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Tabata, Masahisa
Faculty of Mathematics, Kyushu University

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M. Tabata

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Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

ENERGY STABLE FINITE ELEMENT SCHEMES AND THEIR APPLICATIONS TO TWO-FLUID FLOW PROBLEMS

Masahisa Tabata
Kyushu University, Faculty of Mathematics
Hakozaki, Higashi-ku, Fukuoka, 812-8581, Japan
e-mail: tabata@math.kyushu-u.ac.jp
web page: http://www.math.kyushu-u.ac.jp/~tabata/

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Abstract

We present energy stable finite element schemes for two-fluid problems with interfacial tension. The interface is captured by solving numerically ordinary differential equations and the interface tension is brought in into the weak formulation of the finite element scheme. Some numerical results of rising bubble problems show the robustness and the applicability of these schemes.

1 INTRODUCTION

Many numerical schemes have been developed and applied to two-fluid flow problems with surface/interfacial tension, see e.g., [7, 8] and references therein. It is, however, not an easy task to construct numerical schemes, stable and convergent. To the best of our knowledge, there are no numerical schemes whose solutions are proved to converge to the exact one. There are very little discussion even for the stability of schemes. When there is no surface tension, we have developed energy stable finite element schemes from the approach of the density-dependent Navier-Stokes equations and applied them to the Rayleigh-Taylor problem [5, 4]. These schemes are proved to be unconditionally stable in the energy sense. In this formulation the density is treated as a field function solved in the whole domain, and it also works as a level set function.

Here we extend the energy-stable finite element schemes to two-fluid flow problems with interfacial tension. In order to get the correct interfacial tension we need more precise position of the interface. We distribute particles on the interface, and solve numerically ordinary differential equations. The Navier-Stokes equations written in the weak formulation including the interfacial tension are solved in the fixed finite element mesh. A similar idea for the total scheme has been already presented in [8], where the Navier-Stokes equations are solved by a finite volume projection method. We employ the P2/P1 finite element to get the velocity and the pressure, whose convergence theory has been well established [2, 1]. Caused by the effect of interfacial tension, the obtained scheme is not proved to be stable a priori. We give another more stable scheme. In this paper we do not discuss the analysis of these schemes, but present the formulation and show some numerical results obtained from them.

2 TWO-FLUID FLOW PROBLEMS WITH INTERFACIAL TENSION

Let Ω be a bounded domain in $\mathbf{R}^2$ with piecewise smooth boundary Γ, and T be a positive number. At the initial time $t = 0$ the domain Ω is occupied by two immiscible incompressible viscous fluids; each domain is denoted by Ω_k^0, $k = 1, 2$, whose interface $\partial\Omega_1^0 \cap \partial\Omega_2^0$ is denoted by Γ_{12}^0. We assume that Γ_{12}^0 is a closed curve, which means that one fluid, say, fluid 1, is in the interior of the other fluid 2. At $t \in (0, T)$ the two fluids occupy domains $\Omega_k(t)$, $k = 1, 2$, and the interface curve is denoted by $\Gamma_{12}(t)$. Let ρ_k and μ_k, $k = 1, 2$, be the densities and the viscosities of the two fluids. Let

$$u : \Omega \times (0, T) \to \mathbf{R}^2, \quad p : \Omega \times (0, T) \to \mathbf{R}$$

be the velocity and the pressure to be found. The Navier-Stokes equations are satisfied in each domain $\Omega_k(t)$, $k = 1, 2$, $t \in (0, T)$,

$$\rho_k \left\{ \frac{\partial u}{\partial t} + (u \cdot \nabla) u \right\} - \nabla(2\mu_k D(u)) + \nabla p = \rho_k f, \tag{1a}$$

$$\nabla \cdot u = 0, \tag{1b}$$

where $f : \Omega \times (0, T) \to \mathbf{R}^2$ is a given function and $D(u)$ is the strain-rate tensor defined by

$$D_{ij}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

On the interface $\Gamma_{12}(t)$, $t \in (0, T)$, interface conditions

$$[u] = 0, \qquad [\sigma(\mu, u, p)n] = \gamma_0 \kappa n \tag{2}$$

are imposed, where $[\cdot]$ means the difference of the values approached from both sides to the interface, κ is the curvature of the interface, γ_0 is the coefficient of the interfacial tension, n is the unit normal vector, and σ is the stress tensor defined by

$$\sigma(\mu, u, p) = -pI + 2\mu D(u).$$

On the boundary Γ, $t \in (0, T)$, the slip conditions

$$u \cdot n = 0, \qquad \sigma(\mu_2, u, p)n \times n = 0 \tag{3}$$

are imposed. Initial conditions at $t = 0$ for the velocity

$$u = u^0 \tag{4}$$

are given.

