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The intial value problem for a third-order dispersive flow into compact almost Hermitian manifolds

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THE INITIAL VALUE PROBLEM FOR A THIRD-ORDER DISPERSIVE FLOW INTO COMPACT ALMOST HERMITIAN MANIFOLDS

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ABSTRACT. We present a time-local existence theorem of solutions to the initial value problem for a third-order dispersive evolution equation for open curves into compact almost Hermitian manifolds. Our equations geometrically generalize a two-sphere valued physical model describing the motion of vortex filament . These equations cause the so-called loss of one-derivative since the target manifold is not supposed to be a Kähler manifold. We overcome this difficulty by using a gauge transformation of a multiplier on the pull-back bundle to eliminate the bad first order terms essentially.

1. INTRODUCTION

Let (N, J, g) be a compact almost Hermitian manifold with an almost complex structure J and a hermitian metric g, and let ∇ be the Levi-Civita connection with respect to g. X denotes $\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$. Consider the initial value problem of the form

$$u_t = a\,\nabla_x^2 u_x + J_u \nabla_x u_x + b\, g_u(u_x, u_x) u_x \quad \text{in} \quad \mathbb{R} \times X, \tag{1.1}$$

$$u(0, x) = u_0(x) \quad \text{in} \quad X, \tag{1.2}$$

where $a, b \in \mathbb{R}$ are constants, $u(t, x)$ is an N-valued unknown function of $(t, x) \in \mathbb{R} \times X$, $u_t(t, x) = du_{(t,x)}((\partial/\partial t)_{(t,x)})$, $u_x(t, x) = du_{(t,x)}((\partial/\partial x)_{(t,x)})$, $du_{(t,x)} : T_{(t,x)}(\mathbb{R} \times X) \to T_{u(t,x)}N$ is the differential of the mapping u at (t, x), ∇_x is the covariant derivative induced from ∇ with respect to x along the mapping u, and J_u and g_u mean the almost complex structure and the metric at $u \in N$ respectively. The equation (1.1) is an equality of sections of the pull-back bundle $u^{-1}TN$. We call the solution of (1.1) a dispersive flow. In particular, when $a = b = 0$, this is called a one-dimensional Schrödinger map.

Examples of dispersive flows arise in classical mechanics: the motion of vortex filament, the Heisenberg ferromagnetic spin chain and etc. Solutions to these physical models are valued in two-dimensional unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. For $\vec{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$ and $\vec{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$, let

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3, \quad |\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}},$$

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).$$

In [2], Da Rios formulated the equation modeling the motion of vortex filament of the form

$$\vec{u}_t = \vec{u} \times \vec{u}_{xx}, \tag{1.3}$$

where $\vec{u}(t, x) \in \mathbb{S}^2$ denotes the velocity vector along the space curve describing the position of the vortex filament in $\mathbb{R}^3$ at (t, x), t is the time and x is the arc-length in this physical model. See also, e.g., [8] and [10] for physical backgrounds of (1.3). The physical model (1.3) is an

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example of the equation of the one-dimensional Schrödinger map. Our equation (1.1) with $b = a/2$ geometrically generalizes an $\mathbb{S}^2$-valued physical model

$$\vec{u}_t = \vec{u} \times \vec{u}_{xx} + a\left[\vec{u}_{xxx} + \frac{3}{2}\{\vec{u}_x \times (\vec{u} \times \vec{u}_x)\}_x\right] \tag{1.4}$$

describing the motion of vortex filament in $\mathbb{R}^3$ proposed by Fukumoto and Miyazaki in [5].

Here we state the known results on the mathematical analysis of the IVP (1.1)-(1.2). There has been many studies on the existence of solutions to (1.1)-(1.2) both on $X = \mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ only when (N, J, g) is a Kähler manifold. See [1], [3], [9], [11], [12], [13], [19], [21] for $a = 0$ and [16], [17], [18], [22] for $a \neq 0$. Time-local existence theorems were proved by some classical energy estimates with respect to the following quantity like the L^2-energy

$$\|V\|^2_{L^2(X;TN)} = \int_X g_{u(x)}\left(V(x), V(x)\right) dx \quad \text{for} \quad V \in \Gamma(u^{-1}TN).$$

More precisely, if ∇ is a metric connection ($\nabla g = 0$) and g is a Kähler metric ($\nabla J = 0$), then the equation (1.1) behaves like symmetric hyperbolic systems, and the classical energy method works well. This fact is closely related with the geometric studies of the good structure of the equation of dispersive flow into a compact Riemann surface on $\mathbb{R}$. Being inspired with Hasimoto's pioneering work in [8], Chang, Shatah and Uhlenbeck constructed a good moving frame along the map, and rigorously reduced the equation of the one-dimensional Schrödinger map into a compact Riemann surface to a simple form of a complex-valued nonlinear Schrödinger equation in [1]. Using the same idea, the author studied the geometric reduction of the equations of higher-order dispersive flows in [18]. In addition, time-global existence theorems were also studied under some geometric conditions. For the one-dimensional Schrödinger maps, time-global existence holds if (N, J, g) is locally symmetric. See [9], [19], and [21]. For the third-order equation (1.1), Nishiyama and Tani in [16] and [22] proved time-local and time-global existence of solutions when $X = \mathbb{R}$ or $X = \mathbb{R}/\mathbb{Z}$, $N = \mathbb{S}^2$, and the integrability condition $b = a/2$ is satisfied. They made use of some conservation laws to prove the global existence theorem. These conservation laws were discovered by Zakharov and Shabat in the study of the Hirota equation. See [24] for details. In [17] the author generalized these results when $X = \mathbb{R}/\mathbb{Z}$. He proved a time-local existence theorem for (1.1)-(1.2) when N is a compact Kähler manifold, and proved a time-global existence theorem when N is a compact Riemann surface with a constant curvature K, and the condition $b = Ka/2$ holds.

On the other hands, almost Hermitian manifolds do not necessarily satisfy the Kähler condition $\nabla J = 0$. For example, it is well-known that $\mathbb{S}^6$, the Hopf manifold $\mathbb{S}^{2p+1} \times \mathbb{S}^1$, and $\mathbb{S}^{2p+1} \times \mathbb{S}^{2q+1}$ $(p, q = 1, 2, 3, \dots)$ never admit the structure of Kähler manifolds. If the Kähler condition fails to hold, then ∇J causes the so-called loss of one-derivative, and the equation (1.1) behaves like the Cauchy-Riemann equation. In this case, the classical energy method breaks down. The main purpose of this paper is to show the time-local existence theorem of (1.1)-(1.2) without the Kähler condition. To state our results, we here introduce some function spaces for mappings.

Definition 1.1. Let $\mathbb{N}$ be the set of positive integers. For $m \in \mathbb{N} \cup \{0\}$, the Sobolev space of mappings is defined by

$$H^{m+1}(\mathbb{R}; N) = \{u \in C(\mathbb{R}; N) \mid u_x \in H^m(\mathbb{R}; TN)\},$$

where $u_x \in H^m(\mathbb{R}; TN)$ means that u_x satisfies

$$\|u_x\|^2_{H^m(\mathbb{R};TN)} = \sum_{j=0}^{m} \int_{\mathbb{R}} g_{u(x)}(\nabla_x^j u_x(x), \nabla_x^j u_x(x))dx < +\infty.$$

Moreover, let I be an interval in $\mathbb{R}$, and let w be an isometric embedding of (N, J, g) into the standard Euclidean space $(\mathbb{R}^d, g_0)$. We say that $u \in C(I; H^{m+1}(\mathbb{R}; N))$ if $u \in C(I \times \mathbb{R}; N)$ and $(w \circ u)_x \in C(I; H^m(\mathbb{R}; \mathbb{R}^d))$, where $C(I; H^m(\mathbb{R}; \mathbb{R}^d))$ is the set of usual Sobolev space valued continuous functions on I.

Our main results is the following.

Theorem 1.1. *Let (N, J, g) be a compact almost Hermitian manifold, and let $a \neq 0$, $b \in \mathbb{R}$. Then for any $u_0 \in H^{m+1}(\mathbb{R}; N)$ with an integer $m \geqslant 4$, there exists a constant $T > 0$ depending only on a, b, N and $\|u_{0x}\|_{H^4(\mathbb{R};TN)}$ such that the initial value problem* (1.1)-(1.2) *possesses a unique solution $u \in C([-T, T]; H^{m+1}(\mathbb{R}; N))$.*

Roughly speaking, Theorem 1.1 says that (1.1)-(1.2) has a time-local solution in the usual Sobolev space $H^5(\mathbb{R}; \mathbb{R}^d) = (1 - \partial_x^2)^{-5/2} L^2(\mathbb{R}; \mathbb{R}^d)$.

Our idea of the proof comes from the theory of linear dispersive partial differential operators. Consider the initial value problem for linear partial differential equations of the form

$$u_t + u_{xxx} + a(x)u_x + b(x)u = f(t, x) \quad \text{in} \quad \mathbb{R} \times \mathbb{R}, \tag{1.5}$$

where $a(x), b(x) \in \mathscr{B}^\infty(\mathbb{R})$, which is the set of all smooth functions on $\mathbb{R}$ whose derivative of any order are bounded on $\mathbb{R}$, $u(t, x)$ is a complex-valued unknown function, and $f(t, x)$ is a given function. Tarama proved in [23] that the initial value problem for (1.5) is L^2-well-posed if and only if

$$\left| \int_x^y \operatorname{Im} a(s) ds \right| \leqslant C|x - y|^{1/2} \tag{1.6}$$

for any $x, y \in \mathbb{R}$ with some constant $C > 0$. The necessity is proved by the usual method of asymptotic solutions. In order to prove the sufficiency, Tarama first constructed a nice pseudo-differential operators of order zero which is automorphic on $L^2(\mathbb{R}; \mathbb{C})$ under the condition (1.6), and eliminates $\sqrt{-1} \operatorname{Im} a(x)\partial_x$. This is one of the methods of bringing out the local smoothing effect of $e^{-t\partial_x^3}$ on $\mathbb{R}$, and this property breaks down on $\mathbb{R}/\mathbb{Z}$. See e.g., [4]. Tarama also pointed out unofficially that if $\operatorname{Im} a \in L^2(\mathbb{R}; \mathbb{R})$, then (1.6) holds and the proof of sufficiency becomes quite easier than the general case of (1.6). In this case, a gauge transformation defined by

$$u(x) \longmapsto v(x) = u(x) \exp\left(\frac{1}{3} \int_{-\infty}^x \{\operatorname{Im} a(y)\}^2 dy \right) \tag{1.7}$$

is automorphic on $L^2(\mathbb{R}; \mathbb{C})$, and (1.5) becomes

$$v_t + v_{xxx} - \{\operatorname{Im} a(x)\}^2 v_{xx} + \{\tilde{a}(x) + \sqrt{-1} \operatorname{Im} a(x)\} v_x + \tilde{b}(x) v = \tilde{f}(t, x) \tag{1.8}$$

with some $\tilde{a}, \tilde{b} \in \mathscr{B}^\infty(\mathbb{R})$ and $\tilde{f}$, where $\tilde{a}$ is a real-valued. The initial value problem for (1.8) is L^2-well-posed in the positive direction of t since the second-order term $\{\operatorname{Im} a(x)\}^2 \partial_x^2$ dominates the seemingly bad first-order term $\sqrt{-1} \operatorname{Im} a(x)\partial_x$ essentially. In this special case, pseudodifferential calculus is not required.

