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Abstract

We consider a Hankel determinant formula for generic solutions of the Painlevé IV equation. We show that the generating functions for the entries of the Hankel determinants are related to the asymptotic solution at infinity of the isomonodromic problem. Summability of these generating functions is also discussed.

Keywords and Phrases. Painlevé equation, determinant formula, isomonodromic problem
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1 Introduction

In this article, we consider the Painlevé IV equation (P_{IV}),

$$\frac{d^2y}{dt^2} = \frac{1}{2y} \left(\frac{dy}{dt} \right)^2 + \frac{3}{2}y^3 - 2ty^2 + \left(\frac{t^2}{2} - \alpha_1 + \alpha_0 \right)y - \frac{\alpha_2^2}{2y}, \quad (1.1)$$

where α_i ($i = 0, 1, 2$) are parameters satisfying $\alpha_0 + \alpha_1 + \alpha_2 = 1$. We also denote Equation (1.1) as $P_{IV}[\alpha_0, \alpha_1, \alpha_2]$ when it is necessary to specify the parameters explicitly. P_{IV} (1.1) has been studied extensively from various points of view. In particular, it is well-known that it admits symmetries of affine Weyl group of type $A_2^{(1)}$ as a group of Bäcklund transformations[15, 17, 18] (see also [13, 1, 3]). Moreover, it is known that P_{IV} admits two classes of classical solutions for special values of parameters: transcendental classical solutions which are expressed by the parabolic cylinder function[18] and rational solutions [10, 14]. Otherwise the solutions are higher transcendental functions[16, 19]. Among these

solutions, particular attention has been paid to a class of rational solutions which are located on the center of the Weyl chambers. Those solutions are expressed by logarithmic derivatives of certain characteristic polynomials with integer coefficients, which are generated by the Toda equation. Those polynomials are called the Okamoto polynomials[15, 17, 18, 4].

The determinant formulas are useful for understanding the nature of the Okamoto polynomials. In fact, it has been shown that the Okamoto polynomials are nothing but a specialization of the 3-core Schur functions by using the Jacobi-Trudi type determinant formula [12, 15, 17]. Also, there is another determinant formula which expresses the Okamoto polynomials in terms of the Hankel determinant. Therefore it is an intriguing problem to clarify the underlying meaning of the Hankel determinant formula. In order to do this, generating functions for the entries of Hankel determinants have been constructed in [9], and it is shown that they are expressed as logarithmic derivatives of solutions of the Airy equation.

A similar phenomenon has been observed in the study of rational solution of the Painlevé II equation (P_{II})[6]. Namely, the generating function associated with the rational solutions of P_{II} is also expressed as logarithmic derivatives of the Airy function.

Then what do these phenomena mean? In order to answer the question, the Hankel determinant formula for the *generic* solution of P_{II} was considered in [8]. It was shown that the generating functions of the entries of the Hankel determinant formula are related to the solutions of isomonodromic problem of P_{II} [7]. More precisely, the coefficients of asymptotic expansion of the ratio of solutions of the isomonodromic problem at infinity give the entries of Hankel determinant formula. The next natural problem then is to investigate whether such structure can be seen in other Painlevé equations or not.

The purpose of this article is to study the generating functions associated with the Hankel determinant formula for the generic solutions of P_{IV} (1.1). In Section 2, we give a brief review of the symmetries and τ functions for P_{IV} through the theory of the symmetric form of P_{IV} [15]. We then construct the Hankel determinant formula by applying the formula for the Toda equation[11] in Section 3. In Section 4, we consider the generating functions for the entries of the Hankel determinant formula. By linearizing the Riccati equations satisfied by the generating functions, it is shown that the generating functions are related to the isomonodromic problem of P_{IV} . We also show that the formal series for the generating functions are summable. Section 5 is devoted to concluding remarks.

2 Symmetric Form of P_{IV}

In this section we give a brief review of the theory of the symmetric form of P_{IV} . We refer to [15]¹ for details.

¹The convention of composition of transformations is different from what is often used in generating complex exact solutions from simple ones. See the section A.4 of [15] for details.

2.1 Symmetric form and Bäcklund transformations

The symmetric form of P_{IV} (1.1) is given by

$$\begin{aligned} f'_0 &= f_0(f_1 - f_2) + \alpha_0, \\ f'_1 &= f_1(f_2 - f_0) + \alpha_1, \\ f'_2 &= f_2(f_0 - f_1) + \alpha_2, \end{aligned} \quad (2.1)$$

where $' = d/dt$ and

$$\alpha_0 + \alpha_1 + \alpha_2 = 1, \quad f_0 + f_1 + f_2 = t. \quad (2.2)$$

We obtain P_{IV} (1.1) for $y = f_2$ by eliminating f_0 and f_1 . The τ functions τ_i ($i = 0, 1, 2$) are defined by

$$h_0 = \frac{\tau'_0}{\tau_0}, \quad h_1 = \frac{\tau'_1}{\tau_1}, \quad h_2 = \frac{\tau'_2}{\tau_2}, \quad (2.3)$$

where h_i ($i = 0, 1, 2$) are Hamiltonians given by

$$\begin{aligned} h_0 &= f_0 f_1 f_2 + \frac{\alpha_1 - \alpha_2}{3} f_0 + \frac{\alpha_1 + 2\alpha_2}{3} f_1 - \frac{2\alpha_1 + \alpha_2}{3} f_2, \\ h_1 &= f_0 f_1 f_2 - \frac{2\alpha_2 + \alpha_0}{3} f_0 + \frac{\alpha_2 - \alpha_0}{3} f_1 + \frac{\alpha_2 + 2\alpha_0}{3} f_2, \\ h_2 &= f_0 f_1 f_2 + \frac{\alpha_0 + 2\alpha_1}{3} f_0 - \frac{2\alpha_0 + \alpha_1}{3} f_1 + \frac{\alpha_0 - \alpha_1}{3} f_2. \end{aligned} \quad (2.4)$$

The symmetric form of P_{IV} (2.1) admits the following Bäcklund transformations s_i ($i = 0, 1, 2$) and π defined by Table 2.1.

