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https://doi.org/10.5109/12582

出版情報：Bulletin of informatics and cybernetics. 36, pp.105-130, 2004-12. Research Association of Statistical Sciences
バージョン：
権利関係：

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Reprinted from the Bulletin of Informatics and Cybernetics
Research Association of Statistical Sciences, Vol.36

FUKUOKA, JAPAN
2004

Bulletin of Informatics and Cybernetics, Vol. 36, 2004

AN EDGEWORTH EXPANSION OF A CONVEX COMBINATION OF U-STATISTICS BASED ON STUDENTIZATION

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Hajime Yamato[*], **Koichiro Toda**[†], **Toshifumi Nomachi**[‡]
and
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Abstract

As an estimator of an estimable parameter, Toda and Yamato (2001) introduce Y-statistic which is a convex combination of U-statistics including V-statistic and LB-statistic. We give the Edgeworth expansions of studentized Y-statistic about the estimable parameter using a jackknife variance estimator, with remainder $o(n^{-1})$.

Key Words and Phrases: Edgeworth expansion, Convex combination of U-statistics, Studentization.

1. Introduction

Let $\theta(F)$ be an estimable parameter of an unknown distribution F. Let $g(x_1, ..., x_k)$ be the symmetric kernel of degree $k(\geq 2)$ for this parameter $\theta(F)$. In this paper, we assume that the kernel g is not degenerate. Let $X_1, \ldots, X_n$ be a random sample of size n from the distribution F. Let X be a random variable having the distribution F.

As an estimator of $\theta(F)$, a convex combination Y_n of U-statistics is introduced by Toda and Yamato (2001) as follows: Let $w(r_1, \ldots, r_j; k)$ be a nonnegative and symmetric function of positive integers $r_1, \ldots, r_j$ such that $j = 1, \ldots, k$ and $r_1 + \cdots + r_j = k$, where k is the degree of the kernel g and fixed. We assume that at least one of $w(r_1, \ldots, r_j; k)$'s is positive. For $j = 1, \ldots, k$, let $g_{(j)}(x_1, ..., x_j)$ be the kernel given by

$$g_{(j)}(x_1, \ldots, x_j) = \frac{1}{d(k,j)} \sum\nolimits^{+}_{r_1+\cdots+r_j=k} w(r_1, \ldots, r_j; k) g(\underbrace{x_1, \ldots, x_1}_{r_1}, \ldots, \underbrace{x_j, \ldots, x_j}_{r_j}), \tag{1}$$

where the summation $\sum^{+}_{r_1+\cdots+r_j=k}$ is taken over all positive integers $r_1, ..., r_j$ satisfying $r_1 + \cdots + r_j = k$ with j and k fixed and $d(k,j) = \sum^{+}_{r_1+\cdots+r_j=k} w(r_1, \ldots, r_j; k)$ for $j = 1, 2, ..., k$. Let $U_n^{(j)}$ be the U-statistic associated with this kernel $g_{(j)}(x_1, \ldots, x_j)$

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for $j = 1, \ldots, k$. The kernel $g_{(j)}(x_1, \ldots, x_j)$ is symmetric because of the symmetry of $w(r_1, \ldots, r_j; k)$. If $d(k, j)$ is equal to zero for some j, then the associated $w(r_1, \ldots, r_j; k)$'s are equal to zero. In this case, we let the corresponding statistic $U_n^{(j)}$ be zero. The statistic Y_n is given by

$$Y_n = \frac{1}{D(n,k)} \sum_{j=1}^{k} d(k,j) \binom{n}{j} U_n^{(j)}, \tag{2}$$

where $D(n,k) = \sum_{j=1}^{k} d(k,j)\binom{n}{j}$. Since w's are nonnegative and at least one of them is positive, $D(n, k)$ is positive. Note that $U_n^{(k)}$ is equal to the U-statistic U_n given below for $w(1, \ldots, 1; k) > 0$, because of $g_{(k)} = g$.

Another type of a linear combination of U-statistics, L_n, is introduced by (3.3) of Sen (1977). While Y_n and L_n are both linear combination of U-statistics, Y_n is different from L_n in the mean that the weight function w's determines Y_n as an estimator of θ. Since the coefficients of $U_n^{(\cdot)}$ on the right-hand side of (1.2) are non-negative and their sum is equal to one, the linear combination given by (1.2) is also a convex combination.

For example, let w be the function given by $w(1, 1, \ldots, 1; k) = 1$ and $w(r_1, \ldots, r_j; k) = 0$ for positive integers $r_1, \ldots, r_j$ such that $j = 1, \ldots, k-1$ and $r_1 + \cdots + r_j = k$. Then the corresponding statistic Y_n is equal to U-statistic U_n, which is given by $U_n = \binom{n}{k}^{-1} \sum_{1 \le j_1 < \cdots < j_k \le n} g(X_{j_1}, \ldots, X_{j_k})$, where $\sum_{1 \le j_1 < \cdots < j_k \le n}$ denotes the summation over all integers $j_1, \ldots, j_k$ satisfying $1 \le j_1 < \cdots < j_k \le n$.

Let w be the function given by $w(r_1, \ldots, r_j; k) = 1$ for positive integers $r_1, \ldots, r_j$ such that $j = 1, \ldots, k$ and $r_1 + \cdots + r_j = k$. Then the corresponding statistic Y_n is equal to the LB-statistic B_n given by $B_n = \binom{n+k-1}{k}^{-1} \sum_{r_1 + \cdots + r_n = k} g(X_1, \ldots, X_1, \ldots, X_n, \ldots, X_n)$, where the numbers of X_1,...., X_n are r_1,...,r_n, respectively, and $\sum_{r_1 + \cdots + r_n = k}$ denotes the summation over all non-negative integers $r_1, ..., r_n$ satisfying $r_1 + \cdots + r_n = k$.

Let w be the function given by $w(r_1, \ldots, r_j; k) = k!/(r_1! \cdots r_j!)$ for positive integers $r_1, \ldots, r_j$ such that $j = 1, \ldots, k$ and $r_1 + \cdots + r_j = k$. Then the corresponding statistic Y_n is equal to the V-statistic V_n given by $V_n = n^{-k} \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} g(X_{j_1}, \ldots, X_{j_k})$. (See Toda and Yamato (2001).)

Let w be the function given by $w(r_1, \ldots, r_j; k) = k!/(r_1 \cdots r_j)$ for positive integers $r_1, \ldots, r_j$ such that $j = 1, \ldots, k$ and $r_1 + \cdots + r_j = k$. Then, for example, the corresponding statistic Y_n for the third central moment of the distribution F is given by $S_n = n(n^2+1)^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^3$, where $\bar{X}$ is the sample mean of $X_1, \ldots, X_n$ (see Nomachi et al. (2002)).

The Edgeworth expansion of the standardized Y-statistic Y_n about θ is obtained with remainder $o(n^{-1})$ by Yamato et al. (2003). It also gives the Edgeworth expansion of the studentized Y-statistic with remainder $o(n^{-1/2})$. For the studentization of Y-statistic Y_n given by (1.2), we use a jackknife variance estimator. That is, as a variance estimator of $\sqrt{n}Y_n$, we use $\hat{\sigma}_n^2$ given by

$$\hat{\sigma}_n^2 = (n-1)\sum_{i=1}^{n}(Y_n^{(i)} - Y_n)^2 \tag{3}$$

where $Y_n^{(i)}$ is the Y-statistic given by (1.2) computed from a sample of size $n-1$ with X_i left out.

Our purpose is to get an Edgeworth expansion of the studentized statistic Y_n given by (1.2), using the jackknife variance estimator $\hat{\sigma}_n^2$ with remainder term $o(n^{-1})$. For the studentized U-statistic, Helmers (1991) and Maesono (1995) obtained its Edgeworth expansion using a jackknife variance estimator with remainder term $o(n^{-1/2})$. Maesono (1997) get an Edgeworth expansion using a jackknife variance estimator with remainder term $o(n^{-1})$. Maesono (1996) gave an Edgeworth expansion of $\sqrt{n}[L_n - E(L_n)]/\tilde{\sigma}_n$, where $\tilde{\sigma}_n^2$ is a jackknife variance estimator of $\sqrt{n}[L_n - E(L_n)]$.

In Section 2, we give an Edgeworth expansion of $\sqrt{n}[Y_n - E(Y_n)]/\hat{\sigma}_n$, following Maesono (1996). In Section 3, using the result of Section 2 we shall derive another Edgeworth expansion about parameter θ, that is, the expansion of $\sqrt{n}[Y_n - \theta]/\hat{\sigma}_n$. We give some examples in Section 4. In Section 5, we give supplementary propositions necessary for the previous sections.

2. Studentized Y-statistic about its expectation

In the following sections, we assume $d(k,k) > 0$. Then, with $\delta_k = kd(k,k-1)/d(k,k)$ it holds that

$$\frac{d(k,k)}{D(n,k)}\binom{n}{k} = 1 - \frac{\delta_k}{n} + O\Big(\frac{1}{n^2}\Big), \tag{4}$$

and

$$\frac{d(k,k-1)}{D(n,k)}\binom{n}{k-1} = \frac{\delta_k}{n} + O\Big(\frac{1}{n^2}\Big). \tag{5}$$

For the U-statistic U_n, $d(k,k)n^{(k)}/[D(n,k)k!] = 1$ and $\delta_k = 0$. For the V-statistic V_n and the S-statistic S_n, $\delta_k = k(k-1)/2$. For the LB-statistic B_n, $\delta_k = k(k-1)$ (see Nomachi et al. (2002)).