We make use of the idea of the gauge transformation (1.7). Roughly speaking, we see $\nabla_x^m u_x$ satisfies the form

$$\left(\nabla_t - a\nabla_x^3 - \nabla_x J_u \nabla_x\right) \nabla_x^m u_x - m(\nabla_x J_u) \nabla_x \nabla_x^m u_x = \text{harmless terms}, \tag{1.9}$$

where $(\nabla_x J_u)$ is the covariant derivative of the $(1,1)$-tensor field J_u with respect to x along u. The term $m(\nabla_x J_u)\nabla_x \nabla_x^m u_x$ cannot be controlled by the classical energy method since $(\nabla_x J_u)$ behaves as anti-symmetric operator on $L^2(\mathbb{R}; TN)$ in the sense

$$\int_{\mathbb{R}} g((\nabla_x J_u)V, W)dx = -\int_{\mathbb{R}} g(V, (\nabla_x J_u)W)dx, \quad \text{for} \quad V, W \in \Gamma(u^{-1}TN).$$

We introduce a gauge transformation on $u^{-1}TN$ defined by

$$\nabla_x^m u_x(t,x) \longmapsto \nabla_x^m u_x(t,x) \exp\left(-\frac{1}{3a}\int_{-\infty}^{x} g(u_x(t,y), u_x(t,y))dy\right), \tag{1.10}$$

which eliminates the bad term essentially since $(\nabla_x J_u) = O\left(g(u_x, u_x)^{1/2}\right)$. Parabolic regularization and the energy estimates with (1.10) prove Theorem 1.1. The assumption $m \geqslant 4$ is the requirement on the integer for our method to work.

When (N, J, g) is a Kähler manifold, we do not need the regularity $m \geqslant 4$. In this case, the term $m(\nabla_x J_u)\nabla_x \nabla_x^m u_x$ vanishes in (1.9), thus the classical energy method works. Indeed we prove the following.

Theorem 1.2. *Let (N, J, g) be a compact Kähler manifold and let $a \neq 0$ and $b \in \mathbb{R}$. Then for any $u_0 \in H^{m+1}(\mathbb{R}; N)$ with an integer $m \geqslant 2$, there exists a constant $T > 0$ depending only on a, b, N, and $\|u_{0x}\|_{H^2(\mathbb{R};TN)}$ such that the initial value problem* (1.1)-(1.2) *possesses a unique solution $u \in C([-T,T]; H^{m+1}(\mathbb{R}; N))$.*

Theorem 1.3. *Let (N, J, g) be a compact Riemann surface with constant Gaussian curvature K and let $a \neq 0$ and $b = aK/2$. Then for any $u_0 \in H^{m+1}(\mathbb{R}; N)$ with an integer $m \geqslant 2$, there exists a unique solution $u \in C(\mathbb{R}; H^{m+1}(\mathbb{R}; N))$ to* (1.1)-(1.2).

Theorem 1.2 and 1.3 are analogues of the results on $X = \mathbb{R}/\mathbb{Z}$ in [17]. We remark that Theorem 1.3 generalizes the results on $X = \mathbb{R}$ in [16] and [22]. The key idea of the proof is the use of some conserved quantities generalizing what is used in [16]. Examples of Riemann surfaces satisfying the conditions in Theorem 1.3 are not only the two-sphere $\mathbb{S}^2$ $(K = 1)$ and the flat torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ $(K = 0)$, but also closed hyperbolic surfaces $(K = -1)$.

The organization of this paper is as follows. Section 2 is devoted to geometric preliminaries. In Section 3 we construct a sequence of approximate solutions by solving the IVP for a fourth-order parabolic equation. In Section 4 we obtain uniform estimates of approximate solutions. In Section 5 we complete the proof of Theorem 1.1. Finally, in Section 6 we give the sketch of the proof of Theorem 1.2 and 1.3.

2. Geometric Preliminaries

In this section, we introduce some geometric notations used later in our proof. One can refer [15] for the elements of nonlinear geometric analysis.

We will use $C = C(\cdot, \ldots, \cdot)$ to denote a positive constant depending on the certain parameters, geometric properties of N, et al. The partial differentiation is written by ∂, or the subscript, e.g., $\partial_x f$, f_x, to distinguish from the covariant derivative along the curve, e.g., ∇_x.

Throughout this paper, w is fixed as an isometric embedding mapping from (N, J, g) into a standard Euclidean space $(\mathbb{R}^d, g_0)$. Existence of w is ensured by the celebrated works of Nash [14], Gromov and Rohlin [7], and related papers.

For $\delta > 0$, let $(w(N))_\delta$ be a δ-tubular neighbourhood of $w(N) \subset \mathbb{R}^d$ defined by

$$(w(N))_\delta = \left\{ Q = q + X \in \mathbb{R}^d \mid q \in w(N),\ X \in (T_q w(N))^\perp,\ |X| < \delta \right\}$$

where $|\cdot|$ denotes the distance in $\mathbb{R}^d$, and let $\pi : (w(N))_\delta \to w(N)$ be the nearest point projection map defined by $\pi(Q) = q$ for $Q = q + X \in (w(N))_\delta$. Since $w(N)$ is compact, for any sufficiently small δ, π exists and is smooth. We fix such small δ.

Let $u : \mathbb{R} \to N$ be given. $u^{-1}TN = \bigcup_{x\in\mathbb{R}} T_{u(x)}N$ is the pull-back bundle induced from TN by u. V is called a section of $u^{-1}TN$ if $V(x) \in T_{u(x)}N$ for all $x \in \mathbb{R}$. We denote the space of all the sections of $u^{-1}TN$ by $\Gamma(u^{-1}TN)$. For $V, W \in \Gamma(u^{-1}TN)$, define the quantities like L^2-inner product by

$$\int_{\mathbb{R}} g(V,W)dx = \int_{\mathbb{R}} g_{u(x)}(V(x),W(x))dx, \quad \|V\|^2_{L^2(\mathbb{R};TN)} = \int_{\mathbb{R}} g(V,V)dx.$$

Then the quantity $\|u_x\|^2_{H^m(\mathbb{R};TN)}$ defined in Definition 1.1 is written by

$$\|u_x\|^2_{H^m(\mathbb{R};TN)} = \sum_{j=0}^{m} \|\nabla_x^j u_x\|^2_{L^2(\mathbb{R};TN)}.$$

In contrast, the standard L^2-product and L^2-norm are written by

$$\langle V,W\rangle = \int_{\mathbb{R}} g_0(V(x),W(x))dx, \quad \|V\|^2_{L^2(\mathbb{R};\mathbb{R}^d)} = \langle V,V\rangle$$

for $V, W \in L^2(\mathbb{R};\mathbb{R}^d)$, and the quantity $\|V\|^2_{H^m(\mathbb{R};\mathbb{R}^d)}$ is written by

$$\|V\|^2_{H^m(\mathbb{R};\mathbb{R}^d)} = \sum_{j=0}^{m} \|\partial_x^j V\|^2_{L^2(\mathbb{R};\mathbb{R}^d)}.$$

At this time $\|u_x\|_{H^m(\mathbb{R};TN)} < \infty$ if and only if $\|(w\circ u)_x\|_{H^m(\mathbb{R};\mathbb{R}^d)} < \infty$. See, e.g., [20, Section 1] or [11, Proposition 2.5] for this equivalence. Noting this equivalence, we see

$$H^{m+1}(\mathbb{R};N) = \{u \in C(\mathbb{R};N) \mid (w\circ u)_x \in H^m(\mathbb{R};\mathbb{R}^d)\}.$$

Finally, for $\alpha > 0$, $m \in \mathbb{N}\cup\{0\}$ and an interval $I \subset \mathbb{R}$, $C^{0,\alpha}(I;H^m(\mathbb{R};\mathbb{R}^d))$ denotes the usual $H^m(\mathbb{R};\mathbb{R}^d)$-valued α-Hörder space on I. We will make use of fundamental Sobolev space theory of $H^m(\mathbb{R};\mathbb{R}^d)$ later in our proof.

3. Parabolic Regularization

The aim of this section is to obtain a sequence $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ solving

$$u_t = -\varepsilon\,\nabla_x^3 u_x + a\,\nabla_x^2 u_x + J_u\nabla_x u_x + b\,g_u(u_x,u_x)u_x, \quad \text{in} \quad (0,T_\varepsilon)\times\mathbb{R}, \tag{3.1}$$

$$u(0,x) = u_0(x) \quad \text{in} \quad \mathbb{R} \tag{3.2}$$

for each $\varepsilon \in (0,1)$, where $u = u^\varepsilon(t,x)$ is also an N-valued unknown function of $(t,x) \in [0,T_\varepsilon]\times\mathbb{R}$, and u_0 is the same initial data as that of (1.1)-(1.2) independent of $\varepsilon \in (0,1)$. The argument in this section is essentially same as that in [17, Section 3]. In fact, we can show that (3.1)-(3.2) admits a unique solution near the initial data u_0. Define

$$L^\infty_{\delta,T} = \left\{u \in L^\infty((0,T)\times\mathbb{R};N) \mid \|w\circ u - w\circ u_0\|_{L^\infty((0,T)\times\mathbb{R};\mathbb{R}^d)} \leqslant \delta/2\right\}$$

for $T > 0$, where $\delta > 0$ is the fixed constant describing the radius of the tubular neighbourhood of $w(N)$ as stated in the previous section. We show the following.

Proposition 3.1. *Let $u_0 \in H^{k+1}(\mathbb{R};N)$ with an integer $k \geqslant 2$. Then for each $\varepsilon \in (0,1)$, there exists a constant $T_\varepsilon = T(\varepsilon,a,b,N,\|u_{0x}\|_{H^k(\mathbb{R};TN)}) > 0$ and a unique solution $u = u^\varepsilon \in C([0,T_\varepsilon];H^{k+1}(\mathbb{R};N)) \cap L^\infty_{\delta,T_\varepsilon}$ to* (3.1)-(3.2).

Proof of Proposition 3.1. Via the relation $v = w\circ u$, the IVP (3.1)-(3.2) is equivalent to the following problem

$$v_t = -\varepsilon v_{xxxx} + F(v) \quad \text{in} \quad (0, T_\varepsilon) \times \mathbb{R}, \tag{3.3}$$

$$v(0, x) = w\circ u_0(x) \quad \text{in} \quad \mathbb{R}, \tag{3.4}$$

where $v = v^\varepsilon(t, x)$ is a $w(N)$-valued unknown function of $(t, x) \in [0, T_\varepsilon] \times \mathbb{R}$, and $F(v)$ is written by the form

$$\begin{aligned} F(v) = &-\varepsilon\{[A(v)(v_x, v_x)]_{xx} + [A(v)(v_{xx} + A(v)(v_x, v_x), v_x)]_x \\ &\quad + A(v)(v_{xxx} + [A(v)(v_x, v_x)]_x + A(v)(v_{xx} + A(v)(v_x, v_x), v_x), v_x)\} \\ &+ a\{v_{xxx} + [A(v)(v_x, v_x)]_x + A(v)(v_{xx} + A(v)(v_x, v_x), v_x)\} \\ &+ dw_{w^{-1}\circ v} J_{w^{-1}\circ v} dw_v^{-1}(v_{xx} + A(v)(v_x, v_x)) + b|v_x|^2 v_x, \end{aligned}$$

where, $A(v)(\cdot, \cdot) : T_v w(N) \times T_v w(N) \to (T_v w(N))^\perp$ is the second fundamental form of $w(N) \subset \mathbb{R}^d$ at $v \in w(N)$. Note that there exists $G \in C^\infty(\mathbb{R}^{4d}; \mathbb{R}^d)$ such that

$$F(v) = G(v, v_x, v_{xx}, v_{xxx})$$

for $v : \mathbb{R} \to w(N)$, and $G(v, p, q, r)$ satisfies

$$G(v, 0, 0, 0) = 0, \quad \frac{\partial^2 G}{\partial r^2}(v, p, q, r) = 0.$$

The equation (3.3) is a system of fourth-order parabolic evolution equations for $\mathbb{R}^d$-valued function. In place of the IVP (3.1)-(3.2), we will solve the IVP (3.3)-(3.4). The proof consists of the following two steps. First, we construct a solution of (3.3)-(3.4) whose image are contained in $(w(N))_\delta \subset \mathbb{R}^d$. More precisely, we extend (3.3) to an equation for the vector-valued function valued in $(w(N))_\delta$ and construct a unique time-local solution of the IVP for the extended equation in the class

$$Y_T = \{v \in X_T \mid \|v - w\circ u_0\|_{L^\infty((0,T)\times\mathbb{R};\mathbb{R}^d)} \leqslant \delta/2\}$$

for sufficiently small $T > 0$. Here

$$X_T = \{v \in C([0, T] \times \mathbb{R}; \mathbb{R}^d) \mid v_x \in C([0, T]; H^k(\mathbb{R}; \mathbb{R}^d))\}$$

is the Banach space with the following norm

$$\|v\|_{X_T} = \|v\|_{L^\infty([0,T]\times\mathbb{R};\mathbb{R}^d)} + \|v_x\|_{L^\infty(0,T;H^k(\mathbb{R};\mathbb{R}^d))}, \quad v \in X_T.$$

Secondly, we check that this solution is actually $w(N)$-valued by using a kind of maximum principle.

