Construction of Integrals of Higher-Order Mappings

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LETTER TO THE EDITOR

Construction of integrals of higher-order mappings

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Abstract. We find that certain higher-order mappings arise as reductions of the integrable discrete AKP and BKP equations. Finding conservation laws for the AKP and BKP equations, we use these conservation laws to derive integrals of the associated reduced maps.

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1. Introduction

The search for discrete integrable systems has received a lot of attention in the past decade. This has resulted in the discovery of integrable mappings of the second-order, e.g. the QRT mapping [1], and discrete Painlevé equations [2]. Apart from second-order integrable mappings, the results for higher-order integrable mappings are few [3, 4, 5, 6, 7, 8, 9, 10, 11]. Discrete integrable systems have applications to various areas of physics, such as statistical mechanics, quantum gravity, and discrete analogues of integrable systems in classical mechanics and solid state physics. Here we study a novel class of higher-order integrable mappings which have bilinear forms.

As an example, we discuss the following 6th-order mapping:

$$Dx_{n+3}x_{n+2}^2x_{n+1}^3x_n^3x_{n-1}^3x_{n-2}^2x_{n-3}+Ax_{n+2}x_{n+1}^2x_n^2x_{n-1}^2x_{n-2}+Bx_{n+1}x_nx_{n-1}+C=0\,. \tag{1}$$

(Here and below A,B,C,D are arbitrary parameters). How can we obtain integrals for this mapping? In the paper [3], a method for construction of integrals was proposed and integrable third-order mappings which possess two integrals were obtained. However, this method is not applicable to higher-order mappings because this method uses some ansatz at first and needs the help of high performance computers. If we consider 6th-order, 8th-order and higher-order mappings, this method does not work, as current computer power is not sufficient [12].

In this Letter, we propose a systematic method to construct integrals for a class of higher-order integrable mappings without the help of computers. Our method proposed here is based on discrete bilinear forms related to the AKP and BKP soliton equations. Conservation laws for integrable partial difference equations have been studied in [13, 14].

2. Conservation Laws for Discrete Bilinear Forms

Before we discuss conservation laws for discrete systems, let us briefly recall conservation laws for continuous systems [15]. To this end, consider a (scalar) partial differential equation (PDE) $\Delta[x, u^{(i)}] = 0$. A conservation law of such a PDE is a divergence expression

$$\sum_j \frac{\partial P_j}{\partial x_j} = 0$$

which vanishes for all solutions of the given system. It follows that there exists a function Λ (called the *characteristic* of the given conservation law) such that

$$\sum_j \frac{\partial P_j}{\partial x_j} = \Lambda\Delta.$$

Similarly, a conservation law of a scalar partial difference equation $\Delta[n, u_n] = 0$ is an expression

$$\sum_j (S_j - id)P_j = 0\,,$$

which vanishes for all solutions of the discrete system. (Here S_j is a unit shift in the n_j direction, and $\Delta[n, u_n]$ denotes a smooth function depending on n, u_n and finitely many iterates of u_n). It follows again that there exists a function Λ such that

$$\sum_j (S_j - id)P_j = \Lambda\Delta\,. \tag{2}$$

We will call Λ the characteristic of the discrete conservation law.

Here we give a list of characteristics of the discrete AKP and BKP equations.

Discrete BKP equation

The discrete BKP equation [16] is given by

$$A\tau_{k+1,l,m}\tau_{k,l+1,m+1} + B\tau_{k,l+1,m}\tau_{k+1,l,m+1} + C\tau_{k,l,m+1}\tau_{k+1,l+1,m} + D\tau_{k,l,m}\tau_{k+1,l+1,m+1} = 0\,. \tag{3}$$

We have found the following 12 explicit rational characteristics for the discrete BKP equation:

$$\begin{aligned}
\Lambda_1 = {} & A\left(\frac{\tau_{k-1,l+1,m+1}}{\tau_{k,l,m+1}\tau_{k,l+1,m+1}\tau_{k,l+1,m}} - \frac{\tau_{k+2,l,m}}{\tau_{k+1,l,m}\tau_{k+1,l+1,m}\tau_{k+1,l,m+1}}\right) \\
& + D\left(\frac{\tau_{k-1,l,m}}{\tau_{k,l,m}\tau_{k,l+1,m}\tau_{k,l,m+1}} - \frac{\tau_{k+2,l+1,m+1}}{\tau_{k+1,l+1,m+1}\tau_{k+1,l+1,m}\tau_{k+1,l,m+1}}\right), \\
\Lambda_2 = {} & B\left(\frac{\tau_{k+1,l-1,m+1}}{\tau_{k+1,l,m}\tau_{k+1,l,m+1}\tau_{k,l,m+1}} - \frac{\tau_{k,l+2,m}}{\tau_{k,l+1,m}\tau_{k,l+1,m+1}\tau_{k+1,l+1,m}}\right) \\
& + D\left(\frac{\tau_{k,l-1,m}}{\tau_{k,l,m}\tau_{k,l,m+1}\tau_{k+1,l,m}} - \frac{\tau_{k+1,l+2,m+1}}{\tau_{k+1,l+1,m+1}\tau_{k,l+1,m+1}\tau_{k+1,l+1,m}}\right), \\
\Lambda_3 = {} & C\left(\frac{\tau_{k+1,l+1,m-1}}{\tau_{k+1,l+1,m}\tau_{k,l+1,m}\tau_{k+1,l,m}} - \frac{\tau_{k,l,m+2}}{\tau_{k,l,m+1}\tau_{k,l+1,m+1}\tau_{k+1,l,m+1}}\right)
\end{aligned}$$

$$+D\left(\frac{\tau_{k,l,m-1}}{\tau_{k+1,l,m}\tau_{k,l+1,m}\tau_{k,l,m}}-\frac{\tau_{k+1,l+1,m+2}}{\tau_{k+1,l+1,m+1}\tau_{k,l+1,m+1}\tau_{k+1,l,m+1}}\right),$$