We put

$$\psi_c(x_1,\dots,x_c) = E\big[g(X_1,\dots,X_k) \mid X_1 = x_1,\dots,X_c = x_c\big], \quad c = 1,2,3$$

and

$$g^{(1)}(x_1) = \psi_1(x_1) - \theta,$$

for $c = 2,3$

$$g^{(c)}(x_1,\dots,x_c) = \psi_c(x_1,\dots,x_c) - \sum_{i=1}^{c-1}\sum_{1\le l_1<\cdots<l_i\le c} g^{(i)}(x_{l_1},\dots,x_{l_i}) - \theta.$$

For the kernel $g_{(k-1)}(x_1,\dots,x_{k-1})$, we put

$$\theta_{k-1} = Eg_{(j)}(X_1,\dots,X_{k-1}),$$

$$\psi_{(k-1),1}(x_1) = E\big[g_{(k-1)}(X_1,\dots,X_{k-1}) \mid X_1 = x_1\big],$$

and

$$g^{(1)}_{(k-1)}(x_1) = \psi_{(k-1),1}(x_1) - \theta_{k-1}.$$

We put

$$\begin{aligned}
\sigma_1^2 &= E\big[\{g^{(1)}(X)\}^2\big], \quad \sigma_2^2 = (k-1)^2E\big[\{g^{(2)}(X_1,X_2)\}^2\big],\\
\nu &= \sigma_2^2 + \frac{2(k-1)\delta_k}{k}E[g^{(1)}(X)g^{(1)}_{(k-1)}(X)] - 2\delta_k\sigma_1^2,\\
f_1(x) &= \frac{1}{2}\big[\{g^{(1)}(x)\}^2 - \sigma_1^2\big] + (k-1)E\big[g^{(1)}(X_2)g^{(2)}(x,X_2)\big]
\end{aligned}$$

and

$$\begin{aligned}
f_2(x,y) = &- g^{(1)}(x)g^{(1)}(y) + (k-1)\Big\{g^{(2)}(x,y)[g_1(x)+g_1(y)]\\
&- E\big[g^{(2)}(x,X_3)g^{(1)}(X_3)\big] - E\big[g^{(2)}(y,X_3)g^{(1)}(X_3)\big]\Big\}\\
&+ (k-1)^2E\big[g^{(2)}(x,X_3)g^{(2)}(y,X_3)\big]\\
&+ (k-1)(k-2)E\big[g^{(3)}(x,y,X_3)g^{(1)}(X_3)\big],
\end{aligned}$$

which satisfy the relations $Ef_1(X) = 0$ and $E[f_2(X_1,X_2)|X_1] = 0$ a.s. (almost surely), respectively. Furthermore we put

$$\begin{aligned}
\tau &= \frac{3E[f_1^2(X_1)]}{2\sigma_1^4} - \frac{\nu}{2\sigma_1^2},\\
\zeta &= E\big[f_1(X_1)g^{(1)}(X_1)\big]
\end{aligned}$$

and

$$\begin{aligned}
a_1(x) &= \frac{\delta_k}{k}\big[(k-1)g^{(1)}_{(k-1)}(x) - kg^{(1)}(x)\big] + \tau g^{(1)}(x)\\
&\quad -\frac{1}{\sigma_1^2}\Big\{[f_1(x)g^{(1)}(x) - \zeta] + \Big(E\big[f_2(x,X_2)g^{(1)}(X_2)\big] - \frac{3\zeta}{\sigma_1^2}f_1(x)\Big)\\
&\quad +(k-1)E\big[g^{(2)}(x,X_2)f_1(X_2)\big]\Big\},\\
a_2(x,y) &= (k-1)g^{(2)}(x,y) - \frac{1}{\sigma_1^2}\big[f_1(x)g^{(1)}(y) + f_1(y)g^{(1)}(x)\big],\\
a_3(x,y,z) &= (k-1)(k-2)g^{(3)}(x,y,z)\\
&\quad -\frac{1}{\sigma_1^2}\Big\{(k-1)\big[f_1(x)g^{(2)}(y,z) + f_1(y)g^{(2)}(x,z) + f_1(z)g^{(1)}(x,y)\big]\\
&\quad +g^{(1)}(x)\big[f_2(y,z) - \frac{3}{\sigma_1^2}f_1(y)f_1(z)\big] + g^{(1)}(y)\big[f_2(x,z) - \frac{3}{\sigma_1^2}f_1(x)f_1(z)\big]\\
&\quad +g^{(1)}(z)\big[f_2(x,y) - \frac{3}{\sigma_1^2}f_1(x)f_1(y)\big]\Big\},
\end{aligned}$$

which satisfy the relations $E[a_1(X)] = 0$, $E[a_2(X_1, X_2)|X_1] = E[a_2(X_1, X_2)|X_2] = 0$ and $E[a_3(X_1, X_2, X_3)|X_1, X_2] = 0$ a.s. because of $Ef_1(X) = 0$, $E[f_2(X_1, X_2)|X_1] = 0$ and $E[g^{(2)}(X_1, X_2)|X_2] = 0$ a.s. We define

$$\begin{aligned}
\lambda_1 &= E[g^{(1)}(X_1)]^3, \\
\lambda_2 &= E[g^{(1)}(X_1)g^{(1)}(X_2)g^{(2)}(X_1, X_2)], \\
\lambda_3 &= E[g^{(1)}(X_1)]^4, \\
\lambda_4 &= E[\big(g^{(1)}(X_1)\big)^2 g^{(1)}(X_2)g^{(2)}(X_1, X_2)], \\
\lambda_5 &= E[g^{(1)}(X_1)g^{(1)}(X_2)g^{(2)}(X_1, X_3)g^{(2)}(X_2, X_3)], \\
\lambda_6 &= E[g^{(1)}(X_1)g^{(1)}(X_2)g^{(1)}(X_3)g^{(3)}(X_1, X_2, X_3)], \\
\lambda_7 &= E[g^{(1)}(X_1)a_1(X_1)], \\
\kappa_3 &= \sigma_1^{-3}(\lambda_1 + 3\lambda_2), \\
\kappa_4 &= \sigma_1^{-4}(\lambda_3 - 3\sigma_1^4 + 12\lambda_4 + 12\lambda_5 + 4\lambda_6)
\end{aligned}$$

and

$$\begin{aligned}
Q_n(x) \quad = \quad & \Phi(x) - \phi(x)\bigg\{\frac{\kappa_3}{6\sqrt{n}}(x^2 - 1) + \frac{\kappa_4}{24n}(x^3 - 3x) \\
& + \frac{\kappa_3^2}{72n}(x^5 - 10x^3 + 15x) + \frac{x}{n\sigma_1^2}\Big(\lambda_7 + \frac{1}{4}E\big[a_2^2(X_1, X_2)\big]\Big)\bigg\}.
\end{aligned}$$

LEMMA 2.1. (Maesono (1996)) *If* $E|g(X_{i_1}, ..., X_{i_k})|^2 < \infty$ *for* $1 \le i_1 \le \cdots \le i_k \le k$*, and* $E|g(X_1, X_2, X_3, ..., X_k)|^{4+\varepsilon} < \infty$ *and* $E|g(X_1, X_1, X_2, ..., X_k)|^{4+\varepsilon} < \infty$ *for* $\varepsilon > 0$*, then we have*

$$\hat{\sigma}_n^2 = k^2\sigma_1^2 + \frac{2k^2}{n}\sum_{i=1}^{n} f_1(X_i) + \frac{2k^2}{n(n-1)}\sum_{1\le i<j\le n} f_2(X_i, X_j) + \frac{k^2\nu}{n} + o_p^*(n^{-1})$$

and

$$\begin{aligned}
k\sigma_1\hat{\sigma}_n^{-1} = 1 - \frac{1}{n\sigma_1^2}\sum_{i=1}^{n} f_1(X_i) - \frac{1}{n^2\sigma_1^2}\sum_{1\le i<j\le n}\Big[f_2(X_i, X_j) - \frac{3}{\sigma_1^2}f_1(X_i)f_1(X_j)\Big] \\
+ \frac{1}{n}\bigg\{\frac{3E[f_1^2(X_1)]}{2\sigma_1^4} - \frac{\nu}{2\sigma_1^2}\bigg\} + o_p^*(n^{-1}) \qquad (6)
\end{aligned}$$

where $o_p^*(n^{-1})$ *is a quantity satisfying* $P\big(|o_p^*(n^{-1})| \ge cn^{-1}(\log n)^{-1}\big) = o(n^{-1})$ *for a constant* $c > 0$.

Thus by Maesono (1996) we have the following: Assume that $E|g(X_{i_1}, ..., X_{i_k})|^2 < \infty$ for $1 \le i_1 \le \cdots \le i_k \le k$, $E|g(X_1, X_2, X_3, ..., X_k)|^9 < \infty$ and $E|g(X_1, X_1, X_2, ..., X_k)|^{4+\varepsilon} < \infty$ for $\varepsilon > 0$. Then we have

$$\hat{\sigma}_n^{-1}\sqrt{n}\big(Y_n - E[Y_n]\big) = \frac{\sqrt{n}}{\sigma_1}U_n^* - \frac{\zeta}{\sqrt{n}\sigma_1^3} + o_p^*(n^{-1}), \qquad (7)$$

where

$$U_n^* = \frac{1}{n}\sum_{i=1}^{n}\Big\{g^{(1)}(X_i)+\frac{a_1(X_i)}{n}\Big\}+\frac{1}{n^2}\sum_{1\le i<j\le n} a_2(X_i,X_j)+\frac{1}{n^3}\sum_{1\le i<j<l\le n} a_3(X_i,X_j,X_l).$$

For Edgeworth expansion of the statistic, we use the result of Lai and Wang (1993) which needs the following conditions.

Condition (C): $E \mid g^{(2)} \mid^r < \infty$ for some $r > 2$ and there exists K Borel functions $h_j : R \to R$ such that $K(r-2) > 8(4r-5)$, $Eh_j^2(X_1) < \infty$ $(j = 1, \ldots, K)$, and the covariance matrix of $(W_1, \ldots, W_K)$ is positive definite, where $W_j = (Lh_j)(X_1)$ and $(Lh)(x) = E[a_2(x, X_2)h(X_2)]$.

In case of $k \ge 3$, the original condition of Lai and Wang (1993) contains the term $I_{[E|g^{(3)}(X_1,X_2,X_3)|>0]}$, which equals 1 since $g^{(3)}(X_1,\ X_2, X_3)$ is not zero a.s. under the assumption that the kernel g is not degenerate, that is, $\sigma_1^2 > 0$. It also contains $E \mid g^{(2)} \mid^r < \infty$ for some $r > 2$. This condition is satisfied with $r = 4$ under our condition $E[\mid \psi_3(X_1, X_2, X_3) \mid^4] < \infty$, which is necessary for the condition (A4) of Lai and Wang (1993).

Condition (D): There exist constants c_j and Borel functions $h_j : R \to R$ such that $Eh_j(X_1) = 0$, $E \mid h_j(X_1) \mid^r < \infty$ for some $r \ge 5$ and $a_2(X_1, X_2) = \sum_{j=1}^{K} c_j h_j(X_1)h_j(X_2)$ a.s.; moreover, for some $0 < \varepsilon < \min\{1, 2(1 - 11r^{-1}/3)\}$,

$$\limsup_{|t|\to\infty}\ \sup_{|s_1|+\cdots+|s_K|\le|t|^{-\varepsilon}} \Big|E\exp\big(it\big[g^{(1)}(X_1)+\sum_{j=1}^{K} s_j h_j(X_1)\big]\big)\Big| < 1. \tag{8}$$

The asymptotic expansion of the statistic $\sqrt{n}\sigma_1^{-1}U_n^*$ is given by the following.

