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AN EDGEWORTH EXPANSION OF A CONVEX COMBINATION OF U-STATISTICS BASED ON STUDENTIZATION

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Abstract

As an estimator of an estimable parameter, Toda and Yamato (2001) introduce Y-statistic which is a convex combination of U-statistics including V-statistic and LB-statistic. We give the Edgeworth expansions of studentized Y-statistic about the estimable parameter using a jackknife variance estimator, with remainder $o(n^{-1})$.

Key Words and Phrases: Edgeworth expansion, Convex combination of U-statistics, Studentization.

1. Introduction

Let $\theta(F)$ be an estimable parameter of an unknown distribution F . Let $g(x_1, \dots, x_k)$ be the symmetric kernel of degree $k (\geq 2)$ for this parameter $\theta(F)$. In this paper, we assume that the kernel g is not degenerate. Let $X_1, \dots, X_n$ be a random sample of size n from the distribution F . Let X be a random variable having the distribution F .

As an estimator of $\theta(F)$, a convex combination Y_n of U-statistics is introduced by Toda and Yamato (2001) as follows: Let $w(r_1, \dots, r_j; k)$ be a nonnegative and symmetric function of positive integers $r_1, \dots, r_j$ such that $j = 1, \dots, k$ and $r_1 + \dots + r_j = k$, where k is the degree of the kernel g and fixed. We assume that at least one of $w(r_1, \dots, r_j; k)$'s is positive. For $j = 1, \dots, k$, let $g_{(j)}(x_1, \dots, x_j)$ be the kernel given by

$$g_{(j)}(x_1, \dots, x_j) = \frac{1}{d(k, j)} \sum_{r_1 + \dots + r_j = k}^+ w(r_1, \dots, r_j; k) g(\underbrace{x_1, \dots, x_{r_1}}_{r_1}, \dots, \underbrace{x_j, \dots, x_j}_{r_j}), \quad (1)$$

where the summation $\sum_{r_1 + \dots + r_j = k}^+$ is taken over all positive integers $r_1, \dots, r_j$ satisfying $r_1 + \dots + r_j = k$ with j and k fixed and $d(k, j) = \sum_{r_1 + \dots + r_j = k}^+ w(r_1, \dots, r_j; k)$ for $j = 1, 2, \dots, k$. Let $U_n^{(j)}$ be the U-statistic associated with this kernel $g_{(j)}(x_1, \dots, x_j)$

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for $j = 1, \dots, k$. The kernel $g_{(j)}(x_1, \dots, x_j)$ is symmetric because of the symmetry of $w(r_1, \dots, r_j; k)$. If $d(k, j)$ is equal to zero for some j , then the associated $w(r_1, \dots, r_j; k)$'s are equal to zero. In this case, we let the corresponding statistic $U_n^{(j)}$ be zero. The statistic Y_n is given by

$$Y_n = \frac{1}{D(n, k)} \sum_{j=1}^k d(k, j) \binom{n}{j} U_n^{(j)}, \quad (2)$$

where $D(n, k) = \sum_{j=1}^k d(k, j) \binom{n}{j}$. Since w 's are nonnegative and at least one of them is positive, $D(n, k)$ is positive. Note that $U_n^{(k)}$ is equal to the U-statistic U_n given below for $w(1, \dots, 1; k) > 0$, because of $g_{(k)} = g$.

Another type of a linear combination of U-statistics, L_n , is introduced by (3.3) of Sen (1977). While Y_n and L_n are both linear combination of U-statistics, Y_n is different from L_n in the mean that the weight function w 's determines Y_n as an estimator of θ . Since the coefficients of $U_n^{(j)}$ on the right-hand side of (1.2) are non-negative and their sum is equal to one, the linear combination given by (1.2) is also a convex combination.

For example, let w be the function given by $w(1, 1, \dots, 1; k) = 1$ and $w(r_1, \dots, r_j; k) = 0$ for positive integers $r_1, \dots, r_j$ such that $j = 1, \dots, k-1$ and $r_1 + \dots + r_j = k$. Then the corresponding statistic Y_n is equal to U-statistic U_n , which is given by $U_n = \binom{n}{k}^{-1} \sum_{1 \leq j_1 < \dots < j_k \leq n} g(X_{j_1}, \dots, X_{j_k})$, where $\sum_{1 \leq j_1 < \dots < j_k \leq n}$ denotes the summation over all integers $j_1, \dots, j_k$ satisfying $1 \leq j_1 < \dots < j_k \leq n$.

Let w be the function given by $w(r_1, \dots, r_j; k) = 1$ for positive integers $r_1, \dots, r_j$ such that $j = 1, \dots, k$ and $r_1 + \dots + r_j = k$. Then the corresponding statistic Y_n is equal to the LB-statistic B_n given by $B_n = \binom{n+k-1}{k}^{-1} \sum_{r_1 + \dots + r_n = k} g(X_1, \dots, X_1, \dots, X_n, \dots, X_n)$, where the numbers of $X_1, \dots, X_n$ are $r_1, \dots, r_n$, respectively, and $\sum_{r_1 + \dots + r_n = k}$ denotes the summation over all non-negative integers $r_1, \dots, r_n$ satisfying $r_1 + \dots + r_n = k$.

Let w be the function given by $w(r_1, \dots, r_j; k) = k!/(r_1! \dots r_j!)$ for positive integers $r_1, \dots, r_j$ such that $j = 1, \dots, k$ and $r_1 + \dots + r_j = k$. Then the corresponding statistic Y_n is equal to the V-statistic V_n given by $V_n = n^{-k} \sum_{j_1=1}^n \dots \sum_{j_k=1}^n g(X_{j_1}, \dots, X_{j_k})$. (See Toda and Yamato (2001).)

Let w be the function given by $w(r_1, \dots, r_j; k) = k!/(r_1 \dots r_j)$ for positive integers $r_1, \dots, r_j$ such that $j = 1, \dots, k$ and $r_1 + \dots + r_j = k$. Then, for example, the corresponding statistic Y_n for the third central moment of the distribution F is given by $S_n = n(n^2 + 1)^{-1} \sum_{i=1}^n (X_i - \bar{X})^3$, where $\bar{X}$ is the sample mean of $X_1, \dots, X_n$ (see Nomachi et al. (2002)).

The Edgeworth expansion of the standardized Y-statistic Y_n about θ is obtained with remainder $o(n^{-1})$ by Yamato et al. (2003). It also gives the Edgeworth expansion of the studentized Y-statistic with remainder $o(n^{-1/2})$. For the studentization of Y-statistic Y_n given by (1.2), we use a jackknife variance estimator. That is, as a variance

estimator of $\sqrt{n}Y_n$, we use $\hat{\sigma}_n^2$ given by

$$\hat{\sigma}_n^2 = (n-1) \sum_{i=1}^n (Y_n^{(i)} - Y_n)^2 \quad (3)$$

where $Y_n^{(i)}$ is the Y-statistic given by (1.2) computed from a sample of size $n-1$ with X_i left out.

Our purpose is to get an Edgeworth expansion of the studentized statistic Y_n given by (1.2), using the jackknife variance estimator $\hat{\sigma}_n^2$ with remainder term $o(n^{-1})$. For the studentized U-statistic, Helmers (1991) and Maesono (1995) obtained its Edgeworth expansion using a jackknife variance estimator with remainder term $o(n^{-1/2})$. Maesono (1997) get an Edgeworth expansion using a jackknife variance estimator with remainder term $o(n^{-1})$. Maesono (1996) gave an Edgeworth expansion of $\sqrt{n}[L_n - E(L_n)]/\hat{\sigma}_n$, where $\hat{\sigma}_n^2$ is a jackknife variance estimator of $\sqrt{n}[L_n - E(L_n)]$.

In Section 2, we give an Edgeworth expansion of $\sqrt{n}[Y_n - E(Y_n)]/\hat{\sigma}_n$, following Maesono (1996). In Section 3, using the result of Section 2 we shall derive another Edgeworth expansion about parameter θ , that is, the expansion of $\sqrt{n}[Y_n - \theta]/\hat{\sigma}_n$. We give some examples in Section 4. In Section 5, we give supplementary propositions necessary for the previous sections.

2. Studentized Y-statistic about its expectation

In the following sections, we assume $d(k, k) > 0$. Then, with $\delta_k = kd(k, k-1)/d(k, k)$ it holds that

$$\frac{d(k, k)}{D(n, k)} \binom{n}{k} = 1 - \frac{\delta_k}{n} + O\left(\frac{1}{n^2}\right), \quad (4)$$

and

$$\frac{d(k, k-1)}{D(n, k)} \binom{n}{k-1} = \frac{\delta_k}{n} + O\left(\frac{1}{n^2}\right). \quad (5)$$

For the U-statistic U_n , $d(k, k)n^{(k)}/[D(n, k)k!] = 1$ and $\delta_k = 0$. For the V-statistic V_n and the S-statistic S_n , $\delta_k = k(k-1)/2$. For the LB-statistic B_n , $\delta_k = k(k-1)$ (see Nomachi et al. (2002)).

We put

$$\psi_c(x_1, \dots, x_c) = E[g(X_1, \dots, X_k) \mid X_1 = x_1, \dots, X_c = x_c], \quad c = 1, 2, 3$$

and

$$g^{(1)}(x_1) = \psi_1(x_1) - \theta,$$

for $c = 2, 3$

$$g^{(c)}(x_1, \dots, x_c) = \psi_c(x_1, \dots, x_c) - \sum_{i=1}^{c-1} \sum_{1 \leq l_1 < \dots < l_i \leq c} g^{(i)}(x_{l_1}, \dots, x_{l_i}) - \theta.$$