In short, it suffices to show the following two lemmas to complete our proof.

Lemma 3.2. *For each $\varepsilon \in (0, 1)$, there exists a constant $T_\varepsilon > 0$ depending on ε, a, b, N and $\|(w\circ u_0)_x\|_{H^k(\mathbb{R};\mathbb{R}^d)}$ and there exists a unique solution $v = v^\varepsilon \in Y_{T_\varepsilon}$ to*

$$v_t = -\varepsilon v_{xxxx} + F(\pi\circ v) \quad \text{in} \quad (0, T_\varepsilon) \times \mathbb{R}, \tag{3.5}$$

$$v(0, x) = w\circ u_0(x) \quad \text{in} \quad \mathbb{R}. \tag{3.6}$$

Moreover, the map $(w\circ u_0)_x \in H^k(\mathbb{R}; \mathbb{R}^d) \to v_x^\varepsilon \in C([0, T_\varepsilon]; H^k(\mathbb{R}; \mathbb{R}^d))$ is continuous.

Lemma 3.3. *Fix $\varepsilon \in (0, 1)$. Assume that $v = v^\varepsilon \in Y_{T_\varepsilon}$ solves* (3.5)-(3.6)*. Then $v(t, x) \in w(N)$ for all $(t, x) \in [0, T_\varepsilon] \times \mathbb{R}$, thus v solves* (3.3)-(3.4)*.*

Proof of Lemma 3.2. The idea of the proof is due to the contraction mapping argument.

Let L be a nonlinear map defined by

$$\begin{aligned} Lv(t) &= e^{-\varepsilon t\partial_x^4}v_0 + \int_0^t e^{-\varepsilon(t-s)\partial_x^4}F((\pi\circ v)(s))ds \\ &= \int_{\mathbb{R}} E(t,x-y)v_0(y)dy + \int_0^t\int_{\mathbb{R}} E(t-s,x-y)F((\pi\circ v)(s,y))dyds, \end{aligned}$$

where $v_0 = w\circ u_0$, and $E(t,x)$ is the fundamental solution associated to $\partial_t + \varepsilon\partial_x^4$. Note that, if $v \in Y_T$, $\pi\circ v$ takes value in $w(N)$ and thus $F(\pi\circ v)$ makes sense. The IVP (3.5)-(3.6) is equivalent to an integral equation of the form $v = Lv$.

Set $M = \|v_{0x}\|_{H^k(\mathbb{R};\mathbb{R}^d)}$, and define the space

$$Z_T = \{v \in Y_T \mid \|v_x\|_{L^\infty(0,T;H^k(\mathbb{R};\mathbb{R}^d))} \leqslant 2M\}.$$

Z_T is a closed subset of the Banach space X_T. To complete the proof, we have only to show that the map L has a unique fixed point in Z_{T_ε} for sufficiently small $T_\varepsilon > 0$, since the uniqueness in the whole space Y_{T_ε} follows by similar and standard arguments.

First, consider the properties of $e^{-\varepsilon t\partial_x^4}$. Since $u_0 \in H^{k+1}(\mathbb{R};N)$, v_0 is especially bounded and uniformly continuous on $\mathbb{R}$. Thus, it is easy to check that

$$e^{-\varepsilon t\partial_x^4}v_0 \longrightarrow v_0 \quad \text{in} \quad C(\mathbb{R};\mathbb{R}^d) \quad \text{as} \quad t \to 0, \tag{3.7}$$

and

$$\|e^{-\varepsilon t\partial_x^4}v_{0x}\|_{H^k(\mathbb{R};\mathbb{R}^d)} \leqslant \|v_{0x}\|_{H^k(\mathbb{R};\mathbb{R}^d)}. \tag{3.8}$$

Moreover, $e^{-\varepsilon t\partial_x^4}$ gains the regularity of order 3, since $(\varepsilon^{1/4}t^{1/4}|\xi|)^j e^{-\varepsilon t\xi^4}$ is bounded for $j = 0,1,2,3$. In fact, there exists $C_1 > 0$ such that

$$\|e^{-\varepsilon t\partial_x^4}\phi\|_{H^{k+1}(\mathbb{R};\mathbb{R}^d)} \leqslant C_1\varepsilon^{-3/4}t^{-3/4}\|\phi\|_{H^{k-2}(\mathbb{R};\mathbb{R}^d)} \tag{3.9}$$

holds for any $\phi \in H^{k-2}(\mathbb{R};\mathbb{R}^d)$.

Secondly, consider the nonlinear estimates of $F(\pi\circ v)$. If v belongs to the class Z_T, we see $v(t,\cdot) \in C(\mathbb{R};(w(N))_\delta)$ and $\|v_x(t)\|_{H^k(\mathbb{R};\mathbb{R}^d)} \leqslant 2M$ follows for all $t \in [0,T]$. Thus, by observing the form of $F(v)$ and the compactness of $w(N)$, it is easy to check that there exists $C_2 = C_2(a,b,M,N) > 0$ such that

$$\|F(\pi\circ v)(t)\|_{H^{k-2}(\mathbb{R};\mathbb{R}^d)} \leqslant C_2\|v_x(t)\|_{H^k(\mathbb{R};\mathbb{R}^d)}, \tag{3.10}$$

$$\|F(\pi\circ u)(t) - F(\pi\circ v)(t)\|_{H^{k-2}(\mathbb{R};\mathbb{R}^d)} \leqslant C_2\left(\|u(t)-v(t)\|_{L^\infty(\mathbb{R};\mathbb{R}^d)} + \|u_x(t)-v_x(t)\|_{H^k(\mathbb{R};\mathbb{R}^d)}\right) \tag{3.11}$$

for any $u, v \in Z_T$.

Using the properties (3.7), (3.8), (3.9) and the nonlinear estimates (3.10), (3.11), we can prove that L is a contraction mapping from Z_{T_ε} into itself if T_ε is sufficiently small. It is the standard argument, thus we omit the rest of the proof. □

Remark 1. Suppose that $v^\varepsilon \in Y_{T_\varepsilon}$ solves (3.3)-(3.4). Then we can easily check $v^\varepsilon_{xxxx} \in L^2(0,T_\varepsilon;H^1(\mathbb{R};\mathbb{R}^d))$ and $F(\pi\circ v^\varepsilon) \in L^2(0,T_\varepsilon;H^1(\mathbb{R};\mathbb{R}^d))$ from the standard arguments. Thus we see v^ε_t belongs to the same class $L^2(0,T_\varepsilon;H^1(\mathbb{R};\mathbb{R}^d))$, which implies that $v^\varepsilon - v_0$ belongs to the class $C^{0,1/2}([0,T_\varepsilon];H^1(\mathbb{R};\mathbb{R}^d))$.

Proof of Lemma 3.3. Suppose $v \in Y_{T_\varepsilon}$ solves (3.5)-(3.6). Define the map $\rho : (w(N))_\delta \to \mathbb{R}^d$ by $\rho(Q) = Q - \pi(Q)$ for $Q \in (w(N))_\delta$. Then we deduce

$$|\rho\circ v(t,x)| = \min_{q\in w(N)} |v(t,x) - q| \leqslant |v(t,x) - v_0(x)|.$$

Notice that the first equality above is due to the compactness of $w(N)$. In addition, as is stated in Remark 1, $v(t) - v_0$ belongs to $L^2(\mathbb{R};\mathbb{R}^d)$ and thus $\rho\circ v(t)$ makes sense in $L^2(\mathbb{R};\mathbb{R}^d)$ for each t. To obtain that v is $w(N)$-valued, we will show

$$\|\rho\circ v(t)\|^2_{L^2(\mathbb{R};\mathbb{R}^d)} = \langle \rho\circ v(t), \rho\circ v(t)\rangle = 0$$

for all $t \in [0, T_\varepsilon]$. Since $\pi + \rho$ is identity on $(w(N))_\delta$,

$$d\pi_v + d\rho_v = I_d \tag{3.12}$$

holds on $T_v(w(N))_\delta$, where I_d is the identity. By identifying $T_v(w(N))_\delta$ with $\mathbb{R}^d$, we see that $v_t(t,x) \in T_{v(t,x)}(w(N))_\delta$ and $d\pi_v(v_t)(t,x) \in T_{\pi\circ v(t,x)}w(N)$ for each (t,x). Thus it follows that $\langle \rho\circ v, d\pi_v(v_t)\rangle = 0$. Using this relation and (3.12), we deduce

$$\frac{1}{2}\frac{d}{dt}\|\rho\circ v\|^2_{L^2(\mathbb{R};\mathbb{R}^d)} = \langle \rho\circ v, d\rho_v(v_t)\rangle = \langle \rho\circ v, d\rho_v(v_t) + d\pi_v(v_t)\rangle = \langle \rho\circ v, v_t\rangle .$$

Recall here, by the form of the right hand side of (3.3), that $-\varepsilon\tilde{v}_{xxxx} + F(\tilde{v}) \in \Gamma(\tilde{v}^{-1}Tw(N))$ holds for any $\tilde{v} : \mathbb{R} \to w(N)$. Thus we see $(-\varepsilon(\pi\circ v)_{xxxx} + F(\pi\circ v))(t) \in \Gamma((\pi\circ v(t))^{-1}Tw(N))$ since $\pi\circ v(t) \in w(N)$, and thus this is perpendicular to $\rho\circ v(t)$. Noting this and substituting (3.5), we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\rho\circ v\|^2_{L^2(\mathbb{R};\mathbb{R}^d)} &= \langle \rho\circ v, -\varepsilon v_{xxxx} + F(\pi\circ v)\rangle \\
&= \langle \rho\circ v, -\varepsilon(\rho\circ v)_{xxxx} - \varepsilon(\pi\circ v)_{xxxx} + F(\pi\circ v)\rangle \\
&= \langle \rho\circ v, -\varepsilon(\rho\circ v)_{xxxx}\rangle \\
&= -\varepsilon\|(\rho\circ v)_{xx}\|^2_{L^2(\mathbb{R};\mathbb{R}^d)} \leqslant 0,
\end{aligned}$$

which implies $\|\rho\circ v(t)\|^2_{L^2(\mathbb{R};\mathbb{R}^d)} \leqslant \|\rho\circ v_0\|^2_{L^2(\mathbb{R};\mathbb{R}^d)} = 0$. Hence $\rho\circ v(t) \equiv 0$ holds. Thus $v(t)$ is $w(N)$-valued for all t, which completes the proof. □

Set $u = w^{-1}\circ v$ for the solution v in Lemma 3.2. It is now obvious that this u solves (3.1)-(3.2). Thus we complete the proof. □

4. Geometric energy estimates

Let $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ be a sequence of solutions to (3.1)-(3.2) constructed in Section 3 with $k = m \geqslant 4$. We will obtain the uniform estimate of $\{u^\varepsilon_x\}_{\varepsilon\in(0,1)}$ and the existence time. Our goal of this section is the following.

