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A THIRD ORDER DISPERSIVE FLOW FOR CLOSED CURVES INTO ALMOST HERMITIAN MANIFOLDS

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ABSTRACT. We discuss a short-time existence theorem of solutions to the initial value problem for a third order dispersive flow for closed curves into a compact almost Hermitian manifold. Our equations geometrically generalize a physical model describing the motion of vortex filament. The classical energy method cannot work for this problem since the almost complex structure of the target manifold is not supposed to be parallel with respect to the Levi-Civita connection. In other words, a loss of one derivative arises from the covariant derivative of the almost complex structure. To overcome this difficulty, we introduce a bounded pseudodifferential operator acting on sections of the pullback bundle, and eliminate the loss of one derivative from the partial differential equation of the dispersive flow.

1. INTRODUCTION

Let (N, J, h) be a $2n$-dimensional compact almost Hermitian manifold with an almost complex structure J and a Hermitian metric h. Consider the initial value problem for a third order dispersive flow of the form

$$u_t = a\nabla_x^2 u_x + J_u \nabla_x u_x + bh(u_x, u_x)u_x \quad \text{in} \quad \mathbb{R}\times\mathbb{T}, \tag{1}$$

$$u(0, x) = u_0(x) \quad \text{in} \quad \mathbb{T}, \tag{2}$$

where u is an unknown mapping of $\mathbb{R} \times \mathbb{T}$ to N, $(t, x) \in \mathbb{R} \times \mathbb{T}$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $u_t = du(\partial/\partial t)$, $u_x = du(\partial/\partial x)$, du is the differential of the mapping u, u_0 is a given closed curve on N, ∇ is the induced connection, $a, b \in \mathbb{R}$ are constant. $u(t)$ is a closed curve on N for fixed $t \in \mathbb{R}$, and u describes the motion of a closed curve subject to (1). We present local expression of the covariant derivative ∇_x. Let $y^1, \ldots, y^{2n}$ be local coordinates of N, and let $h = \sum_{a,b=1}^{2n} h_{ab}dy^a \otimes dy^b$. We denote by Γ_{bc}^a, $a, b, c = 1, \ldots, 2n$, the Christoffel symbol of (N, J, h). For a smooth closed curve $u : \mathbb{T} \to N$, $\Gamma(u^{-1}TN)$ is the set of all smooth sections of the pullback bundle $u^{-1}TN$. If we express $V \in \Gamma(u^{-1}TN)$ as

$$V(x) = \sum_{a=1}^{2n} V^a(x) \left(\frac{\partial}{\partial y^a}\right)_u,$$

then, $\nabla_x V$ is given by

$$\nabla_x V(x) = \sum_{a=1}^{2n} \left\{\frac{\partial V^a}{\partial x}(x) + \sum_{b,c=1}^{2n} \Gamma_{bc}^a\big(u(x)\big)V^b(x)\frac{\partial u^c}{\partial x}(x)\right\} \left(\frac{\partial}{\partial y^a}\right)_u.$$

The equation (1) geometrically generalizes two-sphere valued partial differential equations modeling the motion of vortex filament. In his celebrated paper [5], Da Rios first formulated the motion of vortex filament as

$$\vec{u}_t = \vec{u} \times \vec{u}_{xx}, \tag{3}$$

where $\vec{u} = (u^1, u^2, u^3)$ is an $\mathbb{S}^2$-valued function of (t, x), $\mathbb{S}^2$ is a unit sphere in $\mathbb{R}^3$ with a center at the origin, and $\times$ is the exterior product in $\mathbb{R}^3$. The physical meanings of $\vec{u}$ and x are the tangent vector and the signed arc length of vortex filament respectively. After eighty five years, a modified model equation of vortex filament

$$\vec{u}_t = \vec{u} \times \vec{u}_{xx} + a \left[\vec{u}_{xxx} + \frac{3}{2} \left\{ \vec{u}_x \times (\vec{u} \times \vec{u}_x) \right\}_x \right] \tag{4}$$

was proposed by Fukumoto and Miyazaki in [9]. When $a, b = 0$, (1) generalizes (3) and solutions to (1) are called one-dimensional Schrödinger maps. When $b = a/2$, (1) generalizes (4).

In recent ten years, physical models such as (3) and (4) have been generalized and studied from a point of view of geometric analysis in mathematics. The relationship between the geometric settings and the structure of such partial differential equations and their solutions has been recently investigated in mathematics.

The reduction of equations to simpler ones leads us to rough understandings of their structure. This idea originated from Hasimoto's transform discovered in [10]. In their pioneering work [1], Chang, Shatah and Uhlenbeck first rigorously studied the PDE structure of (1) when $a = b = 0$, $x \in \mathbb{R}$, and (N, J, h) is a compact Riemann surface. They constructed a good moving frame along the map and reduced (1) to a simple complex-valued semilinear Schrödinger equation under the assumption that $u(t, x)$ has a fixed base point as $x \to +\infty$. Similarly, Onodera reduced (1) with $a \neq 0$ and a one-dimensional fourth order dispersive flow to complex-valued equations in [23]. Generally speaking, these reductions require some restrictions on the range of the mappings, and one cannot make use of them to solve the initial value problem for the original equations without restrictions on the range of the initial data.

How to solve the initial value problem for such geometric dispersive equations is a fundamental question. In his pioneering work [14], Koiso first reformulated (3) geometrically, and proposed the equation (1) with $a, b = 0$ and the Kähler condition $\nabla^N J = 0$, where ∇^N is the Levi-Civita connection of (N, J, h). Moreover, Koiso established the standard short-time existence theorem, and proved that if (N, J, h) is locally symmetric, that is, $\nabla^N R = 0$, then the solution exists globally in time, where R is the Riemannian curvature tensor of N. See [25] also for one-dimensional Schödinger maps. Recently, Onodera studied local and global existence theorems of (1)-(2) in case $a \neq 0$ in [22] and [24]. To be more precise, [22] studied the case $\nabla^N J = 0$, and proved a short-time existence theorem. Moreover, he proved that if (N, J, h) is a compact Riemann surface with a constant sectional curvature K and a condition $b = Ka/2$ is satisfied, then the time-local solution can be extended globally in time. Nishiyama and Tani proved the global existence of solutions to the initial value problem for (4) in [21] and [26]. Since $K = 1$ for $N = \mathbb{S}^2$, the global existence theorem in [22] is the generalization of the results [21] and [26]. [24] studied a short-time existence theorem for (1)-(2) in case that (N, J, h) is a compact almost Hermitian manifold and $x \in \mathbb{R}$. Being inspired by Tarama's beautiful results on the characterization of L^2-well-posedness of the initial value problem for a one-dimensional linear third order dispersive equations in [28] (See also [17]), Onodera introduced a gauge transform on the pullback bundle to make full use of so-called local smoothing effect of $e^{t\partial^3/\partial x^3}$, and proved a short-time existence theorem.

