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# Asymptotic behavior of the semigroup associated with the linearized compressible Navier-Stokes equation in an infinite layer

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## Abstract

Asymptotic behavior of solutions to the linearized compressible Navier-Stokes equation around a given constant state is considered in an infinite layer  $\mathbf{R}^{n-1} \times (0, a)$ ,  $n \geq 2$ , under the no slip boundary condition for the momentum. The  $L^p$  decay estimates of the associated semigroup are established for all  $1 \leq p \leq \infty$ . It is also shown that the time-asymptotic leading part of the semigroup is given by an  $n - 1$  dimensional heat semigroup.

## 1. Introduction

This paper is concerned with the large time behavior of solutions to the following system of equations:

$$(1.1) \quad \partial_t u + Lu = 0,$$

where  $u = \begin{pmatrix} \phi \\ m \end{pmatrix}$  with  $\phi = \phi(x, t) \in \mathbf{R}$  and  $m = {}^T(m^1(x, t), \dots, m^n(x, t)) \in \mathbf{R}^n$ ,  $n \geq 2$ , and  $L$  is an operator defined by

$$L = \begin{pmatrix} 0 & \gamma \operatorname{div} \\ \gamma \nabla & -\nu \Delta I_n - \tilde{\nu} \nabla \operatorname{div} \end{pmatrix}$$

with positive constants  $\nu$  and  $\gamma$  and a nonnegative constant  $\tilde{\nu}$ . Here  $t > 0$  denotes the time variable and  $x \in \mathbf{R}^n$  denotes the space variable; the superscript  ${}^T$  stands for the transposition;  $I_n$  is the  $n \times n$  identity matrix;

and  $\operatorname{div}$ ,  $\nabla$  and  $\Delta$  are the usual divergence, gradient and Laplacian with respect to  $x$ . We consider (1.1) in an infinite layer

$$\Omega = \mathbf{R}^{n-1} \times (0, a) = \left\{ x = \begin{pmatrix} x' \\ x_n \end{pmatrix}; x' \in \mathbf{R}^{n-1}, 0 < x_n < a \right\}$$

under the boundary condition

$$(1.2) \quad m|_{\partial\Omega} = 0,$$

together with the initial condition

$$(1.3) \quad u|_{t=0} = u_0 = \begin{pmatrix} \phi_0 \\ m_0 \end{pmatrix}.$$

Problem (1.1)–(1.3) is obtained by the linearization of the compressible Navier-Stokes equation around a motionless state with a positive constant density, where  $\phi$  is the perturbation of the density and  $m$  is the momentum.

In [6] we showed that  $-L$  generates the analytic semigroup  $\mathcal{U}(t)$  in  $W^{1,p} \times L^p$  for  $1 < p < \infty$ . In this paper we establish the  $L^p$  decay estimates of  $\mathcal{U}(t)$  for all  $1 \leq p \leq \infty$  and derive an asymptotic state of  $\mathcal{U}(t)$  as  $t \rightarrow \infty$ .

One of the primary factors affecting the large time behavior of solutions to (1.1)–(1.3) is that (1.1) is a symmetric hyperbolic-parabolic system. Due to this structure, solutions of (1.1) exhibit characters of solutions of both wave and heat equations. In the case of the Cauchy problem on the whole space  $\mathbf{R}^n$ , detailed descriptions of large time behavior of solutions have been obtained ([4, 5, 10, 12, 13]). Hoff and Zumbrun [4, 5] showed that there appears some interesting interaction of hyperbolic and parabolic aspects of (1.1) in the decay properties of  $L^p$  norms with  $1 \leq p \leq \infty$ . It was shown in [4, 5] that the solution is asymptotically written in the sum of two terms, one is the solution of the heat equation and the other is given by the convolution of the heat kernel and the fundamental solution of the wave equation. The latter one is called the diffusion wave and it decays faster than the heat kernel in  $L^p$  norm for  $p > 2$  while slower for  $p < 2$ . This decay property of the diffusion wave also appears in the exterior domain problem ([11]). In the case of the half space problem, it was shown in [7, 8] that not only the above mentioned behavior of the diffusion wave appears but also some difference to the Cauchy problem appears in the decay property of the spatial derivatives due to the presence of the unbounded boundary.

There is one more factor that affects the large time behavior of solutions to (1.1)–(1.3). In contrast to the domains mentioned above, the infinite layer  $\Omega$  has a finite thickness in the  $x_n$  direction. This implies that the Poincaré

inequality holds. If one considers, for example, the incompressible Navier-Stokes equation under the no slip boundary condition (1.2), then it is easy to see that, by the Poincaré inequality, the  $L^2$  norm of the solution tends to zero exponentially as  $t \rightarrow \infty$ . In the case of the compressible problem (1.1)–(1.3), the Poincaré inequality still holds for  $m$  but not for  $\phi$ . Therefore, some different behavior could be expected to happen.

In this paper we will show that the solution  $u = \mathcal{U}(t)u_0$  of (1.1)–(1.3) satisfies

$$(1.4) \quad \|u(t)\|_{L^p} = O(t^{-\frac{n-1}{2}(1-\frac{1}{p})}), \quad \|u(t) - u^{(0)}(t)\|_{L^p} = O(t^{-\frac{n-1}{2}(1-\frac{1}{p})-\frac{1}{2}})$$

for any  $1 \leq p \leq \infty$  as  $t \rightarrow \infty$ . Here  $u^{(0)} = (\phi^{(0)}(x', t), 0)$  and  $\phi^{(0)}(x', t)$  is a function satisfying

$$\partial_t \phi^{(0)} - \kappa \Delta' \phi^{(0)} = 0, \quad \phi^{(0)}|_{t=0} = \frac{1}{a} \int_0^a \phi_0(x', x_n) dx_n,$$

where  $\kappa = \frac{a^2 \gamma^2}{12\nu}$  and  $\Delta' = \partial_{x_1}^2 + \cdots + \partial_{x_{n-1}}^2$ . We note that the leading part is given by the density component and no hyperbolic feature appears in the leading part. The precise statement will be given in section 2.

The proof of (1.4) is based on a detailed analysis of the resolvent  $(\lambda + L)^{-1}$  associated with (1.1)–(1.3). The resolvent problem in an infinite layer was studied in [1, 2, 3] for the incompressible Stokes equation. They established  $L^p$  estimates of the resolvent for  $1 < p < \infty$ , which yields the exponential decay of the Stokes semigroup in  $L^p$  norms as  $t \rightarrow \infty$ . To obtain the resolvent estimates, they considered the Fourier transform of the resolvent in  $x' \in \mathbf{R}^{n-1}$  and applied the Fourier multiplier theorem.

In order to analyze the compressible problem (1.1)–(1.3) we also consider the Fourier transform  $(\lambda + \widehat{L}_{\xi'})^{-1}$  of the resolvent in  $x' \in \mathbf{R}^{n-1}$ , where  $\xi' \in \mathbf{R}^{n-1}$  denotes the dual variable. The semigroup  $\mathcal{U}(t)$  generated by  $-L$  is then written as  $\mathcal{U}(t) = \mathcal{F}_{\xi'}^{-1} \left[ \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} (\lambda + \widehat{L}_{\xi'})^{-1} d\lambda \right]$ . In contrast to the case of the incompressible problem,  $(\lambda + \widehat{L}_{\xi'})^{-1}$  has different characters between the cases  $|\xi'| \gg 1$  and  $|\xi'| \ll 1$ . We thus decompose the semigroup  $\mathcal{U}(t)$  into the two parts according to the partition:  $|\xi'| \geq r_0$  and  $|\xi'| \leq r_0$  for some  $r_0 > 0$ .

In [6] we established the estimates of  $(\lambda + \widehat{L}_{\xi'})^{-1}$  with  $|\xi'| \geq r_0$ , which will lead to the exponential decay of the corresponding part of  $\mathcal{U}(t)$ . In this paper we study  $(\lambda + \widehat{L}_{\xi'})^{-1}$  with  $|\xi'| \ll 1$ . We regard  $\widehat{L}_{\xi'}$  as a perturbation from  $\widehat{L}_0$  to investigate the spectrum of  $-L$  near  $\lambda = 0$ . Combining the spectral analysis for  $|\xi'| \ll 1$  and the results in [6], we prove the asymptotic behavior of  $u(t) = \mathcal{U}(t)u_0$  described in (1.4).

This paper is organized as follows. In section 2 we introduce some notation and state the main result of this paper. In section 3 we investigate  $(\lambda + \widehat{L}_{\xi'})^{-1}$  with  $|\xi'| \ll 1$ . Section 4 is devoted to the proof of the main result.

## 2. Main Result

We first introduce some notation which will be used throughout the paper. For a domain  $D$  and  $1 \leq p \leq \infty$  we denote by  $L^p(D)$  the usual Lebesgue space on  $D$  and its norm is denoted by  $\|\cdot\|_{L^p(D)}$ . Let  $\ell$  be a nonnegative integer. The symbol  $W^{\ell,p}(D)$  denotes the  $\ell$  th order  $L^p$  Sobolev space on  $D$  with norm  $\|\cdot\|_{W^{\ell,p}(D)}$ . When  $p = 2$ , the space  $W^{\ell,2}(D)$  is denoted by  $H^\ell(D)$  and its norm is denoted by  $\|\cdot\|_{H^\ell(D)}$ .  $C_0^\ell(D)$  stands for the set of all  $C^\ell$  functions which have compact support in  $D$ . We denote by  $W_0^{1,p}(D)$  the completion of  $C_0^1(D)$  in  $W^{1,p}(D)$ . In particular,  $W_0^{1,2}(D)$  is denoted by  $H_0^1(D)$ .

We simply denote by  $L^p(D)$  (resp.,  $W^{\ell,p}(D)$ ,  $H^\ell(D)$ ) the set of all vector fields  $m = {}^T(m^1, \dots, m^n)$  on  $D$  with  $m^j \in L^p(D)$  (resp.,  $W^{\ell,p}(D)$ ,  $H^\ell(D)$ ),  $j = 1, \dots, n$ , and its norm is also denoted by  $\|\cdot\|_{L^p(D)}$  (resp.,  $\|\cdot\|_{W^{\ell,p}(D)}$ ,  $\|\cdot\|_{H^\ell(D)}$ ). For  $u = \begin{pmatrix} \phi \\ m \end{pmatrix}$  with  $\phi \in W^{k,p}(D)$  and  $m = {}^T(m^1, \dots, m^n) \in W^{\ell,q}(D)$ , we define  $\|u\|_{W^{k,p}(D) \times W^{\ell,q}(D)}$  by  $\|u\|_{W^{k,p}(D) \times W^{\ell,q}(D)} = \|\phi\|_{W^{k,p}(D)} + \|m\|_{W^{\ell,q}(D)}$ . When  $k = \ell$  and  $p = q$ , we simply write  $\|u\|_{W^{k,p}(D) \times W^{k,p}(D)} = \|u\|_{W^{k,p}(D)}$ .

In case  $D = \Omega$  we abbreviate  $L^p(\Omega)$  (resp.,  $W^{\ell,p}(\Omega)$ ,  $H^\ell(\Omega)$ ) as  $L^p$  (resp.,  $W^{\ell,p}$ ,  $H^\ell$ ). In particular, the norm  $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_{L^p}$  is denoted by  $\|\cdot\|_p$ .

In case  $D = (0, a)$  we denote the norm of  $L^p(0, a)$  by  $|\cdot|_p$ . The inner product of  $L^2(0, a)$  is denoted by

$$(f, g) = \int_0^a f(x_n) \overline{g(x_n)} dx_n, \quad f, g \in L^2(0, a).$$

Here  $\bar{g}$  denotes the complex conjugate of  $g$ . Furthermore, we define  $\langle \cdot, \cdot \rangle$  and  $\langle \cdot \rangle$  by

$$\langle f, g \rangle = \frac{1}{a}(f, g) \quad \text{and} \quad \langle f \rangle = \langle f, 1 \rangle = \frac{1}{a} \int_0^a f(x_n) dx_n$$

for  $f, g \in L^2(0, a)$ , respectively.