$$\Lambda_4=C\left(\frac{\tau_{k,l-1,m+1}}{\tau_{k+1,l,m+1}\tau_{k,l,m+1}\tau_{k,l,m}}-\frac{\tau_{k+1,l+2,m}}{\tau_{k+1,l+1,m}\tau_{k,l+1,m}\tau_{k+1,l+1,m+1}}\right)$$
$$+A\left(\frac{\tau_{k+1,l-1,m}}{\tau_{k+1,l,m}\tau_{k,l,m}\tau_{k+1,l,m+1}}-\frac{\tau_{k,l+2,m+1}}{\tau_{k,l+1,m+1}\tau_{k,l+1,m}\tau_{k+1,l+1,m+1}}\right),$$

$$\Lambda_5=A\left(\frac{\tau_{k+1,l,m-1}}{\tau_{k+1,l+1,m}\tau_{k+1,l,m}\tau_{k,l,m}}-\frac{\tau_{k,l+1,m+2}}{\tau_{k,l+1,m+1}\tau_{k,l,m+1}\tau_{k+1,l+1,m+1}}\right)$$
$$+B\left(\frac{\tau_{k,l+1,m-1}}{\tau_{k,l+1,m}\tau_{k,l,m}\tau_{k+1,l+1,m}}-\frac{\tau_{k+1,l,m+2}}{\tau_{k+1,l,m+1}\tau_{k,l,m+1}\tau_{k+1,l+1,m+1}}\right),$$

$$\Lambda_6=B\left(\frac{\tau_{k-1,l+1,m}}{\tau_{k,l+1,m+1}\tau_{k,l+1,m}\tau_{k,l,m}}-\frac{\tau_{k+2,l,m+1}}{\tau_{k+1,l,m+1}\tau_{k+1,l,m}\tau_{k+1,l+1,m+1}}\right)$$
$$+C\left(\frac{\tau_{k-1,l,m+1}}{\tau_{k,l,m+1}\tau_{k,l,m}\tau_{k,l+1,m+1}}-\frac{\tau_{k+2,l+1,m}}{\tau_{k+1,l+1,m}\tau_{k+1,l,m}\tau_{k+1,l+1,m+1}}\right),$$

$$\Gamma_1=A(-1)^k\left(\frac{\tau_{k-1,l+1,m+1}}{\tau_{k,l,m+1}\tau_{k,l+1,m+1}\tau_{k,l+1,m}}+\frac{\tau_{k+2,l,m}}{\tau_{k+1,l,m}\tau_{k+1,l+1,m}\tau_{k+1,l,m+1}}\right)$$
$$+D(-1)^k\left(\frac{\tau_{k-1,l,m}}{\tau_{k,l,m}\tau_{k,l+1,m}\tau_{k,l,m+1}}+\frac{\tau_{k+2,l+1,m+1}}{\tau_{k+1,l+1,m+1}\tau_{k+1,l+1,m}\tau_{k+1,l,m+1}}\right),$$

$$\Gamma_2=B(-1)^l\left(\frac{\tau_{k+1,l-1,m+1}}{\tau_{k+1,l,m}\tau_{k+1,l,m+1}\tau_{k,l,m+1}}+\frac{\tau_{k,l+2,m}}{\tau_{k,l+1,m}\tau_{k,l+1,m+1}\tau_{k+1,l+1,m}}\right)$$
$$+D(-1)^l\left(\frac{\tau_{k,l-1,m}}{\tau_{k,l,m}\tau_{k,l,m+1}\tau_{k+1,l,m}}+\frac{\tau_{k+1,l+2,m+1}}{\tau_{k+1,l+1,m+1}\tau_{k,l+1,m+1}\tau_{k+1,l+1,m}}\right),$$

$$\Gamma_3=C(-1)^m\left(\frac{\tau_{k+1,l+1,m-1}}{\tau_{k+1,l+1,m}\tau_{k,l+1,m}\tau_{k+1,l,m}}+\frac{\tau_{k,l,m+2}}{\tau_{k,l,m+1}\tau_{k,l+1,m+1}\tau_{k+1,l,m+1}}\right)$$
$$+D(-1)^m\left(\frac{\tau_{k,l,m-1}}{\tau_{k+1,l,m}\tau_{k,l+1,m}\tau_{k,l,m}}+\frac{\tau_{k+1,l+1,m+2}}{\tau_{k+1,l+1,m+1}\tau_{k,l+1,m+1}\tau_{k+1,l,m+1}}\right),$$

$$\Gamma_4=C(-1)^l\left(\frac{\tau_{k,l-1,m+1}}{\tau_{k+1,l,m+1}\tau_{k,l,m+1}\tau_{k,l,m}}+\frac{\tau_{k+1,l+2,m}}{\tau_{k+1,l+1,m}\tau_{k,l+1,m}\tau_{k+1,l+1,m+1}}\right)$$
$$+A(-1)^l\left(\frac{\tau_{k+1,l-1,m}}{\tau_{k+1,l,m}\tau_{k,l,m}\tau_{k+1,l,m+1}}+\frac{\tau_{k,l+2,m+1}}{\tau_{k,l+1,m+1}\tau_{k,l+1,m}\tau_{k+1,l+1,m+1}}\right),$$

$$\Gamma_5=A(-1)^m\left(\frac{\tau_{k+1,l,m-1}}{\tau_{k+1,l+1,m}\tau_{k+1,l,m}\tau_{k,l,m}}+\frac{\tau_{k,l+1,m+2}}{\tau_{k,l+1,m+1}\tau_{k,l,m+1}\tau_{k+1,l+1,m+1}}\right)$$
$$+B(-1)^m\left(\frac{\tau_{k,l+1,m-1}}{\tau_{k,l+1,m}\tau_{k,l,m}\tau_{k+1,l+1,m}}+\frac{\tau_{k+1,l,m+2}}{\tau_{k+1,l,m+1}\tau_{k,l,m+1}\tau_{k+1,l+1,m+1}}\right),$$