	α_0	α_1	α_2	f_0	f_1	f_2	τ_0	τ_1	τ_2
s_0	$-\alpha_0$	$\alpha_1 + \alpha_0$	$\alpha_2 + \alpha_0$	f_0	$f_1 + \frac{\alpha_0}{f_0}$	$f_2 - \frac{\alpha_0}{f_0}$	$f_0 \frac{\tau_2 \tau_1}{\tau_0}$	τ_1	τ_2
s_1	$\alpha_0 + \alpha_1$	$-\alpha_1$	$\alpha_2 + \alpha_1$	$f_0 - \frac{\alpha_1}{f_1}$	f_1	$f_2 + \frac{\alpha_1}{f_1}$	τ_0	$f_1 \frac{\tau_0 \tau_2}{\tau_1}$	τ_2
s_2	$\alpha_0 + \alpha_2$	$\alpha_1 + \alpha_2$	$-\alpha_2$	$f_0 + \frac{\alpha_2}{f_2}$	$f_1 - \frac{\alpha_2}{f_2}$	f_2	τ_0	τ_1	$f_2 \frac{\tau_1 \tau_0}{\tau_2}$
π	α_1	α_2	α_0	f_1	f_2	f_0	τ_1	τ_2	τ_0

Table 2.1: Table of Bäcklund transformations for P_{IV} .

Theorem 2.1 (i) s_i ($i = 0, 1, 2$) and π commute with derivation.

(ii) s_i ($i = 0, 1, 2$) and π satisfy the following fundamental relations

$$s_i^2 = 1, \quad (s_i s_{i+1})^3 = 1, \quad \pi^3 = 1, \quad \pi s_i = s_{i+1} \pi, \quad i \in \mathbb{Z}/3\mathbb{Z}, \quad (2.5)$$

and thus $\langle s_0, s_1, s_2, \pi \rangle$ form the extended affine Weyl group of type $A_2^{(1)}$.

2.2 Bilinear equations for τ functions

The f -variables and the τ functions are related by

$$\begin{aligned} f_0 &= \frac{\tau'_2}{\tau_2} - \frac{\tau'_1}{\tau_1} + \frac{t}{3} = \frac{s_0(\tau_0)\tau_0}{\tau_2\tau_1}, \\ f_1 &= \frac{\tau'_0}{\tau_0} - \frac{\tau'_2}{\tau_2} + \frac{t}{3} = \frac{s_1(\tau_1)\tau_1}{\tau_0\tau_2}, \\ f_2 &= \frac{\tau'_1}{\tau_1} - \frac{\tau'_0}{\tau_0} + \frac{t}{3} = \frac{s_2(\tau_2)\tau_2}{\tau_1\tau_0}. \end{aligned} \quad (2.6)$$

Moreover, it is shown that τ functions satisfy various bilinear differential equations of Hirota type. For example, we have:

(I)

$$\left(D_t + \frac{t}{3}\right)\tau_2 \cdot \tau_1 = s_0(\tau_0)\tau_0, \quad (2.7)$$

$$\left(D_t + \frac{t}{3}\right)\tau_0 \cdot \tau_2 = s_1(\tau_1)\tau_1, \quad (2.8)$$

$$\left(D_t + \frac{t}{3}\right)\tau_1 \cdot \tau_0 = s_2(\tau_2)\tau_2. \quad (2.9)$$

(II)

$$\left(D_t^2 + \frac{t}{3}D_t - \frac{2}{9}t^2 + \frac{\alpha_0 - \alpha_1}{3}\right)\tau_0 \cdot \tau_1 = 0, \quad (2.10)$$

$$\left(D_t^2 + \frac{t}{3}D_t - \frac{2}{9}t^2 + \frac{\alpha_1 - \alpha_2}{3}\right)\tau_1 \cdot \tau_2 = 0, \quad (2.11)$$

$$\left(D_t^2 + \frac{t}{3}D_t - \frac{2}{9}t^2 + \frac{\alpha_2 - \alpha_0}{3}\right)\tau_2 \cdot \tau_0 = 0. \quad (2.12)$$

(III)

$$\left(\frac{1}{2}D_t^2 - \frac{\alpha_1 - \alpha_2}{3}\right)\tau_0 \cdot \tau_0 = s_1(\tau_1)s_2(\tau_2), \quad (2.13)$$

$$\left(\frac{1}{2}D_t^2 - \frac{\alpha_2 - \alpha_0}{3}\right)\tau_1 \cdot \tau_1 = s_2(\tau_2)s_0(\tau_0), \quad (2.14)$$

$$\left(\frac{1}{2}D_t^2 - \frac{\alpha_0 - \alpha_1}{3}\right)\tau_2 \cdot \tau_2 = s_0(\tau_0)s_1(\tau_1), \quad (2.15)$$

where D_t^n is the Hirota derivative defined by

$$D_t^n f \cdot g = \left(\frac{d}{dt} - \frac{d}{ds}\right)^n f(t)g(s) \Big|_{s=t}. \quad (2.16)$$

2.3 Translations and τ functions on lattice

Define the translation operators T_i ($i = 1, 2, 3$) by

$$T_1 = \pi s_2 s_1, \quad T_2 = \pi T_1 \pi^{-1} = s_1 \pi s_2, \quad T_3 = \pi T_2 \pi^{-1} = s_2 s_1 \pi. \quad (2.17)$$

Then it follows that

$$T_i T_j = T_j T_i \quad (i \neq j), \quad T_1 T_2 T_3 = 1. \quad (2.18)$$

Actions of T_i on parameters are given by

$$\begin{aligned} T_1(\alpha_0) &= \alpha_0 + 1, & T_1(\alpha_1) &= \alpha_1 - 1, & T_1(\alpha_2) &= \alpha_2, \\ T_2(\alpha_0) &= \alpha_0, & T_2(\alpha_1) &= \alpha_1 + 1, & T_2(\alpha_2) &= \alpha_2 - 1, \\ T_3(\alpha_0) &= \alpha_0 - 1, & T_3(\alpha_1) &= \alpha_1, & T_3(\alpha_2) &= \alpha_2 + 1. \end{aligned} \quad (2.19)$$