The norms of  $W^{\ell,p}(0, a)$  and  $H^\ell(0, a)$  are denoted by  $|\cdot|_{W^{\ell,p}}$  and  $|\cdot|_{H^\ell}$ , respectively.

We often write  $x \in \Omega$  as  $x = \begin{pmatrix} x' \\ x_n \end{pmatrix}$ ,  $x' = {}^T(x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$ .

Partial derivatives of a function  $u$  in  $x$ ,  $x'$ ,  $x_n$  and  $t$  are denoted by  $\partial_x u$ ,  $\partial_{x'} u$ ,

$\partial_{x_n} u$  and  $\partial_t u$ , respectively. We also write higher order partial derivatives of  $u$  in  $x$  as  $\partial_x^k u = (\partial_x^\alpha u; |\alpha| = k)$ .

We denote the  $k \times k$  identity matrix by  $I_k$ . In particular, when  $k = n + 1$ , we simply write  $I$  for  $I_{n+1}$ . We also define  $(n + 1) \times (n + 1)$  diagonal matrices  $Q_0$  and  $\tilde{Q}$  by

$$Q_0 = \text{diag}(1, 0, \dots, 0), \quad \tilde{Q} = \text{diag}(0, 1, \dots, 1).$$

We then have, for  $u = \begin{pmatrix} \phi \\ m \end{pmatrix} \in \mathbf{R}^{n+1}$ ,

$$Q_0 u = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \quad \tilde{Q} u = \begin{pmatrix} 0 \\ m \end{pmatrix}.$$

We next introduce some notation about integral operators. For a function  $f = f(x')$  ( $x' \in \mathbf{R}^{n-1}$ ), we denote its Fourier transform by  $\hat{f}$  or  $\mathcal{F}f$ :

$$\hat{f}(\xi') = (\mathcal{F}f)(\xi') = \int_{\mathbf{R}^{n-1}} f(x') e^{-i\xi' \cdot x'} dx'.$$

The inverse Fourier transform is denoted by  $\mathcal{F}^{-1}$ :

$$(\mathcal{F}^{-1}f)(x) = (2\pi)^{-(n-1)} \int_{\mathbf{R}^{n-1}} f(\xi') e^{i\xi' \cdot x'} d\xi'.$$

For a function  $K(x_n, y_n)$  on  $(0, a) \times (0, a)$  we will denote by  $Kf$  the integral operator  $\int_0^a K(x_n, y_n) f(y_n) dy_n$ .

We denote the resolvent set of a closed operator  $A$  by  $\rho(A)$  and the spectrum of  $A$  by  $\sigma(A)$ . For  $\Lambda \in \mathbf{R}$  and  $\theta \in (\frac{\pi}{2}, \pi)$  we will denote the subset  $\{\lambda \in \mathbf{C}; |\arg(\lambda - \Lambda)| \leq \theta\}$  by  $\Sigma(\Lambda, \theta)$ :

$$\Sigma(\Lambda, \theta) = \{\lambda \in \mathbf{C}; |\arg(\lambda - \Lambda)| \leq \theta\}.$$

We now state the main result of this paper. In [6] we showed that  $-L$  generates the analytic semigroup  $\mathcal{U}(t)$  and established the estimates of  $\mathcal{U}(t)$  for  $0 < t \leq 1$ . As for the large time behavior of  $\mathcal{U}(t)$ , we have the following result.

**Theorem 2.1.** *Let  $\mathcal{U}(t)$  be the semigroup generated by  $-L$ . Then the solution  $u = \mathcal{U}(t)u_0$  of problem (1.1)–(1.3) is decomposed as*

$$\mathcal{U}(t)u_0 = \mathcal{U}^{(0)}(t)u_0 + \mathcal{U}^{(\infty)}(t)u_0,$$

where each term on the right-hand side has the following properties.

(i)  $\mathcal{U}^{(0)}(t)u_0$  is written in the form

$$\mathcal{U}^{(0)}(t)u_0 = \mathcal{W}^{(0)}(t)u_0 + \mathcal{R}^{(0)}(t)u_0.$$

Here  $\mathcal{W}^{(0)}(t)u_0 = \begin{pmatrix} \phi^{(0)}(x', t) \\ 0 \end{pmatrix}$  and  $\phi^{(0)}(x', t)$  is a function independent of  $x_n$  and satisfies the following heat equation on  $\mathbf{R}^{n-1}$ :

$$\partial_t \phi^{(0)} - \kappa \Delta' \phi^{(0)} = 0, \quad \phi^{(0)}|_{t=0} = \langle \phi_0(x', \cdot) \rangle,$$

where  $\kappa = \frac{a^2 \gamma^2}{12\nu}$  and  $\Delta' = \partial_{x_1}^2 + \cdots + \partial_{x_{n-1}}^2$ . The function  $\mathcal{R}^{(0)}(t)u_0$  satisfies the following estimate. For any  $1 \leq p \leq \infty$  and  $\ell = 0, 1$ , there exists a positive constant  $C$  such that

$$\|\partial_x^\ell \mathcal{R}^{(0)}(t)u_0\|_p \leq Ct^{-\frac{n-1}{2}(1-\frac{1}{p})-\frac{1}{2}} \|u_0\|_1$$

holds for  $t \geq 1$ . Furthermore, it holds that

$$\|\partial_x \mathcal{R}^{(0)}(t)\tilde{Q}u_0\|_p \leq Ct^{-\frac{n-1}{2}(1-\frac{1}{p})-1} \|\tilde{Q}u_0\|_1$$

and

$$\|\mathcal{R}^{(0)}(t)[\partial_x \tilde{Q}u_0]\|_p \leq Ct^{-\frac{n-1}{2}(1-\frac{1}{p})-\frac{1}{2}} \|\tilde{Q}u_0\|_1.$$

(ii) There exists a positive constant  $c$  such that  $\mathcal{U}^{(\infty)}(t)u_0$  satisfies

$$\|\partial_x^\ell \mathcal{U}^{(\infty)}(t)u_0\|_p \leq Ce^{-ct} \|u_0\|_{W^{\ell,p} \times L^p}, \quad 1 < p < \infty, \quad \ell = 0, 1,$$

for all  $t \geq 1$ . Furthermore, the following estimates

$$\|\partial_x^\ell \mathcal{U}^{(\infty)}(t)u_0\|_\infty \leq Ce^{-ct} \|u_0\|_{H^{[\frac{q}{2}] + 1 + \ell} \times H^{[\frac{q}{2}] + \ell}},$$

$$\|\partial_x^\ell \mathcal{U}^{(\infty)}(t)u_0\|_p \leq Ce^{-ct} \|u_0\|_{W^{\ell+1,p} \times W^{\ell,p}}, \quad p = 1, \infty,$$

hold for all  $t \geq 1$ . Here  $[q]$  denotes the greatest integer less than or equal to  $q$ .

**Remark 2.2.** We have the optimal decay estimate

$$\|\mathcal{U}^{(0)}(t)u_0\|_p \leq Ct^{-\frac{n-1}{2}(1-\frac{1}{p})} \|u_0\|_1$$

since  $\|\mathcal{W}^{(0)}(t)u_0\|_p$  decays exactly in the order  $t^{-\frac{n-1}{2}(1-\frac{1}{p})}$ . We also note that  $\mathcal{W}^{(0)}(t)\tilde{Q}u_0 = 0$ . Therefore, we have the estimate

$$\|\partial_x \mathcal{U}^{(0)}(t)\tilde{Q}u_0\|_p \leq Ct^{-\frac{n-1}{2}(1-\frac{1}{p})-1} \|\tilde{Q}u_0\|_1$$



for  $t \geq 1$ .

We will prove Theorem 2.1 in section 4.

### 3. Spectral analysis for $-L$

The proof of Theorem 2.1 is based on the analysis of the resolvent problem associated with (1.1)–(1.3), which takes the form

$$(3.1) \quad \lambda u + Lu = f,$$

where  $L$  is the operator on  $H^1 \times L^2$  defined in (1.1) with domain of definition  $D(L) = H^1 \times (H^2 \cap H_0^1)$ . To investigate (3.1) we take the Fourier transform in  $x' \in \mathbf{R}^{n-1}$ . We then have the following boundary value problem for functions  $\phi(x_n)$  and  $m(x_n)$  on the interval  $(0, a)$ :

$$(3.2) \quad \lambda u + \widehat{L}_{\xi'} u = f,$$

where  $u = \begin{pmatrix} \phi(x_n) \\ m'(x_n) \\ m^n(x_n) \end{pmatrix}$ ,  $f = \begin{pmatrix} f^0(x_n) \\ f'(x_n) \\ f^n(x_n) \end{pmatrix}$ , and  $\widehat{L}_{\xi'}$  is the operator of the form

$$\widehat{L}_{\xi'} = \begin{pmatrix} 0 & i\gamma^T \xi' & \gamma \partial_{x_n} \\ i\gamma \xi' & \nu(|\xi'|^2 - \partial_{x_n}^2) I_{n-1} + \tilde{\nu} \xi'^T \xi' & -i\tilde{\nu} \xi' \partial_{x_n} \\ \gamma \partial_{x_n} & -i\tilde{\nu}^T \xi' \partial_{x_n} & \nu(|\xi'|^2 - \partial_{x_n}^2) - \tilde{\nu} \partial_{x_n}^2 \end{pmatrix},$$

which is a closed operator on  $H^1(0, a) \times L^2(0, a)$  with domain of definition  $D(\widehat{L}_{\xi'}) = H^1(0, a) \times (H^2(0, a) \cap H_0^1(0, a))$ .

In [6] we studied  $(\lambda + \widehat{L}_{\xi'})^{-1}$  with  $|\xi'| \geq r$  for any  $r > 0$ . In this section we investigate the spectrum of  $-\widehat{L}_{\xi'}$  for  $|\xi'| \ll 1$ . We analyze it regarding the problem as a perturbation from the one with  $\xi' = 0$ .

We write  $\widehat{L}_{\xi'}$  in the following form:

$$\widehat{L}_{\xi'} = \widehat{L}_0 + \sum_{j=1}^{n-1} \xi_j \widehat{L}_j^{(1)} + \sum_{j,k=1}^{n-1} \xi_j \xi_k \widehat{L}_{jk}^{(2)},$$

where  $\xi' = {}^T(\xi_1, \dots, \xi_{n-1})$ ,

$$\widehat{L}_0 = \begin{pmatrix} 0 & 0 & \gamma \partial_{x_n} \\ 0 & -\nu \partial_{x_n}^2 I_{n-1} & 0 \\ \gamma \partial_{x_n} & 0 & -\nu_1 \partial_{x_n}^2 \end{pmatrix}, \quad \nu_1 = \nu + \tilde{\nu},$$

$$\widehat{L}_j^{(1)} = \begin{pmatrix} 0 & i\gamma^T \mathbf{e}'_j & 0 \\ i\gamma \mathbf{e}'_j & 0 & -i\tilde{\nu} \mathbf{e}'_j \partial_{x_n} \\ 0 & -i\tilde{\nu}^T \mathbf{e}'_j \partial_{x_n} & 0 \end{pmatrix},$$

$$\widehat{L}_{jk}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu \delta_{jk} I_{n-1} + \tilde{\nu} \mathbf{e}'_j{}^T \mathbf{e}'_k & 0 \\ 0 & 0 & \nu \delta_{jk} \end{pmatrix}.$$

We will treat  $\widehat{L}_{\xi'}$  as a perturbation from  $\widehat{L}_0$ . We begin with the analysis of (3.2) with  $\xi' = 0$ :

$$(\lambda + \widehat{L}_0)u = f.$$

We introduce some quantities. For  $k = 1, 2, \dots$ , we set  $a_k = k\pi/a$ . We define  $\lambda_{1,k}$  and  $\lambda_{\pm,k}$  by

$$\lambda_{1,k} = -\nu a_k^2$$

and

$$\lambda_{\pm,k} = -\frac{\nu_1}{2} a_k^2 \pm \frac{1}{2} \sqrt{\nu_1^2 a_k^4 - 4\gamma^2 a_k^2}$$

for  $k = 1, 2, \dots$ . An elementary observation shows that  $\lambda_{\pm,k}$  are the two roots of  $\lambda^2 + \nu_1 a_k^2 \lambda + \gamma^2 a_k^2 = 0$ ;  $\lambda_{-,k} = \overline{\lambda_{+,k}}$  with  $\text{Im } \lambda_{+,k} = \gamma a_k \sqrt{1 - \frac{\nu_1^2}{4\gamma^2} a_k^2}$  when  $a_k < 2\gamma/\nu_1$  and  $\lambda_{\pm,k} \in \mathbf{R}$  when  $a_k > 2\gamma/\nu_1$ ; and it holds that

$$(3.3) \quad \lambda_{+,k} = -\frac{\gamma^2}{\nu_1} + O(k^{-2}), \quad \lambda_{-,k} = -\nu_1 a_k^2 + O(1)$$

as  $k \rightarrow \infty$ . (See [6, Remarks 3.2 and 3.5].)

