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Kinoshita, Takehiko
Faculty of Mathematics, Kyushu University

Hashimoto, Koji
Division of Infant Education, Nakamura Gakuen Junior College

Nakao, T Mitsuhiro
Faculty of Mathematics, Kyushu University

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T. Kinoshita , K. Hashimoto
& M.T. Nakao

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Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

# On the $L^2$ a priori error estimates to the finite element solution of elliptic problems with singular adjoint operator

T. Kinoshita[†], K. Hashimoto[‡], and M.T. Nakao[†]

[†] Faculty of Mathematics, Kyushu University, Fukuoka 812-8581, Japan
[‡] Division of Infant Education, Nakamura Gakuen Junior College, Fukuoka 814-0198, Japan

**Abstract.** The Aubin-Nitsche trick for the finite element method of Dirichlet boundary value problem is a well-known technique to obtain a higher order a priori $L^2$ error estimation than $H_0^1$ estimates by considering the regularly dual problem. However, as far as the authors determine, when the dual problem is singular, it was not known at all up to now whether the a priori order of $L^2$ error is still higher than $H_0^1$ error. In this paper, we propose a technique for getting a priori $L^2$ error estimation by some verified numerical computations for the finite element projection. This enables us to obtain the higher order $L^2$ a priori error than $H_0^1$ error, even though the associated dual problem is singular. Note that our results are *not a posteriori estimates* but *the determination of a priori constants*.

## 1 Introduction

We consider the a priori $L^2$ error estimation for finite element solutions of following linear elliptic boundary value problems,

$$-\triangle u + b\cdot\nabla u + cu = f \quad \text{in } \Omega, \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where $\Omega \subset \mathbb{R}^d,\ (d=1,2,3)$ is bounded polygonal (polyhedral) domain, and $b,\ c,\ f$ are arbitrary elements of $L^\infty(\Omega)^d,\ L^\infty(\Omega),\ L^2(\Omega)$, respectively. We define the Sobolev space $X(\Omega) = \left\{u \in H^1(\Omega)\ ;\ \triangle u \in L^2(\Omega)\right\}$, and the differential operator $\mathscr{L}$ is defined as the left-hand side of (1), i.e.

$$\mathscr{L} : X(\Omega) \to L^2(\Omega), \quad \mathscr{L} = -\triangle + b\cdot\nabla + c. \tag{3}$$

Moreover, the bilinear form $a(\cdot,\cdot)$ corresponding to $\mathscr{L}$ is defined by

$$a : H_0^1(\Omega)\times H_0^1(\Omega) \to \mathbb{R}, \qquad a(u,v) = (\nabla u,\nabla v)_{L^2(\Omega)^d} + (b\cdot\nabla u, v)_{L^2(\Omega)} + (cu,v)_{L^2(\Omega)}. \tag{4}$$

We now introduce a finite element space $S_h$ which is a subspace of $H_0^1(\Omega)$ depending on the mesh size parameter $h$ with base functions $\{\phi_i\}_{i=1}^n$, i.e. $n = \dim S_h$. In this paper, we assume that, for each $u \in H_0^1(\Omega)$, there exists a projection $P_{\mathscr{L}}u \in S_h$ which is defined by

$$P_{\mathscr{L}} : H_0^1(\Omega) \to S_h, \qquad a(u - P_{\mathscr{L}}u, v_h) = 0, \quad \forall v_h \in S_h. \tag{5}$$

This assumption implies that the finite element solution for of (1) associated with the bilinear form (4) is uniquely determined. As the main theorem of this paper, Theorem 3.3, we derive a method to get an a priori $L^2$ error estimates $\|u - P_{\mathscr{L}}u\|_{L^2(\Omega)}$ without the Aubin-Nitsche trick but using a computer assisted approach.

In the following section, we summarize the existing research results up to now as well as describe the difficulty in case that $\mathscr{L}$ has singular adjoint operator. In §3, we propose a new a priori $L^2$ error estimation without using the Aubin-Nitsche trick. In §4, we show several numerical results, which prove that the order of the rate of convergence for $L^2$ error is actually higher than $H_0^1$ error even for the case that the dual problem is singular. Since all of the numerical results are verified results, we can use these estimates in the related computer assisted proofs for nonlinear problems such as [4].

## 2 Previous results

The properties described in this section, are basically obtained in [5]. We define the $H_0^1$-projection $P_h$ which satisfies

$$P_h : H_0^1(\Omega) \to S_h, \qquad (u - P_h u, v_h)_{H_0^1(\Omega)} = 0, \quad \forall v_h \in S_h, \tag{6}$$

where $H_0^1$ inner product is defined by $(u,v)_{H_0^1(\Omega)} := (\nabla u, \nabla v)_{L^2(\Omega)^d}$. As the a priori error estimates of $H_0^1$-projection $P_h$, we assume that the following properties hold.

**Assumption 2.1** *There exists a positive constant $C(h)$ which can be numerically estimated satisfying*

$$\|u - P_h u\|_{H_0^1(\Omega)} \le C(h) \|\triangle u\|_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega) \cap X(\Omega), \tag{7}$$

$$\|u - P_h u\|_{L^2(\Omega)} \le C(h) \|u - P_h u\|_{H_0^1(\Omega)}, \quad \forall u \in H_0^1(\Omega). \tag{8}$$

For example, if $\Omega$ is a rectangular domain and the finite element space $S_h$ is defined by piecewise bilinear polynomials (Q1 element) , then $C(h)$ in (7), (8) can be taken as $h/\pi$.

