \\ \boldsymbol{\varphi}_{18}(\mathbf{Z}) = (Z(n)Z(n-4); 4 \le n < \infty). \end{cases} \tag{55}$$

[e] (Generating system) Let us fix any $q \in \mathbf{N}$. For any integer j $(0 \le j \le d_q)$, we define a one-dimensional stochastic process $\mathbf{Z}_j = (Z_j(n); n \in \mathbf{N}^*)$ with time parameter space $\mathbf{N}^*$ by

$$Z_j(n) \equiv \begin{cases} 0 & (0 \le n < \tau(j)), \\ \varphi_j(\mathbf{Z})(n) - E(\varphi_j(\mathbf{Z})(n)) & (\tau(j) \le n < \infty). \end{cases} \tag{56}$$

Then, we define a $(d_q + 1)$-dimensional stochastic process $\mathbf{Z}^{(q)} = (Z^{(q)}(n); n \in \mathbf{N}^*)$ by

$$Z^{(q)}(n) \equiv {}^t(Z_0(n), Z_1(n), \ldots, Z_{d_q}(n)). \tag{57}$$

Concerning the relation among these stochastic processes $\mathbf{Z}^{(q)}$ and the original stochastic process $\mathbf{Z}$, we have

Theorem 4.1 (i) $Z^{(1)}(n) = Z(n) - E(Z(n)) \qquad (n \in \mathbf{N}^*)$.

(ii) *The system* $\{\mathbf{Z}^{(q)}; q \in \mathbf{N}\}$ *has a nest structure, that is,*

$$Z^{(q+1)}(n) = \begin{pmatrix} Z^{(q)}(n) \\ \star \end{pmatrix} \qquad (q \in \mathbf{N}, n \in \mathbf{N}^*).$$

(iii) $\mathbf{N}_0^n(\mathbf{Z}) = [\{1\}] \oplus [\bigcup_{q=1}^{\infty} \mathbf{M}_0^n(\mathbf{Z}^{(q)})] \qquad (n \in \mathbf{N}^*)$.

We call the system $\{\mathbf{Z}^{(q)}; q \in \mathbf{N}\}$ a generating system of polynomial type of the non-linear information spaces $\mathbf{N}_0^n(\mathbf{Z})$ $(n \in \mathbf{N}^*)$ associated with the stochastic process $\mathbf{Z}$.

5. Stochastic process associated with the logistic map

In this section, we shall consider the logistic map $\varphi : [0,1] \to [0,1]$ defined by

$$\varphi(x) \equiv 4x(1-x). \tag{58}$$

Let μ be the Borel measure on $([0,1], \mathcal{B}([0,1]))$ defined by

$$\mu(dx) \equiv (\pi\sqrt{x(1-x)})^{-1}dx. \tag{59}$$

We know that this μ is a unique invariant probability measure of the logistic map φ:

$$\mu(\varphi^{-1}A) = \mu(A) \quad \text{for any } A \in \mathcal{B}([0,1]). \tag{60}$$

We define a one-dimensional stochastic process $\mathbf{X} = (X(n); n \in \mathbf{N}^*)$ defined on the probability space $([0,1], \mathcal{B}([0,1]), \mu)$ by

$$X(n)(x) \equiv \varphi^n(x) \qquad (n \in \mathbf{N}^*, x \in [0,1]). \tag{61}$$

Taking the standardization of the stochastic process $\mathbf{X}$, we define a one-dimensional stochastic process $\mathbf{W} = (W(n); n \in \mathbf{N}^*)$ on the probability space $([0,1], \mathcal{B}([0,1]), \mu)$ by

$$W(n) \equiv 2\sqrt{2}(X(n) - \frac{1}{2}) \quad (n \in \mathbf{N}^*). \tag{62}$$

Then we can show

Theorem 5.1

(i) $W(n+1) = \sqrt{2}(1 - W(n)^2) \qquad (n \in \mathbf{N}^*)$.

(ii) $\mathbf{W}$ *is identically distributed.*

(iii) $\mathbf{W}$ *is not independent.*

(iv) $\mathbf{W}$ *is a strictly stationary process.*

(v) $\mu((W(0), W(1), \ldots, W(n+1)) \in dx_0 dx_1 \cdots dx_{n+1})$
$= \mu((W(0), W(1), \ldots, W(n)) \in dx_0 dx_1 \cdots dx_n)\delta_{\{\sqrt{2}(1-x_n^2)\}}(dx_{n+1}) \quad (n \in \mathbf{N}^*)$.

(vi) $E(W(n)) = 0 \qquad (n \in \mathbf{N}^*)$.

(vii) $E(W(n)^2) = 1 \qquad (n \in \mathbf{N}^*)$.

(viii) $E(W(n)^{2p+1}) = 0 \qquad (n, p \in \mathbf{N}^*)$.

(ix) $E(W(n)^{2p}) = (2p)!/(2^p(p!)^2) \qquad (n, p \in \mathbf{N}^*)$.

(x) $E(W(n)W(m)) = \delta_{nm} \qquad (n, m \in \mathbf{N}^*)$.

(xi) $E(W(0)^{2p_0+1}W(1)^{p_1}W(2)^{p_2} \cdots W(n)^{p_n}) = 0 \qquad (n \in \mathbf{N}, p_0, p_1, \ldots, p_n \in \mathbf{N}^*)$.

Immediately from Theorem 5.1(vi) and (x), we have

Theorem 5.2 *The stochastic process* $\mathbf{W}$ *is a white noise in a broad sense.*

In order to investigate the problem of coexistence of order and choas for the stochastic process $\mathbf{W}$, we shall consider the following 18 two-dimensional stochastic processes $\mathbf{W}_{(0,j)} = (W_{(0,j)}(n); n \in \mathbf{N}^*)$ $(1 \le j \le 18)$ by shifting the time domains of the system of non-linear transformations of rank 6 introduced in (55) and taking their standardization:

$$\begin{cases} W_{(0,1)}(n) \equiv \ {}^t(W(n), \sqrt{2}(W(n)^2 - 1)) & (n \in \mathbf{N}^*), \\ W_{(0,2)}(n) \equiv \ {}^t(W(n), \sqrt{\frac{2}{5}}W(n)^3) & (n \in \mathbf{N}^*), \\ W_{(0,3)}(n) \equiv \ {}^t(W(n+1), W(n+1)W(n)) & (n \in \mathbf{N}^*), \\ W_{(0,4)}(n) \equiv \ {}^t(W(n), 2\sqrt{\frac{2}{17}}(W(n)^4 - \frac{3}{2})) & (n \in \mathbf{N}^*), \\ W_{(0,5)}(n) \equiv \ {}^t(W(n+1), \sqrt{\frac{2}{3}}W(n+1)^2W(n)) & (n \in \mathbf{N}^*), \\ W_{(0,6)}(n) \equiv \ {}^t(W(n+2), W(n+2)W(n)) & (n \in \mathbf{N}^*), \\ W_{(0,7)}(n) \equiv \ {}^t(W(n), \frac{2}{3}\sqrt{\frac{2}{7}}W(n)^5) & (n \in \mathbf{N}^*), \\ W_{(0,8)}(n) \equiv \ {}^t(W(n+1), \sqrt{\frac{2}{5}}W(n+1)^3W(n)) & (n \in \mathbf{N}^*), \\ W_{(0,9)}(n) \equiv \ {}^t(W(n+2), \sqrt{\frac{2}{3}}W(n+2)^2W(n)) & (n \in \mathbf{N}^*), \\ W_{(0,10)}(n) \equiv \ {}^t(W(n+1), \frac{2}{\sqrt{5}}(W(n+1)W(n)^2 + \frac{1}{\sqrt{2}})) & (n \in \mathbf{N}^*), \\ W_{(0,11)}(n) \equiv \ {}^t(W(n+3), W(n+3)W(n)) & (n \in \mathbf{N}^*), \\ W_{(0,12)}(n) \equiv \ {}^t(W(n), \frac{4}{\sqrt{131}}(W(n)^6 - \frac{5}{2})) & (n \in \mathbf{N}^*), \\ W_{(0,13)}(n) \equiv \ {}^t(W(n+1), 2\sqrt{\frac{2}{35}}W(n+1)^4W(n)) & (n \in \mathbf{N}^*), \\ W_{(0,14)}(n) \equiv \ {}^t(W(n+2), \sqrt{\frac{2}{5}}W(n+2)^3W(n)) & (n \in \mathbf{N}^*), \\ W_{(0,15)}(n) \equiv \ {}^t(W(n+1), \frac{2}{\sqrt{7}}(W(n+1)^2W(n)^2 - 1)) & (n \in \mathbf{N}^*), \\ W_{(0,16)}(n) \equiv \ {}^t(W(n+3), \sqrt{\frac{2}{3}}W(n+3)^2W(n)) & (n \in \mathbf{N}^*), \\ W_{(0,17)}(n) \equiv \ {}^t(W(n+2), W(n+2)W(n+1)W(n)) & (n \in \mathbf{N}^*), \\ W_{(0,18)}(n) \equiv \ {}^t(W(n+4), W(n+4)W(n)) & (n \in \mathbf{N}^*). \end{cases} \tag{63}$$