$$\Gamma_6=B(-1)^k\left(\frac{\tau_{k-1,l+1,m}}{\tau_{k,l+1,m+1}\tau_{k,l+1,m}\tau_{k,l,m}}+\frac{\tau_{k+2,l,m+1}}{\tau_{k+1,l,m+1}\tau_{k+1,l,m}\tau_{k+1,l+1,m+1}}\right)$$
$$+C(-1)^k\left(\frac{\tau_{k-1,l,m+1}}{\tau_{k,l,m+1}\tau_{k,l,m}\tau_{k,l+1,m+1}}+\frac{\tau_{k+2,l+1,m}}{\tau_{k+1,l+1,m}\tau_{k+1,l,m}\tau_{k+1,l+1,m+1}}\right).$$

From these characteristics we can obtain the associated conservation laws, using eq.(2). For example, P_1, P_2 and P_3 associated to Λ_1 are

$$\begin{aligned}
P_1 = & -A^2\frac{\tau_{k-1,l+1,m+1}\tau_{k+1,l,m}}{\tau_{k+1,l+1,m}\tau_{k+1,l,m+1}} - D^2\frac{\tau_{k-1,l,m}\tau_{k+1,l+1,m+1}}{\tau_{k,l+1,m}\tau_{k,l,m+1}} - AB\frac{\tau_{k-1,l+1,m}\tau_{k+1,l,m}}{\tau_{k,l,m}\tau_{k,l+1,m}} \\
& - BD\frac{\tau_{k-1,l+1,m}\tau_{k+1,l+1,m+1}}{\tau_{k,l+1,m+1}\tau_{k,l+1,m}} - AC\frac{\tau_{k-1,l,m+1}\tau_{k+1,l,m}}{\tau_{k,l,m}\tau_{k,l,m+1}} - CD\frac{\tau_{k-1,l,m+1}\tau_{k+1,l+1,m+1}}{\tau_{k,l+1,m+1}\tau_{k,l,m+1}} \\
& - AD\left(\frac{\tau_{k-1,l,m}\tau_{k,l+1,m+1}\tau_{k+1,l,m}}{\tau_{k,l,m}\tau_{k,l+1,m}\tau_{k,l,m+1}} + \frac{\tau_{k,l,m}\tau_{k-1,l+1,m+1}\tau_{k+1,l+1,m+1}}{\tau_{k,l+1,m+1}\tau_{k,l+1,m}\tau_{k,l,m+1}}\right), \\
P_2 = & AC\frac{\tau_{k+1,l,m}\tau_{k-1,l,m+1}}{\tau_{k,l,m+1}\tau_{k,l,m}} - BD\frac{\tau_{k-1,l,m}\tau_{k+1,l,m+1}}{\tau_{k,l,m+1}\tau_{k,l,m}}, \\
P_3 = & AB\frac{\tau_{k+1,l,m}\tau_{k-1,l+1,m}}{\tau_{k,l,m}\tau_{k,l+1,m}} - CD\frac{\tau_{k-1,l,m}\tau_{k+1,l+1,m}}{\tau_{k,l+1,m}\tau_{k,l,m}}.
\end{aligned}$$

Discrete AKP equation

The discrete AKP (Hirota-Miwa) equation [17, 16] is given by

$$A\tau_{k+1,l,m}\tau_{k,l+1,m+1} + B\tau_{k,l+1,m}\tau_{k+1,l,m+1} + C\tau_{k,l,m+1}\tau_{k+1,l+1,m} = 0. \tag{4}$$

Note that the discrete AKP equation is the special case $D = 0$ of the discrete BKP equation. The discrete AKP equation inherits the above 12 characteristics (with $D = 0$) from the discrete BKP equation and we have found the following 2 additional characteristics:

$$\Lambda_7 = \frac{\tau_{k,l,m}}{\tau_{k+1,l,m}\tau_{k,l+1,m}\tau_{k,l,m+1}} - \frac{\tau_{k+1,l+1,m+1}}{\tau_{k,l+1,m+1}\tau_{k+1,l,m+1}\tau_{k+1,l+1,m}},$$

$$\Gamma_7 = (-1)^{k+l+m}\left(\frac{\tau_{k,l,m}}{\tau_{k+1,l,m}\tau_{k,l+1,m}\tau_{k,l,m+1}} + \frac{\tau_{k+1,l+1,m+1}}{\tau_{k,l+1,m+1}\tau_{k+1,l,m+1}\tau_{k+1,l+1,m}}\right).$$

3. Reduction to Finite Dimensional Mappings and Construction of their Integrals

First example Consider the following 4th-order mapping:

$$Dx_{n+2}x_{n+1}^2x_n^2x_{n-1}^2x_{n-2} + Ax_{n+1}x_nx_{n-1} + B + \frac{C}{x_n} = 0, \tag{5}$$

Using the transformation of the dependent variable

$$x_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2},$$

we obtain a bilinear form

$$D\tau_{n+3}\tau_{n-3} + A\tau_{n+2}\tau_{n-2} + B\tau_{n+1}\tau_{n-1} + C\tau_n^2 = 0. \tag{6}$$