We now define the compact operator $Q : H_0^1(\Omega) \to H_0^1(\Omega)$ by $Q = \triangle^{-1}(b \cdot \nabla + c)$, where $\triangle^{-1}$ stands for the solution operator of the Poisson equation with homogeneous Dirichlet boundary condition. Furthermore, define the $n \times n$ matrices $G = \left(G_{ij}\right)_{i,j=1}^n$ and $D = \left(D_{ij}\right)_{i,j=1}^n$ by

$$G_{ij} = a(\phi_j, \phi_i), \qquad D_{ij} = (\phi_j, \phi_i)_{H_0^1(\Omega)}, \qquad i,j = 1, \cdots, n,$$

respectively. Note that $G$ is invertible because $P_{\mathscr{L}} u$ is uniquely determined. Moreover, $D$ is symmetric and positive definite, which enables us to find the Cholesky decomposition, i.e., there exists a lower triangular matrix $D^{1/2}$, and its transpose $D^{T/2}$ such that $D = D^{1/2} D^{T/2}$.
The following lemma, obtained in [5], is a starting point to our main theorem. In order to make the argument self-contained, we present a proof.

**Lemma 2.2 (cf. [5])** *Let $M(h) = \left\|D^{T/2} G^{-1} D^{1/2}\right\|_E$, where $\|\cdot\|_E$ means matrix norm induced from the Euclidean norm. Then we have*

$$\|P_h u - P_{\mathscr{L}} u\|_{H_0^1(\Omega)} \le M(h) \|P_h Q(u - P_h u)\|_{H_0^1(\Omega)}, \quad \forall u \in H_0^1(\Omega). \tag{9}$$

**Proof.** For each $u \in H_0^1(\Omega)$, define $u_\perp := u - P_h u$ and $\psi := -Q u_\perp \equiv -\triangle^{-1}(b \cdot \nabla + c) u_\perp$, . Then we have, for an arbitrary $v_h \in S_h$,

$$\begin{aligned} a\big(P_{\mathscr{L}} u - P_h u, v_h\big) &= a\big(P_{\mathscr{L}} u_\perp, v_h\big) \\ &= (\nabla u_\perp, \nabla v_h)_{L^2(\Omega)^d} + ((b \cdot \nabla + c) u_\perp, v_h)_{L^2(\Omega)} \\ &= (-\triangle \psi, v_h)_{L^2(\Omega)} \\ &= (\nabla P_h \psi, \nabla v_h)_{L^2(\Omega)^d}, \end{aligned}$$

which implies that

$$G \vec{u}_{\perp h} = D \vec{\psi}_h,$$

where $\vec{u}_{\perp h}$ and $\vec{\psi}_h$ are coefficient vectors of $P_{\mathscr{L}} u_\perp$ and $P_h \psi$, respectively. Hence, we have

$$\begin{aligned} \|P_h u - P_{\mathscr{L}} u\|_{H_0^1(\Omega)}^2 &= \|P_{\mathscr{L}} u_\perp\|_{H_0^1(\Omega)}^2 \\ &= (\vec{u}_{\perp h})^T D \vec{u}_{\perp h} \\ &= \left(D^{T/2} \vec{u}_{\perp h}\right)^T D^{T/2} G^{-1} D^{1/2} \left(D^{T/2} \vec{\psi}_h\right) \\ &\le \|P_{\mathscr{L}} u_\perp\|_{H_0^1(\Omega)} \left\|D^{T/2} G^{-1} D^{1/2}\right\|_E \|P_h \psi\|_{H_0^1(\Omega)}. \end{aligned}$$

Thus, the proof is completed. □

Note that $M(h)$ is the quantity corresponding to the operator norm of $[I-Q]_h^{-1}$ which is an approximate inverse operator of $\mathscr{L}$. Therefore, if $\mathscr{L}$ is invertible, $M(h)$ is expected to be bounded by a constant, otherwise, i.e., if not invertible, $M(h)\to\infty$ as $h\to 0$. Also the Euclidean norm can be estimated by the verified numerical computation of the largest singular value of a matrix. Therefore, by using this $M(h)$, we obtain the $H_0^1$ and $L^2$ a priori error estimation of $\mathscr{L}$-projection as below.

**Corollary 2.3 (cf. [5, Cor. 7])** *Suppose that $b\in L^\infty(\Omega)^d$ and $c\in L^\infty(\Omega)$. Then we have*

$$\|u-P_{\mathscr{L}}u\|_{H_0^1(\Omega)} \le C(h)\hat{\alpha}\,\|\triangle u\|_{L^2(\Omega)}, \qquad \forall u\in H_0^1(\Omega)\cap X(\Omega), \tag{10}$$

*where $C(h)$ is the constant introduced in Assumption 2.1 and $\hat{\alpha}$ is defined as*

$$\hat{\alpha}=\sqrt{1+\Big(M(h)C_p\big(\|b\|_{L^\infty(\Omega)^d}+C(h)\,\|c\|_{L^\infty(\Omega)}\big)\big(\|b\|_{L^\infty(\Omega)^d}+C_p\,\|c\|_{L^\infty(\Omega)}\big)\Big)^2},$$

*where $C_p$ is a Poincaré constant, and $\|b\|^2_{L^\infty(\Omega)^d}:=\sum_{i=1}^d\|b_i\|^2_{L^\infty(\Omega)}$.*

**Theorem 2.4 (cf. [5, Thm. 5])** *If $b\in W^{1,\infty}(\Omega)^d$ and $c\in L^\infty(\Omega)$, then we have*

$$\|u-P_{\mathscr{L}}u\|_{L^2(\Omega)} \le C(h)\alpha\,\|u-P_hu\|_{H_0^1(\Omega)}, \qquad \forall u\in H_0^1(\Omega), \tag{11}$$

*where*

$$\alpha = 1+C_pM(h)\Big(C_p\,\|div\, b\|_{L^\infty(\Omega)}+\|b\|_{L^\infty(\Omega)^d}+C_p\,\|c\|_{L^\infty(\Omega)}\Big).$$