**Lemma 3.1.** (i) *The spectrum  $\sigma(-\widehat{L}_0)$  is given by*

$$\sigma(-\widehat{L}_0) = \{0\} \cup \{\lambda_{1,k}\}_{k=1}^{\infty} \cup \{\lambda_{+,k}, \lambda_{-,k}\}_{k=1}^{\infty} \cup \{-\frac{\gamma^2}{\nu_1}\}.$$

Here 0 is an eigenvalue.

(ii) *There exist positive numbers  $\eta_0$  and  $\theta_0$  with  $\theta_0 \in (\frac{\pi}{2}, \pi)$  such that the following estimates hold uniformly for  $\lambda \in \rho(-\widehat{L}_0) \cap \Sigma(-\eta_0, \theta_0)$ :*

$$\begin{aligned} |(\lambda + \widehat{L}_0)^{-1} f|_{H^\ell \times L^2} &\leq \frac{C}{|\lambda|} |f|_{H^\ell \times L^2}, \quad \ell = 0, 1, \\ |\partial_{x_n}^\ell \widetilde{Q}(\lambda + \widehat{L}_0)^{-1} f|_2 &\leq \frac{C}{(|\lambda| + 1)^{1-\frac{\ell}{2}}} |f|_{H^{\ell-1} \times L^2}, \quad \ell = 1, 2, \end{aligned}$$

$$\left| \partial_{x_n}^2 Q_0(\lambda + \widehat{L}_0)^{-1} f \right|_2 \leq \frac{C}{(|\lambda| + 1)^{\frac{1}{2}}} |f|_{H^2 \times H^1}.$$

**Proof.** We write (3.2) with  $\xi' = 0$  as

$$(3.4) \quad \lambda m' - \nu \partial_{x_n}^2 m' = f', \quad m'|_{x_n=0,a} = 0,$$

and

$$(3.5) \quad \begin{cases} \lambda \phi + \gamma \partial_{x_n} m^n = f^0, \\ \lambda m^n - \nu_1 \partial_{x_n}^2 m^n + \gamma \partial_{x_n} \phi = f^n, \quad m^n|_{x_n=0,a} = 0. \end{cases}$$

It is easy to see that (3.4) has a unique solution  $m' \in H^2(0, a) \cap H_0^1(0, a)$  for any  $f' \in L^2(0, a)$  if and only if  $\lambda \neq \lambda_{1,k}$  for any  $k = 1, 2, \dots$ . Furthermore, it is also possible to deduce the estimates

$$\left| \partial_{x_n}^\ell m' \right|_2 \leq \frac{C}{(|\lambda| + 1)^{1 - \frac{\ell}{2}}} |f'|_2, \quad \ell = 0, 1, 2,$$

uniformly in  $\lambda = -\frac{\nu\pi^2}{2a^2} + \eta e^{\pm i\theta}$  with  $\eta \geq 0$  and  $\theta \in [0, \theta_0)$ . Here  $\theta_0$  is any fixed constant in  $(\frac{\pi}{2}, \pi)$  and  $C$  is a positive constant depending only on  $\theta_0$ .

We next consider (3.5). Let  $\lambda = 0$  and  $f^0 = f^n = 0$  in (3.5). We see from the first equation of (3.5) that  $\partial_{x_n} m^n = 0$ . Then the boundary condition  $m^n|_{x_n=0,a} = 0$  implies that  $m^n = 0$ . It follows from the second equation of (3.5) that  $\phi$  is a constant. Therefore, 0 is an eigenvalue and the geometric eigenspace is spanned by  $\psi^{(0)} = T(1, 0, \dots, 0)$ .

Let  $\lambda \neq 0$  in (3.5). We then see that problem (3.5) is equivalent to

$$(3.6) \quad \phi = \frac{1}{\lambda} \{ f^0 - \gamma \partial_{x_n} m^n \},$$

$$(3.7) \quad \lambda^2 m^n - (\nu_1 \lambda + \gamma^2) \partial_{x_n}^2 m^n = \lambda f^n - \gamma \partial_{x_n} f^0, \quad m^n|_{x_n=0,a} = 0.$$

In case  $\nu_1 \lambda + \gamma^2 = 0$ , it is easy to see that problem (3.6)–(3.7) has only the trivial solution  $\phi = m^n = 0$  for  $f^0 = f^n = 0$ . For general  $f^0 \in H^1(0, a)$  and  $f^n \in L^2(0, a)$ , (3.7) implies that  $m^n = \lambda^{-2} \{ \lambda f^n - \gamma \partial_{x_n} f^0 \}$  which is not necessarily in  $H^1(0, a)$ . This implies that  $-\frac{\gamma^2}{\nu_1} \in \sigma(-\widehat{L}_0)$ .

Let us consider the case  $\lambda \neq 0$  and  $\nu_1 \lambda + \gamma^2 \neq 0$ . In this case, (3.7) is equivalent to

$$(3.8) \quad \sigma m^n - \partial_{x_n}^2 m^n = \frac{1}{\nu_1 \lambda + \gamma^2} \{ \lambda f^n - \gamma \partial_{x_n} f^0 \}, \quad m^n|_{x_n=0,a} = 0,$$

where  $\sigma = \frac{\lambda^2}{\nu_1 \lambda + \gamma^2}$ . Since  $\lambda f^n - \gamma \partial_{x_n} f^0 \in L^2(0, a)$ , problem (3.8) has a unique solution  $m^n \in H^2(0, a) \cap H_0^1(0, a)$  if and only if  $\sigma \neq -a_k^2$  for any  $k = 1, 2, \dots$ , namely,  $(\lambda - \lambda_{+,k})(\lambda - \lambda_{-,k}) \neq 0$  for any  $k = 1, 2, \dots$ . If (3.8) has a solution  $m^n \in H^2(0, a) \cap H_0^1(0, a)$ , then (3.6) determines  $\phi$  which is in  $H^1(0, a)$ . Consequently we see that  $\sigma(-\widehat{L}_0) = \{0\} \cup \{\lambda_{1,k}\}_{k=1}^\infty \cup \{\lambda_{+,k}, \lambda_{-,k}\}_{k=1}^\infty \cup \{-\frac{\gamma^2}{\nu_1}\}$ .

We next derive estimates for  $\phi$  and  $m^n$  uniformly in  $\lambda \in \rho(-\widehat{L}_0) \cap \Sigma(-\eta_0, \theta_0)$  with suitable  $\eta_0$  and  $\theta_0$ . To do so, we expand the solution  $m^n$  of (3.8) into the Fourier sine series  $m^n = \sum_{k=1}^\infty m_k^n \sin a_k x_n$ . It is easy to see that the Fourier coefficients  $m_k^n$  are given by

$$m_k^n = \frac{1}{\sigma + a_k^2} \frac{1}{\nu_1 \lambda + \gamma^2} \left\{ \lambda f_k^n + \gamma a_k f_k^0 \right\}$$

for  $k = 1, 2, \dots$ , where  $f_k^0$  and  $f_k^n$  are the coefficients of the Fourier cosine and sine series expansion of  $f^0$  and  $f^n$ , respectively.

Since  $(\sigma + a_k^2)(\nu_1 \lambda + \gamma^2) = (\lambda - \lambda_{+,k})(\lambda - \lambda_{-,k})$ , we have

$$|m^n|_2^2 \leq C \sum_{k=1}^\infty \frac{1}{|(\lambda - \lambda_{+,k})(\lambda - \lambda_{-,k})|^2} \left\{ |\lambda|^2 |f_k^n|^2 + a_k^2 |f_k^0|^2 \right\}.$$

It then follows from (3.3) that there are positive numbers  $\eta_0$  and  $\theta_0 \in (\frac{\pi}{2}, \pi)$  such that, for  $\lambda$  with  $|\arg(\lambda + \eta_0)| \leq \theta_0$ ,

$$\begin{aligned} |m^n|_2^2 &\leq C \sum_{k=1}^\infty \frac{1}{(|\lambda| + 1)^2 (|\lambda| + k^2)^2} \left\{ |\lambda|^2 |f_k^n|^2 + a_k^2 |f_k^0|^2 \right\} \\ &\leq \frac{C |f|_2^2}{(|\lambda| + 1)^2}. \end{aligned}$$

This, together with (3.8), then implies that

$$\begin{aligned} \left| \partial_{x_n}^2 m^n \right|_2 &\leq |\sigma| |m^n|_2 + \frac{|\lambda|}{|\nu_1 \lambda + \gamma^2|} |f^n|_2 + \frac{1}{|\nu_1 \lambda + \gamma^2|} \left| \partial_{x_n} f^0 \right|_2 \\ &\leq C |f|_{H^1 \times L^2} \end{aligned}$$

uniformly in  $\lambda$  with  $|\arg(\lambda + \eta_0)| \leq \theta_0$ . Taking the  $L^2$  inner product of (3.8) with  $m^n$  and integrating by parts, we have

$$\begin{aligned} \left| \partial_{x_n} m^n \right|_2^2 &\leq C \left\{ |\sigma| |m^n|_2^2 + |f^n|_2 |m^n|_2 + \frac{1}{|\lambda| + 1} |f^0|_2 \left| \partial_{x_n} m^n \right|_2 \right\} \\ &\leq \frac{C |f|_2^2}{|\lambda| + 1} + \frac{1}{2} \left| \partial_{x_n} m^n \right|_2^2 \end{aligned}$$

uniformly in  $\lambda$  with  $|\arg(\lambda + \eta_0)| \leq \theta_0$ , and hence,  $|\partial_{x_n} m^n|_2 \leq \frac{C|f|_2}{(|\lambda|+1)^{\frac{1}{2}}}$ .  
Consequently, we have

$$(3.9) \quad \left| \partial_{x_n}^\ell m^n \right|_2 \leq \frac{C|f|_{H^{(\ell-1)+\times L^2}}}{(|\lambda|+1)^{1-\frac{\ell}{2}}}$$

for  $\ell = 0, 1, 2$  uniformly in  $\lambda$  with  $|\arg(\lambda + \eta_0)| \leq \theta_0$ . It then follows from (3.6) and (3.9) that

$$|\phi|_2 \leq \frac{1}{|\lambda|} \left\{ |f^0|_2 + \gamma |\partial_{x_n} m^n|_2 \right\} \leq \frac{C}{|\lambda|} |f|_2.$$