This bilinear form is obtained from the discrete BKP equation by applying the reduction $\tau_n \equiv \tau_{Z_1k+Z_2l+Z_3m}$ where $Z_1 = 1, Z_2 = 2, Z_3 = 3$. Using the characteristics of the discrete BKP equation, we obtain the following integrating factors for the discrete bilinear form (6):

$$\Lambda_1 = A\left(\frac{\tau_{n+1}}{\tau_n\tau_{n+2}\tau_{n-1}} - \frac{\tau_{n-1}}{\tau_{n-2}\tau_n\tau_{n+1}}\right) + D\left(\frac{\tau_{n-4}}{\tau_{n-3}\tau_{n-1}\tau_n} - \frac{\tau_{n+4}}{\tau_{n+3}\tau_n\tau_{n+1}}\right) = -\Lambda_6,$$

$$\Lambda_2 = B\left(\frac{\tau_{n-1}}{\tau_{n-2}\tau_{n+1}\tau_n} - \frac{\tau_{n+1}}{\tau_{n-1}\tau_{n+2}\tau_n}\right) + D\left(\frac{\tau_{n-5}}{\tau_{n-3}\tau_n\tau_{n-2}} - \frac{\tau_{n+5}}{\tau_{n+3}\tau_{n+2}\tau_n}\right) = -\Lambda_4,$$

$$\Lambda_3 = C\left(\frac{\tau_{n-3}}{\tau_n\tau_{n-1}\tau_{n-2}} - \frac{\tau_{n+3}}{\tau_n\tau_{n+2}\tau_{n+1}}\right) + D\left(\frac{\tau_{n-6}}{\tau_{n-2}\tau_{n-1}\tau_{n-3}} - \frac{\tau_{n+6}}{\tau_{n+3}\tau_{n+2}\tau_{n+1}}\right) = \frac{A}{D}\Lambda_4 + \frac{B}{D}\Lambda_6\,,$$
$$\Lambda_4 = C\left(\frac{\tau_{n-2}}{\tau_{n+1}\tau_n\tau_{n-3}} - \frac{\tau_{n+2}}{\tau_n\tau_{n-1}\tau_{n+3}}\right) + A\left(\frac{\tau_{n-4}}{\tau_{n-2}\tau_{n-3}\tau_{n+1}} - \frac{\tau_{n+4}}{\tau_{n+2}\tau_{n-1}\tau_{n+3}}\right)\,,$$
$$\Lambda_5 = A\left(\frac{\tau_{n-5}}{\tau_n\tau_{n-2}\tau_{n-3}} - \frac{\tau_{n+5}}{\tau_{n+2}\tau_n\tau_{n+3}}\right) + B\left(\frac{\tau_{n-4}}{\tau_{n-1}\tau_{n-3}\tau_n} - \frac{\tau_{n+4}}{\tau_{n+1}\tau_n\tau_{n+3}}\right) = -\frac{A}{D}\Lambda_4 - \frac{B}{D}\Lambda_6\,,$$
$$\Lambda_6 = B\left(\frac{\tau_{n-2}}{\tau_{n+2}\tau_{n-1}\tau_{n-3}} - \frac{\tau_{n+2}}{\tau_{n+1}\tau_{n-2}\tau_{n+3}}\right) + C\left(\frac{\tau_{n-1}}{\tau_n\tau_{n-3}\tau_{n+2}} - \frac{\tau_{n+1}}{\tau_n\tau_{n-2}\tau_{n+3}}\right)\,.$$

It is confirmed by using the bilinear form (6) that the integrating factors Λ_1, Λ_2, Λ_3 and Λ_5 lead to the indicated linear combinations of Λ_4 and Λ_6. Note that the two integrating factors Λ_4 and Λ_6 are independent and that the characteristics Γ_n of the discrete BKP equation do not reduce to integrating factors of (6). From the above integrating factors, we can make integrating factors in terms of the x-variable:

$$\tilde{\Lambda}_4 = \tau_{n-1}\tau_{n+1}\Lambda_4 = C\left(\frac{1}{x_{n-1}x_{n-2}} - \frac{1}{x_{n+2}x_{n+1}}\right) + A\left(x_{n-2}x_{n-3} - x_{n+3}x_{n+2}\right)\,,$$
$$\begin{aligned}\tilde{\Lambda}_6 &= \tau_{n-1}\tau_{n+1}\Lambda_6\\ &= B\left(\frac{1}{x_{n+1}x_nx_{n-1}x_{n-2}} - \frac{1}{x_{n+2}x_{n+1}x_nx_{n-1}}\right) + C\left(\frac{1}{x_{n+1}x_nx_{n-1}^2x_{n-2}} - \frac{1}{x_{n+2}x_{n+1}^2x_nx_{n-1}}\right)\,.\end{aligned}$$

We then obtain the following two integrals:

$$\begin{aligned}Q_4 &= CDx_{n+2}x_{n+1}^2x_n^2x_{n-1} - ADx_{n+3}x_{n+2}^2x_{n+1}^2x_n^2x_{n-1}^2x_{n-2}\\ &\quad - A^2(x_{n+3}x_{n+2}x_{n+1}x_nx_{n-1} + x_{n+2}x_{n+1}x_nx_{n-1}x_{n-2}) - C^2\left(\frac{1}{x_{n+2}x_{n+1}x_n} + \frac{1}{x_{n+1}x_nx_{n-1}}\right)\\ &\quad - BC\left(\frac{1}{x_{n+2}x_{n+1}} + \frac{1}{x_{n+1}x_n} + \frac{1}{x_nx_{n-1}}\right) - AB\sum_{j=0}^{4}x_{n+3-j}x_{n+2-j} - AC\sum_{j=0}^{3}\frac{x_{n+3-j}x_{n+2-j}}{x_{n-j}}\,,\end{aligned}$$
$$\begin{aligned}Q_6 &= BDx_{n+2}x_{n+1}x_nx_{n-1} + CD(x_{n+2}x_{n+1}x_n + x_{n+1}x_nx_{n-1}) - AB\left(\frac{1}{x_{n+2}} + \frac{1}{x_{n+1}} + \frac{1}{x_n} + \frac{1}{x_{n-1}}\right)\\ &\quad - AC\left(\frac{1}{x_{n+2}x_{n+1}} + \frac{1}{x_{n+1}x_n} + \frac{1}{x_nx_{n-1}}\right) - \frac{B^2}{x_{n+2}x_{n+1}x_nx_{n-1}} - \frac{C^2}{x_{n+2}x_{n+1}^2x_n^2x_{n-1}}\\ &\quad - BC\left(\frac{1}{x_{n+2}x_{n+1}x_n^2x_{n-1}} + \frac{1}{x_{n+2}x_{n+1}^2x_nx_{n-1}}\right)\,.\end{aligned}$$

It is not difficult to show that Q_4 and Q_6 are functionally independent.