In order to derive numerical schemes we reformulate the problem as follows: find functions

$$\chi : [0,1] \times (0,T) \to \mathbf{R}^2, \quad (u,p) : \Omega \times (0,T) \to \mathbf{R}^2 \times \mathbf{R}$$

satisfying for any $t \in (0,T)$

$$\frac{\partial \chi}{\partial t} = u(\chi, t), \qquad (s \in [0,1]) \tag{5}$$

and (1) in $\Omega_k(t)$, $k = 1, 2$, with interface conditions (2), slip boundary conditions (3), and initial conditions (4) and

$$\chi(\cdot, 0) = \chi^0, \tag{6}$$

where $\chi^0 : [0,1] \to \mathbf{R}^2$ is an initial closed curve in Ω. For any t, $\chi(1,t) = \chi(0,t)$ and

$$\{\chi(s,t); s \in [0,1]\}$$

shows a closed curve $\mathcal{C}(t)$ in Ω. $\mathcal{C}(t)$ is nothing but the interface curve at t, and $\Omega_k(t)$, $k = 1, 2$, are defined as the interior and the exterior of $\mathcal{C}(t)$, respectively.

3 ENERGY STABLE FINITE ELEMENT SCHEMES

We now present energy-stable finite element schemes for the problem described in the previous section. We prepare function spaces,

$$\begin{aligned} X &= \{\chi \in H^1(0,1)^2; \chi(1) = \chi(0)\}, \\ V &= \{v \in H^1(\Omega)^2; v \cdot n = 0 \ (x \in \Gamma)\}, \\ Q &= L^2_0(\Omega), \end{aligned}$$

where unknown functions $\chi(t), u(t)$, and $p(t)$ are sought. Let X_h, V_h, and Q_h be finite-dimensional approximation spaces of X, V, and Q. Let Δt be a time increment and $N_T = \lfloor T/\Delta t \rfloor$. We seek approximate solutions χ_h^n, u_h^n, and p_h^n at $t = n\Delta t$ in X_h, V_h, and Q_h, respectively. More precisely, these approximate function spaces are constructed as follows. Dividing the domain Ω into a union of triangles, we use P2 and P1 finite element spaces for V_h and Q_h, respectively. They are fixed for any time step n. On the other hand X_h is composed of functions obtained by the parameterization of polygons. We denote by $\{s_i^n \in [0,1]; i = 0, \cdots, N_x^n\}$ the set of parameter values such that $s_0^n = 0$ and $s_{N_x^n}^n = 1$ and that $\{\chi_h^n(s_i^n); i\}$ are vertices of a polygon. The number N_x^n may change depending on n. We also introduce an auxiliary function space $\Psi = H^1(\Omega)$ and the P1 finite element approximation space Ψ_h. We denote by $\bar{D}_{\Delta t}$ the backward difference operator, e.g.,

$$\bar{D}_{\Delta t} u_h^n = \frac{u_h^n - u_h^{n-1}}{\Delta t}.$$

Scheme I: Find $\{(\chi_h^n, u_h^n, p_h^n) \in X_h \times V_h \times Q_h; n = 1, \cdots, N_T\}$ satisfying

$$\bar{D}_{\Delta t}\chi_h^n = \frac{3}{2}u_h^{n-1}(\chi_h^{n-1}) - \frac{1}{2}u_h^{n-2}(\chi_h^{n-1} - \Delta t u_h^{n-1}(\chi_h^{n-1})), \qquad \forall s_i^n \tag{7a}$$