Lemma 4.1. *Let $u_0 \in H^{m+1}(\mathbb{R};N)$ with an integer $m \geqslant 4$, and let $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ be a sequence of solutions to* (3.1)-(3.2). *Then there exists a constant $T > 0$ depending only on $a, b, N, \|u_{0x}\|_{H^4(\mathbb{R};TN)}$ such that $\{u^\varepsilon_x\}_{\varepsilon\in(0,1)}$ is a bounded sequence in $L^\infty(0,T;H^m(\mathbb{R};TN))$.*

Proof of Lemma 4.1. We define

$$\begin{aligned}
K^\varepsilon(t,x) &= -\frac{1}{3a}\int_{-\infty}^{x} g\left(u^\varepsilon_x(t,y), u^\varepsilon_x(t,y)\right) dy, \\
V^{\varepsilon,(m)}(t,x) &= e^{K^\varepsilon(t,x)}\nabla^m_x u^\varepsilon_x(t,x),
\end{aligned}$$

$$N_m^\varepsilon(t) = \left(\|u_x^\varepsilon(t)\|^2_{H^{m-1}(\mathbb{R};TN)} + \|V^{\varepsilon,(m)}(t)\|^2_{L^2(\mathbb{R};TN)} \right)^{1/2}.$$

We will obtain the differential inequality for $(N_m^\varepsilon(t))^2$. Since $N_4^\varepsilon(0)$ is independent of ε, we set $r_0 = N_4^\varepsilon(0)$ and

$$T_\varepsilon^* = \sup\{T > 0 \mid N_4^\varepsilon(t) \leqslant 2r_0 \text{ for all } t \in [0,T]\}.$$

Lemma 3.2 shows $T_\varepsilon^* > 0$. Moreover, there exists a positive constant $C(a, r_0) > 1$ such that

$$C(a,r_0)^{-1} N_m^\varepsilon(t) \leqslant \|u_x^\varepsilon(t)\|_{H^m(\mathbb{R};TN)} \leqslant C(a,r_0) N_m^\varepsilon(t) \quad \text{for } t \in [0, T_\varepsilon^*].$$

This follows from the relation

$$\left| e^{\pm K^\varepsilon(t,x)} \right| \leqslant 1 + e^{\frac{1}{3|a|}\|u_x^\varepsilon(t)\|^2_{L^2(\mathbb{R};TN)}} \leqslant 1 + e^{\frac{1}{3|a|}\|u_{0x}\|^2_{L^2(\mathbb{R};TN)}}.$$

Note here that the second inequality of the estimate above is due to

$$\|u_x^\varepsilon(t)\|^2_{L^2(\mathbb{R};TN)} \leqslant \|u_{0x}\|^2_{L^2(\mathbb{R};TN)},$$

which follows from the energy inequality of the form

$$\begin{aligned}
\frac{1}{2}\|u_x^\varepsilon\|^2_{L^2(\mathbb{R};TN)} &= \int_{\mathbb{R}} g\left(\nabla_t u_x^\varepsilon, u_x^\varepsilon\right) dx \\
&= \int_{\mathbb{R}} g\left(\nabla_x u_t^\varepsilon, u_x^\varepsilon\right) dx \\
&= \int_{\mathbb{R}} g\left(-\varepsilon \nabla_x^4 u_x^\varepsilon + a\nabla_x^3 u_x^\varepsilon + \nabla_x J_{u^\varepsilon} \nabla_x u_x^\varepsilon + b\nabla_x[g(u_x^\varepsilon, u_x^\varepsilon)u_x^\varepsilon], u_x^\varepsilon\right) dx \\
&= -\varepsilon \|\nabla_x^2 u_x^\varepsilon\|^2_{L^2(\mathbb{R};TN)} \leqslant 0.
\end{aligned}$$

The last equality of the estimate above is easily checked by repeatedly using integration by parts. Especially, we see that

$$\int_{\mathbb{R}} g\left(\nabla_x J_{u^\varepsilon} \nabla_x u_x^\varepsilon, u_x^\varepsilon\right) dx = -\int_{\mathbb{R}} g\left(J_{u^\varepsilon} \nabla_x u_x^\varepsilon, \nabla_x u_x^\varepsilon\right) dx = 0,$$

where the second equality above is due to the fact that (N, J, g) is an almost hermitian manifold.

Having these notations and properties in mind, we show the following.

Proposition 4.2. *There exists a positive constant $C = C(a, b, m, N, r_0) > 0$ and an increasing function $P(\cdot)$ on $[0, +\infty)$ such that*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(N_m^\varepsilon(t)\right)^2 + \frac{\varepsilon}{2}\left(\|\nabla_x^2 V^{\varepsilon,(m)}(t)\|^2_{L^2(\mathbb{R};TN)} + \sum_{l=0}^{m-1} \|\nabla_x^{l+2} u_x^\varepsilon(t)\|^2_{L^2(\mathbb{R};TN)} \right) \\
&\qquad + \frac{1}{2}\| \left(g(u_x^\varepsilon(t), u_x^\varepsilon(t))\right)^{1/2} \nabla_x V^{\varepsilon,(m)}(t)\|^2_{L^2(\mathbb{R};TN)} \\
&\leqslant C(a,b,m,N,r_0) P(N_4^\varepsilon(t) + N_{m-1}^\varepsilon(t)) \left(N_m^\varepsilon(t)\right)^2
\end{aligned} \tag{4.1}$$

follows for all $t \in [0, T_\varepsilon^*]$.

Proof of Proposition 4.2. Throughout the proof of (4.1) we simply write u, J, g, K, $V^{(m)}$ in place of u^ε, J_{u^ε}, g_{u^ε}, K^ε, $V^{\varepsilon,(m)}$ respectively, and write $\|\cdot\|_{H^k} = \|\cdot\|_{H^k(\mathbb{R};TN)}$, $\|\cdot\|_{L^2} = \|\cdot\|_{L^2(\mathbb{R};TN)}$, $\|\cdot\|_{L^\infty} = \|\cdot\|_{L^\infty(\mathbb{R};TN)}$ for $k \in \mathbb{N}$, and sometimes omit to write time variable t.

The main object of the proof is the estimation of

$$\frac{1}{2}\frac{d}{dt}\|V^{(m)}(t)\|^2_{L^2} = \int_{\mathbb{R}} g(\nabla_t V^{(m)}(t), V^{(m)}(t)) dx. \tag{4.2}$$

Thus let us compute the equation of $V^{(m)}$. Operating $e^K\nabla_x^{m+1}$ on (3.1), we have

$$\nabla_t V^{(m)} + \varepsilon\nabla_x^4 V^{(m)} - a\nabla_x^3 V^{(m)} - \nabla_x J\nabla_x V^{(m)} - \varepsilon F_1 - F_2 = F_3, \tag{4.3}$$

where

$$\begin{aligned} F_1 =& 4K_x\nabla_x^3 V^{(m)} + 6(K_{xx} - K_x^2)\nabla_x^2 V^{(m)} \\ &+ 4(K_{xxx} - 3K_xK_{xx} + K_x^3)\nabla_x V^{(m)} \\ &+ (K_{xxxx} - 4K_xK_{xxx} - 3K_{xx}^2 + 6K_x^2K_{xx} - K_x^4)V^{(m)} \\ &+ \sum_{l=0}^{m-1} e^K\nabla_x^l\left[R(u_x,\nabla_x^3u_x)\nabla_x^{m-1-l}u_x\right], \end{aligned} \tag{4.4}$$

$$\begin{aligned} F_2 =& -3aK_x\nabla_x^2V^{(m)} - 3a(K_{xx} - K_x^2)\nabla_x V^{(m)} \\ &- K_x\nabla_x JV^{(m)} - K_xJ\nabla_xV^{(m)} + m\left(\nabla_xJ\right)\nabla_xV^{(m)} \\ &- aR(u_x,\nabla_xV^{(m)})u_x + 2b\,g(\nabla_xV^{(m)},u_x)u_x + b\,g(u_x,u_x)\nabla_xV^{(m)}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} F_3 =& K_tV^{(m)} - a\left(\sum_{l=0}^{m-1} e^K\nabla_x^l\left[R(u_x,\nabla_x^2u_x)\nabla_x^{m-1-l}u_x\right] - R(u_x,\nabla_xV^{(m)})u_x\right) \\ &- \sum_{l=0}^{m-1} e^K\nabla_x^l\left[R(u_x,J\nabla_xu_x)\nabla_x^{m-1-l}u_x\right] \\ &- a(K_{xxx} - 3K_xK_{xx} + K_x^3)V^{(m)} \\ &- (K_{xx} - K_x^2)JV^{(m)} - mK_x\left(\nabla_xJ\right)V^{(m)} \\ &+ \sum_{l=1}^{m}\sum_{j=1}^{l}\frac{l!}{j!(l-j)!}e^K\left(\nabla_x^{j+1}J\right)\nabla_x^{m+1-j}u_x \\ &- 2bK_xg(V^{(m)},u_x)u_x - bK_xg(u_x,u_x)V^{(m)} \\ &+ b\,e^K\sum_{\substack{\alpha+\beta+\gamma=m+1\\ \alpha,\beta,\gamma\geqslant 0\\ \max\{\alpha,\beta,\gamma\}\leqslant m}}\frac{(m+1)!}{\alpha!\beta!\gamma!}g(\nabla_x^\alpha u_x,\nabla_x^\beta u_x)\nabla_x^\gamma u_x. \end{aligned} \tag{4.6}$$

Here R denotes the curvature tensor on (N,J,g), and (∇_xJ) is the covariant derivative of $(1,1)$-tensor field J with respect to x along u defined as

$$\left(\nabla_xJ\right)V = \nabla_xJV - J\nabla_xV \quad \text{for} \quad V\in\Gamma(u^{-1}TN). \tag{4.7}$$

(∇_xJ) is, by definition, a $(1,1)$-tensor field. In the same way, $(\nabla_x^{j+1}J)$ denoting the $(j+1)$-th covariant derivative of J is also $(1,1)$-tensor field along u. See, Appendix, for the precise computations above.

We next obtain the estimate of (4.2) by putting (4.3) into there. To make this estimate be clear or to focus only on the estimation of important parts as possible, we use the notation as follows.

Definition 4.1. For $A,B\in\mathbb{R}$, $A\equiv B$ if and only if there exists a positive constant $C = C(a,b,m,N,r_0)>0$ and an increasing function $P(\cdot)$ on $[0,+\infty)$ such that

$$A - B \leqslant C(a,b,m,N,r_0)P(N_4^\varepsilon(t) + N_{m-1}^\varepsilon(t))\left(N_m^\varepsilon(t)\right)^2$$

follows for $t\in[0,T_\varepsilon^*]$.