[1:$\mathbf{W}_{(0,1)}$] At first, we shall treat the stochastic process $\mathbf{W}_{(0,1)}$. Immediately from Theorem 5.1(i), we have

$$W(n)^2 = 1 - \frac{1}{\sqrt{2}}W(n+1) \qquad (n \in \mathbf{N}^*). \tag{64}$$

Hence, it follows from Theorem 5.1(iv) that the stochastic process $\mathbf{W}_{(0,1)}$ is strictly stationary. In particular, we can see from Theorem 5.1(vi), (vii), (viii), (ix), (x), (xi) and (64) that

Theorem 5.3 *The stochastic process* $\mathbf{W}_{(0,1)}$ *is a degenerate and weakly stationary process whose covariance matrix function* $R(\mathbf{W}_{(0,1)})$ *is given by*

(i) $\quad R(\mathbf{W}_{(0,1)})(0) = I_2,$

(ii) $\quad R(\mathbf{W}_{(0,1)})(1) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},$

(iii) $$R(\mathbf{W}_{(0,1)})(n) = 0 \qquad (|n| \geq 2).$$

At first, we shall construct the system of the minimum KM$_2$O-Langevin matrices associated with the stochastic process $\mathbf{W}_{(0,1)}$. For any fixed $N \in \mathbf{N}$, we restrict the time domain of the stochastic process $\mathbf{W}_{(0,1)}$ to the set $\{0, 1, \ldots, N\}$ and define a two-dimensional flows $\mathbf{Z} = (Z(n); 0 \leq n \leq N)$ in the real inner product space $L^2([0,1], \mathcal{B}([0,1]), \mu)$ by

$$Z(n) \equiv W_{(0,1)}(n) \qquad (0 \leq n \leq N). \tag{65}$$

Since it follows from the fluctuation-dissipation algorithm stated in Section 3 that the system $\mathcal{LM}(\mathbf{Z})$ does not depend upon N, we can construct six matrix functions $\delta^0_\pm(\mathbf{W}_{(0,1)}) = (\delta^0_\pm(\mathbf{W}_{(0,1)})(n); n \geq 1), \gamma^0_\pm(\mathbf{W}_{(0,1)}) = (\gamma^0_\pm(\mathbf{W}_{(0,1)})(m,n); 0 \leq n < m < \infty)$ and $V_\pm(\mathbf{W}_{(0,1)}) = (V_\pm(\mathbf{W}_{(0,1)})(n); n \geq 0)$ such that for any $N \in \mathbf{N}$,

$$\begin{cases} \delta^0_\pm(\mathbf{W}_{(0,1)})(n) = \delta^0_\pm(\mathbf{Z})(n) & (1 \leq n \leq N), \\ \gamma^0_\pm(\mathbf{W}_{(0,1)})(m,n) = \gamma^0_\pm(\mathbf{Z})(m,n) & (0 \leq n < m \leq N), \\ V_\pm(\mathbf{W}_{(0,1)})(n) = V_\pm(\mathbf{Z})(n) & (0 \leq n \leq N). \end{cases} \tag{66}$$

We call the system $\mathcal{LM}(\mathbf{W}_{(0,1)})$ of such matrices the minimum KM$_2$O-Langevin matrix associated with the stochastic process $\mathbf{W}$:

$$\begin{aligned} \mathcal{LM}(\mathbf{W}_{(0,1)}) \equiv \{&\gamma^0_+(\mathbf{W}_{(0,1)})(n,k), \gamma^0_-(\mathbf{W}_{(0,1)})(n,k), V_+(\mathbf{W}_{(0,1)})(m), V_-(\mathbf{W}_{(0,1)})(m); \\ &0 \leq k < n < \infty, 0 \leq m < \infty\}. \end{aligned} \tag{67}$$

Since (64) implies that the flow $\mathbf{Z}$ is degenerate, we shall apply the weight transformation with additive white noise flow to it and obtain the system of the minimum KM$_2$O-Langevin matrices associated with the stationary flow $\mathbf{Z}^w$. For simplicity of the notation, we put, for each $w > 0$,

$$\begin{cases} R^w(n) \equiv R(\mathbf{Z}^w)(n) & (|n| \leq N), \\ \delta^w_\pm(n) \equiv \delta_\pm(\mathbf{Z}^w)(n) & (1 \leq n \leq N), \\ \gamma^w_\pm(m,n) \equiv \gamma_\pm(\mathbf{Z}^w)(m,n) & (0 \leq n < m \leq N), \\ V^w_\pm(n) \equiv V_\pm(\mathbf{Z}^w)(n) & (0 \leq n \leq N). \end{cases} \tag{68}$$

It follows from Lemma 2.2(i) and Theorem 5.3 that

Lemma 5.1 *For any $w > 0$,*

(i) $$R^w(0) = (1 + w^2) I_2,$$

(ii) $$R^w(1) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},$$

(iii) $$R^w(n) = 0 \qquad (|n| \geq 2).$$

It follows from (37), Theorems 3.1(ii) and 5.3 that

Lemma 5.2 *For any $w > 0$,*

$$
\text{(i)} \qquad \begin{cases} \delta_+^w(1) = \frac{1}{1+w^2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ \delta_-^w(1) = \frac{1}{1+w^2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ \delta_+^w(n) = \delta_-^w(n) = 0 \ (n \geq 2), \end{cases}
$$

$$
\text{(ii)} \qquad \begin{cases} V_+^w(0) = V_-^w(0) = (1+w^2)I_2, \\ V_+^w(1) = \frac{1}{1+w^2}\begin{pmatrix} 2w^2+w^4 & 0 \\ 0 & (1+w^2)^2 \end{pmatrix}, \\ V_-^w(1) = \frac{1}{1+w^2}\begin{pmatrix} (1+w^2)^2 & 0 \\ 0 & 2w^2+w^4 \end{pmatrix}. \end{cases}
$$

Therefore, according to the fluctuation-dissipation algorithm stated in Section 3, we can let w tend to 0 in Lemma 5.2 to obtain

Theorem 5.4

$$
\text{(i)} \qquad \begin{cases} \delta_+^0(\mathbf{W}_{(0,1)})(1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ \delta_-^0(\mathbf{W}_{(0,1)})(1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \end{cases}
$$

$$
\text{(ii)} \qquad \delta_+^0(\mathbf{W}_{(0,1)})(1)^2 = \delta_-^0(\mathbf{W}_{(0,1)})(1)^2 = 0,
$$

$$
\text{(iii)} \qquad \delta_+^0(\mathbf{W}_{(0,1)})(n) = \delta_-^0(\mathbf{W}_{(0,1)})(n) = 0 \qquad (n \geq 2),
$$

$$
\text{(iv)} \qquad \begin{cases} V_+(\mathbf{W}_{(0,1)})(0) = V_-(\mathbf{W}_{(0,1)})(0) = I_2, \\ V_+(\mathbf{W}_{(0,1)})(n) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad (n \geq 1), \\ V_-(\mathbf{W}_{(0,1)})(n) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad (n \geq 1). \end{cases}
$$

We note from Theorem 5.3(iii) that the covariance matrix function $R(\mathbf{W}_{(0,1)})$ of the stochastic process $\mathbf{W}_{(0,1)}$ has the same structure as that of the moving average processes of order 1—MA(1)-process. Moreover, we find from Theorem 5.4(iii) that the minimum KM$_2$O-Langevin partial autocorrelation matrix functions $\delta_\pm^0(\mathbf{W}_{(0,1)})$ have the same structure as that of the autoregressive processes of order 1—AR(1)-process.

There exists a close relation between Theorems 5.3(iii) and 5.4(iii) under Theorem 5.4(ii). In fact, we shall show the following general Theorem 5.5.

Theorem 5.5 *Let $\mathbf{Z} = (Z(n); 0 \leq n \leq N)$ be any d-dimensional stationary flow in a real inner product space W with an inner product $(\star, *)$ satisfying*

$$
\delta_+^0(\mathbf{Z})(n) = 0 \quad \textit{for any } n \ (2 \leq n \leq N).
$$

Then, for any $p \in \mathbf{N}^*$ $(0 \leq p \leq N)$, the following two conditions are equivalent to each other:

$$
\text{(i)} \qquad R(\mathbf{Z})(n) = 0 \qquad (p \leq |n| \leq N);
$$

$$
\text{(ii)} \qquad \delta_+^0(\mathbf{Z})(1)^p V_+(\mathbf{Z})(0) = 0.
$$

Proof. It is clear that (i) and (ii) are equivalent to each other for $p = 0$. Let p $(1 \leq p \leq N)$ be any fixed number. It follows from (DDT) in Theorem 3.1 that for any n $(2 \leq n \leq N)$,

$$\gamma_+^0(\mathbf{Z})(n,k) = \begin{cases} \delta_+^0(\mathbf{Z})(1) & (k = n-1), \\ 0 & (0 \leq k < n-1). \end{cases}$$

Substituting these into (32) for $\ell = 0$, we see that for any n $(1 \leq n \leq N)$,

$$\begin{aligned} R(\mathbf{Z})(n) &= -\delta_+^0(\mathbf{Z})(1)R(\mathbf{Z})(n-1) \\ &= (-1)^n \delta_+^0(\mathbf{Z})(1)^n V_+(\mathbf{Z})(0). \end{aligned}$$

Therefore, we find that (i) and (ii) are equivalent to each other. (Q.E.D.)

