Asymptotic behavior of the semigroup associated with the linearized compressible Navier-Stokes equation in an infinite layer

Kagei, Yoshiyuki
Faculty of Mathematics, Kyushu University

https://hdl.handle.net/2324/12559
Asymptotic behavior
of the semigroup associated with
the linearized compressible
Navier-Stokes equation
in an infinite layer

Y. Kagei

MHF 2006-23

(Received June 8, 2006)
Asymptotic behavior of the semigroup associated with the linearized compressible Navier-Stokes equation in an infinite layer

Yoshiyuki Kagei
Faculty of Mathematics, Kyushu University, Fukuoka 812-8581, Japan

Abstract
Asymptotic behavior of solutions to the linearized compressible Navier-Stokes equation around a given constant state is considered in an infinite layer $\mathbb{R}^{n-1} \times (0, a)$, $n \geq 2$, under the no slip boundary condition for the momentum. The $L^p$ decay estimates of the associated semigroup are established for all $1 \leq p \leq \infty$. It is also shown that the time-asymptotic leading part of the semigroup is given by an $n-1$ dimensional heat semigroup.

1. Introduction

This paper is concerned with the large time behavior of solutions to the following system of equations:

\[(1.1) \quad \partial_t u + Lu = 0,\]

where $u = \begin{pmatrix} \phi \\ m \end{pmatrix}$ with $\phi = \phi(x, t) \in \mathbb{R}$ and $m = T(m^1(x, t), \cdots, m^n(x, t)) \in \mathbb{R}^n$, $n \geq 2$, and $L$ is an operator defined by

\[
L = \begin{pmatrix} 0 & \gamma \text{div} \\ \gamma \nabla & -\nu \Delta I_n - \tilde{\nu} \nabla \text{div} \end{pmatrix}
\]

with positive constants $\nu$ and $\gamma$ and a nonnegative constant $\tilde{\nu}$. Here $t > 0$ denotes the time variable and $x \in \mathbb{R}^n$ denotes the space variable; the superscript $^T$ stands for the transposition; $I_n$ is the $n \times n$ identity matrix;
and div, \( \nabla \) and \( \Delta \) are the usual divergence, gradient and Laplacian with respect to \( x \). We consider (1.1) in an infinite layer

\[
\Omega = \mathbb{R}^{n-1} \times (0, a) = \{ x = \left( \begin{array}{c} x' \\ x_n \end{array} \right); \ x' \in \mathbb{R}^{n-1}, \ 0 < x_n < a \}
\]

under the boundary condition

\[
(1.2) \quad m|_{\partial \Omega} = 0,
\]

together with the initial condition

\[
(1.3) \quad u|_{t=0} = u_0 = \left( \begin{array}{c} \phi_0 \\ m_0 \end{array} \right).
\]

Problem (1.1)--(1.3) is obtained by the linearization of the compressible Navier-Stokes equation around a motionless state with a positive constant density, where \( \phi \) is the perturbation of the density and \( m \) is the momentum.

In [6] we showed that \(-L\) generates the analytic semigroup \( \mathcal{U}(t) \) in \( W^{1,p} \times L^p \) for \( 1 < p < \infty \). In this paper we establish the \( L^p \) decay estimates of \( \mathcal{U}(t) \) for all \( 1 \leq p \leq \infty \) and derive an asymptotic state of \( \mathcal{U}(t) \) as \( t \to \infty \).

One of the primary factors affecting the large time behavior of solutions to (1.1)--(1.3) is that (1.1) is a symmetric hyperbolic-parabolic system. Due to this structure, solutions of (1.1) exhibit characters of solutions of both wave and heat equations. In the case of the Cauchy problem on the whole space \( \mathbb{R}^n \), detailed descriptions of large time behavior of solutions have been obtained ([4, 5, 10, 12, 13]). Hoff and Zumbrun [4, 5] showed that there appears some interesting interaction of hyperbolic and parabolic aspects of (1.1) in the decay properties of \( L^p \) norms with \( 1 \leq p \leq \infty \). It was shown in [4, 5] that the solution is asymptotically written in the sum of two terms, one is the solution of the heat equation and the other is given by the convolution of the heat kernel and the fundamental solution of the wave equation. The latter one is called the diffusion wave and it decays faster than the heat kernel in \( L^p \) norm for \( p > 2 \) while slower for \( p < 2 \). This decay property of the diffusion wave also appears in the exterior domain problem ([11]). In the case of the half space problem, it was shown in [7, 8] that not only the above mentioned behavior of the diffusion wave appears but also some difference to the Cauchy problem appears in the decay property of the spatial derivatives due to the presence of the unbounded boundary.

There is one more factor that affects the large time behavior of solutions to (1.1)--(1.3). In contrast to the domains mentioned above, the infinite layer \( \Omega \) has a finite thickness in the \( x_n \) direction. This implies that the Poincaré
inequality holds. If one considers, for example, the incompressible Navier-
Stokes equation under the no slip boundary condition (1.2), then it is easy
to see that, by the Poincaré inequality, the $L^2$ norm of the solution tends to
zero exponentially as $t \to \infty$. In the case of the compressible problem (1.1)–
(1.3), the Poincaré inequality still holds for $m$ but not for $\phi$. Therefore, some
different behavior could be expected to happen.

In this paper we will show that the solution $u = \mathcal{U}(t)u_0$ of (1.1)–(1.3)
satisfies

\begin{equation}
\|u(t)\|_{L^p} = O(t^{-\frac{n-1}{2}(1-\frac{1}{p})}), \quad \|u(t) - u^{(0)}(t)\|_{L^p} = O(t^{-\frac{n-1}{2}(1-\frac{1}{p})-\frac{1}{2}})
\end{equation}

for any $1 \leq p \leq \infty$ as $t \to \infty$. Here $u^{(0)} = (\phi^{(0)}(x', t), 0)$ and $\phi^{(0)}(x', t)$ is a
function satisfying

$$\partial_t \phi^{(0)} - \kappa \Delta' \phi^{(0)} = 0, \quad \phi^{(0)}|_{t=0} = \frac{1}{a} \int_0^a \phi_0(x', x_n) \, dx_n,$$

where $\kappa = \frac{a^2 x_2^2}{12a}$ and $\Delta' = \partial_{x_1}^2 + \cdots + \partial_{x_{n-1}}^2$. We note that the leading part
is given by the density component and no hyperbolic feature appears in the
leading part. The precise statement will be given in section 2.

The proof of (1.4) is based on a detailed analysis of the resolvent $(\lambda + L)^{-1}$
associated with (1.1)–(1.3). The resolvent problem in an infinite layer was
studied in [1, 2, 3] for the incompressible Stokes equation. They established
$L^p$ estimates of the resolvent for $1 < p < \infty$, which yields the exponential
decay of the Stokes semigroup in $L^p$ norms as $t \to \infty$. To obtain the resolvent
estimates, they considered the Fourier transform of the resolvent in $x' \in \mathbb{R}^{n-1}$
and applied the Fourier multiplier theorem.

In order to analyze the compressible problem (1.1)–(1.3) we also consider the
Fourier transform $(\lambda + \hat{L}_{\xi'})^{-1}$ of the resolvent in $x' \in \mathbb{R}^{n-1}$, where $\xi' \in \mathbb{R}^{n-1}$
denotes the dual variable. The semigroup $\mathcal{U}(t)$ generated by $-L$ is then
written as $\mathcal{U}(t) = \mathcal{F}_{\xi'}^{-1} \left[ \frac{1}{2\pi} \int_{\mathbb{R}^{n-1}} e^{i\xi \cdot \lambda} (\lambda + \hat{L}_{\xi'})^{-1} \, d\lambda \right]$. In contrast to the case of
the incompressible problem, $(\lambda + \hat{L}_{\xi'})^{-1}$ has different characters between the
cases $|\xi'| >> 1$ and $|\xi'| << 1$. We thus decompose the semigroup $\mathcal{U}(t)$ into
the two parts according to the partition: $|\xi'| \geq r_0$ and $|\xi'| \leq r_0$ for some
$r_0 > 0$.

In [6] we established the estimates of $(\lambda + \hat{L}_{\xi'})^{-1}$ with $|\xi'| \geq r_0$, which will
lead to the exponential decay of the corresponding part of $\mathcal{U}(t)$. In this paper
we study $(\lambda + \hat{L}_{\xi'})^{-1}$ with $|\xi'| << 1$. We regard $\hat{L}_{\xi'}$ as a perturbation from $\hat{L}_0$
to investigate the spectrum of $-L$ near $\lambda = 0$. Combining the spectral
analysis for $|\xi'| << 1$ and the results in [6], we prove the asymptotic behavior
of $u(t) = \mathcal{U}(t)u_0$ described in (1.4).
This paper is organized as follows. In section 2 we introduce some notation and state the main result of this paper. In section 3 we investigate \((\lambda + \tilde{L}_\xi)^{-1}\) with \(|\xi'| << 1\). Section 4 is devoted to the proof of the main result.

2. Main Result

We first introduce some notation which will be used throughout the paper. For a domain \(D\) and \(1 \leq p \leq \infty\) we denote by \(L^p(D)\) the usual Lebesgue space on \(D\) and its norm is denoted by \(\| \cdot \|_{L^p(D)}\). Let \(\ell\) be a nonnegative integer. The symbol \(W^{\ell,p}(D)\) denotes the \(\ell\) th order \(L^p\) Sobolev space on \(D\) with norm \(\| \cdot \|_{W^{\ell,p}(D)}\). When \(p = 2\), the space \(W^{\ell,2}(D)\) is denoted by \(H^\ell(D)\) and its norm is denoted by \(\| \cdot \|_{H^\ell(D)}\). \(C^0_0(D)\) stands for the set of all \(C^\ell\) functions which have compact support in \(D\). We denote by \(W^{1,p}_0(D)\) the completion of \(C^0_0(D)\) in \(W^{1,p}(D)\). In particular, \(W^{1,2}_0(D)\) is denoted by \(H^1_0(D)\).

We simply denote by \(L^p(D)\) (resp., \(W^{\ell,p}(D), H^\ell(D)\)) the set of all vector fields \(m = T(m^1, \cdots, m^n)\) on \(D\) with \(m^j \in L^p(D)\) (resp., \(W^{\ell,p}(D), H^\ell(D)\)), \(j = 1, \cdots, n\), and its norm is also denoted by \(\| \cdot \|_{L^p(D)}\) (resp., \(\| \cdot \|_{W^{\ell,p}(D)}\), \(\| \cdot \|_{H^\ell(D)}\)). For \(u = \left( \begin{array}{c} \phi \\ m \end{array} \right)\) with \(\phi \in W^{k,p}(D)\) and \(m = T(m^1, \cdots, m^n) \in W^{\ell,q}(D)\), we define \(\|u\|_{W^{k,p}(D) \times W^{\ell,q}(D)}\) by \(\|u\|_{W^{k,p}(D) \times W^{\ell,q}(D)} = \|\phi\|_{W^{k,p}(D)} + \|m\|_{W^{\ell,q}(D)}\). When \(k = \ell\) and \(p = q\), we simply write \(\|u\|_{W^{k,p}(D) \times W^{k,p}(D)} = \|u\|_{W^{k,p}(D)}\).

In case \(D = \Omega\) we abbreviate \(L^p(\Omega)\) (resp., \(W^{\ell,p}(\Omega), H^\ell(\Omega)\)) as \(L^p\) (resp., \(W^{\ell,p}, H^\ell\)). In particular, the norm \(\| \cdot \|_{L^p(\Omega)} = \| \cdot \|_{L^p}\) is denoted by \(\| \cdot \|_p\).

In case \(D = (0,a)\) we denote the norm of \(L^p(0,a)\) by \(| \cdot |_p\). The inner product of \(L^2(0,a)\) is denoted by

\[
(f, g) = \int_0^a f(x_n)\overline{g(x_n)} \, dx_n, \quad f, g \in L^2(0,a).
\]

Here \(\overline{\cdot}\) denotes the complex conjugate of \(\cdot\). Furthermore, we define \(\langle \cdot, \cdot \rangle\) and \(\langle \cdot \rangle\) by

\[
\langle f, g \rangle = \frac{1}{a} (f, g) \quad \text{and} \quad \langle f \rangle = \langle f, 1 \rangle = \frac{1}{a} \int_0^a f(x_n) \, dx_n
\]

for \(f, g \in L^2(0,a)\), respectively.