We define

$$\tau_{l,m,n} = T_1^l T_2^m T_3^n(\tau_0), \quad (2.20)$$

so that

$$\tau_0 = \tau_{0,0,0}, \quad \tau_1 = \tau_{1,0,0}, \quad \tau_2 = \tau_{0,0,-1}. \quad (2.21)$$

We note that $\tau_{l+k,m+k,n+k} = \tau_{l,m,n}$ follows from $T_1 T_2 T_3 = 1$. Applying $T_1^l T_2^m T_3^n$ on the bilinear equations (2.7)-(2.15), we obtain the following equations:

(I)

$$\left(D_t + \frac{t}{3}\right) \tau_{l+1,m+1,n} \cdot \tau_{l+1,m,n} = \tau_{l+1,m,n-1} \tau_{l,m,n}, \quad (2.22)$$

$$\left(D_t + \frac{t}{3}\right) \tau_{l,m,n} \cdot \tau_{l+1,m+1,n} = \tau_{l,m+1,n} \tau_{l+1,m,n}, \quad (2.23)$$

$$\left(D_t + \frac{t}{3}\right) \tau_{l+1,m,n} \cdot \tau_{l,m,n} = \tau_{l,m-1,n} \tau_{l+1,m+1,n}. \quad (2.24)$$

(II)

$$\left(D_t^2 + \frac{t}{3} D_t - \frac{2}{9} t^2 + \frac{\alpha_0 - \alpha_1 + 2l - m - n}{3}\right) \tau_{l,m,n} \cdot \tau_{l+1,m,n} = 0, \quad (2.25)$$

$$\left(D_t^2 + \frac{t}{3} D_t - \frac{2}{9} t^2 + \frac{\alpha_1 - \alpha_2 - l + 2m - n}{3}\right) \tau_{l+1,m,n} \cdot \tau_{l+1,m+1,n} = 0, \quad (2.26)$$

$$\left(D_t^2 + \frac{t}{3} D_t - \frac{2}{9} t^2 + \frac{\alpha_2 - \alpha_0 - l - m + 2n}{3}\right) \tau_{l+1,m+1,n} \cdot \tau_{l,m,n} = 0. \quad (2.27)$$

(III)

$$\left(\frac{1}{2} D_t^2 - \frac{\alpha_1 - \alpha_2 - l + 2m - n}{3}\right) \tau_{l,m,n} \cdot \tau_{l,m,n} = \tau_{l,m+1,n} \tau_{l,m-1,n}, \quad (2.28)$$

$$\left(\frac{1}{2} D_t^2 - \frac{\alpha_2 - \alpha_0 - l - m + 2n}{3}\right) \tau_{l+1,m,n} \cdot \tau_{l+1,m,n} = \tau_{l+1,m,n+1} \tau_{l+1,m,n-1}, \quad (2.29)$$

$$\left(\frac{1}{2} D_t^2 - \frac{\alpha_0 - \alpha_1 + 2l - m - n}{3}\right) \tau_{l+1,m+1,n} \cdot \tau_{l+1,m+1,n} = \tau_{l+2,m+1,n} \tau_{l,m+1,n}. \quad (2.30)$$

Remark 2.2 Suppose $\tau_0 = \tau_{0,0,0}$, $\tau_1 = \tau_{1,0,0}$ and $\tau_2 = \tau_{1,1,0}$ satisfy the bilinear equations (2.10)-(2.12). Then, f_0 , f_1 and f_2 defined by Equations (2.6) satisfy the symmetric form of P_{IV} (2.1). This can be verified as follows. Dividing Equations (2.10) and (2.11) by $\tau_0\tau_1$ and $\tau_1\tau_2$, respectively, we have

$$\begin{aligned}(h_0 + h_1)' + (h_0 - h_1)^2 + \frac{t}{3}(h_0 - h_1) - \frac{2}{9}t^2 + \frac{\alpha_0 - \alpha_1}{3} &= 0, \\ (h_1 + h_2)' + (h_1 - h_2)^2 + \frac{t}{3}(h_1 - h_2) - \frac{2}{9}t^2 + \frac{\alpha_1 - \alpha_2}{3} &= 0.\end{aligned}$$

Subtracting the second equation from the first equation we have

$$\begin{aligned}0 &= (h_0 - h_2)' + (h_0 - h_2)(h_0 - 2h_1 + h_2) + \frac{t}{3}(h_0 - 2h_1 + h_2) + \frac{1}{3} - \alpha_1 \\ &= \left(h_0 - h_2 + \frac{t}{3}\right)' + \left(h_0 - h_2 + \frac{t}{3}\right)(h_0 - 2h_1 + h_2) - \alpha_1 = 0 \\ &= f_1' + f_1(f_0 - f_2) - \alpha_1,\end{aligned}$$

which is the first equation in Equation (2.1). Here we have used the relations

$$h_0 - h_2 = f_1 - \frac{t}{3}, \quad h_1 - h_0 = f_2 - \frac{t}{3}, \quad h_2 - h_1 = f_2 - \frac{t}{3},$$

which follow from Equation (2.4). Other equations in Equation (2.1) can be derived in a similar manner.

