Integration by parts formulae for Wiener measures restricted to subsets in R^d

Hariya, Yuu
Faculty of Mathematics, Kyushu University

https://hdl.handle.net/2324/11853

出版情報：Journal of Functional Analysis. 239 (2), pp.594-610, 2006-10-15. Elsevier
バージョン：
権利関係：

MHF Preprint Series

Kyushu University
21st Century COE Program
Development of Dynamic Mathematics with High Functionality

Integration by parts formulae for the Wiener measure restricted to subsets in $\mathbb{R}^d$

Y. Hariya

MHF 2005-20

(Received May 17, 2005)

Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

Integration by parts formulae for the Wiener measure restricted to subsets in $\mathbb{R}^d$

Yuu Hariya

Abstract

We study an integration by parts formula for the pinned Wiener measure restricted to a space of paths staying within a subset in $\mathbb{R}^d$. The result presented here generalizes the formula in [10] for the case of a half line in $\mathbb{R}$.

1 Introduction and the main result

Let $W = C([0,1];\mathbb{R}^d), d \in \mathbb{N}$, and $H \subset W$ the Cameron-Martin subspace. Let $\Omega \subset \mathbb{R}^d$ be an open region. We denote by W_Ω the set of paths staying within Ω: $W_\Omega = \{w \in W; w(s) \in \Omega, 0 \le s \le 1\}$. In this paper, we study an (infinite-dimensional) integration by parts formula (IbP formula) for the pinned Wiener measure restricted to W_Ω, which is formulated in the following form: For a smooth functional F on W and $h \in H$,

$$\int_{W_\Omega} \partial_h F(w)\, d\mathcal{W}^{a,b}_{[0,1]}(w) = \int_{W_\Omega} F(w)[h](w)\, d\mathcal{W}^{a,b}_{[0,1]}(w) + (\mathrm{BC}) \tag{IbP}$$

with $a, b \in \Omega$. Here $\partial_h F$ denotes the Gâteaux derivative of F, $\mathcal{W}^{a,b}_{[0,1]}$ is the law on W of (d-dimensional) pinned Brownian motion with boundary conditions $w(0) = a, w(1) = b$, $[h](w)$ denotes the Wiener integral: $[h](w) = \int_0^1 h'(s) \cdot dw(s)$, and (BC) represents the boundary contribution, which is an analogue to that in Gauss' divergence formula of finite dimension. Zambotti [10] firstly explored this problem for the case $\Omega = (0,\infty) \subset \mathbb{R}$, in the connection with stochastic partial differential equations with reflection.

In this paper, we present an explicit form of (BC) for more general Ω's in $\mathbb{R}^d$, in terms of hitting times of Brownian motion; in particular, we specify the infinite-dimensional boundary measure appearing in (BC). To state the result, we prepare several notation: For $x = (x_k)_{1\le k\le d}, y = (y_k)_{1\le k\le d} \in \mathbb{R}^d$, we denote by $x \cdot y$ the inner product in $\mathbb{R}^d$: $x \cdot y = \sum_{k=1}^d x_k y_k$, and by $|x|$ the Euclidean norm: $|x| = (x \cdot x)^{1/2}$. We denote by $\overline{\Omega}$ the closure of Ω, and by $\partial\Omega$ the boundary of Ω: $\partial\Omega = \overline{\Omega}\backslash\Omega$. Throughout this paper, Ω is assumed to be a bounded region for which Gauss' divergence theorem holds.

For $a, b \in \Omega$, let $B, \widehat{B}$ be independent d-dimensional Brownian motions with $B_0 = a, \widehat{B}_0 = b$. Let $\tau_\Omega(B)$ and $\tau_\Omega(\widehat{B})$ be the first exit times from Ω of B and of $\widehat{B}$, respectively. Conditionally on $\tau_\Omega(B) + \tau_\Omega(\widehat{B}) = 1$, $B_{\tau_\Omega(B)} = x$, and $\widehat{B}_{\tau_\Omega(\widehat{B})} = x$, define the process $X = \{X_t, 0 \leq t \leq 1\}$ by

$$X_t = \begin{cases} B_t, & 0 \leq t \leq \tau_\Omega(B), \\ \widehat{B}_{\tau_\Omega(B)+\tau_\Omega(\widehat{B})-t}, & \tau_\Omega(B) \leq t \leq \tau_\Omega(B) + \tau_\Omega(\widehat{B}). \end{cases}$$

We denote by $\mathbb{P}^{a,x,b}_{[0,1]}$ the law on $C([0,1]; \overline{\Omega})$ of X. For each element $w \in \operatorname{supp}\mathbb{P}^{a,x,b}_{[0,1]}$, we denote by $S_x(w) \in (0,1)$ the time at which $w(S_x(w)) = x$.

Let Δ_Ω be the Dirichlet Laplacian for Ω. We assume:
(A0) there exists a fundamental solution $p_\Omega(t; x, y)$ to the Cauchy equation $\partial/\partial t + (1/2)\Delta_\Omega = 0$.
Note that $p_\Omega(t; x, y)$, if exists, is unique by the maximum principle; moreover, it is non-negative. We also note that it admits the following probabilistic representation:

$$p_\Omega(t; x, y) = p(t; x, y)\mathcal{W}^{x,y}_{[0,t]}\left(w(s) \in \Omega, 0 \leq s \leq t\right), \tag{1.1}$$

where $p(t; x, y)$ denotes the Gaussian kernel:

$$p(t; x, y) = \frac{1}{\sqrt{(2\pi t)^d}} \exp\left\{-\frac{|x-y|^2}{2t}\right\},$$

and $\mathcal{W}^{x,y}_{[0,t]}$ is the law of pinned Brownian motion over the interval $[0,t]$ starting at x and ending at y. We assume (A0) so that, for every $f \in C(\Omega)$,