Both of the reduction of equations and the study of existence theorem are deeply connected with the relationship between the geometric settings of equations and the theory of linear dispersive partial differential equations. For the latter subject, see, e.g., [3], [6], [16, Lecture VII], [17], [27], [28] and references therein. Being concerned with the compactness of the source space, we need to mention local smoothing effect of dispersive partial differential equations. It is well-known that solutions to the initial value problem for some kinds of dispersive equations gain extra smoothness in comparison with the initial data. In his celebrated work [6], Doi characterized the existence of microlocal smoothing effect of Schrödinger evolution equations on complete Riemannian manifolds according to the global

behavior of the geodesic flow on the unit cotangent sphere bundle over the source manifolds. Roughly speaking, the local smoothing effect occurs if and only if all the geodesics go to "infinity". For more general dispersive equations, the existence or nonexistence of local smoothing effect is determined by the global behavior of the Hamilton flow generated by the principal symbol of the equations. In particular, if the source space is compact, then no smoothing effect occurs since all the integral curves of the Hamilton vector field are trapped. For this reason, it is essential to study the initial value problem (1)-(2) when the source space is $\mathbb{T}$ and not $\mathbb{R}$.

Here we mention the relationship between the Kähler condition $\nabla^N J = 0$ and the structure of the equation (1). All the preceding works on (1) except for [24] assume that (N, J, h) is a Kähler manifold. If $\nabla^N J = 0$, then (1) behaves like symmetric hyperbolic systems, and the short-time existence theorem can be proved by the classical energy method. See [22] for the detail. If $\nabla^N J \neq 0$, then (1) has a first order terms in some sense, and the classical energy method breaks down.

The purpose of the present paper is to show a short-time existence theorem for (1)-(2) without using the Kähler condition and the local smoothing effect. To state our results, we here introduce function spaces of mappings. For a nonnegative integer k, $H^{k+1}(\mathbb{T}; TN)$ is the set of all continuous mappings $u : \mathbb{T} \to N$ satisfying

$$\|u\|_{H^{k+1}}^2 = \sum_{l=0}^{k} \int_{\mathbb{T}} h\left(\nabla_x^k u_x, \nabla_x^k u_x\right) dx < \infty,$$

See e.g., [11] for the Sobolev space of mappings. The Nash embedding theorem shows that there exists an isometric embedding $w \in C^\infty(N; \mathbb{R}^d)$ with some integer $d > 2n$. See [7], [8] and [19] for the Nash embedding theorem. Let I be an interval in $\mathbb{R}$. We denote by $C(I; H^{k+1}(\mathbb{T}; TN))$ the set of all $H^{k+1}(\mathbb{T}; TN)$-valued continuous functions on I, In other words, we define it by the pullback of the function space as $C(I; H^{k+1}(\mathbb{T}; TN)) = C(I; w^* H^{k+1}(\mathbb{T}; \mathbb{R}^d))$, where $H^{k+1}(\mathbb{T}; \mathbb{R}^d)$ is the usual Sobolev space of $\mathbb{R}^d$-valued functions on $\mathbb{T}$.

Here we state our main results.

Theorem 1. *Let k be a positive integer satisfying $k \geqslant 4$. Then, for any $u_0 \in H^{k+1}(\mathbb{T}; TN)$, there exists $T = T(\|u\|_{H^5}) > 0$ such that* (1)-(2) *possesses a unique solution $u \in C([-T, T]; H^{k+1}(\mathbb{T}; TN))$.*

We will prove Theorem 1 by the uniform energy estimates of solutions to a fourth order parabolic regularized equation. To avoid the difficulty arising from $\nabla_x J_u$, we modify the method introduced for the initial value problem for Schrödinger maps of a closed Riemannian manifold to a compact almost Hermitian manifold in [4]. Being inspired by his own previous paper [2], Chihara introduced a transformation of unknown mappings defined by a bounded pseudodifferential operator acting on sections of $\Gamma(u^{-1}TN)$, and eliminated first order terms coming from $\nabla_x J_u$ in [4].

The plan of the present paper is as follows. Section 2 studies the well-posedness of an auxiliary initial value problem for some one-dimensional linear dispersive partial differential equations related with (1)-(2). We believe that Section 2 will be very helpful to understand our idea of the proof of Theorem 1, though the arguments and results there are nonsense from a point of view of the theory of linear partial differential equations. Section 3 proves Theorem 1.

2. An Auxiliary Linear Problem

In this section we study the initial value problem for a one-dimensional third order linear dispersive partial differential equation related with (1) of the form

$$LU \equiv U_t + U_{xxx} + \sqrt{-1}\{a(x)U_x\}_x + b_x(x)U_x + c(x)U = F(t, x) \quad \text{in} \quad \mathbb{R} \times \mathbb{T}, \qquad (5)$$

$$U(0, x) = U_0(x) \quad \text{in} \quad \mathbb{T}, \qquad (6)$$

where U is a complex-valued unknown function of $(t, x) \in \mathbb{R} \times \mathbb{T}$, $a, b, c \in C^\infty(\mathbb{T})$, $\text{Im}\, a = 0$, $U_0(x)$ and $F(t, x)$ are given functions. The operator L is very special in the sense that the coefficient of

the first order term is a derivative of a smooth function. The well-posedness of the initial value problem for third and fourth order dispersive equations on $\mathbb{R}$ or $\mathbb{T}$ was studied in [17], [27] and [28]. In most of cases the well-posedness was characterized by the conditions on the coefficients of differential operators. Let $L^2(\mathbb{T})$ be the standard Lebesgue space of square-integrable functions on $\mathbb{T}$, and let $L^1_{\text{loc}}(\mathbb{R}; L^2(\mathbb{T}))$ be the set of all $L^2(\mathbb{T})$-valued locally integrable functions on $\mathbb{R}$. Mizuhara characterized the well-posedness of the initial value problem for a general third order dispersive equations on $\mathbb{R} \times \mathbb{T}$. In view of his results in [17, Theorem 6.1], one can immediately check that the special initial value problem (5)-(6) is well-posed.

