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Maesono, Yoshihiko
Faculty of Economics, Kyushu University

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Yoshihiko Maesono

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ASYMPTOTIC REPRESENTATIONS OF SKEWNESS ESTIMATORS OF STUDENTIZED U-STATISTICS

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Yoshihiko Maesono*

Abstract

A skewness is a measure of symmetry of a distribution and appears in an Edgeworth expansion of a standardized or studentized statistic. It has been found in simulation studies that jackknife estimators of the skewness have downward biases. Fujioka and Maesono (2000) have obtained a normalizing transformation with residual term $o(n^{-1})$ and they pointed out that in order to construct the normalizing transformation, we need an asymptotic representation of a skewness estimator. Maesono (1998) has obtained the asymptotic representation of the jackknife skewness estimators and discussed their biases. In this paper we propose another skewness estimator of a U-statistic and obtain asymptotic representations of both estimators with remainder term $o_p(n^{-1})$ and discuss the biases theoretically.

Key Words and Phrases: Jackknife estimator, Hoeffding decomposition, Skewness, Studentized U-statistics, Unbiased estimator.

1. Introduction

Let $X_1, \cdots, X_n$ be independently and identically distributed random variables with distribution function F and $h(x_1, \cdots, x_r)$ be a real valued function which is symmetric in its arguments. For $n \geq r$ let us define U-statistic by

$$U_n = \binom{n}{r}^{-1} \sum_{C_{n,r}} h(X_{i_1}, \cdots, X_{i_r})$$

where $\sum_{C_{n,r}}$ denotes that the summation is taken over all integers $i_1, \cdots, i_r$ satisfying $1 \leq i_1 < \cdots < i_r \leq n$. For a standardized U_n, Hoeffding (1948) proved the asymptotic normality, and Callaert, Janssen and Veraverbeke (1980) and Bickel, Goetze and van Zwet (1986) obtained an Edgeworth expansion for the distribution of the standardized U-statistic $\sigma_n^{-1}(U_n - \theta)$, where $\theta = E[h(X_1, \cdots, X_r)]$ and $\sigma_n^2 = Var(U_n)$. Based on these approximations, we can construct confidence intervals. But, as pointed out by Hall (1992, Chap.3), both convergence rates of coverage probabilities of those intervals are $O(n^{-1/2})$. Thus we cannot improve the convergence rates. To improve the rates, we have to construct the confidence interval based on the Edgeworth expansion of a studentized U-statistic

$$S_n = (U_n - \theta)/\hat{\sigma}_n$$

* Faculty of Economics, Kyushu University, Ropponmatsu 4–2–1 Chuo-Ku Fukuoka 810-8560 Japan. tel +81–92–726–4778 maesono@en.kyushu-u.ac.jp

where $\hat{\sigma}_n^2$ is an estimator of the variance σ_n^2. The properties of the jackknife variance estimator $\hat{\sigma}_n^2$ are precisely studied. Arvesen (1969) has obtained the exact representation of $\hat{\sigma}_n^2$, and Efron and Stein (1981) have showed that $\hat{\sigma}_n^2$ has a positive bias. Further Maesono (1997) has obtained an asymptotic representation and an Edgeworth expansion with remainder term $o(n^{-1/2})$.

Maesono (1997) has obtained the Edgeworth expansion of the studentized U-statistic substituting a jackknife variance estimator $\hat{\sigma}_n^2$. The expansion includes the asymptotic skewness κ_3 of S_n. The skewness κ_3 depends on the main terms of the variance and the third moment of the U-statistic U_n. There are also some papers which studied properties of a jackknife estimator of the third moment of the U-statistic. Tu and Gross (1994) discussed bias reduction of the estimator. Maesono (1998a) has obtained an asymptotic representations of the jackknife variance and third moment estimators and then got an asymptotic representation of the jackknife skewness estimator $\hat{\kappa}_3$. On the other hand, Fujioka and Maesono (2000) proposed a higher order normalizing transformation which improve the convergence rates of the probabilities of the confidence intervals. They pointed out that if we want to use a higher order normalizing transformation, we need the asymptotic representation of the skewness estimator $\hat{\kappa}_3$.

In this paper we will discuss asymptotic properties of skewness estimators based on the jackknife estimators and an unbiased estimator of the main term of the asymptotic skewness κ_3. Using the Hoeffding (1961) decomposition (H-decomposition), the asymptotic representations of the skewness estimators $\hat{\kappa}_3$ are established, and the biases of $\hat{\kappa}_3$ are studied theoretically. In Section 2, we review the H-decomposition and the asymptotic representation of the jackknife variance estimator $n\hat{\sigma}_n^2$. In Section 3, the asymptotic representations and the biases of the estimators $\hat{\kappa}_3$ are established. Finally, in the case of variance estimation, we study the biases of the estimators $\hat{\kappa}_3$ in Section 4.

It is desirable to study asymptotic mean squared errors of the skewness estimators $\hat{\kappa}_3$. But to calculate the errors, we should obtain more precise representations of the estimators. So, it may be studied in the future. Hereafter for the sake of simplicity, we will consider the kernel of degree 2. The generalization to the kernel with arbitrary degree will be obtained with notational complications and tedious calculations.