Lemma 2.2. (Maesono (1996)) *Assume that* $E[g^{(1)}(X_1))]^4 < \infty$, $\sigma_1^2 > 0$, $E[|a_1(X_1)|^3 + |a_3(X_1, X_2, X_3)|^4] < \infty$ *and* $\limsup_{|t|\to\infty} |E\big[\exp\{itg^{(1)}(X_1)\}\big]| < 1$. *If either condition* (C) *or* (D) *is satisfied, we have*

$$\sup_{-\infty<x<\infty}\Big|P\big(\sqrt{n}\sigma_1^{-1}U_n^* \le x\big) - Q_n(x)\Big| = o(n^{-1}). \tag{9}$$

For example, by Minkowski's inequality and Schwarz's one, one of the above conditions $E[|a_1(X_1)|^3 + |a_3(X_1, X_2, X_3)|^4] < \infty$ is satisfied if

$$E|g^{(1)}(X_1)|^{12} < \infty, \quad E|g^{(2)}(X_1, X_2)|^{12} < \infty,$$

$$E|g^{(3)}(X_1, X_2, X_3)|^{12} < \infty, \quad E|g^{(1)}_{(k-1)}(X_1)|^3 < \infty.$$

We define

$$\begin{aligned}
e_1 &= E[g^{(1)}(X_1)]^3,\\
e_2 &= (k-1)E[g^{(1)}(X_1)g^{(1)}(X_2)g^{(2)}(X_1,X_2)],\\
e_3 &= E[g^{(1)}(X_1)]^4,\\
e_4 &= (k-1)E[\big(g^{(1)}(X_1)\big)^2 g^{(1)}(X_2)g^{(2)}(X_1,X_2)],\\
e_5 &= (k-1)^2E[g^{(1)}(X_1)g^{(1)}(X_2)g^{(2)}(X_1,X_3)g^{(2)}(X_2,X_3)],\\
e_6 &= (k-1)(k-2)E[g^{(1)}(X_1)g^{(1)}(X_2)g^{(1)}(X_3)g^{(3)}(X_1,X_2,X_3)],\\
v_1 &= \sigma_1^{-3}(2e_1+3e_2),\\
v_2 &= \sigma_1^{-3}(e_1+3e_2),\\
v_3 &= -\sigma_1^{-6}(2e_1+3e_2)^2,\\
v_4 &= 6\sigma_1^{-4}(e_3-6\sigma_1^4+12e_4+6e_5+4e_6)-2\sigma_1^{-6}(2e_1+3e_2)(2e_1+9e_2),\\
v_5 &= 3\sigma_1^{-6}(4e_1^2+12e_1e_2+3e_2^2)+18\sigma_1^{-4}(\sigma_1^2\sigma_2^2-e_3+2\sigma_1^4-4e_4-2e_5).
\end{aligned}$$

Using $f_{12}(x) = E[g^{(1)}(X_2)g^{(2)}(x,X_2)]$ which appears in the second term of f_1, we can write

$$e_2 = (k-1)E[g^{(1)}(X_1)f_{12}(X_1)], \quad e_4 = (k-1)E\big[\{g^{(1)}(X_1)\}^2 f_{12}(X_1)\big],$$

$$e_5 = (k-1)^2E[\{f_{12}(X_1)\}^2].$$

Furthermore, we define

$$H_n(x) = \Phi(x) + \phi(x)\frac{1}{6\sqrt{n}}(v_1x^2+v_2) + \phi(x)\frac{1}{72n}(v_3x^5+v_4x^3+v_5x).$$

Between Q_n and H_n, it holds that

$$Q_n\Big(x + \frac{\zeta}{\sqrt{n}\sigma_1^3}\Big) = H_n(x) + o(n^{-1}). \tag{10}$$

The asymptotic expansion of the statistic $\hat{\sigma}_n^{-1}\sqrt{n}(Y_n - EY_n)$ is given by the following.

Lemma 2.3. (Maesono (1996)) *Assume that* $E|g(X_{i_1},...,X_{i_k})|^2 < \infty$ *for* $1 \le i_1 \le \cdots \le i_k \le k$, $E|g(X_1,X_2,...,X_k)|^9 < \infty$, $E|g(X_1,X_1,X_2,...,X_k)|^{4+\varepsilon} < \infty$ *for* $\varepsilon > 0$, *and* $E[g^{(1)}(X_1))]^4 < \infty$, $\sigma_1^2 > 0$. *Furthermore we assume that* $E[|a_1(X_1)|^3 + |a_3(X_1,X_2,X_3)|^4] < \infty$ *and* $\limsup_{|t|\to\infty}|E\big[\exp\{itg^{(1)}(X_1)\}\big]| < 1$. *If either condition* (C) *or* (D) *is satisfied, we have*

$$\sup_{-\infty<x<\infty}\Big|P\big(\hat{\sigma}_n^{-1}\sqrt{n}(Y_n - EY_n) \le x\big) - H_n(x)\Big| = o(n^{-1}). \tag{11}$$

3. Studentized Y-statistic about θ

At first, we note that

$$\hat{\sigma}_n^{-1}\sqrt{n}(Y_n-\theta)=\hat{\sigma}_n^{-1}\sqrt{n}(Y_n-EY_n)+\hat{\sigma}_n^{-1}\sqrt{n}(EY_n-\theta). \tag{12}$$

By Nomachi et al. (2002), (3.5), we have

$$\sqrt{n}(EY_n-\theta)=\frac{\mu_k}{\sqrt{n}}+O(n^{-3/2}) \tag{13}$$

where $\mu_k=\delta_k(\theta_{k-1}-\theta)$.

If we put $R_{2n}=o_p^*(n^{-1})$, then $\sqrt{n}(EY_n-\theta)R_{2n}=o_p^*(n^{-1})$. Because for a constant $c>0$ we have $P\big(|\sqrt{n}(EY_n-\theta)R_{2n}|\geq cn^{-1}(\log n)^{-1}\big)\leq P\big(|R_{2n}|\geq cn^{-1}(\log n)^{-1}\big)=o(n^{-1})$, since $\sqrt{n}(EY_n-\theta)\leq 1$ for a large n. We multiply (2.3) by (3.2), and use this fact. Then, we get

$$k\sigma_1\hat{\sigma}_n^{-1}\sqrt{n}(EY_n-\theta)=\frac{\mu_k}{\sqrt{n}}\Big\{1-\frac{1}{n\sigma_1^2}\sum_{i=1}^{n}f_1(X_i)\Big\}+R_n^*+o_p^*(n^{-1}) \tag{14}$$

where $E|R_n^*|=O(n^{-3/2})$. Thus from (2.4) , (3.1) and (3.3) we get

$$\hat{\sigma}_n^{-1}\sqrt{n}(Y_n-\theta)=\frac{\sqrt{n}}{\sigma_1}U_n^{**}+\frac{1}{\sqrt{n}}\Big\{-\frac{\zeta}{\sigma_1^3}+\frac{\mu_k}{k\sigma_1}\Big\}+R_n^*+o_p^*(n^{-1}), \tag{15}$$

where

$$U_n^{**}=\frac{1}{n}\sum_{i=1}^{n}\{g^{(1)}(X_i)+\frac{a_1^*(X_i)}{n}\}+\frac{1}{n^2}\sum_{1\leq i<j\leq n}a_2(X_i,X_j)+\frac{1}{n^3}\sum_{1\leq i<j<l\leq n}a_3(X_i,X_j,X_l), \tag{16}$$

and

$$a_1^*(X_i)=a_1(X_i)-\frac{\mu_k}{k\sigma_1^3}f_1(X_i).$$

We can also obtain the expansion (3.4) by multiplying (2.3) and the following (3.6). For the detail of this multiplication, see Appendix.

Lemma 3.1. (Yamato et al. (2003)) *Assume that $d(k,k)>0$ and $E|g(X_{i_1},...,X_{i_k})|^2<\infty$ for $1\leq i_1\leq\cdots\leq i_k\leq k$. Then, we have*

$$\sqrt{n}(Y_n-\theta)=Y_n^{**}+\frac{\mu_k}{\sqrt{n}}+R_n', \tag{17}$$

where $E\mid R_n'\mid^2=O(n^{-3})$ and

$$Y_n^{**}=k\big(1-\frac{\delta_k}{n}\big)\frac{1}{n^{1/2}}\sum_{i=1}^{n}g^{(1)}(X_i)+(k-1)\delta_k\frac{1}{n^{3/2}}\sum_{i=1}^{n}g_{(k-1)}^{(1)}(X_i)$$
$$+k(k-1)\frac{1}{n^{3/2}}\sum_{1\leq i<j\leq n}g^{(2)}(X_i,X_j)+k(k-1)(k-2)\frac{1}{n^{5/2}}\sum_{1\leq i<j<l\leq n}g^{(3)}(X_i,X_j,X_l).$$

In the asymptotic evaluation of of $\hat{\sigma}_n^{-1}\sqrt{n}(Y_n-\theta)$ with remainder term $o(n^{-1})$, we can neglect at first the term $o_p^*(n^{-1})$ of (3.4) by using the relation given by Lemma 5.3 and then the terms R_n^* of (3.4) by using the relation given by Lemma 5.2. Thus, we can get the following.

LEMMA 3.2. *Assume that* $d(k,k)>0$, $E|g(X_{i_1},...,X_{i_k})|^2<\infty$ *for* $1\le i_1\le\cdots\le i_k\le k$, $E|g(X_1,X_2,X_3,...,X_k)|^9<\infty$ *and* $E|g(X_1,X_1,X_2,...,X_k)|^{4+\varepsilon}<\infty$ *for* $\varepsilon>0$. *Then, we have*

$$\sup_{-\infty<x<\infty}\left|P\Big(\hat{\sigma}_n^{-1}\sqrt{n}(Y_n-\theta)\le x\Big)-P\Big(\frac{\sqrt{n}}{\sigma_1}U_n^{**}+\frac{1}{\sqrt{n}}\Big\{-\frac{\zeta}{\sigma_1^3}+\frac{\mu_k}{k\sigma_1}\Big\}\le x\Big)\right|=o(n^{-1}). \tag{18}$$

U_n^{**} is different from U_n^* only in the term a_1^*. Thus the Edgeworth expansion of U_n^{**} is different from U_n^* in the term λ_7. By Lemma 2.2, we get the following.

LEMMA 3.3. *Assume that* $d(k,k)>0$. *Furthermore, we assume that* $E|g(X_{i_1},...,\ X_{i_k})|^2<\infty$ *for* $1\le i_1\le\cdots\le i_k\le k$, $E[g^{(1)}(X_1)]^4<\infty$, $\sigma_1^2>0$, $E[|a_1(X_1)|^3+|a_3(X_1,X_2,X_3)|^4]<\infty$ *and* $\limsup_{|t|\to\infty}|E\big[\exp\{itg^{(1)}(X_1)\}\big]|<1$. *If either condition* (C) *or* (D) *is satisfied, we have*

$$\sup_{-\infty<x<\infty}\left|P\big(\sqrt{n}\sigma_1^{-1}U_n^{**}\le x\big)-Q_n^*(x)\right|=o(n^{-1}) \tag{19}$$

where $Q_n^*(x)$ *is obtained from* $Q_n(x)$ *by replacing* λ_7 *with*

$$\lambda_7^*=\lambda_7-\frac{\mu_k}{k\sigma_1^3}(\frac{1}{2}e_1+e_2).$$

The last term of the above right-hand side is due to the bias of the Y-statistic. We also know that

$$Q_n^*(x)=Q_n(x)-\frac{\mu_k}{nk\sigma_1^5}(\frac{1}{2}e_1+e_2)x\phi(x). \tag{20}$$