For the kernel $g_{(k-1)}(x_1, \dots, x_{k-1})$, we put

$$\theta_{k-1} = E g_{(j)}(X_1, \dots, X_{k-1}),$$

$$\psi_{(k-1),1}(x_1) = E[g_{(k-1)}(X_1, \dots, X_{k-1}) \mid X_1 = x_1],$$

and

$$g_{(k-1)}^{(1)}(x_1) = \psi_{(k-1),1}(x_1) - \theta_{k-1}.$$

We put

$$\begin{aligned} \sigma_1^2 &= E[\{g^{(1)}(X)\}^2], \quad \sigma_2^2 = (k-1)^2 E[\{g^{(2)}(X_1, X_2)\}^2], \\ \nu &= \sigma_2^2 + \frac{2(k-1)\delta_k}{k} E[g^{(1)}(X)g_{(k-1)}^{(1)}(X)] - 2\delta_k\sigma_1^2, \\ f_1(x) &= \frac{1}{2}[\{g^{(1)}(x)\}^2 - \sigma_1^2] + (k-1)E[g^{(1)}(X_2)g^{(2)}(x, X_2)] \end{aligned}$$

and

$$\begin{aligned} f_2(x, y) &= -g^{(1)}(x)g^{(1)}(y) + (k-1)\left\{g^{(2)}(x, y)[g_1(x) + g_1(y)]\right. \\ &\quad \left.- E[g^{(2)}(x, X_3)g^{(1)}(X_3)] - E[g^{(2)}(y, X_3)g^{(1)}(X_3)]\right\} \\ &\quad + (k-1)^2 E[g^{(2)}(x, X_3)g^{(2)}(y, X_3)] \\ &\quad + (k-1)(k-2)E[g^{(3)}(x, y, X_3)g^{(1)}(X_3)], \end{aligned}$$

which satisfy the relations $E f_1(X) = 0$ and $E[f_2(X_1, X_2) \mid X_1] = 0$ a.s. (almost surely), respectively. Furthermore we put

$$\begin{aligned} \tau &= \frac{3E[f_1^2(X_1)]}{2\sigma_1^4} - \frac{\nu}{2\sigma_1^2}, \\ \zeta &= E[f_1(X_1)g^{(1)}(X_1)] \end{aligned}$$

and

$$\begin{aligned} a_1(x) &= \frac{\delta_k}{k}[(k-1)g_{(k-1)}^{(1)}(x) - kg^{(1)}(x)] + \tau g^{(1)}(x) \\ &\quad - \frac{1}{\sigma_1^2}\left\{[f_1(x)g^{(1)}(x) - \zeta] + \left(E[f_2(x, X_2)g^{(1)}(X_2)] - \frac{3\zeta}{\sigma_1^2}f_1(x)\right)\right. \\ &\quad \left.+ (k-1)E[g^{(2)}(x, X_2)f_1(X_2)]\right\}, \\ a_2(x, y) &= (k-1)g^{(2)}(x, y) - \frac{1}{\sigma_1^2}[f_1(x)g^{(1)}(y) + f_1(y)g^{(1)}(x)], \\ a_3(x, y, z) &= (k-1)(k-2)g^{(3)}(x, y, z) \\ &\quad - \frac{1}{\sigma_1^2}\left\{(k-1)[f_1(x)g^{(2)}(y, z) + f_1(y)g^{(2)}(x, z) + f_1(z)g^{(1)}(x, y)]\right. \\ &\quad \left.+ g^{(1)}(x)[f_2(y, z) - \frac{3}{\sigma_1^2}f_1(y)f_1(z)] + g^{(1)}(y)[f_2(x, z) - \frac{3}{\sigma_1^2}f_1(x)f_1(z)]\right. \\ &\quad \left.+ g^{(1)}(z)[f_2(x, y) - \frac{3}{\sigma_1^2}f_1(x)f_1(y)]\right\}, \end{aligned}$$

which satisfy the relations $E[a_1(X)] = 0$, $E[a_2(X_1, X_2)|X_1] = E[a_2(X_1, X_2)|X_2] = 0$ and $E[a_3(X_1, X_2, X_3)|X_1, X_2] = 0$ a.s. because of $E[f_1(X)] = 0$, $E[f_2(X_1, X_2)|X_1] = 0$ and $E[g^{(2)}(X_1, X_2)|X_2] = 0$ a.s. We define

$$\begin{aligned}\lambda_1 &= E[g^{(1)}(X_1)]^3, \\ \lambda_2 &= E[g^{(1)}(X_1)g^{(1)}(X_2)g^{(2)}(X_1, X_2)], \\ \lambda_3 &= E[g^{(1)}(X_1)]^4, \\ \lambda_4 &= E[(g^{(1)}(X_1))^2 g^{(1)}(X_2)g^{(2)}(X_1, X_2)], \\ \lambda_5 &= E[g^{(1)}(X_1)g^{(1)}(X_2)g^{(2)}(X_1, X_3)g^{(2)}(X_2, X_3)], \\ \lambda_6 &= E[g^{(1)}(X_1)g^{(1)}(X_2)g^{(1)}(X_3)g^{(3)}(X_1, X_2, X_3)], \\ \lambda_7 &= E[g^{(1)}(X_1)a_1(X_1)], \\ \kappa_3 &= \sigma_1^{-3}(\lambda_1 + 3\lambda_2), \\ \kappa_4 &= \sigma_1^{-4}(\lambda_3 - 3\sigma_1^4 + 12\lambda_4 + 12\lambda_5 + 4\lambda_6)\end{aligned}$$

and

$$\begin{aligned}Q_n(x) &= \Phi(x) - \phi(x) \left\{ \frac{\kappa_3}{6\sqrt{n}}(x^2 - 1) + \frac{\kappa_4}{24n}(x^3 - 3x) \right. \\ &\quad \left. + \frac{\kappa_3^2}{72n}(x^5 - 10x^3 + 15x) + \frac{x}{n\sigma_1^2} \left(\lambda_7 + \frac{1}{4}E[a_2^2(X_1, X_2)] \right) \right\}.\end{aligned}$$

LEMMA 2.1. (Maesono (1996)) *If $E|g(X_{i_1}, \dots, X_{i_k})|^2 < \infty$ for $1 \leq i_1 \leq \dots \leq i_k \leq k$, and $E|g(X_1, X_2, X_3, \dots, X_k)|^{4+\varepsilon} < \infty$ and $E|g(X_1, X_1, X_2, \dots, X_k)|^{4+\varepsilon} < \infty$ for $\varepsilon > 0$, then we have*

$$\hat{\sigma}_n^2 = k^2\sigma_1^2 + \frac{2k^2}{n} \sum_{i=1}^n f_1(X_i) + \frac{2k^2}{n(n-1)} \sum_{1 \leq i < j \leq n} f_2(X_i, X_j) + \frac{k^2\nu}{n} + o_p^*(n^{-1})$$

and

$$\begin{aligned}k\sigma_1\hat{\sigma}_n^{-1} &= 1 - \frac{1}{n\sigma_1^2} \sum_{i=1}^n f_1(X_i) - \frac{1}{n^2\sigma_1^2} \sum_{1 \leq i < j \leq n} \left[f_2(X_i, X_j) - \frac{3}{\sigma_1^2} f_1(X_i)f_1(X_j) \right] \\ &\quad + \frac{1}{n} \left\{ \frac{3E[f_1^2(X_1)]}{2\sigma_1^4} - \frac{\nu}{2\sigma_1^2} \right\} + o_p^*(n^{-1}) \quad (6)\end{aligned}$$

where $o_p^*(n^{-1})$ is a quantity satisfying $P(|o_p^*(n^{-1})| \geq cn^{-1}(\log n)^{-1}) = o(n^{-1})$ for a constant $c > 0$.

Thus by Maesono (1996) we have the following: Assume that $E|g(X_{i_1}, \dots, X_{i_k})|^2 < \infty$ for $1 \leq i_1 \leq \dots \leq i_k \leq k$, $E|g(X_1, X_2, X_3, \dots, X_k)|^9 < \infty$ and $E|g(X_1, X_1, X_2, \dots, X_k)|^{4+\varepsilon} < \infty$ for $\varepsilon > 0$. Then we have

$$\hat{\sigma}_n^{-1}\sqrt{n}(Y_n - E[Y_n]) = \frac{\sqrt{n}}{\sigma_1}U_n^* - \frac{\zeta}{\sqrt{n}\sigma_1^3} + o_p^*(n^{-1}), \quad (7)$$

where

$$U_n^* = \frac{1}{n} \sum_{i=1}^n \left\{ g^{(1)}(X_i) + \frac{a_1(X_i)}{n} \right\} + \frac{1}{n^2} \sum_{1 \leq i < j \leq n} a_2(X_i, X_j) + \frac{1}{n^3} \sum_{1 \leq i < j < l \leq n} a_3(X_i, X_j, X_l).$$

For Edgeworth expansion of the statistic, we use the result of Lai and Wang (1993) which needs the following conditions.

Condition (C): $E | g^{(2)} |^r < \infty$ for some $r > 2$ and there exists K Borel functions $h_j : R \rightarrow R$ such that $K(r-2) > 8(4r-5)$, $Eh_j^2(X_1) < \infty$ ($j = 1, \dots, K$), and the covariance matrix of $(W_1, \dots, W_K)$ is positive definite, where $W_j = (Lh_j)(X_1)$ and $(Lh)(x) = E[a_2(x, X_2)h(X_2)]$.

In case of $k \geq 3$, the original condition of Lai and Wang (1993) contains the term $I_{[E|g^{(3)}(X_1, X_2, X_3)| > 0]}$, which equals 1 since $g^{(3)}(X_1, X_2, X_3)$ is not zero a.s. under the assumption that the kernel g is not degenerate, that is, $\sigma_1^2 > 0$. It also contains $E | g^{(2)} |^r < \infty$ for some $r > 2$. This condition is satisfied with $r = 4$ under our condition $E[| \psi_3(X_1, X_2, X_3) |^4] < \infty$, which is necessary for the condition (A4) of Lai and Wang (1993).

Condition (D): There exist constants c_j and Borel functions $h_j : R \rightarrow R$ such that $Eh_j(X_1) = 0$, $E | h_j(X_1) |^r < \infty$ for some $r \geq 5$ and $a_2(X_1, X_2) = \sum_{j=1}^K c_j h_j(X_1) h_j(X_2)$ a.s.; moreover, for some $0 < \varepsilon < \min\{1, 2(1 - 11r^{-1}/3)\}$,

$$\limsup_{|t| \rightarrow \infty} \sup_{|s_1| + \dots + |s_K| \leq |t|^{-\varepsilon}} \left| E \exp \left(it \left[g^{(1)}(X_1) + \sum_{j=1}^K s_j h_j(X_1) \right] \right) \right| < 1. \quad (8)$$

The asymptotic expansion of the statistic $\sqrt{n}\sigma_1^{-1}U_n^*$ is given by the following.

