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# Extensions of truncated discrete valuation rings

Dedicated to Professor Jean-Pierre Serre  
on the Occasion of His Eightieth Birthday

Toshiro Hiranouchi<sup>1</sup> and Yuichiro Taguchi

## Abstract

An equivalence is established between the category of at most  $a$ -ramified finite separable extensions of a complete discrete valuation field  $K$  and the category of at most  $a$ -ramified finite extensions of the “length- $a$  truncation”  $\mathcal{O}_K/\mathfrak{m}_K^a$  of the integer ring of  $K$ .

## 1 Introduction

Let  $K$  be a complete discrete valuation field (abbr. cdvf in the following),  $\mathcal{O}_K$  its valuation ring, and  $\mathfrak{m}_K$  its maximal ideal. Let  $a$  be an integer  $\geq 1$ . In this paper, we prove that the category  $\mathcal{FE}_K^{\leq a}$  of finite étale  $K$ -algebras with ramification “bounded by  $a$ ” (cf. Def. 3.1) depends only on  $\mathcal{O}_K/\mathfrak{m}_K^a$ . More precisely, let  $m$  be any rational number such that  $0 < m \leq a$  and put  $A = \mathcal{O}_K/\mathfrak{m}_K^a$ . We give an equivalence of  $\mathcal{FE}_K^{\leq m}$  with a category  $\mathcal{FFP}_A^{\leq m}$  of finite flat principal  $A$ -algebras<sup>2</sup> with ramification “bounded by  $m$ ” (cf. Def. 3.2). The morphisms in  $\mathcal{FFP}_A^{\leq m}$  are defined (cf. Def. 3.3) by using Hattori’s functor ([6]); they are the usual  $A$ -algebra homomorphisms modulo a certain equivalence relation.

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<sup>2</sup>We mean by a *principal*  $A$ -algebra an  $A$ -algebra of which every ideal is generated by one element. All algebras in this paper are commutative.

For each object  $L$  in  $\mathcal{FE}_K^{\leq m}$ , let  $\mathcal{O}_L$  be the integral closure of  $\mathcal{O}_K$  in  $L$ . Then the quotient ring  $T(L) := \mathcal{O}_L/\mathfrak{m}_K^a \mathcal{O}_L$  is an object of  $\mathcal{FFP}_A^{\leq m}$  (Cor. 3.5). This correspondence  $L \mapsto T(L)$  is functorial, and thus we obtain a functor

$$T : \mathcal{FE}_K^{\leq m} \rightarrow \mathcal{FFP}_A^{\leq m}.$$

Our main result in this paper is:

**Theorem 1.1.** *The functor  $T$  is an equivalence of categories.*

*Remarks.* (i) The case of  $a = 1$  in the Theorem is well-known (cf. [12], Chap. III, §5). Indeed, if  $m \leq 1$ , the objects of  $\mathcal{FE}_K^{\leq m}$  are direct products of finite unramified extensions of  $K$ , and the Theorem implies that the objects of  $\mathcal{FFP}_A^{\leq m}$  are étale over  $A$ . Thus our main interest is in the case  $a > 1$ .

(ii) Let  $G_K = \text{Gal}(\bar{K}/K)$  denote the absolute Galois group of  $K$ , and  $G_K^a$  its  $a$ th ramification subgroup defined by Abbes and Saito ([2], [3]). The category  $\mathcal{FE}_K^{\leq m}$  is, and hence  $\mathcal{FFP}_A^{\leq m}$  is also, a Galois category whose fundamental group is  $G_K/G_K^m$  by the very definition of the ramification filtration (cf. Sect. 3). Note that  $\mathcal{FE}_K^{\leq m}$  is equivalent also to the category of coverings of  $\text{Spec}(\mathcal{O}_K)$  with ramification bounded by  $\mathfrak{m}_K^m$  ([7], Def. 2.3); in the terminology of *op. cit.*, we have  $\pi_1(\text{Spec}(\mathcal{O}_K), \mathfrak{m}_K^m) = G_K/G_K^m$ .

A finite étale  $K$ -algebra is the direct product of a finite number of finite separable extension fields of  $K$ . Similarly, a finite flat principal  $A$ -algebra is the direct product of a finite number of local objects (cf. [9], Th. 1.1, Th. 1.2). Since the boundedness of ramification of direct products of  $K$ - and  $A$ -algebras may be considered componentwise, the above Theorem is equivalent with the following Corollary, in which  $\text{FE}_K^{\leq m}$  (resp.  $\text{FFP}_A^{\leq m}$ ) denotes the full subcategory of  $\mathcal{FE}_K^{\leq m}$  (resp.  $\mathcal{FFP}_A^{\leq m}$ ) consisting of local rings.