$$\Big(\rho_h^{n-1}\bar{D}_{\Delta t}u_h^n + \frac{1}{2}u_h^n\bar{D}_{\Delta t}\rho_h^n, v_h\Big) + a_1(\rho_h^n, u_h^{n-1}, u_h^n, v_h) + a_0(\rho_h^n, u_h^n, v_h) + b(v_h, p_h^n)$$

$$= (\rho_h^n \Pi_h f^n, v_h) - d(\chi_h^n, v_h; \mathcal{C}_h^n), \quad \forall v_h \in V_h \tag{7b}$$

$$b(u_h^n, q_h) = 0, \quad \forall q_h \in Q_h \tag{7c}$$

subject to the initial conditions

$$\chi_h^0 = \Pi_h \chi^0, \quad u_h^0 = \Pi_h u^0. \tag{8}$$

Here Π_h is the interpolation operator to the corresponding finite-dimensional space, $(\cdot, \cdot)$ shows the inner product in $L^2(\Omega)^2$, and

$$a_1(\rho, w, u, v) = \int_\Omega \big\{\frac{1}{2}(w\cdot\nabla\rho)u + \frac{1}{2}\rho(\nabla\cdot w)u + \rho(w\cdot\nabla)u\big\}\cdot v \; dx,$$

$$a_0(\rho, u, v) = \int_\Omega 2\mu(\rho)\nabla u^T : \nabla v^T \; dx,$$

$$b(v, q) = -\int_\Omega (\nabla\cdot v)q \; dx,$$

$$d(u, v; \mathcal{C}_h) = \sum_{i=1}^{N_x} \gamma_0 \bar{D}_{\Delta s}u_i \bar{D}_{\Delta s}v_i \; (s_i - s_{i-1})/|\chi_i - \chi_{i-1}|.$$

Let τ be the tangential to the interface $\mathcal{C}$. The bilinear form $d(u, v)$ is an approximation to

$$\int_{\mathcal{C}} \gamma_0 \frac{\partial u}{\partial \tau}\frac{\partial v}{\partial \tau} \; dl,$$

which is derived from the interfacial tension. Once χ_h^n is known, we can define Ω_{hk}^n, $k = 1, 2$, as the interior and the exterior of the polygon, respectively. In (7b) $\rho_h \in \Psi_h$ is an auxiliary function defined by

$$\rho_h(P_i) = \rho_k$$

if the node P_i belongs to Ω_{hk}^n. (7a) is based on the Adams-Bashforth method for the numerical solution of ordinary differential equations. N_x^n shows the number of particles on the interface at time step n. We control N_x^n so that the particles may distribute uniformly in a sense. (7b) and (7c) consist of an energy-stable finite element scheme developed in [5] when there is no term d.

Scheme II: We add the term

$$\Delta t \; d(u_h^n, v_h; \mathcal{C}_h^n) \tag{9}$$

to the left-hand side of (7b). The other parts are same as Scheme I. This means that the curvature of the interface is computed implicitly using the unkown u_h^n. For the idea of the introduction of this term we refer to [6]. Scheme II is more stable than Scheme I, which is recognized in the next section.