2. Preliminaries

At first we prepare the H-decomposition of U-statistic. The H-decomposition or $ANOVA$-decomposition is a basic tool of the analysis of variance, the jackknife inference, etc. Under the assumption that $E|h(X_1, X_2)| < \infty$, let us define

$$g_1(x) = E[h(x, X_2)] - \theta, \qquad g_2(x, y) = h(x, y) - \theta - g_1(x) - g_1(y)$$

$$A_1 = \sum_{i=1}^{n} g_1(X_i) \quad \text{and} \quad A_2 = \sum_{C_{n,2}} g_2(X_i, X_j).$$

Then we have

$$U_n - \theta = \frac{2}{n} A_1 + \frac{2}{n(n-1)} A_2.$$

Note that

$$E[g_2(X_1, X_2)|X_1] = 0 \ \ a.s.$$

If one of $\{i_1, i_2\}$ is not contained in $\{j_1, \cdots, j_\ell\}$, for ℓ-variate function π which satisfies $E|\pi g_2| < \infty$, we get

$$E[g_k(X_{i_1}, X_{i_2})\pi(X_{j_1}, \cdots, X_{j_\ell})] = 0.$$

Using this equation we have the variance σ_n^2 of U_n

$$\sigma_n^2 = \frac{4}{n}\xi_1^2 + \frac{2}{n(n-1)}\xi_2^2$$

where

$$\xi_1^2 = E[g_1^2(X_1)] \quad \text{and} \quad \xi_2^2 = E[g_2^2(X_1, X_2)].$$

Let $U_n^{(i)}$ denote U-statistic computed from a sample of $n-1$ points with X_i left out. Then the jackknife variance estimator $\hat{\sigma}_n^2$ is given by

$$\hat{\sigma}_n^2 = \frac{n-1}{n}\sum_{i=1}^{n}[U_n^{(i)} - U_n]^2. \tag{1}$$

Maesono (1997) has obtained an asymptotic representation of $n\hat{\sigma}_n^2$ as follows. To discuss asymptotic properties of a statistic, it is convenient to obtain an asymptotic representation with remainder term $o_p(n^{-1})$ which means

$$P\{|o_p(n^{-1})| \geq n^{-1}(\log n)^{-1}\} = o(n^{-1}).$$

From Maesono (1997), we have the following lemma.

LEMMA 2.1. *If $E|h(X_1, X_2)|^{4+\varepsilon} < \infty$ for some $\varepsilon > 0$, an asymptotic representation of the jackknife variance estimator $n\hat{\sigma}_n^2$ defined by* (1) *is given by*

$$n\hat{\sigma}_n^2 = n\sigma_n^2 + \frac{2}{n}\sum_{i=1}^{n} f_1(X_i) + \frac{2}{n(n-1)}\sum_{C_{n,2}} f_2(X_i, X_j) + \frac{2\xi_2^2}{n} + o_p(n^{-1})$$

where

$$f_1(x) = 2[g_1^2(x) - \xi_1^2] + 4E[g_1(X_2)g_2(x, X_2)]$$

and

$$\begin{aligned} f_2(x,y) \quad = \quad & -4g_1(x)g_1(y) + 4E[g_2(x, X_3)g_2(y, X_3)] \\ & +4g_2(x,y)\{g_1(x) + g_1(y)\} - 4E[\{g_2(x, X_3) + g_2(y, X_3)\}g_1(X_3)]. \end{aligned}$$

PROOF. See Maesono (1997).

3. Skewness estimator

We will consider the asymptotic skewness κ_3 of the studentized U-statistic S_n. Maesono (1997) has proved an asymptotic representation of S_n. Let us define

$$\tau = \frac{3E[f_1^2(X_1)]}{2\xi_1^4} - \frac{\xi_2^2}{2\xi_1^2}, \qquad \rho = E[f_1(X_1)g_1(X_1)],$$

$$\begin{aligned} \eta_1(x) = \tau g_1(x) - \frac{1}{\xi_1^2}\Big\{&(f_1(x)g_1(x) - \rho) \\ &+\Big(E[f_2(x, X_2)g_1(X_2)] - \frac{3\rho}{\xi_1^2}f_1(x)\Big) + E[g_2(x, X_2)f_1(X_2)]\Big\}, \end{aligned}$$

$$\eta_2(x,y) = g_2(x,y) - \frac{1}{\xi_1^2}\Big[f_1(x)g_1(y) + f_1(y)g_1(x)\Big]$$

and

$$\eta_3(x,y,z) = -\frac{1}{\xi_1^2}\Big\{f_1(x)g_2(y,z)+f_1(y)g_2(x,z)+f_1(z)g_2(x,y)$$
$$+g_1(x)\Big[f_2(y,z)-\frac{3}{\xi_1^2}f_1(y)f_1(z)\Big]+g_1(y)\Big[f_2(x,z)-\frac{3}{\xi_1^2}f_1(x)f_1(z)\Big]$$
$$+g_1(z)\Big[f_2(x,y)-\frac{3}{\xi_1^2}f_1(x)f_1(y)\Big]\Big\}.$$

Then we have the following lemma.

Lemma 3.1. *If* $E|h(X_1,X_2)|^9 < \infty$ *and* $\xi_1^2 > 0$, *for the studentized* U*-statistic* $S_n = (U_n - \theta)/\hat{\sigma}_n$, *we have*

$$S_n = \sqrt{n}U_n^* - \frac{\rho}{\sqrt{n}\xi_1^3} + o_p(n^{-1})$$

where

$$U_n^* = \frac{1}{n\xi_1}\sum_{i=1}^{n}\{g_1(X_i)+\frac{\eta_1(X_i)}{n}\}$$
$$+ \frac{2}{n(n-1)\xi_1}\sum_{C_{n,2}}\eta_2(X_i,X_j)+\frac{2}{n(n-1)(n-2)\xi_1}\sum_{C_{n,3}}\eta_3(X_i,X_j,X_k).$$

Proof. See Maesono (1997).