We next estimate the derivatives of  $\phi$ . Differentiating the first equation of (3.5) we have

$$(3.10) \quad \lambda \partial_{x_n} \phi + \gamma \partial_{x_n}^2 m^n = \partial_{x_n} f^0$$

We see from the second equation of (3.5) that

$$(3.11) \quad -\nu_1 \partial_{x_n}^2 m^n + \gamma \partial_{x_n} \phi = f^n - \lambda m^n.$$

By adding (3.11)  $\times \frac{\gamma}{\nu_1}$  to (3.10) we obtain

$$\left( \lambda + \frac{\gamma^2}{\nu_1} \right) \partial_{x_n}^{\ell+1} \phi = \partial_{x_n}^{\ell+1} f^0 + \frac{\gamma}{\nu_1} \left\{ \partial_{x_n}^\ell f^n - \lambda \partial_{x_n}^\ell m^n \right\}, \quad \ell = 0, 1.$$

This, together with (3.9), implies that

$$\begin{aligned} \left| \partial_{x_n}^{\ell+1} \phi \right|_2 &\leq \frac{C}{|\lambda|+1} \left\{ \left| \partial_{x_n}^{\ell+1} f^0 \right|_2 + \left| \partial_{x_n}^\ell f^n \right|_2 + |\lambda| \left| \partial_{x_n}^\ell m^n \right|_2 \right\} \\ &\leq \frac{C}{(|\lambda|+1)^{1-\frac{\ell}{2}}} |f|_{H^{\ell+1} \times H^\ell}, \quad \ell = 0, 1, \end{aligned}$$

for  $\lambda$  with  $|\arg(\lambda + \eta_0)| \leq \theta_0$ , by changing  $\eta_0 > 0$  and  $\theta_0 \in (\frac{\pi}{2}, \pi)$  suitably if necessary. This completes the proof.

We next investigate the eigenvalue 0 of  $-\widehat{L}_0$ .

**Lemma 3.2.** *The eigenvalue 0 of  $-\widehat{L}_0$  is simple and the associated eigen-projection is given by*

$$\widehat{\Pi}^{(0)} u = \begin{pmatrix} \langle \phi \rangle \\ 0 \end{pmatrix} \quad \text{for } u = \begin{pmatrix} \phi \\ m \end{pmatrix}.$$

**Proof.** To show the simplicity of the eigenvalue 0, let us first consider the problem  $\widehat{L}_0 u = \psi^{(0)}$ , where  $\psi^{(0)} = {}^T(1, 0, \dots, 0)$  is an eigenfunction for the eigenvalue 0. This problem is equivalent to (3.4)–(3.5) with  $\lambda = 0$ ,  $f' = 0$ ,  $f^0 = 1$ ,  $f^n = 0$ . By (3.4), we have  $m' = 0$ , and by the first equation of (3.5), we have  $m^n = \frac{1}{\gamma}x_n + c$  for some constant  $c$ . There is no such  $m^n$  satisfying the boundary condition  $m^n|_{x_n=0,a} = 0$ . Therefore, 0 is a simple eigenvalue.

Let us prove that the eigenprojection  $\widehat{\Pi}^{(0)}$  has the desired form. Since  $\dim \text{Range } \widehat{\Pi}^{(0)} = 1$ , we have  $\widehat{\Pi}^{(0)}u = c_u \psi^{(0)}$  for some  $c_u \in \mathbf{C}$ . It then follows that

$$(3.12) \quad \langle \widehat{\Pi}^{(0)}u, \psi^{(0)} \rangle = c_u.$$

Consider now the formal adjoint problem

$$\lambda u + \widehat{L}_0^* u = 0,$$

where

$$\widehat{L}_0^* = \begin{pmatrix} 0 & 0 & -\gamma \partial_{x_n} \\ 0 & -\nu \partial_{x_n}^2 I_{n-1} & 0 \\ -\gamma \partial_{x_n} & 0 & -\nu_1 \partial_{x_n}^2 \end{pmatrix}.$$

with domain of definition  $D(\widehat{L}_0^*) = D(\widehat{L}_0)$ . Similarly to above, we can see that  $\sigma(-\widehat{L}_0^*) = \sigma(-\widehat{L}_0)$ , and, in particular, 0 is a simple eigenvalue and  $\widehat{L}_0^* \psi^{(0)} = 0$ . Furthermore, let  $\widehat{\Pi}^{(0)*}$  be the eigenprojection for the eigenvalue 0 of  $-\widehat{L}_0^*$ . Then we have

$$\widehat{\Pi}^{(0)}u = \frac{1}{2\pi i} \int_{\Gamma} (\lambda + \widehat{L}_0)^{-1} d\lambda, \quad \widehat{\Pi}^{(0)*}u = \frac{1}{2\pi i} \int_{\Gamma} (\lambda + \widehat{L}_0^*)^{-1} d\lambda,$$

where  $\Gamma$  is a circle with center 0 and sufficiently small radius. Let  $\widehat{G}^{(0)}(\lambda, x_n, y_n)$  be the integral kernel of  $(\lambda + \widehat{L}_0)^{-1}$ . Then it is easy to see that  $\widehat{G}^{(0)*}(\lambda, x_n, y_n) = {}^T \left( \widehat{G}^{(0)}(\overline{\lambda}, x_n, y_n) \right)$  is the integral kernel of  $(\lambda + \widehat{L}_0^*)^{-1}$ . We then obtain

$$\begin{aligned} \langle \widehat{\Pi}^{(0)}u, \psi^{(0)} \rangle &= \frac{1}{a} \int_0^a \left( \frac{1}{2\pi i} \int_{\Gamma} \int_0^a \widehat{G}^{(0)}(\lambda, x_n, y_n) u(y_n) dy_n d\lambda \right) \psi^{(0)}(x_n) dx_n \\ &= \frac{1}{a} \int_0^a u(y_n) \overline{\left( \frac{1}{2\pi i} \int_{\Gamma} \int_0^a \widehat{G}^{(0)*}(\lambda, x_n, y_n) \psi^{(0)}(x_n) dx_n d\lambda \right)} dy_n \\ &= \langle u, \widehat{\Pi}^{(0)*} \psi^{(0)} \rangle = \langle u, \psi^{(0)} \rangle = \langle \phi \rangle \end{aligned}$$

for  $u = \begin{pmatrix} \phi \\ m \end{pmatrix}$ . This, together with (3.12), gives the desired expression of  $\widehat{\Pi}^{(0)}$ . This completes the proof.

We next estimate  $(\lambda + \widehat{L}_{\xi'})^{-1}$  for small  $\xi'$ . Based on Lemma 3.1 we obtain the following estimates.

**Theorem 3.3.** *Let  $\eta_0$  and  $\theta_0$  be the numbers given in Lemma 3.1. Then there exists a positive number  $\tilde{r}_0 = \tilde{r}_0(\eta_0, \theta_0)$  such that the set  $\Sigma(-\eta_0, \theta_0) \cap \{\lambda; |\lambda| \geq \frac{\eta_0}{2}\}$  is in  $\rho(-\widehat{L}_{\xi'})$  for  $|\xi'| \leq \tilde{r}_0$ . Furthermore, the following estimates hold for any multi-index  $\alpha'$  with  $|\alpha'| \leq n$  uniformly in  $\lambda \in \Sigma(-\eta_0, \theta_0) \cap \{\lambda; |\lambda| \geq \frac{\eta_0}{2}\}$  and  $\xi'$  with  $|\xi'| \leq \tilde{r}_0$ :*

$$\begin{aligned} |\partial_{\xi'}^{\alpha'} (\lambda + \widehat{L}_{\xi'})^{-1} f|_{H^\ell \times L^2} &\leq \frac{C}{|\lambda|} |f|_{H^\ell \times L^2}, \quad \ell = 0, 1, \\ |\partial_{\xi'}^{\alpha'} \partial_{x_n}^\ell \tilde{Q}(\lambda + \widehat{L}_{\xi'})^{-1} f|_2 &\leq \frac{C}{(|\lambda| + 1)^{1-\frac{\ell}{2}}} |f|_{H^{\ell-1} \times L^2}, \quad \ell = 1, 2, \\ |\partial_{\xi'}^{\alpha'} \partial_{x_n}^2 Q_0(\lambda + \widehat{L}_{\xi'})^{-1} f|_2 &\leq \frac{C}{(|\lambda| + 1)^{\frac{1}{2}}} |f|_{H^2 \times H^1}. \end{aligned}$$

**Proof.** In the following we will write

$$\widehat{L}^{(1)}(\xi') = \sum_{j=1}^{n-1} \xi_j \widehat{L}_j^{(1)} \quad \text{and} \quad \widehat{L}^{(2)}(\xi') = \sum_{j,k=1}^{n-1} \xi_j \xi_k \widehat{L}_{jk}^{(2)}.$$

We first observe that

$$(3.13) \quad \left| \widehat{L}_j^{(1)} u \right|_{H^\ell \times H^{(\ell-1)_+}} \leq C \left\{ |Q_0 u|_{H^{(\ell-1)_+}} + |\tilde{Q} u|_{H^{(\ell-1)_++1}} \right\}$$

and

$$(3.14) \quad \left| \widehat{L}_{jk}^{(2)} u \right|_{H^\ell \times H^{(\ell-1)_+}} \leq C |\tilde{Q} u|_{H^{(\ell-1)_+}}.$$

It then follows from Lemma 3.1 and (3.14) that

$$(3.15) \quad \left| \widehat{L}_{jk}^{(2)} (\lambda + \widehat{L}_0)^{-1} f \right|_{H^\ell \times L^2} \leq C \left| \tilde{Q} (\lambda + \widehat{L}_0)^{-1} f \right|_2 \leq C |f|_2$$

for  $\ell = 0, 1$  and  $\lambda \in \Sigma(-\eta_0, \theta_0) \cap \{\lambda; |\lambda| \geq \frac{\eta_0}{2}\}$  with  $C = C(\eta_0, \theta_0) > 0$ . Also, by Lemma 3.1 and (3.13), we have

$$(3.16) \quad \left| \widehat{L}_j^{(1)} (\lambda + \widehat{L}_0)^{-1} f \right|_{H^\ell \times L^2} \leq C |f|_2$$

for  $\ell = 0, 1$  and  $\lambda \in \Sigma(-\eta_0, \theta_0) \cap \left\{ \lambda; |\lambda| \geq \frac{\eta_0}{2} \right\}$  with  $C = C(\eta_0, \theta_0) > 0$ . It then follows that there exists a positive number  $\tilde{r}_0$  such that if  $|\xi'| \leq \tilde{r}_0$ , then

$$\left| \left( \widehat{L}^{(1)}(\xi') + \widehat{L}^{(2)}(\xi') \right) (\lambda + \widehat{L}_0)^{-1} f \right|_{H^\ell \times L^2} \leq \frac{1}{2} |f|_2$$

for  $\ell = 0, 1$  and  $\lambda \in \Sigma(-\eta_0, \theta_0) \cap \left\{ \lambda; |\lambda| \geq \frac{\eta_0}{2} \right\}$ . By the Neumann series expansion, we see that  $I + \left( \widehat{L}^{(1)}(\xi') + \widehat{L}^{(2)}(\xi') \right) (\lambda + \widehat{L}_0)^{-1}$  is invertible on  $H^\ell(0, a) \times L^2(0, a)$ ,  $\ell = 0, 1$ , for  $\lambda \in \Sigma(-\eta_0, \theta_0) \cap \left\{ \lambda; |\lambda| \geq \frac{\eta_0}{2} \right\}$  and  $\xi'$  with  $|\xi'| \leq \tilde{r}_0$ . In particular, we conclude that  $\Sigma(-\eta_0, \theta_0) \cap \left\{ \lambda; |\lambda| \geq \frac{\eta_0}{2} \right\} \subset \rho(-\widehat{L}_{\xi'})$  and

$$(3.17) \quad (\lambda + \widehat{L}_{\xi'})^{-1} = (\lambda + \widehat{L}_0)^{-1} \sum_{N=0}^{\infty} (-1)^N \left[ \left( \widehat{L}^{(1)}(\xi') + \widehat{L}^{(2)}(\xi') \right) (\lambda + \widehat{L}_0)^{-1} \right]^N$$

for  $\lambda \in \Sigma(-\eta_0, \theta_0) \cap \left\{ \lambda; |\lambda| \geq \frac{\eta_0}{2} \right\}$  and  $\xi'$  with  $|\xi'| \leq \tilde{r}_0$ . Furthermore, we see from Lemma 3.1, (3.13) and (3.14) that

$$\begin{aligned} & \left| \partial_{\xi'}^{\alpha'} (\lambda + \widehat{L}_{\xi'})^{-1} f \right|_{H^\ell \times L^2} \\ & \leq \frac{C}{|\lambda|} \left| \partial_{\xi'}^{\alpha'} \sum_{N=0}^{\infty} (-1)^N \left[ \left( \widehat{L}^{(1)}(\xi') + \widehat{L}^{(2)}(\xi') \right) (\lambda + \widehat{L}_0)^{-1} \right]^N f \right|_{H^\ell \times L^2} \\ & \leq \frac{C}{|\lambda|} |f|_{H^\ell \times L^2}, \quad \ell = 0, 1. \end{aligned}$$