Remark 2.3 Applying $T_1^l T_2^m T_3^n$ on Equation (2.6), we have

$$\begin{aligned}T_1^l T_2^m T_3^n(f_0) &= \frac{\tau'_{l+1,m+1,n}}{\tau_{l+1,m+1,n}} - \frac{\tau'_{l+1,m,n}}{\tau_{l+1,m,n}} + \frac{t}{3} = \frac{\tau_{l+2,m+1,n}\tau_{l,m,n}}{\tau_{l+1,m+1,n}\tau_{l+1,m,n}}, \\ T_1^l T_2^m T_3^n(f_1) &= \frac{\tau'_{l,m,n}}{\tau_{l,m,n}} - \frac{\tau'_{l+1,m+1,n}}{\tau_{l+1,m+1,n}} + \frac{t}{3} = \frac{\tau_{l,m+1,n}\tau_{l+1,m,n}}{\tau_{l,m,n}\tau_{l+1,m+1,n}}, \\ T_1^l T_2^m T_3^n(f_2) &= \frac{\tau'_{l+1,m,n}}{\tau_{l+1,m,n}} - \frac{\tau'_{l,m,n}}{\tau_{l,m,n}} + \frac{t}{3} = \frac{\tau_{l,m-1,n}\tau_{l+1,m+1,n}}{\tau_{l+1,m,n}\tau_{l,m,n}}.\end{aligned}\tag{2.31}$$

3 Hankel Determinant Formula

Now consider the sequence of τ functions $\tau_{n,0,0}$ ($n \in \mathbb{Z}$), which are τ functions in the direction of T_1 on the line $\alpha_2 = \text{const.}$ in the parameter space. It is possible to regard this sequence as being generated by the Toda equation

$$\left(\frac{1}{2}D_t^2 - \frac{\alpha_0 - \alpha_1 + 2n - 1}{3}\right)\tau_{n,0,0} \cdot \tau_{n,0,0} = \tau_{n+1,0,0}\tau_{n-1,0,0},\tag{3.1}$$

from $\tau_0 = \tau_{0,0,0}$ and $\tau_1 = \tau_{1,0,0}$. We note that Equation (3.1) follows from a specialization of Equation (2.30). Let us introduce the variables κ_n ($n \in \mathbb{Z}$) by

$$\kappa_n = e^{-\frac{1}{3}nt^2} \frac{\tau_{n,0,0}}{\tau_{0,0,0}}, \quad (3.2)$$

and put

$$\kappa_{-1} = \psi_{-1}, \quad \kappa_1 = \psi_1, \quad (3.3)$$

where $\psi_{\pm 1} = \psi_{\pm 1}(t)$. Then, Equation (3.1) can be rewritten as

$$\frac{1}{2} D_t^2 \kappa_n \cdot \kappa_n = \kappa_{n+1} \kappa_{n-1} - \psi_{-1} \psi_1 \kappa_n^2, \quad \kappa_{-1} = \psi_{-1}, \quad \kappa_0 = 1, \quad \kappa_1 = \psi_1, \quad (3.4)$$

by using the identities

$$\begin{aligned} D_t(ab) \cdot (cb) &= b^2 D_t a \cdot c, \\ D_t^2(ab) \cdot (cb) &= (D_t^2 a \cdot c) b^2 + ac(D_t^2 b \cdot b), \end{aligned}$$

and Equation (3.1) with $n = 0$. It is known that κ_n can be expressed by a Hankel determinant as follows[11]:

Theorem 3.1 κ_n is given by

$$\kappa_n = \begin{cases} \det(a_{i+j-2})_{1 \leq i, j \leq n} & n > 0 \\ 1 & n = 0 \\ \det(b_{i+j-2})_{1 \leq i, j \leq |n|} & n < 0 \end{cases} \quad (3.5)$$

where the entries are defined by the recurrence relations,

$$a_n = a'_{n-1} + \psi_{-1} \sum_{k=0}^{n-2} a_k a_{n-2-k}, \quad a_0 = \psi_1, \quad (3.6)$$

$$b_n = b'_{n-1} + \psi_1 \sum_{k=0}^{n-2} b_k b_{n-2-k}, \quad b_0 = \psi_{-1}. \quad (3.7)$$

We note that

$$y_{-1} = -\frac{\psi'_{-1}}{\psi_{-1}} + t, \quad y_0 = \frac{\psi'_1}{\psi_1} + t, \quad (3.8)$$

satisfy $P_{IV}[\alpha_0 - 1, \alpha_1 + 1, \alpha_2]$ and $P_{IV}[\alpha_0, \alpha_1, \alpha_2]$, respectively. Moreover,

$$y_n = \frac{\kappa'_{n+1}}{\kappa_{n+1}} - \frac{\kappa'_n}{\kappa_n} + t, \quad (3.9)$$

satisfies $P_{IV}[\alpha_0 + n, \alpha_1 - n, \alpha_2]$,

$$\frac{d^2 y_n}{dt^2} = \frac{1}{2y_n} \left(\frac{dy_n}{dt} \right)^2 + \frac{3}{2} y_n^3 - 2t y_n^2 + \left(\frac{t^2}{2} - \alpha_1 + \alpha_0 + 2n \right) y_n - \frac{\alpha_2^2}{2y_n}. \quad (3.10)$$

We also note that the determinant formula for the τ sequences in the directions of T_2 and T_3 are formulated in a similar manner.

4 Generating Functions and Isomonodromic Problem

4.1 Riccati Equations

For the determinant formula Theorem 3.1, let us consider the generating functions of the entries

$$F_\infty(t, \lambda) = \sum_{n=0}^{\infty} a_n \lambda^{-n}, \quad G_\infty(t, \lambda) = \sum_{n=0}^{\infty} b_n \lambda^{-n}, \quad (4.1)$$

where a_n and b_n are characterized by the recurrence relations (3.6) and (3.7). Multiplying Equations (3.6) and (3.7) by λ^{-n} , and taking summation over n , we obtain the following Riccati equations for the generating functions F_∞ and G_∞ .

Proposition 4.1 $F_\infty(t, \lambda)$ and $G_\infty(t, \lambda)$ satisfy the Riccati equations

$$\lambda \frac{\partial F}{\partial t} = -\psi_{-1} F^2 + \lambda^2 F - \lambda^2 \psi_1, \quad (4.2)$$

$$\lambda \frac{\partial G}{\partial t} = -\psi_1 G^2 + \lambda^2 G - \lambda^2 \psi_{-1}, \quad (4.3)$$

respectively.

Since F_∞ and G_∞ are defined as formal power series at $\lambda = \infty$, it is convenient to derive differential equations that they satisfy with respect to λ . The following auxiliary recurrence relations for a_n and b_n are useful for this purpose.