In order to get the $L^2$ estimation in Theorem 2.4, a usual Aubin-Nitsche's trick is used. Namely, in the proof of [5, Lem. 4], for each $v\in H_0^1(\Omega)$, setting $\psi:=-Qv\equiv(-\triangle)^{-1}(b\cdot\nabla+c)v$, they used the following estimates

$$\begin{aligned}\|\psi\|^2_{H_0^1} &= (-\triangle\psi,\psi)_{L^2}\\ &= (v,\operatorname{div}(b\psi))_{L^2}+(v,c\psi)_{L^2}\\ &\le (\|\operatorname{div} b\|_{L^\infty}\|\psi\|_{L^2}+\|b\|_{L^\infty}\|\nabla\psi\|_{L^2}+\|c\|_{L^\infty}\|\psi\|_{L^2})\,\|v\|_{L^2}.\end{aligned}$$

The above estimation is essentially equivalent to the usual duality argument using the adjoint problem for the operator $\mathscr{L}^*\phi\equiv-\triangle\phi-\operatorname{div}(b\phi)+c\phi$ of $\mathscr{L}$, which needs the regularity with $b\in W^{1,\infty}(\Omega)^d$ or at least $\operatorname{div} b\in L^\infty(\Omega)$. On the other hand, when $b$ is not smooth, i.e., in case that the adjoint operator becomes singular, the Aubin-Nitsche trick can no longer be applied. As far as the authors know, it is not yet known whether $L^2$ error estimation is still higher order than the $H_0^1$ estimation. Namely, we have no theoretical approaches which resolve such a difficulty up to now. In order to break this situation, in the next section, we will propose a technique to get a priori $L^2$ error estimation of $\mathscr{L}$-projection incorporated with some verified numerical computations.

# 3 A priori $L^2$ error estimation with singular adjoint operator

First, we introduce the following theorem for later use.

**Theorem 3.1** *Let $W$ and $V$ be Hilbert spaces, and let $W^*$ and $V^*$ be their dual spaces. Let $B:W\to V$ be a bounded, closed range linear operator, and let $B^*:V\to W$ be the dual operator of B. Define the bilinear form $\hat{b}:V\times W\to\mathbb{R}$ by $\hat{b}(v,w^*)=(B^*v,w^*)_W$, for any $v\in V$, $w^*\in W$. We assume that bilinear forms $\hat{a}:W\times W\to\mathbb{R}$ and $\hat{b}$ satisfy the following properties.*

- *$\hat{a}$ is continuous on $W\times W$. i.e. there exists a constant $C>0$ s.t.*

$$\hat{a}(w,w^*)\le C\,\|w\|_W\,\|w^*\|_W, \qquad \forall w,w^*\in W. \tag{12}$$

- *$\hat{a}$ is coercive on $N(B) \subset W$. i.e. there exists a constant $K > 0$ s.t.*

$$\hat{a}(w,w) \geq K \|w\|_W^2, \qquad \forall w \in N(B). \tag{13}$$

- *$\hat{b}$ satisfies the inf-sup condition, i.e., there exists a constant $\beta > 0$ s.t.*

$$\inf_{R(B)\ni v\neq 0} \sup_{W\ni w^*\neq 0} \frac{\hat{b}(v,w^*)}{\|v\|_V \|w^*\|_W} \geq \beta. \tag{14}$$

*Here, $N(B)$ is the null set of $B$ in $W$, and $R(B)$ is the range of $B$ in $V$. Then for arbitrary $F \in W^*$ and $G \in V^*$, there exists a unique $(w,v) \in W \times R(B)$ satisfying*

$$\hat{a}(w,w^*) + \hat{b}(v,w^*) = F(w^*), \qquad \forall w^* \in W \tag{15}$$

$$\hat{b}(v^*,w) = G(v^*), \qquad \forall v^* \in V. \tag{16}$$

For the proof of this theorem, see, e.g., [2], [1] etc.

**Corollary 3.2** *Let $\Omega$ be a bounded domain in $\mathbb{R}^d$. Then for each $f \in L^2(\Omega)^d$, there exists a unique $(w,v) \in L^2(\Omega)^d \times H_0^1(\Omega)$ which is a solution of*

$$(w,w^*)_{L^2(\Omega)^d} + (\nabla v, w^*)_{L^2(\Omega)^d} = (f,w^*)_{L^2(\Omega)^d}, \qquad \forall w^* \in L^2(\Omega)^d \tag{17}$$

$$(\nabla v^*, w)_{L^2(\Omega)^d} = 0, \qquad \forall v^* \in H_0^1(\Omega). \tag{18}$$

**Proof.** Theorem 3.1 can be applied for $W = L^2(\Omega)^d$, $V = H_0^1(\Omega)$ and $B^* = \nabla$. In the present case, the constant of (12) is realized by $C = 1$ from $\hat{a}(w,w^*) \equiv (w,w^*)_{L^2(\Omega)^d}$ and the Schwarz inequality. Moreover, since $\hat{a}$ is an inner product of $L^2(\Omega)^d$, the constant of (13) is also realized by $K = 1$.
Then observe that for arbitrary $v \in H_0^1(\Omega)$

$$\|\nabla v\|_{L^2(\Omega)^d} = \sup_{L^2(\Omega)^d \ni w^* \neq 0} \frac{(\nabla v, w^*)_{L^2(\Omega)^d}}{\|w^*\|_{L^2(\Omega)^d}}.$$

Thus we have

$$1 = \inf_{H_0^1(\Omega)\ni v\neq 0} \sup_{L^2(\Omega)^d \ni w^* \neq 0} \frac{(\nabla v, w^*)_{L^2(\Omega)^d}}{\|v\|_{H_0^1(\Omega)} \|w^*\|_{L^2(\Omega)^d}}.$$

Therefore, the constant $\beta$ in (14) is realized by $\beta = 1$. Also it is clear that $N(B^*) = \{0\}$. Since $W$ is decomposed as $W = N(B^*) \oplus R(B)$, we have $R(B) = L^2(\Omega)^d$, i.e. $B$ is a closed range operator.
Thus we can apply Theorem 3.1 and the solution $(w,v)$ of (17), (18) exists and is unique in $L^2(\Omega)^d \times H_0^1(\Omega)$. $\square$

We now define the Hilbert space by $H(\mathrm{div},\Omega) \equiv \left\{ w \in L^2(\Omega)^d \, ; \, \mathrm{div}\, w \in L^2(\Omega) \right\}$. It is well known that the solution $w$ of Corollary 3.2 belongs to $H(\mathrm{div},\Omega)$. Let $W_h$ be a finite element space of $H(\mathrm{div},\Omega)$ with base functions $\chi_i$., i.e., $W_h = \mathrm{span}_{1\leq i\leq m}\{\chi_i\}$.