LEMMA 2.2. (Maesono (1996)) *Assume that $E[g^{(1)}(X_1)]^4 < \infty$, $\sigma_1^2 > 0$, $E[|a_1(X_1)|^3 + |a_3(X_1, X_2, X_3)|^4] < \infty$ and $\limsup_{|t| \rightarrow \infty} |E[\exp\{itg^{(1)}(X_1)\}]| < 1$. If either condition (C) or (D) is satisfied, we have*

$$\sup_{-\infty < x < \infty} \left| P(\sqrt{n}\sigma_1^{-1}U_n^* \leq x) - Q_n(x) \right| = o(n^{-1}). \quad (9)$$

For example, by Minkowski's inequality and Schwarz's one, one of the above conditions $E[|a_1(X_1)|^3 + |a_3(X_1, X_2, X_3)|^4] < \infty$ is satisfied if

$$E|g^{(1)}(X_1)|^{12} < \infty, \quad E|g^{(2)}(X_1, X_2)|^{12} < \infty,$$

$$E|g^{(3)}(X_1, X_2, X_3)|^{12} < \infty, \quad E|g_{(k-1)}^{(1)}(X_1)|^3 < \infty.$$

We define

$$\begin{aligned}
e_1 &= E[g^{(1)}(X_1)]^3, \\
e_2 &= (k-1)E[g^{(1)}(X_1)g^{(1)}(X_2)g^{(2)}(X_1, X_2)], \\
e_3 &= E[g^{(1)}(X_1)]^4, \\
e_4 &= (k-1)E[(g^{(1)}(X_1))^2 g^{(1)}(X_2)g^{(2)}(X_1, X_2)], \\
e_5 &= (k-1)^2 E[g^{(1)}(X_1)g^{(1)}(X_2)g^{(2)}(X_1, X_3)g^{(2)}(X_2, X_3)], \\
e_6 &= (k-1)(k-2)E[g^{(1)}(X_1)g^{(1)}(X_2)g^{(1)}(X_3)g^{(3)}(X_1, X_2, X_3)], \\
v_1 &= \sigma_1^{-3}(2e_1 + 3e_2), \\
v_2 &= \sigma_1^{-3}(e_1 + 3e_2), \\
v_3 &= -\sigma_1^{-6}(2e_1 + 3e_2)^2, \\
v_4 &= 6\sigma_1^{-4}(e_3 - 6\sigma_1^4 + 12e_4 + 6e_5 + 4e_6) - 2\sigma_1^{-6}(2e_1 + 3e_2)(2e_1 + 9e_2), \\
v_5 &= 3\sigma_1^{-6}(4e_1^2 + 12e_1e_2 + 3e_2^2) + 18\sigma_1^{-4}(\sigma_1^2\sigma_2^2 - e_3 + 2\sigma_1^4 - 4e_4 - 2e_5).
\end{aligned}$$

Using $f_{12}(x) = E[g^{(1)}(X_2)g^{(2)}(x, X_2)]$ which appears in the second term of f_1 , we can write

$$\begin{aligned}
e_2 &= (k-1)E[g^{(1)}(X_1)f_{12}(X_1)], \quad e_4 = (k-1)E[\{g^{(1)}(X_1)\}^2 f_{12}(X_1)], \\
e_5 &= (k-1)^2 E[\{f_{12}(X_1)\}^2].
\end{aligned}$$

Furthermore, we define

$$H_n(x) = \Phi(x) + \phi(x)\frac{1}{6\sqrt{n}}(v_1x^2 + v_2) + \phi(x)\frac{1}{72n}(v_3x^5 + v_4x^3 + v_5x).$$

Between Q_n and H_n , it holds that

$$Q_n\left(x + \frac{\zeta}{\sqrt{n}\sigma_1^3}\right) = H_n(x) + o(n^{-1}). \quad (10)$$

The asymptotic expansion of the statistic $\hat{\sigma}_n^{-1}\sqrt{n}(Y_n - EY_n)$ is given by the following.

LEMMA 2.3. (Maesono (1996)) *Assume that $E|g(X_{i_1}, \dots, X_{i_k})|^2 < \infty$ for $1 \leq i_1 \leq \dots \leq i_k \leq k$, $E|g(X_1, X_2, \dots, X_k)|^9 < \infty$, $E|g(X_1, X_1, X_2, \dots, X_k)|^{4+\varepsilon} < \infty$ for $\varepsilon > 0$, and $E[g^{(1)}(X_1)]^4 < \infty$, $\sigma_1^2 > 0$. Furthermore we assume that $E[|a_1(X_1)|^3 + |a_3(X_1, X_2, X_3)|^4] < \infty$ and $\limsup_{|t| \rightarrow \infty} |E[\exp\{itg^{(1)}(X_1)\}]| < 1$. If either condition (C) or (D) is satisfied, we have*

$$\sup_{-\infty < x < \infty} \left| P(\hat{\sigma}_n^{-1}\sqrt{n}(Y_n - EY_n) \leq x) - H_n(x) \right| = o(n^{-1}). \quad (11)$$

3. Studentized Y-statistic about θ

At first, we note that

$$\hat{\sigma}_n^{-1}\sqrt{n}(Y_n - \theta) = \hat{\sigma}_n^{-1}\sqrt{n}(Y_n - EY_n) + \hat{\sigma}_n^{-1}\sqrt{n}(EY_n - \theta). \quad (12)$$

By Nomachi et al. (2002), (3.5), we have

$$\sqrt{n}(EY_n - \theta) = \frac{\mu_k}{\sqrt{n}} + O(n^{-3/2}) \quad (13)$$

where $\mu_k = \delta_k(\theta_{k-1} - \theta)$.

If we put $R_{2n} = o_p^*(n^{-1})$, then $\sqrt{n}(EY_n - \theta)R_{2n} = o_p^*(n^{-1})$. Because for a constant $c > 0$ we have $P(|\sqrt{n}(EY_n - \theta)R_{2n}| \geq cn^{-1}(\log n)^{-1}) \leq P(|R_{2n}| \geq cn^{-1}(\log n)^{-1}) = o(n^{-1})$, since $\sqrt{n}(EY_n - \theta) \leq 1$ for a large n . We multiply (2.3) by (3.2), and use this fact. Then, we get

$$k\sigma_1\hat{\sigma}_n^{-1}\sqrt{n}(EY_n - \theta) = \frac{\mu_k}{\sqrt{n}} \left\{ 1 - \frac{1}{n\sigma_1^2} \sum_{i=1}^n f_1(X_i) \right\} + R_n^* + o_p^*(n^{-1}) \quad (14)$$

where $E|R_n^*| = O(n^{-3/2})$. Thus from (2.4), (3.1) and (3.3) we get

$$\hat{\sigma}_n^{-1}\sqrt{n}(Y_n - \theta) = \frac{\sqrt{n}}{\sigma_1}U_n^{**} + \frac{1}{\sqrt{n}} \left\{ -\frac{\zeta}{\sigma_1^3} + \frac{\mu_k}{k\sigma_1} \right\} + R_n^* + o_p^*(n^{-1}), \quad (15)$$

where

$$U_n^{**} = \frac{1}{n} \sum_{i=1}^n \left\{ g^{(1)}(X_i) + \frac{a_1^*(X_i)}{n} \right\} + \frac{1}{n^2} \sum_{1 \leq i < j \leq n} a_2(X_i, X_j) + \frac{1}{n^3} \sum_{1 \leq i < j < l \leq n} a_3(X_i, X_j, X_l), \quad (16)$$

and

$$a_1^*(X_i) = a_1(X_i) - \frac{\mu_k}{k\sigma_1^3}f_1(X_i).$$

We can also obtain the expansion (3.4) by multiplying (2.3) and the following (3.6). For the detail of this multiplication, see Appendix.

LEMMA 3.1. (Yamato et al. (2003)) *Assume that $d(k, k) > 0$ and $E|g(X_{i_1}, \dots, X_{i_k})|^2 < \infty$ for $1 \leq i_1 \leq \dots \leq i_k \leq k$. Then, we have*

$$\sqrt{n}(Y_n - \theta) = Y_n^{**} + \frac{\mu_k}{\sqrt{n}} + R'_n, \quad (17)$$

where $E|R'_n|^2 = O(n^{-3})$ and

$$\begin{aligned} Y_n^{**} = & k \left(1 - \frac{\delta_k}{n} \right) \frac{1}{n^{1/2}} \sum_{i=1}^n g^{(1)}(X_i) + (k-1)\delta_k \frac{1}{n^{3/2}} \sum_{i=1}^n g_{(k-1)}^{(1)}(X_i) \\ & + k(k-1) \frac{1}{n^{3/2}} \sum_{1 \leq i < j \leq n} g^{(2)}(X_i, X_j) + k(k-1)(k-2) \frac{1}{n^{5/2}} \sum_{1 \leq i < j < l \leq n} g^{(3)}(X_i, X_j, X_l). \end{aligned}$$

In the asymptotic evaluation of $\hat{\sigma}_n^{-1}\sqrt{n}(Y_n - \theta)$ with remainder term $o(n^{-1})$, we can neglect at first the term $o_p^*(n^{-1})$ of (3.4) by using the relation given by Lemma 5.3 and then the terms R_n^* of (3.4) by using the relation given by Lemma 5.2. Thus, we can get the following.

LEMMA 3.2. *Assume that $d(k, k) > 0$, $E|g(X_{i_1}, \dots, X_{i_k})|^2 < \infty$ for $1 \leq i_1 \leq \dots \leq i_k \leq k$, $E|g(X_1, X_2, X_3, \dots, X_k)|^9 < \infty$ and $E|g(X_1, X_1, X_2, \dots, X_k)|^{4+\varepsilon} < \infty$ for $\varepsilon > 0$. Then, we have*

$$\sup_{-\infty < x < \infty} \left| P\left(\hat{\sigma}_n^{-1}\sqrt{n}(Y_n - \theta) \leq x\right) - P\left(\frac{\sqrt{n}}{\sigma_1}U_n^{**} + \frac{1}{\sqrt{n}}\left\{-\frac{\zeta}{\sigma_1^3} + \frac{\mu_k}{k\sigma_1}\right\} \leq x\right) \right| = o(n^{-1}). \quad (18)$$

U_n^{**} is different from U_n^* only in the term a_1^* . Thus the Edgeworth expansion of U_n^{**} is different from U_n^* in the term λ_7 . By Lemma 2.2, we get the following.

LEMMA 3.3. *Assume that $d(k, k) > 0$. Furthermore, we assume that $E|g(X_{i_1}, \dots, X_{i_k})|^2 < \infty$ for $1 \leq i_1 \leq \dots \leq i_k \leq k$, $E[g^{(1)}(X_1)]^4 < \infty$, $\sigma_1^2 > 0$, $E[|a_1(X_1)|^3 + |a_3(X_1, X_2, X_3)|^4] < \infty$ and $\limsup_{|t| \rightarrow \infty} |E[\exp\{itg^{(1)}(X_1)\}]| < 1$. If either condition (C) or (D) is satisfied, we have*

$$\sup_{-\infty < x < \infty} \left| P(\sqrt{n}\sigma_1^{-1}U_n^{**} \leq x) - Q_n^*(x) \right| = o(n^{-1}) \quad (19)$$

where $Q_n^*(x)$ is obtained from $Q_n(x)$ by replacing λ_7 with

$$\lambda_7^* = \lambda_7 - \frac{\mu_k}{k\sigma_1^3} \left(\frac{1}{2}e_1 + e_2 \right).$$

The last term of the above right-hand side is due to the bias of the Y-statistic. We also know that

$$Q_n^*(x) = Q_n(x) - \frac{\mu_k}{nk\sigma_1^5} \left(\frac{1}{2}e_1 + e_2 \right) x\phi(x). \quad (20)$$

By (2.6), we have

$$Q_n\left(x + \frac{\zeta}{\sqrt{n}\sigma_1^3} - \frac{\mu_k}{\sqrt{nk}\sigma_1}\right) = H_n\left(x - \frac{\mu_k}{\sqrt{nk}\sigma_1}\right) + o(n^{-1}) \quad (21)$$

and by Lemma 5.4

$$\begin{aligned} H_n\left(x - \frac{\mu_k}{\sqrt{nk}\sigma_1}\right) &= \Phi(x) + \phi(x) \frac{1}{6\sqrt{n}} \left(v_1x^2 + v_2 - 6\frac{\mu_k}{k\sigma_1} \right) \\ &+ \phi(x) \frac{1}{72n} \left(v_3x^5 + (v_4 + 12\frac{v_1\mu_k}{k\sigma_1})x^3 + \left[v_5 + 12\frac{\mu_k}{k\sigma_1}(v_2 - 2v_1) \right]x - 36\left(\frac{\mu_k}{k\sigma_1}\right)^2 \right) + O(n^{-3/2}). \end{aligned} \quad (22)$$

By (3.9), (3.10), (3.11) and Lemma 5.4, we can get

$$Q_n^*\left(x + \frac{1}{\sqrt{n}}\left\{\frac{\zeta}{\sigma_1^3} - \frac{\mu_k}{k\sigma_1}\right\}\right) = H_n^*(x) + O(n^{-3/2}), \quad (23)$$

where

$$H_n^*(x) = \Phi(x) + \phi(x) \frac{1}{6\sqrt{n}} \left(v_1 x^2 + v_2 - \frac{6\mu_k}{k\sigma_1} \right) + \phi(x) \frac{1}{72n} \left\{ v_3 x^5 + \left(v_4 + \frac{v_1 \mu_k}{k\sigma_1} \right) x^3 \right. \\ \left. + \left[v_5 + 12 \frac{\mu_k}{k\sigma_1} (v_2 - 2v_1) - 72 \frac{\mu_k}{k\sigma_1^5} \left(\frac{1}{2} e_1 + e_2 \right) \right] x - 36 \left(\frac{\mu_k}{k\sigma_1} \right)^2 \right\}.$$

Thus, by (3.7), (3.8), (3.11) and (3.12) we get the following.