In the special case $D = 0$, the fourth-order mapping (5) reduces to the second-order mapping

$$Ax_{n+1}x_nx_{n-1} + B + \frac{C}{x_n} = 0\,, \tag{7}$$

which is a special case of the QRT mapping [1]. Using the transformation of the dependent variable

$$x_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}\,,$$

we obtain a bilinear form

$$A\tau_{n+2}\tau_{n-2} + B\tau_{n+1}\tau_{n-1} + C\tau_n^2 = 0\,. \tag{8}$$

This bilinear form is obtained from the discrete AKP equation by applying the reduction $\tau_n \equiv \tau_{Z_1k+Z_2l+Z_3m}$ where $Z_1 = 1, Z_2 = 2, Z_3 = 3$. Using the characteristics of the discrete AKP equation, we obtain the following integrating factors for the discrete bilinear form (8):

$$\Lambda_1 = \left(\frac{\tau_{n+1}}{\tau_n\tau_{n+2}\tau_{n-1}} - \frac{\tau_{n-1}}{\tau_{n-2}\tau_n\tau_{n+1}}\right),$$

$$\Lambda_2 = \left(\frac{\tau_{n-1}}{\tau_{n-2}\tau_{n+1}\tau_n} - \frac{\tau_{n+1}}{\tau_{n-1}\tau_{n+2}\tau_n}\right) = -\Lambda_1\,,$$

$$\Lambda_3 = \left(\frac{\tau_{n-3}}{\tau_n\tau_{n-1}\tau_{n-2}} - \frac{\tau_{n+3}}{\tau_n\tau_{n+2}\tau_{n+1}}\right) = \frac{C}{A}\Lambda_1\,,$$

$$\Lambda_4 = C\left(\frac{\tau_{n-2}}{\tau_{n+1}\tau_n\tau_{n-3}} - \frac{\tau_{n+2}}{\tau_n\tau_{n-1}\tau_{n+3}}\right) + A\left(\frac{\tau_{n-4}}{\tau_{n-2}\tau_{n-3}\tau_{n+1}} - \frac{\tau_{n+4}}{\tau_{n+2}\tau_{n-1}\tau_{n+3}}\right) = B\Lambda_1\,,$$

$$\Lambda_5 = A\left(\frac{\tau_{n-5}}{\tau_n\tau_{n-2}\tau_{n-3}} - \frac{\tau_{n+5}}{\tau_{n+2}\tau_n\tau_{n+3}}\right) + B\left(\frac{\tau_{n-4}}{\tau_{n-1}\tau_{n-3}\tau_n} - \frac{\tau_{n+4}}{\tau_{n+1}\tau_n\tau_{n+3}}\right) = -\frac{C^2}{A}\Lambda_1\,,$$

$$\Lambda_6 = B\left(\frac{\tau_{n-2}}{\tau_{n+2}\tau_{n-1}\tau_{n-3}} - \frac{\tau_{n+2}}{\tau_{n+1}\tau_{n-2}\tau_{n+3}}\right) + C\left(\frac{\tau_{n-1}}{\tau_n\tau_{n-3}\tau_{n+2}} - \frac{\tau_{n+1}}{\tau_n\tau_{n-2}\tau_{n+3}}\right) = -A\Lambda_1\,,$$

$$\Lambda_7 = \frac{\tau_{n-3}}{\tau_{n-2}\tau_{n-1}\tau_n} - \frac{\tau_{n+3}}{\tau_{n+2}\tau_{n+1}\tau_n} = \frac{C}{A}\Lambda_1\,.$$

There is only one independent integrating factor, Λ_1. From Λ_1, we can make an integrating factors in terms of the x-variable:

$$\tilde{\Lambda}_1 = \tau_{n+1}\tau_{n-1}\Lambda_1 = \frac{1}{x_{n+1}} - \frac{1}{x_{n-1}}\,.$$

We then obtain the following integral:

$$Q_1 = -Ax_{n+1}x_n + B\left(\frac{1}{x_{n+1}} + \frac{1}{x_n}\right) + \frac{C}{x_{n+1}x_n}\,.$$

Second example

As a second example, let us discuss the following 6th-order mapping

$$Dx_{n+3}x_{n+2}^2x_{n+1}^3x_n^3x_{n-1}^3x_{n-2}^2x_{n-3} + Ax_{n+2}x_{n+1}^2x_n^2x_{n-1}^2x_{n-2} + Bx_{n+1}x_nx_{n-1} + C = 0\,. \tag{9}$$

Using the transformation of the dependent variable

$$x_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}\,,$$

we obtain a bilinear form

$$A\tau_{n+3}\tau_{n-3} + B\tau_{n+2}\tau_{n-2} + C\tau_{n+1}\tau_{n-1} + D\tau_{n+4}\tau_{n-4} = 0\,. \tag{10}$$