Proposition 2. (5)-(6) *is L^2-well-posed, that is, for any $U_0 \in L^2(\mathbb{T})$ and for any $F \in L^1_{loc}(\mathbb{R}; L^2(\mathbb{T}))$, (5)-(6) possesses a unique solution $U \in C(\mathbb{R}; L^2(\mathbb{T}))$.*

All the descriptions in the present section are meaningless from a viewpoint of the general theory of linear partial differential equations. However, the purpose of this section is to illustrate our idea of the proof of Theorem 1 by showing the special proof of Proposition 2. In what follows we make use of an elementary theory of pseudodifferential operators on $\mathbb{R}$. See [15] for instance. In view of the idea in [13, Section 2], one can deal with pseudodifferential operators on $\mathbb{T}$ in the same way as those on $\mathbb{R}$ without using the general theory of pseudodifferential operators on manifolds. $C^\infty(\mathbb{T})$ is regarded as the set of all 1-periodic smooth functions on $\mathbb{R}$. Its topological dual is the set of all 1-periodic tempered distributions on $\mathbb{R}$.

Let $p(\xi)$ be a real-valued smooth odd function on $\mathbb{R}$ satisfying $p(\xi) = 1/\xi$ for $\xi \in \mathbb{R} \setminus (-2, 2)$ and $p(\xi) = 0$ for $\xi \in [-1, 1]$. A pseudodifferential operator $p(D_x)$ is defined by an oscillatory integral of the form

$$p(D_x)u(x) = \frac{1}{2\pi} \iint_{\mathbb{R} \times \mathbb{R}} e^{\sqrt{-1}(x-y)\xi} p(\xi) U(y) dy d\xi \quad \text{for} \quad U \in \mathscr{B}^\infty(\mathbb{R}),$$

where $D_x = -\sqrt{-1}\partial/\partial x$, $\mathscr{B}^\infty(\mathbb{R})$ is the set of all bounded C^∞-functions on $\mathbb{R}$ whose derivative of any order is also bounded in $\mathbb{R}$. It is well-known that $p(D_x)$ is well-defined on $\mathscr{B}^\infty(\mathbb{R})$ and extended on the set of all tempered distributions on $\mathbb{R}$. $-\sqrt{-1}p(D_x)$ is an essential realization of the integral over $(-\infty, x]$ by pseudodifferential operators. The important properties of $p(D_x)$ are the following.

Lemma 3. *If $U(x)$ is real-valued and 1-periodic, then so is $\sqrt{-1}p(D_x)U(x)$.*

Proof. Let $U \in C^\infty(\mathbb{T})$. We can easily check that $p(D_x)u(x)$ is 1-periodic by using a translation $x \mapsto x + 1$. Suppose that $U(x)$ is real-valued in addition. Then,

$$\text{Im}\left\{\sqrt{-1}p(D_x)U(x)\right\} = \text{Re}\left\{p(D_x)U(x)\right\}$$

$$= \frac{1}{2\pi} \iint_{\mathbb{R} \times \mathbb{R}} \text{Re}\left\{e^{\sqrt{-1}(x-y)\xi} p(\xi) U(y)\right\} dy d\xi$$

$$= \frac{1}{2\pi} \iint_{\mathbb{R} \times \mathbb{R}} \cos\{(x-y)\xi\} p(\xi) U(y) dy d\xi = 0$$

since the integrand in the last integral above is an odd function in ξ. $\qquad\square$

Our special proof of Proposition 2 uses a bounded pseudodifferential operator defined by

$$\lambda(x, D_x) = 1 - \tilde{\lambda}(x, D_x), \quad \tilde{\lambda}(x, \xi) = \frac{\sqrt{-1}}{3} b(x) p(\xi).$$

Roughly speaking, $\lambda(x, D_x)$ is a linear automorphism on $L^2(\mathbb{T})$. Indeed, it is easy to see that there exists a constant $M > 1$ depending on $b(x)$ and $p(\xi)$ such that

$$M^{-1}\|U\| \leqslant \mathcal{N}(U) \leqslant M\|U\| \quad \text{for any} \quad U \in L^2(\mathbb{T}), \tag{7}$$

where $\mathcal{N}(U)^2 = \|\lambda(x, D_x)U\|^2 + \|\langle D_x \rangle^{-1}U\|^2$, $\langle D_x \rangle = (1 - \partial^2/\partial x^2)^{1/2}$, and $\|\cdot\|$ is the norm of $L^2(\mathbb{T})$. We prove Proposition 2 by using a transform $U \mapsto \lambda(x, D_x)U$ as follows.

Sketch of proof of Proposition 2. It suffices to show forward and backward energy inequalities. See [12, Section 23.1] for instance. We obtain only an energy inequality in the positive direction in t. The backward one can be obtained similarly. A direct computation shows that

$$\lambda(x, D_x)L = (\partial_t + \partial_x^3 + \sqrt{-1}\partial_x a(x)\partial_x)\lambda(x, D_x)$$
$$- [\tilde{\lambda}(x, D_x), \partial_x^3] + b_x(x)\partial_x + r_1(x, D_x), \tag{8}$$
$$r_1(x, D_x) = -[\tilde{\lambda}(x, D_x), \sqrt{-1}\partial_x a(x)p_x] - \tilde{\lambda}(x, D_x)b_x(x)\partial_x + \lambda(x, D_x)c(x),$$
$$\langle D_x \rangle^{-1}L = (\partial_t + \partial_x^3 + \sqrt{-1}\partial_x a(x)\partial_x)\langle D_x \rangle^{-1} + r_2(x, D_x) \tag{9}$$
$$r_2(x, D_x) = [\langle D_x \rangle^{-1}, \sqrt{-1}\partial_x a(x)\partial_x] + \langle D_x \rangle^{-1}(b_x(x)\partial_x + c(x)),$$

where $\partial_t = \partial/\partial t$ and $\partial_x = \partial/\partial x$. $r_1(x, D_x)$ and $r_2(x, D_x)$ are L^2-bounded pseudodifferential operators. We remark that

$$-[\tilde{\lambda}(x, D_x), \partial_x^3] = -b_x(x)\partial_x + r_3(x, D_x),$$

$$r_3(x, D_x) = b_x(x)\partial_x(1 - p(D_x)D_x) - b_{xx}(x)p(D_x)D_x + \frac{\sqrt{-1}}{3}b_{xxx}(x)p(D_x),$$

and $r_3(x, D_x)$ is also an L^2-bounded pseudodifferential operator. Set $r_4 = r_1 + r_3$ for short. Then, (8) becomes

$$\lambda(x, D_x)L = (\partial_t + \partial_x^3 + \sqrt{-1}\partial_x a(x)\partial_x)\lambda(x, D_x) + r_4(x, D_x). \tag{10}$$