THEOREM 3.4. *Assume that $d(k, k) > 0$, $E|g(X_{i_1}, \dots, X_{i_k})|^2 < \infty$ for $1 \leq i_1 \leq \dots \leq i_k \leq k$, $E[g^{(1)}(X_1)]^4 < \infty$, $\sigma_1^2 > 0$, $E|g(X_1, X_2, \dots, X_k)|^9 < \infty$, and $E|g(X_1, X_1, X_2, \dots, X_k)|^{4+\varepsilon} < \infty$ for $\varepsilon > 0$. Furthermore we assume that $\limsup_{|t| \rightarrow \infty} |E[\exp\{itg^{(1)}(X_1)\}]| < 1$ and $E[|a_1(X_1)|^3 + |a_3(X_1, X_2, X_3)|^4] < \infty$. If either condition (C) or (D) is satisfied, we have*

$$\sup_{-\infty < x < \infty} \left| P(\hat{\sigma}_n^{-1} \sqrt{n}(Y_n - \theta) \leq x) - H_n^*(x) \right| = o(n^{-1}). \quad (24)$$

As stated after Lemma 2.2, one of the conditions of Theorem 3.4 $E[|a_1(X_1)|^3 + |a_3(X_1, X_2, X_3)|^4] < \infty$ is satisfied if $E|g^{(1)}(X_1)|^{12} < \infty$, $E|g^{(2)}(X_1, X_2)|^{12} < \infty$, and $E|g^{(3)}(X_1, X_2, X_3)|^{12} < \infty$, $E|g_{(k-1)}^{(1)}(X_1)|^3 < \infty$.

COROLLARY 3.5. *Especially, let the degree k be 2. Assume that $d(k, k) > 0$, $E|g(X_1, X_1)|^{4+\varepsilon} < \infty$ for $\varepsilon > 0$, $E[g^{(1)}(X_1)]^4 < \infty$, $\sigma_1^2 > 0$, and $E|g(X_1, X_2)|^9 < \infty$. Furthermore we assume that $\limsup_{|t| \rightarrow \infty} |E[\exp\{itg^{(1)}(X_1)\}]| < 1$ and $E[|a_1(X_1)|^3 + |a_3(X_1, X_2, X_3)|^4] < \infty$. If either condition (C) or (D) is satisfied, we have (3.13).*

In the case of $k = 2$, the condition $E[|a_1(X_1)|^3 + |a_3(X_1, X_2, X_3)|^4] < \infty$ is satisfied if $E|g^{(1)}(X_1)|^{12} < \infty$, $E|g^{(2)}(X_1, X_2)|^{12} < \infty$, and $E|g_{(k-1)}^{(1)}(X_1)|^3 < \infty$.

The difference of the Edgeworth expansions of the studentized Y-statistic about its expectation and θ is the following.

COROLLARY 3.6.

$$H_n^*(x) = H_n(x) - \phi(x) \frac{\mu_k}{\sqrt{nk}\sigma_1} + \phi(x) \frac{1}{72n} \left\{ \frac{v_1 \mu_k}{k\sigma_1} x^3 \right. \\ \left. + \left[12 \frac{\mu_k}{\sqrt{nk}\sigma_1} (v_2 - 2v_1) - 72 \frac{\mu_k}{\sqrt{nk}\sigma_1^5} \left(\frac{1}{2} e_1 + 2e_2 \right) \right] x - 36 \left(\frac{\mu_k}{\sqrt{nk}\sigma_1} \right)^2 \right\}. \quad (25)$$

Especially, if $\theta_{k-1} = \theta$ then $H_n^*(x) = H_n(x)$.

The condition $\theta_{k-1} = \theta$ above is equivalent to $Eg(X_1, X_1, X_2, X_3, \dots, X_{k-1}) = Eg(X_1, X_2, \dots, X_k)$. The difference between the Edgeworth expansions about its expectation and θ appears at the term related with μ_k which arise from the bias. The value of the difference depends on each Y-statistic. The values of μ_k for V-statistic, S-statistic and LB-statistic are as follows.

$$\mu_k = \delta_k(\theta_{k-1} - \theta), \quad \delta_k = \begin{cases} \frac{k(k-1)}{2} & (\text{V, S-statistic}) \\ k(k-1) & (\text{LB-statistic}). \end{cases}$$

By Remark 3 of Maesono (1996), $H_n(x)$ is equal to the Edgeworth expansion of studentized U-statistic using the jackknife variance estimator. Hence, if $\theta_{k-1} = \theta$, then the Edgeworth expansion of studentized Y-statistic about θ using the jackknife variance estimator is equal to the one of studentized U-statistic using the jackknife variance estimator. This is also read from (3.1) and (3.2).

4. Examples

We give examples of the Edgeworth expansion of the studentized Y-statistic about estimable parameter θ .

Example 4.1 We consider the third central moment $\theta = \int (x - \mu)^3 dF(x)$, where μ is the mean of F . Its kernel $g(x_1, x_2, x_3)$ is given by

$$\frac{1}{3}(x_1^3 + x_2^3 + x_3^3) - \frac{1}{2}(x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2) + 2x_1x_2x_3.$$

For this kernel, we have $g_{(k-1)}(x_1, x_2) = g_{(2)}(x_1, x_2) = 0$ and $g_{(k-2)}(x_1) = g_{(1)}(x_1) = 0$ and so $\theta_{k-1}(= \theta_2) = 0$ $\theta_{k-2}(= \theta_1) = 0$. Therefore, we have

$$Y_n = \frac{d(k, k)}{D(n, k)} \binom{n}{k} U_n = \frac{d(3, 3)n^2}{6D(n, 3)} \sum_{j=1}^n (X_j - \bar{X})^3 \quad (26)$$

where $\bar{X} = \sum_{j=1}^n X_j/n$. We assume that the distribution F has a density. We also assume $E|X|^{27} < \infty$ and denote j th moment of X about the origin by m'_j ($j = 2, 3, \dots, 12$). In order to study the statistical properties of Y_n , by (4.1) the mean μ is assumed to be zero, without loss of generality. Thus, in this case $\theta = m'_3$ and $\mu_k = -\delta_k m'_3$. We consider the two cases that the distribution F is symmetric or not.

Example 4.1.1 We assume that the distribution F is symmetric about zero. In this case, by the symmetry $m'_j = 0$ ($j = 3, 5, \dots, 11$) and $\theta = 0$. Then, we have

$$\begin{aligned} g^{(1)}(x_1) &= \frac{1}{3}(x_1^3 - 3x_1m'_2), \\ g^{(2)}(x_1, x_2) &= \psi_2(x_1, x_2) - g^{(1)}(x_1) - g^{(1)}(x_2) - \theta \\ &= \frac{1}{2}(-x_1^2x_2 - x_1x_2^2 + x_1m'_2 + x_2m'_2), \\ g^{(3)}(x_1, x_2, x_3) &= 2x_1x_2x_3. \end{aligned}$$

By the computation based on these, we get

$$\begin{aligned} e_1 &= 0, \quad e_2 = 0, \\ e_3 &= \frac{1}{81}(m'_{12} - 12m'_{10}m'_2 + 54m'_8m'^2_2 - 108m'_6m'^3_2 + 81m'_4m'^4_2), \\ e_4 &= \frac{1}{27}(-m'_8m'_4 + 3m'_8m'^2_2 + 7m'_6m'_4m'_2 - 21m'_6m'^3_2 \\ &\quad - 15m'^2_4m'^2_2 + 54m'_4m'^4_2 - 27m'^6_2), \\ e_5 &= \frac{1}{9}(m'^3_4 - 7m'^2_4m'^2_2 + 15m'_4m'^4_2 - 9m'^6_2), \\ e_6 &= \frac{4}{27}(m'^3_4 - 9m'^2_4m'^2_2 + 27m'_4m'^4_2 - 27m'^6_2). \end{aligned}$$

Furthermore,

$$\sigma_1^2 = \frac{1}{9}(m'_6 - 6m'_4m'_2 + 9m'^3_2), \quad \sigma_2^2 = 2(m'_4m'_2 - m'^3_2).$$

Thus we get

$$\begin{aligned} v_1 &= 0, \quad v_2 = 0, \quad v_3 = 0, \\ v_4 &= \frac{6}{(m'_6 - 6m'_4m'_2 + 9m'^3_2)^2} \\ &\quad \times \left\{ m'_{12} - 12m'_{10}m'_2 - 36m'_8m'_4 + 162m'_8m'^2_2 - 6m'^2_6 + 324m'_6m'_4m'_2 \right. \\ &\quad \left. - 972m'_6m'^3_2 + 102m'^3_4 - 1566m'^2_4m'^2_2 + 4779m'_4m'^4_2 - 3240m'^6_2 \right\}, \\ v_5 &= \frac{18}{(m'_6 - 6m'_4m'_2 + 9m'^3_2)^2} \\ &\quad \times \left\{ -m'_{12} + 12m'_{10}m'_2 + 12m'_8m'_4 - 90m'_8m'^2_2 + 2m'^2_6 - 90m'_6m'_4m'_2 \right. \\ &\quad \left. + 378m'_6m'^3_2 - 18m'^3_4 + 270m'^2_4m'^2_2 - 945m'_4m'^4_2 + 486m'^6_2 \right\} \end{aligned}$$

and

$$E[g^{(1)}(X_1)f_1(X_1)] = \frac{1}{2}e_1 + e_2 = 0.$$

Now we check the condition (D): We can write $2g^{(2)}(x, y) = (x + y)(m'_2 - xy) = -(x^2 + x - m'_2)(y^2 + y - m'_2) + (x^2 - m'_2)(y^2 - m'_2) + xy$. We can also write

$$f_1(x)g^{(1)}(y) + f_1(y)g^{(1)}(x) = [f_1(x) + g^{(1)}(x)][f_1(y) + g^{(1)}(y)] - f_1(x)f_1(y) - g^{(1)}(x)g^{(1)}(y).$$