**Corollary 1.2.** *The functor  $T$  induces an equivalence  $\text{FE}_K^{\leq m} \simeq \text{FFP}_A^{\leq m}$ .*

This extends a theorem of Deligne ([4], Th. 2.8) to the imperfect residue field case, except that our construction of the category  $\mathcal{FFP}_A^{\leq m}$  for  $A = \mathcal{O}_K/\mathfrak{m}_K^a$  depends on the cdvf  $K$  and hence our result is somewhat weaker than the “true” generalization of Deligne’s theorem. We expect, however, the category  $\mathcal{FFP}_A^{\leq m}$  depends only on the isomorphism class of  $A$  as a ring (such a ring as  $A = \mathcal{O}_K/\mathfrak{m}_K^a$  is called a *truncated discrete valuation ring*; see Section 2). If this is the case, we may define the Galois group  $G_A$  of  $A$  to be  $G_K/G_K^a$  (or equivalently, to be the fundamental group of the Galois category

$\mathcal{FFP}_A^{\leq a}$ ) together with the ramification subgroups  $G_A^m := G_K^m/G_K^a$ , where  $K$  is any cdvf such that  $A \simeq \mathcal{O}_K/\mathfrak{m}_K^a$ . The filtered group  $G_A$  should depend (up to inner automorphisms) only on the isomorphism class of  $A$  as a ring. It is natural to ask the converse:

*Question.* If  $A$  and  $A'$  are two truncated discrete valuation rings of length  $a$  and if there is an isomorphism  $\gamma : G_A \rightarrow G_{A'}$  of groups such that  $\gamma(G_A^m) = G_{A'}^m$  for all  $m \leq a$ , then is it true that  $A \simeq A'$  as a ring?

This problem is a version of the Grothendieck conjecture in anabelian geometry. It will certainly be necessary to assume that the residue fields of  $A$  and  $A'$  are either finite or of some “anabelian” nature. For the case of local fields (or, the case of “ $a = \infty$ ” and finite residue fields), see [10] and [1].

In Section 2, we study basic properties of truncated discrete valuation rings. After recalling some basics of the ramification theory of Abbes-Saito ([2], [3]) and Hattori ([6]), we construct the category  $\mathcal{FFP}_A^{\leq m}$  and prove the Theorem in Section 3.

Throughout this paper,  $K$  is a complete discrete valuation field with residual characteristic  $p > 0$ . We denote by  $\mathcal{O}_K$  the valuation ring of  $K$ ,  $\mathfrak{m}_K$  the maximal ideal of  $\mathcal{O}_K$ ,  $\pi_K$  a uniformizing element of  $K$ , and  $\bar{K}$  a fixed separable closure of  $K$ . For any étale  $K$ -algebra  $L$ , we denote by  $\mathcal{O}_L$  the integral closure of  $\mathcal{O}_K$  in  $L$ . For  $A$ -algebras  $B$  and  $B'$ , we denote by  $\text{Hom}_A(B, B')$  the set of  $A$ -algebra homomorphisms  $B \rightarrow B'$ . We use the following abbreviations:

cdvf := complete discrete valuation field,  
 cdvr := complete discrete valuation ring,  
 tdvr := truncated discrete valuation ring.

It is our pleasure to dedicate this paper to Professor Jean-Pierre Serre, whose mathematical influence on us has been enormous. In particular, the Book *Corps Locaux* has ever been our main source of inspiration in ramification theory.

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## 2 Truncated discrete valuation rings