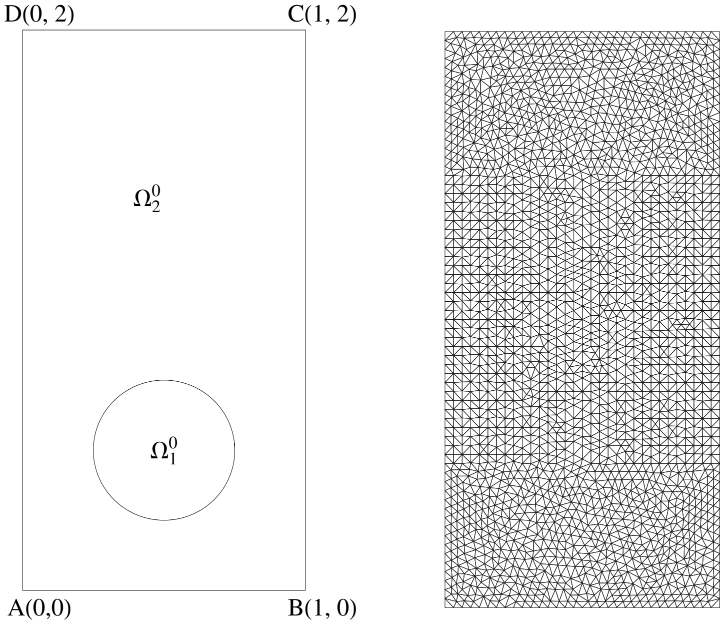

Figure 1: Statement of the problem and a mesh, $N_e = 4,500$

4 NUMERICAL RESULTS

4.1 A rising bubble in a channel

Let $\Omega \equiv (0,1) \times (0,2)$ and $T = 10$. Setting

$$\chi^0(s) = \Big(\frac{1}{2} + \frac{1}{4}\cos 2\pi s, \frac{1}{2} + \frac{1}{4}\sin 2\pi s\Big),$$

we see the initial domains Ω_1^0 and Ω_2^0 shown in Fig. 1. We take the following values,

$$(\rho_1, \mu_1) = (10, 1), \quad (\rho_2, \mu_2) = (100, 2)$$

and the initial velocity and the outer force,

$$u^0 = (0,0)^T, \quad f = (0, -1)^T.$$

We divide the domain Ω into a union of triangles. The sides AB and CD are divided into 32 equal segments, and the other sides are divided into 64. The mesh is made by FreeFEM [9]. The total element number N_e is equal to 4,500.

4.1.1 The comparison of Schemes I and II

We compare the stability property of Schemes I and II. Both schemes are proved to be unconditionally energy-stable, i.e., for any $\Delta t > 0$

$$\max\Big\{ \int_\Omega \rho_h^n |u_h^n|^2 dx \ ; \ n = 0, \cdots, N_T \Big\} < \infty$$

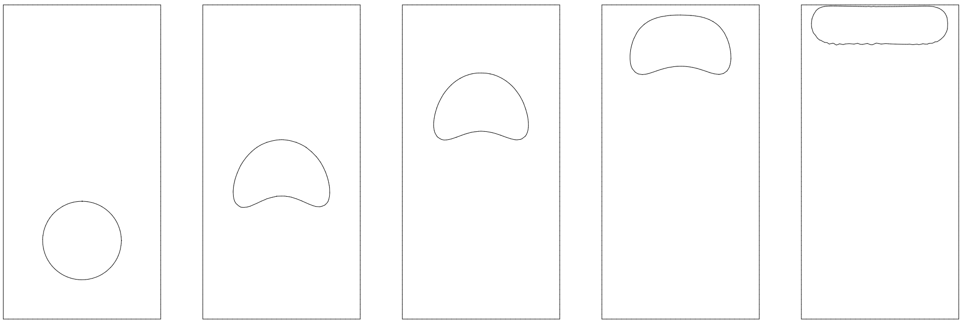

Figure 2: Scheme 1. $\Omega_1(t)$ at $t = 0.0, 2.5, 5.0, 7.5, 10.0$ ($\gamma_0 = 1.0, \Delta t = 1/16$)

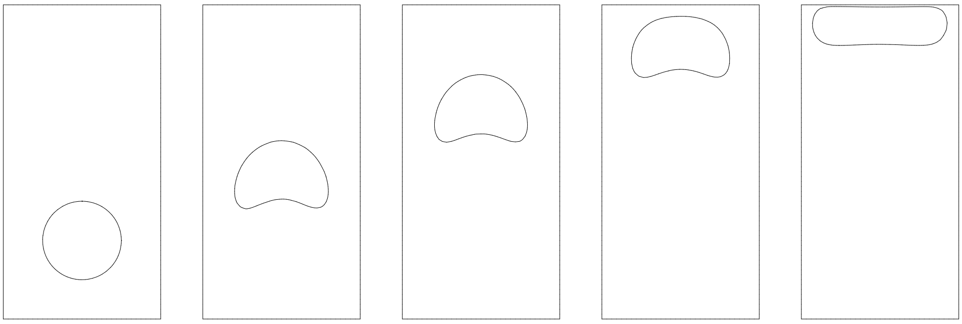

Figure 3: Scheme 2. $\Omega_1(t)$ at $t = 0.0, 2.5, 5.0, 7.5, 10.0$ ($\gamma_0 = 1.0, \Delta t = 1/8$)

if γ_0 is equal to zero [5]. In the case of $\gamma_0 > 0$ we see Scheme II is more stable than Scheme I. Let $\gamma_0 = 1.0$ and $\Delta t = 1/8$. Then we cannot get any solution in Scheme I due to the instability caused by the interfacial tension, but we get the solution in Scheme II. By setting $\Delta t = 1/16$ we can get the solution in Scheme I, though the small oscillation is seen in the interface at $t = 10$, see Figs. 2 and 3.