The norms of \(W^{\ell,p}(0,a)\) and \(H^\ell(0,a)\) are denoted by \(| \cdot |_{W^{\ell,p}}\) and \(| \cdot |_{H^\ell}\), respectively.

We often write \(x \in \Omega\) as \(x = \left( \begin{array}{c} x^t \\ x_n \end{array} \right), x^t = T(x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}\). Partial derivatives of a function \(u\) in \(x, x^t, x_n\) and \(t\) are denoted by \(\partial_x u, \partial_{x^t} u, \partial_{x_n} u, \partial_t u\), respectively.
\[ \frac{\partial}{\partial x_n} u \text{ and } \frac{\partial}{\partial t} u, \text{ respectively. We also write higher order partial derivatives of } u \text{ in } x \text{ as } \frac{\partial^k}{\partial x^k} u = (\frac{\partial^\alpha}{\partial x^\alpha} u; |\alpha| = k). \]

We denote the \( k \times k \) identity matrix by \( I_k \). In particular, when \( k = n + 1 \), we simply write \( I \) for \( I_{n+1} \). We also define \((n+1) \times (n+1)\) diagonal matrices \( Q_0 \) and \( \tilde{Q} \) by

\[
Q_0 = \text{diag}(1, 0, \cdots, 0), \quad \tilde{Q} = \text{diag}(0, 1, \cdots, 1).
\]

We then have, for \( u = \begin{pmatrix} \phi \\ m \end{pmatrix} \in \mathbb{R}^{n+1} \),

\[
Q_0 u = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \quad \tilde{Q} u = \begin{pmatrix} 0 \\ m \end{pmatrix}.
\]

We next introduce some notation about integral operators. For a function \( f = f(x') (x' \in \mathbb{R}^{n-1}) \), we denote its Fourier transform by \( \hat{f} \) or \( \mathcal{F} f \):

\[
\hat{f}(\xi') = (\mathcal{F} f)(\xi') = \int_{\mathbb{R}^{n-1}} f(x') e^{-i\xi' \cdot x'} \, dx'.
\]

The inverse Fourier transform is denoted by \( \mathcal{F}^{-1} \):

\[
(\mathcal{F}^{-1} f)(x) = (2\pi)^{-(n-1)} \int_{\mathbb{R}^{n-1}} f(\xi') e^{i\xi' \cdot x'} \, d\xi'.
\]

For a function \( K(x_n, y_n) \) on \((0, a) \times (0, a)\) we will denote by \( Kf \) the integral operator \( \int_0^a K(x_n, y_n) f(y_n) \, dy_n \).

We denote the resolvent set of a closed operator \( A \) by \( \rho(A) \) and the spectrum of \( A \) by \( \sigma(A) \). For \( A \in \mathbb{R} \) and \( \theta \in (\frac{\pi}{2}, \pi) \) we will denote the subset \( \{ \lambda \in \mathbb{C}; |\arg(\lambda - A)| \leq \theta \} \) by \( \Sigma(A, \theta) \):

\[
\Sigma(A, \theta) = \{ \lambda \in \mathbb{C}; |\arg(\lambda - A)| \leq \theta \}.
\]

We now state the main result of this paper. In [6] we showed that \(-L\) generates the analytic semigroup \( \mathcal{U}(t) \) and established the estimates of \( \mathcal{U}(t) \) for \( 0 < t \leq 1 \). As for the large time behavior of \( \mathcal{U}(t) \), we have the following result.

**Theorem 2.1.** Let \( \mathcal{U}(t) \) be the semigroup generated by \(-L\). Then the solution \( u = \mathcal{U}(t) u_0 \) of problem (1.1)–(1.3) is decomposed as

\[
\mathcal{U}(t) u_0 = \mathcal{U}^{(0)}(t) u_0 + \mathcal{U}^{(\infty)}(t) u_0,
\]

where each term on the right-hand side has the following properties.
(i) \( \mathcal{W}(0)(t)u_0 \) is written in the form

\[
\mathcal{W}(0)(t)u_0 = \mathcal{W}(0)(t)u_0 + \mathcal{R}(0)(t)u_0.
\]

Here \( \mathcal{W}(0)(t)u_0 = \begin{pmatrix} \phi(0)(x', t) \\ 0 \end{pmatrix} \) and \( \phi(0)(x', t) \) is a function independent of \( x_n \) and satisfies the following heat equation on \( \mathbb{R}^{n-1} \):

\[
\partial_t \phi(0) - \kappa \Delta' \phi(0) = 0, \quad \phi(0)|_{t=0} = \langle \phi_0(x', \cdot) \rangle,
\]

where \( \kappa = \frac{a^2 \gamma^2}{12r} \) and \( \Delta' = \partial_{x_1}^2 + \cdots + \partial_{x_{n-1}}^2 \). The function \( \mathcal{R}(0)(t)u_0 \) satisfies the following estimate. For any \( 1 \leq p \leq \infty \) and \( \ell = 0, 1 \), there exists a positive constant \( C \) such that

\[
\| \partial_x \mathcal{R}(0)(t)u_0 \|_p \leq Ct^{-\frac{n+1}{2}(1-\frac{1}{p})-\frac{\ell}{2}} \| u_0 \|_1
\]

holds for \( t \geq 1 \). Furthermore, it holds that

\[
\| \partial_x \mathcal{R}(0)(t)\bar{Q}u_0 \|_p \leq Ct^{-\frac{n+1}{2}(1-\frac{1}{p})-1} \| \bar{Q}u_0 \|_1
\]

and

\[
\| \mathcal{R}(0)(t)[\partial_x \bar{Q}u_0] \|_p \leq Ct^{-\frac{n+1}{2}(1-\frac{1}{p})-\frac{\ell}{2}} \| \bar{Q}u_0 \|_1.
\]

(ii) There exists a positive constant \( c \) such that \( \mathcal{W}(\infty)(t)u_0 \) satisfies

\[
\| \partial_x \mathcal{W}(\infty)(t)u_0 \|_p \leq Ce^{-ct} \| u_0 \|_{W^\ell,p \times L^p}, \quad 1 < p < \infty, \quad \ell = 0, 1,
\]

for all \( t \geq 1 \). Furthermore, the following estimates

\[
\| \partial_x \mathcal{W}(\infty)(t)u_0 \|_\infty \leq Ce^{-ct} \| u_0 \|_{H^{\frac{n+1}{2}+\ell} \times H^{\frac{n}{2}+\ell}},
\]

\[
\| \partial_x \mathcal{W}(\infty)(t)u_0 \|_p \leq Ce^{-ct} \| u_0 \|_{W^{\ell+1,p} \times W^\ell,p}, \quad p = 1, \infty,
\]

hold for all \( t \geq 1 \). Here \([q]\) denotes the greatest integer less than or equal to \( q \).

**Remark 2.2.** We have the optimal decay estimate

\[
\| \mathcal{W}(0)(t)u_0 \|_p \leq Ct^{-\frac{n+1}{2}(1-\frac{1}{p})} \| u_0 \|_1
\]

since \( \| \mathcal{W}(0)(t)u_0 \|_p \) decays exactly in the order \( t^{-\frac{n+1}{2}(1-\frac{1}{p})} \). We also note that \( \mathcal{W}(0)(t)\bar{Q}u_0 = 0 \). Therefore, we have the estimate

\[
\| \partial_x \mathcal{W}(0)(t)\bar{Q}u_0 \|_p \leq Ct^{-\frac{n+1}{2}(1-\frac{1}{p})-1} \| \bar{Q}u_0 \|_1
\]

6
for $t \geq 1$.

We will prove Theorem 2.1 in section 4.

3. Spectral analysis for $-L$

The proof of Theorem 2.1 is based on the analysis of the resolvent problem associated with (1.1)–(1.3), which takes the form

\[ (3.1) \quad \lambda u + Lu = f, \]

where $L$ is the operator on $H^1 \times L^2$ defined in (1.1) with domain of definition $D(L) = H^1 \times (H^2 \cap H^1_0)$. To investigate (3.1) we take the Fourier transform in $x' \in \mathbb{R}^{n-1}$. We then have the following boundary value problem for functions $\phi(x_n)$ and $m(x_n)$ on the interval $(0, a)$:

\[ (3.2) \quad \lambda u + \hat{L}_{\xi'}u = f, \]

where $u = \begin{pmatrix} \phi(x_n) \\ m'(x_n) \\ m^n(x_n) \end{pmatrix}$, $f = \begin{pmatrix} f^0(x_n) \\ f'(x_n) \\ f^n(x_n) \end{pmatrix}$, and $\hat{L}_{\xi'}$ is the operator of the form

\[
\hat{L}_{\xi'} = \begin{pmatrix}
0 & i\gamma^T \xi' & \gamma \partial_{x_n} \\
ixi' & \nu(|\xi'|^2 - \partial_{x_n}^2)I_{n-1} + \tilde{\nu} \xi'^T \xi' & -i\tilde{\nu} \xi' \partial_{x_n} \\
\gamma \partial_{x_n} & -i\tilde{\nu} \xi' \partial_{x_n} & \nu(|\xi'|^2 - \partial_{x_n}^2) - \tilde{\nu} \partial_{x_n}^2
\end{pmatrix},
\]

which is a closed operator on $H^1(0, a) \times L^2(0, a)$ with domain of definition $D(\hat{L}_{\xi'}) = H^1(0, a) \times (H^2(0, a) \cap H^1_0(0, a))$.

In [6] we studied $(\lambda + \hat{L}_{\xi'})^{-1}$ with $|\xi'| \geq r$ for any $r > 0$. In this section we investigate the spectrum of $-\hat{L}_{\xi'}$ for $|\xi'| << 1$. We analyze it regarding the problem as a perturbation from the one with $\xi' = 0$.

We write $\hat{L}_{\xi'}$ in the following form:

\[
\hat{L}_{\xi'} = \hat{L}_0 + \sum_{j=1}^{n-1} \xi_j \hat{L}_j^{(1)} + \sum_{j,k=1}^{n-1} \xi_j \xi_k \hat{L}_{jk}^{(2)},
\]

where $\xi' = T(\xi_1, \ldots, \xi_{n-1})$,

\[
\hat{L}_0 = \begin{pmatrix}
0 & 0 & \gamma \partial_{x_n} \\
0 & -\nu \partial_{x_n}^2 I_{n-1} & 0 \\
\gamma \partial_{x_n} & 0 & -\nu_1 \partial_{x_n}^2
\end{pmatrix}, \quad \nu_1 = \nu + \tilde{\nu},
\]

\[ 7 \]
\[ \hat{L}_j^{(1)} = \begin{pmatrix} 0 & i\gamma e'_j & 0 \\ i\gamma e'_j & 0 & -i\nu e'_j \partial_{x_n} \\ 0 & -i\nu e'_j \partial_{x_n} & 0 \end{pmatrix}, \]

\[ \hat{L}_j^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu\delta_{jk}I_{n-1} + \nu e'_j e'_k & 0 \\ 0 & 0 & \nu\delta_{jk} \end{pmatrix}. \]

We will treat \( \hat{L}_\xi \) as a perturbation from \( \hat{L}_0 \). We begin with the analysis of (3.2) with \( \xi' = 0 \):

\[ (\lambda + \hat{L}_0)u = f. \]

We introduce some quantities. For \( k = 1, 2, \ldots \), we set \( a_k = k\pi/a \). We define \( \lambda_{1,k} \) and \( \lambda_{\pm,k} \) by

\[ \lambda_{1,k} = -\nu a_k^2 \]

and

\[ \lambda_{\pm,k} = -\frac{\nu_1}{2} a_k^2 \pm \frac{1}{2} \sqrt{\nu_1^2 a_k^4 - 4\gamma^2 a_k^2} \]

for \( k = 1, 2, \ldots \). An elementary observation shows that \( \lambda_{\pm,k} \) are the two roots of \( \lambda^2 + \nu a_k^2 \lambda + \gamma^2 a_k^2 = 0 \); \( \lambda_{-k} = \overline{\lambda_{+k}} \) with \( \text{Im} \lambda_{+k} = \gamma a_k \sqrt{1 - \frac{\nu_1^2}{\gamma^2} a_k^2} \) when \( a_k < 2\gamma/\nu_1 \) and \( \lambda_{\pm,k} \in \mathbb{R} \) when \( a_k > 2\gamma/\nu_1 \); and it holds that

\[ \lambda_{+k} = -\frac{\gamma^2}{\nu_1} + O(k^{-2}), \quad \lambda_{-k} = -\nu_1 a_k^2 + O(1) \]

as \( k \to \infty \). (See [6, Remarks 3.2 and 3.5].)