A *tdvr* is an Artinian local ring whose maximal ideal is generated by one element. The *length* of a tdvr  $A$  is the length of  $A$  as an  $A$ -module. It is known that a tdvr  $A$  is principal, and any ideal is of the form  $\mathfrak{m}_A^i$  for some  $i \geq 0$  if  $\mathfrak{m}_A$  is the maximal ideal of  $A$ . Any generator  $\pi_A$  of  $\mathfrak{m}_A$  is said to be a *uniformizer* of  $A$ . Any non-zero element  $x$  of  $A$  can be written as  $x = u\pi_A^i$  with  $u \in A^\times$ ,  $\pi_A$  a uniformizer of  $A$ , and  $0 \leq i < \text{length}(A)$  (with the convention  $0^0 = 1$  if  $\text{length}(A) = 1$ ). If  $\text{length}(A) > 1$  (resp.  $\text{length}(A) = 1$ ), we mean by an *extension*  $B/A$  of tdvr's a local ring homomorphism  $A \rightarrow B$  of tdvr's via which  $B$  is flat over  $A$  (resp. an extension  $B/A$  of fields); thus we refrain from calling a homomorphism such as  $A \hookrightarrow A[t]/(t^a)$  an extension if  $A$  is a field. An extension  $B/A$  is said to be *finite* if  $B$  is finite as an  $A$ -module. If  $a > 1$ , an  $A$ -algebra is a finite extension of  $A$  if and only if it is finite, flat, principal and local. In general, the objects of the category  $\text{FFP}_A^{\leq m}$  are finite extensions of the tdvr  $A$ . The *ramification index*  $e_{B/A}$  of a homomorphism  $f : A \rightarrow B$  of tdvr's is defined to be the integer  $e$  such that  $f(\mathfrak{m}_A)B = \mathfrak{m}_B^e$  (with the convention  $e_{B/A} = 1$  if  $\text{length}(A) = 1$ ). Note that the homomorphism  $f$  is an extension of tdvr's if and only if one has the equality  $\text{length}(B) = e_{B/A} \text{length}(A)$  (cf. [4], Sect. 1.4 and [8], Exer. 22.1).

**Lemma 2.1.** *Let  $B$  and  $C$  be extensions of  $A$ . Then any  $A$ -algebra homomorphism  $f : B \rightarrow C$  is an extension.*

*Proof.* We have to show that  $\text{length}(C) = e_{C/B} \text{length}(B)$ . We may assume that  $\text{length}(A) > 1$ . Let  $\mathfrak{m}_A, \mathfrak{m}_B$  and  $\mathfrak{m}_C$  be respectively the maximal ideals of  $A, B$  and  $C$ . By the definition of ramification index, we have  $\mathfrak{m}_A B = \mathfrak{m}_B^{e_{B/A}}$ ,  $\mathfrak{m}_A C = \mathfrak{m}_C^{e_{C/A}}$ , and  $f(\mathfrak{m}_B)C = \mathfrak{m}_C^{e_{C/B}}$ . The equality  $\mathfrak{m}_C^{e_{C/A}} = f(\mathfrak{m}_B^{e_{B/A}})C (= \text{the ideal generated by } \mathfrak{m}_A)$  implies that  $e_{C/A} = e_{C/B} e_{B/A}$ . Since  $B$  and  $C$  are extensions of  $A$ , we have  $\text{length}(C) = e_{C/A} \text{length}(A) = e_{C/B} e_{B/A} \text{length}(A) = e_{C/B} \text{length}(B)$ .  $\square$

If  $K$  is a cdvf, then  $\mathcal{O}_K/\mathfrak{m}_K^a$  is a tdvr for any integer  $a \geq 1$ . If  $L/K$  is a finite extension of cdvf's, then  $B = \mathcal{O}_L/\mathfrak{m}_K^a \mathcal{O}_L$  is a finite extension of  $A = \mathcal{O}_K/\mathfrak{m}_K^a$ . Conversely, it is known that any tdvr is a quotient of a cdvf ([9], Th. 3.3). More precisely, we have:

**Proposition 2.2.** (i) *Let  $A$  be a tdvr with residue field  $k$  of characteristic  $p \geq 0$ , and let  $a$  be the length of  $A$ . Then there exists a cdvf  $\mathcal{O}$  such that  $A$*

is isomorphic to  $\mathcal{O}/\mathfrak{m}^a$ , where  $\mathfrak{m}$  is the maximal ideal of  $\mathcal{O}$ . If  $pA = 0$ , then this  $\mathcal{O}$  can be taken to be the power series ring  $k[[\pi]]$ ; if  $pA \neq 0$ , then  $\mathcal{O}$  as above must be finite over a Cohen  $p$ -ring ([5], 0<sub>IV</sub>, 19.8) with residue field  $k$ . (If  $pA = 0$  and  $p \neq 0$ , then both types of  $\mathcal{O}$  are possible.)