First, it follows from the repeatedly using of integration by parts that

$$\int_{\mathbb{R}} g(-\varepsilon\nabla_x^4 V^{(m)}, V^{(m)})dx = -\varepsilon\|\nabla_x^2 V^{(m)}\|_{L^2}^2, \tag{4.8}$$

$$\int_{\mathbb{R}} g(a\nabla_x^3 V^{(m)}, V^{(m)})dx = -a\int_{\mathbb{R}} g(\nabla_x^2 V^{(m)}, \nabla_x V^{(m)})dx = 0, \tag{4.9}$$

$$\int_{\mathbb{R}} g(\nabla_x J\nabla_x V^{(m)}, V^{(m)})dx = -\int_{\mathbb{R}} g(J\nabla_x V^{(m)}, \nabla_x V^{(m)})dx = 0. \tag{4.10}$$

Next, let us go to the estimation of F_2. The following four terms

$$\begin{gathered} -3a(K_{xx} - K_x^2)\nabla_x V^{(m)}, \quad -aR(u_x, \nabla_x V^{(m)})u_x, \\ 2b\, g(\nabla_x V^{(m)}, u_x)u_x, \quad b\, g(u_x, u_x)\nabla_x V^{(m)} \end{gathered}$$

are easily controlled by a use of integration by parts. Indeed, we have

$$\begin{aligned} &\int_{\mathbb{R}} g(-3a(K_{xx} - K_x^2)\nabla_x V^{(m)}, V^{(m)})dx \\ &= -\frac{3a}{2}\int_{\mathbb{R}} g((K_{xx} - K_x^2)\nabla_x V^{(m)}, V^{(m)})dx \\ &\quad + \frac{3a}{2}\int_{\mathbb{R}} g((K_{xx} - K_x^2)V^{(m)}, \nabla_x V^{(m)})dx \\ &\quad + \frac{3a}{2}\int_{\mathbb{R}} g((K_{xx} - K_x^2)_x V^{(m)}, V^{(m)})dx \\ &= \frac{3a}{2}\int_{\mathbb{R}} g((K_{xx} - K_x^2)_x V^{(m)}, V^{(m)})dx \\ &\equiv 0, \end{aligned} \tag{4.11}$$

$$\begin{aligned} &\int_{\mathbb{R}} g(-aR(u_x, \nabla_x V^{(m)})u_x, V^{(m)})dx \\ &= -\frac{a}{2}\int_{\mathbb{R}} g(R(u_x, \nabla_x V^{(m)})u_x, V^{(m)})dx \\ &\quad + \frac{a}{2}\int_{\mathbb{R}} g(R(u_x, V^{(m)})u_x, \nabla_x V^{(m)})dx \\ &\quad + \frac{a}{2}\int_{\mathbb{R}} g(R(u_x, V^{(m)})\nabla_x u_x, V^{(m)})dx \\ &\quad + \frac{a}{2}\int_{\mathbb{R}} g(R(\nabla_x u_x, V^{(m)})u_x, V^{(m)})dx \\ &\quad + \frac{a}{2}\int_{\mathbb{R}} g((\nabla_x R)(u_x, V^{(m)})u_x, V^{(m)})dx \\ &= a\int_{\mathbb{R}} g(R(u_x, V^{(m)})\nabla_x u_x, V^{(m)})dx \\ &\quad + \frac{a}{2}\int_{\mathbb{R}} g((\nabla_x R)(u_x, V^{(m)})u_x, V^{(m)})dx \\ &\equiv 0, \end{aligned} \tag{4.12}$$

$$\begin{aligned}
&\int_{\mathbb{R}} g(2b\, g(\nabla_x V^{(m)}, u_x)u_x, V^{(m)})dx \\
&= -2b \int_{\mathbb{R}} g(g(V^{(m)}, \nabla_x u_x)u_x, V^{(m)})dx \\
&\equiv 0
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
&\int_{\mathbb{R}} g(b\, g(u_x, u_x)\nabla_x V^{(m)}, V^{(m)})dx \\
&= -b \int_{\mathbb{R}} g(g(\nabla_x u_x, u_x)V^{(m)}, V^{(m)})dx \\
&\equiv 0.
\end{aligned} \tag{4.14}$$

Notice that the second equality of (4.12) follows from the fundamental property of the Riemannian curvature tensor R such as

$$g\left(R(X,Y)Z, W\right) = g\left(R(Z,W)X, Y\right) \quad \text{for} \quad X, Y, Z, W \in \Gamma(u^{-1}TN).$$

The estimates of the rest terms of F_2 are demonstrated as follows. For the estimate related to the term $-3aK_x\nabla_x^2 V^{(m)}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}} g(-3aK_x\nabla_x^2 V^{(m)}, V^{(m)})dx \\
&= \int_{\mathbb{R}} g(g(u_x, u_x)\nabla_x^2 V^{(m)}, V^{(m)})dx \\
&= -\int_{\mathbb{R}} g(g(u_x, u_x)\nabla_x V^{(m)}, \nabla_x V^{(m)})dx - 2\int_{\mathbb{R}} g(g(\nabla_x u_x, u_x)\nabla_x V^{(m)}, V^{(m)})dx \\
&= -\| \left(g(u_x, u_x)\right)^{1/2} \nabla_x V^{(m)}\|_{L^2}^2 + \int_{\mathbb{R}} g([g(\nabla_x u_x, u_x)]_x V^{(m)}, V^{(m)})dx \\
&\equiv -\| \left(g(u_x, u_x)\right)^{1/2} \nabla_x V^{(m)}\|_{L^2}^2.
\end{aligned} \tag{4.15}$$

As for the term $m\left(\nabla_x J\right)\nabla_x V^{(m)}$, note first that there exists a positive constant $C_1 = C_1(N) > 0$ such that

$$|(\nabla_x J)|\,(x) \leqslant C_1(N)\left(g(u_x(x), u_x(x))\right)^{1/2} \tag{4.16}$$

holds uniformly with respect to x. Thus we have

$$\begin{aligned}
&\int_{\mathbb{R}} g(m\left(\nabla_x J\right)\nabla_x V^{(m)}, V^{(m)})dx \\
&\leqslant m\| \left(\nabla_x J\right)\nabla_x V^{(m)}\|_{L^2}\|V^{(m)}\|_{L^2} \\
&\leqslant mC_1(N)\| \left(g(u_x, u_x)\right)^{1/2}\nabla_x V^{(m)}\|_{L^2}\|V^{(m)}\|_{L^2} \\
&\leqslant \rho\| \left(g(u_x, u_x)\right)^{1/2}\nabla_x V^{(m)}\|_{L^2}^2 + \frac{m^2C_1^2}{4\rho}\|V^{(m)}\|_{L^2}^2 \\
&\equiv \rho\| \left(g(u_x, u_x)\right)^{1/2}\nabla_x V^{(m)}\|_{L^2}^2
\end{aligned} \tag{4.17}$$

for any $\rho > 0$. Note that the third inequality above is due to the Schwartz inequality.

In the same way, as for the term $-K_x\nabla_x JV^{(m)}$ and $-K_xJ\nabla_xV^{(m)}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}} g(-K_x\nabla_x JV^{(m)}, V^{(m)})dx + \int_{\mathbb{R}} g(-K_xJ\nabla_xV^{(m)}, V^{(m)})dx \\
&= \int_{\mathbb{R}} g(K_{xx}JV^{(m)}, V^{(m)})dx + 2\int_{\mathbb{R}} g(K_xJV^{(m)}, \nabla_xV^{(m)})dx \\
&= 2\int_{\mathbb{R}} g(K_xJV^{(m)}, \nabla_xV^{(m)})dx \\
&= -\frac{2}{3a}\int_{\mathbb{R}} g((g(u_x,u_x))^{1/2}\, JV^{(m)}, (g(u_x,u_x))^{1/2}\, \nabla_xV^{(m)})dx \\
&\leqslant \rho\| (g(u_x,u_x))^{1/2}\, \nabla_xV^{(m)}\|_{L^2}^2 + \left(\frac{2}{3|a|}\right)^2 \frac{1}{4\rho}\| (g(u_x,u_x))^{1/2}\, JV^{(m)}\|_{L^2}^2 \\
&\equiv \rho\| (g(u_x,u_x))^{1/2}\, \nabla_xV^{(m)}\|_{L^2}^2
\end{aligned} \tag{4.18}$$

for any $\rho > 0$.

By combining (4.11), (4.12), (4.13), (4.14), (4.15), (4.17) and (4.18), and by taking $\rho = 1/4$, we deduce

$$\int_{\mathbb{R}} g(F_2(t), V^{(m)}(t))dx \equiv -\frac{1}{2}\| (g(u_x(t),u_x(t)))^{1/2}\, \nabla_xV^{(m)}(t)\|_{L^2}^2. \tag{4.19}$$

Thirdly, we consider F_3. There never appear the terms containing higher ordered derivative like $\nabla_x^{m+l}u_x$ with $l \in \mathbb{N}$ in F_3. Hence it is easy to obtain that

$$\begin{aligned}
\int_{\mathbb{R}} g(F_3(t), V^{(m)}(t))dx &\leqslant C(a,b,m,N,r_0)P(N_4^\varepsilon(t) + N_{m-1}^\varepsilon(t))N_m^\varepsilon(t)\|V^{(m)}(t)\|_{L^2} \\
&\equiv 0.
\end{aligned} \tag{4.20}$$

Here we add some comments on the estimation. The curvature tensor is estimated as follows: for $l \geqslant 0$ (resp. $j \geqslant 1$) and $U, V, W \in \Gamma(u^{-1}TN)$, there exists a positive constant $C(N,l) > 0$ (resp. $C(N,j) > 0$) such that

$$\begin{aligned}
\left|\nabla_x^l\left[R(U,V)W\right]\right|(x) &\leqslant C(N,l) \sum_{\substack{p+q+r+j=l \\ p,q,r,j\geqslant 0}} \left|(\nabla_x^j R)\right| |\nabla_x^p U|\, |\nabla_x^q V|\, |\nabla_x^r W|\,(x), \\
\left|(\nabla_x^j R)\right|(x) &\leqslant C(N,j)\sum_{\alpha=1}^{j} \sum_{\substack{\alpha+\sum_{h=1}^{\alpha} p_h = j \\ p_h\geqslant 0}} |\nabla_x^{p_1}u_x|\cdots|\nabla_x^{p_\alpha}u_x|\,(x)
\end{aligned} \tag{4.21}$$

uniformly with respect to x, where $|\cdot| = (g(\cdot,\cdot))^{1/2}$. Similarly, the $(1,1)$-tensor field $(\nabla_x^{j+1}J)$ with $j \geqslant 0$ is estimated as

$$\left|(\nabla_x^{j+1}J)\right|(x) \leqslant C(N,j)\sum_{\alpha=1}^{j+1} \sum_{\substack{\alpha+\sum_{h=1}^{\alpha} p_h = j+1 \\ p_h\geqslant 0}} |\nabla_x^{p_1}u_x|\cdots|\nabla_x^{p_\alpha}u_x|\,(x) \tag{4.22}$$

for some positive constant $C(N,j) > 0$. Observing them, we can see that higher ordered derivatives never appear in F_3 and thus (4.20) is obtained. Note also $K_tV^{(m)}$ is contained in F_3. The requirement $m \geqslant 4$ comes to control this term. In other words, the L^∞-norm of K_t is bounded by some positive constant $C = C(a, r_0)$. Hence $K_tV^{(m)}$ is also harmless in the estimation (4.20).

Finally we consider the term εF_1. By repeatedly using integration by parts and the Schwartz inequality as before, it is easy to check that

$$\int_{\mathbb{R}} g(\varepsilon F_1(t), V^{(m)}(t))dx \equiv \rho\varepsilon\|\nabla_x^2 V^{(m)}(t)\|_{L^2}^2 \tag{4.23}$$

for any $\rho > 0$. Thus, by taking $\rho = 1/2$, it follows from (4.8) and (4.23) that

$$\int_{\mathbb{R}} g(-\varepsilon\nabla_x^4 V^{(m)}(t) + \varepsilon F_1(t), V^{(m)}(t))dx \equiv -\frac{\varepsilon}{2}\|\nabla_x^2 V^{(m)}(t)\|_{L^2}^2. \tag{4.24}$$

Consequently, (4.9), (4.10), (4.19), (4.20), and (4.24) yield that (4.2) is estimated as follows:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|V^{(m)}(t)\|_{L^2}^2 + \frac{\varepsilon}{2}\|\nabla_x^2 V^{(m)}(t)\|_{L^2}^2 + \frac{1}{2}\|\left(g(u_x(t), u_x(t))\right)^{1/2}\nabla_x V^{(m)}(t)\|_{L^2}^2 \\ &\leqslant C(a,b,m,N,r_0)P(N_4^\varepsilon(t) + N_{m-1}^\varepsilon(t))\left(N_m^\varepsilon(t)\right)^2\end{aligned} \tag{4.25}$$

for some $C(a,b,m,N,r_0) > 0$ and increasing function $P(\cdot)$.