Finally in [1:$\mathbf{W}_{(0,1)}$], we shall show that the inequalities in Theorem 2.3 are tight.

Lemma 5.3 *For any $w > 0$ and any $n \in \mathbf{N}$,*

$$\begin{cases} \|\Gamma_\pm(\mathbf{W}_{(0,1)}^w)(n) - \Gamma_\pm(\mathbf{W}_{(0,1)})(n)\| = \frac{w^2}{w^2+1}, \\ \|\Gamma_\pm(\mathbf{W}_{(0,1)})(n)\| = 1. \end{cases}$$

Proof. It follows from Theorem 3.1 and Lemma 5.2 that $\gamma_+^w(1,0) = \frac{1}{1+w^2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and for any $n \geq 2$, $\gamma_+^w(n, n-k) = \gamma_+^w(1,0)$ $(k=1), \gamma_+^w(n,n-k) = 0$ $(2 \leq k \leq n)$. Therefore, we can see that for any $n \in \mathbf{N}$,

$$\Gamma_+(\mathbf{W}_{(0,1)}^w)(n) = \begin{cases} \frac{1}{1+w^2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & (n=1), \\ \frac{1}{1+w^2}\,{}^t\begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix} & (n \geq 2), \end{cases}$$

$$\Gamma_+(\mathbf{W}_{(0,1)})(n) = \begin{cases} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & (n=1), \\ {}^t\begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix} & (n \geq 2). \end{cases}$$

Thus, we see that the plus part holds. Similarly, the minus part is proved. (Q.E.D.)

Lemma 5.4 *For any $n \in \mathbf{N}$, $\lambda_\pm(\mathbf{W}_{(0,1)})(n) = 1$.*

Proof. It follows from (31) and Theorem 5.3 that

$$T_+(\mathbf{W}_{(0,1)})(n) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We put $f_n(\lambda) \equiv \det(T_+(\mathbf{W}_{(0,1)})(n) - \lambda I_{2n})$ $(\lambda \in \mathbf{R})$. Using the expansion formula of the determinant $\det(T_+(\mathbf{W}_{(0,1)})(n) - \lambda I_{2n})$ with respect to the first row, we have

$$f_n(\lambda) = (1-\lambda) g_{n-1}(\lambda),$$

where $g_{n-1}(\lambda)$ is given by

$$g_{n-1}(\lambda) \equiv \det \begin{pmatrix} 1-\lambda & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & 1-\lambda & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\lambda & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1-\lambda & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\lambda & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1-\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1-\lambda & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1-\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1-\lambda \end{pmatrix}.$$

Since $f_1(\lambda) = (1-\lambda)^2$, we note that $g_0(\lambda) = 1-\lambda$. Using the expansion formula of the determinant, we have

$$g_{n-1}(\lambda) = (1-\lambda) f_{n-1}(\lambda) - g_{n-2}(\lambda).$$

Using these algorithms, we can see that

$$f_n(\lambda) = \lambda^{n-1}(\lambda-1)^2(\lambda-2)^{n-1} \qquad (n \in \mathbf{N}).$$

Hence, we find that $\lambda_+(\mathbf{W}_{(0,1)})(n) = 1$ and so the plus part holds. Similarly, the minus part is proved. (Q.E.D.)

Consequently, we see from Theorem 2.3, Lemmas 5.3 and 5.4 that

Theorem 5.6 *The inequalities in Theorem 2.3 are tight, that is, for any $w > 0$ and any $n \in \mathbf{N}$,*

$$\text{(i)} \quad \frac{\|\Gamma_+(\mathbf{W}^w_{(0,1)})(n) - \Gamma_+(\mathbf{W}_{(0,1)})(n)\|}{\|\Gamma_+(\mathbf{W}_{(0,1)})(n)\|} = \frac{w^2}{w^2 + \lambda_+(\mathbf{W}_{(0,1)})(n)},$$

$$\text{(ii)} \quad \frac{\|\Gamma_-(\mathbf{W}^w_{(0,1)})(n) - \Gamma_-(\mathbf{W}_{(0,1)})(n)\|}{\|\Gamma_-(\mathbf{W}_{(0,1)})(n)\|} = \frac{w^2}{w^2 + \lambda_-(\mathbf{W}_{(0,1)})(n)}.$$

[2:$\mathbf{W}_{(0,2)}$] We shall treat the stochastic process $\mathbf{W}_{(0,2)}$. It follows from Theorem 5.1(ix) that

$$E(W(n)^4) = \frac{3}{2} \qquad (n \in \mathbf{N}^*), \tag{69}$$

$$E(W(n)^6) = \frac{5}{2} \qquad (n \in \mathbf{N}^*). \tag{70}$$

Therefore, it follows from Theorem 5.1(x) and (xi) that

Theorem 5.7 *The stochastic process $\mathbf{W}_{(0,2)}$ is a weakly stationary process whose covariance matrix function $R(\mathbf{W}_{(0,2)})$ is given by*

$$\text{(i)} \qquad R(\mathbf{W}_{(0,2)})(0) = \begin{pmatrix} 1 & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & 1 \end{pmatrix},$$

$$\text{(ii)} \qquad R(\mathbf{W}_{(0,2)})(n) = 0 \qquad (|n| \geq 1).$$

[3:$\mathbf{W}_{(0,3)}$] We shall treat the stochastic process $\mathbf{W}_{(0,3)}$. It follows from Theorem 5.1(vi), (vii), (viii) and (64) that

$$E(W(n+1)W(n)^2) = -\frac{1}{\sqrt{2}} \qquad (n \in \mathbf{N}^*), \tag{71}$$

$$E(W(n+1)^2 W(n)^2) = 1 \qquad (n \in \mathbf{N}^*). \tag{72}$$

Therefore, we see from Theorem 5.1(x) and (xi) that

Theorem 5.8 *The stochastic process $\mathbf{W}_{(0,3)}$ is a weakly stationary process whose covariance matrix function $R(\mathbf{W}_{(0,3)})$ is given by*

$$\text{(i)} \qquad R(\mathbf{W}_{(0,3)})(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\text{(ii)} \qquad R(\mathbf{W}_{(0,3)})(1) = \begin{pmatrix} 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix},$$

$$\text{(iii)} \qquad R(\mathbf{W}_{(0,3)})(n) = 0 \qquad (|n| \geq 2).$$

[4:$\mathbf{W}_{(0,4)}$] We shall treat the stochastic process $\mathbf{W}_{(0,4)}$. It follows from Theorem 5.1(ix) that

$$E(W(n)^8) = \frac{35}{8} \qquad (n \in \mathbf{N}^*). \tag{73}$$

We note from (64) that

$$W(n)^4 = 1 - \sqrt{2}W(n+1) + \frac{1}{2}W(n+1)^2 \qquad (n \in \mathbf{N}^*). \tag{74}$$

Using this and (64) again, we see from Theorem 5.1(vi), (vii), (viii) and (x) that

$$E(W(n)W(0)^4) = \begin{cases} 0 & (n=0), \\ -\sqrt{2} & (n=1), \\ -\frac{1}{2\sqrt{2}} & (n=2), \\ 0 & (n \geq 3). \end{cases} \tag{75}$$

Further, it follows from (64), (69), (70), (73), (74) and Theorem 5.1(viii) that

$$E(W(n)^4 W(0)^4) = \begin{cases} \frac{35}{8} & (n=0), \\ \frac{11}{4} & (n=1), \\ \frac{9}{4} & (n \geq 2). \end{cases} \tag{76}$$

Therefore, we see from Theorem 5.1(x) and (xi) that

Theorem 5.9 *The stochastic process $\mathbf{W}_{(0,4)}$ is a weakly stationary process whose covariance matrix function $R(\mathbf{W}_{(0,4)})$ is given by*

(i) $$R(\mathbf{W}_{(0,4)})(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

(ii) $$R(\mathbf{W}_{(0,4)})(1) = \begin{pmatrix} 0 & -\frac{4}{\sqrt{17}} \\ 0 & \frac{4}{17} \end{pmatrix},$$

(iii) $$R(\mathbf{W}_{(0,4)})(2) = \begin{pmatrix} 0 & -\frac{1}{\sqrt{17}} \\ 0 & 0 \end{pmatrix},$$

(iv) $$R(\mathbf{W}_{(0,4)})(n) = 0 \qquad (|n| \geq 3).$$

[5:$\mathbf{W}_{(0,5)}$] We shall treat the stochastic process $\mathbf{W}_{(0,5)}$. We see from (64), (69), (72), Theorem 5.1(x) and (xi) that

