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Mikoda, Akihiro
Graduate School of Mathematics, Kyushu University

Inokuchi, Shuichi
Faculty of Mathematics, Kyushu University

Mizoguchi, Yoshihiro
Faculty of Mathematics, Kyushu University

Fujio, Mitsuhiko
Department of Systems Innovation and Informatics, Kyushu Institute of Technology

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A. Mikoda, S. Inokuchi
Y. Mizoguchi, M. Fujio

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Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

The number of orbits of periodic box-ball systems

Akihiro Mikoda[1], Shuichi Inokuchi[2], Yoshihiro Mizoguchi[2], and Mitsuhiko Fujio[3]

[1] Graduate School of Mathematics, Kyushu University, JAPAN,
[2] Faculty of Mathematics, Kyushu University, JAPAN,
ym@math.kyushu-u.ac.jp,
[3] Department of Systems Innovation and Informatics,
Kyushu Institute of Technology, JAPAN

Abstract. A box-ball system is a kind of cellular automata obtained by the ultradiscrete Lotka-Volterra equation. Similarities and differences between behavious of discete systems (cellular automata) and continuous systems (differential equations) are investigated using techniques of ultradiscretizations. Our motivations is to take advantage of behavious of box-ball systems for new kinds of computations. Especially, we tried to find out useful periodic box-ball systems(pBBS) for random number generations. Applicable pBBS systems should have long fundamental cycles. We focus on pBBS with at most two kinds of solitons and investigate their behaviours, especially, the length of cycles and the number of orbits. We showed some relational equations of soliton sizes, a box size and the number of orbits. Varying a box size, we also found out some simulation results of the periodicity of orbits of pBBS with same kinds of solitons.

1 Introduction

In 1990, Takahashi and Satsuma introduced a soliton cellular automaton(SCA)[7]. The SCA is now called a box and ball system(BBS) because they explained transitions of the system using an infinite array of boxes and a finite number of balls. BBS has a property of solitons because of its transition being obtained by the ultradiscrete Lotka-Volterra equation[6, 8].

In 1997, a new soliton cellular automaton is proposed by Takahashi et al[6]. That system is called box and ball system with a carrier(BBSC). BBSC can be considered as a kind of abstract model of Hyper-Threading(HT) Technology. HT Technology is a recent attractive CPU hardware technology. The main aim of HT Technology brings out the parallel efficiency of CPUs and improves the performance of a system. We hope that we could make a connection between a study of BBSC and the HT Technology in the future.

Recently, the researche areas using ultradiscritizations is extending and it contains crystal formulations, combinatorics, stochastic cellular automata and algorithms[1, 2, 4, 5].

In 2003, the notion of periodic box-ball system(pBBS) is introduced by Yoshihara et al[9]. They have shown a formula to determine the fundamental cycle of a pBBS for a given initial state. In the same year, Habu et al.[3] investigated properties about randomness and autocorrelations of configurations of pBBS and compared with Gold sequences. They showed some experimental results about their properties for a fixed system size varying the number of balls and the size of solitons.

In this paper, we focus on pBBS with at most two kinds of solitons. We re-formulate the pBBS and define sets of configurations precisely. A set of configurations with a same type is divided into some disjoint same size of orbits. We investigate the size of the configuration set and the number of orbits for designing a pBBS with a longer fundamental cycle. According to the result of Yoshihara et al.[9], we reformulate the equation of the fundamental cycles. Further, we induce the equation of the number of orbits and prove that its upperbound is not depened on the size of boxes. Finally, we show some experimental results between a size of boxes and the number of orbits.

2 periodic box-ball systems (pBBS)

Let $Q = \{0, 1\}$, N a natural number, $\bar{N} = \{1, 2, \ldots, N\}$ and $\overline{2N} = \{1, 2, \ldots, 2N\}$. We define three functions $dbl : Q^{\bar{N}} \to Q^{\overline{2N}}$, $snd : Q^{\overline{2N}} \to Q^{\bar{N}}$ and $trs : Q^{\overline{2N}} \to Q^{\overline{2N}}$ by $dbl(c)_j = c_{((j-1) \mod N)+1}$, $snd(c)_j = c_{N+j}$ and $trs(c)_j = \min\left(1 - c_j, \sum_{i=1}^{j-1}(c_i - trs(c)_i)\right)$. The shift function $sft_\alpha : Q^{\bar{N}} \to Q^{\bar{N}}$ is defined by $sft_\alpha(c)_j = c_{((j-1+\alpha) \mod N)+1}$ $(\alpha = 0, \cdots, N-1)$.

Definition 1 (N-pBBS). *The periodic box-ball system with the size N (N-pBBS) is the dynamical system (C, f), where $C = \{c \in Q^{\bar{N}} | \sum_{j=1}^{N} c_j < \dfrac{N}{2}\}$ and the transition function $f : C \to C$ is defined by $f = snd \circ trs \circ dbl$.*

The definition of the N-$pBBS$ is well-defined. It is guaranteed by the next proposition.