Since S_n is an asymptotic U-statistic, the asymptotic skewness $\kappa_3 = n^2E(U_n^*)^3$ follows from Maesono (1998a). Let us define

$$e_1 = E[g_1^3(X_1)], \quad e_2 = E[g_1(X_1)f_{12}(X_1)], \quad e_3 = E[g_1(X_1)g_2^2(X_1,X_2)],$$
$$e_4 = E[g_2(X_1,X_2)g_2(X_1,X_3)g_2(X_2,X_3)], \quad e_5 = E[g_1^4(X_1)],$$
$$e_6 = E[g_1^2(X_1)f_{12}(X_1)], \quad e_7 = E[f_{12}^2(X_1)], \quad e_8 = E[g_1^5(X_1)],$$
$$e_9 = E[g_1^2(X_1)g_1^2(X_2)g_2(X_1,X_2)], \quad e_{10} = E[g_1^3(X_1)f_{12}(X_1)],$$
$$e_{11} = E[g_1^2(X_1)g_2(X_1,X_2)f_{12}(X_2)], \quad e_{12} = E[g_1(X_1)f_{12}^2(X_1)],$$
$$e_{13} = E[g_2(X_1,X_2)f_{12}(X_1)f_{12}(X_2)]$$

where

$$f_{12}(x) = E[g_1(X_2)g_2(x,X_2)].$$

From direct computations, we have an asymptotic skewness κ_3.

Lemma 3.2. *If* $E|h(X_1,\cdots,X_r)|^9 < \infty$ *and* $\xi_1^2 > 0$, *we have*

$$\kappa_3 = n^2E(U_n^*)^3 = \frac{1}{\xi_1^3}(-2e_1-3e_2)$$
$$+ n^{-1}\Big\{\frac{1}{\xi_1^3}\Big(-\frac{39}{8}e_1-\frac{3}{2}e_2-3e_3-2e_4\Big)$$

$$+\frac{1}{\xi_1^5}\Big[-3\xi_2^2(e_1+2e_2)-\frac{3}{4}e_8+\frac{3}{2}e_9-3e_{10}-3e_{11}-9e_{12}-6e_{13}\Big]$$
$$+\frac{1}{\xi_1^7}\Big[3e_1\Big(\frac{e_5}{8}+\frac{9}{2}e_6+\frac{11}{2}e_7\Big)+3e_2\Big(\frac{e_5}{2}+10e_6+12e_7\Big)\Big]$$
$$-\frac{5}{2\xi_1^9}\Big(e_1+e_2\Big)^3\Big\}+O(n^{-2}). \qquad (2)$$

PROOF. See Maesono (1998a).

Let $U_n^{(i,j)}$ denote U-statistic computed from a sample of $n-2$ points with X_i and X_j left out. The jackknife skewness estimator $\hat{\kappa}_{3J}$ of κ_3 is given by

$$\hat{\kappa}_{3J}=\frac{\hat{\nu}_n}{(n\hat{\sigma}_n^2)^{3/2}} \qquad (3)$$

where

$$\hat{\nu}_n=\frac{2(n-1)^3}{n}\sum_{i=1}^{n}(U_n^{(i)}-U_n)^3 \qquad (4)$$
$$-\frac{3(n-1)^2}{n}\sum_{i\neq j}(U_n^{(i)}-U_n)(U_n^{(j)}-U_n)[nU_n-(n-1)(U_n^{(i)}+U_n^{(j)})+U_n^{(i,j)}].$$

The skewness κ_3 is a coefficient of $n^{-1/2}$ term in an Edgeworth expansion of S_n. So, the estimator of the skewness plays an important role when obtaining an approximate upper α-quantile or constructing a confidence interval based on the Edgeworth expansion. Beran (1984), and Hinkley and Wei (1984) have discussed the jackknife estimation of the skewness. The simulation studies by Beran (1984), Schemper (1987), and Tu and Zhang (1992) show that the jackknife skewness estimators have large downward biases. Beran (1984) further has found that the biases in skewness estimators have a significant impact on the accuracy of the jackknifed Edgeworth approximation and the correctness of confidence intervals based on this approximation.

Using Lemma 4 and 5 in Maesono (1998a), we can obtain the asymptotic representation of $\hat{\nu}_n$. Let us define

$$\lambda_1(x)=-8\{g_1^3(x)-e_1\}-24\{g_1(x)E[g_1(X_2)g_2(x,X_2)]-e_2\}$$
$$-24E[g_1^2(X_2)g_2(x,X_2)]+24\xi_1^2g_1(x)-24E[g_1(X_2)g_2(x,X_3)g_2(X_2,X_3)],$$