**Lemma 3.1.** (i) The spectrum \( \sigma(-\hat{L}_0) \) is given by

\[ \sigma(-\hat{L}_0) = \{0\} \cup \{\lambda_{1,k}\}_{k=1}^{\infty} \cup \{\lambda_{+,k}, \lambda_{-,k}\}_{k=1}^{\infty} \cup \{-\frac{\gamma^2}{\nu_1}\}. \]

Here 0 is an eigenvalue.

(ii) There exist positive numbers \( \eta_0 \) and \( \theta_0 \) with \( \theta_0 \in \left(\frac{\pi}{2}, \pi\right) \) such that the following estimates hold uniformly for \( \lambda \in \rho(-\hat{L}_0) \cap \Sigma(-\eta_0, \theta_0) \):

\[ \left| (\lambda + \hat{L}_0)^{-1} f \right|_{H^\ell \times L^2} \leq \frac{C}{|\lambda|} |f|_{H^\ell \times L^2}, \quad \ell = 0, 1, \]

\[ \left| \partial_{x_n}^\ell \overline{Q}(\lambda + \hat{L}_0)^{-1} f \right|_2 \leq \frac{C}{(|\lambda| + 1)^{1-\frac{1}{2}}} |f|_{H^{\ell-1} \times L^2}, \quad \ell = 1, 2, \]

(8)
\[
|\partial_{x_n}^2 Q_0(\lambda + \hat{L}_0)^{-1} f|_2 \leq \frac{C}{(|\lambda| + 1)^{1+\frac{n}{2}}} |f|_{H^2 \times H^1}.
\]

**Proof.** We write (3.2) with \( \xi' = 0 \) as

\[
(3.4) \quad \lambda m' - \nu \partial_{x_n}^2 m' = f', \quad m'|_{x_n=0,a} = 0,
\]

and

\[
(3.5) \quad \begin{cases} 
\lambda \phi + \gamma \partial_{x_n} m^n = f^0, \\
\lambda m^n - \nu_1 \partial_{x_n}^2 m^n + \gamma \partial_{x_n} \phi = f^n, \quad m^n|_{x_n=0,a} = 0.
\end{cases}
\]

It is easy to see that (3.4) has a unique solution \( m' \in H^2(0,a) \cap H_0^1(0,a) \) for any \( f' \in L^2(0,a) \) if and only if \( \lambda \neq \lambda_{1,k} \) for any \( k = 1, 2, \ldots \). Furthermore, it is also possible to deduce the estimates

\[
|\partial_{x_n}^\ell m'|_2 \leq \frac{C}{(|\lambda| + 1)^{1-\frac{\ell}{2}}} |f'|_2, \quad \ell = 0, 1, 2,
\]

uniformly in \( \lambda = -\frac{x^2}{2a^2} + \eta e^{\pm i\theta} \) with \( \eta \geq 0 \) and \( \theta \in [0, \theta_0) \). Here \( \theta_0 \) is any fixed constant in \( (\frac{\pi}{2}, \pi) \) and \( C \) is a positive constant depending only on \( \theta_0 \).

We next consider (3.5). Let \( \lambda = 0 \) and \( f^0 = f^n = 0 \) in (3.5). We see from the first equation of (3.5) that \( \partial_{x_n} m^n = 0 \). Then the boundary condition \( m^n|_{x_n=0,a} = 0 \) implies that \( m^n = 0 \). It follows from the second equation of (3.5) that \( \phi \) is a constant. Therefore, 0 is an eigenvalue and the geometric eigenspace is spanned by \( \psi^{(0)} = T(1, 0, \ldots, 0) \).

Let \( \lambda \neq 0 \) in (3.5). We then see that problem (3.5) is equivalent to

\[
(3.6) \quad \phi = \frac{1}{\lambda} \left\{ f^0 - \gamma \partial_{x_n} m^n \right\},
\]

\[
(3.7) \quad \lambda^2 m^n - (\nu_1 \lambda + \gamma^2) \partial_{x_n}^2 m^n = \lambda f^n - \gamma \partial_{x_n} f^0, \quad m^n|_{x_n=0,a} = 0.
\]

In case \( \nu_1 \lambda + \gamma^2 = 0 \), it is easy to see that problem (3.6)–(3.7) has only the trivial solution \( \phi = m^n = 0 \) for \( f^0 = f^n = 0 \). For general \( f^0 \in H^1(0,a) \) and \( f^n \in L^2(0,a) \), (3.7) implies that \( m^n = \lambda^{-2} \left\{ \lambda f^n - \gamma \partial_{x_n} f^0 \right\} \) which is not necessarily in \( H^1(0,a) \). This implies that \( -\frac{\gamma^2}{\nu_1} \in \sigma(\hat{L}_0) \).

Let us consider the case \( \lambda \neq 0 \) and \( \nu_1 \lambda + \gamma^2 \neq 0 \). In this case, (3.7) is equivalent to

\[
(3.8) \quad \sigma m^n - \partial_{x_n}^2 m^n = \frac{1}{\nu_1 \lambda + \gamma^2} \left\{ \lambda f^n - \gamma \partial_{x_n} f^0 \right\}, \quad m^n|_{x_n=0,a} = 0,
\]
where \( \sigma = \frac{\lambda^2}{\nu_1 \lambda + \gamma} \). Since \( \lambda f^n - \gamma \partial_{x_n} f^0 \in L^2(0, a) \), problem (3.8) has a unique solution \( m^n \in H^2(0, a) \cap H_0^1(0, a) \) if and only if \( \sigma \neq -a_k^2 \) for any \( k = 1, 2, \ldots \), namely, \( (\lambda - \lambda_{+,k})(\lambda - \lambda_{-,k}) \neq 0 \) for any \( k = 1, 2, \ldots \). If (3.8) has a solution \( m^n \in H^2(0, a) \cap H_0^1(0, a) \), then (3.6) determines \( \phi \) which is in \( H^1(0, a) \). Consequently we see that \( \sigma(-\hat{L}_0) = \{0\} \cup \{\lambda_{1,k}\}_{k=1}^\infty \cup \{\lambda_{+,k}, \lambda_{-,k}\}_{k=1}^\infty \cup \{-\frac{\gamma}{\nu_1}\} \).

We next derive estimates for \( \phi \) and \( m^n \) uniformly in \( \lambda \in \rho(-\hat{L}_0) \cap \Sigma(-\eta_0, \theta_0) \) with suitable \( \eta_0 \) and \( \theta_0 \). To do so, we expand the solution \( m^n \) of (3.8) into the Fourier sine series \( m^n = \sum_{k=1}^\infty m_k \sin \omega_k x_n \). It is easy to see that the Fourier coefficients \( m_k^n \) are given by

\[
m_k^n = \frac{1}{\sigma + a_k^2} \frac{1}{\nu_1 \lambda + \gamma^2} \left\{ \lambda f_k^n + \gamma a_k f_k^0 \right\}
\]

for \( k = 1, 2, \ldots \), where \( f_k^0 \) and \( f_k^n \) are the coefficients of the Fourier cosine and sine series expansion of \( f^0 \) and \( f^n \), respectively.

Since \( (\sigma + a_k^2)(\nu_1 \lambda + \gamma^2) = (\lambda - \lambda_{+,k})(\lambda - \lambda_{-,k}) \), we have

\[
|m_k^n|^2 \leq C \sum_{k=1}^\infty \frac{1}{|\lambda - \lambda_{+,k}|(\lambda - \lambda_{-,k})|^2} \left\{ \lambda^2 |f_k^n|^2 + a_k^2 |f_k^0|^2 \right\}.
\]

It then follows from (3.3) that there are positive numbers \( \eta_0 \) and \( \theta_0 \in (\pi, \pi) \) such that, for \( \lambda \) with \( \text{arg} (\lambda + \eta_0) \leq \theta_0 \),

\[
|m_k^n|^2 \leq C \sum_{k=1}^\infty \frac{1}{(|\lambda| + 1)^2(|\lambda| + k^2)^2} \left\{ \lambda^2 |f_k^n|^2 + a_k^2 |f_k^0|^2 \right\}
\]

\[
\leq \frac{C|f|^2_2}{(|\lambda| + 1)^2}.
\]

This, together with (3.8), then implies that

\[
|\partial_{x_n}^2 m^n|_2 \leq |\sigma| |m^n|_2 + \frac{|\lambda|}{|\nu_1 \lambda + \gamma^2|} |f^n|_2 + \frac{1}{|\nu_1 \lambda + \gamma^2|} |\partial_{x_n} f^0|_2
\]

\[
\leq C\|f\|_{H^1 \times L^2}
\]

uniformly in \( \lambda \) with \( \text{arg} (\lambda + \eta_0) \leq \theta_0 \). Taking the \( L^2 \) inner product of (3.8) with \( m^n \) and integrating by parts, we have

\[
|\partial_{x_n} m^n|_2^2 \leq C \left\{ |\sigma| |m^n|_2^2 + |f^n|_2 |m^n|_2 + \frac{1}{|\lambda| + 1} |f^0|_2 |\partial_{x_n} m^n|_2 \right\}
\]

\[
\leq \frac{C|f|^2_2}{|\lambda| + 1} + \frac{1}{2} |\partial_{x_n} m^n|_2^2
\]
uniformly in $\lambda$ with $|\text{arg}(\lambda + \eta_0)| \leq \theta_0$, and hence, $|\partial_{x_n}m^n|_2 \leq \frac{C|f|_2}{(|\lambda|+1)^{1-\frac{\ell}{2}}}$. Consequently, we have

\begin{equation}
|\partial_{x_n}^\ell m^n|_2 \leq \frac{C|f|_{H^{(\ell-1)+\times L^2}}}{(|\lambda|+1)^{1-\frac{\ell}{2}}}
\end{equation}

for $\ell = 0, 1, 2$ uniformly in $\lambda$ with $|\text{arg}(\lambda + \eta_0)| \leq \theta_0$. It then follows from (3.6) and (3.9) that

$$|\phi|_2 \leq \frac{1}{|\lambda|} \left\{ |f^0|_2 + \gamma |\partial_{x_n}m^n|_2 \right\} \leq \frac{C}{|\lambda|}|f|_2.$$

We next estimate the derivatives of $\phi$. Differentiating the first equation of (3.5) we have

\begin{equation}
\lambda \partial_{x_n}\phi + \gamma \partial_{x_n}^2 m^n = \partial_{x_n} f^0
\end{equation}

We see from the second equation of (3.5) that

\begin{equation}
-\nu_1 \partial_{x_n}^2 m^n + \gamma \partial_{x_n} \phi = f^n - \lambda m^n.
\end{equation}

By adding (3.11) $\times \frac{\gamma}{\nu_1}$ to (3.10) we obtain

$$\left(\lambda + \frac{\gamma^2}{\nu_1}\right) \partial_{x_n}^{\ell+1} \phi = \partial_{x_n}^{\ell+1} f^0 + \frac{\gamma}{\nu_1} \left\{ \partial_{x_n}^{\ell} f^n - \lambda \partial_{x_n}^{\ell} m^n \right\}, \ \ell = 0, 1.$$

This, together with (3.9), implies that

$$|\partial_{x_n}^{\ell+1} \phi|_2 \leq \frac{C}{|\lambda|+1} \left\{ |\partial_{x_n}^{\ell+1} f^0|_2 + |\partial_{x_n}^{\ell} f^n|_2 + |\lambda| |\partial_{x_n}^{\ell} m^n|_2 \right\} \leq \frac{C}{(|\lambda|+1)^{1-\frac{\ell}{2}}} |f|_{H^{\ell+1}\times H^\ell}, \ \ell = 0, 1,$$

for $\lambda$ with $|\text{arg}(\lambda + \eta_0)| \leq \theta_0$, by changing $\eta_0 > 0$ and $\theta_0 \in (\frac{\pi}{2}, \pi)$ suitably if necessary. This completes the proof.

We next investigate the eigenvalue 0 of $-\hat{L}_0$.