By (2.6), we have

$$Q_n\Big(x+\frac{\zeta}{\sqrt{n}\sigma_1^3}-\frac{\mu_k}{\sqrt{n}k\sigma_1}\Big)=H_n\Big(x-\frac{\mu_k}{\sqrt{n}k\sigma_1}\Big)+o(n^{-1}) \tag{21}$$

and by Lemma 5.4

$$H_n\Big(x-\frac{\mu_k}{\sqrt{n}k\sigma_1}\Big)=\Phi(x)+\phi(x)\frac{1}{6\sqrt{n}}\Big(v_1x^2+v_2-6\frac{\mu_k}{k\sigma_1}\Big)$$
$$+\phi(x)\frac{1}{72n}\left(v_3x^5+\big(v_4+12\frac{v_1\mu_k}{k\sigma_1}\big)x^3+\Big[v_5+12\frac{\mu_k}{k\sigma_1}(v_2-2v_1)\Big]x-36\big(\frac{\mu_k}{k\sigma_1}\big)^2\right)+O(n^{-3/2}). \tag{22}$$

By (3.9), (3.10), (3.11) and Lemma 5.4, we can get

$$Q_n^*\Big(x+\frac{1}{\sqrt{n}}\{\frac{\zeta}{\sigma_1^3}-\frac{\mu_k}{k\sigma_1}\}\Big)=H_n^*(x)+O(n^{-3/2}), \tag{23}$$

where

$$H_n^*(x) = \Phi(x) + \phi(x)\frac{1}{6\sqrt{n}}\Big(v_1x^2 + v_2 - \frac{6\mu_k}{k\sigma_1}\Big) + \phi(x)\frac{1}{72n}\Big\{v_3x^5 + \big(v_4 + \frac{v_1\mu_k}{k\sigma_1}\big)x^3 + \Big[v_5 + 12\frac{\mu_k}{k\sigma_1}(v_2 - 2v_1) - 72\frac{\mu_k}{k\sigma_1^5}\big(\frac{1}{2}e_1 + e_2\big)\Big]x - 36\big(\frac{\mu_k}{k\sigma_1}\big)^2\Big\}.$$

Thus, by (3.7), (3.8), (3.11) and (3.12) we get the following.

THEOREM 3.4. *Assume that* $d(k,k) > 0$, $E|g(X_{i_1},...,X_{i_k})|^2 < \infty$ *for* $1 \le i_1 \le \cdots \le i_k \le k$, $E[g^{(1)}(X_1)]^4 < \infty$, $\sigma_1^2 > 0$, $E|g(X_1,X_2,...,X_k)|^9 < \infty$, *and* $E|g(X_1,X_1,X_2,...,X_k)|^{4+\varepsilon} < \infty$ *for* $\varepsilon > 0$. *Furthermore we assume that* $\limsup_{|t|\to\infty} |E\big[\exp\{itg^{(1)}(X_1)\}\big]| < 1$ *and* $E[|a_1(X_1)|^3 + |a_3(X_1,X_2,X_3)|^4] < \infty$. *If either condition* (C) *or* (D) *is satisfied, we have*

$$\sup_{-\infty<x<\infty}\Big|P\big(\hat{\sigma}_n^{-1}\sqrt{n}(Y_n - \theta) \le x\big) - H_n^*(x)\Big| = o(n^{-1}). \tag{24}$$

As stated after Lemma 2.2, one of the conditions of Theorem 3.4 $E[|a_1(X_1)|^3 + |a_3(X_1,X_2,X_3)|^4] < \infty$ is satisfied if $E|g^{(1)}(X_1)|^{12} < \infty$, $E|g^{(2)}(X_1,X_2)|^{12} < \infty$, and $E|g^{(3)}(X_1,X_2,X_3)|^{12} < \infty$, $E|g^{(1)}_{(k-1)}(X_1)|^3 < \infty$.

COROLLARY 3.5. *Especially, let the degree k be* 2. *Assume that* $d(k,k) > 0$, $E|g(X_1,X_1)|^{4+\varepsilon} < \infty$ *for* $\varepsilon > 0$, $E[g^{(1)}(X_1)]^4 < \infty$, $\sigma_1^2 > 0$, *and* $E|g(X_1,X_2)|^9 < \infty$. *Furthermore we assume that* $\limsup_{|t|\to\infty} |E\big[\exp\{itg^{(1)}(X_1)\}\big]| < 1$ *and* $E[|a_1(X_1)|^3 + |a_3(X_1,X_2,X_3)|^4] < \infty$. *If either condition* (C) *or* (D) *is satisfied, we have* (3.13).

In the case of $k = 2$, the condition $E[|a_1(X_1)|^3 + |a_3(X_1,X_2,X_3)|^4] < \infty$ is satisfied if $E|g^{(1)}(X_1)|^{12} < \infty$, $E|g^{(2)}(X_1,X_2)|^{12} < \infty$, and $E|g^{(1)}_{(k-1)}(X_1)|^3 < \infty$.

The difference of the Edgeworth expansions of the studentized Y-statistic about its expectation and θ is the following.

COROLLARY 3.6.

$$H_n^*(x) = H_n(x) - \phi(x)\frac{\mu_k}{\sqrt{n}k\sigma_1} + \phi(x)\frac{1}{72n}\Big\{\frac{v_1\mu_k}{k\sigma_1}x^3 + \Big[12\frac{\mu_k}{\sqrt{n}k\sigma_1}(v_2 - 2v_1) - 72\frac{\mu_k}{\sqrt{n}k\sigma_1^5}\big(\frac{1}{2}e_1 + 2_2\big)\Big]x - 36\Big(\frac{\mu_k}{\sqrt{n}k\sigma_1}\Big)^2\Big\}. \tag{25}$$

Especially, if $\theta_{k-1} = \theta$ *then* $H_n^*(x) = H_n(x)$.

The condition $\theta_{k-1} = \theta$ above is equivalent to $Eg(X_1,X_1,X_2,X_3,\ldots,X_{k-1}) = Eg(X_1,\ X_2,\ldots,X_k)$. The difference between the Edgeworth expansions about its expectation and θ appears at the term related with μ_k which arise from the bias. The value of the difference depends on each Y-statistic. The values of μ_k for V-statistic, S-statistic and LB-statistic are as follows.

$$\mu_k = \delta_k(\theta_{k-1} - \theta), \quad \delta_k = \begin{cases} \dfrac{k(k-1)}{2} & (\mathrm{V,S-statistic}) \\ k(k-1) & (\mathrm{LB-statistic}). \end{cases}$$

By Remark 3 of Maesono (1996), $H_n(x)$ is equal to the Edgeworth expansion of studentized U-statistic using the jackknife variance estimator. Hence, if $\theta_{k-1} = \theta$, then the Edgeworth expansion of studentized Y-statistic about θ using the jackknife variance estimator is equal to the one of studentized U-statistic using the jackknife variance estimator. This is also read from (3.1) and (3.2).

4. Examples

We give examples of the Edgeworth expansion of the studentized Y-statistic about estimable parameter θ.

Example 4.1 We consider the third central moment $\theta = \int (x-\mu)^3 dF(x)$, where μ is the mean of F. Its kernel $g(x_1, x_2, x_3)$ is given by

$$\frac{1}{3}(x_1^3 + x_2^3 + x_3^3) - \frac{1}{2}(x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2) + 2x_1 x_2 x_3.$$

For this kernel, we have $g_{(k-1)}(x_1, x_2) = g_{(2)}(x_1, x_2) = 0$ and $g_{(k-2)}(x_1) = g_{(1)}(x_1) = 0$ and so $\theta_{k-1}(= \theta_2) = 0$ $\theta_{k-2}(= \theta_1) = 0$. Therefore, we have

$$Y_n = \frac{d(k,k)}{D(n,k)} \binom{n}{k} U_n = \frac{d(3,3)n^2}{6D(n,3)} \sum_{j=1}^{n} (X_j - \bar{X})^3 \tag{26}$$

where $\bar{X} = \sum_{j=1}^n X_j/n$. We assume that the distribution F has a density. We also assume $E|X|^{27} < \infty$ and denote jth moment of X about the origin by m'_j $(j = 2, 3, ..., 12)$. In order to study the statistical properties of Y_n, by (4.1) the mean μ is assumed to be zero, without loss of generality. Thus, in this case $\theta = m'_3$ and $\mu_k = -\delta_k m'_3$. We consider the two cases that the distribution F is symmetric or not.

Example 4.1.1 We assume that the distribution F is symmetric about zero. In this case, by the symmetry $m'_j = 0$ $(j = 3, 5, ..., 11)$ and $\theta = 0$. Then, we have

$$\begin{aligned}
g^{(1)}(x_1) &= \frac{1}{3}(x_1^3 - 3x_1 m'_2), \\
g^{(2)}(x_1, x_2) &= \psi_2(x_1, x_2) - g^{(1)}(x_1) - g^{(1)}(x_2) - \theta \\
&= \frac{1}{2}(-x_1^2 x_2 - x_1 x_2^2 + x_1 m'_2 + x_2 m'_2), \\
g^{(3)}(x_1, x_2, x_3) &= 2x_1 x_2 x_3.
\end{aligned}$$

By the computation based on these, we get

$$\begin{aligned}
e_1 &= 0, \quad e_2 = 0, \\
e_3 &= \frac{1}{81}(m'_{12} - 12m'_{10}m'_2 + 54m'_8 m'^2_2 - 108m'_6 m'^3_2 + 81m'_4 m'^4_2), \\
e_4 &= \frac{1}{27}(-m'_8 m'_4 + 3m'_8 m'^2_2 + 7m'_6 m'_4 m'_2 - 21m'_6 m'^3_2 \\
&\quad -15m'^2_4 m'^2_2 + 54m'_4 m'^4_2 - 27m'^6_2), \\
e_5 &= \frac{1}{9}(m'^3_4 - 7m'^2_4 m'^2_2 + 15m'_4 m'^4_2 - 9m'^6_2), \\
e_6 &= \frac{4}{27}(m'^3_4 - 9m'^2_4 m'^2_2 + 27m'_4 m'^4_2 - 27m'^6_2).
\end{aligned}$$

Furthermore,

$$\sigma_1^2 = \frac{1}{9}(m'_6 - 6m'_4m'_2 + 9m'^3_2), \quad \sigma_2^2 = 2(m'_4m'_2 - m'^3_2).$$

Thus we get

$$\begin{aligned} v_1 &= 0, \quad v_2 = 0, \quad v_3 = 0, \\ v_4 &= \frac{6}{(m'_6 - 6m'_4m'_2 + 9m'^3_2)^2} \\ &\quad \times \Big\{ m'_{12} - 12m'_{10}m'_2 - 36m'_8m'_4 + 162m'_8m'^2_2 - 6m'^2_6 + 324m'_6m'_4m'_2 \\ &\qquad -972m'_6m'^3_2 + 102m'^3_4 - 1566m'^2_4m'^2_2 + 4779m'_4m'^4_2 - 3240m'^6_2 \Big\}, \\ v_5 &= \frac{18}{(m'_6 - 6m'_4m'_2 + 9m'^3_2)^2} \\ &\quad \times \Big\{ -m'_{12} + 12m'_{10}m'_2 + 12m'_8m'_4 - 90m'_8m'^2_2 + 2m'^2_6 - 90m'_6m'_4m'_2 \\ &\qquad +378m'_6m'^3_2 - 18m'^3_4 + 270m'^2_4m'^2_2 - 945m'_4m'^4_2 + 486m'^6_2 \Big\} \end{aligned}$$

and

$$E[g^{(1)}(X_1)f_1(X_1)] = \frac{1}{2}e_1 + e_2 = 0.$$

Now we check the condition (D): We can write $2g^{(2)}(x,y) = (x+y)(m_2' - xy) = -(x^2 + x - m_2')(y^2 + y - m_2') + (x^2 - m_2')(y^2 - m_2') + xy$. We can also write

$$f_1(x)g^{(1)}(y) + f_1(y)g^{(1)}(x) = [f_1(x) + g^{(1)}(x)][f_1(y) + g^{(1)}(y)] - f_1(x)f_1(y) - g^{(1)}(x)g^{(1)}(y).$$