**Theorem 3.3** *Let us assume $b \in L^\infty(\Omega)^d$. For each $\psi_h \in S_h$, we assume that there exists a unique solution $(w_h,v_h) \in W_h \times S_h$ of the following problem*

$$(w_h,w_h^*)_{L^2(\Omega)^d} + (\nabla v_h, w_h^*)_{L^2(\Omega)^d} = (b\psi_h, w_h^*)_{L^2(\Omega)^d}, \quad \forall w_h^* \in W_h, \tag{19}$$

$$(w_h, \nabla v_h^*)_{L^2(\Omega)^d} = 0, \qquad \forall v_h^* \in S_h. \tag{20}$$

*And define $\sigma_0(h)$ and $\sigma_1(h)$ as follows*

$$\sigma_0(h) \equiv \sup_{S_h\ni\psi_h\neq 0} \frac{\|w_h + \nabla v_h - b\psi_h\|_{L^2(\Omega)^d}}{\|\nabla \psi_h\|_{L^2(\Omega)^d}}, \quad \sigma_1(h) \equiv \sup_{S_h\ni\psi_h\neq 0} \frac{\|div\, w_h\|_{L^2(\Omega)}}{\|\nabla \psi_h\|_{L^2(\Omega)^d}}. \tag{21}$$

*Then, we have*

$$\|P_h Q(u-P_h u)\|_{H_0^1(\Omega)} \le K(h)\,\|u-P_h u\|_{H_0^1(\Omega)}\,, \qquad \forall u \in H_0^1(\Omega), \tag{22}$$

*where* $K(h) = \sigma_0(h) + C(h)\sigma_1(h) + C_p C(h)\,\|c\|_{L^\infty(\Omega)}$.

**Proof.** Let $u_\perp := u - P_h u,\ \psi := Q u_\perp$. By virtue of (8), (21), we have

$$\begin{aligned}
\|P_h Q u_\perp\|^2_{H_0^1(\Omega)} &= (\nabla P_h Q u_\perp, \nabla P_h \psi)_{L^2(\Omega)^d} \\
&= (\nabla Q u_\perp, \nabla P_h \psi)_{L^2(\Omega)^d} \\
&= -\,(b\cdot\nabla u_\perp + c u_\perp, P_h\psi)_{L^2(\Omega)} \\
&= (\nabla u_\perp, w_h + \nabla v_h - b P_h \psi)_{L^2(\Omega)^d} - (\nabla u_\perp, w_h)_{L^2(\Omega)^d} - (c u_\perp, P_h \psi)_{L^2(\Omega)} \\
&\le \|\nabla u_\perp\|_{L^2(\Omega)^d}\,\|w_h + \nabla v_h - b P_h \psi\|_{L^2(\Omega)^d} \\
&\qquad + \|u_\perp\|_{L^2(\Omega)}\,\|\mathrm{div}\, w_h\|_{L^2(\Omega)} + \|c\|_{L^\infty(\Omega)}\,\|u_\perp\|_{L^2(\Omega)}\,\|P_h\psi\|_{L^2(\Omega)} \\
&\le \Big(\sigma_0(h) + C(h)\sigma_1(h) + C_p C(h)\,\|c\|_{L^\infty(\Omega)}\Big)\,\|\nabla u_\perp\|_{L^2(\Omega)^d}\,\|\nabla P_h \psi\|_{L^2(\Omega)^d}\,.
\end{aligned} \tag{23}$$

Therefore, the proof is completed. □

If $W_h$ and $S_h$ satisfy the inf-sup condition, Theorem 3.1 assures the existence and uniqueness of the solution to (19), (20). In general, $K(h)$ depends on the choice of bases of $W_h$ and $S_h$. Usually, the combination of bases is very important in case that we use the mixed finite element method. Note that the unique solvability of (19) and (20) can be automatically proved by the computational results using the self-validating numerical methods such as INTLAB [7]. Therefore, we don't need to discuss about the inf-sup condition of $W_h$ and $S_h$ in this paper. And as described later, the verified computation of $K(h)$ is also possible by some finite procedures.

On the expected order of $K(h)$, we have the following theorem.

**Theorem 3.4** *Let $\Omega$ be a bounded domain in $\mathbb{R}^d$, and let $0 \le r,\ s \le 1$ be real numbers. We assume that the convergence order of $C(h)$ in Assumption 2.1 is $O(h^s)$ and that there exists a constant $C > 0$, independent of $h$, satisfying*

$$\|u - P_h u\|_{H_0^1(\Omega)} \le C h^r\,\|u\|_{H^{1+r}(\Omega)}\,, \quad \forall u \in H_0^1(\Omega) \cap H^{1+r}(\Omega). \tag{24}$$

*For given $b \in L^\infty(\Omega)^d$ and $\psi \in H_0^1(\Omega)$, let $(w,v) \in L^2(\Omega)^d \times H_0^1(\Omega)$ be a solution of* (17), (18) *with $f = b\psi$. If $v \in H^{1+r}(\Omega)$ and the estimation*

$$\|v\|_{H^{1+r}(\Omega)} \le C_b\,\|\psi\|_{H_0^1(\Omega)} \tag{25}$$

*holds for a constant $C_b > 0$, then $K(h)$ in* (22) *has the convergence rate of $O(h^{\min\{r,s\}})$.*