Thus we have $a_2(x, y) = \sum_{j=1}^6 c_j h_j(x) h_j(y)$ where

$$h_1(x) = x^2 + x - m'_2, \quad h_2(x) = x^2 - m'_2, \quad h_3(x) = x,$$

$$h_4(x) = f_1(x) + g^{(1)}(x), \quad h_5(x) = f_1(x), \quad h_6(x) = g^{(1)}(x),$$

and

$$c_1 = -\frac{1}{2}(k-1), \quad c_2 = \frac{1}{2}(k-1), \quad c_3 = \frac{1}{2}(k-1), \quad c_4 = -\frac{1}{\sigma_1^2}, \quad c_5 = c_6 = \frac{1}{\sigma_1^2}.$$

In this example we can write f_1 as follows:

$$f_1(x) = \frac{1}{18}[(x^3 - 3xm'_2)^2 - 9\sigma_1^2 + 6(m'_4 - 3m'^2_2)(m'_2 - x^2)].$$

Thus for any $s_1, \dots, s_K$ ($-\infty < s_1, \dots, s_K < \infty$), $g^{(1)}(x) + \sum_{j=1}^K s_j h_j(x)$ is a polynomial of degree 6 with respect to x . Therefore the distribution of $g^{(1)}(X) + \sum_{j=1}^K s_j h_j(X)$ has the density for any $s_1, \dots, s_K$ ($-\infty < s_1, \dots, s_K < \infty$) and by Lemma 5.1 the condition (D) is satisfied. We note that the check of the condition (D) of the example (1) of 5 of Yamato et al. (2003) is corrected and may be done like as the above.

Under our assumption, $\mu_k = \delta_k(\theta_{k-1} - \theta) = 0$ and the Edgeworth expansion of the studentized Y-statistic about estimable parameter θ is given by

$$H_n^*(x) = \Phi(x) + \phi(x) \frac{1}{\sqrt{2n}} \{v_4 x^3 + v_5 x\}.$$

That is, there is no difference of the Edgeworth expansions of the studentized Y-statistic about its expectation and θ . Thus, if the distribution F is symmetric then there is no difference among the Edgeworth expansions of the studentized Y-statistic about its expectation and θ . By Remark 3 of Maesono (1996), this expansion $H_n^*(x)$ is also equal to the Edgeworth expansion of studentized U-statistic using a jackknife variance estimator.

Example 4.1.2 We assume that the distribution F is not symmetric about zero. In this case, $\theta = m'_3$ and $\mu_k = -\delta_k m'_3$. Now, we have

$$\begin{aligned} g^{(1)}(x_1) &= \frac{1}{3}x_1^3 - m'_2 x_1 - m'_3, \\ g^{(2)}(x_1, x_2) &= \frac{1}{2}(-x_1^2 x_2 - x_1 x_2^2 + x_1 m'_2 + x_2 m'_2), \\ g^{(3)}(x_1, x_2, x_3) &= 2x_1 x_2 x_3. \end{aligned}$$

Thus, by the same reason as in Example 4.1.1, Condition (D) is satisfied. By the computation based on these functions, we get

$$\begin{aligned} e_1 &= \frac{1}{27}(m'_9 - 9m'_7 m'_2 - 3m'_6 m'_3 + 27m'_5 m'^2_2 \\ &\quad + 18m'_4 m'_3 m'_2 + 2m'^3_3 - 54m'_3 m'^3_2), \\ e_2 &= \frac{2}{9}(-m'_5 m'_4 + 3m'_5 m'^2_2 + 4m'_4 m'_3 m'_2 - 12m'_3 m'^3_2), \\ e_3 &= \frac{1}{81}(m'_{12} - 12m'_{10} m'_2 - 4m'_9 m'_3 + 54m'_8 m'^2_2 + 36m'_7 m'_3 m'_2 \\ &\quad + 6m'_6 m'^2_3 - 108m'_6 m'^3_2 - 108m'_5 m'_3 m'^2_2 - 36m'_4 m'^2_3 m'_2 \\ &\quad + 81m'^4_4 m'^2_2 - 3m'^4_3 + 162m'^2_3 m'^3_2), \\ e_4 &= \frac{1}{27}(-m'_8 m'_4 + 3m'_8 m'^2_2 - m'_7 m'_5 + 4m'_7 m'_3 m'_2 + 7m'_6 m'_4 m'_2 \\ &\quad - 21m'_6 m'^3_2 + 6m'^2_5 m'_2 + 4m'_5 m'_4 m'_3 - 45m'_5 m'_3 m'^2_2 \\ &\quad - 15m'^2_4 m'^2_2 - 16m'_4 m'^2_3 m'_2 + 54m'_4 m'^4_2 + 84m'^2_3 m'^3_2 - 27m'^6_2), \\ e_5 &= \frac{1}{9}(m'^2_5 m'_2 + 2m'_5 m'_4 m'_3 - 14m'_5 m'_3 m'^2_2 + m'^3_4 - 7m'^2_4 m'^2_2 \\ &\quad - 8m'_4 m'^2_3 m'_2 + 15m'_4 m'^4_2 + 40m'^2_3 m'^3_2 - 9m'^6_2), \\ e_6 &= \frac{4}{27}(m'^3_4 - 9m'^2_4 m'^2_2 + 27m'_4 m'^4_2 - 27m'^6_2) \end{aligned}$$

Furthermore,

$$\sigma_1^2 = \frac{1}{9}(m'_6 - 6m'_4 m'_2 - m'^2_3 + 9m'^3_2), \quad \sigma_2^2 = 2(m'_4 m'_2 + m'^2_3 - m'^3_2).$$

Thus we can get v_1, v_2, v_3, v_4 , and v_5 , which are tedious and we omit to write them.

We also have

$$\begin{aligned} \frac{1}{2}e_1 + e_2 &= \frac{1}{54}(m'_9 - 9m'_7m'_2 - 3m'_6m'_3 - 12m'_5m'_4 + 63m'_5m'^2_2 \\ &\quad + 66m'_4m'_3m'_2 + 2m'^3_3 - 198m'_3m'^3_2). \end{aligned}$$

The Edgeworth expansion $H_n^*(x)$ is given by (3.12) with

$$\mu_k = \begin{cases} -\frac{k(k-1)}{2}m'_3 & (\text{V, S - statistic}) \\ -k(k-1)m'_3 & (\text{LB - statistic}). \end{cases}$$

In the relation (3.14), $H_n^*(x)$ is different from $H_n(x)$ with this μ_k .

Example 4.2 We consider the kernel $g(x_1, x_2, \dots, x_k) = x_1x_2 \cdots x_k$ ($k \geq 3$). This kernel yields estimable parameter $\theta(F) = \mu^k$, where μ is the mean of the distribution F . We assume that the distribution F has the density. We also assume that F is symmetric about the mean μ (> 0), and $E|X|^9 < \infty$ in case of $k = 3, 4$ and $E|X|^{2k+\varepsilon} < \infty$ in case of $k \geq 5$. We shall denote the central moments about the mean by m_j ($j = 2, 4$). Now, we have

$$\begin{aligned} g^{(1)}(x_1) &= \mu^{k-1}(x_1 - \mu), \\ g^{(2)}(x_1, x_2) &= \mu^{k-2}(x_1 - \mu)(x_2 - \mu), \\ g^{(3)}(x_1, x_2, x_3) &= \mu^{k-3}(x_1 - \mu)(x_2 - \mu)(x_3 - \mu). \end{aligned}$$

The computation based on these values yields

$$\begin{aligned} e_1 &= 0, \quad e_2 = (k-1)\mu^{3k-4}m_2^2, \quad e_3 = \mu^{4k-4}m_4, \quad e_4 = 0, \\ e_5 &= (k-1)^2\mu^{4k-6}m_2^3, \quad e_6 = (k-1)(k-2)\mu^{4k-6}m_2^3. \end{aligned}$$

Furthermore,

$$\sigma_1^2 = \mu^{2k-2}m_2, \quad \sigma_2^2 = (k-1)^2\mu^{2k-4}m_2^2.$$

Thus we get

$$\begin{aligned} v_1 &= \frac{3(k-1)\sqrt{m_2}}{\mu}, \quad v_2 = \frac{3(k-1)\sqrt{m_2}}{\mu}, \quad v_3 = -\frac{9(k-1)^2m_2}{\mu^2}, \\ v_4 &= \frac{6\xi_1^{-4}(e_3 - 6\xi_1^4 + 12e_4 + 6e_5 + 4e_6) - 2\xi_1^{-6}(2e_1 + 3e_2)(2e_1 + 9e_2)}{\mu^2m_2^2}, \\ v_5 &= \frac{3\xi_1^{-6}(4e_1^2 + 12e_1e_2 + 3e_2^2) + 18\xi_1^{-4}(\xi_1^2\xi_2^2 - e_3 + 2\xi_1^4 - 4e_4 - 2e_5)}{\mu^2m_2^2}. \end{aligned}$$

Now we check the condition (D): By the relation derived in Example 4.1, we have $a_2(x, y) = \sum_{j=1}^4 c_j h_j(x) h_j(y)$ where

$$h_1(x) = x - \mu, \quad h_2(x) = f_1(x) + g^{(1)}(x), \quad h_3(x) = f_1(x), \quad h_4(x) = g^{(1)}(x),$$

and

$$c_1 = -\frac{1}{2}\mu^{k-2}(k-1), \quad c_2 = -\frac{1}{\sigma_1^2}, \quad c_3 = c_4 = \frac{1}{\sigma_1^2}.$$

In this example we can write f_1 as follows:

$$f_1(x) = \frac{1}{2}\mu^{2k-3}[\mu(x-\mu)^2 + 2(k-1)m_2(x-\mu) - \mu m_2].$$

Thus for any $s_1, \dots, s_K$ ($-\infty < s_1, \dots, s_K < \infty$), $g^{(1)}(x) + \sum_{j=1}^K s_j h_j(x)$ is a polynomial of degree 2 with respect to x . Therefore the distribution of $g^{(1)}(X) + \sum_{j=1}^K s_j h_j(X)$ has the density for any $s_1, \dots, s_K$ ($-\infty < s_1, \dots, s_K < \infty$) and by Lemma 5.1 the condition (D) is satisfied.

The values of μ_k for V-statistic, S-statistic and LB-statistic are

$$\mu_k = \begin{cases} \frac{k(k-1)}{2}(m_2^2 + \mu^2 - \mu^3)\mu^{k-3} & (\text{V, S - statistic}) \\ k(k-1)(m_2^2 + \mu^2 - \mu^3)\mu^{k-3} & (\text{LB - statistic}). \end{cases}$$

These values give the difference among the Edgeworth expansions $H_n^*(x)$. The Edgeworth expansions $H_n^*(x)$ are given by (3.14) with the above values.