**Lemma 3.2.** The eigenvalue 0 of $-\hat{L}_0$ is simple and the associated eigenprojection is given by

$$\hat{P}^{(0)} u = \begin{pmatrix} \langle \phi \rangle \\ 0 \end{pmatrix} \text{ for } u = \begin{pmatrix} \phi \\ m \end{pmatrix}. $$
Proof. To show the simplicity of the eigenvalue 0, let us first consider the problem
\[ \hat{L}_0 u = \psi^{(0)}, \]
where \( \psi^{(0)} = \hat{T}(1, 0, \cdots, 0) \) is an eigenfunction for the eigenvalue 0. This problem is equivalent to (3.4)–(3.5) with \( \lambda = 0, f' = 0, f^0 = 1, f^n = 0 \). By (3.4), we have \( m' = 0 \), and by the first equation of (3.5), we have \( m^n = \frac{1}{\gamma} x_n + c \) for some constant \( c \). There is no such \( m^n \) satisfying the boundary condition \( m^n|_{x_n=a} = 0 \). Therefore, 0 is a simple eigenvalue.

Let us prove that the eigenprojection \( \hat{\Pi}^{(0)} \) has the desired form. Since \( \dim \text{Range} \hat{\Pi}^{(0)} = 1 \), we have
\[ \hat{\Pi}^{(0)} u = cu \psi^{(0)} \]
for some \( c \in \mathbb{C} \). It then follows that
\[ \langle \hat{\Pi}^{(0)} u, \psi^{(0)} \rangle = cu. \]

Consider now the formal adjoint problem
\[ \lambda u + \hat{L}_0^* u = 0, \]
where
\[ \hat{L}_0^* = \begin{pmatrix} 0 & 0 & -\gamma \partial x_n \\ 0 & -\nu \partial^2 x_{n-1} & 0 \\ -\gamma \partial x_n & 0 & -\nu_1 \partial^2 x_n \end{pmatrix}. \]
with domain of definition \( D(\hat{L}_0^*) = D(\hat{L}_0) \). Similarly to above, we can see that \( \sigma(-\hat{L}_0^*) = \sigma(-\hat{L}_0) \), and, in particular, 0 is a simple eigenvalue and \( \hat{L}_0^* \psi^{(0)} = 0 \). Furthermore, let \( \hat{\Pi}^{(0)*} \) be the eigenprojection for the eigenvalue 0 of \( -\hat{L}_0^* \). Then we have
\[ \langle \hat{\Pi}^{(0)} u, \psi^{(0)} \rangle = \frac{1}{a} \int_0^a \left( \frac{1}{2\pi i} \int_G \hat{G}^{(0)}(\lambda, x_n, y_n) u(y_n) dy_n d\lambda \right) \psi^{(0)}(x_n) dx_n = \langle \hat{\Pi}^{(0)*} \psi^{(0)} \rangle = \langle \hat{\Pi}^0 \psi^{(0)} \rangle = \langle \phi \rangle = \langle \phi \rangle. \]
for \( u = \left( \begin{array}{c} \phi \\ m \end{array} \right) \). This, together with (3.12), gives the desired expression of \( \hat{H}^{(0)} \). This completes the proof.

We next estimate \((\lambda + \tilde{L}_{\xi'})^{-1}\) for small \( \xi' \). Based on Lemma 3.1 we obtain the following estimates.

**Theorem 3.3.** Let \( \eta_0 \) and \( \theta_0 \) be the numbers given in Lemma 3.1. Then there exists a positive number \( \tilde{r}_0 = \tilde{r}_0(\eta_0, \theta_0) \) such that the set \( \Sigma(-\eta_0, \theta_0) \cap \{ \lambda; |\lambda| \geq \frac{\eta_0}{2} \} \) is in \( \rho(-\tilde{L}_{\xi'}) \) for \( |\xi'| \leq \tilde{r}_0 \). Furthermore, the following estimates hold for any multi-index \( \alpha' \) with \( |\alpha'| \leq n \) uniformly in \( \lambda \in \Sigma(-\eta_0, \theta_0) \cap \{ \lambda; |\lambda| \geq \frac{\eta_0}{2} \} \) and \( \xi' \) with \( |\xi'| \leq \tilde{r}_0 \):

\[
|\partial_{\xi'}^\alpha (\lambda + \tilde{L}_{\xi'})^{-1} f|_{H^\ell \times L^2} \leq \frac{C}{|\lambda|} |f|_{H^\ell \times L^2}, \quad \ell = 0, 1,
\]

\[
|\partial_{\xi'}^\alpha \partial_{x_n} \tilde{Q} (\lambda + \tilde{L}_{\xi'})^{-1} f|_2 \leq \frac{C}{(|\lambda| + 1)^{1+\frac{\ell}{2}}} |f|_{H^{\ell-1} \times L^2}, \quad \ell = 1, 2,
\]

\[
|\partial_{\xi'}^\alpha \partial_{x_n}^2 Q_0 (\lambda + \tilde{L}_{\xi'})^{-1} f|_2 \leq \frac{C}{(|\lambda| + 1)^{\frac{\ell}{2}}} |f|_{H^2 \times H^1}.
\]

**Proof.** In the following we will write

\[
\hat{L}^{(1)}(\xi') = \sum_{j=1}^{n-1} \xi_j \hat{L}_j^{(1)} \quad \text{and} \quad \hat{L}^{(2)}(\xi') = \sum_{j,k=1}^{n-1} \xi_j \xi_k \hat{L}_{jk}^{(2)}.
\]

We first observe that

\[
|\hat{L}_j^{(1)} u|_{H^\ell \times H^{(\ell-1)+}} \leq C \left\{ |Q_0 u|_{H^{(\ell-1)+}} + |\tilde{Q} u|_{H^{(\ell-1)+}} \right\}
\]

and

\[
|\hat{L}_{jk}^{(2)} u|_{H^\ell \times H^{(\ell-1)+}} \leq C |\tilde{Q} u|_{H^{(\ell-1)+}}.
\]

It then follows from Lemma 3.1 and (3.14) that

\[
|\hat{L}_{jk}^{(2)} (\lambda + \tilde{L}_0)^{-1} f|_{H^\ell \times L^2} \leq C |\tilde{Q} (\lambda + \tilde{L}_0)^{-1} f|_2 \leq C |f|_2
\]

for \( \ell = 0, 1 \) and \( \lambda \in \Sigma(-\eta_0, \theta_0) \cap \{ \lambda; |\lambda| \geq \frac{\eta_0}{2} \} \) with \( C = C(\eta_0, \theta_0) > 0 \). Also, by Lemma 3.1 and (3.13), we have

\[
|\hat{L}_j^{(1)} (\lambda + \tilde{L}_0)^{-1} f|_{H^\ell \times L^2} \leq C |f|_2
\]
for \( \ell = 0, 1 \) and \( \lambda \in \Sigma(-\eta_0, \theta_0) \cap \{ \lambda; |\lambda| \geq \frac{\eta_0}{2} \} \) with \( C = C(\eta_0, \theta_0) > 0 \). It then follows that there exists a positive number \( \tilde{r}_0 \) such that if \( |\xi'| \leq \tilde{r}_0 \), then

\[
\left| \left( \hat{\Delta}^{(1)}(\xi') + \hat{\Delta}^{(2)}(\xi') \right) (\lambda + \hat{\Delta}_0)^{-1} f \right|_{H^{\ell} \times L^2} \leq \frac{1}{2} |f|_2
\]

for \( \ell = 0, 1 \) and \( \lambda \in \Sigma(-\eta_0, \theta_0) \cap \{ \lambda; |\lambda| \geq \frac{\eta_0}{2} \} \). By the Neumann series expansion, we see that \( I + \left( \hat{\Delta}^{(1)}(\xi') + \hat{\Delta}^{(2)}(\xi') \right) (\lambda + \hat{\Delta}_0)^{-1} \) is invertible on \( H^0(0, a) \times L^2(0, a) \), \( \ell = 0, 1 \), for \( \lambda \in \Sigma(-\eta_0, \theta_0) \cap \{ \lambda; |\lambda| \geq \frac{\eta_0}{2} \} \) and \( \xi' \) with \( |\xi'| \leq \tilde{r}_0 \). In particular, we conclude that \( \Sigma(-\eta_0, \theta_0) \cap \{ \lambda; |\lambda| \geq \frac{\eta_0}{2} \} \subset \rho(-\hat{\Delta}_{\xi'}) \) and

\[
(3.17) \quad (\lambda + \hat{\Delta}_{\xi'})^{-1} = (\lambda + \hat{\Delta}_0)^{-1} \sum_{N=0}^{\infty} (-1)^N \left[ \left( \hat{\Delta}^{(1)}(\xi') + \hat{\Delta}^{(2)}(\xi') \right) (\lambda + \hat{\Delta}_0)^{-1} \right]^N f
\]

for \( \lambda \in \Sigma(-\eta_0, \theta_0) \cap \{ \lambda; |\lambda| \geq \frac{\eta_0}{2} \} \) and \( \xi' \) with \( |\xi'| \leq \tilde{r}_0 \). Furthermore, we see from Lemma 3.1, (3.13) and (3.14) that

\[
\left| \partial_{\xi'} (\lambda + \hat{\Delta}_{\xi'})^{-1} f \right|_{H^{\ell} \times L^2} \leq \frac{C}{|\lambda|} \left| \partial_{\xi'} \sum_{N=0}^{\infty} (-1)^N \left[ \left( \hat{\Delta}^{(1)}(\xi') + \hat{\Delta}^{(2)}(\xi') \right) (\lambda + \hat{\Delta}_0)^{-1} \right]^N f \right|_{H^{\ell} \times L^2}
\]

\[
\leq \frac{C}{|\lambda|} |f|_{H^{\ell} \times L^2}, \quad \ell = 0, 1.
\]

Similarly, we have, for \( \ell = 1, 2 \),

\[
\left| \partial_{\xi'} \partial_{\xi_n} Q (\lambda + \hat{\Delta}_{\xi'})^{-1} f \right|_2 \leq \frac{C}{(|\lambda| + 1)^{1 + \frac{\ell}{2}}} \left| \partial_{\xi'} \sum_{N=0}^{\infty} (-1)^N \left[ \left( \hat{\Delta}^{(1)}(\xi') + \hat{\Delta}^{(2)}(\xi') \right) (\lambda + \hat{\Delta}_0)^{-1} \right]^N f \right|_{H^{\ell-1} \times L^2} \leq \frac{C}{(|\lambda| + 1)^{1 + \frac{\ell}{2}}} |f|_{H^{\ell-1} \times L^2}.
\]

Let us estimate \( \partial_{\xi_n}^2 Q_0 (\lambda + \hat{\Delta}_{\xi'})^{-1} f \). We see from Lemma 3.1, (3.13) and (3.14) that

\[
\left| \hat{\Delta}_{j1} (\lambda + \hat{\Delta}_0)^{-1} f \right|_{H^2 \times H^1} \leq C |f|_{H^1 \times L^2}
\]

and

\[
\left| \hat{\Delta}_{j2} (\lambda + \hat{\Delta}_0)^{-1} f \right|_{H^2 \times H^1} \leq C |f|_2
\]
uniformly for \( \lambda \in \Sigma(-\eta_0, \theta_0) \cap \{ \lambda; |\lambda| \geq \frac{\eta_0}{2} \} \). Therefore, taking \( \bar{r}_0 \) smaller if necessary, we have

\[
\left| (\tilde{L}^{(1)}(\xi') + \tilde{L}^{(2)}(\xi')) (\lambda + \hat{L}_0)^{-1} f \right|_{H^2 \times H^1} \leq \frac{1}{2} |f|_{H^2 \times L^2}
\]

for \( \lambda \in \Sigma(-\eta_0, \theta_0) \cap \{ \lambda; |\lambda| \geq \frac{\eta_0}{2} \} \) and \( \xi' \) with \( |\xi'| \leq \bar{r}_0 \). It then follows from Lemma 3.1 and (3.17) that

\[
\left| \partial_{\xi'} \partial_{\xi_n}^2 Q_0(\lambda + \hat{L}_{\xi'})^{-1} f \right|_2 \leq \frac{C}{(|\lambda| + 1)^{\frac{1}{2}}} \left| \partial_{\xi'} \sum_{N=0}^{\infty} (-1)^N \left[ (\tilde{L}^{(1)}(\xi') + \tilde{L}^{(2)}(\xi')) (\lambda + \hat{L}_0)^{-1} \right]^N f \right|_{H^2 \times H^1}
\]

This completes the proof.