This bilinear form is obtained from the discrete BKP equation by applying the reduction $\tau_n \equiv \tau_{Z_1k+Z_2l+Z_3m}$ where $Z_1 = 1, Z_2 = 2, Z_3 = 5$ or $Z_1 = 1, Z_2 = 3, Z_3 = 4$. Using the characteristics of the discrete BKP equation, we obtain the following integrating factors for the discrete bilinear form (10):

$$\Lambda_1 = A\left(\frac{\tau_{n+2}}{\tau_{n+1}\tau_{n+3}\tau_{n-2}} - \frac{\tau_{n-2}}{\tau_{n-3}\tau_{n-1}\tau_{n+2}}\right) + D\left(\frac{\tau_{n-5}}{\tau_{n-4}\tau_{n-2}\tau_{n+1}} - \frac{\tau_{n+5}}{\tau_{n-1}\tau_{n+2}\tau_{n+4}}\right) = -\Lambda_6\,,$$
$$\Lambda_2 = B\left(\frac{\tau_{n}}{\tau_{n-3}\tau_{n+2}\tau_{n+1}} - \frac{\tau_{n}}{\tau_{n-2}\tau_{n+3}\tau_{n-1}}\right) + D\left(\frac{\tau_{n-6}}{\tau_{n-4}\tau_{n-3}\tau_{n+1}} - \frac{\tau_{n+6}}{\tau_{n-1}\tau_{n+3}\tau_{n+4}}\right) = \Lambda_4\,,$$
$$\Lambda_3 = C\left(\frac{\tau_{n-6}}{\tau_{n-1}\tau_{n-2}\tau_{n-3}} - \frac{\tau_{n+6}}{\tau_{n+1}\tau_{n+3}\tau_{n+2}}\right) + D\left(\frac{\tau_{n-9}}{\tau_{n-4}\tau_{n-3}\tau_{n-2}} - \frac{\tau_{n+9}}{\tau_{n+2}\tau_{n+3}\tau_{n+4}}\right) = -\Lambda_5\,,$$
$$\Lambda_4 = C\left(\frac{\tau_{n-1}}{\tau_{n+2}\tau_{n+1}\tau_{n-4}} - \frac{\tau_{n+1}}{\tau_{n-1}\tau_{n-2}\tau_{n+4}}\right) + A\left(\frac{\tau_{n-5}}{\tau_{n-3}\tau_{n-4}\tau_{n+2}} - \frac{\tau_{n+5}}{\tau_{n+3}\tau_{n-2}\tau_{n+4}}\right)\,,$$
$$\Lambda_5 = A\left(\frac{\tau_{n-8}}{\tau_{n-1}\tau_{n-3}\tau_{n-4}} - \frac{\tau_{n+8}}{\tau_{n+3}\tau_{n+1}\tau_{n+4}}\right) + B\left(\frac{\tau_{n-7}}{\tau_{n-2}\tau_{n-4}\tau_{n-1}} - \frac{\tau_{n+7}}{\tau_{n+2}\tau_{n+1}\tau_{n+4}}\right)\,,$$
$$\Lambda_6 = B\left(\frac{\tau_{n-3}}{\tau_{n+3}\tau_{n-2}\tau_{n-4}} - \frac{\tau_{n+3}}{\tau_{n+2}\tau_{n-3}\tau_{n+4}}\right) + C\left(\frac{\tau_{n}}{\tau_{n+1}\tau_{n-4}\tau_{n+3}} - \frac{\tau_{n}}{\tau_{n-1}\tau_{n-3}\tau_{n+4}}\right)\,.$$

It is confirmed by using the bilinear form (10) that integrating factors Λ_1, Λ_2 and Λ_3 lead to Λ_6, Λ_4 and Λ_5 respectively. Note that the 3 integrating factors Λ_4, Λ_5 and Λ_6 are independent and that the characteristics Γ_n of the discrete BKP equation do not reduce to integrating factors of (10). From the above integrating factors, we can make integrating factors in terms of the x-variable:

$$\begin{aligned}\tilde\Lambda_4 &= \tau_{n+1}\tau_{n-1}\Lambda_4\\ &= \frac{C}{x_{n+1}x_n^2x_{n-1}}\left(\frac{1}{x_{n-1}^2x_{n-2}^2x_{n-3}} - \frac{1}{x_{n+3}x_{n+2}^2x_{n+1}^2}\right) + \frac{A}{x_{n+1}x_nx_{n-1}}\left(x_{n-3}x_{n-4} - x_{n+4}x_{n+3}\right)\,,\end{aligned}$$
$$\begin{aligned}\tilde\Lambda_5 = \tau_{n+1}\tau_{n-1}\Lambda_5 &= Ax_n(x_{n-1}^2x_{n-2}^3x_{n-3}^4x_{n-4}^4x_{n-5}^3x_{n-6}^2x_{n-7} - x_{n+7}x_{n+6}^2x_{n+5}^3x_{n+4}^4x_{n+3}^4x_{n+2}^3x_{n+1}^2)\\ &\quad + Bx_n(x_{n-1}^2x_{n-2}^3x_{n-3}^3x_{n-4}^2x_{n-5}x_{n-6} - x_{n+6}x_{n+5}^2x_{n+4}^3x_{n+3}^3x_{n+2}^2x_{n+1})\,,\end{aligned}$$
$$\begin{aligned}\tilde\Lambda_6 = \tau_{n+1}\tau_{n-1}\Lambda_6 &= \frac{B}{x_{n+2}x_{n+1}^2x_n^2x_{n-1}^2x_{n-2}}\left(\frac{1}{x_{n-3}} - \frac{1}{x_{n+3}}\right)\\ &\quad + \frac{C}{x_{n+2}x_{n+1}^2x_n^3x_{n-1}^2x_{n-2}}\left(\frac{1}{x_{n-1}x_{n-2}x_{n-3}} - \frac{1}{x_{n+3}x_{n+2}x_{n+1}}\right)\,.\end{aligned}$$