(ii) Let  $K$  be a cdvf and let  $A = \mathcal{O}_K/\mathfrak{m}_K^a$  with  $a \geq 1$ . For any finite extension  $B/A$  of tdiv's, there exist a finite separable extension  $L/K$  and an isomorphism  $\psi : \mathcal{O}_L/\mathfrak{m}_K^a \mathcal{O}_L \rightarrow B$  such that the diagram

$$(1) \quad \begin{array}{ccc} \mathcal{O}_L/\mathfrak{m}_K^a \mathcal{O}_L & \xrightarrow{\psi} & B \\ \uparrow & & \uparrow \\ \mathcal{O}_K/\mathfrak{m}_K^a & \xlongequal{\quad} & A \end{array}$$

is commutative, where the left vertical arrow is the one induced by  $\mathcal{O}_K \hookrightarrow \mathcal{O}_L$ .

*Proof.* (i) Let  $W$  be a Cohen  $p$ -ring with residue field  $k$ . The reduction map  $W \rightarrow k$  lifts by the formal smoothness of  $W$  to a local ring homomorphism  $W \rightarrow A$  ([5], 0<sub>IV</sub>, 19.8.6).

If  $pA = 0$ , the map  $W \rightarrow A$  factors through the residue field  $k$ , which makes  $A$  a  $k$ -algebra. Then there exists a surjective  $A$ -algebra homomorphism  $k[[\pi]] \rightarrow A$  which maps  $\pi$  to  $\pi_A$ , where  $\pi_A$  is a uniformizer of  $A$ . Hence  $A$  is isomorphic to  $k[[\pi]]/(\pi^a)$  (cf. [9], Th. 3.1).

In the general case, we can write  $A$  as a quotient of the polynomial ring  $W[X]$  by sending  $X$  to  $\pi_A$ . Then we obtain a surjection onto  $A$  from a cdvf  $\mathcal{O}$  which is finite over  $W$  by the same procedure as in the proof of (ii) below.

(ii) Since  $B$  is finite over  $A = \mathcal{O}_K/\mathfrak{m}_K^a$ , there exists a surjective  $\mathcal{O}_K$ -algebra homomorphism  $\phi : R \rightarrow B$  from a polynomial ring  $R = \mathcal{O}_K[X_1, \dots, X_n]$  onto  $B$ . Let  $\mathfrak{m} = \phi^{-1}(\mathfrak{m}_B)$  and  $R_{\mathfrak{m}}$  the localization of  $R$  at the maximal ideal  $\mathfrak{m}$ . Then  $R_{\mathfrak{m}}$  is a regular local ring of Krull dimension  $n+1$  ([5], 0<sub>IV</sub>, 17.3.7), and  $\phi$  extends to a surjective  $\mathcal{O}_K$ -algebra homomorphism  $\varphi : R_{\mathfrak{m}} \rightarrow B$ . By abuse of notation, we denote also by  $\mathfrak{m}$  the maximal ideal of  $R_{\mathfrak{m}}$ . Put  $\mathfrak{n} = \text{Ker}(\varphi)$ . We identify the residue field  $k'$  of  $R_{\mathfrak{m}}$  with that of  $B$  via  $\varphi$ . Since  $\varphi(\mathfrak{m}^2) = \mathfrak{m}_B^2$ , the map  $\varphi$  induces a surjective  $k'$ -linear map  $\mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{m}_B/\mathfrak{m}_B^2$  and its kernel is  $(\mathfrak{n} + \mathfrak{m}^2)/\mathfrak{m}^2 \simeq \mathfrak{n}/(\mathfrak{n} \cap \mathfrak{m}^2)$ . Thus we have an exact sequence

$$0 \rightarrow \mathfrak{n}/(\mathfrak{n} \cap \mathfrak{m}^2) \rightarrow \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{m}_B/\mathfrak{m}_B^2 \rightarrow 0.$$

Assume  $a \geq 2$ , as the case  $a = 1$  can be treated similarly and more easily. Then  $\dim_{k'}(\mathfrak{m}_B/\mathfrak{m}_B^2) = 1$  and  $\dim_{k'}(\mathfrak{n}/(\mathfrak{n} \cap \mathfrak{m}^2)) = n$ . Choose a regular