We next investigate the spectrum of \(-\hat{L}_{\xi'}\) near \( \lambda = 0 \).

**Theorem 3.4.** Let \( \eta_0 \) and \( \bar{r}_0 \) be the numbers given in Theorem 3.3. Then there exists a positive number \( r_0 \) with \( r_0 \leq \bar{r}_0 \) such that for each \( \xi' \) with \( |\xi'| \leq \bar{r}_0 \) it holds that

\[
\sigma(-\tilde{L}_{\xi'}) \cap \{ \lambda; |\lambda| \leq \eta_0 \} = \{ \lambda_0(\xi') \},
\]

where \( \lambda_0(\xi') \in \mathbb{R} \) and \( \lambda_0(\xi') \) is a simple eigenvalue of \(-\tilde{L}_{\xi'}\) that has the form

\[
\lambda_0(\xi') = -\frac{a^2 \gamma^2}{12 \nu} |\xi'|^2 + O(|\xi'|^4)
\]
as \( |\xi'| \to 0 \).

**Proof.** By Theorem 3.3, (3.13) and (3.14), we see that if \( |\lambda| = \eta_0 \), then \( \lambda \in \rho(-\tilde{L}_{\xi'}) \) for \( |\xi'| \leq \bar{r}_0 \). In particular,

\[
\tilde{H}(\xi') = \frac{1}{2\pi i} \int_{|\lambda| = \eta_0} (\lambda + \hat{L}_{\xi'})^{-1} d\lambda
\]
is the eigenprojection for the eigenvalues of \(-\tilde{L}_{\xi'}\) lying inside the circle \( |\lambda| = \eta_0 \). The perturbation theory then implies that dim Range \( \tilde{H}(\xi') \) = dim Range \( \tilde{H}^{(0)} \) = 1. Therefore, we see from Lemma 3.2 that \( \sigma(-\tilde{L}_{\xi'}) \cap \{ \lambda; |\lambda| \leq \eta_0 \} \) consists of only one simple eigenvalue, say \( \lambda_0(\xi') \).
To show that $\lambda_0(\xi')$ has the desired asymptotic form, we first observe that $\lambda$ is an eigenvalue of $-\hat{L}_{\xi'}$ if and only if it is an eigenvalue of $-\hat{L}_{T^2\xi'}$ for any $(n-1) \times (n-1)$ orthogonal matrix $T'$, since $\hat{L}_{\xi'} = T^{-1}\hat{L}_{T^2\xi'}T$, where

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & T' & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

It then follows that $\lambda_0(\xi')$ is a function of $|\xi'|$, and hence, it suffices to consider $\hat{L}_{\xi'}$ with $\xi' = \eta e_1'$, where $\eta \in \mathbb{R}$ and $|\eta| = |\xi'|$.

We write $\hat{L}_{\xi'}$ with $\xi' = \eta e_1'$ as $\tilde{L}_\eta$, and $\tilde{L}_\eta = \hat{L}_0 + \eta \hat{L}_1^{(1)} + \eta^2 \hat{L}_1^{(2)}$. We also denote the corresponding eigenvalue by $\lambda_0(\eta)$. With this $\tilde{L}_\eta$, taking $T' = -I_{n-1}$, we see that $\lambda_0(\eta) = \lambda_0(-\eta)$ since $\lambda_0(\xi')$ is simple. Furthermore, we have a relation $\overline{L_\eta u} = \tilde{L}_{-\eta} \overline{u}$, which implies that $\lambda_0(\eta) = \lambda_0(-\eta) = \lambda_0(\eta)$. This means that $\lambda_0(\eta) \in \mathbb{R}$.

In view of (3.13) and (3.14) we can apply the analytic perturbation theory [9, Chap. 2 and 7] to see that

$$\lambda_0(\eta) = \lambda^{(0)} + \eta \lambda^{(1)} + \eta^2 \lambda^{(2)} + \eta^3 \lambda^{(3)} + O(\eta^4)$$

with $\lambda^{(0)} = 0$. Since $\lambda_0(\eta) = \lambda_0(-\eta)$, we have $\lambda^{(1)} = \lambda^{(3)} = 0$. The coefficient $\lambda^{(2)}$ of $\eta^2$ is given by

$$\lambda^{(2)} = -\langle \tilde{L}_1^{(2)} \psi^{(0)}, \psi^{(0)} \rangle + \langle \tilde{L}_1^{(1)} S \tilde{L}_1^{(1)} \psi^{(0)}, \psi^{(0)} \rangle,$$

where $S = [(I - \hat{P}^{(0)}) \hat{L}_0 (I - \hat{P}^{(0)})]^{-1}$. It is easy to see that $\tilde{L}_1^{(2)} \psi^{(0)} = 0$.

Let us compute $\langle \tilde{L}_1^{(1)} S \tilde{L}_1^{(1)} \psi^{(0)}, \psi^{(0)} \rangle$. Since $\tilde{L}_1^{(1)} \psi^{(0)} = i\gamma \begin{pmatrix} 0 \\ e_1' \\ 0 \end{pmatrix}$, we have

$$S \tilde{L}_1^{(1)} \psi^{(0)} = \frac{i\gamma}{\nu} \begin{pmatrix} 0 \\ e_1' \\ 0 \end{pmatrix} (-\partial_{x_n}^2)^{-1} \cdot 1,$$

and hence,

$$\tilde{L}_1^{(1)} S \tilde{L}_1^{(1)} \psi^{(0)} = -\begin{pmatrix} \gamma^2 (\partial_{x_n}^2)^{-1} \cdot 1 \\ 0 \\ -\frac{\nu}{\nu} \partial_{x_n} (\partial_{x_n}^2)^{-1} \cdot 1 \end{pmatrix}.$$

Here $(-\partial_{x_n}^2)^{-1}$ denotes the inverse of $-\partial_{x_n}^2$ under the 0-Dirichlet boundary condition at $x_n = 0, a$. We thus obtain

$$\langle \tilde{L}_1^{(1)} S \tilde{L}_1^{(1)} \psi^{(0)}, \psi^{(0)} \rangle = -\frac{\gamma^2}{\nu} \langle (\partial_{x_n}^2)^{-1} \cdot 1 \rangle = -\frac{\gamma^2 a^2}{12 \nu}.$$

16
Consequently, we obtain
\[ \lambda_0(\eta) = -\frac{a^2\gamma^2}{12\nu} \eta^2 + O(\eta^4). \]
This completes the proof.

We next investigate the eigenprojection \( \tilde{\Pi}(\xi') \) associated with \( \lambda_0(\xi') \). To do so, we will consider the formal adjoint problem
\[ \lambda u + \tilde{L}_\xi^* u = f, \]
where \( \tilde{L}_\xi^* \) is an operator of the form
\[
\tilde{L}_\xi^* = \tilde{L}_0^* + \sum_{j=1}^{n-1} \xi_j \tilde{L}_j^{(1)*} + \sum_{j,k=1}^{n-1} \xi_j \xi_k \tilde{L}_j^{(2)*}
\]
with domain of definition \( D(\tilde{L}_\xi^*) = D(\tilde{L}_\xi') \). Here \( \xi' = (\xi_1, \cdots, \xi_{n-1}) \),
\[
\tilde{L}_0^* = \begin{pmatrix} 0 & 0 & -\gamma \partial_{x_n} \\ 0 & -\nu \partial_{x_n}^2 I_{n-1} & 0 \\ -\gamma \partial_{x_n} & 0 & -\nu \partial_{x_n}^2 \end{pmatrix},
\]
\[
\tilde{L}_j^{(1)*} = \begin{pmatrix} 0 & -i\gamma \tilde{e}_j' \\ -i\gamma \tilde{e}_j' & 0 & -i\tilde{\nu} \tilde{e}_j' \partial_{x_n} \\ 0 & -i\tilde{\nu} \tilde{e}_j' \partial_{x_n} & 0 \end{pmatrix},
\]
\[
\tilde{L}_j^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ \nu \delta_{jk} I_{n-1} + \tilde{\nu} \tilde{e}_j' \tilde{e}_j' & 0 \\ 0 & 0 & \nu \delta_{jk} \end{pmatrix}.
\]

**Theorem 3.5.** Let \( \tilde{\Pi}(\xi') \) be the eigenprojection associated with \( \lambda_0(\xi') \). Then there exists a positive number \( r_0 \) such that, for any \( \xi' \) with \( |\xi'| \leq r_0 \), \( \tilde{\Pi}(\xi') \) is written in the form
\[
\tilde{\Pi}(\xi')u = \int_0^a \tilde{\Pi}(\xi', x_n, y_n)u(y_n) \, dy_n
\]
with
\[
\tilde{\Pi}(\xi', x_n, y_n) = \tilde{\Pi}^{(0)} + \sum_{j=1}^{n-1} \xi_j \tilde{\Pi}_{j}^{(1)}(x_n, y_n) + \tilde{\Pi}_{j}^{(2)}(\xi', x_n, y_n).
\]
Here \( \hat{\Pi}^{(0)} = \frac{1}{a} Q_0 \); \( \hat{\Pi}^{(1)}_j \in W^{1,\infty}((0,a) \times (0,a)) \), \( j = 1, \ldots, n - 1; \) and \( \hat{\Pi}^{(2)}(\xi', x_n, y_n) \) satisfies
\[
|\partial_{\xi'}^{\alpha'} \hat{\Pi}^{(2)}(\xi', \cdot')|_{W^{1,\infty}((0,a) \times (0,a))} \leq C|\xi'|^{2-|\alpha'|}
\]
for any multi-index \( \alpha' \) with \(|\alpha'| \leq n \) uniformly in \( \xi' \) with \(|\xi'| \leq r_0 \). Furthermore, \( \hat{\Pi}(\xi') \) has the properties
\[
\hat{\Pi}(\xi') [\partial_{x_n} \bar{Q} u] = -\sum_{j=1}^{n-1} \xi_j \left( \partial_{y_n} \hat{\Pi}^{(1)}_j(\xi') \right) \left[ \bar{Q} u \right] - \left( \partial_{y_n} \hat{\Pi}^{(2)}(\xi') \right) \left[ \bar{Q} u \right]
\]
and
\[
\partial_{x_n} \hat{\Pi}(\xi') \bar{Q} u = \partial_{x_n} \hat{\Pi}^{(2)}(\xi') \bar{Q} u.
\]