$$\lambda_2(x,y)=-24\{g_1(x)g_1(y)g_2(x,y)+e_2$$
$$-E[(g_1(x)g_2(x,X_2)+g_1(y)g_2(y,X_2))g_1(X_2)]\}$$
$$+24\{g_1^2(x)g_1(y)+g_1^2(y)g_1(x)-\xi_1^2g_1(x)-\xi_1^2g_1(y)\}$$
$$-24\{[g_1^2(x)+g_1^2(y)]g_2(x,y)-E[g_1^2(X_2)\{g_2(x,X_2)+g_2(y,X_2)\}]\}$$
$$+48\xi_1^2g_2(x,y)+72E[(g_1(x)g_2(y,X_3)+g_1(y)g_2(x,X_3))g_1(X_3)]$$
$$-48E[g_1(X_3)g_2(x,X_3)g_2(y,X_3)]$$
$$-24\{E[(g_1(x)+g_1(y))g_2(x,X_3)g_2(y,X_3)$$
$$+g_1(X_3)g_2(x,y)(g_2(x,X_3)+g_2(y,X_3))]$$
$$-2E[(g_2(x,X_3)+g_2(y,X_3))g_1(X_2)g_2(X_2,X_3)]\}$$
$$-24E[g_2(x,X_3)g_2(y,X_4)g_2(X_3,X_4)]$$

and

$$\delta = 6e_1 + 12e_2 - 12e_3 - 3e_4.$$

Similarly as $n\hat{\sigma}_n^2$, Maesono (1998a) obtained the following asymptotic representation of $\hat{\nu}_n$.

Lemma 3.3. *If $E|h(X_1, X_2)|^{6+\varepsilon} < \infty$ for some $\varepsilon > 0$, an asymptotic representation of $\hat{\nu}_n$ defined by* (4) *is*

$$\hat{\nu}_n = 8(-2e_1 - 3e_2) + \frac{2}{n}\sum_{i=1}^{n} \lambda_1(X_i) + \frac{2}{n(n-1)}\sum_{C_{n,2}} \lambda_2(X_i, X_j) + \frac{8\delta}{n} + o_p(n^{-1}).$$

Proof. See Maesono (1998a).

Using Lemma 2.1, 3.2 and 3.3, we can obtain the asymptotic representation of $\hat{\kappa}_{3J}$.

Theorem 3.4. *If $E|h(X_1, X_2)|^{10+\varepsilon} < \infty$ for some $\varepsilon > 0$ and $\xi_1^2 > 0$, an asymptotic representation of the jackknife skewness estimator $\hat{\kappa}_{3J}$ in* (3) *is given by*

$$\hat{\kappa}_{3J} = \kappa_3 + \frac{2}{n\xi_1^3}\sum_{i=1}^{n} \zeta_1(X_i) + \frac{2}{n(n-1)\xi_1^3}\sum_{C_{n,2}} \zeta_2(X_i, X_j) + \frac{d}{n\xi_1^3} + o_p(n^{-1})$$

where

$$\begin{aligned}
\zeta_1(x) &= \frac{1}{8}\lambda_1(x) - \frac{3(e_1 + 3e_2)}{8\xi_1^2} f_1(x), \\
\zeta_2(x, y) &= \frac{1}{8}\lambda_2(x, y) - \frac{3}{16\xi_1^2}\Big\{ f_1(x)\lambda_1(y) + f_1(y)\lambda_1(x) \Big\} \\
&\quad - \frac{3(e_1 + 3e_2)}{8\xi_1^2} f_2(x, y) + \frac{5(e_1 + 3e_2)}{8\xi_1^4} f_1(x) f_1(y)
\end{aligned}$$

and

$$d = \delta + (e_1 + 3e_2)\Big\{ \frac{15E[f_1^2(X_1)]}{32\xi_1^4} - \frac{3\xi_2^2}{4\xi_1^2} \Big\} - \frac{3}{16\xi_1^2} E[f_1(X_1)\lambda_1(X_1)].$$

Proof. See Maesono (1998a).

It is possible to construct another type estimator of the asymptotic skewness κ_3. Substituting an unbiased estimator to e_1 and e_2, we propose new skewness estimator. From the definition of $e_1, e_2, g_1(x)$ and $g_2(x, y)$, we have

$$e_1 = \tau_1 - 3\tau_2 + 2\tau_3 \qquad \text{and} \qquad e_2 = \tau_4 - 2\tau_2 + \tau_3$$

where

$$\begin{aligned}
\tau_1 &= E[h(X_1, X_2)h(X_1, X_3)h(X_1, X_4)], \\
\tau_2 &= E[h(X_1, X_2)h(X_3, X_4)h(X_3, X_5)], \\
\tau_3 &= E[h(X_1, X_2)h(X_3, X_4)h(X_5, X_6)]
\end{aligned}$$

and

$$\tau_4 = E[h(X_1, X_2)h(X_3, X_4)h(X_1, X_3)].$$

Since τ_1, τ_2, τ_3 and τ_4 are estimable parameters, we can make unbiased estimators based on the theory of U-statistics. The kernel of the U-statistic should be symmetric in its arguments and so, considering every combination of the indices, let us define

$$\begin{aligned} & h_{\tau_1}(x_1, x_2, x_3, x_4) \\ = \; & \frac{1}{4}\Big\{h(x_1, x_2)h(x_1, x_3)h(x_1, x_4) + h(x_1, x_2)h(x_2, x_3)h(x_2, x_4) \\ & + h(x_1, x_3)h(x_2, x_3)h(x_3, x_4) + h(x_1, x_4)h(x_2, x_4)h(x_3, x_4)\Big\}, \end{aligned}$$

$$\begin{aligned} & h_{\tau_2}(x_1, x_2, x_3, x_4, x_5) \\ = \; & \frac{1}{30}\Big\{h(x_1, x_2)h(x_3, x_4)h(x_3, x_5) + h(x_1, x_2)h(x_3, x_4)h(x_4, x_5) \\ & + \cdots\cdots\cdots \\ & + h(x_4, x_5)h(x_1, x_2)h(x_2, x_3) + h(x_4, x_5)h(x_1, x_3)h(x_2, x_3)\Big\}, \end{aligned}$$