Proposition 1. *Asuume $\#\{i \in \bar{N} | c_i = 1\} \le \frac{N}{2}$ for $c \in Q^{\bar{N}}$.*

(1) $\sharp\{i \in \bar{N} | c_i = 1\} = \sharp\{i \in \bar{N} | (snd \circ trs \circ dbl(c))_i = 1\}$, *where $\sharp S$ is the size of the set S.*
(2) $(snd \circ trs \circ dbl) \circ sft_\alpha(c) = sft_\alpha \circ (snd \circ trs \circ dbl)(c)$ $(\alpha = 0, 1, \cdots, N-1)$.

The proposition is proved using the following lemma.

Lemma 1. *For $c \in Q^{\overline{N}}$, we put $\delta_j = \sum_{i=1}^{j}(dbl(c)_i - trs(dbl(c))_i)$,*

$\Delta_j = \sum_{i=1}^{j} \left(dbl(c)_i - \overline{dbl(c)_i} \right)$, *where* $\overline{x}$ *denote the conmplement* $1 - x$ *for* $x \in Q$. *Then we have*

(1) $\delta_j = \Delta_j + \max_{1 \le i \le j} \{dbl(c)_i - \Delta_i\}$ $(j = 1, 2, \ldots, 2N)$.
(2) $\Delta_{N+j} = \Delta_N + \Delta_j$ $(j = 1, 2, \ldots, N)$.
(3) $\delta_{N+j} = \max\{\delta_N + \Delta_j,\, \delta_j\}$ $(j = 1, 2, \ldots, 2N)$.

The proof of Lemma 1 and Proposition 1 is listed in an appendix.

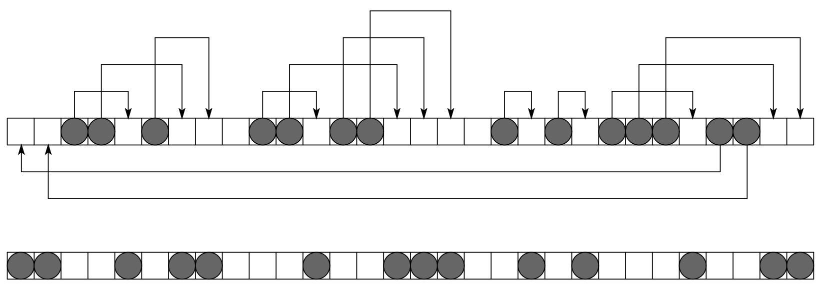


Fig. 1. Transition of pBBS

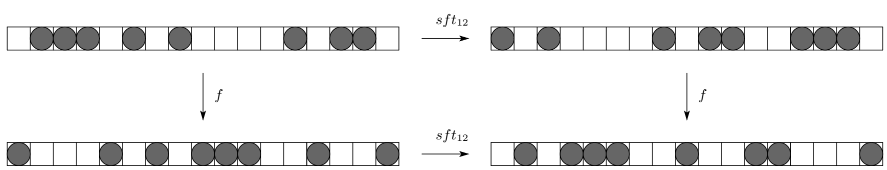


Fig. 2. commutative diagram

Example 1. Fig. 1 is an example of a transition of pBBS with size 30. Fig. 2 is an example transition ($f \circ sft_{12} = sft_{12} \circ f$) to confirm Proposition 1(2).

Definition 2 (Fundamental cycle of a pBBS). *Let* (C, f) *be a pBBS with size* N. *The fundamental cycle of a configuration* $c \in C$ *is defined by* $l(c) = \min\{t | f^t(c) = c, t > 0\}$.

Yoshihara et al. classified configurations of pBBS using size of solitons $L_1, \cdots, L_s$ and introduced an equation to compute the fundamental cycle of it.

Theorem 1 (Yoshihara 2003[9]). *Let (C, f) be a pBBS with size N. If a configuration $c \in C$ has a type $(L_1, L_2, \cdots, L_s)$, then the fundamental cycle T of the configuration c is*

$$T = L.C.M\left(\frac{N_s N_{s-1}}{l_s l_0}, \frac{N_{s-1}N_{s-2}}{l_{s-1} l_0}, \cdots, \frac{N_1 N_0}{l_1 l_0}, 1\right),$$

where $l_j = L_j - L_{j+1} \ \ (j = 1, 2, \cdots, s-1)$ *and* $N_j = l_0 + 2\sum_{i=1}^{j} n_i \left(L_i - L_{j+1}\right)$. $\square$