4.1.2 The effect of the interfacial tension coefficient

We use Scheme II and set $\Delta t = 1/8$. We take $\gamma_0 = 0.0, 1.0, 2.0$. Their interface curves at $t = 0, 2.5, 5.0, 7.5, 10.0$ are shown in Figs. 4, 3, and 5. As γ_0 becomes larger, the shape of the interface becomes more round because of larger interfacial tension. Fig. 6 shows the elevations of the pressure at $t = 5.0$ in those cases. In these figures the front side is CD. When γ_0 becomes large, the pressure in Ω_1 increases.

4.2 A rising bubble in a swayed channel

Our scheme is based on the finite element method, which has an excellent geometrical flexibility. Fig. 7 shows a numerical result of a rising bubble in a swayed channel. The values ρ_i and μ_i, $i = 1, 2$, and the positions of four corners are same as those of

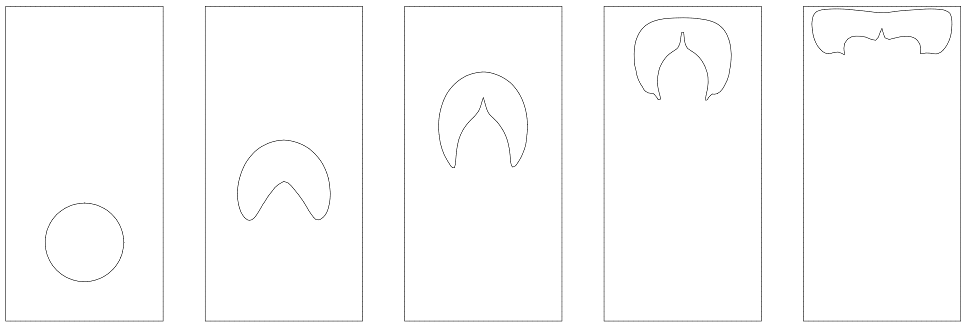

Figure 4: $\Omega_1(t)$ at $t = 0.0, 2.5, 5.0, 7.5, 10.0$ ($\gamma_0 = 0.0$)

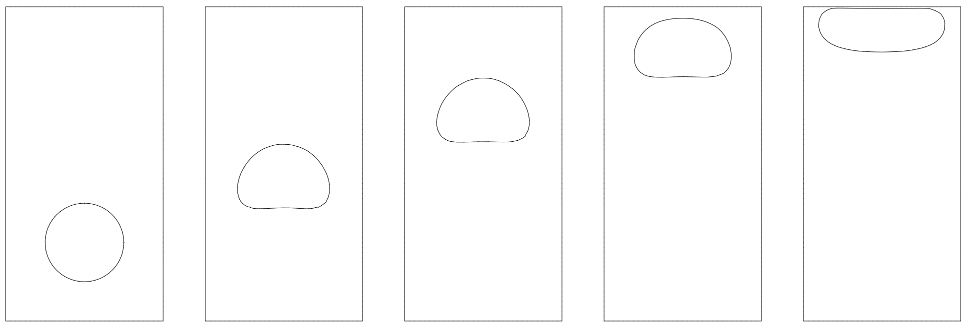

Figure 5: $\Omega_1(t)$ at $t = 0.0, 2.5, 5.0, 7.5, 10.0$ ($\gamma_0 = 2.0$)

the previous problem, and both swayed walls are expressed by

$$x_1 = a \sin \pi x_2,\ 1 + a \sin \pi x_2, \qquad a = 0.2.$$

The initial interface curve is defined by

$$\chi^0(s) = \big(\frac{1}{2} + \frac{1}{5}\cos 2\pi s, \frac{1}{5} + \frac{1}{5}\sin 2\pi s\big).$$

The domain is divided into a union of triangles. The total element number N_e is equal to 5,914 and Δt is set to be 1/8.