$$\begin{aligned} & h_{\tau_3}(x_1, x_2, x_3, x_4, x_5, x_6) \\ = \; & \frac{1}{15}\Big\{h(x_1, x_2)h(x_3, x_4)h(x_5, x_6) + h(x_1, x_2)h(x_3, x_5)h(x_4, x_6) \\ & + \cdots\cdots\cdots \\ & + h(x_1, x_6)h(x_2, x_4)h(x_3, x_5) + h(x_1, x_6)h(x_2, x_5)h(x_3, x_4)\Big\}, \end{aligned}$$

and

$$\begin{aligned} & h_{\tau_4}(x_1, x_2, x_3, x_4) \\ = \; & \frac{1}{24}\Big\{h(x_1, x_2)h(x_1, x_3)h(x_3, x_4) + h(x_1, x_2)h(x_1, x_4)h(x_3, x_4) \\ & + \cdots\cdots\cdots \\ & + h(x_3, x_4)h(x_2, x_4)h(x_1, x_2) + h(x_3, x_4)h(x_1, x_4)h(x_1, x_2)\Big\}. \end{aligned}$$

Then unbiased estimators $\hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3$ and $\hat{\tau}_4$ of τ_1, τ_2, τ_3 and τ_4 are given by

$$\begin{aligned} \hat{\tau}_1 &= \binom{n}{4}^{-1} \sum_{C_{n,4}} h_{\tau_1}(X_{i_1}, \cdots, X_{i_4}), \\ \hat{\tau}_2 &= \binom{n}{5}^{-1} \sum_{C_{n,5}} h_{\tau_2}(X_{i_1}, \cdots, X_{i_5}), \\ \hat{\tau}_3 &= \binom{n}{6}^{-1} \sum_{C_{n,6}} h_{\tau_3}(X_{i_1}, \cdots, X_{i_6}) \end{aligned}$$

and

$$\hat{\tau}_4 = \binom{n}{4}^{-1} \sum_{C_{n,4}} h_{\tau_4}(X_{i_1}, \cdots, X_{i_4}).$$

Let us define

$$\hat{\mu}_n = 8\{-2(\hat{\tau}_1 - 3\hat{\tau}_2 + \hat{\tau}_3) - 3(\hat{\tau}_4 - 2\hat{\tau}_2 + \hat{\tau}_3)\}. \tag{5}$$

Then the estimator of the skewness κ_3 based on the unbiased estimation is given by

$$\hat{\kappa}_{3U} = \frac{\hat{\mu}_n}{(n\hat{\sigma}_n^2)^{3/2}}.$$

Since $\hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3$ and $\hat{\tau}_4$ are U-statistics, we can apply the H-decomposition and obtain an asymptotic representation as follows.

Lemma 3.5. *If $E|h(X_1, X_2)|^{6+\varepsilon} < \infty$ for some $\varepsilon > 0$, an asymptotic representation of $\hat{\mu}_n$ defined by* (5) *is*

$$\hat{\mu}_n = 8(-2e_1 - 3e_2) + \frac{2}{n}\sum_{i=1}^{n} \lambda_1(X_i) + \frac{2}{n(n-1)} \sum_{C_{n,2}} \lambda_2(X_i, X_j) + o_p(n^{-1}).$$

Proof. See Appendix.

Thus, similary as $\hat{\kappa}_{3J}$, we can obtain an asymptotic representation of the estimator $\hat{\kappa}_{3U}$ as follows.

Theorem 3.6. *If $E|h(X_1, X_2)|^{10+\varepsilon} < \infty$ for some $\varepsilon > 0$, we have*

$$\hat{\kappa}_{3U} = \kappa_3 + \frac{2}{n\xi_1^3}\sum_{i=1}^{n} \zeta_1(X_i) + \frac{2}{n(n-1)\xi_1^3} \sum_{C_{n,2}} \zeta_2(X_i, X_j) + \frac{\delta_U}{n\xi_1^3} + o_p(n^{-1}) \tag{6}$$

where

$$\begin{aligned} \delta_U \;=\; & \frac{1}{\xi_1^3}\left(\frac{39}{8}e_1 + \frac{3}{2}e_2 + 3e_3 + 2e_4\right) \\ & + \frac{1}{\xi_1^5}\Bigg(6\xi_2^2 e_1 + \frac{21}{2}\xi_2^2 e_1 - 12\xi_1^2 e_1 - 27\xi_1^2 e_2 + \frac{15}{4}e_8 \\ & \quad - \frac{15}{2}e_9 + 18e_{10} + 30e_{11} + 27e_{12} + 24e_{13}\Bigg) \\ & - \frac{1}{\xi_1^7}\Bigg\{e_1\left(\frac{3}{16}e_5 + \frac{3}{16} + \frac{51}{4}e_6 + \frac{63}{4}e_7\right) \\ & \quad + e_2\left(\frac{39}{32}e_5 + \frac{9}{32}\xi_1^4 + \frac{231}{8}e_6 + \frac{279}{8}e_7\right)\Bigg\} \\ & + \frac{5}{2\xi_1^9}(e_1 + e_2)^3. \end{aligned}$$

PROOF. $\hat{\kappa}_{3U}$ is an ratio statistic. For the ratio statistic, Maesono (1998b) has obtained an asymptotic representation with residual term $o_p(n^{-1})$. Applying his result, we can show the equation (6).