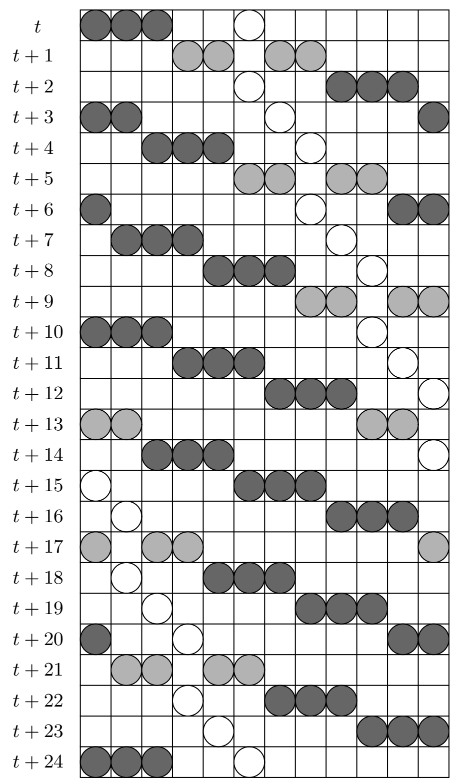


Fig. 3. Time evolution rule of pBBS

3 The number of orbits of a pBBS

In this section, we restrict the number of solitons up to 2. We re-formulate the class of configurations and imply a simple equation of the fundamental cycle.

We also introduce an equation of the total number of all configurations and the number of orbits.

Definition 3. *Let (C, f) be pBBS with size N. All configurations with two solitons is defined by*

$$C_2 = \{c \in C | c = 0^{x_1}1^{l_1}0^{x_2}1^{l_2}0^{x_3},\ 0 \le x_1, x_3,\ 1 \le l_1, l_2, x_2,\ x_1+l_1+x_2+l_2+x_3 = N\}.$$

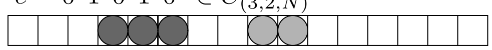
For numbers L_1 and L_2 ($L_1 + L_2 < \dfrac{N}{2}$, $L_1 \ge L_2$), we define a set $C_{(L_1,L_2,N)}$ of configurations with a type (L_1, L_2, N) as follows:

(a) *If $c = 0^{x_1}1^{l_1}0^{x_2}1^{l_2}0^{x_3}$ and $(l_1 \ge l_2,\ l_2 \le x_2)$ then $c \in C_{(l_1,l_2,N)}$ and $sft_\alpha(c) \in C_{(l_1,l_2,N)}$ for $\alpha = 0, 1, \cdots, N-1$.*
(b) *If $c = 0^{x_1}1^{l_1}0^{x_2}1^{l_2}0^{x_3}$ and $(l_1 \ge l_2,\ x_2 < l_2)$ then $c \in C_{(l_1+l_2-x_2,x_2,N)}$, and $sft_\alpha(c) \in C_{(l_1+l_2-x_2,x_2,N)}$ for $\alpha = 0, 1, \cdots, N-1$.*

We note that we can find some number L_1 and L_2 for a configuration $c = 0^{x_1}1^{l_1}0^{x_2}1^{l_2}0^{x_3}$ $(l_1 < l_2)$ to belong in $C_{(L_1,L_2,N)}$ using above Definition and sft_α.

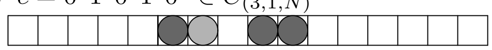
Example 2. Let $N = 16$.

(a) $c = 0^3 1^3 0^2 1^2 0^6 \in C_{(3,2,N)}$



$\cdots L_1 = 3, L_2 = 2.$

(b) $c = 0^5 1^2 0^1 1^2 0^6 \in C_{(3,1,N)}$



$\cdots L_1 = 3, L_2 = 1.$

Definition 4 ((L_1, L_2, N)-pBBS). *We define a subsystem (L_1, L_2, N)-pBBS of pBBS (C, f) with size N by a dynamical system $(C_{(L_1,L_2,N)}, f)$. The fundamental cycles for all $c \in C_{(L_1,L_2,N)}$ are the same number T. We call T as the fundamental cycle of $C_{(L_1,L_2,N)}$.*

The definition of the (L_1, L_2, N)-$pBBS$ is well-defined. It is guaranteed by the next proposition.

Proposition 2. **(1)** *$f(c) \in C_{(L_1,L_2,N)}$ for any $c \in C_{(L_1,L_2,N)}$.*
(2) *If $c_0, c_1 \in C_{(L_1,L_2,N)}$ then $l(c_0) = l(c_1)$.*
(3) *Let $\alpha = L_1 + L_2, \beta = L_1 - L_2, N = 2(L_1 + L_2) + n$. The number of configurations of (L_1, L_2, N)-pBBS is $(2\alpha + n)(2\beta + n)$.*

□

We denote the number $S = (2\alpha + n)(2\beta + n)$ in Proposition 2(3) by S.