Thus we have $a_2(x,y) = \sum_{j=1}^{6} c_j h_j(x) h_j(y)$ where

$$h_1(x) = x^2 + x - m_2', \;\; h_2(x) = x^2 - m_2', \;\; h_3(x) = x,$$

$$h_4(x) = f_1(x) + g^{(1)}(x), \;\; h_5(x) = f_1(x), \;\; h_6(x) = g^{(1)}(x),$$

and

$$c_1 = -\frac{1}{2}(k-1), \;\; c_2 = \frac{1}{2}(k-1), \;\; c_3 = \frac{1}{2}(k-1), \;\; c_4 = -\frac{1}{\sigma_1^2}, \;\; c_5 = c_6 = \frac{1}{\sigma_1^2}.$$

In this example we can write f_1 as follows:

$$f_1(x) = \frac{1}{18}\big[(x^3 - 3xm_2')^2 - 9\sigma_1^2 + 6(m'_4 - 3m'^2_2)(m_2' - x^2)\big].$$

Thus for any $s_1, ..., s_K$ $(-\infty < s_1, ..., s_K < \infty)$, $g^{(1)}(x) + \sum_{j=1}^{K} s_j h_j(x)$ is a polynomial of degree 6 with respect to x. Therefore the distribution of $g^{(1)}(X) + \sum_{j=1}^{K} s_j h_j(X)$ has the density for any $s_1, ..., s_K$ $(-\infty < s_1, ..., s_K < \infty)$ and by Lemma 5.1 the condition (D) is satisfied. We note that the check of the condition (D) of the example (1) of 5 of Yamato et al. (2003) is corrected and may be done like as the above.

Under our assumption, $\mu_k = \delta_k(\theta_{k-1} - \theta) = 0$ and the Edgeworth expansion of the studentized Y-statistic about estimable parameter θ is given by

$$H_n^*(x) = \Phi(x) + \phi(x)\frac{1}{72n}\{v_4x^3 + v_5x\}.$$

That is, there is no difference of the Edgeworth expansions of the studentized Y-statistic about its expectation and θ. Thus, if the distribution F is symmetric then there is no difference among the Edgeworth expansions of the studentized Y-statistic about its expectation and θ. By Remark 3 of Maesono (1996), this expansion $H_n^*(x)$ is also equal to the Edgeworth expansion of studentized U-statistic using a jackknife variance estimator.

Example 4.1.2 We assume that the distribution F is not symmetric about zero. In this case, $\theta = m_3'$ and $\mu_k = -\delta_k m_3'$. Now, we have

$$\begin{aligned}
g^{(1)}(x_1) &= \frac{1}{3}x_1^3 - m'_2x_1 - m_3',\\
g^{(2)}(x_1,x_2) &= \frac{1}{2}(-x_1^2x_2 - x_1x_2^2 + x_1m'_2 + x_2m'_2),\\
g^{(3)}(x_1,x_2,x_3) &= 2x_1x_2x_3.
\end{aligned}$$

Thus, by the same reason as in Example 4.1.1, Condition (D) is satisfied. By the computation based on these functions, we get

$$\begin{aligned}
e_1 &= \frac{1}{27}(m'_9 - 9m'_7m'_2 - 3m'_6m'_3 + 27m'_5m'^2_2\\
&\quad +18m'_4m'_3m'_2 + 2m'^3_3 - 54m'_3m'^3_2),\\
e_2 &= \frac{2}{9}(-m'_5m'_4 + 3m'_5m'^2_2 + 4m'_4m'_3m'_2 - 12m'_3m'^3_2),\\
e_3 &= \frac{1}{81}(m'_{12} - 12m'_{10}m'_2 - 4m'_9m'_3 + 54m'_8m'^2_2 + 36m'_7m'_3m'_2\\
&\quad +6m'_6m'^2_3 - 108m'_6m'^3_2 - 108m'_5m'_3m'^2_2 - 36m'_4m'^2_3m'_2\\
&\quad +81m'_4m'^4_2 - 3m'^4_3 + 162m'^2_3m'^3_2),\\
e_4 &= \frac{1}{27}(-m'_8m'_4 + 3m'_8m'^2_2 - m'_7m'_5 + 4m'_7m'_3m'_2 + 7m'_6m'_4m'_2\\
&\quad -21m'_6m'^3_2 + 6m'^2_5m'_2 + 4m'_5m'_4m'_3 - 45m'_5m'_3m'^2_2\\
&\quad -15m'^2_4m'^2_2 - 16m'_4m'^2_3m'_2 + 54m'_4m'^4_2 + 84m'^2_3m'^3_2 - 27m'^6_2),\\
e_5 &= \frac{1}{9}(m'^2_5m'_2 + 2m'_5m'_4m'_3 - 14m'_5m'_3m'^2_2 + m'^3_4 - 7m'^2_4m'^2_2\\
&\quad -8m'_4m'^2_3m'_2 + 15m'_4m'^4_2 + 40m'^2_3m'^3_2 - 9m'^6_2),\\
e_6 &= \frac{4}{27}(m'^3_4 - 9m'^2_4m'^2_2 + 27m'_4m'^4_2 - 27m'^6_2)
\end{aligned}$$

Furthermore,

$$\sigma_1^2 = \frac{1}{9}(m'_6 - 6m'_4m'_2 - m'^2_3 + 9m'^3_2), \quad \sigma_2^2 = 2(m'_4m'_2 + m'^2_3 - m'^3_2).$$

Thus we can get v_1, v_2, v_3, v_4, and v_5, which are tedious and we omit to write them.

We also have

$$\begin{aligned} \frac{1}{2}e_1 + e_2 &= \frac{1}{54}(m'_9 - 9m'_7m'_2 - 3m'_6m'_3 - 12m'_5m'_4 + 63m'_5m'^2_2 \\ &\qquad +66m'_4m'_3m'_2 + 2m'^3_3 - 198m'_3m'^3_2). \end{aligned}$$

The Edgeworth expansion $H_n^*(x)$ is given by (3.12) with

$$\mu_k = \begin{cases} -\dfrac{k(k-1)}{2}m'_3 & (\mathrm{V, S-statistic}) \\ -k(k-1)m'_3 & (\mathrm{LB-statistic}). \end{cases}$$

In the relation (3.14), $H_n^*(x)$ is different from $H_n(x)$ with this μ_k.

Example 4.2 We consider the kernel $g(x_1, x_2, ..., x_k) = x_1x_2\cdots x_k$ $(k \geq 3)$. This kernel yields estimable parameter $\theta(F) = \mu^k$, where μ is the mean of the distribution F. We assume that the distribution F has the density. We also assume that F is symmetric about the mean μ (> 0), and $E|X|^9 < \infty$ in case of $k = 3, 4$ and $E|X^{2k+\varepsilon}| < \infty$ in case of $k \geq 5$. We shall denote the central moments about the mean by m_j $(j = 2, 4)$. Now, we have

$$\begin{aligned} g^{(1)}(x_1) &= \mu^{k-1}(x_1 - \mu), \\ g^{(2)}(x_1, x_2) &= \mu^{k-2}(x_1 - \mu)(x_2 - \mu), \\ g^{(3)}(x_1, x_2, x_3) &= \mu^{k-3}(x_1 - \mu)(x_2 - \mu)(x_3 - \mu). \end{aligned}$$

The computation based on these values yields

$$\begin{aligned} e_1 &= 0, \quad e_2 = (k-1)\mu^{3k-4}m_2^2, \quad e_3 = \mu^{4k-4}m_4, \quad e_4 = 0, \\ e_5 &= (k-1)^2\mu^{4k-6}m_2^3, \quad e_6 = (k-1)(k-2)\mu^{4k-6}m_2^3. \end{aligned}$$

Furthermore,

$$\sigma_1^2 = \mu^{2k-2}m_2, \quad \sigma_2^2 = (k-1)^2\mu^{2k-4}m_2^2.$$

Thus we get

$$\begin{aligned} v_1 &= \frac{3(k-1)\sqrt{m_2}}{\mu}, \quad v_2 = \frac{3(k-1)\sqrt{m_2}}{\mu}, \quad v_3 = -\frac{9(k-1)^2m_2}{\mu^2}, \\ v_4 &= 6\xi_1^{-4}(e_3 - 6\xi_1^4 + 12e_4 + 6e_5 + 4e_6) - 2\xi_1^{-6}(2e_1 + 3e_2)(2e_1 + 9e_2) \\ &= \frac{6\{\mu^2m_4 - 6\mu^2m_2^2 + (k-1)(k-5)m_2^3\}}{\mu^2m_2^2}, \\ v_5 &= 3\xi_1^{-6}(4e_1^2 + 12e_1e_2 + 3e_2^2) + 18\xi_1^{-4}(\xi_1^2\xi_2^2 - e_3 + 2\xi_1^4 - 4e_4 - 2e_5) \\ &= -\frac{9\{2\mu^2m_4 - 4\mu^2m_2^2 + (k-1)(2k-3)m_2^3\}}{\mu^2m_2^2}. \end{aligned}$$

Now we check the condition (D): By the relation derived in Example 4.1, we have $a_2(x, y) = \sum_{j=1}^4 c_jh_j(x)h_j(y)$ where

$$h_1(x) = x - \mu, \;\; h_2(x) = f_1(x) + g^{(1)}(x), \;\; h_3(x) = f_1(x), \;\; h_4(x) = g^{(1)}(x),$$

and

$$c_1 = -\frac{1}{2}\mu^{k-2}(k-1), \;\; c_2 = -\frac{1}{\sigma_1^2}, \;\; c_3 = c_4 = \frac{1}{\sigma_1^2}.$$

In this example we can write f_1 as follows:

$$f_1(x) = \frac{1}{2}\mu^{2k-3}\big[\mu(x-\mu)^2 + 2(k-1)m_2(x-\mu) - \mu m_2\big].$$

Thus for any $s_1, ..., s_K$ $(-\infty < s_1, ..., s_K < \infty)$, $g^{(1)}(x) + \sum_{j=1}^{K} s_j h_j(x)$ is a polynomial of degree 2 with respect to x. Therefore the distribution of $g^{(1)}(X) + \sum_{j=1}^{K} s_j h_j(X)$ has the density for any $s_1, ..., s_K$ $(-\infty < s_1, ..., s_K < \infty)$ and by Lemma 5.1 the condition (D) is satisfied.