system of parameters  $(w, f_1, \dots, f_n)$  of  $R_{\mathfrak{m}}$  such that  $\varphi(w)$  gives a basis of  $\mathfrak{m}_B/\mathfrak{m}_B^2$  and  $f_1, \dots, f_n \in \mathfrak{n}$  give a basis of  $\mathfrak{n}/(\mathfrak{n} \cap \mathfrak{m}^2)$ . Let  $\mathfrak{p}$  be the ideal of  $R_{\mathfrak{m}}$  generated by  $f_1, \dots, f_n$ . Then by [5], 0IV, Prop. 17.1.7, the quotient ring  $\mathcal{O} = R_{\mathfrak{m}}/\mathfrak{p}$  is a regular local ring of dimension 1 and hence a discrete valuation ring. It contains  $\mathcal{O}_K$  since  $\varphi$  maps  $\pi_K$  to a non-zero non-unit in  $B$ , and is finite over  $\mathcal{O}_K$ . Hence it is a cdvr. Since  $\mathfrak{n} \supset \mathfrak{p}$ , the map  $\varphi$  factors through  $\mathcal{O}$ . Thus we see the diagram (1) commutes (with  $\mathcal{O}$  in place of  $\mathcal{O}_L$ ). Since  $B$  is flat over  $A$ , the induced homomorphism  $\psi$  is bijective.

To make the fraction field  $L$  of  $\mathcal{O}$  separable over  $K$ , we “deform” the prime ideal  $\mathfrak{p}$  if necessary. By multiplying the  $f_i$  with some  $u \in R \setminus \mathfrak{m}$ , we may assume that all  $f_i$  are in the polynomial ring  $R$ . Note that the composite map  $R \hookrightarrow R_{\mathfrak{m}} \rightarrow R_{\mathfrak{m}}/\mathfrak{p} = \mathcal{O}$  is surjective by Nakayama’s lemma, since its image generates  $B = \mathcal{O}/\mathfrak{m}_K^a \mathcal{O}$ . Let  $\mathfrak{q}$  be its kernel, so that  $\mathcal{O} = R/\mathfrak{q}$ . We have  $\mathfrak{q}R_{\mathfrak{m}} = \mathfrak{p}$ , *i.e.*,  $\mathfrak{q}$  is generated by  $f_1, \dots, f_n$  locally at  $\mathfrak{m}$ . By the Jacobian criterion ([11], V, Sect. 2, Th. 5), the  $K$ -algebra  $L$  is separable (*i.e.*, the  $\mathcal{O}_K$ -algebra  $\mathcal{O}$  is étale at the generic point of  $\text{Spec}(\mathcal{O})$ ) if and only if the Jacobian  $\det \left( \frac{\partial f_i}{\partial X_j} \right)_{1 \leq i, j \leq n} \not\equiv 0 \pmod{\mathfrak{q}}$ . Let  $g_i := f_i + xX_i$  with  $x \in \mathfrak{m}_K^a$ . Then, since  $g_i \in \mathfrak{n}$  and  $g_i \equiv f_i \pmod{\mathfrak{n} \cap \mathfrak{m}^2}$ , the ideal  $\mathfrak{p}' = (g_1, \dots, g_n)$  of  $R_{\mathfrak{m}}$  has similar properties as  $\mathfrak{p}$  so that the quotient ring  $\mathcal{O}' := R_{\mathfrak{m}}/\mathfrak{p}'$  is a cdvr which contains  $\mathcal{O}_K$  and surjects onto  $B$ . Moreover, if  $J := \left( \frac{\partial f_i}{\partial X_j} \right)_{1 \leq i, j \leq n}$ , we have

$$\det \left( \frac{\partial g_i}{\partial X_j} \right)_{1 \leq i, j \leq n} = \det(xI_n + J) = x^n + \text{Tr}(J)x^{n-1} + \dots + \det(J).$$

Considering this modulo  $\mathfrak{q}$  and noticing that  $\mathcal{O}_K \subset \mathcal{O} = R/\mathfrak{q}$ , we find an  $x \in \mathfrak{m}_K^a$  such that  $\det \left( \frac{\partial g_i}{\partial X_j} \right) \not\equiv 0 \pmod{\mathfrak{q}}$ . Then the fraction field of  $\mathcal{O}'$  is separable over  $K$ .  $\square$

### 3 Ramification

Let  $G_K$  be the absolute Galois group of  $K$ . A. Abbes and T. Saito ([2], [3]) defined a decreasing filtration  $(G_K^m)_{m \geq 0}$  by closed normal subgroups  $G_K^m$  of  $G_K$  indexed with rational numbers  $m \geq 0$ , in such a way that  $\cap_{m \geq 0} G_K^m = 1$ ,  $G_K^0 = G_K$  and  $G_K^1$  is the inertia subgroup of  $G_K$ . The filtration coincides