**Proof.** For any $u \in H_0^1(\Omega)$, setting $u_\perp := u - P_h u$, by the similar derivation to (23), we have

$$\begin{aligned}
\|P_h Q u_\perp\|^2_{H_0^1(\Omega)} &= (\nabla u_\perp, w + \nabla v - b\psi)_{L^2(\Omega)^d} - (\nabla u_\perp, w)_{L^2(\Omega)^d} - (\nabla u_\perp, \nabla v)_{L^2(\Omega)^d} - (c u_\perp, \psi)_{L^2(\Omega)} \\
&= -\,(\nabla u_\perp, \nabla v)_{L^2(\Omega)^d} - (c u_\perp, \psi)_{L^2(\Omega)} \\
&= -\,(\nabla u_\perp, \nabla v - \nabla v_h)_{L^2(\Omega)^d} - (c u_\perp, \psi)_{L^2(\Omega)}\,, \quad \forall v_h \in S_h \\
&\le \left(\inf_{v_h \in S_h}\|\nabla v - \nabla v_h\|_{L^2(\Omega)^d} + C(h)\,\|c\|_{L^\infty(\Omega)}\,\|\psi\|_{L^2(\Omega)}\right)\|\nabla u_\perp\|_{L^2(\Omega)^d}\,.
\end{aligned}$$

By (24) and (25), we get

$$\begin{aligned}
\inf_{v_h \in S_h}\|\nabla v - \nabla v_h\|_{L^2(\Omega)^d} &\le C h^r\,\|v\|_{H^{1+r}(\Omega)} \\
&\le C C_b h^r\,\|\psi\|_{H_0^1(\Omega)}\,.
\end{aligned}$$

Therefore, we obtain the estimate

$$\|P_h Q u_\perp\|_{H_0^1(\Omega)} \leq \left( C C_b h^r + C_p C(h)\,\|c\|_{L^\infty(\Omega)} \right) \|\nabla u_\perp\|_{L^2(\Omega)^d}\,,$$

which proves the theorem. □

For example, let assume that $\operatorname{div} b \in L^\infty(\Omega)$. Then, by taking the test function $w^*$ as $\nabla v^* \in L^2(\Omega)^d$, the solution $(w,v)$ of (17) and (18) satisfies

$$(\nabla v, \nabla v^*)_{L^2(\Omega)} = -\left(\operatorname{div}(b\psi), v^*\right)_{L^2(\Omega)}\,, \quad \forall v^* \in H_0^1(\Omega).$$

This means that $v$ is a solution of the Poisson equation. Therefore, if the domain $\Omega$ is convex and piecewise smooth, then $v$ belongs to $H^2(\Omega)$, and (25) holds because we have

$$|v|_{H^2(\Omega)} \leq \|\operatorname{div}(b\psi)\|_{L^2(\Omega)} \leq \left( C_p \,\|\operatorname{div} b\|_{L^\infty(\Omega)} + \|b\|_{L^\infty(\Omega)^d} \right) \|\nabla \psi\|_{L^2(\Omega)^d}\,.$$

Hence, $K(h)$ has the $O(h)$ property in this case.

**Remark 3.5** *The determination of $\sigma_0(h)$ and $\sigma_1(h)$ in (21) are finite dimensional problems. That is, these values can be estimated by the computation of the largest eigenvalue or singular value of certain matrices.*

Below, we briefly explain the eigenvalue problem in the above remark. Let $\vec{w}_h = (w_1, \cdots, w_m)^T,\ \vec{v}_h = (v_1, \cdots, v_n)^T,\ \vec{\psi}_h = (\psi_1, \cdots, \psi_n)^T$ be coefficient vectors for $w_h \in W_h,\ v_h,\ \psi_h \in S_h$, satisfying

$$w_h = \sum_{i=1}^m w_i \chi_i, \qquad v_h = \sum_{i=1}^n v_i \phi_i, \qquad \psi_h = \sum_{i=1}^n \psi_i \phi_i.$$

Then (19) and (20) can be represented by the following simultaneous linear equations.

$$\begin{pmatrix} S_A & S_B^T \\ S_B & 0 \end{pmatrix} \begin{pmatrix} \vec{w}_h \\ \vec{v}_h \end{pmatrix} = \begin{pmatrix} S_L \\ 0 \end{pmatrix} \vec{\psi}_h, \tag{26}$$

where $S_A,\ S_B,\ S_C,\ S_L$ are the following matrices.

$$\mathbb{R}^{m\times m} \ni S_A = \left( (\chi_j, \chi_i)_{L^2} \right)_{1\leq i,j\leq m}, \qquad \mathbb{R}^{n\times m} \ni S_B = \left( (\chi_j, \nabla \phi_i)_{L^2} \right)_{1\leq i\leq n, 1\leq j\leq m},$$
$$\mathbb{R}^{m\times n} \ni S_L = \left( (b\phi_j, \chi_i)_{L^2} \right)_{1\leq i\leq m, 1\leq j\leq n}, \qquad \mathbb{R}^{n\times n} \ni S_C = S_B S_A^{-1} S_B^T.$$

Thus we have

$$\begin{aligned} \begin{pmatrix} \vec{w}_h \\ \vec{v}_h \end{pmatrix} &= \begin{pmatrix} S_A^{-1} - S_A^{-1} S_B^T S_C^{-1} S_B S_A^{-1} & S_A^{-1} S_B^T S_C^{-1} \\ S_C^{-1} S_B S_A^{-1} & -S_C^{-1} \end{pmatrix} \begin{pmatrix} S_L \\ 0 \end{pmatrix} \vec{\psi}_h \\ &= \begin{pmatrix} S_A^{-1} S_L - S_A^{-1} S_B^T S_C^{-1} S_B S_A^{-1} S_L \\ S_C^{-1} S_B S_A^{-1} S_L \end{pmatrix} \vec{\psi}_h. \end{aligned} \tag{27}$$