Fix arbitrary $T > 0$. Suppose that $U \in C([0, T]; H^3(\mathbb{T})) \cap C^1([0, T]; L^2(\mathbb{T}))$. By using (9) and (10), one can easily show that there exists a positive constant C_0 depending on a, b, c and p such that

$$\frac{d\mathcal{N}(U(t))^2}{dt} \leqslant C_0 \left(\mathcal{N}(U(t)) + \mathcal{N}(LU(t))\right)\mathcal{N}(U(t)),$$

which implies a desired energy inequality

$$\|U(t)\| \leqslant C_1 \left\{ \|U(0)\| + \int_0^t \|LU(s)\|ds \right\} \quad \text{for} \quad t \in [0, T],$$

where C_1 is a positive constant depending only on a, b, c and p. $\qquad\square$

3. PROOF OF THEOREM 1

We shall prove Theorem 1 by the uniform energy estimates of solutions to the initial value problem for semilinear parabolic equations of the form

$$u_t^\varepsilon = -\varepsilon \nabla_x^3 u_x^\varepsilon + a\nabla_x^2 u_x^\varepsilon + J_{u^\varepsilon}\nabla_x u_x^\varepsilon + bh(u_x^\varepsilon, u_x^\varepsilon)u_x^\varepsilon \quad \text{in} \quad (0, \infty) \times \mathbb{T}, \tag{11}$$
$$u^\varepsilon(0, x) = u_0(x) \quad \text{in} \quad \mathbb{T}, \tag{12}$$

where $\varepsilon \in (0, 1]$ is a parameter. The existence of solutions to (11)-(12) was proved as follows.

Lemma 4 ([22, Proposition 3.1])**.** *Let k be a positive integer satisfying $k \geqslant 2$. Then, for any $u_0 \in H^{k+1}(\mathbb{T}; TN)$, there exists $T_\varepsilon = T(\varepsilon, \|u\|_{H^3}) >$ such that (11)-(12) possesses a unique solution $u^\varepsilon \in C([0, T_\varepsilon]; H^{k+1}(\mathbb{T}; TN))$.*

The proof of Lemma 4 given in [22] does not depend on the Kähler condition at all. Lemma 4 is proved by the standard arguments: the contraction mapping theorem and some kind of maximum principle. Firstly, we push forward (11)-(12) into $\mathbb{R}^d$ by the Nash embedding w, and construct a solution taking values in a small tubular neighborhood of $w(N)$. Secondly, we check that the value of the solution remains in $w(N)$. See [22, Section 3] for the detail.

We split the proof of Theorem 1 into three steps. Firstly, we construct a solution by the uniform energy estimates and the standard compactness argument. Secondly, we check the uniqueness of solutions. Finally, we recover the continuity in time of solutions.

Construction of Solutions. Let u^ε be a unique solution to (11)-(12) with a parameter $\varepsilon \in (0,1]$. It suffices to show that there exists $T > 0$ which is independent of $\varepsilon \in (0,1]$, such that $\{u^\varepsilon\}_{\varepsilon \in (0,1]}$ is bounded in $L^\infty(0,T; H^{k+1}(\mathbb{T}; TN))$, which is the set of all H^{k+1}-valued essentially bounded functions on $(0,T)$. Indeed, if this is true, then the standard compactness argument shows that there exist u and a subsequence $\{u^\varepsilon\}_{\varepsilon \in (0,1]}$ such that

$$u^\varepsilon \longrightarrow u \quad \text{in} \quad C([0,T]; H^k(\mathbb{T}; TN)),$$
$$u^\varepsilon \longrightarrow u \quad \text{in} \quad L^\infty(0,T; H^{k+1}(\mathbb{T}; TN)) \quad \text{weakly star},$$

as $\varepsilon \downarrow 0$, and u solves (1)-(2) and is H^{k+1}-valued weakly continuous in time.

Set $u = u^\varepsilon$ for short. Any confusion will not occur. We actually evaluate

$$\mathcal{N}_{k+1}(u)^2 = \|u\|_{H^k}^2 + \|\Lambda \nabla_x^k u_x\|^2,$$

where $\Lambda = \Lambda_\varepsilon(t,x,u)$ is a bounded pseudodifferential operator acting on $\Gamma((u^\varepsilon)^{-1}TN)$ defined later, and $\|\cdot\|$ is a norm of $L^2(\mathbb{T}; TN)$ defined by

$$\|V\|^2 = \int_{\mathbb{T}} h(V,V)dx \quad \text{for} \quad V : \mathbb{T} \to TN.$$

Set

$$T_\varepsilon^* = \sup\{T > 0 \mid \mathcal{N}_{k+1}(u(t)) \leqslant 2\mathcal{N}_{k+1}(u_0) \text{ for } t \in [0,T]\}.$$

We need to compute

$$\nabla_x^{l+1}\left(u_t + \varepsilon\nabla_x^3 u_x - a\nabla_x^2 u_x - J_u\nabla_x u_x - bh(u_x, u_x)u_x\right) = 0, \quad l = 0, \dots, k.$$

Main tools of the computation are

$$\nabla_X du(Y) = \nabla_Y du(X) + du([X,Y]) = \nabla_Y du(X), \tag{13}$$
$$\nabla_X \nabla_Y V = \nabla_Y \nabla_X V + \nabla_{[X,Y]}V + R\big(du(X), du(Y)\big)V$$
$$= \nabla_Y \nabla_X V + R\big(du(X), du(Y)\big)V. \tag{14}$$

for $X, Y \in \{\partial_t, \partial_x\}$ and $V \in \Gamma(u^{-1}TN)$. We make use of basic techniques of geometric analysis of nonlinear problems. See [20] for instance.

In view of (13) and (14), we have

$$\nabla_x u_t = \nabla_t u_x, \tag{15}$$
$$\nabla_x^2 u_t = \nabla_t \nabla_x u_x + R(u_x, u_t)u_x,$$
$$\nabla_x^{l+1} u_t = \nabla_t \nabla_x^l u_x + \sum_{m=0}^{l-1} \nabla_x^{l-1-m}\left\{R(u_x, u_t)\nabla_x^m u_x\right\}$$
$$= \nabla_t \nabla_x^l u_x + \sum_{m=0}^{l-1} \nabla_x^{l-1-m}$$
$$\times \left\{R\big(u_x, -\varepsilon\nabla_x^3 u_x + a\nabla_x^2 u_x + J_u\nabla_x u_x + bh(u_x,u_x)u_x\big)\nabla_x^m u_x\right\}$$
$$= \nabla_t \nabla_x^l u_x + aR(u_x, \nabla_x^{l+1}u_x) - \varepsilon P_{1,l+1} - Q_{1,l+1}, \tag{16}$$
$$P_{1,l+1} = \sum_{m=0}^{l-1} \nabla_x^{l-1-m}\left\{R\big(u_x, \nabla_x^3 u_x\big)\nabla_x^m u_x\right\},$$
$$Q_{1,l+1} = -a\sum_{m=0}^{l-1} \nabla_x^{l-1-m}\left\{R\big(u_x, \nabla_x^2 u_x\big)\nabla_x^m u_x\right\} + aR(u_x, \nabla_x^{l+1}u_x)$$

$$-\sum_{m=0}^{l-1} \nabla_x^{l-1-m}\left\{R\big(u_x, J_u\nabla_x u_x + bh(u_x, u_x)u_x\big)\nabla_x^m u_x\right\}.$$