Theorem 5.10 *The stochastic process $\mathbf{W}_{(0,5)}$ is a weakly stationary process whose covariance matrix function $R(\mathbf{W}_{(0,5)})$ is given by*

(i) $$R(\mathbf{W}_{(0,5)})(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

(ii) $$R(\mathbf{W}_{(0,5)})(1) = \begin{pmatrix} 0 & 0 \\ \sqrt{\frac{2}{3}} & 0 \end{pmatrix},$$

(iii) $$R(\mathbf{W}_{(0,5)})(n) = 0 \qquad (|n| \geq 2).$$

[6:$\mathbf{W}_{(0,6)}$] We shall treat the stochastic process $\mathbf{W}_{(0,6)}$. It follows from (64), Theorem 5.1(x) and (xi) that

Theorem 5.11 *The stochastic process* $\mathbf{W}_{(0,6)}$ *is a weakly stationary process whose covariance matrix function* $R(\mathbf{W}_{(0,6)})$ *is given by*

$$\text{(i)} \qquad R(\mathbf{W}_{(0,6)})(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\text{(ii)} \qquad R(\mathbf{W}_{(0,6)})(n) = 0 \qquad (|n| \geq 1).$$

[7:$\mathbf{W}_{(0,7)}$] We shall treat the stochastic process $\mathbf{W}_{(0,7)}$. It follows from Theorem 5.1(ix) that

$$E(W(n)^{10}) = \frac{63}{8} \qquad (n \in \mathbf{N}^*). \tag{77}$$

Using this and (70), we see from Theorem 5.1(x) and (xi) that

Theorem 5.12 *The stochastic process* $\mathbf{W}_{(0,7)}$ *is a weakly stationary process whose covariance matrix function* $R(\mathbf{W}_{(0,7)})$ *is given by*

$$\text{(i)} \qquad R(\mathbf{W}_{(0,7)})(0) = \begin{pmatrix} 1 & \frac{5}{3}\sqrt{\frac{2}{7}} \\ \frac{5}{3}\sqrt{\frac{2}{7}} & 1 \end{pmatrix},$$

$$\text{(ii)} \qquad R(\mathbf{W}_{(0,7)})(n) = 0 \qquad (|n| \geq 1).$$

[8:$\mathbf{W}_{(0,8)}$] We shall treat the stochastic process $\mathbf{W}_{(0,8)}$. It follows from (64), (69), Theorem 5.1(viii), (x) and (xi) that

Theorem 5.13 *The stochastic process* $\mathbf{W}_{(0,8)}$ *is a weakly stationary process whose covariance matrix function* $R(\mathbf{W}_{(0,8)})$ *is given by*

$$\text{(i)} \qquad R(\mathbf{W}_{(0,8)})(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\text{(ii)} \qquad R(\mathbf{W}_{(0,8)})(1) = \begin{pmatrix} 0 & 0 \\ -\frac{3}{2\sqrt{5}} & 0 \end{pmatrix},$$

$$\text{(iii)} \qquad R(\mathbf{W}_{(0,8)})(n) = 0 \qquad (|n| \geq 2).$$

[9:$\mathbf{W}_{(0,9)}$] We shall treat the stochastic process $\mathbf{W}_{(0,9)}$. It follows from (64), (69), Theorem 5.1(viii), (x) and (xi) that

Theorem 5.14 *The stochastic process* $\mathbf{W}_{(0,9)}$ *is a weakly stationary process whose covariance matrix function* $R(\mathbf{W}_{(0,9)})$ *is given by*

$$\text{(i)} \qquad R(\mathbf{W}_{(0,9)})(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\text{(ii)} \qquad R(\mathbf{W}_{(0,9)})(1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\text{(iii)} \qquad R(\mathbf{W}_{(0,9)})(2) = \begin{pmatrix} 0 & 0 \\ \sqrt{\frac{2}{3}} & 0 \end{pmatrix},$$

(iv) $$R(\mathbf{W}_{(0,9)})(n) = 0 \qquad (|n| \geq 3).$$

[10:$\mathbf{W}_{(0,10)}$] We shall treat the stochastic process $\mathbf{W}_{(0,10)}$. It follows from (64), (72), (74), Theorem 5.1(vi), (vii), (viii), (x) and (xi) that

Theorem 5.15 *The stochastic process* $\mathbf{W}_{(0,10)}$ *is a weakly stationary process whose covariance matrix function* $R(\mathbf{W}_{(0,10)})$ *is given by*

(i) $$R(\mathbf{W}_{(0,10)})(0) = \begin{pmatrix} 1 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 1 \end{pmatrix},$$

(ii) $$R(\mathbf{W}_{(0,10)})(1) = \begin{pmatrix} 0 & \frac{1}{\sqrt{5}} \\ 0 & \frac{2}{5} \end{pmatrix},$$

(iii) $$R(\mathbf{W}_{(0,10)})(n) = 0 \qquad (|n| \geq 2).$$

[11:$\mathbf{W}_{(0,11)}$] We shall treat the stochastic process $\mathbf{W}_{(0,11)}$. We see from (64), Theorem 5.1(vi), (vii), (x) and (xi) that

Theorem 5.16 *The stochastic process* $\mathbf{W}_{(0,11)}$ *is a weakly stationary process whose covariance matrix function* $R(\mathbf{W}_{(0,11)})$ *is given by*

(i) $$R(\mathbf{W}_{(0,11)})(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

(ii) $$R(\mathbf{W}_{(0,11)})(n) = 0 \qquad (|n| \geq 1).$$

[12:$\mathbf{W}_{(0,12)}$] We shall treat the stochastic process $\mathbf{W}_{(0,12)}$. It follows from Theorem 5.1(ix) that

$$E(W(n)^{12}) = \frac{231}{16} \qquad (n \in \mathbf{N}^*). \tag{78}$$

We note from (64) that

$$W(n)^6 = 1 - \frac{3}{\sqrt{2}}W(n+1) + \frac{3}{2}W(n+1)^2 - \frac{1}{2\sqrt{2}}W(n+1)^3 \qquad (n \in \mathbf{N}^*). \tag{79}$$

Hence, we see from Theorem 5.1(vi), (vii), (viii), (x), (xi), (64), (69) and (70) that

Theorem 5.17 *The stochastic process* $\mathbf{W}_{(0,12)}$ *is a weakly stationary process whose covariance matrix function* $R(\mathbf{W}_{(0,12)})$ *is given by*

(i) $$R(\mathbf{W}_{(0,12)})(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

(ii) $$R(\mathbf{W}_{(0,12)})(1) = \begin{pmatrix} 0 & -\frac{15}{\sqrt{262}} \\ 0 & \frac{45}{131} \end{pmatrix},$$

(iii) $$R(\mathbf{W}_{(0,12)})(2) = \begin{pmatrix} 0 & -3\sqrt{\frac{2}{131}} \\ 0 & 0 \end{pmatrix},$$

(iv) $$R(\mathbf{W}_{(0,12)})(n) = 0 \qquad (|n| \geq 3).$$

[13:$\mathbf{W}_{(0,13)}$] We shall treat the stochastic process $\mathbf{W}_{(0,13)}$. It follows from Theorem 5.1(viii), (x), (xi), (69) and (73) that

Theorem 5.18 *The stochastic process $\mathbf{W}_{(0,13)}$ is a weakly stationary process whose covariance matrix function $R(\mathbf{W}_{(0,13)})$ is given by*

(i) $$R(\mathbf{W}_{(0,13)})(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

(ii) $$R(\mathbf{W}_{(0,13)})(1) = \begin{pmatrix} 0 & 0 \\ 3\sqrt{\frac{2}{35}} & 0 \end{pmatrix},$$

(iii) $$R(\mathbf{W}_{(0,13)})(n) = 0 \qquad (|n| \geq 2).$$

[14:$\mathbf{W}_{(0,14)}$] We shall treat the stochastic process $\mathbf{W}_{(0,14)}$. It follows from (64), (70), Theorem 5.1(viii), (x) and (xi) that

Theorem 5.19 *The stochastic process $\mathbf{W}_{(0,14)}$ is a weakly stationary process whose covariance matrix function $R(\mathbf{W}_{(0,14)})$ is given by*

(i) $$R(\mathbf{W}_{(0,14)})(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

(ii) $$R(\mathbf{W}_{(0,14)})(n) = 0 \qquad (|n| \geq 1).$$

[15:$\mathbf{W}_{(0,15)}$] We shall treat the stochastic process $\mathbf{W}_{(0,15)}$. It follows from Theorem 5.1(vi), (vii), (viii), (x), (xi), (69), (70), (72) and (74) that

Theorem 5.20 *The stochastic process $\mathbf{W}_{(0,15)}$ is a weakly stationary process whose covariance matrix function $R(\mathbf{W}_{(0,15)})$ is given by*

(i) $$R(\mathbf{W}_{(0,15)})(0) = \begin{pmatrix} 1 & -\frac{3}{\sqrt{14}} \\ -\frac{3}{\sqrt{14}} & 1 \end{pmatrix},$$