Definition 5 (Orbits of pBBS). *Configuration c and d are on the same orbit if and only if $d = f^i(c)$ for some i.(cf. Fig. 4)*

$C_{(L_1,L_2,N)}$ is covered by several disjoint orbits like $\{f^i(c) | i \ge 0\}$. By Proposition 2(2), each orbits contains T elements, where T is the fundamental cycle of $C_{(L_1,L_2,N)}$.

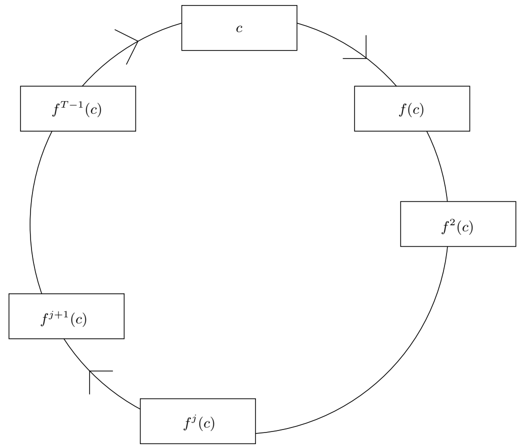


Fig. 4. The orbits of (L_1, L_2, N)-pBBS

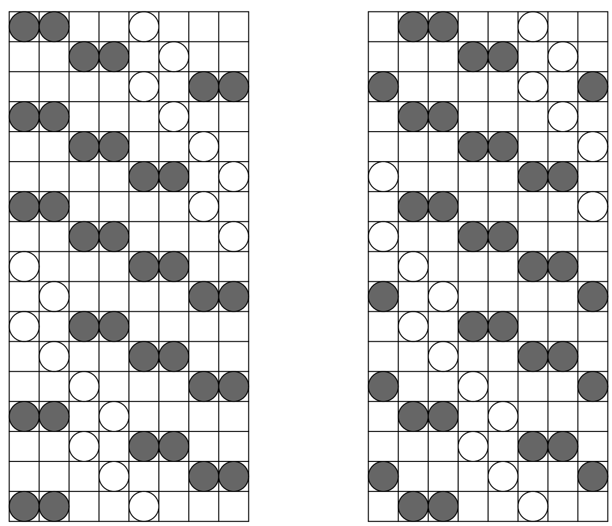


Fig. 5. The orbits of $(2, 1, 8)$-pBBS

Example 3. Fig. 5 is an example of orbit sets. $C_{(2,1,8)}$ is covered by two orbit sets where each set contains 17 elements. The fundamental cycle of c is 17 for any $c \in C_{(2,1,8)}$. $S = 34$, $T = 17$ and $K = 2$.

Theorem 2 (The number of orbits). *Let* $\alpha = L_1 + L_2, \beta = L_1 - L_2, N = 2(L_1 + L_2) + n$.

(1) *The fundamental cycle* T *of* $(L_1, L_2, 2(L_1 + L_2) + n)$*-pBBS is*
$T = L.C.M\left(\frac{(2\alpha+n)(2\beta+n)}{G.C.D\left((2\alpha+n)(2\beta+n), \frac{\alpha-\beta}{2}n\right)}, \frac{2\beta+n}{G.C.D(2\beta+n,\beta)}\right)$,

(2) *The number of orbits* K *of* $(L_1, L_2, 2(L_1 + L_2) + n)$*-pBBS is*
$K = G.C.D\left((2\alpha+n)(2\beta+n), (2\alpha+n)\beta, \frac{\alpha-\beta}{2}n\right)$.

Proof. (1) is immediately induced by Theorem 1. Since $T = L.C.M\left(\frac{N_2 N_1}{l_2 l_0}, \frac{N_1 N_0}{l_1 l_0}, 1\right)$, $N_2 = N$, $N_1 = N - 4L_2$, $N_0 = l_0$, $l_1 = L_1 - L_2$ and $l_2 = L_2$, we have $T = L.C.M\left(\frac{N(N-4L_2)}{L_2(N-2L_1-2L_2)}, \frac{N-4L_2}{L_1-L_2}, 1\right)$. Since $\alpha = L_1 + L_2, \beta = L_1 - L_2$, we have $T = L.C.M\left(\frac{(2\alpha+n)(2\beta+n)}{G.C.D\left((2\alpha+n)(2\beta+n), \frac{\alpha-\beta}{2}n\right)}, \frac{2\beta+n}{G.C.D(2\beta+n,\beta)}\right)$.