5 CONCLUDING REMARKS

We have developed stable finite element schemes for two-fluid flow problems with interfacial tension and presented some numerical results for rising bubble problems. These are robust and mathematically sound schemes. Their stability analyis will be given in a forth coming paper.

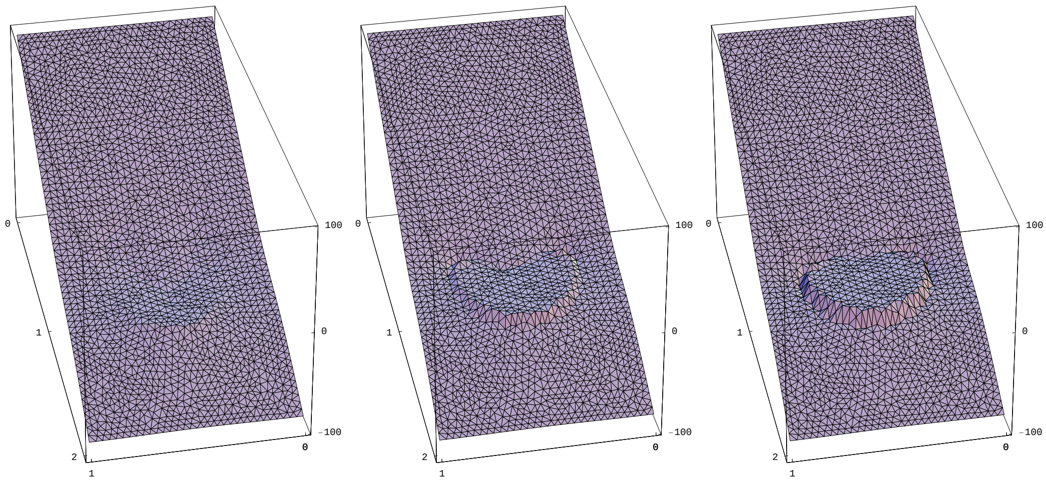

Figure 6: Elevations of the pressure at $t = 5.0$ ($\gamma_0 = 0.0, 1.0, 2.0$)

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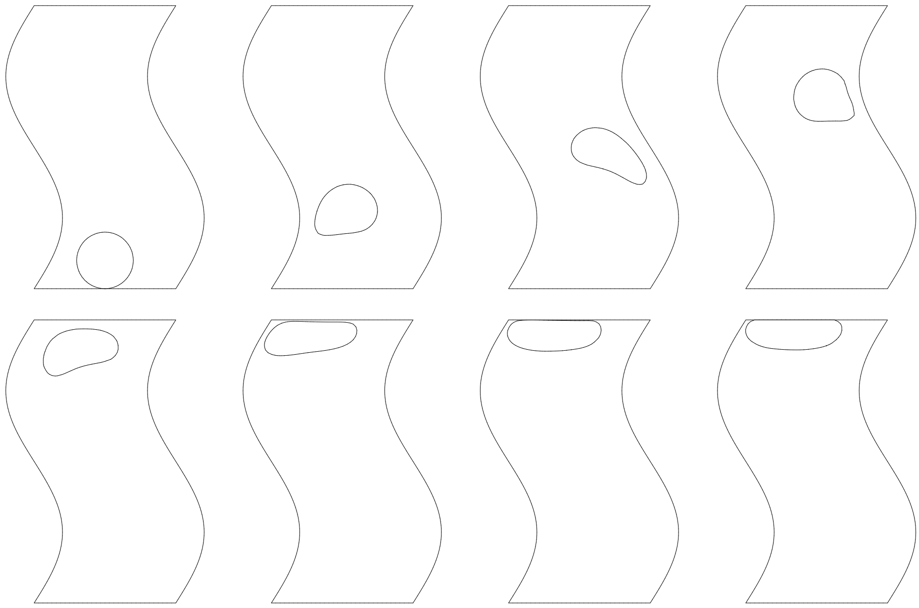

Figure 7: A rising bubble in a swayed channel at $t = 0.0, 2.5, \cdots, 17.5$ ($\gamma_0 = 1.0$)

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