(ii) $$R(\mathbf{W}_{(0,15)})(1) = \begin{pmatrix} 0 & -\sqrt{\frac{2}{7}} \\ 0 & \frac{3}{7} \end{pmatrix},$$

(iii) $$R(\mathbf{W}_{(0,15)})(n) = 0 \qquad (|n| \geq 2).$$

[16:$\mathbf{W}_{(0,16)}$] We shall treat the stochastic process $\mathbf{W}_{(0,16)}$. It follows from Theorem 5.1(vii), (x), (xi), (64) and (69) that

Theorem 5.21 *The stochastic process $\mathbf{W}_{(0,16)}$ is a weakly stationary process whose covariance matrix function $R(\mathbf{W}_{(0,16)})$ is given by*

(i) $$R(\mathbf{W}_{(0,16)})(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

(ii) $$R(\mathbf{W}_{(0,16)})(1) = R(\mathbf{W}_{(0,16)})(2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

(iii) $$R(\mathbf{W}_{(0,16)})(3) = \begin{pmatrix} 0 & 0 \\ \sqrt{\frac{2}{3}} & 0 \end{pmatrix},$$

(iv) $$R(\mathbf{W}_{(0,16)})(n) = 0 \qquad (|n| \geq 4).$$

[17:$\mathbf{W}_{(0,17)}$] We shall treat the stochastic process $\mathbf{W}_{(0,17)}$. It follows from Theorem 5.1(vi), (vii), (viii), (x), (xi), (64) and (72) that

Theorem 5.22 *The stochastic process $\mathbf{W}_{(0,17)}$ is a weakly stationary process whose covariance matrix function $R(\mathbf{W}_{(0,17)})$ is given by*

(i) $$R(\mathbf{W}_{(0,17)})(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

(ii) $$R(\mathbf{W}_{(0,17)})(1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

(iii) $$R(\mathbf{W}_{(0,17)})(2) = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix},$$

(iv) $$R(\mathbf{W}_{(0,17)})(n) = 0 \qquad (|n| \geq 3).$$

[18:$\mathbf{W}_{(0,18)}$] We shall treat the stochastic process $\mathbf{W}_{(0,18)}$. It follows from Theorem 5.1(vi), (vii), (x), (xi) and (64) that

Theorem 5.23 *The stochastic process $\mathbf{W}_{(0,18)}$ is a weakly stationary process whose covariance matrix function $R(\mathbf{W}_{(0,18)})$ is given by*

(i) $$R(\mathbf{W}_{(0,18)})(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

(ii) $$R(\mathbf{W}_{(0,18)})(n) = 0 \qquad (|n| \geq 1).$$

As stated before for the stochastic process $\mathbf{W}_{(0,1)}$, we find from Theorem 5.3 and Theorem 5.7-Theorem 5.23 that all the covariance matrix functions $R(\mathbf{W}_{(0,j)})$ of the stochastic processes $\mathbf{W}_{(0,j)}$ $(1 \leq j \leq 18)$ have the same structure as that of the moving average processes, that is, there exists a natural number m_j such that $R(\mathbf{W}_{(0,j)})(n) = 0$ for any $n \geq m_j$ $(1 \leq j \leq 18)$.

On the other hand, similarly as in Theorem 5.4 for the stochastic process $\mathbf{W}_{(0,1)}$, applying the fluctuation-dissipation algorithm stated in Section 3 to the covariance matrix functions $R(\mathbf{W}_{(0,j)})$ $(1 \leq j \leq 18)$ calculated in Theorem 5.3 and Theorem 5.7-Theorem 5.23, we can show that the minimum KM$_2$O-Langevin partial autocorrelation matrix functions $\delta^0_{\pm}(\mathbf{W}_{(0,j)}) = (\delta^0_{\pm}(\mathbf{W}_{(0,j)})(n); n \in \mathbf{N})$ except $j = 4, 10$ have the same structure as that of the autoregressive processes, that is, there exists a natural number ℓ_j such that $\delta^0_{\pm}(\mathbf{W}_{(0,j)})(n) = 0$ for any $n \geq \ell_j$ $(1 \leq j \leq 18, j \neq 4, 10)$. The calculation for the cases where $j = 4, 10$ will be given in Section 7.

6. Stochastic process associated with the tent map

In this section, we shall consider the tent map $\psi : [0,1] \to [0,1]$ defined by

$$\psi(x) \equiv \begin{cases} 2x & (0 \le x \le \frac{1}{2}), \\ 2(1-x) & (\frac{1}{2} \le x \le 1). \end{cases} \tag{80}$$

Let ν be the Lebesgue measure on $([0,1], \mathcal{B}([0,1]))$, that is,

$$\nu(dx) \equiv dx. \tag{81}$$

We define a one-dimensional stochastic process $\mathbf{X} = (X(n); n \in \mathbf{N}^*)$ on the probability space $([0,1], \mathcal{B}([0,1]), \nu)$ by

$$X(n)(x) = \psi^n(x) \qquad (n \in \mathbf{N}^*, x \in [0,1]). \tag{82}$$

At first, we shall show that the probability measure ν is the unique invariant measure for the tent map ψ.

Lemma 6.1 *For any bounded Borel function $f : [0,1] \to \mathbf{R}$,*

$$E(f(X(n))) = E(f(X(0))) = \int_0^1 f(x)dx \qquad (n \in \mathbf{N}^*).$$

Proof. Let us denote by $(*_n)$ the equality to be proved. We shall show it by mathematical induction with respect to n. It is clear that $(*_n)$ holds for $n = 0$. Let us assume that $(*_n)$ holds for some n (≥ 0).

$$E(f(X(n+1))) = \int_0^{1/2} f(X(n)(2x))dx + \int_{1/2}^1 f(X(n)(2(1-x)))dx.$$

Taking the changes of variables $y = 2x$ and $y = 2(1-x)$ in the first term and the second term of the right-hand side of the above equation, respectively, we find that both the first term and the second term become $\frac{1}{2}\int_0^1 f(X(n)(y))dy$. Therefore, $E(f(X(n+1))) = E(f(X(n)))$ and so $(*_{n+1})$ holds. Hence, by mathematical induction, we see that $(*_n)$ holds for any $n \in \mathbf{N}^*$. (Q.E.D.)

Immediately from Lemma 6.1, we have

Lemma 6.2 *For any bounded Borel function $f : [0,1] \to \mathbf{R}$,*

$$E(f(\psi^n)) = E(f(X(0))) = \int_0^1 f(x)dx \qquad (n \in \mathbf{N}^*).$$

By putting $n = 1$ and $f = \chi_A$ $(A \in \mathcal{B}([0,1]))$, we have

Lemma 6.3 $\nu(\psi^{-1}A) = \nu(A)$ *for any $A \in \mathcal{B}([0,1])$.*

Lemma 6.3 implies that the probability measure ν is an invariant measure for the tent map ψ. We know that this probability measure ν is a unique invariant probability measure of the tent map ψ.

Moreover, we introduce a one-dimensional stochastic process $\mathbf{W} = (W(n); n \in \mathbf{N}^*)$ defined on the probability space $([0,1], \mathcal{B}([0,1]), \nu)$ by

$$W(n) \equiv 2\sqrt{3}(X(n) - \frac{1}{2}) \quad (n \in \mathbf{N}^*). \tag{83}$$

We call the stochastic process $\mathbf{W}$ a stochastic process associated with the tent map ψ.

A direct calculation gives us the following Theorem 6.1.

Theorem 6.1

(i) $\mathbf{W}$ *is identically distributed.*

(ii) $\mathbf{W}$ *is not independent.*

(iii) $\mathbf{W}$ *is a strictly stationary process.*

(iv) $E(W(n)) = 0 \qquad (n \in \mathbf{N}^*)$.

(v) $E(W(n)^2) = 1 \qquad (n \in \mathbf{N}^*)$.

(vi) $E((W(n)^2 - 1)^2) = \frac{4}{5} \qquad (n \in \mathbf{N}^*)$.

(vii) $E(W(0)^2 W(1)) = -\frac{\sqrt{3}}{2}$.

(viii) $E(W(0)^2 W(1)^2) = \frac{6}{5}$.

(ix) $E(W(n)^{2p+1}) = 0 \qquad (n, p \in \mathbf{N}^*)$.

(x) $E(W(0)^{2p+1} W(n)^q) = 0 \qquad (n \in \mathbf{N}, p, q \in \mathbf{N}^*)$.

(xi) $E(W(0)^{2p_0+1} W(1)^{p_1} W(2)^{p_2} \cdots W(n)^{p_n}) = 0 \qquad (n \in \mathbf{N}, p_0, p_1, \ldots, p_n \in \mathbf{N}^*)$.