We then obtain the following three integrals ‡:

$$\begin{aligned}Q_4 &= CD(x_{n+3}x_{n+2}^2x_{n+1}^2x_n + x_{n+2}x_{n+1}^2x_n^2x_{n-1} + x_{n+1}x_n^2x_{n-1}^2x_{n-2})\\ &\quad - ADx_{n+4}x_{n+3}^2x_{n+2}^2x_{n+1}^2x_n^2x_{n-1}^2x_{n-2}^2x_{n-3} - A^2x_{n+3}x_{n+2}x_{n+1}x_nx_{n-1}x_{n-2}(x_{n+4}+x_{n-3})\\ &\quad - BC\sum_{j=0}^{2}\frac{1}{x_{n+3-j}x_{n+2-j}^2x_{n+1-j}^2x_{n-j}} - AB\sum_{j=0}^{6}x_{n+4-j}x_{n+3-j}\\ &\quad - C^2\left(\frac{1}{x_{n+3}x_{n+2}^2x_{n+1}^3x_n^2x_{n-1}} + \frac{1}{x_{n+2}x_{n+1}^2x_n^3x_{n-1}^2x_{n-2}}\right) - AC\sum_{j=0}^{3}\frac{x_{n+4-j}x_{n+3-j}}{x_{n+1-j}x_{n-j}x_{n-1-j}}\,,\end{aligned}$$
$$Q_5 = AD\sum_{j=0}^{3}x_{n+7-j}x_{n+6-j}^2x_{n+5-j}^3x_{n+4-j}^4x_{n+3-j}^5x_{n+2-j}^5x_{n+1-j}^5x_{n-j}^4x_{n-1-j}^3x_{n-2-j}^2x_{n-3-j}$$

‡ Note that the map (9) can be used to eliminate e.g. x_{n+4} and x_{n-3} from Q_4. Similarly for Q_5 and Q_6.

$$+BD\sum_{j=0}^{2} x_{n+6-j}x_{n+5-j}^2x_{n+4-j}^3x_{n+3-j}^4x_{n+2-j}^4x_{n+1-j}^5x_{n-j}^4x_{n-1-j}^3x_{n-2-j}^2x_{n-3-j}$$
$$+A^2\sum_{j=0}^{4} x_{n+7-j}x_{n+6-j}^2x_{n+5-j}^3x_{n+4-j}^4x_{n+3-j}^4x_{n+2-j}^4x_{n+1-j}^4x_{n-j}^3x_{n-1-j}^2x_{n-2-j}$$
$$+AB\sum_{j=0}^{4} x_{n+6-j}x_{n+5-j}^2x_{n+4-j}^3x_{n+3-j}^3x_{n+2-j}^4x_{n+1-j}^4x_{n-j}^3x_{n-1-j}^2x_{n-2-j}$$
$$+AB\sum_{j=0}^{4} x_{n+7-j}x_{n+6-j}^2x_{n+5-j}^3x_{n+4-j}^4x_{n+3-j}^4x_{n+2-j}^3x_{n+1-j}^3x_{n-j}^2x_{n-1-j}$$
$$+B^2\sum_{j=0}^{5} x_{n+6-j}x_{n+5-j}^2x_{n+4-j}^3x_{n+3-j}^3x_{n+2-j}^3x_{n+1-j}^3x_{n-j}^2x_{n-1-j}$$
$$+BC\sum_{j=0}^{5} x_{n+6-j}x_{n+5-j}^2x_{n+4-j}^3x_{n+3-j}^3x_{n+2-j}^3x_{n+1-j}^2x_{n-j}$$
$$+AC\sum_{j=0}^{6} x_{n+7-j}x_{n+6-j}^2x_{n+5-j}^3x_{n+4-j}^4x_{n+3-j}^4x_{n+2-j}^3x_{n+1-j}^2x_{n-j}\,,$$

$$Q_6 = BDx_{n+3}x_{n+2}x_{n+1}x_nx_{n-1}x_{n-2} + CD\sum_{j=0}^{3} x_{n+3-j}x_{n+2-j}x_{n+1-j}$$
$$-AB\sum_{j=0}^{5}\frac{1}{x_{n+3-j}} - AC\sum_{j=0}^{2}\frac{1}{x_{n+3-j}x_{n+2-j}x_{n+1-j}x_{n-j}}$$
$$-\frac{B^2}{x_{n+3}x_{n+2}x_{n+1}x_nx_{n-1}x_{n-2}} - \frac{C^2}{x_{n+3}x_{n+2}^2x_{n+1}^3x_n^3x_{n-1}^2x_{n-2}}$$
$$-\frac{BC}{x_{n+3}x_{n+2}x_{n+1}^2x_n^2x_{n-1}^2x_{n-2}} - \frac{BC}{x_{n+3}x_{n+2}^2x_{n+1}^2x_n^2x_{n-1}x_{n-2}}\,.$$

Using e.g. Mathematica, one can show that Q_4, Q_5 and Q_6 are functionally independent.

We consider the 4th-order mapping

$$Ax_{n+2}x_{n+1}^2x_n^2x_{n-1}^2x_{n-2} + Bx_{n+1}x_nx_{n-1} + C = 0\,. \tag{11}$$

This mapping is the special case $D = 0$ of the 6th-order mapping (9). Applying

$$x_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}\,,$$

we have a bilinear form

$$A\tau_{n+3}\tau_{n-3} + B\tau_{n+2}\tau_{n-2} + C\tau_{n+1}\tau_{n-1} = 0\,. \tag{12}$$