with the classical upper numbering ramification filtration shifted by one if the residue field of  $K$  is perfect (see [12], Chap. IV, §3, for the classical case). It is defined by using certain functors  $F$  and  $F^m$  from the category  $\mathcal{FE}_K$  of finite étale  $K$ -algebras to the category  $\mathcal{S}_K$  of finite  $G_K$ -sets. We recall here the definition of  $F$  and  $F^m$  assuming for simplicity that  $m$  is a positive integer. Let  $L$  be a finite étale  $K$ -algebra, and let  $\mathcal{O}_L$  be the integral closure of  $\mathcal{O}_K$  in  $L$ . We define  $F(L) := \text{Hom}_K(L, \bar{K}) = \text{Hom}_{\mathcal{O}_K}(\mathcal{O}_L, \mathcal{O}_{\bar{K}})$ . The functor  $F$  gives an anti-equivalence of  $\mathcal{FE}_K$  with  $\mathcal{S}_K$ , thereby making  $\mathcal{FE}_K$  a Galois category. To define  $F^m$ , we proceed as follows: An *embedding* of  $\mathcal{O}_L$  is a pair  $(\mathbb{B}, \mathbb{B} \rightarrow \mathcal{O}_L)$  consisting of an  $\mathcal{O}_K$ -algebra  $\mathbb{B}$  which is formally of finite type and formally smooth over  $\mathcal{O}_K$  and a surjection  $\mathbb{B} \rightarrow \mathcal{O}_L$  of  $\mathcal{O}_K$ -algebras which induces an isomorphism  $\mathbb{B}/\mathfrak{m}_{\mathbb{B}} \rightarrow \mathcal{O}_L/\mathfrak{m}_L$ , where  $\mathfrak{m}_{\mathbb{B}}$  and  $\mathfrak{m}_L$  are respectively the radicals of  $\mathbb{B}$  and  $\mathcal{O}_L$  (cf. [3], Def. 1.1). Let  $I$  be the kernel of the surjection  $\mathbb{B} \rightarrow \mathcal{O}_L$ . Define an affinoid algebra  $\mathbf{B}^m$  over  $K$  by  $\mathbf{B}^m = \mathbb{B}[I/\pi_K^m]^\wedge \otimes_{\mathcal{O}_K} K$ , where  $\wedge$  means the  $\pi_K$ -adic completion. Let  $X^m(\mathbb{B} \rightarrow \mathcal{O}_L)$  be the affinoid variety  $\text{Sp}(\mathbf{B}^m)$  associated with  $\mathbf{B}^m$ . For any affinoid variety  $X$  over  $K$ , let  $\pi_0(X_{\bar{K}})$  denote the set  $\varprojlim_{K'} \pi_0(X \otimes_K K')$  of geometric connected components, where  $K'$  runs through the finite separable extensions of  $K$ . Then we define the functor  $F^m$  by

$$F^m(L) := \varprojlim_{(\mathbb{B} \rightarrow \mathcal{O}_L)} \pi_0(X^m(\mathbb{B} \rightarrow \mathcal{O}_L)_{\bar{K}}),$$

where  $(\mathbb{B} \rightarrow \mathcal{O}_L)$  runs through the category of embeddings of  $\mathcal{O}_L$ . The projective system in the right-hand side is constant. The finite set  $F(L)$  can be identified with a subset of  $X^m(\mathbb{B} \rightarrow \mathcal{O}_L)_{\bar{K}}$ , and this causes a natural surjective map  $F(L) \rightarrow F^m(L)$ . The  $m$ th ramification subgroup  $G_K^m$  is characterized by the property that  $F(L)/G_K^m = F^m(L)$  for all  $L$ .

**Definition 3.1** ([2], Def. 6.3). Let  $L$  be a finite étale  $K$ -algebra. We say that the *ramification of  $L$  is bounded by  $m$*  if  $F(L) \rightarrow F^m(L)$  is bijective.

Thus the category  $\mathcal{FE}_K^{\leq m}$  of finite étale  $K$ -algebras with ramification bounded by  $m$  forms a Galois full-subcategory of  $\mathcal{FE}_K$  whose fundamental group is  $G_K/G_K^m$  ([2], Prop. 2.1) as noted in the Introduction. Note that the above definition of “ramification bounded by  $m$ ” coincides with Deligne’s one in [4] when  $L$  is a field and  $\mathcal{O}_L$  is monogenic over  $\mathcal{O}_K$  (cf. [2], Prop. 6.7).