The values of μ_k for V-statistic, S-statistic and LB-statistic are

$$\mu_k = \begin{cases} \dfrac{k(k-1)}{2}(m_2^2 + \mu^2 - \mu^3)\mu^{k-3} & (\mathrm{V, S-statistic}) \\ k(k-1)(m_2^2 + \mu^2 - \mu^3)\mu^{k-3} & (\mathrm{LB-statistic}). \end{cases}$$

These values give the difference among the Edgeworth expansions $H_n^*(x)$. The Edgeworth expansions $H_n^*(x)$ are given by (3.14) with the above values.

Example 4.3 We consider the kernel

$$g(x_1, x_2, x_3) = \frac{1}{3}\{I(x_1 > x_2 + x_3) + I(x_2 > x_1 + x_3) + I(x_3 > x_1 + x_2)\}, \qquad (27)$$

where $I(A)$ is the indicator function of an event A. This kernel yields the estimable parameter $\theta(F) = E\big[1 - F(X_1 + X_2)\big]$ which measures the degree to which a life distribution F has the NBU (new better than used) property. If X_1 and X_2 are random variables having the life distribution F, the NBU property is denoted by $P(X_1 > x) \geq P(X_2 > x + y | X_2 > y)$ for x, $y > 0$. (See, Hollander and Proschan (1972), and Lee (1990)). We note that

$$\psi_1(x) = \frac{1}{3}E\big[F(x - X_3)\big] + \frac{2}{3}E\big[1 - F(x + X_3)\big],$$
$$\psi_2(x_1, x_2) = \frac{1}{3}\big[F(|x_2 - x_1|) + 1 - F(x_1 + x_2)\big].$$

For the corresponding U-statistic, we shall derive Edgeworth expansion in cases that F are the uniform distribution U(0,1) and the exponential distribution $e(1)$ with parameter 1. Since the kernel (4.2) is scale invariant, the Edgeworth expansion for the uniform distribution U(0,1) is equal to the one for the uniform distribution U(0, α), $\alpha > 0$. The Edgeworth expansion for the exponential distribution $e(1)$ is also equal to the one for the exponential distribution $e(\alpha)$, $\alpha > 0$.

Example 4.3.1 We assume that F is the uniform distribution U(0,1). Then $\theta(F) =$

1/6, and

$$g^{(1)}(x_1) = \frac{1}{2}x_1^2 - \frac{2}{3}x_1 + \frac{1}{6} \quad (0 < x < 1),$$

$$g^{(2)}(x_1, x_2) = \begin{cases} \frac{1}{3}\left[|x_1 - x_2| - (x_1 + x_2)\right] - \frac{1}{2}(x_1^2 + x_2^2) + \frac{2}{3}(x_1 + x_2) - \frac{1}{6} & (0 < x_1 + x_2 < 1) \\ \frac{1}{3}|x_1 - x_2| - \frac{1}{2}(x_1^2 + x_2^2) + \frac{2}{3}(x_1 + x_2) - \frac{1}{2} & (x_1 + x_2 > 1). \end{cases}$$

By using the expression of $g^{(1)}$ and $g^{(2)}$ to

$$E[g^{(1)}(X_2)g^{(2)}(X_1, X_2)|X_1 = x_1] = \int_{0<x_2<1, 0<x_1+x_2<1} g^{(1)}(x_2)g^{(2)}(x_1, x_2)dx_2 + \int_{0<x_2<1, x_1+x_2 \geq 1} g^{(1)}(x_2)g^{(2)}(x_1, x_2)dx_2,$$

we get

$$\begin{aligned} E[g^{(1)}(X_2)g^{(2)}(X_1, X_2)|X_1 = x_1] &= \frac{1}{3}\left\{2x_1 \int_0^{x_1} g^{(1)}(x_2)dx_2 - \int_0^{x_1} x_2 g^{(1)}(x_2)dx_2 + \int_{x_1}^1 x_2 g^{(1)}(x_2)dx_2\right\} \\ &- \int_0^1 [g^{(1)}(x_2)]^2 dx_2 + \frac{1}{3}\left\{(1 - x_1)\int_0^{1-x_1} g^{(1)}(x_2)dx_2 - \int_0^{1-x_1} x_2 g^{(1)}(x_2)dx_2\right\}. \end{aligned}$$

Thus we get

$$E[g^{(1)}(X_2)g^{(2)}(X_1, X_2)|X_1 = x_1] = \frac{1}{24}x_1^4 - \frac{5}{54}x_1^3 + \frac{1}{18}x_1^2 - \frac{1}{270}.$$

By the computation based on these functions using Mathematica ver. 4.0, we get

$$\sigma_1^2 = \frac{1}{270} \doteq 0.0037, \quad \sigma_2^2 = \frac{1}{135} \doteq 0.0074$$

and

$$e_1 = \frac{1}{140} \doteq 0.00714, \quad e_2 \doteq -\frac{1}{4536} = -0.00022, \quad e_3 = \frac{1}{280} \doteq 0.00357,$$

$$e_4 = -\frac{19}{1360800} \doteq -0.00001, \quad e_5 = \frac{1}{72900} \doteq 0.000014.$$

Since we can write

$$e_6 = (k-1)(k-2)E[g^{(1)}(X_1)h(X_1)]$$

where $h(x) = \int_{0<y+z<x} g^{(1)}(y)g^{(1)}(z)dydz$, we have

$$e_6 = -\frac{1}{272160} \doteq -0.0000037.$$

Thus we get

$$v_1 = \frac{309\sqrt{30}}{28} \doteq 60.44510, \quad v_2 = \frac{21\sqrt{30}}{4} \doteq 28.7554, \quad v_3 = -\frac{1432215}{292} \doteq -4904.8459,$$

$$v_4 = -\frac{4878216261}{960400} \doteq -5079.35887, \quad v_5 = \frac{2495655}{392} \doteq 6366.4668.$$

Now we check the condition (C): We take $h(y) = y^l$ $(0 < y < 1,\ l = 1, 2, ...)$ for $(Lh)(x) = E[a_2(x, X_2)h(X_2)]$ $(0 < x < 1)$. Since f_1 and $g^{(1)}$ are polynomials of degrees 4 and 2, respectively, the term related to f_1 and $g^{(1)}$ in $E[a_2(x, X_2)h(X_2)]$ is a polynomial of degree 4. Among the terms of $g^{(2)}$, $|x - y|$ yields the integral

$$\int_0^1 |x-y|y^l dy = \frac{2}{(l+1)(l+2)}x^{l+2} - \frac{1}{l+1}x + \frac{1}{l+2} \quad (0 < x < 1)$$

which is a polynomial of degree $l + 2$. Among the terms of $g^{(2)}$, the other term yields a polynomial of degree 2. That is, $(Lh)(x)$ $(0 < x < 1)$ is a polynomial of degree $l + 2$ for $h(y) = y^l$ $(0 < y < 1,\ l = 2, 3, ...)$. Thus if we choose $h_j(y) = y^j$ for $j = 2, 3, ..., K$, then $h_1(x_1)$,...,$h_K(x_1)$ are linearly independent and the covariance matrix of $(h_1(X_1),...,h_K(X_1))$ is positive definite. Thus the condition (C) is satisfied.

The values of μ_k for V-statistic, S-statistic and LB-statistic are

$$\mu_k = \begin{cases} \frac{1}{4} & (\mathrm{V, S-statistic}) \\ \frac{1}{2} & (\mathrm{LB-statistic}). \end{cases}$$

These values give the difference among the Edgeworth expansions $H_n^*(x)$. The Edgeworth expansions $H_n^*(x)$ are given by (3.14) with the above values.

Example 4.3.2 We assume that F is the exponential distribution $e(1)$. Then $\theta(F) = 1/4$, and

$$\begin{aligned} g^{(1)}(x_1) &= \frac{1}{12} - \frac{1}{3}x_1 e^{-x_1}, \quad (x_1 > 0) \\ g^{(2)}(x_1, x_2) &= -\frac{1}{12} + \frac{1}{3}\left[x_1 e^{-x_1} + x_2 e^{-x_2} - e^{-|x_1 - x_2|} + e^{-(x_1+x_2)}\right] \quad (x_1, x_2 > 0). \end{aligned}$$

By using the expression of $g^{(1)}(x_1)$ and $g^{(2)}(x_1, x_2)$ to

$$\begin{aligned} & E[g^{(1)}(X_2)g^{(2)}(X_1, X_2)|X_1 = x_1] \\ &= \int_0^{x_1} g^{(1)}(x_2)g^{(2)}(x_1, x_2)dx_2 + \int_{x_1}^{\infty} g^{(1)}(x_2)g^{(2)}(x_1, x_2)dx_2, \end{aligned}$$

we get

$$E[g^{(1)}(X_2)g^{(2)}(X_1, X_2)|X_1 = x_1] = -\frac{5}{3888} - \frac{8}{81}e^{-2x_1} + \frac{8}{81}e^{-x_1} - \frac{2}{27}x_1 e^{-2x_1} - \frac{1}{36}x_1 e^{-x_1}.$$

By the computation based on these functions using Mathematica ver. 4.0, we get

$$\sigma_1^2 = \frac{5}{3888} \doteq 0.001286, \quad \sigma_2^2 = \frac{251}{972} \doteq 0.25823$$

and

$$e_1 = \frac{1}{31104} \doteq 0.000032, \quad e_2 = -\frac{5}{69984} \doteq -0.000071, \quad e_3 = \frac{2171}{583200000} \doteq 0.0000037,$$

$$e_4 = \frac{-10127}{9447840000} \doteq -0.000001, \quad e_5 = \frac{2083}{157464000} \doteq 0.000013.$$

By the method similar to Example 4.3.1, we have

$$e_6 = \frac{1}{1119744} \doteq 0.00000089.$$

Thus we get

$$v_1 = -\frac{21\sqrt{3}}{5\sqrt{5}} \doteq -3.253306, \quad v_2 = -\frac{51\sqrt{3}}{10\sqrt{5}} \doteq -3.95044, \quad v_3 = -\frac{1323}{125} \doteq -10.584,$$

$$v_4 = -\frac{787836542389301}{134369280000000} \doteq -5.86322, \quad v_5 = \frac{7274727}{78125} \doteq 93.11651.$$

Now we check the condition (C): We take $h(y) = e^{-ly}$ $(y > 0,\, l = 1, 2, ...)$ for $(Lh)(x) = E[a_2(x, X_2)h(X_2)]$ $(x > 0)$. The terms related to f_1 and $g^{(1)}$ in $E[a_2(x, X_2)h(X_2)]$ contain exponential functions e^{-x} or e^{-2x}. Among the terms of $g^{(2)}$, $e^{-|x-y|}$ yields the integral

$$\int_0^\infty e^{-|x-y|}e^{-ly}dy = \frac{1}{(l-1)}e^{-x} - \frac{2}{(l-1)(l+1)}e^{-lx} \quad (x > 0)$$

which contain exponents e^{-x} and e^{-lx}. Among the terms of $g^{(2)}$, the other terms are constant or contain a exponential function e^{-x}. That is, $(Lh)(x)$ contains exponent e^{-x}, e^{-2x} and e^{-lx} for $h(y) = e^{-ly}$ $(l = 2, 3, ...)$. Thus if we choose $h_j(y) = e^{-(j+1)y}$ $(y > 0)$ for $j = 1, 2, ..., K$, then $h_1(x_1),...,h_K(x_1)$ are linearly independent and the covariance matrix of $(h_1(X_1),...,h_K(X_1))$ is positive definite. Thus the condition (C) is satisfied.