Immediately from Theorem 6.1(iii), (iv), (v) and (x), we have

Theorem 6.2 *The stochastic process* $\mathbf{W}$ *is a white noise in a broad sense.*

In order to investigate the problem of coexistence of order and chaos for the stochastic process $\mathbf{W}$ associated with the tent map ψ, similarly as in the previous section, we introduce a two-dimensional stochastic process $\mathbf{W}_{(0,1)} = (W_{(0,1)}(n); n \in \mathbf{N}^*)$ defined on the probability space $([0,1], \mathcal{B}([0,1]), \nu)$ by

$$W_{(0,1)}(n) \equiv \ {}^t(W(n), \frac{\sqrt{5}}{2}(W^2(n) - 1)) \qquad (n \in \mathbf{N}^*). \tag{84}$$

It follows from Theorem 6.1(iii) that the stochastic process $\mathbf{W}_{(0,1)}$ is a weakly stationary process with mean vector 0. We shall calculate the covariance matrix function $R(\mathbf{W}_{(0,1)}) = (R(\mathbf{W}_{(0,1)})(n); n \in \mathbf{Z})$ for the stochastic process $\mathbf{W}_{(0,1)}$.

Theorem 6.3 *The stochastic process* $\mathbf{W}_{(0,1)}$ *is a weakly stationary process whose covariance matrix function* $R(\mathbf{W}_{(0,1)})$ *is given by*

(i) $R(\mathbf{W}_{(0,1)})(0) = I_2$,

(ii) $R(\mathbf{W}_{(0,1)})(1) = \begin{pmatrix} 0 & -\frac{\sqrt{15}}{4} \\ 0 & \frac{1}{4} \end{pmatrix}$,

(iii) $R(\mathbf{W}_{(0,1)})(n) = \frac{1}{4}R(\mathbf{W}_{(0,1)})(n-1) = \frac{1}{4^{n-1}}R(\mathbf{W}_{(0,1)})(1) \qquad (n \geq 2)$.

Proof. From Theorem 6.1(iv), (v), (vi), (ix) and (x), we see that (i) and (ii) are proved. In order to prove (iii), we shall show the following relation.

$$E((W(0)^2-1)W(n)) = \frac{1}{4}E((W(0)^2-1)W(n-1)) \qquad (n \geq 2). \tag{85}$$

It follows from Theorem 6.1 that

$$\begin{aligned} E((W(0)^2-1)W(n)) &= E(W(0)^2W(n)) \\ &= \sqrt{12}^3(J_1+J_2), \end{aligned}$$

where

$$J_1 \equiv \int_0^{1/2}(x-\frac{1}{2})^2(X(n-1)(\psi(x))-\frac{1}{2})dx \quad \text{and} \quad J_2 \equiv \int_{1/2}^1(x-\frac{1}{2})^2(X(n-1)(\psi(x))-\frac{1}{2})dx.$$

By the definition of the tent map ψ, we see that

$$J_1 = \frac{1}{8}\int_0^1(y-1)^2(X(n-1)(y)-\frac{1}{2})dy \quad \textit{and} \quad J_2 = \frac{1}{8}\int_0^1(y-1)^2(X(n-1)(y)-\frac{1}{2})dy.$$

Therefore

$$\begin{aligned} E((W(0)^2-1)W(n)) &= \frac{\sqrt{12}^3}{4}\int_0^1(y-1)^2(X(n-1)(y)-\frac{1}{2})dy \\ &= \frac{\sqrt{12}^3}{4}\int_0^1((y-\frac{1}{2})^2-(y-\frac{1}{2}-\frac{1}{4}))(X(n-1)(y)-\frac{1}{2})dy \\ &= \frac{\sqrt{12}^3}{4}\{E((X(0)-\frac{1}{2})^2(X(n-1)-\frac{1}{2}))- \\ &\quad -E((X(0)-\frac{1}{2})(X(n-1)-\frac{1}{2}))+\frac{1}{4}E(X(n-1)-\frac{1}{2})\} \\ &= \frac{1}{4}E(W(0)^2W(n-1)), \end{aligned}$$

which implies that (85) holds.

Similarly as in the proof of (85), we can show the following relation.

$$E((W(0)^2-1)(W(n)^2-1)) = \frac{1}{4}E((W(0)^2-1)(W(n-1)^2-1)) \quad (n \geq 2). \tag{86}$$

Thus, it follows from Theorem 6.1(iv), (x), (85) and (86) that (iii) holds. (Q.E.D.)

Next, we shall calculate the system of the minimum KM_2O-Langevin matrices for the stochastic process $\mathbf{W}_{(0,1)}$ associated with the tent map. Similarly as in the logistic map, for any fixed natural number N, we restrict the time domain of the stochastic process $\mathbf{W}_{(0,1)}$ to the set $\{0,1,\ldots,N\}$ and define a two-dimensional flow $\mathbf{Z}=(Z(n);0\leq n\leq N)$ in the real inner product space $L^2([0,1],\mathcal{B}([0,1]),\nu)$ by

$$Z(n)\equiv W_{(0,1)}(n) \qquad (0\leq n\leq N). \tag{87}$$

Lemma 6.4 *The stochastic process $\mathbf{W}_{(0,1)}$ is degenerate.*

Proof. By noting $R(\mathbf{W}_{(0,1)})(0)=I_2$, we can let w tend to 0 in (37) to see that $\delta^0_+(\mathbf{W}_{(0,1)})(1)=-R(\mathbf{W}_{(0,1)})(1)$ and $\delta^0_-(\mathbf{W}_{(0,1)})(1)=-R(\mathbf{W}_{(0,1)})(-1)$, which with Theorem 6.3(ii) implies that

$$\delta^0_+(\mathbf{W}_{(0,1)})(1) = -\begin{pmatrix} 0 & -\frac{\sqrt{15}}{4} \\ 0 & \frac{1}{4}\end{pmatrix}, \tag{88}$$

$$\delta^0_-(\mathbf{W}_{(0,1)})(1) = -\begin{pmatrix} 0 & 0 \\ -\frac{\sqrt{15}}{4} & \frac{1}{4}\end{pmatrix}. \tag{89}$$

On the other hand, it follows from Theorems 3.1(ii), 6.3(i), (88) and (89) that

$$V_+(\mathbf{W}_{(0,1)})(1) = \frac{1}{16}\begin{pmatrix} 1 & \sqrt{15} \\ \sqrt{15} & 15\end{pmatrix}, \tag{90}$$

$$V_-(\mathbf{W}_{(0,1)})(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}. \tag{91}$$

In particular, we have

$$\det V_+(\mathbf{W}_{(0,1)})(1)=\det V_-(\mathbf{W}_{(0,1)})(1)=0, \tag{92}$$

which implies that the stochastic process $\mathbf{W}_{(0,1)}$ is degenerate. (Q.E.D.)

Since it follows from Lemma 6.4 that the flow $\mathbf{Z}$ is degenerate, we shall apply the weight transformation to it and obtain the system $\mathcal{LM}(\mathbf{Z})$ of the minimum KM$_2$O-Langevin matrices associated with the flow $\mathbf{Z}^w$.

For simplicity of the notation, we adopt the same notation as in (68) and (66) with $\mathbf{Z}$ in (65) replaced by $\mathbf{Z}$ in (87). It follows from Theorem 6.3 that

Lemma 6.5 *For any $n(1\leq n\leq N)$,*

$$R(\mathbf{Z})(n)=R^n,$$

where the matrix R of order 2 is given by

$$R=\begin{pmatrix} 0 & -\frac{\sqrt{15}}{4} \\ 0 & \frac{1}{4}\end{pmatrix}.$$

Proof. We have only to note that $R^2=\frac{1}{4}R$ and $R=R(1)$. (Q.E.D.)

It follows from Lemma 2.2(i), Theorem 6.3 and Lemma 6.5 that

Lemma 6.6 *For any $w>0$,*

(i) $\quad R^w(0) = (1+w^2)I_2,$

(ii) $\quad \begin{cases} R^w(n) = R^n & (1 \le n \le N), \\ R^w(-n) = ({}^tR)^n & (1 \le n \le N). \end{cases}$

Using (32), we shall show

Lemma 6.7 *For any $w > 0$ and any n $(1 \le n \le N-1)$,*

(i) $\delta_+^w(n+1) = w^2 \delta_+^w(n) R V_-^w(n)^{-1}$,

(ii) $\delta_-^w(n+1) = w^2 \delta_-^w(n)\ {}^tR V_+^w(n)^{-1}$.

Proof. It follows from (37), Lemmas 6.5 and 6.6 that

$$\begin{aligned} \delta_+^w(n+1)V_-^w(n) &= -(R^{n+1} + \sum_{k=0}^{n-1} \gamma_+^w(n,k)R^{k+1}) \\ &= -(R^w(n) + \sum_{k=0}^{n-1} \gamma_+^w(n,k)R^w(k) - w^2\delta_+^w(n))R. \end{aligned}$$

Therefore, applying (32) to the first term of the right-hand side of the above equation, we see that (i) holds. We note that the matrix $V_-^w(n)$ is invertible, because the stochastic process $\mathbf{Z}^w$ is non-degenerate. Similarly, (ii) can be proved. (Q.E.D.)