Let  $a$  be an integer  $\geq 1$ , and put  $A = \mathcal{O}_K/\mathfrak{m}_K^a$ . For each rational number  $0 < m \leq a$ , Hattori ([6]) defined another functor  $\mathcal{F}^m$  from the category of

finite flat  $A$ -algebras to the category  $\mathcal{S}_K$  of finite  $G_K$ -sets. We next recall the definition of  $\mathcal{F}^m$  assuming for simplicity that  $m$  is a positive integer. Let  $B$  be a finite flat  $A$ -algebra. An *embedding* of  $B$  is a pair  $(\mathbb{B}, \mathbb{B} \rightarrow B)$  consisting of an  $\mathcal{O}_K$ -algebra  $\mathbb{B}$  which is formally of finite type and formally smooth over  $\mathcal{O}_K$  and a surjection  $\mathbb{B} \rightarrow B$  of  $\mathcal{O}_K$ -algebras which induces an isomorphism  $\mathbb{B}/\mathfrak{m}_{\mathbb{B}} \rightarrow B/\mathfrak{m}_B$ , where  $\mathfrak{m}_{\mathbb{B}}$  and  $\mathfrak{m}_B$  are respectively the radicals of  $\mathbb{B}$  and  $B$ . Let  $\mathcal{I}$  be the kernel of the surjection  $\mathbb{B} \rightarrow B$ . Define an affinoid algebra  $\mathcal{B}^m$  over  $K$  by  $\mathcal{B}^m = \mathbb{B}[\mathcal{I}/\pi_K^m]^\wedge \otimes_{\mathcal{O}_K} K$ . Let  $\mathcal{X}^m(\mathbb{B} \rightarrow B)$  be the affinoid variety  $\mathrm{Sp}(\mathcal{B}^m)$  associated with  $\mathcal{B}^m$ . Then we define the functor  $\mathcal{F}^m$  by

$$\mathcal{F}^m(B) := \varprojlim_{(\mathbb{B} \rightarrow B)} \pi_0(\mathcal{X}^m(\mathbb{B} \rightarrow B)_{\bar{K}}),$$

where  $(\mathbb{B} \rightarrow B)$  runs through the category of embeddings of  $B$ . In general, we have  $\sharp \mathcal{F}^m(B) \leq \mathrm{rank}_A(B)$ . Two key definitions in this paper are the following:

**Definition 3.2.** Let  $B$  be a finite flat  $A$ -algebra. We say that the *ramification* of  $B$  is bounded by  $m$  if  $\sharp \mathcal{F}^m(B) = \mathrm{rank}_A(B)$ .

**Definition 3.3.** For any rational number  $m$  with  $0 < m \leq a$ , we define  $\mathcal{FFP}_A^{\leq m}$  to be the category whose objects are finite flat principal  $A$ -algebras with ramification bounded by  $m$  and whose morphisms are defined as follows: For any  $B$  and  $B'$  in  $\mathcal{FFP}_A^{\leq m}$ , set

$$(2) \quad \mathrm{Hom}_{\mathcal{FFP}_A^{\leq m}}(B, B') := \mathrm{Hom}_{\mathcal{S}_K}(\mathcal{F}^m(B'), \mathcal{F}^m(B)).$$

We also define  $\mathrm{FFP}_A^{\leq m}$  to be the full-subcategory of  $\mathcal{FFP}_A^{\leq m}$  consisting of local objects.

To prove Theorem 1.1, we recall the following lemma due to Hattori ([6], Lemma 1):

**Lemma 3.4.** *Let  $L$  be a finite étale  $K$ -algebra, and  $a$  an integer  $\geq 1$ . If  $B = \mathcal{O}_L/\mathfrak{m}_K^a \mathcal{O}_L$ , then we have  $\mathcal{F}^m(B) = F^m(L)$  as an object of  $\mathcal{S}_K$  for any rational number  $0 < m \leq a$ .*

This is because one may choose a common  $\mathbb{B}$  in the embeddings  $(\mathbb{B}, \mathbb{B} \rightarrow \mathcal{O}_L)$  and  $(\mathbb{B}, \mathbb{B} \rightarrow B)$ , so that, if  $m \leq a$ , we have  $\mathcal{X}^m(\mathbb{B} \rightarrow \mathcal{O}_L) = \mathcal{X}^m(\mathbb{B} \rightarrow B)$ .