Similarly, we have, for  $\ell = 1, 2$ ,

$$\begin{aligned} & \left| \partial_{\xi'}^{\alpha'} \partial_{x_n}^\ell \widetilde{Q}(\lambda + \widehat{L}_{\xi'})^{-1} f \right|_2 \\ & \leq \frac{C}{(|\lambda| + 1)^{1 - \frac{\ell}{2}}} \left| \partial_{\xi'}^{\alpha'} \sum_{N=0}^{\infty} (-1)^N \left[ \left( \widehat{L}^{(1)}(\xi') + \widehat{L}^{(2)}(\xi') \right) (\lambda + \widehat{L}_0)^{-1} \right]^N f \right|_{H^{\ell-1} \times L^2} \\ & \leq \frac{C}{(|\lambda| + 1)^{1 - \frac{\ell}{2}}} |f|_{H^{\ell-1} \times L^2}. \end{aligned}$$

Let us estimate  $\partial_{x_n}^2 Q_0(\lambda + \widehat{L}_{\xi'})^{-1} f$ . We see from Lemma 3.1, (3.13) and (3.14) that

$$\left| \widehat{L}_j^{(1)}(\lambda + \widehat{L}_0)^{-1} f \right|_{H^2 \times H^1} \leq C |f|_{H^1 \times L^2}$$

and

$$\left| \widehat{L}_{jk}^{(2)}(\lambda + \widehat{L}_0)^{-1} f \right|_{H^2 \times H^1} \leq C |f|_2$$



uniformly for  $\lambda \in \Sigma(-\eta_0, \theta_0) \cap \{\lambda; |\lambda| \geq \frac{\eta_0}{2}\}$ . Therefore, taking  $\tilde{r}_0$  smaller if necessary, we have

$$\left| \left( \widehat{L}^{(1)}(\xi') + \widehat{L}^{(2)}(\xi') \right) (\lambda + \widehat{L}_0)^{-1} f \right|_{H^2 \times H^1} \leq \frac{1}{2} |f|_{H^1 \times L^2}$$

for  $\lambda \in \Sigma(-\eta_0, \theta_0) \cap \{\lambda; |\lambda| \geq \frac{\eta_0}{2}\}$  and  $\xi'$  with  $|\xi'| \leq \tilde{r}_0$ . It then follows from Lemma 3.1 and (3.17) that

$$\begin{aligned} & \left| \partial_{\xi'}^{\alpha'} \partial_{x_n}^2 Q_0(\lambda + \widehat{L}_{\xi'})^{-1} f \right|_2 \\ & \leq \frac{C}{(|\lambda| + 1)^{\frac{1}{2}}} \left| \partial_{\xi'}^{\alpha'} \sum_{N=0}^{\infty} (-1)^N \left[ \left( \widehat{L}^{(1)}(\xi') + \widehat{L}^{(2)}(\xi') \right) (\lambda + \widehat{L}_0)^{-1} \right]^N f \right|_{H^2 \times H^1} \\ & \leq \frac{C}{(|\lambda| + 1)^{\frac{1}{2}}} |f|_{H^2 \times H^1}. \end{aligned}$$

This completes the proof.

We next investigate the spectrum of  $-\widehat{L}_{\xi'}$  near  $\lambda = 0$ .

**Theorem 3.4.** *Let  $\eta_0$  and  $\tilde{r}_0$  be the numbers given in Theorem 3.3. Then there exists a positive number  $r_0$  with  $r_0 \leq \tilde{r}_0$  such that for each  $\xi'$  with  $|\xi'| \leq r_0$  it holds that*

$$\sigma(-\widehat{L}_{\xi'}) \cap \{\lambda; |\lambda| \leq \eta_0\} = \{\lambda_0(\xi')\},$$

where  $\lambda_0(\xi') \in \mathbf{R}$  and  $\lambda_0(\xi')$  is a simple eigenvalue of  $-\widehat{L}_{\xi'}$  that has the form

$$\lambda_0(\xi') = -\frac{a^2 \gamma^2}{12\nu} |\xi'|^2 + O(|\xi'|^4)$$

as  $|\xi'| \rightarrow 0$ .

**Proof.** By Theorem 3.3, (3.13) and (3.14), we see that if  $|\lambda| = \eta_0$ , then  $\lambda \in \rho(-\widehat{L}_{\xi'})$  for  $|\xi'| \leq \tilde{r}_0$ . In particular,

$$\widehat{\Pi}(\xi') = \frac{1}{2\pi i} \int_{|\lambda|=\eta_0} (\lambda + \widehat{L}_{\xi'})^{-1} d\lambda$$

is the eigenprojection for the eigenvalues of  $-\widehat{L}_{\xi'}$  lying inside the circle  $|\lambda| = \eta_0$ . The perturbation theory then implies that  $\dim \text{Range } \widehat{\Pi}(\xi') = \dim \text{Range } \widehat{\Pi}^{(0)} = 1$ . Therefore, we see from Lemma 3.2 that  $\sigma(-\widehat{L}_{\xi'}) \cap \{\lambda; |\lambda| \leq \eta_0\}$  consists of only one simple eigenvalue, say  $\lambda_0(\xi')$ .

To show that  $\lambda_0(\xi')$  has the desired asymptotic form, we first observe that  $\lambda$  is an eigenvalue of  $-\widehat{L}_{\xi'}$  if and only if it is an eigenvalue of  $-\widehat{L}_{T'\xi'}$  for any  $(n-1) \times (n-1)$  orthogonal matrix  $T'$ , since  $\widehat{L}_{\xi'} = T^{-1}\widehat{L}_{T'\xi'}T$ , where

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & T' & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It then follows that  $\lambda_0(\xi')$  is a function of  $|\xi'|$ , and hence, it suffices to consider  $\widehat{L}_{\xi'}$  with  $\xi' = \eta\mathbf{e}'_1$ , where  $\eta \in \mathbf{R}$  and  $|\eta| = |\xi'|$ .

We write  $\widehat{L}_{\xi'}$  with  $\xi' = \eta\mathbf{e}'_1$  as  $\widehat{L}_\eta$ , and  $\widehat{L}_\eta = \widehat{L}_0 + \eta\widehat{L}_1^{(1)} + \eta^2\widehat{L}_{1,1}^{(2)}$ . We also denote the corresponding eigenvalue by  $\lambda_0(\eta)$ . With this  $\widehat{L}_\eta$ , taking  $T' = -I_{n-1}$ , we see that  $\lambda_0(\eta) = \lambda_0(-\eta)$  since  $\lambda_0(\xi')$  is simple. Furthermore, we have a relation  $\overline{\widehat{L}_\eta u} = \widehat{L}_{-\eta}\bar{u}$ , which implies that  $\overline{\lambda_0(\eta)} = \lambda_0(-\eta) = \lambda_0(\eta)$ . This means that  $\lambda_0(\eta) \in \mathbf{R}$ .

In view of (3.13) and (3.14) we can apply the analytic perturbation theory [9, Chap. 2 and 7] to see that

$$\lambda_0(\eta) = \lambda^{(0)} + \eta\lambda^{(1)} + \eta^2\lambda^{(2)} + \eta^3\lambda^{(3)} + O(\eta^4)$$

with  $\lambda^{(0)} = 0$ . Since  $\lambda_0(\eta) = \lambda_0(-\eta)$ , we have  $\lambda^{(1)} = \lambda^{(3)} = 0$ . The coefficient  $\lambda^{(2)}$  of  $\eta^2$  is given by

$$\lambda^{(2)} = -\langle \widehat{L}_{1,1}^{(2)}\psi^{(0)}, \psi^{(0)} \rangle + \langle \widehat{L}_1^{(1)}S\widehat{L}_1^{(1)}\psi^{(0)}, \psi^{(0)} \rangle,$$

where  $S = [(I - \widehat{H}^{(0)})\widehat{L}_0(I - \widehat{H}^{(0)})]^{-1}$ . It is easy to see that  $\widehat{L}_{1,1}^{(2)}\psi^{(0)} = 0$ .

Let us compute  $\langle \widehat{L}_1^{(1)}S\widehat{L}_1^{(1)}\psi^{(0)}, \psi^{(0)} \rangle$ . Since  $\widehat{L}_1^{(1)}\psi^{(0)} = i\gamma \begin{pmatrix} 0 \\ \mathbf{e}'_1 \\ 0 \end{pmatrix}$ , we have

$$S\widehat{L}_1^{(1)}\psi^{(0)} = \frac{i\gamma}{\nu} \begin{pmatrix} 0 \\ \mathbf{e}'_1 \\ 0 \end{pmatrix} (-\partial_{x_n}^2)^{-1} \cdot 1,$$

and hence,

$$\widehat{L}_1^{(1)}S\widehat{L}_1^{(1)}\psi^{(0)} = - \begin{pmatrix} \frac{\gamma^2}{\nu}(-\partial_{x_n}^2)^{-1} \cdot 1 \\ 0 \\ -\frac{\tilde{\nu}\gamma}{\nu}\partial_{x_n}(-\partial_{x_n}^2)^{-1} \cdot 1 \end{pmatrix}.$$

Here  $(-\partial_{x_n}^2)^{-1}$  denotes the inverse of  $-\partial_{x_n}^2$  under the 0-Dirichlet boundary condition at  $x_n = 0, a$ . We thus obtain

$$\langle \widehat{L}_1^{(1)}S\widehat{L}_1^{(1)}\psi^{(0)}, \psi^{(0)} \rangle = -\frac{\gamma^2}{\nu} \langle (-\partial_{x_n}^2)^{-1} \cdot 1 \rangle = -\frac{a^2\gamma^2}{12\nu}.$$

Consequently, we obtain

$$\lambda_0(\eta) = -\frac{a^2\gamma^2}{12\nu}\eta^2 + O(\eta^4).$$

This completes the proof.