**Proof.** By (3.13)–(3.17) we see that \((\lambda + \hat{L}_{\xi'})^{-1} \) has the form
\[
(\lambda + \hat{L}_{\xi'})^{-1} = (\lambda + \hat{L}_0)^{-1} + \sum_{j=1}^{n-1} \xi_j (\lambda + \hat{L}_0)^{-1} \hat{L}^{(1)}_j (\lambda + \hat{L}_0)^{-1} + \tilde{R}(\lambda, \xi'),
\]
where
\[
\tilde{R}(\lambda, \xi')
\]
\[
= (\lambda + \hat{L}_0)^{-1} \hat{L}^{(2)}(\xi')(\lambda + \hat{L}_0)^{-1}
\]
\[
+ (\lambda + \hat{L}_0)^{-1} \sum_{N=2}^{\infty} (-1)^N \left[ \left( \hat{L}^{(1)}_0(\xi') + \hat{L}^{(2)}(\xi') \right)(\lambda + \hat{L}_0)^{-1} \right]^N
\]
and \( \tilde{R}(\lambda, \xi') \) satisfies
\[
|\partial_{\xi'}^{\alpha'} \tilde{R}(\lambda, \xi') f|_{H^2} \leq C|\xi'|^{2-|\alpha'|} |f|_{H^2 \times H^1}.
\]
Similarly, one can prove that
\[
(\lambda + \hat{L}_{\xi'}^*)^{-1} = (\lambda + \hat{L}_0^*)^{-1} + \sum_{j=1}^{n-1} \xi_j (\lambda + \hat{L}_0^*)^{-1} \hat{L}^{(1)}_j^* (\lambda + \hat{L}_0^*)^{-1} + \tilde{R}^*(\lambda, \xi'),
\]
where
\[
\tilde{R}^*(\lambda, \xi')
\]
\[
= (\lambda + \hat{L}_0^*)^{-1} \hat{L}^{(2)*}(\xi')(\lambda + \hat{L}_0^*)^{-1}
\]
\[
+ (\lambda + \hat{L}_0^*)^{-1} \sum_{N=2}^{\infty} (-1)^N \left[ \left( \hat{L}^{(1)*}_0(\xi') + \hat{L}^{(2)*}(\xi') \right)(\lambda + \hat{L}_0^*)^{-1} \right]^N
\]
and \( \tilde{R}^*(\lambda, \xi') \) satisfies
\[
(3.21) \quad \left| \partial_{\xi'}^\alpha \tilde{R}^*(\lambda, \xi') \right|_{H^2} \leq C|\xi'|^{2-|\alpha'|} \left| f \right|_{H^2 \times H^1}.
\]
We now define \( \psi(\xi', x_n) \) and \( \tilde{\psi}^*(\xi', x_n) \) by
\[
\psi(\xi', x_n) = \frac{1}{2\pi i} \int_{|\lambda| = \eta_0} (\lambda + \tilde{L}_{\xi'})^{-1} \psi^{(0)} d\lambda
\]
and
\[
\tilde{\psi}^*(\xi', x_n) = \frac{1}{2\pi i} \int_{|\lambda| = \eta_0} (\lambda + \tilde{L}_{\xi'})^{-1} \psi^{(0)} d\lambda,
\]
where \( \psi^{(0)} = T(1,0,\ldots,0) \). It then follows from (3.18)–(3.21) that \( \psi \) and \( \tilde{\psi}^* \) have the form
\[
\psi(\xi', x_n) = \psi^{(0)} + \sum_{j=1}^{n-1} \xi_j \psi_j^{(1)}(x_n) + \psi^{(2)}(\xi', x_n),
\]
(3.22)
\[
\tilde{\psi}^*(\xi', x_n) = \psi^{(0)} + \sum_{j=1}^{n-1} \xi_j \tilde{\psi}_j^{(1)*}(x_n) + \tilde{\psi}^{(2)*}(\xi', x_n),
\]
where \( \psi_j^{(1)}, \tilde{\psi}_j^{(1)*}, \psi^{(2)} \) and \( \tilde{\psi}^{(2)*} \) satisfy
\[
\left| \psi_j^{(1)} \right|_{H^2} + \left| \tilde{\psi}_j^{(1)*} \right|_{H^2} \leq C, \quad j = 1, \ldots, n - 1,
\]
\[
\left| \partial_{\xi'}^\alpha \psi^{(2)}(\xi') \right|_{H^2} + \left| \partial_{\xi'}^\alpha \tilde{\psi}^{(2)*}(\xi') \right|_{H^2} \leq C|\xi'|^{2-|\alpha'|}.
\]
Therefore, we have
\[
\left| \psi_j^{(1)} \right|_{W^{1,\infty}} + \left| \tilde{\psi}_j^{(1)*} \right|_{W^{1,\infty}} \leq C, \quad j = 1, \ldots, n - 1,
\]
and
\[
(3.23) \quad \left| \partial_{\xi'}^\alpha \psi^{(2)}(\xi') \right|_{W^{1,\infty}} + \left| \partial_{\xi'}^\alpha \tilde{\psi}^{(2)*}(\xi') \right|_{W^{1,\infty}} \leq C|\xi'|^{2-|\alpha'|}.
\]
We note that \( \langle \psi(\xi'), \tilde{\psi}^*(\xi') \rangle \) is analytic in \( \xi' \) and
\[
\langle \psi(\xi'), \tilde{\psi}^*(\xi') \rangle = 1 + \sum_{j=1}^{n-1} \xi_j \left\{ \langle \psi^{(0)}, \tilde{\psi}_j^{(1)*} \rangle + \langle \psi_j^{(1)}, \psi^{(0)} \rangle \right\} + \tilde{\psi}^{(2)}(\xi'),
\]
where \( \tilde{\psi}^{(2)}(\xi') \) satisfies \( \left| \partial_{\xi'}^\alpha \tilde{\psi}^{(2)}(\xi') \right| \leq C|\xi'|^{2-|\alpha'|} \). In particular, taking \( r_0 \) smaller if necessary, we see that
\[
\left| \langle \psi(\xi'), \tilde{\psi}^*(\xi') \rangle \right| \geq \frac{1}{2}
\]
for $|\xi'| \leq r_0$.

We set
\[
\psi^*(\xi', x_n) = \frac{1}{\langle \psi(\xi'), \psi^*(\xi') \rangle} \tilde{\psi}^*(\xi', x_n).
\]

Then we have
\[
\langle \psi(\xi'), \psi^*(\xi') \rangle = 1
\]
and
\[
\psi^*(\xi', x_n) = \psi^{(0)}(x_n) + \sum_{j=1}^{n-1} \xi_j \psi_j^{(1)*}(x_n) + \psi^{(2)*}(\xi', x_n),
\]
where $\psi_j^{(1)*}$ and $\psi^{(2)*}$ satisfy
\[
|\psi_j^{(1)*}|_{W^{1,\infty}} \leq C, \quad j = 1, \cdots, n-1,
\]
\[
|\partial_{\alpha'} \psi^{(2)*}(\xi')|_{W^{1,\infty}} \leq C|\xi'|^2-|\alpha'|.
\]

It is not difficult to see that $\langle u, \psi^*(\xi') \rangle \psi(\xi')$ is the eigenprojection $\hat{\Pi}(\xi')$ associated with $\lambda_0(\xi')$.

Setting
\[
\hat{\Pi}^{(0)} = \frac{1}{a}Q_0,
\]
\[
\hat{\Pi}_j^{(1)}(x_n, y_n) = \psi^{(0)}(x_n)^T \psi_j^{(1)*}(y_n) + \psi_j^{(1)}(x_n)^T \psi^{(0)}(y_n),
\]
\[
\hat{\Pi}_j^{(2)}(\xi', x_n, y_n) = \psi(\xi', x_n)^T \psi_j^{(2)*}(\xi', y_n) + \psi_j^{(2)}(x_n)^T \psi^*(\xi', y_n),
\]
we see from (3.22)–(3.25) that the integral kernel $\hat{\Pi}(\xi', x_n, y_n)$ of $\hat{\Pi}(\xi')$ is written as
\[
\hat{\Pi}(\xi', x_n, y_n) = \psi(\xi', x_n)^T \psi^*(\xi', y_n) = \hat{\Pi}^{(0)} + \sum_{j=1}^{n-1} \xi_j \hat{\Pi}_j^{(1)}(x_n, y_n) + \hat{\Pi}_j^{(2)}(\xi', x_n, y_n)
\]
with $\hat{\Pi}_j^{(1)} \in W^{1,\infty}((0, a) \times (0, a))$, $j = 1, \cdots, n-1$, and
\[
|\partial_{\alpha'}^{\alpha'} \hat{\Pi}_j^{(2)}(\xi', \cdot, \cdot)|_{W^{1,\infty}((0, a) \times (0, a))} \leq C|\xi'|^2-|\alpha'|.
\]

We thus conclude that $\hat{\Pi}(\xi')$ is written in the desired form.

We finally show that $\hat{\Pi}(\xi') \left[ \partial_{x_n} \tilde{Q} \bar{u} \right] (x_n)$ and $\partial_{x_n} \hat{\Pi}(\xi') \tilde{Q} \bar{u}$ have the desired forms. Since $\psi^*(\xi', y_n)$ is an eigenfunction of $L_{\xi'}$, we have $\tilde{Q} \psi^*|_{y_n=0, a} = 0$, 
which implies that \( \tilde{H}(\xi', x_n, y_n) \tilde{Q} \big|_{y_n = 0, \alpha} = 0 \). An integration by parts then yields

\[
\tilde{H}(\xi') \left[ \partial_{x_n} \tilde{Q} u \right] (x_n) = \int_0^a \tilde{H}(\xi', x_n, y_n) \partial_{y_n} \tilde{Q} u(y_n) \, dy_n
\]

\[
= - \int_0^a \partial_{y_n} \tilde{H}(\xi', x_n, y_n) \tilde{Q} u(y_n) \, dy_n
\]

\[
= - \left( \partial_{y_n} \tilde{H}(\xi') \right) \tilde{Q} u (x_n).
\]

Since \( \partial_{y_n} \tilde{H}^{(0)} = 0 \), we have the desired form of \( \tilde{H}(\xi') \left[ \partial_{x_n} \tilde{Q} u \right] \). Furthermore, since \( \partial_{x_n} \psi^{(0)} = 0 \) and \( \tilde{Q} \psi^{(0)} = 0 \), we have \( \partial_{x_n} \tilde{H}^{(1)}(x_n, y_n) \tilde{Q} = 0 \), and hence, \( \partial_{x_n} \tilde{H}(\xi') \tilde{Q} u = \partial_{x_n} \tilde{H}^{(2)}(\xi') \tilde{Q} u \). This completes the proof.

We next consider \( (\lambda + \tilde{L}_\xi)^{-1} \) with \( |\xi'| \geq r_0 \). The analysis of \( (\lambda + \tilde{L}_\xi)^{-1} \) with \( |\xi'| \geq r \) for any \( r > 0 \) is given in [6]. Applying [6, Theorems 2.5–2.7], we obtain the following estimates.

Let \( r_0 \) be the number given in Theorem 3.5. We take a cut-off function \( \chi(\xi') \in C^\infty(\mathbb{R}^{n-1}) \) satisfying 0 \( \leq \chi \leq 1 \) on \( \mathbb{R}^{n-1} \), \( \chi(\xi') = 1 \) for \( |\xi'| \leq \frac{r_0}{2} \) and \( \chi(\xi') = 0 \) for \( |\xi'| \geq r_0 \). We set

\[
\chi^{(0)}(\xi') = \chi(\xi'), \quad \chi^{(1)}(\xi') = 1 - \chi(\xi').
\]

We define the operators \( R^{(j)}(\lambda), j = 0, 1, \) by

\[
R^{(j)}(\lambda)f = \mathcal{F}_\xi^{-1} \left[ \chi^{(j)}(\xi') (\lambda + \tilde{L}_\xi)^{-1} f \right], \quad j = 0, 1.
\]

By [6, Theorems 2.5–2.7] we have the following estimates.

**Theorem 3.6.** Let \( r_0 \) be the positive number given in Theorem 3.5.

(i) There exist positive numbers \( \bar{\eta} \) and \( \bar{\theta} \) with \( \bar{\theta} \in (\frac{\pi}{2}, \pi) \) such that \( \Sigma(-\bar{\eta}, \bar{\theta}) \subset \rho(-\tilde{L}_\xi) \) for \( |\xi'| \geq \frac{r_0}{2} \).

(ii) Let \( 1 < p < \infty \) and define \( R^{(1)}(\lambda) \) as above. Then the following estimates hold uniformly in \( \lambda \in \Sigma(-\bar{\eta}, \bar{\theta}) \):

\[
\| \partial^k_x R^{(1)}(\lambda)f \|_p \leq \left\{ \frac{\| Q_0 f \|_{W^{k,p}}}{|\lambda| + 1} + \frac{\| \tilde{Q} f \|_p}{(|\lambda| + 1)^{1-\frac{\pi}{2}}} \right\}, \quad k = 0, 1.
\]

**Theorem 3.7** Let \( \bar{\eta} \) and \( \bar{\theta} \) be the numbers as in Theorem 3.6. Then the following estimates hold uniformly in \( \lambda \in \Sigma(-\bar{\eta}, \bar{\theta}) \):

\[
\| \partial^k_x Q_0 R^{(1)}(\lambda)f \|_\infty \leq C \left\{ \frac{\| Q_0 f \|_{H^{\left[\|\xi'\|+1\right]}^{1+k}}}{|\lambda| + 1} + \frac{\| \tilde{Q} f \|_{H^{\left[\|\xi'\|+1\right]}^{1+k}}}{{(|\lambda| + 1)^{\frac{\pi}{2}}}} \right\}, \quad k = 0, 1,
\]

21
and
\[
\|\partial_k^k \tilde{Q} R^{(1)}(\lambda) f\|_\infty \leq C \left\{ \frac{\|Q_0 f\|_{H^{|k+1}}}{|\lambda| + 1} + \frac{\|\tilde{Q} f\|_{H^{1-k+1}}}{|\lambda| + 1} \right\}, \quad k = 0, 1.
\]

Here \( \varepsilon \) is some number satisfying \( 0 < \varepsilon < \frac{1}{3} \).