(2) By Proposition 2(3) and above results, we have

$$\begin{aligned}
K &= \frac{(2\alpha+n)(2\beta+n)}{L.C.M\left(\frac{(2\alpha+n)(2\beta+n)}{G.C.D\left((2\alpha+n)(2\beta+n), \frac{\alpha-\beta}{2}n\right)}, \frac{2\beta+n}{G.C.D(2\beta+n,\beta)}\right)} \\
&= \frac{2\alpha+n}{L.C.M\left(\frac{2\alpha+n}{G.C.D\left((2\alpha+n)(2\beta+n), \frac{\alpha-\beta}{2}n\right)}, \frac{1}{G.C.D(2\beta+n,\beta)}\right)} \\
&= \frac{(2\alpha+n)G.C.D\,(2\beta+n,\beta)}{L.C.M\left(\frac{(2\alpha+n)G.C.D(2\beta+n,\beta)}{G.C.D\left((2\alpha+n)(2\beta+n), \frac{\alpha-\beta}{2}n\right)}, 1\right)} \\
&= \frac{(2\alpha+n)G.C.D\,(2\beta+n,\beta)}{\frac{(2\alpha+n)G.C.D(2\beta+n,\beta)}{G.C.D\left((2\alpha+n)(2\beta+n),(2\alpha+n)\beta, \frac{\alpha-\beta}{2}n\right)}} \\
&= G.C.D\left((2\alpha+n)(2\beta+n), (2\alpha+n)\beta, \frac{\alpha-\beta}{2}n\right)
\end{aligned}$$

□

The next theorem shows some relations between the box-size n and the number of orbits K, especially the upper bound of the number of orbits K.

Theorem 3. *Let* $\alpha = L_1 + L_2, \beta = L_1 - L_2$.

(1) $gcd(L_1 - L_2, n)|K$,
(2) $gcd(2, n)|K$,
(3) $gcd(L_1 + L_2, n)|K$, *and*

(4) $K | \dfrac{\alpha\beta(\alpha-\beta)}{G.C.D\,(L_1, L_2)}$.

Proof. **(1)** Let $L_1 - L_2 = ka, n = ma$. We have

$$\begin{aligned} K &= G.C.D\left((2\alpha+n)(2\beta+n), (2\alpha+n)\beta, \frac{\alpha-\beta}{2}n\right) \\ &= G.C.D\left((2\alpha+ma)(2ka+ma), (2\alpha+ma)ka, \frac{\alpha-ka}{2}ma\right) \\ &= a \times G.C.D\left((2\alpha+ma)(2k+m), (2\alpha+ma)k, \frac{\alpha-ka}{2}m\right). \end{aligned}$$

(2) Let $n = 2k$. We have

$$\begin{aligned} K &= G.C.D\left((2\alpha+n)(2\beta+n), (2\alpha+n)\beta, \frac{\alpha-\beta}{2}n\right) \\ &= G.C.D\left((2\alpha+2k)(2\beta+2k), (2\alpha+2k)\beta, \frac{\alpha-\beta}{2}2k\right) \\ &= 2 \times G.C.D\left(2(\alpha+k)(\beta+k), (\alpha+k)\beta, \frac{\alpha-\beta}{2}k\right). \end{aligned}$$

(3) Let $L_1 + L_2 = ka$, $n = ma$. We have

$$\begin{aligned} K &= G.C.D\left((2\alpha+n)(2\beta+n), (2\alpha+n)\beta, \frac{\alpha-\beta}{2}n\right) \\ &= G.C.D\left((2ka+ma)(2\beta+ma), (2ka+ma)\beta, \frac{ka-\beta}{2}ma\right) \\ &= a \times G.C.D\left((2k+m)(2\beta+ma), (2k+m)\beta, \frac{ka-\beta}{2}m\right). \end{aligned}$$

(4) Let $g = G.C.D\,(an+b, cn)$. Since $g|cn$ and $cn = G.C.D\,(a,c) \times \frac{c}{G.C.D(a,c)} \times n$, we can set $g = g_a g_c g_n$ where $g_a | G.C.D\,(a,c)$, $g_c | \frac{c}{G.C.D(a,c)}$ and $g_n | n$.

Since $g_a g_n | an$ and $g_a g_n | (an+b)$, we have $g_a g_n | b$. So we can induce $g_a g_c g_n \mid \frac{bc}{G.C.D(a,c)}$.

Let $a = \beta$, $b = 2\alpha\beta$ and $c = \frac{\alpha-\beta}{2}$.
Then we have $g = G.C.D\left(2\alpha\beta + \beta n, \frac{\alpha-\beta}{2}n\right) | \frac{2\alpha\beta\cdot\frac{\alpha-\beta}{2}}{G.C.D(\beta, \frac{\alpha-\beta}{2})}$.