This bilinear form is obtained from the discrete AKP equation by applying the reduction $\tau_n \equiv \tau_{Z_1k+Z_2l+Z_3m}$ where $Z_1 = 1, Z_2 = 2, Z_3 = 5$ or $Z_1 = 1, Z_2 = 3, Z_3 = 4$. Using the characteristics of the discrete AKP equation, we obtain the following integrating factors for the discrete bilinear form (12):

$$\Lambda_1 = \frac{\tau_{n+2}}{\tau_{n+1}\tau_{n+3}\tau_{n-2}} - \frac{\tau_{n-2}}{\tau_{n-3}\tau_{n-1}\tau_{n+2}}\,,$$

$$\Lambda_2 = \frac{\tau_n}{\tau_{n-3}\tau_{n+2}\tau_{n+1}} - \frac{\tau_n}{\tau_{n-2}\tau_{n+3}\tau_{n-1}}\,,$$
$$\Lambda_3 = \frac{\tau_{n-6}}{\tau_{n-1}\tau_{n-2}\tau_{n-3}} - \frac{\tau_{n+6}}{\tau_{n+1}\tau_{n+3}\tau_{n+2}} = -\frac{C^2}{A^2}\Lambda_1 - \frac{B^2C}{A^3}\Lambda_2\,,$$
$$\Lambda_4 = C\left(\frac{\tau_{n-1}}{\tau_{n+2}\tau_{n+1}\tau_{n-4}} - \frac{\tau_{n+1}}{\tau_{n-1}\tau_{n-2}\tau_{n+4}}\right) + A\left(\frac{\tau_{n-5}}{\tau_{n-3}\tau_{n-4}\tau_{n+2}} - \frac{\tau_{n+5}}{\tau_{n+3}\tau_{n-2}\tau_{n+4}}\right) = -B\Lambda_2\,,$$
$$\Lambda_5 = A\left(\frac{\tau_{n-8}}{\tau_{n-1}\tau_{n-3}\tau_{n-4}} - \frac{\tau_{n+8}}{\tau_{n+3}\tau_{n+1}\tau_{n+4}}\right) + B\left(\frac{\tau_{n-7}}{\tau_{n-2}\tau_{n-4}\tau_{n-1}} - \frac{\tau_{n+7}}{\tau_{n+2}\tau_{n+1}\tau_{n+4}}\right)$$
$$= \frac{C^3}{A^2}\Lambda_1 + \frac{B^2C^2}{A^3}\Lambda_2\,,$$
$$\Lambda_6 = B\left(\frac{\tau_{n-3}}{\tau_{n+3}\tau_{n-2}\tau_{n-4}} - \frac{\tau_{n+3}}{\tau_{n+2}\tau_{n-3}\tau_{n+4}}\right) + C\left(\frac{\tau_n}{\tau_{n+1}\tau_{n-4}\tau_{n+3}} - \frac{\tau_n}{\tau_{n-1}\tau_{n-3}\tau_{n+4}}\right) = -A\Lambda_1\,,$$
$$\Lambda_7 = \frac{\tau_{n-4}}{\tau_{n-3}\tau_{n-2}\tau_{n+1}} - \frac{\tau_{n+4}}{\tau_{n+3}\tau_{n+2}\tau_{n-1}} = -\frac{C}{A}\Lambda_2\,.$$

It is confirmed by using the bilinear form (12) that the two integrating factors Λ_1 and Λ_2 are independent and the characteristics Γ_n of the discrete AKP equation do not reduce to integrating factors of (12). From the above integrating factors, we can make integrating factors in terms of the x-variable:

$$\tilde{\Lambda}_1 = \tau_{n+1}\tau_{n-1}\Lambda_1 = \frac{1}{x_{n+1}x_nx_{n-1}}\left(\frac{1}{x_{n+2}} - \frac{1}{x_{n-2}}\right),$$
$$\tilde{\Lambda}_2 = \tau_{n+1}\tau_{n-1}\Lambda_2 = \frac{1}{x_{n+1}x_n^2x_{n-1}}\left(\frac{1}{x_{n-1}x_{n-2}} - \frac{1}{x_{n+2}x_{n+1}}\right).$$

We then obtain the following two integrals:

$$Q_1 = -Ax_{n+2}x_{n+1}x_nx_{n-1} + B\left(\frac{1}{x_{n+2}} + \frac{1}{x_{n+1}} + \frac{1}{x_n} + \frac{1}{x_{n-1}}\right) + \frac{C}{x_{n+2}x_{n+1}x_nx_{n-1}}\,,$$
$$Q_2 = A(x_{n+2}x_{n+1} + x_{n+1}x_n + x_nx_{n-1}) - B\left(\frac{1}{x_{n+2}x_{n+1}x_n} + \frac{1}{x_{n+1}x_nx_{n-1}}\right) - \frac{C}{x_{n+2}x_{n+1}^2x_n^2x_{n-1}}\,.$$

We note that the map (11) preserves the symplectic structure

$$\begin{pmatrix} 0 & x_{n-2}x_{n-1} & -x_{n-2}x_n & x_{n-2}x_{n+1} \\ -x_{n-1}x_{n-2} & 0 & x_{n-1}x_n & -x_{n-1}x_{n+1} \\ x_nx_{n-2} & -x_nx_{n-1} & 0 & x_nx_{n+1} \\ -x_{n+1}x_{n-2} & x_{n+1}x_{n-1} & -x_{n+1}x_n & 0 \end{pmatrix},$$

and that the two integrals Q_1 and Q_2 are in involution w.r.t. this structure, giving an independent confirmation of the integrability of the map (11).

4. Conclusions

We have studied a class of integrable mappings which have bilinear forms. We have proposed a method to construct integrals of these higher-order integrable maps. The key to the construction are the conservation laws of the discrete bilinear forms of the associated AKP and BKP equations. Note that, generalizing the examples in this Letter, we can construct a family of higher-order mappings from the discrete AKP and BKP equations, by applying the reduction $\tau_n \equiv \tau_{Z_1k+Z_2l+Z_3m}$ for any Z_1, Z_2 and Z_3.

We hope to discuss details of our methods and higher-order mappings in the class given here in a forthcoming paper.

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