Now, in order to get $\sigma_0(h)$, we need a matrix representation of the norm $\|w_h + \nabla v_h - b\psi_h\|_{L^2(\Omega)^d}$ as follows:

$$\begin{aligned} \|w_h + \nabla v_h - b\psi_h\|_{L^2(\Omega)^d}^2 &= (w_h + \nabla v_h - b\psi_h, w_h + \nabla v_h - b\psi_h)_{L^2(\Omega)^d} \\ &= (w_h, w_h)_{L^2(\Omega)^d} + 2\,(w_h, \nabla v_h)_{L^2(\Omega)^d} - 2\,(w_h, b\psi_h)_{L^2(\Omega)^d} \\ &\quad + (\nabla v_h, \nabla v_h)_{L^2(\Omega)^d} - 2\,(\nabla v_h, b\psi_h)_{L^2(\Omega)^d} + (b\psi_h, b\psi_h)_{L^2(\Omega)^d} \\ &= \vec{w}_h^T S_A \vec{w}_h + 2\vec{w}_h^T S_B \vec{v}_h - 2\vec{w}_h^T S_L^T \vec{\psi}_h \\ &\quad + \vec{v}_h^T D \vec{v}_h - 2\vec{v}_h^T S_B^T \vec{\psi}_h + \vec{\psi}_h^T S_K \vec{\psi}_h \\ &= \vec{\psi}_h^T M_0 \vec{\psi}_h, \end{aligned} \tag{28}$$

where we set $S_K := \left((b\phi_j, b\phi_i)_{L^2}\right)_{1\le i,j\le n}$, and $M_0$ is a matrix obtained by substituting $\vec{\psi}_h$ for $\vec{w}_h, \vec{v}_h$ in (28) by using the relation (27). We also note that $\|\nabla\psi_h\|^2_{L^2(\Omega)^d} = \vec{\psi}_h^T D\vec{\psi}_h$. Thus we have the following estimates

$$\begin{aligned}
\sup_{S_h\ni\psi_h\neq 0} \frac{\|w_h+\nabla v_h - b\psi_h\|^2_{L^2(\Omega)^d}}{\|\nabla\psi_h\|^2_{L^2(\Omega)^d}} &= \sup_{\mathbb{R}^n\ni\vec{\psi}_h\neq 0} \frac{\vec{\psi}_h^T M_0\vec{\psi}_h}{\vec{\psi}_h^T D\vec{\psi}_h} \\
&= \sup_{\mathbb{R}^n\ni\vec{\psi}_h\neq 0} \frac{\left(D^{T/2}\vec{\psi}_h\right)^T D^{-1/2}M_0 D^{-T/2}\left(D^{T/2}\vec{\psi}_h\right)}{\left(D^{T/2}\vec{\psi}_h\right)^T\left(D^{T/2}\vec{\psi}_h\right)} \\
&\le \left\|D^{-1/2}M_0D^{-T/2}\right\|_E.
\end{aligned}$$

Therefore, we get $\sigma_0(h) = \|D^{-1/2}M_0D^{-T/2}\|_E^{1/2}$.

Similarly, $\sigma_1(h)$ can also be bounded by solving the largest eigenvalue problem as follows:

$$\begin{aligned}
\|\operatorname{div} w_h\|^2_{L^2(\Omega)} &= (\operatorname{div} w_h, \operatorname{div} w_h)_{L^2(\Omega)} \\
&= \vec{w}_h^T S_D \vec{w}_h \\
&= \vec{\psi}_h^T M_1\vec{\psi}_h,
\end{aligned}$$

where we set $S_D := \left((\operatorname{div}\chi_j, \operatorname{div}\chi_i)_{L^2}\right)_{1\le i,j\le m}$, and $M_1$ is a matrix obtained by substituting $\vec{\psi}_h$ for $\vec{w}_h$ by using the relation (27). Thus we have the following estimates

$$\begin{aligned}
\sup_{S_h\ni\psi_h\neq 0} \frac{\|\operatorname{div} w_h\|^2_{L^2(\Omega)}}{\|\nabla\psi_h\|^2_{L^2(\Omega)^d}} &= \sup_{\mathbb{R}^n\ni\vec{\psi}_h\neq 0} \frac{\vec{\psi}_h^T M_1\vec{\psi}_h}{\vec{\psi}_h^T D\vec{\psi}_h} \\
&= \sup_{\mathbb{R}^n\ni\vec{\psi}_h\neq 0} \frac{\left(D^{T/2}\vec{\psi}_h\right)^T D^{-1/2}M_1 D^{-T/2}\left(D^{T/2}\vec{\psi}_h\right)}{\left(D^{T/2}\vec{\psi}_h\right)^T\left(D^{T/2}\vec{\psi}_h\right)} \\
&\le \left\|D^{-1/2}M_1D^{-T/2}\right\|_E.
\end{aligned}$$

Therefore, we obtain $\sigma_1(h) = \|D^{-1/2}M_1D^{-T/2}\|_E^{1/2}$.

Now, we obtain the following a priori $L^2$ error estimation without Aubin-Nitsche's trick, which is our desired result in this paper.

**Theorem 3.6** *Let $b \in L^\infty(\Omega)^d,\ c\in L^\infty(\Omega)$. Under the Assumption 2.1, the following estimate holds,*

$$\|u - P_{\mathscr{L}}u\|_{L^2(\Omega)} \le \left(C(h) + C_pM(h)K(h)\right)\|u-P_hu\|_{H_0^1(\Omega)}, \qquad \forall u \in H_0^1(\Omega). \tag{29}$$

**Proof.** By Lemma 2.2, we have

$$\begin{aligned}
\|P_hu - P_{\mathscr{L}}u\|_{L^2(\Omega)} &\le C_p\|P_hu - P_{\mathscr{L}}u\|_{H_0^1(\Omega)} \\
&\le C_pM(h)\|P_hQ(u-P_hu)\|_{H_0^1(\Omega)}.
\end{aligned}$$

Therefore, by using Theorem 3.3, we get the estimates

$$\begin{aligned}
\|u - P_{\mathscr{L}}u\|_{L^2(\Omega)} &\le \|u-P_hu\|_{L^2(\Omega)} + \|P_hu - P_{\mathscr{L}}u\|_{L^2(\Omega)} \\
&\le \|u-P_hu\|_{L^2(\Omega)} + C_pM(h)\|P_hQ(u-P_hu)\|_{H_0^1(\Omega)} \\
&\le \left(C(h)+C_pM(h)K(h)\right)\|u-P_hu\|_{H_0^1(\Omega)},
\end{aligned}$$

which yields the desired estimates. □

Note that, if the constants $M(h)$ and $K(h)$ are computed with guaranteed accuracy, the a priori error estimation in Theorem 3.3 can also be used in the computer assisted proofs such as [3]. Moreover, the verification condition of invertibility of the elliptic operator derived in [4] is rewritten as follows, which presents a more efficient condition than the original form.