On the other hands, it is easy to prove

$$\frac{1}{2}\frac{d}{dt}\|u_x(t)\|_{H^{m-1}}^2 + \frac{\varepsilon}{2}\sum_{l=0}^{m-1}\|\nabla_x^{l+2}u_x(t)\|_{L^2}^2 \equiv 0. \tag{4.26}$$

By adding (4.25) and (4.26), we obtain the desired estimate (4.1). □

Lemma 4.1 follows immediately from Proposition 4.2 in the following way. If $m = 4$, then (4.1) implies that

$$(N_4^\varepsilon(t))^2 \leqslant r_0^2\exp\left(2C(a,b,4,N,r_0)t\right) \quad \text{for} \quad t \in [0, T_\varepsilon^*].$$

If we set $t = T_\varepsilon^*$, then this becomes

$$4r_0^2 = (N_4^\varepsilon(T_\varepsilon^*))^2 \leqslant r_0^2\exp\left(2C(a,b,4,N,r_0)T_\varepsilon^*\right),$$

which implies

$$T_\varepsilon^* \geqslant T \equiv \frac{2C(a,b,4,N,r_0)}{\log 4}.$$

Clearly T depends only on $a, b, N, \|u_{0x}\|_{H^4}$, being independent of $\varepsilon \in (0,1)$, and $\{u_x^\varepsilon\}_{\varepsilon\in(0,1)}$ is a bounded sequence in $L^\infty(0,T;H^4(\mathbb{R};TN))$. Then, by using the Gronwall inequality for $m = 5, 6, \ldots$ inductively, we obtain that $\{u_x^\varepsilon\}_{\varepsilon\in(0,1)}$ is a bounded sequence in $L^\infty(0,T;H^m(\mathbb{R};TN))$. □

Remark 2. $\{u_x^\varepsilon\}_{\varepsilon\in(0,1)}$ gains the regularity in the following sense: By integrating (4.1) on $[0,T]$, we obtain

$$\frac{\varepsilon}{2}\left(\|\nabla_x^2 V^{\varepsilon,(m)}\|_{L^2((0,T)\times\mathbb{R};TN)}^2 + \sum_{l=0}^{m-1}\|\nabla_x^{l+2}u_x^\varepsilon\|_{L^2((0,T)\times\mathbb{R};TN)}^2\right) \leqslant C$$

for some constant $C = C(a,b,N,\|u_{0x}\|_{H^m},T) > 0$ independent of $\varepsilon \in (0,1)$. This implies that the sequence $\{\varepsilon^{1/2}\nabla_x^m u_x^\varepsilon\}_{\varepsilon\in(0,1)}$ is bounded in $L^2(0,T;H^2(\mathbb{R};TN))$. From this and Lemma 4.1 it is obvious that $\{u_t^\varepsilon\}_{\varepsilon\in(0,1)}$ is also a bounded sequence in $L^2(0,T;H^{m-2}(\mathbb{R};TN))$. We will use this property in the compactness argument in the next section.

5. Proof of Theorem 1.1

Proof of Theorem 1.1. We are now in a position to complete the proof of Theorem 1.1. We have only to solve (1.1)-(1.2) in the positive direction of the time variable.

Proof of existence. Suppose that $u_0 \in H^{m+1}(\mathbb{R}; N)$ with the integer $m \geqslant 4$ is given. By applying Proposition 3.1 as $k = m$, we construct a sequence $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ solving (3.1)-(3.2) for each $\varepsilon > 0$. Recall that Lemma 4.1 implies that there exists $T = T(a, b, N, \|u_{0x}\|_{H^4(\mathbb{R};TN)}) > 0$ which is independent of $\varepsilon \in (0, 1)$ such that $\{u^\varepsilon_x\}_{\varepsilon\in(0,1)}$ is bounded in $L^\infty(0, T; H^m(\mathbb{R}; TN))$. Recall also, as stated in Remark 2 in the previous section, $\{u^\varepsilon_t\}_{\varepsilon\in(0,1)}$ is bounded in the class $L^2(0, T; H^{m-2}(\mathbb{R}; TN))$. Having them in mind, define $v^\varepsilon = w\circ u^\varepsilon$. Then the boundnesses above imply respectively that $\{v^\varepsilon_x\}_{\varepsilon\in(0,1)}$ is bounded in $L^\infty(0, T; H^m(\mathbb{R}; \mathbb{R}^d))$ and $\{v^\varepsilon_t\}_{\varepsilon\in(0,1)}$ is bounded in $L^2(0, T; H^{m-2}(\mathbb{R}; \mathbb{R}^d))$. Especially, this boundness of $\{v^\varepsilon_t\}_{\varepsilon\in(0,1)}$ yields that $\{v^\varepsilon_x\}_{\varepsilon\in(0,1)}$ is bounded in the class $C^{0,1/2}([0, T]; H^{m-3}(\mathbb{R}; \mathbb{R}^d))$. Then the standard compactness arguments imply that there exists a subsequence $\{v^j\}_{j\in\mathbb{N}}$ and v such that

$$v^j_x \xrightarrow{w^\star} v_x \quad \text{in} \quad L^\infty(0, T; H^m(\mathbb{R}; \mathbb{R}^d)) \qquad \text{as} \quad j \to \infty, \tag{5.1}$$

$$v^j_x \longrightarrow v_x \quad \text{in} \quad C([0, T]; H^{m-1}_{loc}(\mathbb{R}; \mathbb{R}^d)) \quad \text{as} \quad j \to \infty, \tag{5.2}$$

$$v^j \longrightarrow v \quad \text{in} \quad C([0, T] \times \overline{B(0, R)}; \mathbb{R}^d)) \quad \text{as} \quad j \to \infty \tag{5.3}$$

for any $R > 0$, where $\overline{B(0, R)} = \{x \in \mathbb{R} \mid |x| \leqslant R\}$. In particular, (5.3) implies that $v \in C([0, T] \times \mathbb{R}; w(N))$ and $w^{-1}\circ v$ satisfies the initial condition (1.2). Furthermore, it is easy to check that v satisfies (3.3) with $\varepsilon = 0$. At this time, notice that $v_x \in L^\infty(0, T; H^m(\mathbb{R}; \mathbb{R}^d)) \cap C([0, T]; H^{m-1}(\mathbb{R}; \mathbb{R}^d))$ follows. As a consequence, we have $u = w^{-1}\circ v \in C([0, T] \times \mathbb{R}; N)$ with

$$u_x \in L^\infty(0, T; H^m(\mathbb{R}; TN)) \cap C([0, T]; H^{m-1}(\mathbb{R}; TN)) \tag{5.4}$$

which solves (1.1) with the initial data u_0. Thus we complete the proof of the existence of time-local solutions. □

Remark 3. For the solution $u = w^{-1}\circ v$, since $v_x \in L^\infty(0, T; H^m(\mathbb{R}; \mathbb{R}^d))$, v_t belongs to $L^\infty(0, T; H^{m-2}(\mathbb{R}; \mathbb{R}^d))$, and thus we see that $v - w\circ u_0$ belongs to $C^{0,1}([0, T]; H^{m-2}(\mathbb{R}; \mathbb{R}^d))$.

Proof of uniqueness. Let $u, v \in C([0, T] \times \mathbb{R}; N)$ be solutions of (1.1)-(1.2) with (5.4), and let $u(0, x) = v(0, x)$. Identify u, v with $w\circ u, w\circ v$. Then u and v satisfy

$$v_t - av_{xxx} = f(v, v_x, v_{xx}),$$

where

$$\begin{aligned} f(v, v_x, v_{xx}) =& a\left\{[A(v)(v_x, v_x)]_x + A(v)(v_{xx} + A(v)(v_x, v_x), v_x)\right\} \\ &+ dw_{w^{-1}\circ v} J_{w^{-1}\circ v} dw_v^{-1}(v_{xx} + A(v)(v_x, v_x)) + b\,|v_x|^2\, v_x \end{aligned}$$

for $v : \mathbb{R} \to N$. As is stated in Remark 3, both $u - w\circ u_0$ and $v - w\circ u_0$ belong to the class $C^{0,1}([0, T]; H^{m-2}(\mathbb{R}; \mathbb{R}^d))$ and thus $z = u - v$ is well-defined as a $\mathbb{R}^d$-valued function. Taking the difference between two equations, we have

$$z_t - az_{xxx} = f(u, u_x, u_{xx}) - f(v, v_x, v_{xx}),$$

To prove that $z = 0$, we can show that there exists a constant $C > 0$ depending only on a, b, N, $\|u_x\|_{L^\infty(0,T;H^2(\mathbb{R};\mathbb{R}^d))}$, and $\|v_x\|_{L^\infty(0,T;H^2(\mathbb{R};\mathbb{R}^d))}$ such that

$$\frac{d}{dt}\|z(t)\|^2_{H^1(\mathbb{R};\mathbb{R}^d)} \leqslant C\|z(t)\|^2_{H^1(\mathbb{R};\mathbb{R}^d)}. \tag{5.5}$$

This estimate can be obtained by completely same calculation as that in the proof of the uniqueness in [17]. Note, though the only case that (N, J, g) is a Kähler manifold is discussed in [17], the argument proving the uniqueness works also when (N, J, g) is a compact almost Hermitian manifold. Thus we omit the proof of (5.5). □

Proof of the continuity in time of $\nabla_x^m u_x$ *in* $L^2(\mathbb{R}; TN)$. We have already proved the existence of a unique solution $u \in C([0,T] \times \mathbb{R}; N)$ with (5.4). Thus the proof of $\nabla_x^m u_x \in C([0,T]; L^2(\mathbb{R}; TN)$ is left. Let $v = w \circ u$. To obtain this continuity, it suffices to show that $dw_u(V^{(m)})$ belongs to $C([0,T]; L^2(\mathbb{R}; \mathbb{R}^d))$.

First of all, the energy estimate (4.1) implies $(d/dt)\left(N_m^\varepsilon(t)\right)^2 \leqslant C$ for some $C > 0$ which is independent of $\varepsilon \in (0,1)$. Hence we deduce

$$\begin{aligned}\|V^{\varepsilon,(m)}(t)\|^2_{L^2(\mathbb{R};TN)} + \|u_x^\varepsilon(t)\|^2_{H^{m-1}(\mathbb{R};TN)} \\ \leqslant \|V^{\varepsilon,(m)}(0)\|^2_{L^2(\mathbb{R};TN)} + \|u_x^\varepsilon(0)\|^2_{H^{m-1}(\mathbb{R};TN)} + Ct.\end{aligned}$$

Letting $\varepsilon \downarrow 0$, we see that $V^{(m)}(t) = (e^K \nabla_x^m u_x)(t) \in L^2(\mathbb{R}; \mathbb{R}^d)$ makes sense for all $t \in [0,T]$, and

$$\begin{aligned}\|V^{(m)}(t)\|^2_{L^2(\mathbb{R};TN)} + \|u_x(t)\|^2_{H^{m-1}(\mathbb{R};TN)} \\ \leqslant \|V^{(m)}(0)\|^2_{L^2(\mathbb{R};TN)} + \|u_x(0)\|^2_{H^{m-1}(\mathbb{R};TN)} + Ct.\end{aligned}$$

Noting that $u_x \in C([0,T]; H^{m-1}(\mathbb{R}; TN))$, we have

$$\limsup_{t\to 0} \|V^{(m)}(t)\|^2_{L^2(\mathbb{R};TN)} \leqslant \|V^{(m)}(0)\|^2_{L^2(\mathbb{R};TN)}. \tag{5.6}$$

Since w is the isometric embedding, (5.6) is equivalent to

$$\limsup_{t\to 0} \|dw_u(V^{(m)})(t)\|^2_{L^2(\mathbb{R};\mathbb{R}^d)} \leqslant \|dw_u(V^{(m)})(0)\|^2_{L^2(\mathbb{R};\mathbb{R}^d)}. \tag{5.7}$$

Moreover, since $v_x \in L^\infty(0,T; H^m(\mathbb{R}; \mathbb{R}^d)) \cap C([0,T]; H^{m-1}(\mathbb{R}; \mathbb{R}^d))$, we see $dw_u(V^{(m)})(t)$ is weakly continuous in $L^2(\mathbb{R}; \mathbb{R}^d)$. Hence it follows that

$$\|dw_u(V^{(m)})(0)\|^2_{L^2(\mathbb{R};\mathbb{R}^d)} \leqslant \liminf_{t\to 0} \|dw_u(V^{(m)})(t)\|^2_{L^2(\mathbb{R};\mathbb{R}^d)}. \tag{5.8}$$

From (5.7) and (5.8), we obtain

$$\lim_{t\to 0} \|dw_u(V^{(m)})(t)\|^2_{L^2(\mathbb{R};\mathbb{R}^d)} = \|dw_u(V^{(m)})(0)\|^2_{L^2(\mathbb{R};\mathbb{R}^d)}. \tag{5.9}$$

Consequently, (5.9) and the weak continuity of $dw_u(V^{(m)})(t)$ in the class $L^2(\mathbb{R}; \mathbb{R}^d)$ imply that $dw_u(V^{(m)})(t)$ is strongly continuous in $L^2(\mathbb{R}; \mathbb{R}^d)$ at $t = 0$. By the uniqueness of u, we see $dw_u(V^{(m)})(t)$ is strongly continuous at each $t \in [0,T]$ in the same way. Thus we complete the proof. □

□

6. Sketch of the proof of Theorem 1.2 and 1.3

This section is devoted to the outline of the proof of Theorem 1.2 and 1.3. Recall in both cases, N is supposed to be a compact Kähler manifold.