We next investigate the eigenprojection  $\widehat{\Pi}(\xi')$  associated with  $\lambda_0(\xi')$ . To do so, we will consider the formal adjoint problem

$$\lambda u + \widehat{L}_{\xi'}^* u = f,$$

where  $\widehat{L}_{\xi'}^*$  is an operator of the form

$$\widehat{L}_{\xi'}^* = \widehat{L}_0^* + \sum_{j=1}^{n-1} \xi_j \widehat{L}_j^{(1)*} + \sum_{j,k=1}^{n-1} \xi_j \xi_k \widehat{L}_{jk}^{(2)*}$$

with domain of definition  $D(\widehat{L}_{\xi'}^*) = D(\widehat{L}_{\xi'})$ . Here  $\xi' = {}^T(\xi_1, \dots, \xi_{n-1})$ ,

$$\begin{aligned} \widehat{L}_0^* &= \begin{pmatrix} 0 & 0 & -\gamma \partial_{x_n} \\ 0 & -\nu \partial_{x_n}^2 I_{n-1} & 0 \\ -\gamma \partial_{x_n} & 0 & -\nu_1 \partial_{x_n}^2 \end{pmatrix}, \\ \widehat{L}_j^{(1)*} &= \begin{pmatrix} 0 & -i\gamma^T \mathbf{e}'_j & 0 \\ -i\gamma \mathbf{e}'_j & 0 & -i\tilde{\nu} \mathbf{e}'_j \partial_{x_n} \\ 0 & -i\tilde{\nu}^T \mathbf{e}'_j \partial_{x_n} & 0 \end{pmatrix}, \\ \widehat{L}_{jk}^{(2)*} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu \delta_{jk} I_{n-1} + \tilde{\nu} \mathbf{e}'_k{}^T \mathbf{e}'_j & 0 \\ 0 & 0 & \nu \delta_{jk} \end{pmatrix}. \end{aligned}$$

**Theorem 3.5.** *Let  $\widehat{\Pi}(\xi')$  be the eigenprojection associated with  $\lambda_0(\xi')$ . Then there exists a positive number  $r_0$  such that, for any  $\xi'$  with  $|\xi'| \leq r_0$ ,  $\widehat{\Pi}(\xi')$  is written in the form*

$$\widehat{\Pi}(\xi') u = \int_0^a \widehat{\Pi}(\xi', x_n, y_n) u(y_n) dy_n$$

with

$$\widehat{\Pi}(\xi', x_n, y_n) = \widehat{\Pi}^{(0)} + \sum_{j=1}^{n-1} \xi_j \widehat{\Pi}_j^{(1)}(x_n, y_n) + \widehat{\Pi}^{(2)}(\xi', x_n, y_n).$$

Here  $\widehat{\Pi}^{(0)} = \frac{1}{a}Q_0$ ;  $\widehat{\Pi}_j^{(1)} \in W^{1,\infty}((0,a) \times (0,a))$ ,  $j = 1, \dots, n-1$ ; and  $\widehat{\Pi}^{(2)}(\xi', x_n, y_n)$  satisfies

$$\left| \partial_{\xi'}^{\alpha'} \widehat{\Pi}^{(2)}(\xi', \cdot, \cdot) \right|_{W^{1,\infty}((0,a) \times (0,a))} \leq C |\xi'|^{2-|\alpha'|}$$

for any multi-index  $\alpha'$  with  $|\alpha'| \leq n$  uniformly in  $\xi'$  with  $|\xi'| \leq r_0$ . Furthermore,  $\widehat{\Pi}(\xi')$  has the properties

$$\widehat{\Pi}(\xi') [\partial_{x_n} \tilde{Q}u] = - \sum_{j=1}^{n-1} \xi_j (\partial_{y_n} \widehat{\Pi}_j^{(1)}(\xi')) [\tilde{Q}u] - (\partial_{y_n} \widehat{\Pi}^{(2)}(\xi')) [\tilde{Q}u]$$

and

$$\partial_{x_n} \widehat{\Pi}(\xi') \tilde{Q}u = \partial_{x_n} \widehat{\Pi}^{(2)}(\xi') \tilde{Q}u.$$

**Proof.** By (3.13)–(3.17) we see that  $(\lambda + \widehat{L}_{\xi'})^{-1}$  has the form

$$(\lambda + \widehat{L}_{\xi'})^{-1} = (\lambda + \widehat{L}_0)^{-1} + \sum_{j=1}^{n-1} \xi_j (\lambda + \widehat{L}_0)^{-1} L_j^{(1)} (\lambda + \widehat{L}_0)^{-1} + \tilde{R}(\lambda, \xi'),$$

where

$$\begin{aligned} & \tilde{R}(\lambda, \xi') \\ (3.18) \quad &= (\lambda + \widehat{L}_0)^{-1} \widehat{L}^{(2)}(\xi') (\lambda + \widehat{L}_0)^{-1} \\ & + (\lambda + \widehat{L}_0)^{-1} \sum_{N=2}^{\infty} (-1)^N [(\widehat{L}^{(1)}(\xi') + \widehat{L}^{(2)}(\xi')) (\lambda + \widehat{L}_0)^{-1}]^N \end{aligned}$$

and  $\tilde{R}(\lambda, \xi')$  satisfies

$$(3.19) \quad \left| \partial_{\xi'}^{\alpha'} \tilde{R}(\lambda, \xi') f \right|_{H^2} \leq C |\xi'|^{2-|\alpha'|} |f|_{H^2 \times H^1}.$$

Similarly, one can prove that

$$(\lambda + \widehat{L}_{\xi'}^*)^{-1} = (\lambda + \widehat{L}_0^*)^{-1} + \sum_{j=1}^{n-1} \xi_j (\lambda + \widehat{L}_0^*)^{-1} L_j^{(1)*} (\lambda + \widehat{L}_0^*)^{-1} + \tilde{R}^*(\lambda, \xi'),$$

where

$$\begin{aligned} & \tilde{R}^*(\lambda, \xi') \\ (3.20) \quad &= (\lambda + \widehat{L}_0^*)^{-1} \widehat{L}^{(2)*}(\xi') (\lambda + \widehat{L}_0^*)^{-1} \\ & + (\lambda + \widehat{L}_0^*)^{-1} \sum_{N=2}^{\infty} (-1)^N [(\widehat{L}^{(1)*}(\xi') + \widehat{L}^{(2)*}(\xi')) (\lambda + \widehat{L}_0^*)^{-1}]^N \end{aligned}$$

and  $\tilde{R}^*(\lambda, \xi')$  satisfies

$$(3.21) \quad \left| \partial_{\xi'}^{\alpha'} \tilde{R}^*(\lambda, \xi') f \right|_{H^2} \leq C |\xi'|^{2-|\alpha'|} |f|_{H^2 \times H^1}.$$

We now define  $\psi(\xi', x_n)$  and  $\tilde{\psi}^*(\xi', x_n)$  by

$$\psi(\xi', x_n) = \frac{1}{2\pi i} \int_{|\lambda|=\eta_0} (\lambda + \widehat{L}_{\xi'})^{-1} \psi^{(0)} d\lambda$$

and

$$\tilde{\psi}^*(\xi', x_n) = \frac{1}{2\pi i} \int_{|\lambda|=\eta_0} (\lambda + \widehat{L}_{\xi'}^*)^{-1} \psi^{(0)} d\lambda,$$

where  $\psi^{(0)} = {}^T(1, 0, \dots, 0)$ . It then follows from (3.18)–(3.21) that  $\psi$  and  $\tilde{\psi}^*$  have the form

$$(3.22) \quad \begin{aligned} \psi(\xi', x_n) &= \psi^{(0)} + \sum_{j=1}^{n-1} \xi_j \psi_j^{(1)}(x_n) + \psi^{(2)}(\xi', x_n), \\ \tilde{\psi}^*(\xi', x_n) &= \psi^{(0)} + \sum_{j=1}^{n-1} \xi_j \tilde{\psi}_j^{(1)*}(x_n) + \tilde{\psi}^{(2)*}(\xi', x_n), \end{aligned}$$

where  $\psi_j^{(1)}$ ,  $\tilde{\psi}_j^{(1)*}$ ,  $\psi^{(2)}$  and  $\tilde{\psi}^{(2)*}$  satisfy

$$\begin{aligned} \left| \psi_j^{(1)} \right|_{H^2} + \left| \tilde{\psi}_j^{(1)*} \right|_{H^2} &\leq C, \quad j = 1, \dots, n-1, \\ \left| \partial_{\xi'}^{\alpha'} \psi^{(2)}(\xi') \right|_{H^2} + \left| \partial_{\xi'}^{\alpha'} \tilde{\psi}^{(2)*}(\xi') \right|_{H^2} &\leq C |\xi'|^{2-|\alpha'|}. \end{aligned}$$

Therefore, we have

$$\left| \psi_j^{(1)} \right|_{W^{1,\infty}} + \left| \tilde{\psi}_j^{(1)*} \right|_{W^{1,\infty}} \leq C, \quad j = 1, \dots, n-1,$$

and

$$(3.23) \quad \left| \partial_{\xi'}^{\alpha'} \psi^{(2)}(\xi') \right|_{W^{1,\infty}} + \left| \partial_{\xi'}^{\alpha'} \tilde{\psi}^{(2)*}(\xi') \right|_{W^{1,\infty}} \leq C |\xi'|^{2-|\alpha'|}.$$

We note that  $\langle \psi(\xi'), \tilde{\psi}^*(\xi') \rangle$  is analytic in  $\xi'$  and

$$\langle \psi(\xi'), \tilde{\psi}^*(\xi') \rangle = 1 + \sum_{j=1}^{n-1} \xi_j \left\{ \langle \psi^{(0)}, \tilde{\psi}_j^{(1)*} \rangle + \langle \psi_j^{(1)}, \psi^{(0)} \rangle \right\} + \tilde{\Psi}^{(2)}(\xi'),$$

where  $\tilde{\Psi}^{(2)}(\xi')$  satisfies  $\left| \partial_{\xi'}^{\alpha'} \tilde{\Psi}^{(2)}(\xi') \right| \leq C |\xi'|^{2-|\alpha'|}$ . In particular, taking  $r_0$  smaller if necessary, we see that

$$\left| \langle \psi(\xi'), \tilde{\psi}^*(\xi') \rangle \right| \geq \frac{1}{2}$$

for  $|\xi'| \leq r_0$ .

We set

$$\psi^*(\xi', x_n) = \frac{1}{\langle \psi(\xi'), \tilde{\psi}^*(\xi') \rangle} \tilde{\psi}^*(\xi', x_n).$$

Then we have

$$\langle \psi(\xi'), \psi^*(\xi') \rangle = 1$$

and

$$(3.24) \quad \psi^*(\xi', x_n) = \psi^{(0)} + \sum_{j=1}^{n-1} \xi_j \psi_j^{(1)*}(x_n) + \psi^{(2)*}(\xi', x_n),$$

where  $\psi_j^{(1)*}$  and  $\psi^{(2)*}$  satisfy

$$(3.25) \quad \begin{aligned} |\psi_j^{(1)*}|_{W^{1,\infty}} &\leq C, \quad j = 1, \dots, n-1, \\ |\partial_{\xi'}^{\alpha'} \psi_j^{(2)*}(\xi')|_{W^{1,\infty}} &\leq C |\xi'|^{2-|\alpha'|}. \end{aligned}$$

It is not difficult to see that  $\langle u, \psi^*(\xi') \rangle \psi(\xi')$  is the eigenprojection  $\widehat{\Pi}(\xi')$  associated with  $\lambda_0(\xi')$ .