By Definitions 3.1 and 3.2, we have:

**Corollary 3.5.** *For any rational number  $0 < m \leq a$ , the ramification of  $B$  is bounded by  $m$  if and only if the ramification of  $L$  is bounded by  $m$ .*

Now we can prove Theorem 1.1. The essential surjectivity of the functor  $T : \mathcal{FE}_{\bar{K}}^{\leq m} \rightarrow \mathcal{FFP}_A^{\leq m}$  follows from (ii) of Proposition 2.2 and Corollary 3.5, since any object of  $\mathcal{FFP}_A^{\leq m}$  is a direct product of finite extensions of  $A$ . To prove the full-faithfulness of  $T$ , let  $L$  and  $L'$  be two objects in  $\mathcal{FE}_{\bar{K}}^{\leq m}$ , and let  $B = T(L)$  and  $B' = T(L')$ . Since the functor  $F^m$  gives an anti-equivalence of the Galois category  $\mathcal{FE}_{\bar{K}}^{\leq m}$  with a full-subcategory of  $\mathcal{S}_K$ , we have

$$\mathrm{Hom}_{\mathcal{FE}_{\bar{K}}^{\leq m}}(L, L') \simeq \mathrm{Hom}_{\mathcal{S}_K}(F^m(L'), F^m(L)).$$

By Lemma 3.4, we have

$$\mathrm{Hom}_{\mathcal{S}_K}(F^m(L'), F^m(L)) = \mathrm{Hom}_{\mathcal{S}_K}(\mathcal{F}^m(B'), \mathcal{F}^m(B)).$$

It follows from our definition (2) of  $\mathrm{Hom}$  in  $\mathcal{FFP}_A^{\leq m}$  that

$$\mathrm{Hom}_{\mathcal{FE}_{\bar{K}}^{\leq m}}(L, L') = \mathrm{Hom}_{\mathcal{FFP}_A^{\leq m}}(B, B').$$

This completes the proof of the Theorem.

*Remark.* The relation of  $\mathrm{Hom}_A(B, B')$  to the  $\mathrm{Hom}$  sets appearing in the above proof is summarized by the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_K(L, L') & \xrightarrow[\mathcal{F}^m]{\simeq} & \mathrm{Hom}_{\mathcal{S}_K}(F^m(L'), F^m(L)) \\ \downarrow & & \parallel \\ \mathrm{Hom}_A(B, B') & \xrightarrow[\mathcal{F}^m]{} & \mathrm{Hom}_{\mathcal{S}_K}(\mathcal{F}^m(B'), \mathcal{F}^m(B)), \end{array}$$

where the left vertical arrow is the reduction mod  $\mathfrak{m}_K^a$  of  $\mathrm{Hom}_K(L, L') = \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_L, \mathcal{O}_{L'})$ . This shows that the map  $\mathcal{F}^m : \mathrm{Hom}_A(B, B') \rightarrow \mathrm{Hom}_{\mathcal{FFP}_A^{\leq m}}(B, B')$  is surjective and compatible with the composition of morphisms. It can be shown that this map identifies the set  $\mathrm{Hom}_{\mathcal{FFP}_A^{\leq m}}(B, B')$  with the quotient of  $\mathrm{Hom}_A(B, B')$  by an equivalence relation  $\simeq^m$  defined as follows: Put  $\bar{A} = \mathcal{O}_{\bar{K}}/\mathfrak{m}_K^a \mathcal{O}_{\bar{K}}$  and let  $\mathcal{X}^m$  be the affinoid variety associated with an embedding of  $B$ . Recall that there exists a natural surjective map  $\mathcal{X}^m(\bar{K}) \rightarrow \mathrm{Hom}_A(B, \bar{A})$  with connected fibers ([2], Lem. 3.2), so that its inverse yields a well-defined map  $\xi : \mathrm{Hom}_A(B, \bar{A}) \rightarrow \pi_0(\mathcal{X}_{\bar{K}}^m)$ . Then we have a map

$$\mathrm{Hom}_A(B, B') \times \mathrm{Hom}_A(B', \bar{A}) \rightarrow \pi_0(\mathcal{X}_{\bar{K}}^m)$$

which maps  $(f, \alpha)$  to  $\xi(\alpha \circ f)$ . For  $f$  and  $f'$  in  $\text{Hom}_A(B, B')$ , define

$$f \simeq f' \iff \xi(\alpha \circ f) = \xi(\alpha \circ f') \quad \text{for all } \alpha \in \text{Hom}_A(B', \bar{A}).$$

It can also be shown that, if  $B'$  is local, then for given  $f$  and  $f'$ , the equality  $\xi(\alpha \circ f) = \xi(\alpha \circ f')$  holds for all  $\alpha \in \text{Hom}_A(B', \bar{A})$  if it holds for some  $\alpha$ .

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