The Sobolev embeddings show that

$$\|P_{1,l+1}\| \leqslant C_k\|u\|_{H^{l+3}}, \quad \|Q_{1,l+1}\| \leqslant C_k\|u\|_{H^{l+1}} \tag{17}$$

for $t \in [0, T_\varepsilon^*]$, where $C_k > 1$ is a constant depending only on a, b and $\|u_0\|_{H^{k+1}}$ and not on $\varepsilon \in (0, 1]$. Such constants are denoted by the same notation C_k below. Using (13) and (14) again, we have

$$\nabla_x^{l+1}(J_u\nabla_x u_x) = \nabla_x J_u\nabla_x\nabla_x^l u_x + l(\nabla_x J_u)\nabla_x\nabla_x^l u_x + Q_{2,l+1}, \tag{18}$$

$$\nabla_x^{l+1}\{h(u_x, u_x)u_x\} = h(u_x, u_x)\nabla_x\nabla_x^l u_x + 2\{h(\nabla_x^l u_x, u_x)\}_x u_x + Q_{3,l+1}, \tag{19}$$

$$Q_{2,l+1} = \sum_{m=0}^{l-1}\frac{(l+1)!}{m!(l+1-m)!}(\nabla_x^{l+1-m}J_u)\nabla_x^{m+1}u_x,$$

$$Q_{3,l+1} = \sum_{\substack{\alpha+\beta+\gamma=l+1\\ \alpha,\beta,\gamma\leqslant l}}\frac{(l+1)!}{\alpha!\beta!\gamma!}h(\nabla_x^\alpha u_x, \nabla_x^\beta u_x)\nabla_x^\gamma u_x - 2h(\nabla_x^l u_x, \nabla_x u_x)u_x.$$

$Q_{2,l+1}$ and $Q_{3,l+1}$ have the same estimates as $Q_{1,l+1}$. Combining (15), (16), (17), (18) and (19), we obtain

$$\left\{\nabla_t + \varepsilon\nabla_x^4 - a\nabla_x^3 - \nabla_x J_u\nabla_x - l(\nabla_x J_u)\nabla_x - bh(u_x, u_x)\nabla_x\right\}\nabla_x^l u_x$$
$$= -aR(u_x, \nabla_x^{l+1}u_x)u_x + 2b\{h(\nabla_x^l u_x, u_x)\}_x u_x + \varepsilon P_{l+1} + Q_{l+1}, \tag{20}$$

$$\|P_{l+1}\| \leqslant C_k\|u\|_{H^{l+3}}, \quad \|Q_{l+1}\| \leqslant C_k\|u\|_{H^{l+1}} \quad \text{for} \quad t \in [0, T_\varepsilon^*]. \tag{21}$$

By using (20), we have

$$\frac{d}{dt}\|u\|_{H^k}^2 = 2\sum_{l=0}^{k-1}\int_{\mathbb{T}} h(\nabla_t\nabla_x^l u_x, \nabla_x^l u_x)dx$$

$$= -2\varepsilon\sum_{l=0}^{k-1}\int_{\mathbb{T}} h(\nabla_x^4\nabla_x^l u_x, \nabla_x^l u_x)dx \tag{22}$$

$$+ 2a\sum_{l=0}^{k-1}\int_{\mathbb{T}} h(\nabla_x^3\nabla_x^l u_x, \nabla_x^l u_x)dx \tag{23}$$

$$+ 2\sum_{l=0}^{k-1}\int_{\mathbb{T}} h(\nabla_x J_u\nabla_x\nabla_x^l u_x, \nabla_x^l u_x)dx \tag{24}$$

$$+ 2\sum_{l=0}^{k-1}\int_{\mathbb{T}} h\big((\nabla_x J_u)\nabla_x\nabla_x^l u_x, \nabla_x^l u_x\big)dx \tag{25}$$

$$+ 2b\sum_{l=0}^{k-1}\int_{\mathbb{T}} h(u_x, u_x)h(\nabla_x\nabla_x^l u_x, \nabla_x^l u_x)dx \tag{26}$$

$$- 2a\sum_{l=0}^{k-1}\int_{\mathbb{T}} h\big(R(u_x, \nabla_x^{l+1}u_x)u_x, \nabla_x^l u_x\big)dx \tag{27}$$

$$+ 4b\sum_{l=0}^{k-1}\int_{\mathbb{T}}\{h(\nabla_x^l u_x, u_x)\}_x h(u_x, \nabla_x^l u_x)dx \tag{28}$$

$$+ 2\sum_{l=0}^{k-1} \int_{\mathbb{T}} h(\varepsilon P_{l+1} + Q_{l+1}, \nabla_x^l u_x)dx. \tag{29}$$

Using integration by parts and the properties of h and J, we deduce that (22), (23), (24), (26), (28) respectively become

$$(22) = -2\varepsilon \sum_{l=0}^{k-1} \int_{\mathbb{T}} h(\nabla_x^{l+2} u_x, \nabla_x^{l+2} u_x)dx, \tag{30}$$

$$(23) = -2a \sum_{l=0}^{k-1} \int_{\mathbb{T}} h(\nabla_x \nabla_x^{l+1} u_x, \nabla_x^{l+1} u_x)dx$$

$$= -a \sum_{l=0}^{k-1} \int_{\mathbb{T}} \left\{ h(\nabla_x^{l+1} u_x, \nabla_x^{l+1} u_x) \right\}_x dx = 0, \tag{31}$$

$$(24) = -2 \sum_{l=0}^{k-1} \int_{\mathbb{T}} h(J_u \nabla_x^{l+1} u_x, \nabla_x^{l+1} u_x)dx = 0, \tag{32}$$

$$(26) = b \sum_{l=0}^{k-1} \int_{\mathbb{T}} h(u_x, u_x) \left\{ h(\nabla_x^l u_x, \nabla_x^l u_x) \right\}_x dx$$

$$= -b \sum_{l=0}^{k-1} \int_{\mathbb{T}} \{h(u_x, u_x)\}_x h(\nabla_x^l u_x, \nabla_x^l u_x)dx, \tag{33}$$

$$(28) = 2b \sum_{l=0}^{k-1} \int_{\mathbb{T}} \left\{ h(\nabla_x^l u_x, u_x)^2 \right\}_x dx = 0. \tag{34}$$