Proof of Theorem 1.2. Since N is a compact Kähler manifold, the procedures of the proof is almost parallel to that in [17]. There is a difference to the proof of Theorem 1.1 in the

energy estimate. Due to the Kähler condition, the classical energy method works effectively. In other words, we do not need to use the gauge transformation of $\nabla_x^m u_x$ used in the proof of Theorem 1.1. This is the reason that this theorem holds for $m \geqslant 2$. Indeed, we can obtain the following.

Lemma 6.1. *Let $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ be a sequence of solution of* (3.1)-(3.2) *constructed in Proposition* 3.1 *as $k = m \geqslant 2$. Then there exists a constant $T > 0$ depending only on a, b, N, and $\|u_{0x}\|_{H^2(\mathbb{R};TN)}$ such that $\{u_x^\varepsilon\}_{\varepsilon\in(0,1)}$ is bounded in $L^\infty(0,T;H^m(\mathbb{R};TN))$.*

Proof of Lemma 6.1. By the completely same calculus as that in [17, Lemma 4.1], we can show that

$$\frac{d}{dt}\|u_x^\varepsilon(t)\|^2_{H^2(\mathbb{R};TN)} \leqslant C(a,b,N)\sum_{r=4}^{8}\|u_x^\varepsilon(t)\|^r_{H^2(\mathbb{R};TN)}, \tag{6.1}$$

$$\frac{d}{dt}\|u_x^\varepsilon(t)\|^2_{H^k(\mathbb{R};TN)} \leqslant C(a,b,N,\|u_x^\varepsilon(t)\|_{H^{k-1}(\mathbb{R};TN)})\|u_x^\varepsilon(t)\|^2_{H^k(\mathbb{R};TN)} \tag{6.2}$$

for $3 \leqslant k \leqslant m$ hold for all $t \in [0, T_\varepsilon]$. From (6.1) and (6.2), the desired boundness is immediately obtained. See [17, Lemma 4.1] for details. □

The other parts of the proof of Theorem 1.2 are same as that was discussed in Theorem 1.1. Thus we omit the detail. □

Next, let (N, J, g) be a compact Riemann surface with constant Gaussian curvature K, and assume that $a \neq 0$ and $b = aK/2$. Theorem 1.2 tells us that, given a initial data $u_0 \in H^{m+1}(\mathbb{R};N)$, there exists $T = T(a,b,N,\|u_{0x}\|_{H^2(\mathbb{R};TN)}) > 0$ such that the IVP (1.1)-(1.2) admits a unique time-local solution $u \in C([0,T);H^{m+1}(\mathbb{R};N))$.

In what follows we will extend the existence time of u over $[0,\infty)$. For this, we have the following energy conversation laws.

Lemma 6.2. *For $u \in C([0,T);H^{m+1}(\mathbb{R};N))$ solving* (1.1)-(1.2)*, the following quantities*

$$\begin{aligned}
&\|u_x(t)\|^2_{L^2(\mathbb{R};TN)},\\
&E(u(t)) = \|\nabla_x^2 u_x(t)\|^2_{L^2(\mathbb{R};TN)} + \frac{K^2}{8}\int_{\mathbb{R}}(g(u_x(t),u_x(t)))^3\,dx\\
&\qquad - K\int_{\mathbb{R}}(g(u_x(t),\nabla_x u_x(t)))^2\,dx\\
&\qquad - \frac{3K}{2}\int_{\mathbb{R}} g(u_x(t),u_x(t))g(\nabla_x u_x(t),\nabla_x u_x(t))dx
\end{aligned}$$

are preserved with respect to $t \in [0,T)$.

Proof of Lemma 6.2. The proof is also same as that was discussed in [17, Lemma 6.1]. Thus we omit the detail. □

Proof of Theorem 1.3. Let $u \in C([0,T);H^{m+1}(\mathbb{R};N))$ be a time-local solution of (1.1)-(1.2) which exists on the maximal time interval $[0,T)$. If $T = \infty$, Theorem 1.3 holds true. Thus we only need to consider the case $T < \infty$. From Lemma 6.2, we know that

$$\|u_x(t)\|^2_{L^2(\mathbb{R};TN)} = \|u_{0x}\|^2_{L^2(\mathbb{R};TN)}, \quad E(u(t)) = E(u_0). \tag{6.3}$$

Hence it follows that

$$\|\nabla_x^2 u_x(t)\|^2_{L^2(\mathbb{R};TN)} = E(u_0) - \frac{K^2}{8}\int_{\mathbb{R}}(g(u_x(t),u_x(t)))^3\,dx$$

$$
\begin{aligned}
&+ K \int_{\mathbb{R}} \left(g(u_x(t), \nabla_x u_x(t))\right)^2 dx \\
&\quad + \frac{3K}{2} \int_{\mathbb{R}} g(u_x(t), u_x(t)) g(\nabla_x u_x(t), \nabla_x u_x(t)) dx \\
\leqslant & E(u_0) + C|K| \|u_x(t)\|^2_{L^\infty(\mathbb{R};TN)} \|\nabla_x u_x(t)\|^2_{L^2(\mathbb{R};TN)}.
\end{aligned}
$$

The second term of the right hand side of the above is estimated as follows. At first, we have

$$
\begin{aligned}
\|\nabla_x u_x(t)\|^2_{L^2(\mathbb{R};TN)} &= -\int_{\mathbb{R}} g\left(u_x(t), \nabla_x^2 u_x(t)\right) dx \\
&\leqslant \|u_x(t)\|_{L^2(\mathbb{R};TN)} \|\nabla_x^2 u_x(t)\|_{L^2(\mathbb{R};TN)} \\
&= \|u_{0x}\|_{L^2(\mathbb{R};TN)} \|\nabla_x^2 u_x(t)\|_{L^2(\mathbb{R};TN)}. \qquad (6.4)
\end{aligned}
$$

Next, note that $dw_u(\nabla_x u_x) = v_{xx} + A(v)(v_x, v_x)$ holds for $v = w \circ u$ by the definition of the covariant derivative along the mapping u. By noting this and by using (6.4) and Sobolev's inequality, we obtain

$$
\begin{aligned}
&\|u_x(t)\|^2_{L^\infty(\mathbb{R};TN)} \\
&= \|v_x(t)\|^2_{L^\infty(\mathbb{R};\mathbb{R}^d)} \\
&\leqslant C \|v_x(t)\|_{L^2(\mathbb{R};\mathbb{R}^d)} \|v_{xx}(t)\|_{L^2(\mathbb{R};\mathbb{R}^d)} \\
&\leqslant C \|v_x(t)\|_{L^2(\mathbb{R};\mathbb{R}^d)} \\
&\qquad \times \left(\|v_{xx}(t) + A(v)(v_x, v_x)(t)\|_{L^2(\mathbb{R};\mathbb{R}^d)} + \|A(v)(v_x, v_x)(t)\|_{L^2(\mathbb{R};\mathbb{R}^d)} \right) \\
&\leqslant C \|v_x(t)\|_{L^2(\mathbb{R};\mathbb{R}^d)} \\
&\qquad \times \Big(\|v_{xx}(t) + A(v)(v_x, v_x)(t)\|_{L^2(\mathbb{R};\mathbb{R}^d)} \\
&\qquad\qquad\qquad + C(N) \|v_x(t)\|_{L^\infty(\mathbb{R};\mathbb{R}^d)} \|v_x(t)\|_{L^2(\mathbb{R};\mathbb{R}^d)} \Big) \\
&= C \|u_x(t)\|_{L^2(\mathbb{R};TN)} \\
&\qquad \times \left(\|\nabla_x u_x(t)\|_{L^2(\mathbb{R};TN)} + C(N) \|u_x(t)\|_{L^\infty(\mathbb{R};TN)} \|u_x(t)\|_{L^2(\mathbb{R};TN)} \right) \\
&\leqslant C \|u_{0x}\|_{L^2(\mathbb{R};TN)} \\
&\qquad \times \Big(\|u_{0x}\|^{1/2}_{L^2(\mathbb{R};TN)} \|\nabla_x^2 u_x(t)\|^{1/2}_{L^2(\mathbb{R};TN)} \\
&\qquad\qquad\qquad + C(N) \|u_x(t)\|_{L^\infty(\mathbb{R};TN)} \|u_{0x}\|_{L^2(\mathbb{R};TN)} \Big) \\
&= C \|u_{0x}\|^{3/2}_{L^2(\mathbb{R};TN)} \|\nabla_x^2 u_x(t)\|^{1/2}_{L^2(\mathbb{R};TN)} \\
&\qquad\qquad + C(N) \|u_x(t)\|_{L^\infty(\mathbb{R};TN)} \|u_{0x}\|^2_{L^2(\mathbb{R};TN)},
\end{aligned}
$$

which implies

$$
\|u_x(t)\|_{L^\infty(\mathbb{R};TN)} \leqslant C(N, \|u_{0x}\|_{L^2(\mathbb{R};TN)}) \left(1 + \|\nabla_x^2 u_x(t)\|^{1/4}_{L^2(\mathbb{R};TN)}\right). \qquad (6.5)
$$

From (6.3), (6.4) and (6.5), we deduce

$$
\|\nabla_x^2 u_x(t)\|^2_{L^2(\mathbb{R};TN)}
$$

$$\leqslant E(u_0) + C(K, N, \|u_{0x}\|_{L^2(\mathbb{R};TN)}) \times \left(1 + \|\nabla_x^2 u_x(t)\|_{L^2(\mathbb{R};TN)}^{1/2}\right) \|\nabla_x^2 u_x(t)\|_{L^2(\mathbb{R};TN)}.$$

Thus $X = X(t) = 1 + \|\nabla_x^2 u_x(t)\|_{L^2(\mathbb{R};TN)}^2$ satisfies

$$X \leqslant 1 + E(u_0) + C(K, N, \|u_{0x}\|_{L^2(\mathbb{R};TN)}) X^{3/4},$$

which implies that $X(t)$ is bounded, and thus

$$\sup_{t\in[0,T)} \|\nabla_x^2 u_x(t)\|_{L^2(\mathbb{R};TN)} \leqslant C(K, N, \|u_{0x}\|_{H^2(\mathbb{R};TN)}) \tag{6.6}$$

for some $C = C(K, N, \|u_{0x}\|_{H^2(\mathbb{R};TN)}) > 0$. Interpolating (6.3) and (6.6) we have

$$\sup_{t\in[0,T)} \|u_x(t)\|_{H^2(\mathbb{R};TN)} \leqslant C(K, N, \|u_{0x}\|_{H^2(\mathbb{R};TN)}).$$

Once we obtain the $H^2(\mathbb{R};TN)$-boundness of u_x, the desired $H^m(\mathbb{R};TN)$-boundness of u_x follows from the use of (6.2) inductively with respect to $k = 3, \dots, m$. Thus the existence time of u can be extended beyond T. □

7. Appendix

In this section, we check (4.3) used in Section 4. Operating $e^K \nabla_x^{m+1}$ on the equation (3.1), we have

$$e^K \nabla_x^{m+1} u_t = -\varepsilon\, e^K \nabla_x^{m+4} u_x + a\, e^K \nabla_x^{m+3} u_x + e^K \nabla_x^{m+1} J \nabla_x u_x + b\, e^K \nabla_x^{m+1} g(u_x, u_x) u_x. \tag{7.1}$$