Lemma 4.2 a_n and b_n satisfy

$$\frac{d}{dt} [\psi_{-1} a_n - (\psi'_{-1} - t\psi_{-1}) a_{n-1}] + n\psi_{-1} a_{n-1} = 0, \quad (4.4)$$

and

$$\frac{d}{dt} [\psi_1 b_n - (\psi'_1 + t\psi_1) b_{n-1}] - n\psi_1 b_{n-1} = 0, \quad (4.5)$$

respectively.

The proof of Lemma 4.2 is achieved by tedious but straight-forward induction by noticing the relations

$$\psi_1'' + t\psi_1' + 2\psi_1^2\psi_{-1} + (\alpha_0 - \alpha_1)\psi_1 = 0, \quad (4.6)$$

$$\psi_{-1}'' - t\psi_{-1}' + 2\psi_1\psi_{-1}^2 + (\alpha_0 - \alpha_1 - 2)\psi_{-1} = 0, \quad (4.7)$$

which follow from the bilinear equation (2.25) with $(l, m, n) = (0, 0, 0)$ and $(-1, 0, 0)$, respectively. Lemma 4.2 yields the following linear partial differential equations for F_∞ and G_∞ :

Lemma 4.3 $F_\infty(t, \lambda)$ and $G_\infty(t, \lambda)$ satisfy the linear differential equations

$$\begin{aligned} & \left(\lambda + t - \frac{\psi'_{-1}}{\psi_{-1}} \right) \frac{\partial F}{\partial t} - \lambda \frac{\partial F}{\partial \lambda} \\ &= - \left[(\lambda + t) \frac{\psi'_{-1}}{\psi_{-1}} - \frac{\psi''_{-1}}{\psi_{-1}} + 2 \right] F + \frac{\lambda}{\psi_{-1}} (\psi_{-1} \psi_1)', \end{aligned} \quad (4.8)$$

$$\begin{aligned} & \left(-\lambda + t + \frac{\psi'_1}{\psi_1} \right) \frac{\partial G}{\partial t} - \lambda \frac{\partial G}{\partial \lambda} \\ &= - \left[(-\lambda + t) \frac{\psi'_1}{\psi_1} + \frac{\psi''_1}{\psi_1} + 2 \right] G - \frac{\lambda}{\psi_1} (\psi_{-1} \psi_1)', \end{aligned} \quad (4.9)$$

respectively.

Eliminating t -derivatives from Equations (4.2) and (4.8), and from Equations (4.3) and (4.9), respectively, we obtain the following Riccati equations with respect to λ .

Proposition 4.4 $F_\infty(t, \lambda)$ and $G_\infty(t, \lambda)$ satisfy the following Riccati equations

$$\begin{aligned} \lambda^2 \frac{\partial F}{\partial \lambda} &= - \left(\lambda + t - \frac{\psi'_{-1}}{\psi_{-1}} \right) \psi_{-1} F^2 \\ &+ \lambda \left(\lambda^2 + \lambda t + 2\psi_1 \psi_{-1} + \alpha_0 - \alpha_1 \right) F - \lambda^2 [(\lambda + t) \psi_1 + \psi'_1], \end{aligned} \quad (4.10)$$

$$\begin{aligned} \lambda^2 \frac{\partial G}{\partial \lambda} &= - \left(-\lambda + t + \frac{\psi'_1}{\psi_1} \right) \psi_1 G^2 \\ &- \lambda \left(\lambda^2 - \lambda t + 2\psi_1 \psi_{-1} + \alpha_0 - \alpha_1 - 2 \right) G - \lambda^2 [(-\lambda + t) \psi_{-1} - \psi'_{-1}], \end{aligned} \quad (4.11)$$

respectively.

4.2 Isomonodromic Problem

The Riccati equations in Proposition 4.1 and Proposition 4.4 can be linearized into second order linear differential equations by the standard technique, which yields isomonodromic problems associated with P_{IV} .

Theorem 4.5 (i) *It is possible to introduce the functions Y_1 and Y_2 consistently as*

$$\begin{aligned} F_\infty(t, \lambda) &= \frac{\lambda}{\psi_{-1}} \left(\frac{1}{Y_1} \frac{\partial Y_1}{\partial t} + \frac{\lambda}{2} \right) \\ &= \frac{\lambda^2}{\psi_{-1}} \frac{1}{\lambda + t - \frac{\psi'_{-1}}{\psi_{-1}}} \\ &\quad \times \left(\frac{1}{Y_1} \frac{\partial Y_1}{\partial \lambda} + \frac{\lambda + t}{2} + \frac{\psi_{-1} \psi_1 + \alpha_0 - 1}{\lambda} + \frac{\alpha_2}{2\lambda} \right), \end{aligned} \quad (4.12)$$

$$Y_2 = \frac{1}{\psi_{-1}} \left(\frac{\partial Y_1}{\partial t} + \frac{\lambda}{2} Y_1 \right). \quad (4.13)$$

Then the Riccati equations (4.2) and (4.10) are linearized to:

$$\frac{\partial}{\partial \lambda} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = A \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \quad \frac{\partial}{\partial t} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = B \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \quad (4.14)$$

$$A = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \lambda + \begin{pmatrix} -\frac{t}{2} & u \\ -\frac{z + \alpha_1 + \alpha_2}{u} & \frac{t}{2} \end{pmatrix} + \begin{pmatrix} -\left(z + \frac{\alpha_2}{2}\right) & uy_{-1} \\ -\frac{z(z + \alpha_2)}{uy_{-1}} & z + \frac{\alpha_2}{2} \end{pmatrix} \lambda^{-1}, \quad (4.15)$$

$$B = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \lambda + \begin{pmatrix} 0 & u \\ -\frac{z + \alpha_1 + \alpha_2}{u} & 0 \end{pmatrix}, \quad (4.16)$$

where

$$u = \psi_{-1}, \quad y_{-1} = -\frac{\psi'_{-1}}{\psi_{-1}} + t, \quad z = \psi_{-1}\psi_1 + \alpha_0 - 1 = \psi_{-1}\psi_1 - \alpha_1 - \alpha_2. \quad (4.17)$$

Conversely, if Y_1 and Y_2 are the solutions of linear system (4.14)-(4.16), then

$$F = \lambda \frac{Y_2}{Y_1}, \quad (4.18)$$

satisfies the Riccati equations (4.2) and (4.10).