Recall the property of the Riemannian curvature tensor R: $h(R(X,Y)Z,W) = h(R(Z,W)X,Y)$ for any vector fields X, Y, X, W on N. Using this and integration by parts, we deduce

$$(27) = -2a \sum_{l=0}^{k-1} \int_{\mathbb{T}} h\big(R(u_x, \nabla_x^l u_x)u_x, \nabla_x^{l+1} u_x\big)dx$$

$$= 2a \sum_{l=0}^{k-1} \int_{\mathbb{T}} h\big(R(u_x, \nabla_x^{l+1} u_x)u_x, \nabla_x^l u_x\big)dx$$

$$+ 2a \sum_{l=0}^{k-1} \int_{\mathbb{T}} h\big((\nabla^N R)(u_x, u_x, \nabla_x^l u_x)u_x, \nabla_x^l u_x\big)dx$$

$$+ 2a \sum_{l=0}^{k-1} \int_{\mathbb{T}} h\big(R(\nabla_x u_x, \nabla_x^l u_x)u_x, \nabla_x^l u_x\big)dx$$

$$+ 2a \sum_{l=0}^{k-1} \int_{\mathbb{T}} h\big(R(u_x, \nabla_x^l u_x)\nabla_x u_x, \nabla_x^l u_x\big)dx,$$

which implies

$$(27) = a \sum_{l=0}^{k-1} \int_{\mathbb{T}} h\big((\nabla^N R)(u_x, u_x, \nabla_x^l u_x)u_x, \nabla_x^l u_x\big)dx$$

$$+ a \sum_{l=0}^{k-1} \int_{\mathbb{T}} h\big(R(\nabla_x u_x, \nabla_x^l u_x)u_x, \nabla_x^l u_x\big)dx$$

$$+ a \sum_{l=0}^{k-1} \int_{\mathbb{T}} h\big(R(u_x, \nabla_x^l u_x)\nabla_x u_x, \nabla_x^l u_x\big)dx. \tag{35}$$

Applying the Schwarz inequality to (33), (35) and (29), we have

$$|(26)|, |(27)| \leqslant C_k \|u\|_{H^k}^2, \tag{36}$$

$$|(29)| \leqslant C_k \varepsilon \|u\|_{H^{k+2}} \|u\|_{H^k} + C_k \|u\|_{H^k}^2$$

$$\leqslant 2\varepsilon \sum_{l=0}^{k-1} \int_{\mathbb{T}} h(\nabla_x^{l+2} u_x, \nabla_x^{l+2} u_x)dx + C_k \|u\|_{H^k}^2. \tag{37}$$

Similarly, (25) is estimated as

$$|(25)| \leqslant C_k \|u\|_{H^{k+1}} \|u\|_{H^k}, \tag{38}$$

Combining (30), (31), (32), (34), (36), (37) and (38), we obtain

$$\frac{d}{dt} \|u\|_{H^k}^2 \leqslant C_k \|u\|_{H^{k+1}} \|u\|_{H^k}. \tag{39}$$

Next we estimate $\Lambda \nabla_x^k u_x$. Here we define the pseudodifferential operator Λ. Let $\{N_\alpha\}$ be the set of local coordinate neighborhood of N, and let $y_\alpha^1, \ldots, y_\alpha^{2n}$ be the local coordinates of N_α. Pick up a partition of unity $\{\Phi_\alpha\}$ subordinated to $\{N_\alpha\}$, and pick up $\{\Psi_\alpha\} \subset C_0^\infty(N)$ so that

$$\Psi_\alpha = 1 \quad \text{in} \quad \text{supp}[\Phi_\alpha], \quad \text{supp}[\Psi_\alpha] \subset N_\alpha,$$

where $C_0^\infty(N)$ is the set of all compactly supported C^∞-functions on N. We define a properly supported pseudodifferential operator Λ acting on $\Gamma(u^{-1}TN)$ by

$$\Lambda = 1 - \tilde{\Lambda}, \quad \tilde{\Lambda} = \frac{\sqrt{-1}k}{3a} J_u \sum_\alpha \Phi_\alpha(u)p(D_x)\Psi_\alpha(u).$$

If

$$V(x) = \sum_{a=1}^{2n} V^a(x) \left(\frac{\partial}{\partial y_\alpha^a}\right)_u \in \Gamma(u^{-1}TN)$$

is supported in $u^{-1}(N_\alpha)$, then

$$\Phi_\alpha(u)p(D_x)V(x) = \sum_{a=1}^{2n} \{\Phi_\alpha(u)p(D_x)V^a(x)\} \left(\frac{\partial}{\partial y_\alpha^a}\right)_u$$

is well-defined and supported in $u^{-1}(N_\alpha)$. Then, each term in $\tilde{\Lambda}$ can be treated as a pseudodifferential operator acting on $\mathbb{R}^d$-valued functions, and we can make use of pseudodifferential operators with nonsmooth symbols. In other words, we can deal with $\tilde{\Lambda}$ as if it were a pseudodifferential operator with a smooth symbol. See [2, Section 2] and [18] for the detail. Symbolic calculus below is valid since the Sobolev embedding shows that $u(t) \in C^{4+\delta}(\mathbb{T})$ for $\delta \in (0, 1/2)$. It is easy to see that there exists $C_k > 1$ such that

$$C_k^{-1}\mathcal{N}_{k+1}(u) \leqslant \|u\|_{H^{k+1}} \leqslant C_k \mathcal{N}_{k+1}(u) \quad \text{for} \quad t \in [0, T_\varepsilon^*].$$

We compute

$$0 = \Lambda \nabla_x^{k+1} \big(u_t + \varepsilon \nabla_x^3 u_x - a\nabla_x^2 u_x - J_u \nabla_x u_x - bh(u_x, u_x)u_x\big)$$

$$= \Lambda \big\{\nabla_t + \varepsilon \nabla_x^4 - a\nabla_x^3 - \nabla_x J_u \nabla_x - k(\nabla_x J_u)\nabla_x - bh(u_x, u_x)\nabla_x\big\} \nabla_x^k u_x$$

$$- \Lambda \left\{ -aR(u_x, \nabla_x^{k+1} u_x)u_x + 2b\{h(\nabla_x^k u_x, u_x)\}_x u_x + \varepsilon P_{k+1} + Q_{k+1} \right\}.$$

A direct computation shows that

$$\Lambda \nabla_t = \nabla_t \Lambda - \frac{\partial \Lambda}{\partial t} = \nabla_t \Lambda + \frac{\partial \tilde{\Lambda}}{\partial t}, \tag{40}$$

$$\left\| \frac{\partial \tilde{\Lambda}}{\partial t} \nabla_x^k u_x \right\| \leqslant C_k \|u\|_{H^k}.$$