**Theorem 3.8.** Let \( p = 1, \infty \) and let \( \tilde{\eta} \) and \( \tilde{\theta} \) be the numbers as in Theorem 3.6. Then the following estimates hold uniformly in \( \lambda \in \Sigma(-\tilde{\eta}, \tilde{\theta}) \):
\[
\|\partial_k^k Q_0 R^{(1)}(\lambda) f\|_p \leq C \frac{|\lambda| + 1}{|\lambda| + 1} \|f\|_{W^{k+1,p} \times W^{k,p}}, \quad k = 0, 1,
\]
and
\[
\|\partial_k^k \tilde{Q} R^{(1)}(\lambda) f\|_p \leq C \left\{ \frac{\|Q_0 f\|_{W^{k,p}}}{|\lambda| + 1} + \frac{\|\tilde{Q} f\|_p}{(|\lambda| + 1)^{1-\frac{k}{2}}} \right\}, \quad k = 0, 1.
\]

**4. Proof of Theorem 2.1**

In this section we prove Theorem 2.1 by applying Theorems 3.3–3.8.

**Proof of Theorem 2.1.** Let \( \eta > 0 \) be a positive number. By Theorem 2.1 in [6] there exists a number \( \theta \in \left( \frac{\pi}{2}, \pi \right) \) such that \( \mathcal{U}(t) u_0 \) is written as
\[
\mathcal{U}(t) u_0 = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} (\lambda + L)^{-1} u_0 d\lambda,
\]
where \( \Gamma = \{ \lambda = \eta + se^{\pm i\theta}; \ s \geq 0 \} \).

We decompose \( \mathcal{U}(t) u_0 \) into the following form:
\[
\mathcal{U}(t) u_0 = U^{(0)}(t) u_0 + U^{(1)}(t) u_0,
\]
where \( U^{(j)}(t) u_0, \ j = 0, 1, \) are defined by
\[
U^{(j)}(t) u_0 = \mathcal{F}^{-1} \left[ \frac{1}{2\pi i} \int_{\Gamma} \chi^{(j)}(\xi') (\lambda + \hat{L})^{-1} d\lambda \right], \quad j = 0, 1,
\]
with \( \chi^{(j)}(\xi') \) defined in (3.26).

We first consider \( U^{(1)}(t) u_0 \). In view of Theorem 3.6, we can deform the contour \( \Gamma \) into \( \Gamma_\infty = \{ \lambda = -\tilde{\eta} + se^{\pm i\tilde{\theta}}; \ s \geq 0 \} \), where \( \tilde{\eta} \) and \( \tilde{\theta} \) are the numbers given in Theorem 3.6. We then obtain
\[
U^{(1)}(t) u_0 = \frac{1}{2\pi i} \int_{\Gamma_\infty} e^{\lambda t} R^{(1)}(\lambda) u_0 d\lambda,
\]
where \( R^{(1)}(\lambda) \) is the operator defined in (3.27). It follows from Theorems 3.6–3.8 that

\[
\left\| \partial_\xi^\ell U^{(1)}(t)u_0 \right\|_p \leq Ce^{-ct}\|u_0\|_{W^\ell,p \times L^p}, \quad 1 < p < \infty, \quad \ell = 0, 1,
\]

\[
\left\| \partial_\xi^\ell U^{(1)}(t)u_0 \right\|_\infty \leq Ce^{-ct}\|u_0\|_{H^{\ell\frac{d}{2}} \times H^{\ell\frac{d}{2} + \epsilon}}, \quad \ell = 0, 1,
\]

\[
\left\| \partial_\xi^\ell U^{(1)}(t)u_0 \right\|_p \leq Ce^{-ct}\|u_0\|_{W^{\ell+1,p} \times W^{\ell,p}}, \quad p = 1, \infty, \quad \ell = 0, 1,
\]

for \( t \geq 1 \).

We next consider \( U^{(0)}(t)u_0 \). By Theorem 3.3, we can deform the contour \( \Gamma \) into \( \Gamma_0 \cup \tilde{\Gamma} \), where

\[
\Gamma_0 = \{ \lambda = -\eta_0 + is; |s| \leq s_0 \}, \quad \tilde{\Gamma} = \{ \lambda = \eta + se^{\pm i\theta}; s \geq \tilde{s}_0 \}.
\]

Here we choose positive numbers \( s_0 \) and \( \tilde{s}_0 \) so that \( \Gamma_0 \) connects with \( \tilde{\Gamma} \) at the end points of \( \Gamma_0 \). It then follows from Theorems 3.4, 3.5 and the residue theorem that \( U^{(0)}(t)u_0 \) is written as

\[
U^{(0)}(t)u_0 = W^{(0)}(t)u_0 + W^{(1)}(t)u_0,
\]

where

\[
W^{(0)}(t)u_0 = \mathcal{F}^{-1} \left[ \chi^{(0)}(\xi') e^{\lambda_0(\xi')t} \widehat{\Pi}(\xi') \widehat{u}_0 \right]
\]

and

\[
W^{(1)}(t)u_0 = \mathcal{F}^{-1} \left[ \frac{1}{2\pi i} \int_{\Gamma_0 \cup \tilde{\Gamma}} e^{\lambda t} \chi^{(0)}(\xi')(\lambda + \widehat{L}_{\xi'})^{-1} \widehat{u}_0 d\lambda \right].
\]

Similarly to the case of \( U^{(1)}(t)u_0 \), by using the integral representation of \( \lambda + \widehat{L}_{\xi'} \) given in [6, Theorem 3.8], one can see that \( W^{(1)}(t)u_0 \) has the same estimates as those for \( U^{(1)}(t)u_0 \).

Let us consider \( W^{(0)}(t)u_0 \). We write it as

\[
W^{(0)}(t)u_0 = \mathcal{W}^{(0)}(t)u_0 + \mathcal{R}^{(0)}(t)u_0,
\]

where

\[
\mathcal{W}^{(0)}(t)u_0 = \mathcal{F}^{-1} \left[ e^{-\kappa|\xi'|^2t} \widehat{\Pi}^{(0)}(\xi') \widehat{u}_0 \right], \quad \kappa = -\frac{a^2-\gamma^2}{12\nu},
\]

and

\[
\mathcal{R}^{(0)}(t)u_0 = \mathcal{W}^{(1)}(t)u_0 + \mathcal{R}^{(0)}_1(t)u_0 + \mathcal{R}^{(0)}_2(t)u_0 + \mathcal{R}^{(0)}_3(t)u_0.
\]

Here

\[
\mathcal{W}^{(1)}(t)u_0 = \mathcal{F}^{-1} \left[ \left( \chi^{(0)}(\xi') - 1 \right) e^{\kappa|\xi'|^2t} \widehat{\Pi}^{(0)}(\xi') \widehat{u}_0 \right],
\]

\[
\mathcal{R}^{(0)}_1(t)u_0 = \mathcal{F}^{-1} \left[ \chi^{(0)}(\xi') e^{\kappa|\xi'|^2t} \widehat{\Pi}^{(1)}(\xi') \widehat{u}_0 \right],
\]

\[
\mathcal{R}^{(0)}_2(t)u_0 = \mathcal{F}^{-1} \left[ \chi^{(0)}(\xi') e^{-\kappa|\xi'|^2t} \widehat{\Pi}^{(1)}(\xi') \widehat{u}_0 \right],
\]

\[
\mathcal{R}^{(0)}_3(t)u_0 = \mathcal{F}^{-1} \left[ \chi^{(0)}(\xi') e^{-\kappa|\xi'|^2t} \widehat{\Pi}^{(2)}(\xi') \widehat{u}_0 \right].
\]
\[ \mathcal{R}_2^{(0)}(t)u_0 = \mathcal{F}^{-1} \left[ \chi^{(0)}(\xi')e^{-\kappa|\xi'|^2 t} \widehat{H}^{(2)}(\xi') \hat{u}_0 \right] \]

and

\[ \mathcal{R}_3^{(0)}(t)u_0 = \mathcal{F}^{-1} \left[ \chi^{(0)}(\xi')(e^{\lambda_0(\xi')t} - e^{-\kappa|\xi'|^2 t}) \widehat{H}(\xi') \hat{u}_0 \right] \]

with \( \kappa = -\frac{a^2}{12b} \).

Clearly, \( \mathcal{W}^{(0)}(t)u_0 = \begin{pmatrix} \phi^{(0)}(t) \\ 0 \end{pmatrix} \) and \( \phi^{(0)} \) satisfies

\[ \partial_t \phi^{(0)} - \Delta' \phi^{(0)} = 0, \quad \phi^{(0)}|_{t=0} = \langle \phi_0 \rangle. \]

It is easy to see that

\[ \left\| \partial^\ell_{\xi'} \mathcal{W}^{(1)}(t)u_0 \right\|_p \leq C e^{-ct} \| u_0 \|_1, \quad \ell = 0, 1. \]

By Theorem 3.5, we easily deduce that

\[ \left\| \partial^\ell_x \mathcal{R}^{(0)}(t)u_0 \right\|_p \leq C t^{-\frac{a-1}{2} - \frac{1}{2}} \| u_0 \|_1, \quad \ell = 0, 1. \]

Let us consider \( \mathcal{B}_2^{(0)}(t)u_0 \). We will estimate it based on the Riemann-Lebesgue lemma as in the estimates for solutions of the Cauchy problem given in [12]. Since

\[ \mathcal{B}_2^{(0)}(t)u_0 = \int_{\mathbb{R}^{n-1}} \int_0^a \mathcal{B}_2^{(0)}(t, x' - y', x_n, y_n)u(y', y_n) dy' dy_n \]

with

\[ \mathcal{B}_2^{(0)}(t, x', x_n, y_n) = \mathcal{F}^{-1} \left[ \chi^{(0)}(\xi')e^{-\kappa|\xi'|^2 t} \widehat{H}^{(2)}(\xi', x_n, y_n) \right] (x') \]

(4.1)

we have

\[ \left\| \partial^\ell_x \mathcal{B}_2^{(0)}(t)u_0 \right\|_1 \leq \sup_{0 \leq y_n \leq a} \left\| \partial^\ell_x \mathcal{B}_2^{(0)}(t, \cdot, \cdot, y_n) \right\|_1 \| u_0 \|_1. \]

By Theorem 3.5, we see that

\[ \sup_{0 \leq y_n \leq a} \left| \partial^\beta'_{\xi'} \left( e^{i\beta' \cdot x'} \partial^j_{x_n} \chi^{(0)}(\xi')e^{-\kappa|\xi'|^2 t} \widehat{H}^{(2)}(\xi', x_n, y_n) \right) \right|_{L^1_{x_n}} \leq C |\xi'|^{2-|\alpha'|} e^{-\frac{a}{2} |\xi'|^2 t} \]

for \( |\beta'| + j \leq 1 \). Therefore, since \( e^{i\xi' \cdot x'} = \sum_{j=1}^{n-1} \frac{x_j}{|x'|^2} \partial_{x_j} e^{i\xi' \cdot x'} \), we perform the integration by parts in (4.1) to obtain, for any \( k = 0, 1, 2, \ldots \),

\[ \sup_{0 \leq y_n \leq a} \left| \partial^\beta'_{\xi'} \partial^j_{x_n} \mathcal{B}_2^{(0)}(t, x', \cdot, y_n) \right|_1 \leq C |x'|^{-k} \int_{\mathbb{R}^{n-1}} |\xi'|^{2+|\alpha'| - k} e^{-\frac{a}{2} |\xi'|^2 t} \ d\xi' \]