(ii) It is possible to introduce the functions Z_1 and Z_2 consistently as

$$\begin{aligned} G_\infty(t, \lambda) &= \frac{\lambda}{\psi_1} \left(\frac{1}{Z_1} \frac{\partial Z_1}{\partial t} + \frac{\lambda}{2} \right) \\ &= \frac{\lambda^2}{\psi_1 - \lambda + t + \frac{\psi'_1}{\psi_1}} \\ &\quad \times \left(\frac{1}{Z_1} \frac{\partial Z_1}{\partial \lambda} + \frac{-\lambda + t}{2} + \frac{\psi_{-1}\psi_1 + \alpha_0 - 1}{s} + \frac{\alpha_2}{2\lambda} \right), \end{aligned} \quad (4.19)$$

$$Z_2 = \frac{1}{\psi_1} \left(\frac{\partial Z_1}{\partial t} + \frac{\lambda}{2} Z_1 \right). \quad (4.20)$$

Then the Riccati equations (4.3) and (4.11) are linearized to:

$$\frac{\partial}{\partial \lambda} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = C \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \quad \frac{\partial}{\partial t} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = D \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \quad (4.21)$$

$$C = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \lambda + \begin{pmatrix} -\frac{t}{2} & -v \\ \frac{w + \alpha_1 + \alpha_2}{v} & \frac{t}{2} \end{pmatrix} \quad (4.22)$$

$$+ \begin{pmatrix} w + \frac{\alpha_2}{2} & vy_0 \\ -\frac{w(w + \alpha_2)}{vy_0} & -w - \frac{\alpha_2}{2} \end{pmatrix} \lambda^{-1}, \quad (4.23)$$

$$D = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \lambda + \begin{pmatrix} 0 & v \\ -\frac{w + \alpha_1 + \alpha_2}{v} & 0 \end{pmatrix}, \quad (4.24)$$

where

$$v = \psi_1, \quad y_0 = \frac{\psi_1'}{\psi_1} + t, \quad w = \psi_{-1}\psi_1 + \alpha_0 - 1 = \psi_{-1}\psi_1 - \alpha_1 - \alpha_2. \quad (4.25)$$

Conversely, if Z_1 and Z_2 are the solutions of linear system (4.21)-(4.24), then

$$G = \lambda \frac{Z_2}{Z_1}, \quad (4.26)$$

satisfies the Riccati equations (4.3) and (4.11).

Remark 4.6 The linear systems (4.14)-(4.16) and (4.21)-(4.24) are nothing but the isomonodromic problem for $P_{IV}[\alpha_0 - 1, \alpha_1 + 1, \alpha_2]$ and $P_{IV}[\alpha_0, \alpha_1, \alpha_2]$, respectively[7]. In fact, compatibility condition of the linear system (4.14)-(4.16)

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial \lambda} + AB - BA = 0, \quad (4.27)$$

gives

$$\frac{dz}{dt} = \frac{z^2}{y_{-1}} + \left(\frac{\alpha_2}{y_{-1}} - y_{-1} \right) z - (\alpha_1 + \alpha_2)y_{-1}, \quad (4.28)$$

$$\frac{dy_{-1}}{dt} = 2z + y_{-1}^2 - ty_{-1} + \alpha_2, \quad (4.29)$$

$$\frac{du}{dt} = (-y_{-1} + t)u. \quad (4.30)$$

Eliminating z , we have $P_{IV}[\alpha_0 - 1, \alpha_1 + 1, \alpha_2]$ for y_{-1}

$$y_{-1}'' = \frac{(y_{-1}')^2}{2y_{-1}} + \frac{3}{2}y_{-1}^3 - 2ty_{-1}^2 + \left(\frac{t^2}{2} - \alpha_1 + \alpha_0 - 2 \right) y_{-1} - \frac{\alpha_2^2}{2y_{-1}}. \quad (4.31)$$

This fact also establishes the consistency of two expressions of $F_\infty(t, \lambda)$ in terms of Y_1 in Equation (4.12). A similar remark holds true for $G_\infty(t, \lambda)$ and Z_1 .

Remark 4.7 From Equation (4.12), Y_1 can be formally expressed as

$$Y_1 = \text{const.} \times \exp\left(-\frac{\lambda^2}{4} - \frac{\lambda t}{2}\right) \lambda^{\alpha_1 + \alpha_2/2} \exp\left(-\sum_{n=1}^{\infty} \lambda^{-n} \int \psi_{-1} a_{n-1} dt\right), \quad (4.32)$$

which coincides with the known asymptotic behavior of Y_1 around $\lambda \sim \infty$ [7].

4.3 Solutions of Isomonodromic Problems and Determinant Formula

We have investigated the generating functions F_∞ and G_∞ of entries of the Hankel determinant formula and shown that they formally satisfy the Riccati equations (4.2), (4.10) and (4.3), (4.11), respectively, and that those Riccati equations are linearized into isomonodromic problems (4.14)-(4.16) and (4.21)-(4.24) for P_{IV} .

Now let us start from the isomonodromic problem (4.14)-(4.16). We have two linearly independent solutions around $\lambda \sim \infty$, one of which is related to the generating function F_∞ by $F = \lambda Y_2/Y_1$. So let us consider another solution. It is known that the linear system (4.14)-(4.16) admits the following formal solutions [7] around $\lambda \sim \infty$

$$\begin{aligned} \begin{pmatrix} Y_1^{(1)} \\ Y_2^{(1)} \end{pmatrix} &= \exp\left(-\frac{\lambda^2}{4} - \frac{\lambda t}{2}\right) \lambda^{\alpha_1 + \alpha_2/2} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} y_{11}^{(1)} \\ y_{21}^{(1)} \end{pmatrix} \lambda^{-1} + \dots \right], \\ \begin{pmatrix} Y_1^{(2)} \\ Y_2^{(2)} \end{pmatrix} &= \exp\left(\frac{\lambda^2}{4} + \frac{\lambda t}{2}\right) \lambda^{-\alpha_1 - \alpha_2/2} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} y_{11}^{(2)} \\ y_{21}^{(2)} \end{pmatrix} \lambda^{-1} + \dots \right]. \end{aligned}$$