REMARK. The differences between the estimators $\hat{\kappa}_{3J}$ and $\hat{\kappa}_{3U}$ are the bias term, and $\zeta_1(x)$ and $\zeta_2(x,y)$ are same. Fujioka and Maesono (2000) discussed a normalizing transformation with remainder term $o(n^{-1})$ and their transformation depends on $\zeta_1(x)$. In the paper Fujioka and Maesono (2000), the transformation is based on the jackknife estimator $\hat{\kappa}_{3J}$, and so we do not need to change the normalizing transformation when we use $\hat{\kappa}_{3U}$ instead of $\hat{\kappa}_{3J}$.

REMARK. Lai and Wang (1993) have established the Edgeworth expansion for the asymptotic U-statistic. Thus it is possible to obtain the Edgeworth expansion of the standardized skewness estimator $(\hat{\kappa}_3 - \kappa_3)/\sqrt{V(\hat{\kappa}_3)}$.

4. Example

Let us consider the unbiased sample variance which is a U-statistic with kernel $h(x,y) = (x-y)^2/2$. It is easy to see that

$$U_n = \frac{2}{n(n-1)} \sum_{C_{n,2}} \frac{1}{2}(X_i - X_j)^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

Thus U_n is the unbiased estimator of $\theta = \sigma^2 = V(X_1)$. Applying the H-decomposition, we have

$$g_1(x) = \frac{1}{2}\{(x-\mu)^2 - \sigma^2)\} \quad \text{and} \quad g_2(x,y) = -(x-\mu)(y-\mu).$$

where $\mu = E(X_1)$. For the sake of simplicity, let us assume that the distribution $F(x)$ is symmetric about origin. Let us define

$$m_k = E[(X_1 - \mu)^k],$$

and then if k is a odd number, $m_k = 0$. Since $f_{12}(x) = 0$, from direct computation, we have

$$\begin{aligned}
&\xi_1^2 = \frac{1}{4}(m_4 - \sigma^4), \quad \xi_2^2 = \sigma^4, \quad e_1 = \frac{1}{8}(m_6 - 3\sigma^2 m_4 + 2\sigma^6), \\
&e_2 = 0, \quad e_3 = \frac{1}{2}(\sigma^2 m_4 - \sigma^6), \quad e_4 = -\sigma^6, \\
&e_5 = \frac{1}{16}(m_8 - 4\sigma^2 m_6 + 6\sigma^4 m_4 - 3\sigma^8), \\
&e_8 = \frac{1}{32}(m_{10} - 5\sigma^2 m_8 + 10\sigma^4 m_6 - 10\sigma^6 m_4 + 4\sigma^{10}), \\
&e_6 = e_7 = e_9 = e_{10} = e_{11} = e_{12} = e_{13} = 0.
\end{aligned}$$

For the normal, logistic and double exponential distributions, we have the following table.

Table

	κ_3	δ_J/ξ_1^3	δ_U/ξ_1^3
Normal	$-5.66 - n^{-1}146.37$	209.22	425.32
Logistic	$-10.22 - n^{-1}923.78$	1517.35	3510.74
Double exp.	$-13.24 - n^{-1}1922.95$	3168.54	7421.82

The order of the biases δ_J/ξ_1^3 and δ_U/ξ_1^3 are $O(n^{-1})$ and all biases are downward.

5. Appendix

Let us review the H-decomposition of the U-statistic with kernel degree r. Let us define

$$\begin{aligned} a_k(x_1,\cdots,x_k) &= E[h(X_1,\cdots,X_r)|X_1=x_1,\cdots,X_k=x_k]-E(h) \\ h_1(x_1) &= a_1(x_1), \\ h_2(x_1,x_2) &= a_2(x_1,x_2)-h_1(x_1)-h_1(x_2), \\ &\cdots \\ h_r(x_1,\cdots,x_r) &= a_r(x_1,\cdots,x_r)-\sum_{k=1}^{r-1}\sum_{C_{r,k}} h_k(x_{i_1},\cdots,x_{i_k}) \end{aligned}$$

and

$$A_k = \sum_{C_{n,k}} h_k(X_{i_1},\cdots,X_{i_k}).$$

Then we have

$$U_n - E(U_n) = \binom{n}{r}^{-1}\sum_{k=1}^{r}\binom{n-k}{r-k}A_k \tag{7}$$

and

$$E[h_k(X_1,\cdots,X_k)|X_1,\cdots,X_{k-1}] = 0 \quad a.s. \tag{8}$$

Let us consider ℓ-variate function $\pi(x_1,\cdots,x_\ell)$, which satisfies $E[|g_k\pi|]<\infty$, and the set of indecies $\{i_1,\cdots,i_k\}$. If there exists an index $i_m \in \{i_1,\cdots,i_k\}$ and $i_m \notin \{j_1,\cdots,j_\ell\}$. It follows from (8) that

$$E[h_k(X_{i_1},\cdots,X_{i_k})\pi(X_{j_1},\cdots,X_{j_\ell})] = 0.$$

Further, using the moment evaluations of a martingale by von Bahr and Esséen (1965) and Dharmadhikari, Fabian and Jogdeo (1968), we have the following inequalities: If $E[|h_k(X_1,\cdots,X_k)|^p]<\infty$ for $1\le p<2$, we have

$$E(|A_k|^p) \le C_h n^k.$$

If $E[|h_k(X_1,\cdots,X_k)|^p]<\infty$ for $2\le p$, we have

$$E(|A_k|^p) \le C n^{\frac{pk}{2}}.$$

C_h is a constant and does not depend on n. Thus for $k\ge 3$ and $\varepsilon>0$, we have

$$E\left|\binom{n}{r}^{-1}\binom{n-k}{r-k}A_k\right|^{2+\varepsilon} = O(n^{-3-3\varepsilon/2}).$$

Therefore it follows from (7) that

$$U_n - E(U_n) = \frac{r}{n}A_1 + \frac{r(r-1)}{n(n-1}A_2 + o_p(n^{-1}). \tag{9}$$

Using H-decomposition for the U-statistic, we will prove Lemma 3.5.