The values of μ_k for V-statistic, S-statistic and LB-statistic are

$$\mu_k = \begin{cases} -\dfrac{5}{12} & (\mathrm{V, S-statistic}) \\ -\dfrac{5}{6} & (\mathrm{LB-statistic}). \end{cases}$$

These values give the difference among the Edgeworth expansions $H_n^*(x)$. The Edgeworth expansions $H_n^*(x)$ are given by (3.14) with the above values.

Next, we consider about the kernel of degree 2.

Example 4.4 We consider the variance $\theta = \int(x-\mu)^2 dF(x)$. Its kernel $g(x_1, x_2)$ is given by

$$\frac{1}{2}\big(x_1^2 + x_2^2 - 2x_1x_2\big).$$

For this kernel, we have $g_{(k-1)}(x_1) = g_{(1)}(x_1) = 0$ and so $\theta_{k-1}(=\theta_1) = 0$. Therefore, we have

$$Y_n = \frac{d(k,k)}{D(n,k)}\binom{n}{k}U_n = \frac{d(2,2)n}{2D(n,2)}\sum_{j=1}^{n}(X_j - \bar{X})^2. \tag{28}$$

We assume that the distribution F has a density. We also assume $E \mid X \mid^{18} < \infty$ and denote jth moment of X about the origin by m'_j $(j = 2, 3, ..., 6)$.
In order to study the statistical properties of Y_n, by (4.3) the mean μ is assumed to be zero, without loss of generality. Thus, in this case $\theta = m'_2$ and $\mu_k = -\delta_k m'_2$. We consider the two cases that the distribution F is symmetric or not.

Example 4.4.1 We assume that the distribution F is symmetric about zero. In this case, by the symmetry $m'_j = 0$ $(j = 3, 5, 7)$. Then, we have

$$g^{(1)}(x_1) = \frac{1}{2}(x_1^2 - m'_2), \quad g^{(2)}(x_1, x_2) = -x_1 x_2, \quad f_{12}(x_1) = 0.$$

By the computation based on these, we get

$$\begin{aligned}
e_1 &= \frac{1}{8}(m'_6 - 3m'_4 m'_2 + 2m'^3_2), \\
e_2 &= 0, \\
e_3 &= \frac{1}{16}(m'_8 - 4m'_6 m'_2 + 6m'_4 m'^2_2 - 3m'^4_2), \\
e_4 &= 0, \qquad e_5 = 0, \qquad e_6 = 0.
\end{aligned}$$

Furthermore,

$$\sigma_1^2 = \frac{1}{4}(m'_4 - m'^2_2), \qquad \sigma_2^2 = m'^2_2.$$

Thus we get

$$\begin{aligned}
v_1 &= \frac{2}{(m'_4 - m'^2_2)^{3/2}}\left\{m'_6 - 3m'_4 m'_2 + 2m'^3_2\right\}, \\
v_2 &= \frac{1}{(m'_4 - m'^2_2)^{3/2}}\left\{m'_6 - 3m'_4 m'_2 + 2m'^3_2\right\}, \\
v_3 &= -\frac{4}{(m'_4 - m'^2_2)^3}\left\{m'_6 - 3m'_4 m'_2 + 2m'^3_2\right\}^2, \\
v_4 &= \frac{2}{(m'_4 - m'^2_2)^3}\left\{3m'_8 m'_4 - 3m'_8 m'^2_2 - 4m'^2_6 + 12m'_6 m'_4 m'_2 - 4m'_6 m'^3_2\right. \\
&\qquad \left. -18m'^3_4 + 36m'^2_4 m'^2_2 - 33m'_4 m'^4_2 + 11m'^6_2\right\}, \\
v_5 &= \frac{3}{(m'_4 - m'^2_2)^3}\left\{-6m'_8 m'_4 + 6m'_8 m'^2_2 + 4m'^2_6 - 8m'_6 m'^3_2\right. \\
&\qquad \left. +12m'^3_4 - 12m'^2_4 m'^2_2 - 6m'_4 m'^4_2 + 10m'^6_2\right\}
\end{aligned}$$

and

$$\frac{1}{2}e_1 + e_2 = \frac{1}{16}(m'_6 - 3m'_4 m'_2 + 2m'^3_2).$$

Now we check the condition (D): Since $g^{(2)}(x_1, x_2) = -x_1 x_2$, by the same reason stated at the Example 4.1.1 we have $a_2(x, y) = \sum_{j=1}^{6} c_j h_j(x) h_j(y)$ where

$$h_1(x) = x, \quad h_2(x) = f_1(x) + g^{(1)}(x), \quad h_3(x) = f_1(x), \quad h_4(x) = g^{(1)}(x),$$

and

$$c_1 = -1, \quad c_2 = -\frac{1}{\sigma_1^2}, \quad c_3 = c_4 = \frac{1}{\sigma_1^2}.$$

In this example we can write f_1 as follows:

$$f_1(x) = \frac{1}{8}\left\{(x^2 - {m'}_2)^2 - ({m'}_4 - {m'}_2^2)\right\}$$

Thus for any $s_1, ..., s_K$ $(-\infty < s_1, ..., s_K < \infty)$, $g^{(1)}(x) + \sum_{j=1}^{K} s_j h_j(x)$ is a polynomial of degree 4 with respect to x. Therefore the distribution of $g^{(1)}(X) + \sum_{j=1}^{K} s_j h_j(X)$ has the density for any $s_1, ..., s_K$ $(-\infty < s_1, ..., s_K < \infty)$ and by Lemma 5.1 the condition (D) is satisfied.

Example 4.4.2 We assume that the distribution F is not symmetric about zero. Then, we have

$$g^{(1)}(x_1) = \frac{1}{2}(x_1^2 - {m'}_2), \quad g^{(2)}(x_1, x_2) = -x_1 x_2, \quad f_{12}(x_1) = -\frac{1}{2} m'_3 x_1.$$

By the computation based on these, we get

$$\begin{aligned}
e_1 &= \frac{1}{8}({m'}_6 - 3{m'}_4{m'}_2 + 2{m'}_2^3), \\
e_2 &= -\frac{1}{4}{m'}_3^2, \\
e_3 &= \frac{1}{16}({m'}_8 - 4{m'}_6{m'}_2 + 6{m'}_4{m'}_2^2 - 3{m'}_2^4), \\
e_4 &= \frac{1}{8}(-{m'}_5{m'}_3 + 2{m'}_3^2{m'}_2), \\
e_5 &= \frac{1}{4}{m'}_3^2{m'}_2, \qquad e_6 = 0.
\end{aligned}$$

Furthermore

$$\sigma_1^2 = \frac{1}{4}({m'}_4 - {m'}_2^2), \qquad \sigma_2^2 = {m'}_2^2.$$

Thus we get

$$
\begin{aligned}
v_1 &= \frac{2}{(m'_4 - m'^2_2)^{3/2}} \left\{ m'_6 - 3m'_4 m'_2 - 3m'^2_3 + 2m'^3_2 \right\}, \\
v_2 &= \frac{1}{(m'_4 - m'^2_2)^{3/2}} \left\{ m'_6 - 3m'_4 m'_2 - 6m'^2_3 + 2m'^3_2 \right\}, \\
v_3 &= -\frac{4}{(m'_4 - m'^2_2)^{3}} \left\{ m'_6 - 3m'_4 m'_2 - 3m'^2_3 + 2m'^3_2 \right\}^2, \\
v_4 &= \frac{2}{(m'_4 - m'^2_2)^{3}} \\
&\quad \times \Big\{ 3m'_8 m'_4 - 3m'_8 m'^2_2 - 4m'^2_6 + 12m'_6 m'_4 m'_2 + 48m'_6 m'^2_3 - 4m'_6 m'^3_2 \\
&\qquad -72m'_5 m'_4 m'_3 + 72m'_5 m'_3 m'^2_2 - 18m'^3_4 + 36m'^2_4 m'^2_2 + 72m'_4 m'^2_3 m'_2 \\
&\qquad -33m'_4 m'^4_2 - 108m'^4_3 - 120m'^2_3 m'^3_2 + 11m'^6_2 \Big\}, \\
v_5 &= \frac{3}{(m'_4 - m'^2_2)^{3}} \\
&\quad \times \Big\{ -6m'_8 m'_4 + 6m'_8 m'^2_2 + 4m'^2_6 - 24m'_6 m'^2_3 - 8m'_6 m'^3_2 + 48m'_5 m'_4 m'_3 \\
&\qquad -48m'_5 m'_3 m'^2_2 + 12m'^3_4 - 12m'^2_4 m'^2_2 - 72m'_4 m'^2_3 m'_2 - 6m'_4 m'^4_2 \\
&\qquad +12m'^4_3 + 96m'^2_3 m'^3_2 + 10m'^6_2 \Big\}
\end{aligned}
$$

and

$$\frac{1}{2}e_1 + e_2 = \frac{1}{16}(m'_6 - 3m'_4 m'_2 - 4m'^2_3 + 2m'^3_2).$$

Now we check the condition (D): Since

$$f_1(x) = \frac{1}{8}\left\{ (x^2 - m'_2)^2 - (m'_4 - m'^2_2) \right\} - \frac{1}{2}m'_3 x,$$

by the same reason as Example 4.4.1, the condition (D) is satisfied.

Example 4.5 We consider the kernel $g(x_1, x_2) = x_1 x_2$. This kernel yields estimable parameter $\theta(F) = \mu^2$. We assume that the distribution F has the density. We also assume that F is symmetric about the mean μ (> 0) and $EX^9 < \infty$. The values e_2, e_3 and e_5 are given by putting $k = 2$ in Example 4.2 and, $e_1 = e_4 = e_6 = 0$.