These solutions give

$$\begin{aligned} F^{(1)}(t, \lambda) &= \lambda \frac{Y_2^{(1)}}{Y_1^{(1)}} = \lambda \times \frac{y_{21}^{(1)} \lambda^{-1} + \dots}{1 + y_{11}^{(1)} \lambda^{-1} + \dots} = a_0 + a_1 \lambda^{-1} + \dots, \\ F^{(2)}(t, \lambda) &= \lambda \frac{Y_2^{(2)}}{Y_1^{(2)}} = \lambda \times \frac{1 + y_{11}^{(1)} \lambda^{-1} + \dots}{y_{21}^{(1)} \lambda^{-1} + \dots} = \lambda^2 (c_0 + c_1 \lambda^{-1} + \dots), \end{aligned}$$

respectively. Theorem 4.5 states that both $F^{(1)}(t, \lambda)$ and $F^{(2)}(t, \lambda)$ satisfy the Riccati equations (4.2) and (4.10). Conversely, the above two possibilities of power-series solutions for the Riccati equations are verified directly.

Proposition 4.8 *The Riccati equations (4.2) and (4.10) admit only the following two kinds of power-series solutions around $\lambda \sim \infty$:*

$$F^{(1)}(t, \lambda) = \sum_{n=0}^{\infty} a_n \lambda^{-n}, \quad F^{(2)}(t, \lambda) = \lambda^2 \sum_{n=0}^{\infty} c_n \lambda^{-n}. \quad (4.33)$$

The proof of Proposition 4.8 is achieved simply by plugging the power-series solution

$$F = \lambda^\rho \sum_{n=0}^{\infty} u_n \lambda^{-n},$$

into the Riccati equations (4.2) and (4.10), and investigating the balance of leading terms. Then we find that $\rho = 0, 2$. A similar result can be shown for the Riccati equations (4.3) and (4.11).

Proposition 4.9 *The Riccati equations (4.3) and (4.11) admit only the following two kinds of power-series solutions around $\lambda \sim \infty$:*

$$G^{(1)}(t, \lambda) = \sum_{n=0}^{\infty} b_n \lambda^{-n}, \quad G^{(2)}(t, \lambda) = \lambda^2 \sum_{n=0}^{\infty} d_n \lambda^{-n}. \quad (4.34)$$

It is obvious that $F^{(1)}(t, \lambda)$ and $G^{(1)}(t, \lambda)$ are nothing but $F_\infty(t, \lambda)$ and $G_\infty(t, \lambda)$, respectively. Therefore it is an important problem to investigate $F^{(2)}(t, \lambda)$ and $G^{(2)}(t, \lambda)$. Now we present two observations regarding this problem. The first observation is that there are quite simple relations among those functions:

Proposition 4.10 *The following relations hold:*

$$F^{(2)}(t, \lambda) = \frac{\lambda^2}{G^{(1)}(t, -\lambda)}, \quad G^{(2)}(t, \lambda) = \frac{\lambda^2}{F^{(1)}(t, -\lambda)}. \quad (4.35)$$

Proof. Substituting $F(t, \lambda) = \lambda^2/g(t, \lambda)$ into the Riccati equations (4.2) and (4.10), we obtain Equations (4.3) and (4.11), respectively, for $G(t, \lambda) = g(t, -\lambda)$. Choosing $g(t, \lambda) = G^{(1)}(t, \lambda)$, $F(t, \lambda)$ must be $F^{(2)}(t, \lambda)$, since its leading order is λ^2 . We obtain the second equation by the similar argument. \square

Secondly, $F^{(2)}(t, \lambda)$ and $G^{(2)}(t, \lambda)$ can be also interpreted as generating functions of the Hankel determinant formula for P_{IV} . Recall that the determinant formula in Theorem 3.1 is for the τ sequence $\kappa_n = e^{-nt^2/3} \tau_{n,0,0}/\tau_{0,0,0}$. The following Proposition states that $F^{(2)}(t, \lambda)$ and $G^{(2)}(t, \lambda)$ correspond to different normalizations of the τ sequence:

Proposition 4.11 *Let*

$$F^{(2)}(t, \lambda) = -\frac{\lambda^2}{\psi_{-1}^2} \sum_{n=0}^{\infty} c_n (-\lambda)^{-n}, \quad (4.36)$$

$$G^{(2)}(t, \lambda) = -\frac{\lambda^2}{\psi_1^2} \sum_{n=0}^{\infty} d_n (-\lambda)^{-n}, \quad (4.37)$$

be formal solutions of the Riccati equations (4.2), (4.10) and (4.3), (4.11), respectively. Then we have the following:

(i) $c_0 = -\psi_{-1}$ and $c_1 = \psi'_{-1}$. For $n \geq 2$, c_n 's are characterized by the recursion relation

$$c_{n+1} = c'_n + \frac{1}{\psi_{-1}} \sum_{k=2}^{n-1} c_k c_{n+1-k}, \quad c_2 = \frac{\psi''_{-1} \psi_{-1} - (\psi'_{-1})^2 + \psi_{-1}^3 \psi_1}{\psi_{-1}}. \quad (4.38)$$

(ii) $d_0 = -\psi_1$ and $d_1 = \psi'_1$. For $n \geq 2$, d_n 's are characterized by the recursion relation

$$d_{n+1} = d'_n + \frac{1}{\psi_1} \sum_{k=2}^{n-1} d_k d_{n+1-k}, \quad d_2 = \frac{\psi''_1 \psi_1 - (\psi'_1)^2 + \psi_{-1} \psi_1^3}{\psi_1}. \quad (4.39)$$

(iii) We put

$$\sigma_{-n-1} = \det(c_{i+j})_{i,j=1,\dots,n} \quad (n > 0), \quad \sigma_{-1} = 1, \quad (4.40)$$

$$\theta_{n+1} = \det(d_{i+j})_{i,j=1,\dots,n} \quad (n > 0), \quad \theta_1 = 1. \quad (4.41)$$