Proof of Lemma 3.5.
We will obtain an asymptotic representation of $\hat{\tau}_1$ and the others are similarly obtained. Let us define

$$b(x) = E[h(X_1, X_2)|X_1 = x].$$

Then, from H-decomposition of the U-statistic, we obtain

$$\begin{aligned} & E[h_{\tau_1}(X_1, X_2, X_3, X_4)|X_1 = x] \\ = & \frac{1}{4}\Big\{E[h(x, X_2)h(x, X_2)h(x, X_3)] + E[h(x, X_2)h(X_2, X_3)h(X_2, X_4)] \\ & + E[h(x, X_3)h(X_2, X_3)h(X_3, X_4)] + E[h(x, X_4)h(X_2, X_4)h(X_3, X_4)]\Big\} \\ = & \frac{1}{4}b^3(x) + \frac{3}{4}E[h(x, X_2)b^2(X_2)]. \end{aligned}$$

Further from the definition of g_1 and g_2, we have

$$b(x) = g_1(x) + \theta \quad \text{and} \quad h(x, y) = g_2(x, y) + g_1(x) + \theta.$$

We also have

$$\tau_1 = E[h_{\tau_1}(X_1, X_2, X_3, X_4)] = e_1 + 3\theta\xi_1^2 + \theta^3.$$

Thus we can obtain the first term of the H-decomposition

$$\begin{aligned} h_{1;1}(x) &= E[h_{\tau_1}(X_1, X_2, X_3, X_4)|X_1 = x] - \tau_1 \\ &= \frac{1}{4}\Big\{g_1^3(x) - e_1 + 3\theta[g_1^2(x) - \xi_1^2] + 3E[g_2(x, X_2)g_1^2(X_2)] \\ &\quad + 3\xi_1^2 g_1(x) + 6\theta E[g_2(x, X_2)g_1(X_2)] + 6\theta^2 g_1(x)\Big\}. \end{aligned}$$

which corresponds to $h_1(x)$

From long but direct computation, we can show that

$$\begin{aligned} & E[h_{\tau_1}(X_1, X_2, X_3, X_4)|X_1 = x, X_2 = y] \\ = & \frac{1}{4}\Big\{h(x, y)[b^2(x) + b^2(y)] + 2E[h(x, X_3)h(y, X_3)h(X_3, X_4)]\Big\} \\ = & \frac{1}{4}\Big\{g_2(x, y)[g_1^2(x) + g_1^2(y)] + 2\theta g_2(x, y)[g_1(x) + g_1(y)] + 2\theta^2 g_2(x, y) \\ & + g_1^3(x) + g_1^3(y) + g_1(x)g_1^2(y) + g_1^2(x)g_1(y) + 3\theta g_1^2(x) + 3\theta g_1^2(y) \\ & + 4\theta g_1(x)g_1(y) + 4\theta^2 g_1(x) + 4\theta^2 g_1(y) + 2\theta^3\Big\} \\ & + \frac{1}{2}\Big\{E[g_2(x, X_3)g_2(y, X_3)g_1(X_3)] + \theta E[g_2(x, X_3)g_2(y, X_3)] \\ & + g_1(x)E[g_2(y, X_3)g_1(X_3)] + g_1(y)E[g_1(X_3)g_2(x, X_3)] \end{aligned}$$

$$
\begin{aligned}
&+E[g_2(x,X_3)g_1^2(X_3)]+E[g_2(y,X_3)g_1^2(X_3)]\\
&+2\theta E[g_2(x,X_3)g_1(X_3)]+2\theta E[g_2(y,X_3)g_1(X_3)]\\
&+\theta g_1(x)g_1(y)+[\xi_1^2+\theta^2]g_1(x)+[\xi_1^2+\theta^2]g_1(y)\\
&+e_1+3\theta\xi_1^2+\theta^3\Big\}.
\end{aligned}
$$

From the above equations, we have

$$
\begin{aligned}
&h_{1;2}(x,y)\\
=\;& E[h_{\tau_1}(X_1,X_2,X_3,X_4)|X_1=x,X_2=y]-\tau_1-h_{1;1}(x)-h_{1;1}(y)\\
=\;& \frac{1}{4}\Big\{g_2(x,y)\Big(g_1^2(x)+g_1^2(y)+2\theta[g_1(x)+g_1(y)]+2\theta^2\Big)\\
&+g_1(x)g_1^2(y)+g_1^2(x)g_1(y)+6\theta g_1(x)g_1(y)\\
&+2E[g_2(x,X_3)g_2(y,X_3)h_1(X_3)]+2\theta E[g_2(x,X_3)g_2(y,X_3)]\\
&+2g_1(x)E[g_1(X_3)g_2(y,X_3)]+2g_1(y)E[g_1(X_3)g_2(x,X_3)]\\
&-E[g_2(x,X_2)g_1^2(X_2)]-E[g_2(y,X_2)g_1^2(X_2)]-2\theta E[g_2(x,X_2)g_1(X_2)]\\
&-2\theta E[g_2(y,X_2)g_1(X_2)]-\xi_1^2[g_1(x)+g_1(y)]\Big\}.
\end{aligned}
$$