Example 4.3 We consider the kernel

$$g(x_1, x_2, x_3) = \frac{1}{3}\{I(x_1 > x_2 + x_3) + I(x_2 > x_1 + x_3) + I(x_3 > x_1 + x_2)\}, \quad (27)$$

where $I(A)$ is the indicator function of an event A . This kernel yields the estimable parameter $\theta(F) = E[1 - F(X_1 + X_2)]$ which measures the degree to which a life distribution F has the NBU (new better than used) property. If X_1 and X_2 are random variables having the life distribution F , the NBU property is denoted by $P(X_1 > x) \geq P(X_2 > x + y | X_2 > y)$ for $x, y > 0$. (See, Hollander and Proschan (1972), and Lee (1990)). We note that

$$\begin{aligned} \psi_1(x) &= \frac{1}{3}E[F(x - X_3)] + \frac{2}{3}E[1 - F(x + X_3)], \\ \psi_2(x_1, x_2) &= \frac{1}{3}[F(|x_2 - x_1|) + 1 - F(x_1 + x_2)]. \end{aligned}$$

For the corresponding U-statistic, we shall derive Edgeworth expansion in cases that F are the uniform distribution $U(0,1)$ and the exponential distribution $e(1)$ with parameter 1. Since the kernel (4.2) is scale invariant, the Edgeworth expansion for the uniform distribution $U(0,1)$ is equal to the one for the uniform distribution $U(0, \alpha)$, $\alpha > 0$. The Edgeworth expansion for the exponential distribution $e(1)$ is also equal to the one for the exponential distribution $e(\alpha)$, $\alpha > 0$.

Example 4.3.1 We assume that F is the uniform distribution $U(0,1)$. Then $\theta(F) =$

1/6, and

$$\begin{aligned}
 & g^{(1)}(x_1) \\
 &= \frac{1}{2}x_1^2 - \frac{2}{3}x_1 + \frac{1}{6} \quad (0 < x < 1), \\
 & g^{(2)}(x_1, x_2) \\
 &= \begin{cases} \frac{1}{3}[|x_1 - x_2| - (x_1 + x_2)] - \frac{1}{2}(x_1^2 + x_2^2) + \frac{2}{3}(x_1 + x_2) - \frac{1}{6} & (0 < x_1 + x_2 < 1) \\ \frac{1}{3}|x_1 - x_2| - \frac{1}{2}(x_1^2 + x_2^2) + \frac{2}{3}(x_1 + x_2) - \frac{1}{2} & (x_1 + x_2 > 1). \end{cases}
 \end{aligned}$$

By using the expression of $g^{(1)}$ and $g^{(2)}$ to

$$\begin{aligned}
 E[g^{(1)}(X_2)g^{(2)}(X_1, X_2)|X_1 = x_1] &= \int_{0 < x_2 < 1, 0 < x_1 + x_2 < 1} g^{(1)}(x_2)g^{(2)}(x_1, x_2)dx_2 \\
 &\quad + \int_{0 < x_2 < 1, x_1 + x_2 \geq 1} g^{(1)}(x_2)g^{(2)}(x_1, x_2)dx_2,
 \end{aligned}$$

we get

$$\begin{aligned}
 & E[g^{(1)}(X_2)g^{(2)}(X_1, X_2)|X_1 = x_1] \\
 &= \frac{1}{3} \left\{ 2x_1 \int_0^{x_1} g^{(1)}(x_2)dx_2 - \int_0^{x_1} x_2 g^{(1)}(x_2)dx_2 + \int_{x_1}^1 x_2 g^{(1)}(x_2)dx_2 \right\} \\
 &\quad - \int_0^1 [g^{(1)}(x_2)]^2 dx_2 + \frac{1}{3} \left\{ (1 - x_1) \int_0^{1-x_1} g^{(1)}(x_2)dx_2 - \int_0^{1-x_1} x_2 g^{(1)}(x_2)dx_2 \right\}.
 \end{aligned}$$

Thus we get

$$E[g^{(1)}(X_2)g^{(2)}(X_1, X_2)|X_1 = x_1] = \frac{1}{24}x_1^4 - \frac{5}{54}x_1^3 + \frac{1}{18}x_1^2 - \frac{1}{270}.$$

By the computation based on these functions using Mathematica ver. 4.0, we get

$$\sigma_1^2 = \frac{1}{270} \doteq 0.0037, \quad \sigma_2^2 = \frac{1}{135} \doteq 0.0074$$

and

$$\begin{aligned}
 e_1 &= \frac{1}{140} \doteq 0.00714, \quad e_2 = -\frac{1}{4536} = -0.00022, \quad e_3 = \frac{1}{280} \doteq 0.00357, \\
 e_4 &= -\frac{19}{1360800} \doteq -0.00001, \quad e_5 = \frac{1}{72900} \doteq 0.000014.
 \end{aligned}$$

Since we can write

$$e_6 = (k-1)(k-2)E[g^{(1)}(X_1)h(X_1)]$$

where $h(x) = \int_{0 < y+z < x} g^{(1)}(y)g^{(1)}(z)dydz$, we have

$$e_6 = -\frac{1}{272160} \doteq -0.0000037.$$

Thus we get

$$\begin{aligned} v_1 &= \frac{309\sqrt{30}}{28} \doteq 60.44510, & v_2 &= \frac{21\sqrt{30}}{4} \doteq 28.7554, & v_3 &= -\frac{1432215}{292} \doteq -4904.8459, \\ v_4 &= -\frac{4878216261}{960400} \doteq -5079.35887, & v_5 &= \frac{2495655}{392} \doteq 6366.4668. \end{aligned}$$

Now we check the condition (C): We take $h(y) = y^l$ ($0 < y < 1$, $l = 1, 2, \dots$) for $(Lh)(x) = E[a_2(x, X_2)h(X_2)]$ ($0 < x < 1$). Since f_1 and $g^{(1)}$ are polynomials of degrees 4 and 2, respectively, the term related to f_1 and $g^{(1)}$ in $E[a_2(x, X_2)h(X_2)]$ is a polynomial of degree 4. Among the terms of $g^{(2)}$, $|x - y|$ yields the integral

$$\int_0^1 |x - y|y^l dy = \frac{2}{(l+1)(l+2)}x^{l+2} - \frac{1}{l+1}x + \frac{1}{l+2} \quad (0 < x < 1)$$

which is a polynomial of degree $l + 2$. Among the terms of $g^{(2)}$, the other term yields a polynomial of degree 2. That is, $(Lh)(x)$ ($0 < x < 1$) is a polynomial of degree $l + 2$ for $h(y) = y^l$ ($0 < y < 1$, $l = 2, 3, \dots$). Thus if we choose $h_j(y) = y^j$ for $j = 2, 3, \dots, K$, then $h_1(x_1), \dots, h_K(x_1)$ are linearly independent and the covariance matrix of $(h_1(X_1), \dots, h_K(X_1))$ is positive definite. Thus the condition (C) is satisfied.

The values of μ_k for V-statistic, S-statistic and LB-statistic are

$$\mu_k = \begin{cases} \frac{1}{4} & (\text{V, S - statistic}) \\ \frac{1}{2} & (\text{LB - statistic}). \end{cases}$$

These values give the difference among the Edgeworth expansions $H_n^*(x)$. The Edgeworth expansions $H_n^*(x)$ are given by (3.14) with the above values.

Example 4.3.2 We assume that F is the exponential distribution $e(1)$. Then $\theta(F) = 1/4$, and

$$\begin{aligned} g^{(1)}(x_1) &= \frac{1}{12} - \frac{1}{3}x_1e^{-x_1}, \quad (x_1 > 0) \\ g^{(2)}(x_1, x_2) &= -\frac{1}{12} + \frac{1}{3}[x_1e^{-x_1} + x_2e^{-x_2} - e^{-|x_1-x_2|} + e^{-(x_1+x_2)}] \quad (x_1, x_2 > 0). \end{aligned}$$

By using the expression of $g^{(1)}(x_1)$ and $g^{(2)}(x_1, x_2)$ to

$$\begin{aligned} &E[g^{(1)}(X_2)g^{(2)}(X_1, X_2)|X_1 = x_1] \\ &= \int_0^{x_1} g^{(1)}(x_2)g^{(2)}(x_1, x_2)dx_2 + \int_{x_1}^{\infty} g^{(1)}(x_2)g^{(2)}(x_1, x_2)dx_2, \end{aligned}$$

we get

$$E[g^{(1)}(X_2)g^{(2)}(X_1, X_2)|X_1 = x_1] = -\frac{5}{3888} - \frac{8}{81}e^{-2x_1} + \frac{8}{81}e^{-x_1} - \frac{2}{27}x_1e^{-2x_1} - \frac{1}{36}x_1e^{-x_1}.$$

By the computation based on these functions using Mathematica ver. 4.0, we get

$$\sigma_1^2 = \frac{5}{3888} \doteq 0.001286, \quad \sigma_2^2 = \frac{251}{972} \doteq 0.25823$$

and

$$e_1 = \frac{1}{31104} \doteq 0.000032, \quad e_2 = -\frac{5}{69984} \doteq -0.000071, \quad e_3 = \frac{2171}{58320000} \doteq 0.0000037,$$

$$e_4 = \frac{-10127}{9447840000} \doteq -0.000001, \quad e_5 = \frac{2083}{157464000} \doteq 0.000013.$$

By the method similar to Example 4.3.1, we have

$$e_6 = \frac{1}{1119744} \doteq 0.00000089.$$

Thus we get

$$v_1 = -\frac{21\sqrt{3}}{5\sqrt{5}} \doteq -3.253306, \quad v_2 = -\frac{51\sqrt{3}}{10\sqrt{5}} \doteq -3.95044, \quad v_3 = -\frac{1323}{125} \doteq -10.584,$$

$$v_4 = -\frac{787836542389301}{13436928000000} \doteq -5.86322, \quad v_5 = \frac{7274727}{78125} \doteq 93.11651.$$

Now we check the condition (C): We take $h(y) = e^{-ly}$ ($y > 0$, $l = 1, 2, \dots$) for $(Lh)(x) = E[a_2(x, X_2)h(X_2)]$ ($x > 0$). The terms related to f_1 and $g^{(1)}$ in $E[a_2(x, X_2)h(X_2)]$ contain exponential functions e^{-x} or e^{-2x} . Among the terms of $g^{(2)}$, $e^{-|x-y|}$ yields the integral

$$\int_0^\infty e^{-|x-y|} e^{-ly} dy = \frac{1}{(l-1)} e^{-x} - \frac{2}{(l-1)(l+1)} e^{-lx} \quad (x > 0)$$

which contain exponents e^{-x} and e^{-lx} . Among the terms of $g^{(2)}$, the other terms are constant or contain an exponential function e^{-x} . That is, $(Lh)(x)$ contains exponent e^{-x} , e^{-2x} and e^{-lx} for $h(y) = e^{-ly}$ ($l = 2, 3, \dots$). Thus if we choose $h_j(y) = e^{-(j+1)y}$ ($y > 0$) for $j = 1, 2, \dots, K$, then $h_1(x_1), \dots, h_K(x_1)$ are linearly independent and the covariance matrix of $(h_1(X_1), \dots, h_K(X_1))$ is positive definite. Thus the condition (C) is satisfied.