Let I_{2n} be the $2n \times 2n$ identity matrix. If we use a local expression $\nabla_x^4 = \partial_x^4 + A_3 \partial_x^3 + A_2 \partial_x^2 + A_1 \partial_x + A_0$ with $2n \times 2n$ matrices A_j, $j = 0, 1, 2, 3$, we deduce that

$$\varepsilon \Lambda \nabla_x^4 = \varepsilon \nabla_x^4 \Lambda + \varepsilon[\Lambda, \nabla_x^4] = \varepsilon \nabla_x^4 \Lambda - \varepsilon[\tilde{\Lambda}, \nabla_x^4], \tag{41}$$

$$[\tilde{\Lambda}, \nabla_x^4] = \left[\frac{\sqrt{-1}k}{3a} J_u p(D_x), I_{2n} \partial_x^4 + \cdots \right], \quad \|[\tilde{\Lambda}, \nabla_x^4] \nabla_x^k u_x\| \leqslant C_k \|u\|_{H^{k+3}},$$

since the matrices of principal symbols $J_u p(D_x)$ and ∇_x^4 commute with each other. Next computation is the most crucial part of the proof of Theorem 1. In the same way as $\varepsilon \Lambda \nabla_x^4$, we have

$$-a\Lambda \nabla_x^3 = -a \nabla_x^3 \Lambda + a[\tilde{\Lambda}, \nabla_x^3].$$

We see the commutator above in detail. A direct computation shows that

$$\begin{aligned}
a[\tilde{\Lambda}, \nabla_x^3] &= \frac{\sqrt{-1}k}{3} \sum_\alpha J_u \Phi_\alpha(u) p(D_x) \Psi_\alpha(u) \nabla_x^3 \\
&\quad - \frac{\sqrt{-1}k}{3} \sum_\alpha \nabla_x^3 J_u \Phi_\alpha(u) p(D_x) \Psi_\alpha(u) \\
&= \frac{\sqrt{-1}k}{3} \sum_\alpha J_u \Phi_\alpha(u) p(D_x) \nabla_x^3 \Psi_\alpha(u) \\
&\quad - \frac{\sqrt{-1}k}{3} \sum_\alpha \nabla_x^3 J_u \Phi_\alpha(u) p(D_x) \Psi_\alpha(u) \\
&\quad + \frac{\sqrt{-1}k}{3} \sum_\alpha J_u \Phi_\alpha(u) p(D_x) [\Psi_\alpha(u), \nabla_x^3].
\end{aligned}$$

The last term above is a smoothing operator since $\mathrm{supp}[\{\Psi_\alpha(u)\}_x] \cap \mathrm{supp}[\Phi_\alpha(u)] = \emptyset$. If we compute the commutator in the framework of modulo L^2-bounded operators, we deduce

$$\begin{aligned}
a[\tilde{\Lambda}, \nabla_x^3] &\equiv \frac{\sqrt{-1}k}{3} \sum_\alpha \left\{ J_u \Phi_\alpha(u) p(D_x) \nabla_x^3 - \nabla_x^3 J_u \Phi_\alpha(u) p(D_x) \right\} \Psi_\alpha(u) \\
&= \frac{\sqrt{-1}k}{3} \sum_\alpha J_u \Phi_\alpha(u) [p(D_x), \nabla_x^3] \Psi_\alpha(u) \\
&\quad - \sqrt{-1}k \sum_\alpha \left[\nabla_x \{J_u \Phi_\alpha(u)\} \right] \nabla_x^2 p(D_x) \Psi_\alpha(u) \\
&\quad - \sqrt{-1}k \sum_\alpha \left[\nabla_x^2 \{J_u \Phi_\alpha(u)\} \right] \nabla_x p(D_x) \Psi_\alpha(u) \\
&\equiv -\sqrt{-1}k \sum_\alpha \left[\nabla_x \{J_u \Phi_\alpha(u)\} \right] \nabla_x^2 p(D_x) \Psi_\alpha(u)
\end{aligned}$$

$$= k \sum_\alpha \left[\nabla_x \{ J_u \Phi_\alpha(u) \} \right] \nabla_x^2 \frac{p(D_x)}{\sqrt{-1}} \Psi_\alpha(u)$$

$$\equiv k \sum_\alpha \left[\nabla_x \{ J_u \Phi_\alpha(u) \} \right] \nabla_x \Psi_\alpha(u)$$

$$= k \sum_\alpha \left[\nabla_x \{ J_u \Phi_\alpha(u) \} \right] \nabla_x$$

$$= k(\nabla_x J_u) \sum_\alpha \Phi_\alpha(u) \nabla_x + k J_u \left\{ \sum_\alpha \Phi_\alpha(u) \right\}_x \nabla_x$$

$$= k(\nabla_x J_u) \nabla_x.$$

Thus,

$$-a\Lambda\nabla_x^3 \equiv -a\nabla_x^3\Lambda + k(\nabla_x J_u)\nabla_x \tag{42}$$

modulo L^2-bounded operators. In the same way as above, we deduce

$$\Lambda\nabla_x J_u \nabla_x \equiv \nabla_x J_u \nabla_x \Lambda, \tag{43}$$

$$-k\Lambda(\nabla_x J_u)\nabla_x \equiv -k(\nabla_x J_u)\nabla_x, \tag{44}$$

$$-b\Lambda h(u_x, u_x)\nabla_x \equiv -bh(u_x, u_x)\nabla_x\Lambda \tag{45}$$

modulo L^2-bounded operators. By using $1 = \Lambda + \tilde{\Lambda}$, we deduce

$$- a\Lambda \left\{ R(u_x, \nabla_x^{k+1} u_x) u_x \right\}$$

$$= - aR(u_x, \nabla_x^{k+1} u_x)u_x + a\tilde{\Lambda} \left\{ R(u_x, \nabla_x^{k+1} u_x) u_x \right\}$$

$$= - aR(u_x, \nabla_x\Lambda\nabla_x^k u_x)u_x - aR(u_x, \nabla_x\tilde{\Lambda}\nabla_x^k u_x)u_x$$

$$\quad + a\tilde{\Lambda}\nabla_x \left\{ R(u_x, \nabla_x^k u_x)u_x \right\} - a\tilde{\Lambda} \left\{ (\nabla^N R)(u_x, u_x, \nabla_x^k u_x) \right\}$$

$$\quad - a\tilde{\Lambda} \left\{ R(\nabla_x u_x, \nabla_x^k u_x)u_x \right\} - a\tilde{\Lambda} \left\{ R(u_x, \nabla_x^k u_x)\nabla_x u_x \right\}$$