Example 4.6 We consider the kernel

$$g(x_1, x_2) = I(x_1 + x_2 > 0),$$

which appears in the Wilcoxon one-sample statistic. We assume that the distribution F has the density and symmetric about zero. Then the value of the estimable parameter θ is equal to $E[I(X_1 + X_2 > 0)] = 1/2$. We have also

$$g_{(1)}(x_1) = I(x_1 > 0), \quad \theta_1 (= \theta_{k-1}) = EI(X_1 > 0) = \frac{1}{2}.$$

Therefore $\mu_2 (= \mu_k) = 0$. We note that $1 - F(-x) = F(x)$ and $F(X)$ has the uniform distribution U(0,1). We have

$$g^{(1)}(x_1) = F(x_1) - \frac{1}{2}, \quad g^{(2)}(x_1, x_2) = I(x_1 + x_2 > 0) - F(x_1) - F(x_2) + \frac{1}{2},$$

and

$$f_1(x_1) = 0, \quad f_{12}(x_1) = \frac{1}{2}\{F(x_1) - F^2(x_1) - \frac{1}{6}\},$$

where we use the relation $E[F(X_2)I(x_1 + X_2 > 0)] = \int_{-x_1}^{\infty} F(x_2)dF(x_2) = [1 - F^2(-x_1)]/2 = [2F(x_1) - F^2(x_1)]/2$. Furthermore, using the relation $E[F(X_2)I(X_1 + X_2 > 0)] = E[2F(X_1) - F^2(X_1)]/2 = 1/3$, we get

$$\sigma_1^2 = \frac{1}{12}, \quad \sigma_2^2 = \frac{1}{12}.$$

Thus we get

$$e_1 = e_2 = 0, \quad e_3 = \frac{1}{80}, \quad e_4 = -\frac{1}{360}, \quad e_5 = \frac{1}{720}, \quad e_6 = 0$$

and

$$v_1 = v_2 = v_3 = 0, \quad v_4 = -\frac{234}{5}, \quad v_5 = \frac{216}{5}.$$

Now we check the condition (C): We assume $E|g^{(2)}|^r < \infty$ $(r > 2)$ and take K such that $K > 8(4r-5)/(r-2)$. We take $h(y) = y^l$ $(l = 1, 2, ..., K)$ for $(Lh)(x) = E[a_2(x, X_2)h(X_2)]$. Under the condition that F has the K-th moment, $\int_{-x}^{\infty} y^l dF(y)$, $l = 1, ..., K$, are linearly independent. Since $a_2(x, y) = g^{(2)}(x_1, x_2) = I(x_1 + x_2 > 0) - F(x_1) - F(x_2) + \frac{1}{2}$, under the same condition, $(Lh_1)(x_1)$,..., $(Lh_K)(x_1)$ are linearly independent and the covariance matrix of $(Lh_1)(X_1)$,..., $(Lh_K)(X_1)$ is positive definite, where $h_l(y) = y^l, l = 1, ..., K$.

Example 4.7 We consider the kernel

$$g(x_1, x_2) = \frac{1}{2}\max(x_1, x_2) = \frac{1}{2}[x_1 I(x_1 \geq x_2) + x_2 I(x_1 < x_2)],$$

which gives the probability weighted moment

$$\theta = \beta_1 = \frac{1}{2}E[\max(X_1, X_2)] = E[XF(X)].$$

We assume that the distribution F has the uniform distribution U(0,1). Then we have $\beta_1 = 1/3$, $g_{(1)}(x_1) = x_1/2$, and $\theta_1 = 1/4$. Furthermore, we have

$$2\psi_1(x_1) = E[x_1 I(x_1 \geq X_2) + X_2 I(x_1 < X_2)] = \frac{1}{4}(1 + x_1^2),$$

$$g^{(1)}(x_1) = \frac{1}{4}x_1^2 - \frac{1}{12},$$

$$g^{(2)}(x_1, x_2) = \frac{1}{2}\max(x_1, x_2) - \frac{1}{4}(x_1^2 + x_2^2) - \frac{1}{6}$$

and

$$f_1(x_1) = \frac{1}{24}x_1^4 - \frac{1}{24}x_1^2 + \frac{1}{180}, \quad f_{12}(x_1) = \frac{1}{96}x_1^4 - \frac{1}{48}x_1^2 + \frac{7}{1440}.$$

Therefore we have

$$\sigma_1^2 = \frac{1}{180}, \quad \sigma_2^2 = \frac{1}{360}.$$

Thus we get

$$e_1 = \frac{1}{3780}, \quad e_2 = -\frac{1}{3780}, \quad e_3 = \frac{1}{15120}, \quad e_4 = -\frac{1}{113400}, \quad e_5 = \frac{1}{75600}, \quad e_6 = 0$$

and

$$v_1 = -\frac{2\sqrt{5}}{7}, \quad v_2 = -\frac{4\sqrt{5}}{7}, \quad v_3 = -\frac{20}{49}, \quad v_4 = -34, \quad v_5 = \frac{267}{49} \doteq 5.45.$$

Now we check the condition (C): We take $h(y) = y^{l+1}$ $(l = 1, 2, ..., K)$ for $(Lh)(x) = E[a_2(x, X_2)h(X_2)]$ and a suitably large $K(> 56)$. Since $\int_0^1 y^{l+1} \max(x, y)dy$ is a polynomial of degree $l + 3$ in x, $(Lh_1)(x_1)$,..., $(Lh_K)(x_1)$ are linearly independent. Hence the covariance matrix of $(Lh_1)(X_1)$,..., $(Lh_K)(X_1)$ is positive definite.

5. Appendix

About the condition (D) by Lai and Wang (1993) for Edgeworth expansion, we give a sufficient condition.

Lemma 5.1. *We assume that the distribution of* $g^{(1)}(X) + \sum_{j=1}^K s_j h_j(X)$ *has the density for any* $s_1, ..., s_K$ $(-\infty < s_1, ..., s_K < \infty)$. *Then, the relation* (2.5) *holds, provided the assumptions of Condition* (D) *preceding* (2.5).

Proof. Since the distribution of $g^{(1)}(X) + \sum_{j=1}^K s_j h_j(X)$ has the density for any $s_1, ..., s_K$, we get

$$E \exp\big(it\big[g^{(1)}(X) + \sum_{j=1}^K s_j h_j(X)\big]\big) \to 0 \quad \text{as} \quad |t| \to \infty \quad \text{for} -\infty < s_1, ..., s_K < \infty.$$

Thus,

$$\sup_{|s_1|+\cdots+|s_K|\le 1} |\, E \exp\big(it\big[g^{(1)}(X) + \sum_{j=1}^K s_j h_j(X)\big]\big) \,| \to 0 \quad \text{as} \quad |t| \to \infty.$$

On the other hand, for a sufficiently large $|t|$ satisfying $|t|^{-\varepsilon} < 1$ with ε given in Condition (D),

$$\begin{aligned} 0 \le & \sup_{|s_1|+\cdots+|s_K|\le |t|^{-\varepsilon}} |\, E \exp\big(it\big[g^{(1)}(X) + \sum_{j=1}^K s_j h_j(X)\big]\big) \,| \\ \le & \sup_{|s_1|+\cdots+|s_K|\le 1} |\, E \exp\big(it\big[g^{(1)}(X) + \sum_{j=1}^K s_j h_j(X)\big]\big) \,|, \end{aligned}$$

which converges to 0 as $|t| \to \infty$, because of the previous reason. Thus, (2.5) holds.

From the proof of Lemma 1.3 in p. 261 of Shorack (2000), we have the following.

LEMMA 5.2. (Shorack (2000)) *For any random variables W and Δ, it holds that*

$$\sup_x \mid P(W+\Delta \le x) - P(W \le x) \mid \le 4(E \mid W\Delta \mid + E \mid \Delta \mid).$$

Lemma 1.7 of Petrov (1994) yields the following.

LEMMA 5.3. (Lemma 3 of Maesono (1996)) *Let H be a bounded function and δ be a positive constant. For any random variables W and Δ, it holds that*

$$\sup_x \mid P(W+\Delta \le x) - H(x) \mid \le \sup_x \mid P(W \le x) - H(x) \mid + P(|\Delta| \ge \delta) + \sup_x |H(x+\delta) - H(x)|.$$

By the Taylor expansion, we can get the following lemma.

LEMMA 5.4. *For a positive constant c, the following relations hold uniformly with respect to $x \in (-\infty, \infty)$.*

$$\Phi(x - \frac{c}{\sqrt{n}}) = \Phi(x) - \frac{c}{\sqrt{n}}\phi(x) - \frac{c^2}{2n}x\phi(x) + O(\frac{1}{n^{3/2}}),$$

$$\phi(x - \frac{c}{\sqrt{n}}) = \phi(x) + \frac{c}{\sqrt{n}}x\phi(x) + \frac{c^2}{2n}(x^2-1)\phi(x) + O(\frac{1}{n^{3/2}}),$$

$$(x - \frac{c}{\sqrt{n}})\phi(x - \frac{c}{\sqrt{n}}) = x\phi(x) + \frac{c}{\sqrt{n}}(x^2-1)\phi(x) + \frac{c^2}{2n}(x^3-3x)\phi(x) + O(\frac{1}{n^{3/2}}),$$

$$(x - \frac{c}{\sqrt{n}})^2\phi(x - \frac{c}{\sqrt{n}}) = x^2\phi(x) + \frac{c}{\sqrt{n}}(x^3-2x)\phi(x) + \frac{c^2}{2n}(x^4-5x^2+2)\phi(x) + O(\frac{1}{n^{3/2}}),$$

$$(x - \frac{c}{\sqrt{n}})^3\phi(x - \frac{c}{\sqrt{n}}) = x^3\phi(x) + \frac{c}{\sqrt{n}}(x^4-3x^2)\phi(x) + \frac{c^2}{2n}(x^5-7x^3+6x)\phi(x) + O(\frac{1}{n^{3/2}}),$$

$$(x - \frac{c}{\sqrt{n}})^5\phi(x - \frac{c}{\sqrt{n}}) = x^5\phi(x) + \frac{c}{\sqrt{n}}(x^6-5x^4)\phi(x) + \frac{c^2}{2n}(x^7-11x^5+20x^3)\phi(x) + O(\frac{1}{n^{3/2}}).$$

Multiplication of (2.3) and (3.6). The term associated with μ_k of (3.4) is obtained by multiplying the second term of (3.6) and the first term of (2.3). The last term of a_1^* is obtained by multiplying the second terms of (3.6) and (2.3). The terms of the expansion associated with $a_2(x,y)$, $a_3(x,y,z)$ are obtained directly by multiplication of the right-hand sides of (2.3) and (3.6). Next we consider the first three terms of $a_1(x)$, which are

$$\frac{\delta_k}{k}\big[(k-1)g^{(1)}_{(k-1)}(x) - kg^{(1)}(x)\big], \quad \tau g^{(1)}(x), \quad -\frac{1}{\sigma_1^2}\big[f_1(x)g^{(1)}(x) - \zeta\big].$$

The terms of the expansion associated with these are also obtained by the same method as the above. Since the third term of a_1 is subtracted the constant ζ, consequently the constant term associated with ζ appears in the second term of (3.4).

Now we consider the terms associated with the last two terms of $a_1(x)$ which are

$$-\frac{1}{\sigma_1^2}\Big(E\big[f_2(x,X_2)g^{(1)}(X_2)\big]-\frac{3\zeta}{\sigma_1^2}f_1(x)\Big),\quad -\frac{1}{\sigma_1^2}(k-1)E\big[g^{(2)}(x,X_2)f_1(X_2)\big].$$

These are obtained by taking the first terms of H-decompositions of

$$\frac{1}{n^{5/2}}\sum_{i<j}[g^{(1)}(X_i)+g^{(1)}(X_j)][f_2(X_i,X_j)-\frac{3}{\sigma_1^2}f_1(X_i)f_1(X_j)] \tag{29}$$

and

$$\frac{1}{n^{5/2}}\sum_{i<j}g^{(2)}(X_i,X_j)[f_1(X_i)+f_1(X_j)], \tag{30}$$

respectively. These (5.1) and (5.2) are obtained directly by multiplication of the right-hand sides of (2.3) and (3.6).

Acknowledgement

The authors would like to express their thanks to the referee for his kind comments.

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Received October 6, 2003
Revised May 8, 2004