**Theorem 3.7** *Let $b \in L^\infty(\Omega)^d$, $c \in L^\infty(\Omega)$. If it holds that*

$$\kappa := C(h)\big(C_1 M(h)K(h) + C_2\big) < 1,$$

*then $\mathscr{L}$ is invertible. Here,*

$$C_1 = \|b\|_{L^\infty(\Omega)^d} + C_p\,\|c\|_{L^\infty(\Omega)}, \qquad C_2 = \|b\|_{L^\infty(\Omega)^d} + C(h)\,\|c\|_{L^\infty(\Omega)}.$$

Since the proof is almost the same as in [4], it is omitted. Although the verification condition in [4] is similar form as in Theorem 3.7, the corresponding constant to $K(h)$ has no order in $h$. Therefore, we can say the present technique is more efficient in the actual verification of the invertibility compared with the method in [4].

# 4 Numerical results

For simplicity, in the present section, the domain $\Omega$ is fixed as the unit square $(0,1)\times(0,1) \subset \mathbb{R}^2$. The extension to more general domains should be straightforward. Let's assume that the finite element partition of $\Omega$ is a uniform rectangular mesh and that the basis of $S_h$ is a set of piecewise bilinear polynomials (Q1 element). Therefore, the assumption 2.1 is realized by $C(h) = h/\pi$. Thus, there exists a possibility that the optimal order in Theorem 3.6 is $O(h)$, i.e., linear order in $h$. We studied the order of $K(h)$ by calculating $\sigma_0(h)$ and $\sigma_1(h)$ for several kinds of function $b$.

Since (19) and (20) is a saddle point type problem, the selection of basis for of $W_h$ and $S_h$ is essentially important. For example, when we use Q1/Q1 element, it was actually observed that $\sigma_0(h)$ has a negative order in $h$.

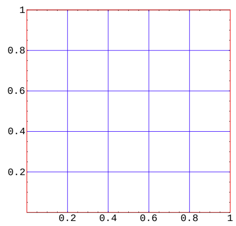


Figure 1: Domain $\Omega$ with uniform mesh $(h = 1/5)$.

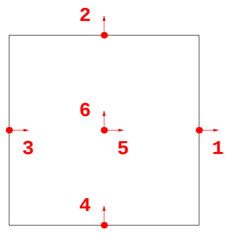


Figure 2: Q2H(div) element.

Below, we mainly used a basis of $W_h$ with piecewise biquadratic polynomials as shown in Figure 2. This base vector function $\chi_i$ has an unknown on the edge or side of element in two and three dimensions, respectively. Moreover, $\chi_i$ satisfies $\chi_i \in H(\mathrm{div},\Omega)$ but $\chi_i \notin C^0(\Omega)^d$. Henceforth, we denote this finite element subspace by Q2H(div). This element has less degree of freedom than the so-called Nedelec element in [6]. Here we compare Q2H(div) with Nedelec elements of quadratic case. The quadratic Nedelec element has 12 degree of freedom on the rectangular element. There exist 8 degree of freedom on the edge of the element and 4 in the interior of the rectangular element. But, as for the numerical efficiency described later, almost no differences were observed, which suggests that Q2H(div) seems to be more suitable for the large-scale computation than the Nedelec element.

**example 1.**
We show a computational result in case that the first component of $b$ is discontinuous (see Figure 3). The computed results are shown in Table 1.

Table 1 shows the result of guaranteed estimations of $\sigma_0(h)$ and $\sigma_1(h)$ with several kinds of combinations for base functions of $W_h$ and $S_h$. Here, Q2H(div) $\subset H(\mathrm{div},\Omega)$ is defined as above, Q2 $\subset H^1(\Omega)$ a piecewise biquadratic element, and Q3Hermite $\subset H^2(\Omega)$ a piecewise bicubic element. Namely, we used the bases with different regularity. As shown in the table 1, all of the result for $\sigma_1(h)$ were small but negative order. In this example, the combination of Q2H(div)/Q1 attains the highest order with $K(h) \approx O(h^{0.93})$.

$$b_1(x,y) = \begin{cases} 1, & (x,y) \in (1/2,1)\times(1/2,1) \\ 0, & \text{otherwise} \end{cases},$$
$$b_2(x,y) \equiv 0.$$

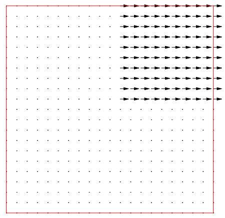

Figure 3: vector field $b$ of example 1.