$$
\begin{aligned}
K &= G.C.D\left((2\alpha+n)(2\beta+n),(2\alpha+n)\beta,\frac{\alpha-\beta}{2}n\right)\\
&= G.C.D\left((2\alpha+n)(2\beta+n),G.C.D\left(2\alpha\beta+\beta n,\frac{\alpha-\beta}{2}n\right)\right)\\
&\mid\ G.C.D\left((2\alpha+n)(2\beta+n),\frac{\alpha\beta(\alpha-\beta)}{G.C.D\left(\beta,\frac{\alpha-\beta}{2}\right)}\right)\\
&\mid\ \frac{\alpha\beta(\alpha-\beta)}{G.C.D\left(L_1-L_2,L_2\right)}\\
&= \frac{\alpha\beta(\alpha-\beta)}{G.C.D\left(L_1,L_2\right)}.
\end{aligned}
$$

$\square$

4 Simulations

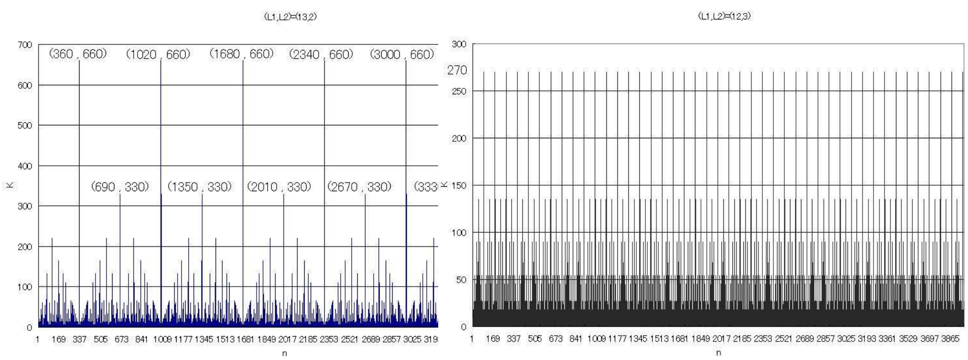


Fig. 6. Simulation results

The lefthand side of Fig. 6 is a graph of n and K for $C_{(13,2,2(13+2)+n)}$. A peak of K is 660 and $\frac{\alpha\beta(\alpha-\beta)}{G.C.D(L_1,L_2)} = \frac{15\cdot 13\cdot 2\cdot 2}{G.C.D(13,2)} = 660$. The righthand side of Fig. 6 is a graph of n and K for $C_{(12,3,2(12+3)+n)}$. A peak of K is 270 and $\frac{\alpha\beta(\alpha-\beta)}{G.C.D(L_1,L_2)} = \frac{15\cdot 9\cdot 2\cdot 3}{G.C.D(12,3)} = 270$.

In Theorem 3 we showed an upperbound of K. By the simulation results $\frac{\alpha\beta(\alpha-\beta)}{G.C.D(L_1,L_2)}$ may not only be an upper boud but also the maximum value of K.

Finally we have another conjecture from experimental resuts. pBBS with the number of orbits $K = 1$ must have a longer fundamental cycle, so the next conjecture may be useful to design a pBBS with a longer fundamental cycle.

Conjecture 1. Let K be the number of orbits for $C_{(L_1,L_2,2(L_1+L_2)+n)}$. If $gcd(L_1 - L_2, n) = 1$, $gcd(2, n) = 1$ and $gcd(L_1 + L_2, n) = 1$ then $K = 1$.

5 Concluding remarks

We re-formulate the pBBS with up to 2 kinds of solitons using precise equations. We showed the formula for the fundamental cycle and the number of orbits for pBBS. Further we proved the number of orbits is bounded some constant defined by the type of solitons. This means that we can design pBBS with longer fundamental cycle if we can choose larger box size pBBS. Future works contain to investigate a expression of orbits and behaviour of orbits when we increase sorts and number of solitons.

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Appendix

Proof. (of Lemma 1)

(1) Since for any c, $trs(c)_1 = 0$, we have for $j = 1$, (left hand side) $= c_1 - trs(c)_1 = c_1 =$ (right hand side). Now suppose that the equality holds for some j. Then it follows that

$$\begin{aligned}
\delta_{j+1} &= \delta_j + dbl(c)_{j+1} - trs(dbl(c))_{j+1} \\
&= \delta_j + dbl(c)_{j+1} - \min\left\{\overline{dbl(c)_{j+1}},\, \delta_j\right\} \\
&= \delta_j + \max\left\{dbl(c)_{j+1} - \overline{dbl(c)_{j+1}},\, dbl(c)_{j+1} - \delta_j\right\} \\
&= \delta_j + \max\left\{\mathit{\Delta}_{j+1} - \mathit{\Delta}_j,\, dbl(c)_{j+1} - \delta_j\right\} \\
&= \max\left\{\delta_j - \mathit{\Delta}_j + \mathit{\Delta}_{j+1},\, dbl(c)_{j+1}\right\} \\
&= \max\left\{\max_{1\le i\le j}\{dbl(c)_i - \mathit{\Delta}_i\} + \mathit{\Delta}_{j+1},\, dbl(c)_{j+1}\right\} \\
&= \mathit{\Delta}_{j+1} + \max\left\{\max_{1\le i\le j}\{dbl(c)_i - \mathit{\Delta}_i\},\, dbl(c)_{j+1} - \mathit{\Delta}_{j+1}\right\} \\
&= \mathit{\Delta}_{j+1} + \max_{1\le i\le j+1}\{dbl(c)_i - \mathit{\Delta}_i\},
\end{aligned}$$

which establishes the equality for $j+1$.