Combining above equations, it follows from (9) that

$$
\hat{\tau}_1 = \tau_1+\frac{4}{n}\sum_{i=1}^{n}h_{1;1}(X_i)+\frac{12}{n(n-1)}\sum_{C_{n,2}}h_{1;2}(X_i,X_j)+o_p(n^{-1}).
$$

For the other estimators $\hat{\tau}_2,\hat{\tau}_3$ and $\hat{\tau}_4$, we can obtain asymptotic representations. Let us define

$$
h_{2;1}(x) = \frac{1}{5}\Big\{\theta[g_1^2(x)-\xi_1^2]+6\theta^2 g_1(x)+2\theta E[g_1(X_2)g_2(x,X_2)]+2\xi_1^2 g_1(x)\Big\},
$$

$$
\begin{aligned}
&h_{2;2}(x,y)\\
=\;& \frac{1}{10}\Big\{\theta g_2(x,y)[g_1(x)+g_1(y)]+(\xi_1^2+3\theta^2)g_2(x,y)+11\theta g_1(x)g_1(y)\\
&-\theta E[g_2(x,X_2)g_1(X_2)]-\theta E[g_2(y,X_2)g_1(X_2)]+g_1^2(x)g_1(y)\\
&+g_1(x)g_2^2(y)+\theta E[g_2(x,X_3)g_2(y,X_3)]+2g_1(x)E[g_2(y,X_2)g_1(X_2)]\\
&+2g_1(y)E[g_2(x,X_2)g_1(X_2)]-\xi_1^2[g_1(x)+g_1(y)]\Big\},
\end{aligned}
$$

$$
h_{3;1}(x) = \theta^2 g_1(x),
$$

$$
h_{3;2}(x,y) = \frac{1}{5}[\theta^2 g_2(x,y)+4\theta g_1(x)g_1(y)],
$$

$$
\begin{aligned}
h_{4;1}(x) =\;& \frac{1}{2}\Big\{g_1(x)E[g_1(X_2)g_2(x,X_2)]-e_2+2\xi_1^2 g_1(x)+\theta[g_1^2(x)-\xi_1^2]\\
&+3\theta^2 g_1(x)+2\theta E[g_1(X_2)g_2(x,X_2)]\\
&+E[g_1(X_2)g_2(x,X_3)g_2(X_2,X_3)]\Big\}
\end{aligned}
$$

and

$$\begin{aligned} h_{4;2}(x,y) &= \frac{1}{6}\Big(g_2(x,y)E[\{g_2(x,X_3)+g_2(y,X_3)\}g_1(X_3)] \\ &\quad -2E[\{g_2(x,X_2)+g_2(y,X_2)\}g_1(X_3)g_2(X_2,X_3)] \\ &\quad +(2\xi_1^2+3\theta^2)g_2(x,y)+2\theta g_2(x,y)[g_1(x)+g_1(y)] \\ &\quad +3g_1(x)E[g_1(X_3)g_2(y,X_3)]+3g_1(y)E[g_1(X_3)g_2(x,X_3)] \\ &\quad +g_1(x)g_1(y)g_2(x,y)-e_2+10\theta g_1(x)g_1(y) \\ &\quad -2\theta E[g_1(X_2)\{g_2(x,X_2)+g_2(y,X_2)\}] \\ &\quad -\{g_1(x)E[g_1(X_2)g_2(x,X_2)]-e_2\} \\ &\quad -\{g_1(y)E[g_1(X_2)g_2(y,X_2)]-e_2\} \\ &\quad +2g_1^2(x)g_1(y)+2g_1(x)g_1^2(y)-2\xi_1^2[g_1(x)+g_1(y)] \\ &\quad +g_1(x)E[g_2(x,X_3)g_2(y,X_3)]+g_1(y)E[g_2(x,X_3)g_2(y,X_3)] \\ &\quad +2\theta E[g_2(x,X_3)g_2(y,X_3)]+E[g_2(x,X_3)g_2(y,X_4)g_2(X_3,X_4)]\Big). \end{aligned}$$

Then we have

$$\begin{aligned} \hat{\tau}_2 &= \tau_2+\frac{5}{n}\sum_{i=1}^{n}h_{2;1}(X_i)+\frac{20}{n(n-1)}\sum_{C_{n,2}}h_{2;2}(X_i,X_j)+o_p(n^{-1}), \\ \hat{\tau}_3 &= \tau_3+\frac{6}{n}\sum_{i=1}^{n}h_{3;1}(X_i)+\frac{30}{n(n-1)}\sum_{C_{n,2}}h_{3;2}(X_i,X_j)+o_p(n^{-1}) \end{aligned}$$

and

$$\hat{\tau}_4=\tau_4+\frac{4}{n}\sum_{i=1}^{n}h_{4;1}(X_i)+\frac{12}{n(n-1)}\sum_{C_{n,2}}h_{4;2}(X_i,X_j)+o_p(n^{-1}).$$

Combining the above decompositions, we have the asymptotic representation.

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