By a direct calculation, we can show from (37), Theorem 3.1(ii), Lemmas 6.6 and 6.7 that

Lemma 6.8 *For any $w > 0$,*

(i) $\quad V_+^w(0) = V_-^w(0) = (1+w^2)I_2,$

(ii) $\quad \begin{cases} \delta_+^w(1) = -\frac{1}{1+w^2}R, \\ \delta_-^w(1) = -\frac{1}{1+w^2}\ {}^tR, \end{cases}$

(iii) $\quad \begin{cases} V_+^w(1) = \frac{1}{16(1+w^2)} \begin{pmatrix} 16(1+w^2)^2 - 15 & \sqrt{15} \\ \sqrt{15} & 16(1+w^2)^2 - 1 \end{pmatrix}, \\ V_-^w(1) = \frac{1}{1+w^2} \begin{pmatrix} (1+w^2)^2 & 0 \\ 0 & (1+w^2)^2 - 1 \end{pmatrix}, \end{cases}$

(iv) $\quad \begin{cases} \delta_+^w(2) = -\frac{1}{4(2+w^2)}R, \\ \delta_-^w(2) = -\frac{1}{4(2+w^2)}\ {}^tR. \end{cases}$

We shall show

Lemma 6.9 *For any $w > 0$ and any n $(1 \le n \le N)$, there exists a real number $\alpha_n(w) \ne 0$ such that*

$$\begin{cases} \delta_+^w(n) = (\prod_{k=1}^n \alpha_k(w))R & (1 \le n \le N), \\ \delta_-^w(n) = (\prod_{k=1}^n \alpha_k(w))\ {}^tR & (1 \le n \le N). \end{cases}$$

In particular,

$$\alpha_1(w) = -\frac{1}{1+w^2} \quad \textit{and} \quad \alpha_2(w) = \frac{1+w^2}{4(2+w^2)}.$$

Proof. We shall show Lemma 6.9 by mathematical induction with respect to n. It follows from Lemma 6.8(ii) and (iv) that Lemma 6.9 holds for $n = 1, 2$. For any fixed natural number n_0 $(2 \le n_0 \le N-1)$, let us assume that Lemma 6.9 holds for any natural number n $(1 \le n \le n_0)$. It follows from Lemma 6.7(i) and Theorem 3.1(ii) that

$$\begin{aligned}
\delta_+^w(n_0+1) &= \delta_+^w(n_0)(w^2R)V_-^w(n_0)^{-1} \\
&= \delta_+^w(n_0)(w^2R)((I_2 - \delta_-^w(n_0)\delta_+^w(n_0))V_-^w(n_0-1))^{-1} \\
&= \delta_+^w(n_0)(w^2R)V_-^w(n_0-1)^{-1}(I_2 - \delta_-^w(n_0)\delta_+^w(n_0))^{-1}.
\end{aligned}$$

Using the hypothesis in mathematical induction and Lemma 6.7(i), we have

$$\begin{aligned}
\delta_+^w(n_0+1) &= \alpha_{n_0}(w)\delta_+^w(n_0-1)(w^2R)V_-^w(n_0-1)^{-1}(I_2 - \delta_-^w(n_0)\delta_+^w(n_0))^{-1} \\
&= \alpha_{n_0}(w)\delta_+^w(n_0)(I_2 - \delta_-^w(n_0)\delta_+^w(n_0))^{-1} \\
&= \alpha_{n_0}(w)(\prod_{k=1}^{n_0}\alpha_k(w))R(I_2 - \prod_{k=1}^{n_0}(\alpha_k^2(w))\ {}^tRR)^{-1}.
\end{aligned}$$

Noting the matrix R in Lemma 6.5, we see from a direct calculation that

$$R(I_2 - \prod_{k=1}^{n_0}(\alpha_k^2(w))\ {}^tRR)^{-1} = \frac{1}{1 - \prod_{k=1}^{n_0}\alpha_k^2(w)}R.$$

Therefore, using the hypothesis in mathematical induction again, we have

$$\begin{aligned}
\delta_+^w(n_0+1) &= \alpha_{n_0}(w)(\prod_{k=1}^{n_0}\alpha_k(w))\frac{1}{1 - \prod_{k=1}^{n_0}\alpha_k^2(w)}R \\
&= \frac{\alpha_{n_0}(w)}{1 - \prod_{k=1}^{n_0}\alpha_k^2(w)}(\prod_{k=1}^{n_0}\alpha_k(w))R \\
&= \frac{\alpha_{n_0}(w)}{1 - \prod_{k=1}^{n_0}\alpha_k^2(w)}\delta_+^w(n_0).
\end{aligned}$$

Similarly, we have

$$\delta_-^w(n_0+1) = \frac{\alpha_{n_0}(w)}{1 - \prod_{k=1}^{n_0}\alpha_k^2(w)}\delta_-^w(n_0).$$

Therefore, putting $\alpha_{n_0+1}(w) \equiv \frac{\alpha_{n_0}(w)}{1-\prod_{k=1}^{n_0}\alpha_k^2(w)}$, we see that Lemma 6.9 holds for $n = n_0 + 1$.

By mathematical induction, we have proved Lemma 6.9. (Q.E.D.)

We can see from the proof of Lemma 6.9 that

Lemma 6.10 *For any $w > 0$ and any n $(2 \le n \le N-1)$,*

$$\alpha_{n+1}(w) = \frac{\alpha_n(w)}{1 - \prod_{k=1}^{n}\alpha_k^2(w)}.$$

We shall show

Lemma 6.11 *For any $w > 0$ and any n $(3 \le n \le N-1)$,*

$$\frac{1}{\alpha_{n+1}(w)} + \alpha_n(w) = \frac{1}{\alpha_n(w)} + \alpha_{n-1}(w).$$

Proof. It follows from Lemma 6.10 that

$$\alpha_{n+1}(w) = \frac{\alpha_n(w)}{1 - \alpha_n^2(w)\prod_{k=1}^{n-1}\alpha_k^2(w)} \qquad (2 \le n \le N-1). \tag{93}$$

Using Lemma 6.10 again, we see that for any natural number n $(3 \le n \le N-1)$,

$$\alpha_n(w)(1 - \prod_{k=1}^{n-1}\alpha_k^2(w)) = \alpha_{n-1}(w).$$

Therefore

$$\prod_{k=1}^{n-1}\alpha_k^2(w) = 1 - \frac{\alpha_{n-1}(w)}{\alpha_n(w)}. \tag{94}$$

Substituting (94) into (93), we have

$$\alpha_{n+1}(w) = \frac{\alpha_n(w)}{1 - \alpha_n^2(w) + \alpha_n(w)\alpha_{n-1}(w)},$$

which implies that Lemma 6.11 holds. (Q.E.D.)

Using Lemmas 6.9 and 6.10, we see from a direct calculation that

Lemma 6.12 *For any $w > 0$,*

$$\frac{1}{\alpha_3(w)} + \alpha_2(w) = \frac{32 + 17w^2}{4(1+w^2)}.$$

It follows from Lemmas 6.11 and 6.12 that

Lemma 6.13 *For any $w > 0$ and any n $(2 \le n \le N-1)$,*

$$\alpha_{n+1}(w) = \frac{1}{\frac{32+17w^2}{4(1+w^2)} - \alpha_n(w)}.$$

We shall show

Lemma 6.14 (i) $\lim_{w\to 0}\alpha_1(w) = -1$;

(ii) *For any n $(2 \le n \le N)$, there exists a $\lim_{w\to 0}\alpha_n(w)$ with $|\lim_{w\to 0}\alpha_n(w)| < 1$.*

Proof. (i) comes from Lemma 6.9. We shall show (ii) by mathematical induction with respect to n. It follows from Lemma 6.9 that (ii) holds for $n = 2$. For any fixed natural number $n_0 \geq 2$, let us assume that (ii) holds for $n = n_0$. Since it follows that

$$\begin{aligned} |\lim_{w\to 0}(\frac{32+17w^2}{4(1+w^2)} - \alpha_{n_0}(w))| &= |8 - \lim_{w\to 0}\alpha_{n_0}(w)| \\ &\geq 8 - |\lim_{w\to 0}\alpha_{n_0}(w)| > 7, \end{aligned}$$

we see from Lemma 6.13 that there exists a $\lim_{w\to 0}\alpha_{n_0+1}(w)$ and it satisfies the following:

$$|\lim_{w\to 0}\alpha_{n_0+1}(w)| = \frac{1}{|\lim_{w\to 0}(\frac{32+17w^2}{4(1+w^2)} - \alpha_{n_0}(w))|} < 1,$$

which implies that (ii) holds for $n = n_0 + 1$. By mathematical induction, we see that (ii) holds. (Q.E.D.)

After the above preparations, we shall show the following theorem.