$$= - aR(u_x, \nabla_x\Lambda\nabla_x^k u_x)u_x + Q'_{1,k+1}, \tag{46}$$

$$2b\Lambda \left[\left\{ h(\nabla_x^k u_x, u_x) \right\}_x u_x \right]$$

$$= 2b \left\{ h(\nabla_x^k u_x, u_x) \right\}_x u_x - 2b\tilde{\Lambda} \left[\left\{ h(\nabla_x^k u_x, u_x) \right\}_x u_x \right]$$

$$= 2b \left\{ h(\Lambda\nabla_x^k u_x, u_x) \right\}_x u_x + 2b \left\{ h(\tilde{\Lambda}\nabla_x^k u_x, u_x) \right\}_x u_x$$

$$\quad - 2b\tilde{\Lambda}\nabla_x \left\{ h(\nabla_x^k u_x, u_x)u_x \right\} + 2b\tilde{\Lambda} \left\{ h(\nabla_x^k u_x, u_x)\nabla_x u_x \right\}$$

$$= 2b \left\{ h(\Lambda\nabla_x^k u_x, u_x) \right\}_x u_x + Q'_{2,k+1}, \tag{47}$$

$$\|Q'_{1,k+1}\|, \|Q'_{2,k+1}\| \leqslant C_k \|u\|_{H^{k+1}}.$$

Combining (40), (41), (42), (43), (44), (45), (46) and (47), we obtain

$$\left\{ \nabla_t + \varepsilon\nabla_x^4 - a\nabla_x^3 - \nabla_x J_u \nabla_x - bh(u_x, u_x)\nabla_x \right\} \Lambda\nabla_x^k u_x$$

$$= - aR(u_x, \nabla_x\Lambda\nabla_x^k u_x)u_x + 2b \left\{ h(\Lambda\nabla_x^k u_x, u_x) \right\}_x u_x + \varepsilon P'_{k+1} + Q'_{k+1}, \tag{48}$$

$$\|P'_{l+1}\| \leqslant C_k\|u\|_{H^{k+3}}, \quad \|Q'_{k+1}\| \leqslant C_k\|u\|_{H^{k+1}} \quad \text{for} \quad t \in [0, T_\varepsilon^*]. \tag{49}$$

Here we remark that $-k(\nabla_x J_u)\nabla_x$ is canceled out in the left hand side of (48) by $a[\tilde{\Lambda}, \nabla_x^3]$. By computations similar to (30), (31), (32), (34), (36), (37) and not to (38), we can deduce from (48) and

(49) that

$$\frac{d}{dt}\|\Lambda\nabla_x^k u_x\|^2 \leqslant C_k \mathcal{N}_{k+1}(u)^2. \tag{50}$$

Combining (39) and (50), we obtain

$$\frac{d}{dt}\mathcal{N}_{k+1}(u) \leqslant C_k \mathcal{N}_{k+1}(u) \quad \text{for} \quad t \in [0, T_\varepsilon^*]. \tag{51}$$

If we take $t = T_\varepsilon^*$, then we have $2\mathcal{N}_{k+1}(u_0) \leqslant \mathcal{N}_{k+1}(u_0)e^{C_k T_\varepsilon^*}$, which implies $T_\varepsilon^* \geqslant T = \log 2/C_k > 0$.

Thus $\{u^\varepsilon\}_{\varepsilon \in (0,1]}$ is bounded in $L^\infty(0, T; H^{k+1}(\mathbb{T}; TN))$. This completes the proof. $\square$

Uniqueness of Solutions. The uniqueness of solutions was proved in [22, Section 5]. The proof given there does not depend on the Kähler condition at all. We prove the uniqueness by H^1-energy estimates of the difference of two solutions with the same initial data in $\mathbb{R}^d$. The symmetry of the second fundamental form of the mapping $w \circ u$ plays a crucial role. See [22, Section 5] for the detail. $\square$

Recovery of Continuity in Time. Let $u \in L^\infty(0, T; H^{k+1}(\mathbb{T}; TN))$ be the unique solution to (1)-(2). Following [4, Section 3], we prove that $\nabla_x^k u_x$ is strongly continuous in time. We remark that we have already known that $u \in C([0, T]; H^k(\mathbb{T}; TN))$ and $\nabla_x^k u_x$ is a weakly continuous $L^2(\mathbb{T}; TN)$-valued function on $[0, T]$. We identify N and $w(N)$ below. Let $\{u^\varepsilon\}_{\varepsilon \in (0,1]}$ be a sequence of solutions to (11)-(12), which approximates u. We can easily check that for any $\phi \in C^\infty([0, T] \times \mathbb{T}; \mathbb{R}^d)$,

$$\Lambda_\varepsilon^* \phi \longrightarrow \Lambda^* \phi \quad \text{in} \quad L^2((0, T) \times \mathbb{T}; \mathbb{R}^d),$$

$$\Lambda_\varepsilon \nabla_x^k u_x^\varepsilon \longrightarrow \tilde{u} \quad \text{in} \quad L^2((0, T) \times \mathbb{T}; \mathbb{R}^d) \quad \text{weakly star},$$

as $\varepsilon \downarrow 0$ with some $\tilde{u}$. Then, $\tilde{u} = \Lambda \nabla_x^k u_x$ in the sense of distributions. We denote by $\mathscr{L}(\mathscr{H})$ the set of all bounded linear operators of a Hilbert space $\mathscr{H}$ to itself. The time-continuity of $\nabla_x^k u_x$ is equivalent to that of $\Lambda \nabla_x^k u_x$ since $\Lambda \in C([0, T]; \mathscr{L}(L^2(\mathbb{T}; \mathbb{R}^d)))$.

It suffices to show that

$$\lim_{t \downarrow 0} \Lambda(t)\nabla_x^k u_x(t) = \Lambda(0)\nabla_x^k u_{0x} \quad \text{in} \quad L^2(\mathbb{T}; \mathbb{R}^d), \tag{52}$$

since the other cases can be proved in the same way. (51) and the lower semicontinuity of L^2-norm imply

$$\sum_{l=0}^{k-1}\|\nabla_x^l u_x(t)\|^2 + \|\Lambda(t)\nabla_x^k u_x(t)\|^2 \leqslant \sum_{l=0}^{k-1}\|\nabla_x^l u_{0x}\|^2 + \|\Lambda(0)\nabla_x^k u_{0x}\|^2 + C_k \mathcal{N}_{k+1}(u_0)^2 t$$

provided that $\varepsilon \downarrow 0$. Letting $t \downarrow 0$, we have

$$\limsup_{t \downarrow 0}\|\Lambda(t)\nabla_x^k u_x(t)\|^2 \leqslant \|\Lambda(0)\nabla_x^k u_{0x}\|^2$$

which implies (52). This completes the proof. $\square$

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