(2) It follows from the fact that $dbl(c)_{N+i} = dbl(c)_i$ $(i = 1, 2, \ldots, N)$.

(3) By virtue of (1) and (2),

$$\begin{aligned}
\delta_{N+j} &= \mathit{\Delta}_{N+j} + \max_{1\le i\le N+j}\{dbl(c)_i - \mathit{\Delta}_i\} \\
&= \mathit{\Delta}_{N+j} + \max\left\{\max_{1\le i\le N}\{dbl(c)_i - \mathit{\Delta}_i\},\, \max_{1\le i\le j}\{dbl(c)_{N+i} - \mathit{\Delta}_{N+i}\}\right\} \\
&= \mathit{\Delta}_N + \mathit{\Delta}_j \\
&\qquad + \max\left\{\max_{1\le i\le N}\{dbl(c)_i - \mathit{\Delta}_i\},\, \max_{1\le i\le j}\{dbl(c)_i - \mathit{\Delta}_N - \mathit{\Delta}_i\}\right\} \\
&= \max\Bigg\{\mathit{\Delta}_j + \mathit{\Delta}_N + \max_{1\le i\le N}\{dbl(c)_i - \mathit{\Delta}_i\}, \\
&\qquad\qquad \mathit{\Delta}_j + \max_{1\le i\le j}\{dbl(c)_i - \mathit{\Delta}_i\}\Bigg\} \\
&= \max\left\{\mathit{\Delta}_j + \delta_N,\, \delta_j\right\}.
\end{aligned}$$

□

Proof. (of Proposition 1).

(1) By (3) of Lemma 1, $\delta_{2N} = \max\{\delta_N + \mathit{\Delta}_N,\, \delta_N\}$. On the other hand, by the assumption, $\mathit{\Delta}_N = \sum_{i=1}^{N}\left(dbl(c)_i - \overline{dbl(c)}_i\right) = \left(\sum_{i=1}^{N} 2dbl(c)_i\right) - N = 2\left(\sum_{i=1}^{N} c_i\right) -$

$N \leq 0$. Hence we have $\delta_{2N} = \delta_N$. This implies that

$$
\begin{aligned}
&\#\{i \in \overline{N} | c_i = 1\} - \#\{i \in \overline{N} | (snd \circ trs \circ dbl(c))_i = 1\} \\
&= \left(\sum_{i=1}^{N} c_i\right) - \left(\sum_{i=1}^{N} (snd \circ trs \circ dbl(c))_i\right) \\
&= \left(\sum_{i=N+1}^{2N} d_i\right) - \left(\sum_{i=N+1}^{2N} trs(d)_i\right) \\
&= \sum_{i=N+1}^{2N} (d_i - trs(d)_i) \\
&= s_{2N} - s_N \\
&= 0.
\end{aligned}
$$

(2) Since $sft_\alpha = \underbrace{sft_1 \circ \cdots \circ sft_1}_{\alpha}$, it suffices to show this for $\alpha = 1$. For the sake of simplicity, we put $d = dbl(c)$ and $e = dbl(sft_1(c))$. Then the equations are rewritten as

$$
sft_1(snd(trs(d)))_j = snd(trs(e))_j \quad (j = 1, 2, \ldots, N) \tag{1}
$$

Furthermore, to describe the effect of shift, we put $\delta_j = \sum_{i=1}^{j}(d_i - trs(d_i))$, $\varepsilon_j = \sum_{i=1}^{j}(e_i - trs(e_i))$, $\Delta_j = \sum_{i=1}^{j} \left(d_i - \overline{d}_i\right)$, $E_j = \sum_{i=1}^{j} (e_i - \overline{e}_i)$.

These variables are related as $\Delta_{j+1} = E_j + (c_1 - \overline{c}_1)$, $\delta_{j+1} = \max\{\varepsilon_j,\, E_j + c_1\}$ for $j = 1, 2, \ldots, 2N-1$. In fact, $\Delta_{j+1} = \sum_{i=1}^{j+1} \left(d_i - \overline{d}_i\right) = (d_1 - \overline{d}_1) + \sum_{i=1}^{j} (e_i - \overline{e}_i) = E_j + (c_1 - \overline{c}_1)$.