Setting

$$\begin{aligned} \widehat{\Pi}^{(0)} &= \frac{1}{a} Q_0 \\ \widehat{\Pi}_j^{(1)}(x_n, y_n) &= \psi^{(0)}(x_n)^T \psi_j^{(1)*}(y_n) + \psi_j^{(1)}(x_n)^T \psi^{(0)}(y_n), \\ \widehat{\Pi}^{(2)}(\xi', x_n, y_n) &= \psi(\xi', x_n)^T \psi^{(2)*}(\xi', y_n) + \psi^{(2)}(\xi', x_n)^T \psi^*(\xi', y_n), \end{aligned}$$

we see from (3.22)–(3.25) that the integral kernel  $\widehat{\Pi}(\xi', x_n, y_n)$  of  $\widehat{\Pi}(\xi')$  is written as

$$\widehat{\Pi}(\xi', x_n, y_n) = \psi(\xi', x_n)^T \psi^*(\xi', y_n) = \widehat{\Pi}^{(0)} + \sum_{j=1}^{n-1} \xi_j \widehat{\Pi}_j^{(1)}(x_n, y_n) + \widehat{\Pi}^{(2)}(\xi', x_n, y_n)$$

with  $\widehat{\Pi}_j^{(1)} \in W^{1,\infty}((0, a) \times (0, a))$ ,  $j = 1, \dots, n-1$ , and

$$\left| \partial_{\xi'}^{\alpha'} \widehat{\Pi}^{(2)}(\xi', \cdot, \cdot) \right|_{W^{1,\infty}((0,a) \times (0,a))} \leq C |\xi'|^{2-|\alpha'|}.$$

We thus conclude that  $\widehat{\Pi}(\xi')$  is written in the desired form.

We finally show that  $\widehat{\Pi}(\xi') [\partial_{x_n} \tilde{Q}u](x_n)$  and  $\partial_{x_n} \widehat{\Pi}(\xi') \tilde{Q}u$  have the desired forms. Since  $\psi^*(\xi', y_n)$  is an eigenfunction of  $\widehat{L}_{\xi'}^*$ , we have  $\tilde{Q}\psi^* \Big|_{y_n=0,a} = 0$ ,

which implies that  $\widehat{\Pi}(\xi', x_n, y_n)\tilde{Q}\big|_{y_n=0,a} = 0$ . An integration by parts then yields

$$\begin{aligned}\widehat{\Pi}(\xi') [\partial_{x_n}\tilde{Q}u](x_n) &= \int_0^a \widehat{\Pi}(\xi', x_n, y_n)\partial_{y_n}\tilde{Q}u(y_n) dy_n \\ &= - \int_0^a \partial_{y_n}\widehat{\Pi}(\xi', x_n, y_n)\tilde{Q}u(y_n) dy_n \\ &= - (\partial_{y_n}\widehat{\Pi}(\xi')) [\tilde{Q}u](x_n).\end{aligned}$$

Since  $\partial_{y_n}\widehat{\Pi}^{(0)} = 0$ , we have the desired form of  $\widehat{\Pi}(\xi') [\partial_{x_n}\tilde{Q}u]$ . Furthermore, since  $\partial_{x_n}\psi^{(0)} = 0$  and  $\tilde{Q}\psi^{(0)} = 0$ , we have  $\partial_{x_n}\widehat{\Pi}_j^{(1)}(x_n, y_n)\tilde{Q} = 0$ , and hence,  $\partial_{x_n}\widehat{\Pi}(\xi')\tilde{Q}u = \partial_{x_n}\widehat{\Pi}^{(2)}(\xi')\tilde{Q}u$ . This completes the proof.

We next consider  $(\lambda + \widehat{L}_{\xi'})^{-1}$  with  $|\xi'| \geq r_0$ . The analysis of  $(\lambda + \widehat{L}_{\xi'})^{-1}$  with  $|\xi'| \geq r$  for any  $r > 0$  is given in [6]. Applying [6, Theorems 2.5–2.7], we obtain the following estimates.

Let  $r_0$  be the number given in Theorem 3.5. We take a cut-off function  $\chi(\xi') \in C^\infty(\mathbf{R}^{n-1})$  satisfying  $0 \leq \chi \leq 1$  on  $\mathbf{R}^{n-1}$ ,  $\chi(\xi') = 1$  for  $|\xi'| \leq \frac{r_0}{2}$  and  $\chi(\xi') = 0$  for  $|\xi'| \geq r_0$ . We set

$$(3.26) \quad \chi^{(0)}(\xi') = \chi(\xi'), \quad \chi^{(1)}(\xi') = 1 - \chi(\xi').$$

We define the operators  $R^{(j)}(\lambda)$ ,  $j = 0, 1$ , by

$$(3.27) \quad R^{(j)}(\lambda)f = \mathcal{F}_{\xi'}^{-1} [\chi^{(j)}(\xi')(\lambda + \widehat{L}_{\xi'})^{-1}\widehat{f}], \quad j = 0, 1.$$

By [6, Theorems 2.5–2.7] we have the following estimates.

**Theorem 3.6.** *Let  $r_0$  be the positive number given in Theorem 3.5.*

(i) *There exist positive numbers  $\tilde{\eta}$  and  $\tilde{\theta}$  with  $\tilde{\theta} \in (\frac{\pi}{2}, \pi)$  such that  $\Sigma(-\tilde{\eta}, \tilde{\theta}) \subset \rho(-\widehat{L}_{\xi'})$  for  $|\xi'| \geq \frac{r_0}{2}$ .*

(ii) *Let  $1 < p < \infty$  and define  $R^{(1)}(\lambda)$  as above. Then the following estimates hold uniformly in  $\lambda \in \Sigma(-\tilde{\eta}, \tilde{\theta})$ :*

$$\|\partial_x^k R^{(1)}(\lambda)f\|_p \leq \left\{ \frac{\|Q_0f\|_{W^{k,p}}}{|\lambda| + 1} + \frac{\|\tilde{Q}f\|_p}{(|\lambda| + 1)^{1-\frac{k}{2}}} \right\}, \quad k = 0, 1.$$

**Theorem 3.7** *Let  $\tilde{\eta}$  and  $\tilde{\theta}$  be the numbers as in Theorem 3.6. Then the following estimates hold uniformly in  $\lambda \in \Sigma(-\tilde{\eta}, \tilde{\theta})$ :*

$$\|\partial_x^k Q_0 R^{(1)}(\lambda)f\|_\infty \leq C \left\{ \frac{\|Q_0f\|_{H^{[\frac{3}{2}] + 1 + k}}}{|\lambda| + 1} + \frac{\|\tilde{Q}f\|_{H^{[\frac{3}{2}] + k}}}{(|\lambda| + 1)^{\frac{3}{4}}} \right\}, \quad k = 0, 1,$$

and

$$\|\partial_x^k \tilde{Q} R^{(1)}(\lambda) f\|_\infty \leq C \left\{ \frac{\|Q_0 f\|_{H^{[\frac{n}{2}] + k}}}{(|\lambda| + 1)^{\frac{3}{4}}} + \frac{\|\tilde{Q} f\|_{H^{[\frac{n}{2}] - 1 + k}}}{(|\lambda| + 1)^{\frac{\varepsilon}{4}}} \right\}, \quad k = 0, 1.$$

Here  $\varepsilon$  is some number satisfying  $0 < \varepsilon < \frac{1}{3}$ .

**Theorem 3.8.** *Let  $p = 1, \infty$  and let  $\tilde{\eta}$  and  $\tilde{\theta}$  be the numbers as in Theorem 3.6. Then the following estimates hold uniformly in  $\lambda \in \Sigma(-\tilde{\eta}, \tilde{\theta})$ :*

$$\|\partial_x^k Q_0 R^{(1)}(\lambda) f\|_p \leq \frac{C}{|\lambda| + 1} \|f\|_{W^{k+1,p} \times W^{k,p}}, \quad k = 0, 1,$$

and

$$\|\partial_x^k \tilde{Q} R^{(1)}(\lambda) f\|_p \leq C \left\{ \frac{\|Q_0 f\|_{W^{k,p}}}{|\lambda| + 1} + \frac{\|\tilde{Q} f\|_p}{(|\lambda| + 1)^{1 - \frac{k}{2}}} \right\}, \quad k = 0, 1.$$

#### 4. Proof of Theorem 2.1

In this section we prove Theorem 2.1 by applying Theorems 3.3–3.8.

**Proof of Theorem 2.1.** Let  $\eta > 0$  be a positive number. By Theorem 2.1 in [6] there exists a number  $\theta \in (\frac{\pi}{2}, \pi)$  such that  $\mathcal{U}(t)u_0$  is written as

$$\mathcal{U}(t)u_0 = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda + L)^{-1} u_0 d\lambda,$$

where  $\Gamma = \{\lambda = \eta + se^{\pm i\theta}; s \geq 0\}$ .

We decompose  $\mathcal{U}(t)u_0$  into the following form:

$$\mathcal{U}(t)u_0 = U^{(0)}(t)u_0 + U^{(1)}(t)u_0,$$

where  $U^{(j)}(t)u_0$ ,  $j = 0, 1$ , are defined by

$$U^{(j)}(t)u_0 = \mathcal{F}^{-1} \left[ \frac{1}{2\pi i} \int_\Gamma \chi^{(j)}(\xi') (\lambda + \hat{L}_{\xi'})^{-1} d\lambda \right], \quad j = 0, 1,$$

with  $\chi^{(j)}(\xi')$  defined in (3.26).

We first consider  $U^{(1)}(t)u_0$ . In view of Theorem 3.6, we can deform the contour  $\Gamma$  into  $\Gamma_\infty = \{\lambda = -\tilde{\eta} + se^{\pm i\tilde{\theta}}; s \geq 0\}$ , where  $\tilde{\eta}$  and  $\tilde{\theta}$  are the numbers given in Theorem 3.6. We then obtain

$$U^{(1)}(t)u_0 = \frac{1}{2\pi i} \int_{\Gamma_\infty} e^{\lambda t} R^{(1)}(\lambda) u_0 d\lambda,$$



where  $R^{(1)}(\lambda)$  is the operator defined in (3.27). It follows from Theorems 3.6–3.8 that

$$\begin{aligned} \left\| \partial_x^\ell U^{(1)}(t)u_0 \right\|_p &\leq Ce^{-ct} \|u_0\|_{W^{\ell,p} \times L^p}, \quad 1 < p < \infty, \quad \ell = 0, 1, \\ \left\| \partial_x^\ell U^{(1)}(t)u_0 \right\|_\infty &\leq Ce^{-ct} \|u_0\|_{H^{[\frac{p}{2}] + 1 + \ell} \times H^{[\frac{p}{2}] + \ell}}, \quad \ell = 0, 1, \\ \left\| \partial_x^\ell U^{(1)}(t)u_0 \right\|_p &\leq Ce^{-ct} \|u_0\|_{W^{\ell+1,p} \times W^{\ell,p}}, \quad p = 1, \infty, \quad \ell = 0, 1, \end{aligned}$$

for  $t \geq 1$ .

We next consider  $U^{(0)}(t)u_0$ . By Theorem 3.3, we can deform the contour  $\Gamma$  into  $\Gamma_0 \cup \tilde{\Gamma}$ , where

$$\Gamma_0 = \{\lambda = -\eta_0 + is; |s| \leq s_0\}, \quad \tilde{\Gamma} = \{\lambda = \eta + se^{\pm i\theta}; s \geq \tilde{s}_0\}.$$

Here we choose positive numbers  $s_0$  and  $\tilde{s}_0$  so that  $\Gamma_0$  connects with  $\tilde{\Gamma}$  at the end points of  $\Gamma_0$ . It then follows from Theorems 3.4, 3.5 and the residue theorem that  $U^{(0)}(t)u_0$  is written as

$$U^{(0)}(t)u_0 = W^{(0)}(t)u_0 + W^{(1)}(t)u_0,$$

where

$$W^{(0)}(t)u_0 = \mathcal{F}^{-1} \left[ \chi^{(0)}(\xi') e^{\lambda_0(\xi')t} \widehat{\Pi}(\xi') \widehat{u}_0 \right]$$

and

$$W^{(1)}(t)u_0 = \mathcal{F}^{-1} \left[ \frac{1}{2\pi i} \int_{\Gamma_0 \cup \tilde{\Gamma}} e^{\lambda t} \chi^{(0)}(\xi') (\lambda + \widehat{L}_{\xi'})^{-1} \widehat{u}_0 d\lambda \right].$$

Similarly to the case of  $U^{(1)}(t)u_0$ , by using the integral representation of  $(\lambda + \widehat{L}_{\xi'})^{-1}$  given in [6, Theorem 3.8], one can see that  $W^{(1)}(t)u_0$  has the same estimates as those for  $U^{(1)}(t)u_0$ .