Table 1: Numerical result of example 1.

| | $W_h$ :Q2H(div), $S_h$ :Q1 (verified computation) | | $W_h$ :Q2, $S_h$ :Q1 (not verified) | | $W_h$ :Q3Hermite, $S_h$ :Q1 (not verified) | |
|---|---|---|---|---|---|---|
| $1/h$ | $\sigma_0$ | $\sigma_1$ | $\sigma_0$ | $\sigma_1$ | $\sigma_0$ | $\sigma_1$ |
| 10 | 0.02811 | 1.06042 | 0.02683 | 1.07983 | 0.02050 | 1.03415 |
| 20 | 0.01437 | 1.18381 | 0.01866 | 1.19133 | 0.01433 | 1.19786 |
| 30 | 0.00961 | 1.21179 | 0.01507 | 1.22202 | 0.01162 | 1.28366 |
| 40 | 0.00734 | 1.23069 | Out of Memory | | Out of Memory | |
| order | 0.93 | -0.04 | 0.52 | -0.06 | 0.51 | -0.17 |

**example 2.**
We show numerical results in case that both $b_1$ and $b_2$ are discontinuous (see Figure 4). The computed results are shown in Table 2.

$$b_1(x,y) = \begin{cases} 1, & (x,y) \in (0,1/2)\times(0,1/2) \\ -1, & (x,y) \in (1/2,1)\times(1/2,1) \\ 0, & \text{otherwise} \end{cases},$$
$$b_2(x,y) = \begin{cases} 1, & (x,y) \in (1/2,1)\times(0,1/2) \\ -1, & (x,y) \in (0,1/2)\times(1/2,1) \\ 0, & \text{otherwise} \end{cases}.$$

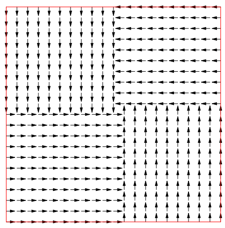

Figure 4: vector field $b$ of example 2.

Table 2: Numerical result of example 2.

| | $W_h$ :Q2H(div), $S_h$ :Q1 | | | |
|---|---|---|---|---|
| $1/h$ | order | $\sigma_0$ | order | $\sigma_1$ |
| 10 | | 0.0291558 | | 1.096629 |
| 20 | 0.951 | 0.0150855 | -0.114 | 1.186535 |
| 30 | 0.959 | 0.0102250 | -0.067 | 1.219194 |
| 40 | 0.963 | 0.0077503 | -0.034 | 1.231223 |

**example 3.**
This is the case that $b$ is given as $b = \nabla\varphi$ for some function $\varphi \in H^1(\Omega)$. In this example, $b$ is discontinuous at $x = 1/2$ and $y = 1/2$ as shown in Figure 5. The computed results are in Table 3.

$$\varphi = \begin{cases} 4xy, & (x,y) \in (0,1/2)\times(0,1/2), \\ 2y, & (x,y) \in (1/2,1)\times(0,1/2), \\ 2x, & (x,y) \in (0,1/2)\times(0,1/2), \\ 1, & (x,y) \in (1/2,1)\times(1/2,1). \end{cases}$$

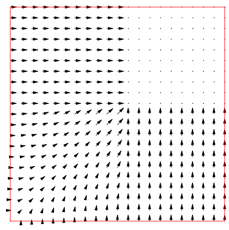

Figure 5: vector field $b$ of example 3.

Table 3: Numerical result of example 3.

| | $W_h$ :Q2H(div), $S_h$ :Q1 | | | |
|---|---|---|---|---|
| $1/h$ | order | $\sigma_0$ | order | $\sigma_1$ |
| 10 | | 0.0574975 | | 2.217311 |
| 20 | 0.897 | 0.0308681 | -0.111 | 2.394029 |
| 30 | 0.911 | 0.0213311 | -0.057 | 2.449684 |
| 40 | 0.921 | 0.0163673 | -0.027 | 2.469006 |

**example 4.**

We also present numerical results for continuous $b$ in Figure 6. In this case, since $b \in W^{1,\infty}(\Omega)^2$, it can be estimated by the framework of [5]. And, naturally, numerical results in Table 4 imply the order estimates in Theorem 3.3 is valid even such a case.

$$b_1(x,y) = xy, \quad (x,y) \in \Omega,$$
$$b_2(x,y) = 0, \quad (x,y) \in \Omega.$$

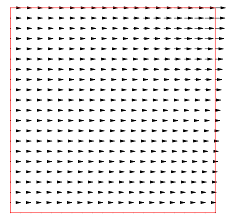

Figure 6: vector field $b$ of example 4.

Table 4: Numerical result of example 4.

| | $W_h$ :Q2H(div), $S_h$ :Q1 | | | |
|---|---|---|---|---|
| $1/h$ | order | $\sigma_0$ | order | $\sigma_1$ |
| 10 | | 0.0283752 | | 1.116333 |
| 20 | 0.977 | 0.0144122 | -0.103 | 1.198800 |
| 30 | 0.985 | 0.0096659 | -0.051 | 1.223655 |
| 40 | 0.988 | 0.0072745 | -0.030 | 1.234204 |

As far as the computational results of the above, it turned out that $K(h)$ are over $O(h^{0.9})$ with guaranteed error bounds. Moreover, it was proved that using discontinuous basis for $W_h$ is more efficient than the continuous one. Since the present method for getting a priori $L^2$ error estimation by Theorem 3.3 has to solve a saddle point problem, the calculation cost is, of course, large. We strongly notice that our estimates are not *a posteriori sense* but *a priori sense*.

All computations are carried out on a Dell Precision 390 Workstation Intel Core2 CPU 2.66GHz by using INTLAB 5.3, a toolbox in MATLAB 7.0.1 developed by Rump [7] for self-validating algorithms. Therefore, all numerical values in these tables are verified data in the sense of strictly rounding error control.

**Conclusion.**
As far as we concerned in this paper, even for the case that the elliptic problem has a singular adjoint operator, the rate of convergence for the a priori $L^2$ error is surely higher than the $H_0^1$ error. This fact is proved by the numerical approach with mathematically rigorous sense. Namely, we proved that, even for such singular cases, there actually exists some finite element subspaces for which the order of a priori $L^2$ error estimates is almost one order higher than a priori $H_0^1$ estimates. This is the first verified result in this field, because there were no such arguments in any papers nor textbooks up to now. In that sense, we can say it is significant and useful in the mathematical theory of finite element methods, as a result obtained by the computer assisted proof. Finally, we note again our results are *not a posteriori error estimates* but *the determination of a priori constants*.

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