The values of μ_k for V-statistic, S-statistic and LB-statistic are

$$\mu_k = \begin{cases} -\frac{5}{12} & (\text{V, S - statistic}) \\ -\frac{5}{6} & (\text{LB - statistic}). \end{cases}$$

These values give the difference among the Edgeworth expansions $H_n^*(x)$. The Edgeworth expansions $H_n^*(x)$ are given by (3.14) with the above values.

Next, we consider about the kernel of degree 2.

Example 4.4 We consider the variance $\theta = \int (x - \mu)^2 dF(x)$. Its kernel $g(x_1, x_2)$ is given by

$$\frac{1}{2}(x_1^2 + x_2^2 - 2x_1x_2).$$

For this kernel, we have $g_{(k-1)}(x_1) = g_{(1)}(x_1) = 0$ and so $\theta_{k-1}(= \theta_1) = 0$. Therefore, we have

$$Y_n = \frac{d(k, k)}{D(n, k)} \binom{n}{k} U_n = \frac{d(2, 2)n}{2D(n, 2)} \sum_{j=1}^n (X_j - \bar{X})^2. \quad (28)$$

We assume that the distribution F has a density. We also assume $E | X |^{18} < \infty$ and denote j th moment of X about the origin by m'_j ($j = 2, 3, \dots, 6$). In order to study the statistical properties of Y_n , by (4.3) the mean μ is assumed to be zero, without loss of generality. Thus, in this case $\theta = m'_2$ and $\mu_k = -\delta_k m'_2$. We consider the two cases that the distribution F is symmetric or not.

Example 4.4.1 We assume that the distribution F is symmetric about zero. In this case, by the symmetry $m'_j = 0$ ($j = 3, 5, 7$). Then, we have

$$g^{(1)}(x_1) = \frac{1}{2}(x_1^2 - m'_2), \quad g^{(2)}(x_1, x_2) = -x_1 x_2, \quad f_{12}(x_1) = 0.$$

By the computation based on these, we get

$$\begin{aligned} e_1 &= \frac{1}{8}(m'_6 - 3m'_4 m'_2 + 2m'^3_2), \\ e_2 &= 0, \\ e_3 &= \frac{1}{16}(m'_8 - 4m'_6 m'_2 + 6m'_4 m'^2_2 - 3m'^4_2), \\ e_4 &= 0, \quad e_5 = 0, \quad e_6 = 0. \end{aligned}$$

Furthermore,

$$\sigma_1^2 = \frac{1}{4}(m'_4 - m'^2_2), \quad \sigma_2^2 = m'^2_2.$$

Thus we get

$$\begin{aligned} v_1 &= \frac{2}{(m'_4 - m'^2_2)^{3/2}} \{m'_6 - 3m'_4 m'_2 + 2m'^3_2\}, \\ v_2 &= \frac{1}{(m'_4 - m'^2_2)^{3/2}} \{m'_6 - 3m'_4 m'_2 + 2m'^3_2\}, \\ v_3 &= -\frac{4}{(m'_4 - m'^2_2)^3} \{m'_6 - 3m'_4 m'_2 + 2m'^3_2\}^2, \\ v_4 &= \frac{2}{(m'_4 - m'^2_2)^3} \{3m'_8 m'_4 - 3m'_8 m'^2_2 - 4m'^2_6 + 12m'_6 m'_4 m'_2 - 4m'_6 m'^3_2 \\ &\quad - 18m'^3_4 + 36m'^2_4 m'^2_2 - 33m'_4 m'^4_2 + 11m'^6_2\}, \\ v_5 &= \frac{3}{(m'_4 - m'^2_2)^3} \{-6m'_8 m'_4 + 6m'_8 m'^2_2 + 4m'^2_6 - 8m'_6 m'^3_2 \\ &\quad + 12m'^3_4 - 12m'^2_4 m'^2_2 - 6m'_4 m'^4_2 + 10m'^6_2\} \end{aligned}$$

and

$$\frac{1}{2}e_1 + e_2 = \frac{1}{16}(m'_6 - 3m'_4 m'_2 + 2m'^3_2).$$

Now we check the condition (D): Since $g^{(2)}(x_1, x_2) = -x_1x_2$, by the same reason stated at the Example 4.1.1 we have $a_2(x, y) = \sum_{j=1}^6 c_j h_j(x) h_j(y)$ where

$$h_1(x) = x, \quad h_2(x) = f_1(x) + g^{(1)}(x), \quad h_3(x) = f_1(x), \quad h_4(x) = g^{(1)}(x),$$

and

$$c_1 = -1, \quad c_2 = -\frac{1}{\sigma_1^2}, \quad c_3 = c_4 = \frac{1}{\sigma_1^2}.$$

In this example we can write f_1 as follows:

$$f_1(x) = \frac{1}{8} \left\{ (x^2 - m'_2)^2 - (m'_4 - m'^2_2) \right\}$$

Thus for any $s_1, \dots, s_K$ ($-\infty < s_1, \dots, s_K < \infty$), $g^{(1)}(x) + \sum_{j=1}^K s_j h_j(x)$ is a polynomial of degree 4 with respect to x . Therefore the distribution of $g^{(1)}(X) + \sum_{j=1}^K s_j h_j(X)$ has the density for any $s_1, \dots, s_K$ ($-\infty < s_1, \dots, s_K < \infty$) and by Lemma 5.1 the condition (D) is satisfied.

Example 4.4.2 We assume that the distribution F is not symmetric about zero. Then, we have

$$g^{(1)}(x_1) = \frac{1}{2}(x_1^2 - m'_2), \quad g^{(2)}(x_1, x_2) = -x_1x_2, \quad f_{12}(x_1) = -\frac{1}{2}m'_3x_1.$$

By the computation based on these, we get

$$\begin{aligned} e_1 &= \frac{1}{8}(m'_6 - 3m'_4m'_2 + 2m'^3_2), \\ e_2 &= -\frac{1}{4}m'^2_3, \\ e_3 &= \frac{1}{16}(m'_8 - 4m'_6m'_2 + 6m'_4m'^2_2 - 3m'^4_2), \\ e_4 &= \frac{1}{8}(-m'_5m'_3 + 2m'^2_3m'_2), \\ e_5 &= \frac{1}{4}m'^2_3m'_2, \quad e_6 = 0. \end{aligned}$$

Furthermore

$$\sigma_1^2 = \frac{1}{4}(m'_4 - m'^2_2), \quad \sigma_2^2 = m'^2_2.$$

Thus we get

$$\begin{aligned}
 v_1 &= \frac{2}{(m'_4 - m'^2_2)^{3/2}} \left\{ m'_6 - 3m'_4 m'_2 - 3m'^2_3 + 2m'^3_2 \right\}, \\
 v_2 &= \frac{1}{(m'_4 - m'^2_2)^{3/2}} \left\{ m'_6 - 3m'_4 m'_2 - 6m'^2_3 + 2m'^3_2 \right\}, \\
 v_3 &= -\frac{4}{(m'_4 - m'^2_2)^3} \left\{ m'_6 - 3m'_4 m'_2 - 3m'^2_3 + 2m'^3_2 \right\}^2, \\
 v_4 &= \frac{2}{(m'_4 - m'^2_2)^3} \\
 &\quad \times \left\{ 3m'_8 m'_4 - 3m'_8 m'^2_2 - 4m'^2_6 + 12m'_6 m'_4 m'_2 + 48m'_6 m'^2_3 - 4m'_6 m'^3_2 \right. \\
 &\quad \left. - 72m'_5 m'_4 m'_3 + 72m'_5 m'_3 m'^2_2 - 18m'^3_4 + 36m'^2_4 m'^2_2 + 72m'_4 m'^2_3 m'_2 \right. \\
 &\quad \left. - 33m'_4 m'^4_2 - 108m'^4_3 - 120m'^2_3 m'^3_2 + 11m'^6_2 \right\}, \\
 v_5 &= \frac{3}{(m'_4 - m'^2_2)^3} \\
 &\quad \times \left\{ -6m'_8 m'_4 + 6m'_8 m'^2_2 + 4m'^2_6 - 24m'_6 m'^2_3 - 8m'_6 m'^3_2 + 48m'_5 m'_4 m'_3 \right. \\
 &\quad \left. - 48m'_5 m'_3 m'^2_2 + 12m'^3_4 - 12m'^2_4 m'^2_2 - 72m'_4 m'^2_3 m'_2 - 6m'_4 m'^4_2 \right. \\
 &\quad \left. + 12m'^4_3 + 96m'^2_3 m'^3_2 + 10m'^6_2 \right\}
 \end{aligned}$$

and

$$\frac{1}{2}e_1 + e_2 = \frac{1}{16}(m'_6 - 3m'_4 m'_2 - 4m'^2_3 + 2m'^3_2).$$

Now we check the condition (D): Since

$$f_1(x) = \frac{1}{8} \left\{ (x^2 - m'_2)^2 - (m'_4 - m'^2_2) \right\} - \frac{1}{2} m'_3 x,$$

by the same reason as Example 4.4.1, the condition (D) is satisfied.

Example 4.5 We consider the kernel $g(x_1, x_2) = x_1 x_2$. This kernel yields estimable parameter $\theta(F) = \mu^2$. We assume that the distribution F has the density. We also assume that F is symmetric about the mean μ (> 0) and $EX^9 < \infty$. The values e_2, e_3 and e_5 are given by putting $k = 2$ in Example 4.2 and, $e_1 = e_4 = e_6 = 0$.