Theorem 6.4 *The minimum KM$_2$O-Langevin partial autocorrelation matrices can be obtained according to the following algorithm*:

(i) $\delta^0_+(\mathbf{W}_{(0,1)})(n) = {}^t\delta^0_-(\mathbf{W}_{(0,1)})(n) \quad (n \geq 1)$,

(ii) $\delta^0_+(\mathbf{W}_{(0,1)})(1) = \begin{pmatrix} 0 & \frac{\sqrt{15}}{4} \\ 0 & -\frac{1}{4} \end{pmatrix}$,

(iii) $\delta^0_\pm(\mathbf{W}_{(0,1)})(n) = \alpha_n\delta^0_\pm(\mathbf{W}_{(0,1)})(n-1) \qquad (n \geq 2)$,
where

$$\begin{cases} \alpha_2 = \frac{1}{8} \\ \alpha_n = \frac{1}{8-\alpha_{n-1}} \qquad (n \geq 3). \end{cases}$$

Proof. By virtue of Lemma 6.14, we can define $\alpha_n \equiv \lim_{w\to 0}\alpha_n(w)$. It then follows from Lemmas 6.9 and 6.13 that Theorem 6.4 holds. (Q.E.D.)

7. Discussion

We have treated the logistic map and the tent map in Section 5 and Section 6, respectively. These dynamical systems whose initial distributions are governed by their unique invariant probability measures have strictly stationarity property from the viewpoint of stochastic processes (Theorems 5.1 and 6.1). In particular, the weakly stationarity property of these stochastic processes $\mathbf{W}$ can be characterized as the fluctuation-dissipation theorem, which is represented as the relation among the system of the minimum KM$_2$O-Langevin matrices associated with the stochastic process (Theorem 3.1).

By taking out 18 kinds of two-dimensional stochastic processes $\mathbf{W}_{(0,j)}$ $(1 \leq j \leq 18)$ from 19 kinds of stochastic processes of rank 6 of the generating system of polynomial type for non-linear information spaces of the stochastic process $\mathbf{W}$ associated with the logistic map, we have calculated covariance matrix functions: Theorems 5.3 and 5.7 - 5.23 for the logistic map. On the other hand, for the tent map, we have used a two-dimensional stochastic process $\mathbf{W}_{(0,1)}$ taken out from the generating system of rank 6 of polynomial type, and have calculated its covariance matrix function: Theorems 6.3.

Further, in order to obtain a mathematical representation for a philosophical concept of the coexistence of order and chaos for the chaotic maps above from the theory of $\mathrm{KM_2O}$-Langevin equations, we have found certain new relations—Theorem 5.4 for the logistic map and Theorem 6.4 for the tent map—besides the fluctuation-dissipation theorem. In fact, as stated in the last paragraph of Section 5, we have found that all the covariance matrix functions for two-dimensional stochastic processes $\mathbf{W}_{(0,j)}$ $(1 \le j \le 18)$ have the same structure as that of the moving average processes, but the minimum $\mathrm{KM_2O}$-Langevin partial autocorrelation matrix functions $\delta^0_{\pm}(\mathbf{W}_{(0,j)})$ except $j = 4, 10$ have the same structure as that of the autoregressive processes.

On the other hand, also for $j = 4, 10$, by taking the proof of Theorem 6.4 for the tent map into deep consideration, we can show that the minimum $\mathrm{KM_2O}$-Langevin partial autocorrelation matrix functions $\delta^0_{\pm}(\mathbf{W}_{(0,j)})$ for $j = 4, 10$ have the same structure as that of $\mathbf{W}_{(0,1)}$ for the tent map. In fact, we can show the following theorem.

Theorem 7.1 *For the logistic map, the minimum KM_2O-Langevin partial autocorrelation matrix functions $\delta^0_{\pm}(\mathbf{W}_{(0,j)})$ for $j = 4, 10$ can be obtained according to the following algorithm*:

(i) (a) $$\delta^0_{+}(\mathbf{W}_{(0,4)})(1) = \begin{pmatrix} 0 & \frac{4}{\sqrt{17}} \\ 0 & -\frac{4}{17} \end{pmatrix} \quad \text{and} \quad \delta^0_{-}(\mathbf{W}_{(0,4)})(1) = \begin{pmatrix} 0 & 0 \\ \frac{4}{\sqrt{17}} & -\frac{4}{17} \end{pmatrix},$$

(b) $$\delta^0_{+}(\mathbf{W}_{(0,4)})(2) = \begin{pmatrix} 0 & \sqrt{17} \\ 0 & 16 \end{pmatrix} \quad \text{and} \quad \delta^0_{-}(\mathbf{W}_{(0,4)})(2) = \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{17}} & 0 \end{pmatrix},$$

(c) $$\delta^0_{\pm}(\mathbf{W}_{(0,4)})(n) = \beta_n \delta^0_{\pm}(\mathbf{W}_{(0,4)})(n-1) \qquad (n \ge 3),$$
where
$$\begin{cases} \beta_3 = -\frac{2}{17} \\ \beta_n = -\frac{2}{17+2\beta_{n-1}} \qquad (n \ge 4). \end{cases}$$

(ii) (a) $$\delta^0_{+}(\mathbf{W}_{(0,10)})(1) = \begin{pmatrix} 2 & -\sqrt{5} \\ \frac{4}{\sqrt{5}} & -2 \end{pmatrix} \quad \text{and} \quad \delta^0_{-}(\mathbf{W}_{(0,10)})(1) = \begin{pmatrix} 0 & 0 \\ -\frac{1}{\sqrt{5}} & 0 \end{pmatrix},$$

(b) $$\delta^0_{\pm}(\mathbf{W}_{(0,10)})(n) = \gamma_n \delta^0_{\pm}(\mathbf{W}_{(0,10)})(n-1) \qquad (n \ge 2),$$
where
$$\begin{cases} \gamma_2 = -\frac{1}{5} \\ \gamma_n = -\frac{1}{5+\gamma_{n-1}} \qquad (n \ge 3). \end{cases}$$

We shall in Table 7.1 show the results that the covariance matrix functions and the minimum $\mathrm{KM_2O}$-Langevin partial autocorrelation matrix functions for 18 kinds of two-dimensional stochastic processes $\mathbf{W}_{(0,j)}$ $(1 \le j \le 18)$ have the same structure as that of MA-process and that of AR-process, respectively. The "recursive" in $\delta^0_{\pm}(\mathbf{W}_{(0,j)})(n)$ for $j = 4, 10$ implies the behavior of the minimum $\mathrm{KM_2O}$-Langevin partial autocorrelation matrix functions in Theorem 7.1, which is similar to the one in Theorem 6.4 for the tent map.

j	$R(\mathbf{W}_{(0,j)})(n)$	$\delta^0_{\pm}(\mathbf{W}_{(0,j)})(n)$
1	$0\ (\lvert n\rvert \geq 2)$	$0\ (n \geq 2)$
2	$0\ (\lvert n\rvert \geq 1)$	$0\ (n \geq 1)$
3	$0\ (\lvert n\rvert \geq 2)$	$0\ (n \geq 2)$
4	$0\ (\lvert n\rvert \geq 3)$	recursive
5	$0\ (\lvert n\rvert \geq 2)$	$0\ (n \geq 2)$
6	$0\ (\lvert n\rvert \geq 1)$	$0\ (n \geq 1)$
7	$0\ (\lvert n\rvert \geq 1)$	$0\ (n \geq 1)$
8	$0\ (\lvert n\rvert \geq 2)$	$0\ (n \geq 2)$
9	$0\ (\lvert n\rvert \geq 3)$	$0\ (n \geq 3)$

j	$R(\mathbf{W}_{(0,j)})(n)$	$\delta^0_{\pm}(\mathbf{W}_{(0,j)})(n)$
10	$0\ (\lvert n\rvert \geq 2)$	recursive
11	$0\ (\lvert n\rvert \geq 1)$	$0\ (n \geq 1)$
12	$0\ (\lvert n\rvert \geq 3)$	$0\ (n \geq 3)$
13	$0\ (\lvert n\rvert \geq 2)$	$0\ (n \geq 2)$
14	$0\ (\lvert n\rvert \geq 1)$	$0\ (n \geq 1)$
15	$0\ (\lvert n\rvert \geq 2)$	$0\ (n \geq 2)$
16	$0\ (\lvert n\rvert \geq 4)$	$0\ (n \geq 4)$
17	$0\ (\lvert n\rvert \geq 3)$	$0\ (n \geq 3)$
18	$0\ (\lvert n\rvert \geq 1)$	$0\ (n \geq 1)$

Table 7.1 The covariance matrix functions and the minimum KM_2O-Langevin partial autocorrelation matrix functions for 18 kinds of two-dimensional stochastic processes $\mathbf{W}_{(0,j)}$ $(1 \leq j \leq 18)$ associated with the logistic map

By taking account of the form of the covariance matrix functions of 18 kinds of two-dimensional stochastic processes $\mathbf{W}_{(0,j)}$ $(1 \leq j \leq 18)$ in Theorems 5.3 and 5.7 - 5.23 and their proofs, we can take out certain algorithms determining the covariance matrix functions and show the behavior of the minimum KM_2O-Langevin partial autocorrelation matrix functions only from their algorithms, apart from the logistic map, which will appear in subsequent paper Okabe and Matsuura (in preparation). As its application, we shall treat 17 kinds of two-dimensional stochastic processes $\mathbf{W}_{(0,j)}$ except $j = 1$ associated with the tent map. Further, we shall investigate a mathematical representation for a philosophical concept of the coexistence of order and chaos for both the logistic map and the tent map.

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