\[ \leq C |x'|^{-k} t^\frac{k}{2} \frac{n-k}{2} \frac{1}{2} - 1, \quad |\beta'| + j \leq 1. \]
This implies that
\[
\sup_{0 \leq y_n \leq a} \left\| \partial_x^\ell \mathcal{R}_2^{(0)}(t, \cdot, \cdot, y_n) \right\|_1 \\
\leq C \int_{|x'| \leq \frac{t}{2}} t^{-\frac{n-1}{2}} dx' + \int_{|x'| \geq \frac{t}{2}} |x'|^{-n} t^{\frac{n-1}{2}} dx' \\
\leq Ct^{-1}
\]
for \( \ell = 0, 1 \). Similarly, one can estimate \( \mathcal{R}_3^{(0)}(t) u_0 \). In fact, by Theorem 3.4, we have \( \lambda_0(\xi') = -\kappa |\xi'|^2 + \lambda^{(4)}(\xi') \), where \( \lambda^{(4)}(\xi') \) is analytic in \( \xi' \) and \( |\lambda^{(4)}(\xi')| \leq C|\xi'|^4 \). Since
\[
e^{\lambda_0(\xi') t} - e^{-\kappa |\xi'|^2 t} = \lambda^{(4)}(\xi') t e^{-\kappa |\xi'|^2 t} \int_0^1 e^{\theta \lambda^{(4)}(\xi') t} d\theta,
\]
we see from Theorem 3.5 that
\[
\sup_{0 \leq y_n \leq a} \left| \partial_x^{\ell} \left[ e^{\beta^j} \partial_{x^n} \lambda^{(0)}(\xi')(e^{\lambda_0(\xi') t} - e^{-\kappa |\xi'|^2 t}) \tilde{H}(\xi', \cdot, y_n) \right] \right|_1 \leq C|\xi'|^{2-|\alpha|} e^{-\frac{3}{2}|\xi'|^2 t}
\]
for \( |\beta| + j \leq 1 \). Similarly to above, one can obtain \( \sup_{0 \leq y_n \leq a} \left\| \partial_x^\ell \mathcal{R}_3^{(0)}(t, \cdot, \cdot, y_n) \right\|_1 \leq Ct^{-1} \) for \( \ell = 0, 1 \). Consequently, we have
\[
\left\| \partial_x^\ell \mathcal{R}_2^{(0)}(t) u_0 \right\|_1 \leq Ct^{-\frac{1}{2}} \| u_0 \|_1, \quad \ell = 0, 1.
\]
On the other hand, it is easy to see that
\[
\left\| \partial_x^\ell \mathcal{R}_2^{(0)}(t) u_0 \right\|_\infty \leq Ct^{-\frac{n-1}{2}-\frac{1}{2}} \| u_0 \|_1, \quad \ell = 0, 1.
\]
Therefore, by interpolation, we have
\[
\left\| \partial_x^\ell \mathcal{R}_2^{(0)}(t) u_0 \right\|_p \leq Ct^{-\frac{n-1}{2}(1-\frac{1}{p})-\frac{1}{2}} \| u_0 \|_1, \quad \ell = 0, 1.
\]
By Theorem 3.5, we have \( \tilde{H}^{(0)} \tilde{Q} = 0 \), \( \partial_{x^n} \tilde{H}^{(0)} = 0 \) and \( \partial_{x^n} \tilde{H}^{(1)}(\xi') \tilde{Q} = 0 \). It then follows that
\[
\left\| \partial_{x^n} \mathcal{R}^{(0)}(t) \tilde{Q} u_0 \right\|_p = \left\| \partial_{x^n} \left( \mathcal{R}_2^{(0)}(t) + \mathcal{R}_3^{(0)}(t) \right) \tilde{Q} u_0 \right\|_p \\
\leq Ct^{-\frac{n-1}{2}(1-\frac{1}{p})-\frac{1}{2}} \| \tilde{Q} u_0 \|_1.
\]
Since \( \tilde{H}^{(j)}(\xi') \left[ \partial_{x^n} \tilde{Q} u_0 \right] = -\left( \partial_{y_n} \tilde{H}^{(j)}(\xi') \right) \left[ \tilde{Q} u_0 \right], j = 1, 2 \), we see that
\[
\left\| \mathcal{R}^{(0)}(t) \left[ \partial_{x^n} \tilde{Q} u_0 \right] \right\|_p = \left\| \left( \partial_{y_n} \mathcal{R}^{(0)}(t) \right) \tilde{Q} u_0 \right\|_p \\
\leq Ct^{-\frac{n-1}{2}(1-\frac{1}{p})-\frac{1}{2}} \| \tilde{Q} u_0 \|_1.
\]
25
Clearly, \( \partial_{x'} R^{(0)}(t) \tilde{Q} u_0 = R^{(0)}(t) \left[ \partial_{x'} \tilde{Q} u_0 \right] \) and
\[
\| \partial_{x'} R^{(0)}(t) \tilde{Q} u_0 \|_p \leq C t^{-\frac{1}{2} \left( 1 - \frac{1}{p} \right) - 1} \| \tilde{Q} u_0 \|_1.
\]

The desired results of Theorem 2.1 are thus obtained by setting \( U^{(0)}(t) = \tilde{W}^{(0)}(t) + R^{(0)}(t) \) and \( U^{(\infty)}(t) = U^{(1)}(t) + W^{(1)}(t) \). This completes the proof.

References


List of MHF Preprint Series, Kyushu University
21st Century COE Program
Development of Dynamic Mathematics with High Functionality

MHF2005-1 Hideki KOSAKI
Matrix trace inequalities related to uncertainty principle

MHF2005-2 Masahisa TABATA
Discrepancy between theory and real computation on the stability of some finite element schemes

MHF2005-3 Yuko ARAKI & Sadanori KONISHI
Functional regression modeling via regularized basis expansions and model selection

MHF2005-4 Yuko ARAKI & Sadanori KONISHI
Functional discriminant analysis via regularized basis expansions

MHF2005-5 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Point configurations, Cremona transformations and the elliptic difference Painlevé equations

MHF2005-6 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Construction of hypergeometric solutions to the \( q \)-Painlevé equations

MHF2005-7 Hiroki MASUDA
Simple estimators for non-linear Markovian trend from sampled data: I. ergodic cases

MHF2005-8 Hiroki MASUDA & Nakahiro YOSHIDA
Edgeworth expansion for a class of Ornstein-Uhlenbeck-based models

MHF2005-9 Masayuki UCHIDA
Approximate martingale estimating functions under small perturbations of dynamical systems

MHF2005-10 Ryo MATSUZAKI & Masayuki UCHIDA
One-step estimators for diffusion processes with small dispersion parameters from discrete observations

MHF2005-11 Junichi MATSUKUBO, Ryo MATSUZAKI & Masayuki UCHIDA
Estimation for a discretely observed small diffusion process with a linear drift

MHF2005-12 Masayuki UCHIDA & Nakahiro YOSHIDA
AIC for ergodic diffusion processes from discrete observations
MHF2005-13 Hiromichi GOTO & Kenji KAJIWARA
Generating function related to the Okamoto polynomials for the Painlevé IV equation

MHF2005-14 Masato KIMURA & Shin-ichi NAGATA
Precise asymptotic behaviour of the first eigenvalue of Sturm-Liouville problems with large drift

MHF2005-15 Daisuke TAGAMI & Masahisa TABATA
Numerical computations of a melting glass convection in the furnace

MHF2005-16 Raimundas VIDŪNAS
Normalized Leonard pairs and Askey-Wilson relations

MHF2005-17 Raimundas VIDŪNAS
Askey-Wilson relations and Leonard pairs

MHF2005-18 Kenji KAJIWARA & Atsushi MUKAIHIRA
Soliton solutions for the non-autonomous discrete-time Toda lattice equation

MHF2005-19 Yuu HARIYA
Construction of Gibbs measures for 1-dimensional continuum fields

MHF2005-20 Yuu HARIYA
Integration by parts formulae for the Wiener measure restricted to subsets in $\mathbb{R}^d$

MHF2005-21 Yuu HARIYA
A time-change approach to Kotani’s extension of Yor’s formula

MHF2005-22 Tadahisa FUNAKI, Yuu HARIYA & Mark YOR
Wiener integrals for centered powers of Bessel processes, I

MHF2005-23 Masahisa TABATA & Satoshi KAIHU
Finite element schemes for two-fluids flow problems

MHF2005-24 Ken-ichi MARUNO & Yasuhiro OHTA
Determinant form of dark soliton solutions of the discrete nonlinear Schrödinger equation

MHF2005-25 Alexander V. KITAEV & Raimundas VIDŪNAS
Quadratic transformations of the sixth Painlevé equation

MHF2005-26 Toru FUJII & Sadanori KONISHI
Nonlinear regression modeling via regularized wavelets and smoothing parameter selection

MHF2005-27 Shuichi INOKUCHI, Kazumasa HONDA, Hyen Yeal LEE, Tatsuro SATO, Yoshihiro MIZOGUCHI & Yasuo KAWAHARA
On reversible cellular automata with finite cell array
MHF2005-28 Toru KOMATSU
Cyclic cubic field with explicit Artin symbols

MHF2005-29 Mitsuhiro T. NAKAO, Kouji HASHIMOTO & Kaori NAGATOU
A computational approach to constructive a priori and a posteriori error estimates for finite element approximations of bi-harmonic problems

MHF2005-30 Kaori NAGATOU, Kouji HASHIMOTO & Mitsuhiro T. NAKAO
Numerical verification of stationary solutions for Navier-Stokes problems

MHF2005-31 Hidefumi KAWASAKI
A duality theorem for a three-phase partition problem

MHF2005-32 Hidefumi KAWASAKI
A duality theorem based on triangles separating three convex sets

MHF2005-33 Takeaki FUCHIKAMI & Hidefumi KAWASAKI
An explicit formula of the Shapley value for a cooperative game induced from the conjugate point

MHF2005-34 Hideki MURAKAWA
A regularization of a reaction-diffusion system approximation to the two-phase Stefan problem

MHF2006-1 Masahisa TABATA
Numerical simulation of Rayleigh-Taylor problems by an energy-stable finite element scheme

MHF2006-2 Ken-ichi MARUNO & G R W QUISPEL
Construction of integrals of higher-order mappings

MHF2006-3 Setsuo TANIGUCHI
On the Jacobi field approach to stochastic oscillatory integrals with quadratic phase function

MHF2006-4 Kouji HASHIMOTO, Kaori NAGATOU & Mitsuhiro T. NAKAO
A computational approach to constructive a priori error estimate for finite element approximations of bi-harmonic problems in nonconvex polygonal domains

MHF2006-5 Hidefumi KAWASAKI
A duality theory based on triangular cylinders separating three convex sets in $\mathbb{R}^n$

MHF2006-6 Raimundas VIDUNAS
Uniform convergence of hypergeometric series

MHF2006-7 Yuji KODAMA & Ken-ichi MARUNO
N-Soliton solutions to the DKP equation and Weyl group actions
MHF2006-8 Toru KOMATSU  
Potentially generic polynomial

MHF2006-9 Toru KOMATSU  
Generic sextic polynomial related to the subfield problem of a cubic polynomial

MHF2006-10 Shu TEZUKA & Anargyros PAPAGEORGIOU  
Exact cubature for a class of functions of maximum effective dimension

MHF2006-11 Shu TEZUKA  
On high-discrepancy sequences

MHF2006-12 Raimundas VIDŪNAS  
Detecting persistent regimes in the North Atlantic Oscillation time series

MHF2006-13 Toru KOMATSU  
Tamely Eisenstein field with prime power discriminant

MHF2006-14 Nalini JOSHI, Kenji KAJIWARA & Marta MAZZOCCO  
Generating function associated with the Hankel determinant formula for the solutions of the Painlevé IV equation

MHF2006-15 Raimundas VIDŪNAS  
Darboux evaluations of algebraic Gauss hypergeometric functions

MHF2006-16 Masato KIMURA & Isao WAKANO  
New mathematical approach to the energy release rate in crack extension

MHF2006-17 Toru KOMATSU  
Arithmetic of the splitting field of Alexander polynomial

MHF2006-18 Hiroki MASUDA  
Likelihood estimation of stable Lévy processes from discrete data

MHF2006-19 Hiroshi KAWABI & Michael RÖCKNER  
Essential self-adjointness of Dirichlet operators on a path space with Gibbs measures via an SPDE approach

MHF2006-20 Masahisa TABATA  
Energy stable finite element schemes and their applications to two-fluid flow problems

MHF2006-21 Yuzuru INAHAMA & Hiroshi KAWABI  
Asymptotic expansions for the Laplace approximations for Itô functionals of Brownian rough paths

MHF2006-22 Yoshiyuki KAGEI  
Resolvent estimates for the linearized compressible Navier-Stokes equation in an infinite layer
Asymptotic behavior of the semigroup associated with the linearized compressible Navier-Stokes equation in an infinite layer