Let us consider  $W^{(0)}(t)u_0$ . We write it as

$$W^{(0)}(t)u_0 = \mathcal{W}^{(0)}(t)u_0 + \mathcal{R}^{(0)}(t)u_0,$$

where

$$\mathcal{W}^{(0)}(t)u_0 = \mathcal{F}^{-1} \left[ e^{-\kappa|\xi'|^2 t} \widehat{\Pi}^{(0)} \widehat{u}_0 \right], \quad \kappa = -\frac{a^2 \gamma^2}{12\nu},$$

and

$$\mathcal{R}^{(0)}(t)u_0 = \mathcal{W}^{(1)}(t)u_0 + \mathcal{R}_1^{(0)}(t)u_0 + \mathcal{R}_2^{(0)}(t)u_0 + \mathcal{R}_3^{(0)}(t)u_0.$$

Here

$$\begin{aligned} \mathcal{W}^{(1)}(t)u_0 &= \mathcal{F}^{-1} \left[ (\chi^{(0)}(\xi') - 1) e^{-\kappa|\xi'|^2 t} \widehat{\Pi}^{(0)} \widehat{u}_0 \right], \\ \mathcal{R}_1^{(0)}(t)u_0 &= \mathcal{F}^{-1} \left[ \chi^{(0)}(\xi') e^{-\kappa|\xi'|^2 t} \widehat{\Pi}^{(1)}(\xi') \widehat{u}_0 \right], \end{aligned}$$

$$\mathcal{R}_2^{(0)}(t)u_0 = \mathcal{F}^{-1} \left[ \chi^{(0)}(\xi') e^{-\kappa|\xi'|^2 t} \widehat{\Pi}^{(2)}(\xi') \widehat{u}_0 \right]$$

and

$$\mathcal{R}_3^{(0)}(t)u_0 = \mathcal{F}^{-1} \left[ \chi^{(0)}(\xi') (e^{\lambda_0(\xi')t} - e^{-\kappa|\xi'|^2 t}) \widehat{\Pi}(\xi') \widehat{u}_0 \right]$$

with  $\kappa = -\frac{a^2 \gamma^2}{12\nu}$ .

Clearly,  $\mathcal{W}^{(0)}(t)u_0 = \begin{pmatrix} \phi^{(0)}(t) \\ 0 \end{pmatrix}$  and  $\phi^{(0)}$  satisfies

$$\partial_t \phi^{(0)} - \Delta' \phi^{(0)} = 0, \quad \phi^{(0)} \Big|_{t=0} = \langle \phi_0 \rangle.$$

It is easy to see that

$$\left\| \partial_x^\ell \mathcal{W}^{(1)}(t)u_0 \right\|_p \leq C e^{-ct} \|u_0\|_1, \quad \ell = 0, 1.$$

By Theorem 3.5, we easily deduce that

$$\left\| \partial_x^\ell \mathcal{R}_1^{(0)}(t)u_0 \right\|_p \leq C t^{-\frac{n-1}{2}(1-\frac{1}{p})-\frac{1}{2}} \|u_0\|_1, \quad \ell = 0, 1.$$

Let us consider  $\mathcal{R}_2^{(0)}(t)u_0$ . We will estimate it based on the Riemann-Lebesgue lemma as in the estimates for solutions of the Cauchy problem given in [12]. Since

$$\mathcal{R}_2^{(0)}(t)u_0 = \int_{\mathbf{R}^{n-1}} \int_0^a \mathcal{R}_2^{(0)}(t, x' - y', x_n, y_n) u(y', y_n) dy' dy_n$$

with

$$\begin{aligned} \mathcal{R}_2^{(0)}(t, x', x_n, y_n) &= \mathcal{F}^{-1} \left[ \chi^{(0)}(\xi') e^{-\kappa|\xi'|^2 t} \widehat{\Pi}^{(2)}(\xi', x_n, y_n) \right] (x') \\ (4.1) \quad &= \int_{\mathbf{R}^{n-1}} \chi^{(0)}(\xi') e^{-\kappa|\xi'|^2 t} \widehat{\Pi}^{(2)}(\xi', x_n, y_n) e^{i\xi' \cdot x} d\xi', \end{aligned}$$

we have

$$\left\| \partial_x^\ell \mathcal{R}_2^{(0)}(t)u_0 \right\|_1 \leq \sup_{0 \leq y_n \leq a} \left\| \partial_x^\ell \mathcal{R}_2^{(0)}(t, \cdot, \cdot, y_n) \right\|_1 \|u_0\|_1.$$

By Theorem 3.5, we see that

$$\sup_{0 \leq y_n \leq a} \left| \partial_{\xi'}^{\alpha'} \left( \xi'^{\beta'} \partial_{x_n}^j \chi^{(0)}(\xi') e^{-\kappa|\xi'|^2 t} \widehat{\Pi}^{(2)}(\xi', x_n, y_n) \right) \right|_{L_{x_n}^1} \leq C |\xi'|^{2-|\alpha'|} e^{-\frac{\kappa}{2}|\xi'|^2 t}$$

for  $|\beta'| + j \leq 1$ . Therefore, since  $e^{i\xi' \cdot x'} = \sum_{j=1}^{n-1} \frac{x_j}{i|x'|^2} \partial_{\xi_j} e^{i\xi' \cdot x'}$ , we perform the integration by parts in (4.1) to obtain, for any  $k = 0, 1, 2, \dots$ ,

$$\begin{aligned} \sup_{0 \leq y_n \leq a} \left| \partial_{x'}^{\beta'} \partial_{x_n}^j \mathcal{R}_2^{(0)}(t, x', \cdot, y_n) \right|_1 &\leq C |x'|^{-k} \int_{\mathbf{R}^{n-1}} |\xi'|^{2+|\alpha'|-k} e^{-\frac{\kappa}{2}|\xi'|^2 t} d\xi' \\ &\leq C |x'|^{-k} t^{\frac{k}{2}} t^{-\frac{n-1}{2}-1}, \quad |\beta'| + j \leq 1. \end{aligned}$$

This implies that

$$\begin{aligned}
& \sup_{0 \leq y_n \leq a} \left\| \partial_x^\ell \mathcal{R}_2^{(0)}(t, \cdot, \cdot, y_n) \right\|_1 \\
& \leq C \int_{|x'| \leq t^{\frac{1}{2}}} t^{-\frac{n-1}{2}-1} dx' + \int_{|x'| \geq t^{\frac{1}{2}}} |x'|^{-n} t^{\frac{n}{2}} t^{-\frac{n-1}{2}-1} dx' \\
& \leq Ct^{-1}
\end{aligned}$$

for  $\ell = 0, 1$ . Similarly, one can estimate  $\mathcal{R}_3^{(0)}(t)u_0$ . In fact, by Theorem 3.4, we have  $\lambda_0(\xi') = -\kappa|\xi'|^2 + \lambda^{(4)}(\xi')$ , where  $\lambda^{(4)}(\xi')$  is analytic in  $\xi'$  and  $|\lambda^{(4)}(\xi')| \leq C|\xi'|^4$ . Since

$$e^{\lambda_0(\xi')t} - e^{-\kappa|\xi'|^2t} = \lambda^{(4)}(\xi')te^{-\kappa|\xi'|^2t} \int_0^1 e^{\theta\lambda^{(4)}(\xi')t} d\theta,$$

we see from Theorem 3.5 that

$$\sup_{0 \leq y_n \leq a} \left| \partial_{\xi'}^{\alpha'} \left[ \xi'^{\beta'} \partial_{x_n}^j \chi^{(0)}(\xi') (e^{\lambda_0(\xi')t} - e^{-\kappa|\xi'|^2t}) \widehat{\Pi}(\xi', \cdot, y_n) \right] \right|_1 \leq C|\xi'|^{2-|\alpha'|} e^{-\frac{\kappa}{2}|\xi'|^2t}$$

for  $|\beta'| + j \leq 1$ . Similarly to above, one can obtain  $\sup_{0 \leq y_n \leq a} \left\| \partial_x^\ell \mathcal{R}_3^{(0)}(t, \cdot, \cdot, y_n) \right\|_1 \leq Ct^{-1}$  for  $\ell = 0, 1$ . Consequently, we have

$$\left\| \partial_x^\ell \mathcal{R}^{(0)}(t)u_0 \right\|_1 \leq Ct^{-\frac{1}{2}} \|u_0\|_1, \quad \ell = 0, 1.$$

On the other hand, it is easy to see that

$$\left\| \partial_x^\ell \mathcal{R}^{(0)}(t)u_0 \right\|_\infty \leq Ct^{-\frac{n-1}{2}-\frac{1}{2}} \|u_0\|_1, \quad \ell = 0, 1.$$

Therefore, by interpolation, we have

$$\left\| \partial_x^\ell \mathcal{R}^{(0)}(t)u_0 \right\|_p \leq Ct^{-\frac{n-1}{2}(1-\frac{1}{p})-\frac{1}{2}} \|u_0\|_1, \quad \ell = 0, 1.$$

By Theorem 3.5, we have  $\widehat{\Pi}^{(0)}\tilde{Q} = 0$ ,  $\partial_{x_n}\widehat{\Pi}^{(0)} = 0$  and  $\partial_{x_n}\widehat{\Pi}^{(1)}(\xi')\tilde{Q} = 0$ . It then follows that

$$\begin{aligned}
\left\| \partial_{x_n} \mathcal{R}^{(0)}(t)\tilde{Q}u_0 \right\|_p &= \left\| \partial_{x_n} \left( \mathcal{R}_2^{(0)}(t) + \mathcal{R}_3^{(0)}(t) \right) \tilde{Q}u_0 \right\|_p \\
&\leq Ct^{-\frac{n-1}{2}(1-\frac{1}{p})-1} \left\| \tilde{Q}u_0 \right\|_1.
\end{aligned}$$

Since  $\widehat{\Pi}^{(j)}(\xi') \left[ \partial_{x_n} \tilde{Q}\hat{u}_0 \right] = - \left( \partial_{y_n} \widehat{\Pi}^{(j)}(\xi') \right) \left[ \tilde{Q}\hat{u}_0 \right]$ ,  $j = 1, 2$ , we see that

$$\begin{aligned}
\left\| \mathcal{R}^{(0)}(t) \left[ \partial_{x_n} \tilde{Q}u_0 \right] \right\|_p &= \left\| \left( \partial_{y_n} \mathcal{R}^{(0)}(t) \right) \tilde{Q}u_0 \right\|_p \\
&\leq Ct^{-\frac{n-1}{2}(1-\frac{1}{p})-\frac{1}{2}} \left\| \tilde{Q}u_0 \right\|_1.
\end{aligned}$$

Clearly,  $\partial_{x'} \mathcal{R}^{(0)}(t) \tilde{Q}u_0 = \mathcal{R}^{(0)}(t) [\partial_{x'} \tilde{Q}u_0]$  and

$$\left\| \partial_{x'} \mathcal{R}^{(0)}(t) \tilde{Q}u_0 \right\|_p \leq Ct^{-\frac{n-1}{2}(1-\frac{1}{p})-1} \left\| \tilde{Q}u_0 \right\|_1.$$

The desired results of Theorem 2.1 are thus obtained by setting  $\mathcal{U}^{(0)}(t) = \mathcal{W}^{(0)}(t) + \mathcal{R}^{(0)}(t)$  and  $\mathcal{U}^{(\infty)}(t) = U^{(1)}(t) + W^{(1)}(t)$ . This completes the proof.

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