Example 4.6 We consider the kernel

$$g(x_1, x_2) = I(x_1 + x_2 > 0),$$

which appears in the Wilcoxon one-sample statistic. We assume that the distribution F has the density and symmetric about zero. Then the value of the estimable parameter θ is equal to $E[I(X_1 + X_2 > 0)] = 1/2$. We have also

$$g_{(1)}(x_1) = I(x_1 > 0), \quad \theta_1 (= \theta_{k-1}) = EI(X_1 > 0) = \frac{1}{2}.$$

Therefore $\mu_2(=\mu_k) = 0$. We note that $1 - F(-x) = F(x)$ and $F(X)$ has the uniform distribution $U(0,1)$. We have

$$g^{(1)}(x_1) = F(x_1) - \frac{1}{2}, \quad g^{(2)}(x_1, x_2) = I(x_1 + x_2 > 0) - F(x_1) - F(x_2) + \frac{1}{2},$$

and

$$f_1(x_1) = 0, \quad f_{12}(x_1) = \frac{1}{2}\{F(x_1) - F^2(x_1) - \frac{1}{6}\},$$

where we use the relation $E[F(X_2)I(x_1 + X_2 > 0)] = \int_{-x_1}^{\infty} F(x_2)dF(x_2) = [1 - F^2(-x_1)]/2 = [2F(x_1) - F^2(x_1)]/2$. Furthermore, using the relation $E[F(X_2)I(X_1 + X_2 > 0)] = E[2F(X_1) - F^2(X_1)]/2 = 1/3$, we get

$$\sigma_1^2 = \frac{1}{12}, \quad \sigma_2^2 = \frac{1}{12}.$$

Thus we get

$$e_1 = e_2 = 0, \quad e_3 = \frac{1}{80}, \quad e_4 = -\frac{1}{360}, \quad e_5 = \frac{1}{720}, \quad e_6 = 0$$

and

$$v_1 = v_2 = v_3 = 0, \quad v_4 = -\frac{234}{5}, \quad v_5 = \frac{216}{5}.$$

Now we check the condition (C): We assume $E|g^{(2)}|^r < \infty$ ($r > 2$) and take K such that $K > 8(4r - 5)/(r - 2)$. We take $h(y) = y^l$ ($l = 1, 2, \dots, K$) for $(Lh)(x) = E[a_2(x, X_2)h(X_2)]$. Under the condition that F has the K -th moment, $\int_{-x}^{\infty} y^l dF(y)$, $l = 1, \dots, K$, are linearly independent. Since $a_2(x, y) = g^{(2)}(x_1, x_2) = I(x_1 + x_2 > 0) - F(x_1) - F(x_2) + \frac{1}{2}$, under the same condition, $(Lh_1)(x_1), \dots, (Lh_K)(x_1)$ are linearly independent and the covariance matrix of $(Lh_1)(X_1), \dots, (Lh_K)(X_1)$ is positive definite, where $h_l(y) = y^l$, $l = 1, \dots, K$.

Example 4.7 We consider the kernel

$$g(x_1, x_2) = \frac{1}{2} \max(x_1, x_2) = \frac{1}{2}[x_1 I(x_1 \geq x_2) + x_2 I(x_1 < x_2)],$$

which gives the probability weighted moment

$$\theta = \beta_1 = \frac{1}{2}E[\max(X_1, X_2)] = E[XF(X)].$$

We assume that the distribution F has the uniform distribution $U(0,1)$. Then we have $\beta_1 = 1/3$, $g_{(1)}(x_1) = x_1/2$, and $\theta_1 = 1/4$. Furthermore, we have

$$2\psi_1(x_1) = E[x_1 I(x_1 \geq X_2) + X_2 I(x_1 < X_2)] = \frac{1}{4}(1 + x_1^2),$$

$$g^{(1)}(x_1) = \frac{1}{4}x_1^2 - \frac{1}{12},$$

$$g^{(2)}(x_1, x_2) = \frac{1}{2} \max(x_1, x_2) - \frac{1}{4}(x_1^2 + x_2^2) - \frac{1}{6}$$

and

$$f_1(x_1) = \frac{1}{24}x_1^4 - \frac{1}{24}x_1^2 + \frac{1}{180}, \quad f_{12}(x_1) = \frac{1}{96}x_1^4 - \frac{1}{48}x_1^2 + \frac{7}{1440}.$$

Therefore we have

$$\sigma_1^2 = \frac{1}{180}, \quad \sigma_2^2 = \frac{1}{360}.$$

Thus we get

$$e_1 = \frac{1}{3780}, \quad e_2 = -\frac{1}{3780}, \quad e_3 = \frac{1}{15120}, \quad e_4 = -\frac{1}{113400}, \quad e_5 = \frac{1}{75600}, \quad e_6 = 0$$

and

$$v_1 = -\frac{2\sqrt{5}}{7}, \quad v_2 = -\frac{4\sqrt{5}}{7}, \quad v_3 = -\frac{20}{49}, \quad v_4 = -34, \quad v_5 = \frac{267}{49} \doteq 5.45.$$

Now we check the condition (C): We take $h(y) = y^{l+1}$ ($l = 1, 2, \dots, K$) for $(Lh)(x) = E[a_2(x, X_2)h(X_2)]$ and a suitably large $K (> 56)$. Since $\int_0^1 y^{l+1} \max(x, y) dy$ is a polynomial of degree $l + 3$ in x , $(Lh_1)(x_1), \dots, (Lh_K)(x_1)$ are linearly independent. Hence the covariance matrix of $(Lh_1)(X_1), \dots, (Lh_K)(X_1)$ is positive definite.

5. Appendix

About the condition (D) by Lai and Wang (1993) for Edgeworth expansion, we give a sufficient condition.

LEMMA 5.1. *We assume that the distribution of $g^{(1)}(X) + \sum_{j=1}^K s_j h_j(X)$ has the density for any $s_1, \dots, s_K$ ($-\infty < s_1, \dots, s_K < \infty$). Then, the relation (2.5) holds, provided the assumptions of Condition (D) preceding (2.5).*

PROOF. Since the distribution of $g^{(1)}(X) + \sum_{j=1}^K s_j h_j(X)$ has the density for any $s_1, \dots, s_K$, we get

$$E \exp \left(it \left[g^{(1)}(X) + \sum_{j=1}^K s_j h_j(X) \right] \right) \rightarrow 0 \quad \text{as } |t| \rightarrow \infty \quad \text{for } -\infty < s_1, \dots, s_K < \infty.$$

Thus,

$$\sup_{|s_1| + \dots + |s_K| \leq 1} \left| E \exp \left(it \left[g^{(1)}(X) + \sum_{j=1}^K s_j h_j(X) \right] \right) \right| \rightarrow 0 \quad \text{as } |t| \rightarrow \infty.$$

On the other hand, for a sufficiently large $|t|$ satisfying $|t|^{-\varepsilon} < 1$ with ε given in Condition (D),

$$\begin{aligned} 0 &\leq \sup_{|s_1| + \dots + |s_K| \leq |t|^{-\varepsilon}} \left| E \exp \left(it \left[g^{(1)}(X) + \sum_{j=1}^K s_j h_j(X) \right] \right) \right| \\ &\leq \sup_{|s_1| + \dots + |s_K| \leq 1} \left| E \exp \left(it \left[g^{(1)}(X) + \sum_{j=1}^K s_j h_j(X) \right] \right) \right|, \end{aligned}$$

which converges to 0 as $|t| \rightarrow \infty$, because of the previous reason. Thus, (2.5) holds.

From the proof of Lemma 1.3 in p. 261 of Shorack (2000), we have the following.

LEMMA 5.2. (Shorack (2000)) *For any random variables W and Δ , it holds that*

$$\sup_x |P(W + \Delta \leq x) - P(W \leq x)| \leq 4(E|W\Delta| + E|\Delta|).$$

Lemma 1.7 of Petrov (1994) yields the following.

LEMMA 5.3. (Lemma 3 of Maesono (1996)) *Let H be a bounded function and δ be a positive constant. For any random variables W and Δ , it holds that*

$$\begin{aligned} \sup_x |P(W + \Delta \leq x) - H(x)| &\leq \sup_x |P(W \leq x) - H(x)| + P(|\Delta| \geq \delta) \\ &\quad + \sup_x |H(x + \delta) - H(x)|. \end{aligned}$$

By the Taylor expansion, we can get the following lemma.

LEMMA 5.4. *For a positive constant c , the following relations hold uniformly with respect to $x \in (-\infty, \infty)$.*

$$\begin{aligned} \Phi(x - \frac{c}{\sqrt{n}}) &= \Phi(x) - \frac{c}{\sqrt{n}}\phi(x) - \frac{c^2}{2n}x\phi(x) + O(\frac{1}{n^{3/2}}), \\ \phi(x - \frac{c}{\sqrt{n}}) &= \phi(x) + \frac{c}{\sqrt{n}}x\phi(x) + \frac{c^2}{2n}(x^2 - 1)\phi(x) + O(\frac{1}{n^{3/2}}), \\ (x - \frac{c}{\sqrt{n}})\phi(x - \frac{c}{\sqrt{n}}) &= x\phi(x) + \frac{c}{\sqrt{n}}(x^2 - 1)\phi(x) + \frac{c^2}{2n}(x^3 - 3x)\phi(x) + O(\frac{1}{n^{3/2}}), \\ (x - \frac{c}{\sqrt{n}})^2\phi(x - \frac{c}{\sqrt{n}}) &= x^2\phi(x) + \frac{c}{\sqrt{n}}(x^3 - 2x)\phi(x) + \frac{c^2}{2n}(x^4 - 5x^2 + 2)\phi(x) \\ &\quad + O(\frac{1}{n^{3/2}}), \\ (x - \frac{c}{\sqrt{n}})^3\phi(x - \frac{c}{\sqrt{n}}) &= x^3\phi(x) + \frac{c}{\sqrt{n}}(x^4 - 3x^2)\phi(x) + \frac{c^2}{2n}(x^5 - 7x^3 + 6x)\phi(x) \\ &\quad + O(\frac{1}{n^{3/2}}), \\ (x - \frac{c}{\sqrt{n}})^5\phi(x - \frac{c}{\sqrt{n}}) &= x^5\phi(x) + \frac{c}{\sqrt{n}}(x^6 - 5x^4)\phi(x) + \frac{c^2}{2n}(x^7 - 11x^5 + 20x^3)\phi(x) \\ &\quad + O(\frac{1}{n^{3/2}}). \end{aligned}$$

Multiplication of (2.3) and (3.6). The term associated with μ_k of (3.4) is obtained by multiplying the second term of (3.6) and the first term of (2.3). The last term of a_1^* is obtained by multiplying the second terms of (3.6) and (2.3). The terms of the expansion associated with $a_2(x, y)$, $a_3(x, y, z)$ are obtained directly by multiplication of the right-hand sides of (2.3) and (3.6). Next we consider the first three terms of $a_1(x)$, which are

$$\frac{\delta_k}{k} [(k-1)g_{(k-1)}^{(1)}(x) - kg^{(1)}(x)], \quad \tau g^{(1)}(x), \quad -\frac{1}{\sigma_1^2} [f_1(x)g^{(1)}(x) - \zeta].$$

The terms of the expansion associated with these are also obtained by the same method as the above. Since the third term of a_1 is subtracted the constant ζ , consequently the constant term associated with ζ appears in the second term of (3.4).

Now we consider the terms associated with the last two terms of $a_1(x)$ which are

$$-\frac{1}{\sigma_1^2} \left(E[f_2(x, X_2)g^{(1)}(X_2)] - \frac{3\zeta}{\sigma_1^2} f_1(x) \right), \quad -\frac{1}{\sigma_1^2} (k-1) E[g^{(2)}(x, X_2)f_1(X_2)].$$

These are obtained by taking the first terms of H-decompositions of

$$\frac{1}{n^{5/2}} \sum_{i < j} [g^{(1)}(X_i) + g^{(1)}(X_j)][f_2(X_i, X_j) - \frac{3}{\sigma_1^2} f_1(X_i)f_1(X_j)] \quad (29)$$

and

$$\frac{1}{n^{5/2}} \sum_{i < j} g^{(2)}(X_i, X_j)[f_1(X_i) + f_1(X_j)], \quad (30)$$

respectively. These (5.1) and (5.2) are obtained directly by multiplication of the right-hand sides of (2.3) and (3.6).

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