First, to compute each term of (7.1), we use the following relation

$$e^K \nabla_x^{m+k} u_x = \nabla_x \left(e^K \nabla_x^{m+k-1} u_x\right) - K_x e^K \nabla_x^{m+k-1} u_x \quad \text{for} \quad k \in \mathbb{N}. \tag{7.2}$$

By using this relation repeatedly, we deduce

$$e^K \nabla_x^{m+1} u_x = \nabla_x V^{(m)} - K_x V^{(m)}, \tag{7.3}$$

$$e^K \nabla_x^{m+2} u_x = \nabla_x^2 V^{(m)} - 2K_x \nabla_x V^{(m)} - \left(K_{xx} - K_x^2\right) V^{(m)}, \tag{7.4}$$

$$e^K \nabla_x^{m+3} u_x = \nabla_x^3 V^{(m)} - 3K_x \nabla_x^2 V^{(m)} - 3\left(K_{xx} - K_x^2\right) \nabla_x V^{(m)} - \left(K_{xxx} - 3K_x K_{xx} + K_x^3\right) V^{(m)}, \tag{7.5}$$

$$e^K \nabla_x^{m+4} u_x = \nabla_x^4 V^{(m)} - 4K_x \nabla_x^3 V^{(m)} - 6\left(K_{xx} - K_x^2\right) \nabla_x^2 V^{(m)} - 4\left(K_{xxx} - 3K_x K_{xx} + K_x^3\right) \nabla_x V^{(m)} - \left(K_{xxxx} - 4K_x K_{xxx} - 3K_{xx}^2 + 6K_x^2 K_{xx} - K_x^4\right) V^{(m)}. \tag{7.6}$$

Moreover, (7.3) and the Leibniz rule yield that

$$e^K \nabla_x^{m+1} \left[g(u_x, u_x) u_x\right] \tag{7.7}$$

$$= 2e^K g(\nabla_x^{m+1} u_x, u_x) u_x + e^K g(u_x, u_x) \nabla_x^{m+1} u_x \tag{7.8}$$

$$+ \sum_{\substack{\alpha+\beta+\gamma=m+1 \\ \alpha,\beta,\gamma\geqslant 0 \\ \max\{\alpha,\beta,\gamma\}\leqslant m}} \frac{(m+1)!}{\alpha!\beta!\gamma!} e^K g(\nabla_x^\alpha u_x, \nabla_x^\beta u_x) \nabla_x^\gamma u_x \tag{7.9}$$

$$= 2g(\nabla_x V^{(m)}, u_x) u_x + g(u_x, u_x) \nabla_x V^{(m)} \tag{7.10}$$

$$- 2g(K_x V^{(m)}, u_x)u_x - g(u_x, u_x)K_x V^{(m)} \tag{7.11}$$

$$+ \sum_{\substack{\alpha+\beta+\gamma=m+1 \\ \alpha,\beta,\gamma\geqslant 0 \\ \max\{\alpha,\beta,\gamma\}\leqslant m}} \frac{(m+1)!}{\alpha!\beta!\gamma!} e^K g(\nabla_x^\alpha u_x, \nabla_x^\beta u_x)\nabla_x^\gamma u_x. \tag{7.12}$$

Next, we compute $e^K \nabla_x^{m+1} u_t$. Note that

$$\nabla_t u_x = \nabla_x u_t \quad \text{and} \quad \nabla_x \nabla_t u_x = \nabla_t \nabla_x u_x + R(u_x, u_t)u_x$$

follow from the definition of the Levi-Civita connection. Using these commutative relations inductively, we have

$$\nabla_x^{m+1} u_t = \nabla_t \nabla_x^m u_x + \sum_{l=0}^{m-1} \nabla_x^l \left[R(u_x, u_t)\nabla_x^{m-(l+1)} u_x \right]. \tag{7.13}$$

By multiplying e^K with (7.13), we have

$$e^K \nabla_x^{m+1} u_t = e^K \nabla_t \nabla_x^m u_x + \sum_{l=0}^{m-1} e^K \nabla_x^l \left[R(u_x, u_t)\nabla_x^{m-(l+1)} u_x \right]. \tag{7.14}$$

By noting $e^K \nabla_t \nabla_x^m u_x = \nabla_t \left(e^K \nabla_x^m u_x \right) - K_t \nabla_x^m u_x = \nabla_t V^{(m)} - K_t V^{(m)}$, and by substituting (3.1) into the second term of (7.14), we deduce

$$e^K \nabla_x^{m+1} u_t = \nabla_t V^{(m)} - K_t V^{(m)} - \varepsilon \sum_{l=0}^{m-1} e^K \nabla_x^l \left[R(u_x, \nabla_x^3 u_x)\nabla_x^{m-(l+1)} u_x \right] \tag{7.15}$$

$$+ a \sum_{l=0}^{m-1} e^K \nabla_x^l \left[R(u_x, \nabla_x^2 u_x)\nabla_x^{m-(l+1)} u_x \right] \tag{7.16}$$

$$+ \sum_{l=0}^{m-1} e^K \nabla_x^l \left[R(u_x, J\nabla_x u_x)\nabla_x^{m-(l+1)} u_x \right]. \tag{7.17}$$

(Note that $R(u_x, b\, g(u_x, u_x)u_x)\nabla_x^{m-(l+1)} u_x = 0$ since $R(u_x, u_x) = 0$.) The fourth term of the right hand side of (7.17) is decompose as

$$a \sum_{l=0}^{m-1} e^K \nabla_x^l \left[R(u_x, \nabla_x^2 u_x)\nabla_x^{m-(l+1)} u_x \right] \tag{7.18}$$

$$= a \left(\sum_{l=0}^{m-1} e^K \nabla_x^l \left[R(u_x, \nabla_x^2 u_x)\nabla_x^{m-1-l} u_x \right] - R(u_x, \nabla_x V^{(m)})u_x \right) \tag{7.19}$$

$$+ a\, R(u_x, \nabla_x V^{(m)})u_x. \tag{7.20}$$

Note the term $\nabla_x^{m+1} u_x$ never appear in the first term of the right hand side of (7.20).

Let us move to the computation of $e^K \nabla_x^{m+1} J \nabla_x u_x$. First, it follows from the definition that

$$(\nabla_x J)\, V = \nabla_x JV - J\nabla_x V \quad \text{for} \quad V \in \Gamma(u^{-1}TN), \tag{7.21}$$

where $(\nabla_x J)$ is the covariant derivative of $(1,1)$-tensor J with respect to x along u and is also $(1,1)$-tensor field along u. We will write $(\nabla_x J)\, V$ not to be confused with $\nabla_x JV$. In the same

way, $(\nabla_x^{j+1}J)$ with $j \geqslant 1$, which is the $(j+1)$-th covariant derivative of $(1,1)$-tensor field J, is also $(1,1)$-tensor field along u defined inductively by the form

$$\left(\nabla_x^{j+1}J\right)V = \nabla_x\left(\nabla_x^{j}J\right)V - \left(\nabla_x^{j}J\right)\nabla_x V \quad \text{for} \quad V \in \Gamma(u^{-1}TN),$$

where $(\nabla_x^1 J) = (\nabla_x J)$. Using (7.21) repeatedly, we deduce

$$e^K \nabla_x^{m+1} J \nabla_x u_x = e^K \nabla_x J \nabla_x^{m+1} u_x + e^K \sum_{l=1}^{m} \nabla_x^l \left(\nabla_x J\right) \nabla_x^{m+1-l} u_x \tag{7.22}$$

For the first term of the right hand side of (7.22), (7.3) and $e^K J = J e^K$ yield

$$\begin{aligned} & e^K \nabla_x J \nabla_x^{m+1} u_x && (7.23)\\ =& \nabla_x\left(J e^K \nabla_x^{m+1} u_x\right) - K_x J e^K \nabla_x^{m+1} u_x && (7.24)\\ =& \nabla_x J \nabla_x V^{(m)} - K_x \nabla_x J V^{(m)} - K_x J \nabla_x V^{(m)} - (K_{xx} - K_x^2) J V^{(m)}. && (7.25) \end{aligned}$$

For the second term of the right hand side of (7.22), by regarding $(\nabla_x J)$ and $\nabla_x^{m+1-l} u_x$ as a $(1,1)$-tensor field and a $(1,0)$-tensor field respectively, we deduce

$$\begin{aligned} & e^K \sum_{l=1}^{m} \nabla_x^l \left(\nabla_x J\right) \nabla_x^{m+1-l} u_x && (7.26)\\ &= e^K \sum_{l=1}^{m} \nabla_x^l C_1^2 \left(\left(\nabla_x J\right) \otimes \nabla_x^{m+1-l} u_x\right) && (7.27)\\ &= e^K \sum_{l=1}^{m} C_1^2 \nabla_x^l \left(\left(\nabla_x J\right) \otimes \nabla_x^{m+1-l} u_x\right) && (7.28)\\ &= e^K \sum_{l=1}^{m} C_1^2 \left\{ \sum_{j=0}^{l} \frac{l!}{j!(l-j)!} \left(\nabla_x^{j+1} J\right) \otimes \nabla_x^{m+1-l+(l-j)} u_x \right\} && (7.29)\\ &= e^K \sum_{l=1}^{m} \sum_{j=0}^{l} \frac{l!}{j!(l-j)!} \left(\nabla_x^{j+1} J\right) \nabla_x^{m+1-j} u_x && (7.30)\\ &= m\, e^K \left(\nabla_x J\right) \nabla_x^{m+1} u_x + e^K \sum_{l=1}^{m} \sum_{j=1}^{l} \frac{l!}{j!(l-j)!} \left(\nabla_x^{j+1} J\right) \nabla_x^{m+1-j} u_x, && (7.31) \end{aligned}$$

where $C_1^2 : T_u N \otimes T_u N \otimes T_u^* N \to T_u N$ is a contraction which maps $x_i \otimes x_j \otimes y_k^*$ into $\sum_{j,k} y_k^*(x_j) x_i$. Notice that the second equality of (7.31) holds since the covariant derivative commutes with the contraction, and the third equality of (7.31) is due to the fact that

$$\nabla_x \left(S \otimes T\right) = \left(\nabla_x S\right) \otimes T + S \otimes \left(\nabla_x T\right)$$

holds for any tensor S and T. See, e.g., [6] for these properties. Moreover, by noting that $f\left(\nabla_x J\right) = \left(\nabla_x J\right) f$ holds for any scalar function f and by using (7.3), we deduce

$$\begin{aligned} m\, e^K \left(\nabla_x J\right) \nabla_x^{m+1} u_x &= m\, \left(\nabla_x J\right) e^K \nabla_x^{m+1} u_x && (7.32)\\ &= m\, \left(\nabla_x J\right) \nabla_x V^{(m)} - m K_x \left(\nabla_x J\right) V^{(m)}. && (7.33) \end{aligned}$$

Combining (7.22),(7.25), (7.31), and (7.33), we obtain

$$\begin{aligned} & e^K \nabla_x^{m+1} J \nabla_x u_x && (7.34)\\ =& \nabla_x J \nabla_x V^{(m)} - K_x \nabla_x J V^{(m)} - K_x J \nabla_x V^{(m)} - (K_{xx} - K_x^2) J V^{(m)} && (7.35) \end{aligned}$$

$$+ m\left(\nabla_x J\right)\nabla_x V^{(m)} - mK_x\left(\nabla_x J\right)V^{(m)} \tag{7.36}$$

$$+ e^K \sum_{l=1}^{m}\sum_{j=1}^{l}\frac{l!}{j!(l-j)!}\left(\nabla_x^{j+1}J\right)\nabla_x^{m+1-j}u_x. \tag{7.37}$$

Consequently, by substituting (7.5),(7.6), (7.12),(7.17), (7.20) and (7.37) into (7.1), we deduce the desired equality (4.3).

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