For the second one, by Lemma 1 (1),

$$\begin{aligned}
\delta_{j+1} &= \varDelta_{j+1} + \max_{1\le i\le j+1}\{d_i - \varDelta_i\} \\
&= \varDelta_{j+1} + \max\left\{d_1 - \varDelta_1,\ \max_{2\le i\le j+1}\{d_i - \varDelta_i\}\right\} \\
&= E_j + (c_1 - \overline{c}_1) + \max\left\{\overline{c}_1,,\ \max_{1\le i\le j}\{d_{i+1} - \varDelta_{i+1}\}\right\} \\
&= E_j + \max\left\{c_1,\ \max_{1\le i\le j}\{d_{i+1} - \varDelta_{i+1} + (c_1 - \overline{c}_1)\}\right\} \\
&= E_j + \max\left\{c_1,\ \max_{1\le i\le j}\{e_i - E_i\}\right\} \\
&= \max\left\{E_j + c_1,\ E_j + \max_{1\le i\le j}\{e_i - E_i\}\right\} \\
&= \max\{E_j + c_1,\ \varepsilon_j\}\,.
\end{aligned}$$

Next, we claim that $sft_1(snd(trs(d)))_j$ and $snd(trs(e))_j$ are related by

$$sft_1(snd(trs(d)))_j = \max\{snd(trs(e))_j,\ \min\{\overline{e}_j,\ E_{N+j-1} + c_1\}\} \tag{2}$$

for $j = 1, 2, \ldots, N$. In fact, if $j < N$,

$$\begin{aligned}
sft_1(snd(trs(d)))_j &= snd(trs(d))_{j+1} \\
&= trs(d)_{N+j+1} \\
&= \min\{\overline{d}_{N+j+1},\ \delta_{N+j}\} \\
&= \min\{\overline{e}_{N+j},\ \max\{\varepsilon_{N+j-1},\ E_{N+j-1} + c_1\}\} \\
&= \max\{\min\{\overline{e}_{N+j},\ \varepsilon_{N+j-1}\},\ \min\{\overline{e}_{N+j},\ E_{N+j-1} + c_1\}\} \\
&= \max\{trs(e)_{N+j},\ \min\{\overline{e}_{N+j},\ E_{N+j-1} + c_1\}\} \\
&= \max\{snd(trs(e))_j,\ \min\{\overline{e}_j,\ E_{N+j-1} + c_1\}\}\,.
\end{aligned}$$

For $j = N$,

$$\begin{aligned}
sft_1(snd(trs(d)))_N &= snd(trs(d))_1 \\
&= trs(d)_{N+1} \\
&= \min\{\overline{d}_{N+1},\ \delta_N\} \\
&= \min\{\overline{e}_N,\ \max\{\varepsilon_{N-1},\ E_{N-1} + c_1\}\} \\
&= \max\{\min\{\overline{e}_N,\ \varepsilon_{N-1}\},\ \min\{\overline{e}_N,\ E_{N-1} + c_1\}\} \\
&= \max\{trs(e)_N,\ \min\{\overline{e}_N,\ E_{N-1} + c_1\}\} \\
&= \max\{snd(trs(e))_N,\ \min\{\overline{e}_N,\ E_{N-1} + c_1\}\}\,.
\end{aligned}$$

Now all we have to show is that

$$snd(trs(e))_j \ge \min\{\overline{e}_j,\ E_{N+j-1} + c_1\} \quad (j = 1, 2, \ldots, N). \tag{3}$$

In fact, by combining this with the relation (2), we obtain (1).
To show (3), we apply similar argment about δ_j's and Δ_j's to ε_j's and E_j's. Recall that, from the assumption of c, it follows that $E_N \leq 0$. By Lemma 1 (3), we have $\varepsilon_N = \max\{\varepsilon_N + E_N, \varepsilon_N\} = \varepsilon_{2N}$. On the other hand, by Lemma 1 (1),

$$\varepsilon_{2N} = E_{2N} + \max_{1 \leq i \leq 2N}\{e_i - E_i\} \;\geq\; E_{2N} + e_{2N} - E_{2N} \;=\; e_{2N} \;=\; c_1.$$

Thus we have $\varepsilon_N \geq c_1$. From this it follows that

$$\begin{aligned}
\varepsilon_{N+j-1} &= \max\{\varepsilon_{j-1},\, \varepsilon_N + E_{j-1}\} \\
&\geq \varepsilon_N + E_{j-1} \\
&\geq c_1 + E_{j-1} \\
&\geq c_1 + E_{j-1} + E_N \\
&= E_{N+j-1} + c_1.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
snd(trs(e))_j &= trs(e)_{N+j} \\
&= \min\{\overline{e}_{N+j},\, \varepsilon_{N+j-1}\} \\
&\geq \min\{\overline{e}_{N+j},\, E_{N+j-1} + c_1\} \\
&= \min\{\overline{e}_j,\, E_{N+j-1} + c_1\},
\end{aligned}$$

that is, the inequality (3). $\square$

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