Then σ_n and θ_n are related to $\tau_{n,0,0}$ as

$$\sigma_n = \frac{\kappa_n}{\kappa_{-1}} = e^{-\frac{1}{3}(n+1)t^2} \frac{\tau_{n,0,0}}{\tau_{-1,0,0}} \quad (n < 0), \quad (4.42)$$

$$\theta_n = \frac{\kappa_n}{\kappa_1} = e^{-\frac{1}{3}(n-1)t^2} \frac{\tau_{n,0,0}}{\tau_{1,0,0}} \quad (n > 0). \quad (4.43)$$

Proof. (i) and (ii) can be proved easily by substituting Equations (4.36) and (4.37) into the Riccati equations (4.2) and (4.3), respectively, and collecting the coefficients of powers of λ . For (iii), we notice that from the Toda equation (3.4) with $n = -1$, namely

$$\psi''_{-1} \psi_{-1} - (\psi'_{-1})^2 = \kappa_{-2} - \psi_{-1}^3 \psi_1,$$

we have

$$c_2 = \frac{\psi''_{-1} \psi_{-1} - (\psi'_{-1})^2 + \psi_{-1}^3 \psi_1}{\psi_{-1}} = \frac{\kappa_{-2}}{\kappa_{-1}} = \sigma_{-2}.$$

Moreover, the coefficient of the quadratic term in Equation (4.38) can be regarded as

$$\frac{1}{\psi_{-1}} = \frac{\kappa_0}{\kappa_{-1}} = \sigma_0.$$

Applying Theorem 3.1, we find that $\sigma_{-n-1} = \det(c_{i+j})_{i,j=1,\dots,n}$ ($n > 0$). The statement for d_n can be shown by a similar argument. \square

4.4 Summability of the Generating Functions

To study the growth as $n \rightarrow \infty$ of the coefficients $a_n(t)$ (or $b_n(t)$) in Equation (4.1), we use a theorem proved in [5].

Theorem 4.12 (Hsieh and Sibuya [5], Theorem XIII-8-3) *Consider the following non-linear differential equation in the variable s*

$$s^{k+1} \frac{dH}{ds} = c(s)H + s b(s, H) \quad (4.44)$$

where k is a positive integer, $c(s)$ is holomorphic in the neighbourhood of $s = 0$ and $c(0) \neq 0$, and $b(s, H)$ is holomorphic in the neighbourhood of $(s, H) = (0, 0)$. Then equation (4.44) admits one and only one formal solution $H_f(s)$ of the form $H_f(s) = \sum_{n=1}^{\infty} a_n s^n$. Moreover, H_f is k -summable in any direction $\arg(s) = \vartheta$ except a finite number of values ϑ . Furthermore, the sum of $H_f(s)$ in the direction $\arg(s) = \vartheta$ is a solution of Equation (4.44).

Equation (4.10) can be put into the form (4.44) by changing variables to $\lambda = 1/s$ and taking $H = F - a_0 = F - \psi_1$. We then obtain Equation (4.44) with $k = 2$, $c(s) \equiv -1$ and

$$\begin{aligned} b(s, H) = & -tH + \psi'_1 + s(\psi_{-1}H^2 - (\alpha_0 - \alpha_1)(H + \psi_1)) \\ & + s^2(t\psi_{-1} - \psi'_{-1})(H + \psi_1)^2 \end{aligned}$$

Applying Theorem 4.12, we deduce that Equation (4.10) admits one and only one formal solution $F_{\infty}(\lambda)$ of the form $\sum_{n=0}^{\infty} a_n \lambda^{-n}$. This formal solution is 2-summable in any direction $\arg(\lambda) = \vartheta$ except a finite number of values ϑ and its sum in the direction $\arg(\lambda) = \vartheta$ is a solution of Equation (4.10).

The definition of k -summability implies that $F_{\infty}(\lambda)$ is of Gevrey order 2, namely, for each t , there exist positive numbers $C(t)$ and $K(t)$ such that

$$|a_n(t)| < C(t)(n!)^2 K(t)^n, \quad \text{for all } n \geq 1.$$

Clearly, one can prove a similar result for the coefficients b_n of the formal solutions G_{∞} in Equation (4.1), as well as the coefficients c_n, d_n of the formal series in Equations (4.33) and (4.34). For the formal solutions $F^{(2)}(t, \lambda)$ (or $G^{(2)}(t, \lambda)$), we need to apply Theorem 4.12 to a new $H(s) = s^2 F^{(2)} - c_0$ (or $s^2 G^{(2)} - d_0$).

We also note that summability of those series implies that they are expressible in terms of inverse Laplace transformation of certain analytic functions. For details, we refer to [2].

5 Concluding Remarks

In this article, we have constructed a Hankel determinant formula for the τ sequence of P_{IV} in the direction of translation T_1 . Then we have shown that the generating functions of

the entries are closely related to the solutions of isomonodromic problems. More precisely, coefficients of asymptotic expansion of the ratio of solutions for isomonodromic problem give the entries of Hankel determinant formula. Moreover, we have shown that there exist simple but mysterious relations among those generating functions. We also discussed the summability of the generating functions.

Let us finally give some remarks. Firstly, in this article we have considered only the τ sequences in the direction of T_1 . It is also possible to consider the directions T_2, T_3 by the $\tilde{W}(A_2^{(1)})$ symmetry of P_{IV} . Secondly, it is surprising that the results obtained in this article are completely parallel to the P_{II} case [8], although concrete computations depend on the specific situation for each case. In particular, it is remarkable that many formulas have exactly the same form as the P_{II} case, such as Equation (4.32) (except for the dominant exponential factor), or formulas in Section 4.3. This coincidence may imply that (i) the phenomena observed for P_{II} and P_{IV} could be universal; at least it can be seen for other Painlevé equations, (ii) the underlying mathematical structure may originate from the Toda equation, rather than the Painlevé equations themselves.

These points will be explored in forthcoming articles.

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