$$\frac{\partial}{\partial t}\int_\Omega f(y)p_\Omega(t; x, y)\,dy = \frac{1}{2}\Delta_\Omega \int_\Omega f(y)p_\Omega(t; x, y)\,dy,$$

and that the following commutations are also valid: for $1 \leq i, j \leq d$,

$$\frac{\partial}{\partial x_i}\int_\Omega f(y)p_\Omega(t; x, y)\,dy = \int_\Omega f(y)\frac{\partial}{\partial x_i}p_\Omega(t; x, y)\,dy, \tag{1.2}$$

$$\frac{\partial^2}{\partial x_i \partial x_j}\int_\Omega f(y)p_\Omega(t; x, y)\,dy = \int_\Omega f(y)\frac{\partial^2}{\partial x_i \partial x_j}p_\Omega(t; x, y)\,dy. \tag{1.3}$$

Following [1, Theorem A.3.2], we also assume:
(A1) for each fixed $t > 0$ and $y \in \Omega$, $p_\Omega(t; \cdot, y)$ is C^1 up to the boundary;
(A2) the restrictions to $\partial\Omega$ of functions which are harmonic on Ω, and C^1 up to boundary, are dense in $C(\partial\Omega)$.
Under these conditions, they proved that the joint distribution of $\tau_\Omega(B)$ and $B_{\tau_\Omega(B)}$ is given by, for every $a \in \Omega$,

$$P_a(\tau_\Omega(B) \in dt, B_{\tau_\Omega(B)} \in dx) = \frac{1}{2}\frac{\partial}{\partial n_x}p_\Omega(t; a, x)\,\sigma(dx)\,dt, \tag{1.4}$$

where σ is the surface measure on $\partial\Omega$, n_x is the inward normal vector and $\partial/\partial n_x$ denotes the normal derivative at $x \in \partial\Omega$. This formula will be often referred to in this paper.

We endow W with the sup-norm: $|w|_W := \sup_{0\le t\le 1} |w(t)|, w \in W$. Let $\mathcal{B}(W)$ be the corresponding Borel σ-field. Let W^* be the topological dual of W and $\langle \cdot, \cdot \rangle$ the natural coupling between W^* and W. For $l \in W^*$, we denote by $|l|_{W^*}$ its operator norm. Let $\mathcal{F}C_b^1$ be the set of the functionals F of the form

$$F(w) = f(\langle l_1, w\rangle, \cdots, \langle l_N, w\rangle), \quad w \in W,$$

for $N \in \mathbb{N}$, $l_i \in W^*$ $(1 \le i \le N)$ and $f \in C_b^1(\mathbb{R}^N)$. Here $C_b^1(\mathbb{R}^N)$ denotes the set of the bounded, continuously differentiable functions on $\mathbb{R}^N$ with bounded derivatives.

Finally, as was mentioned above, we denote by H the Cameron-Martin subspace; that is, H consists of the elements $h = (h_k)_{1\le k\le d} \in W$ that are absolutely continuous and satisfy $h(0) = h(1) = 0$ and $\|h\|_H^2 := \int_0^1 |h'(s)|^2\, ds < \infty$. Here $h'(s) = (h_k'(s))_{1\le k\le d}$. We recall that H is continuously embedded in W; indeed, $|h|_W \le \|h\|_H$ for all $h \in H$.

Now we are prepared to state the main result of this paper:

Theorem 1.1. *Assume* (A0)–(A2). *Then, for every $F \in \mathcal{F}C_b^1$ and $h \in H$,* (IbP) *holds with* (BC) *given by:*

$$(\mathrm{BC}) = -\frac{1}{2p(1;a,b)} \int_{\partial\Omega} \sigma(dx)\, \mathbb{E}_{[0,1]}^{a,x,b}[(n_x \cdot h(S_x(w)))F(w)] \times \int_0^1 du\, \frac{\partial}{\partial n_x} p_\Omega(u;a,x) \frac{\partial}{\partial n_x} p_\Omega(1-u;b,x). \tag{1.5}$$

Remark 1.1. (i) The assumptions (A0) and (A1) are fulfilled if $\partial\Omega$ is of class C^3; we refer to [4, 5], where the validity of the commutative relations (1.2) and (1.3) is also shown.

(ii) For the assumption (A2), the following fact is known: If $\partial\Omega$ is of class C^∞ and $f \in C^{2,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, then the Dirichlet problem:

$$\Delta u = 0 \text{ in } \Omega \quad \text{and} \quad u = f \text{ on } \partial\Omega$$

possesses a unique solution u of class $C^{2,\alpha}(\overline{\Omega})$. Here $C^{2,\alpha}(\overline{\Omega})$ denotes the set of functions in $C^2(\overline{\Omega})$ whose second derivatives are Hölder continuous of order α. This is a particular case of Kellogg's theorem; see, e.g., [7, Theorem 10.3.1].

Remark 1.2. The expression (1.5) of (BC) would also be true even if $\partial\Omega$ has a finite number of corners, at least if it satisfies a certain cone condition; for instance, we may easily see that (1.5) holds when Ω is a box in $\mathbb{R}^d$. The existence of the fundamental solution $p_\Omega(t;x,y)$ for such Ω's is discussed in, e.g., [6, Chapter 3, Section 16].

We may compare (1.5) with the formula in [10] for the case $\Omega = (0,\infty) \subset \mathbb{R}$; we refer to it with a slight generalization of boundary conditions: when $\Omega = (0,\infty)$, (BC) is expressed as, for $a, b > 0$,

$$-\int_0^1 du\, h(u) \frac{\sqrt{2}abe^{(a-b)^2/2}}{\sqrt{\pi u^3(1-u)^3}} \exp\left\{-\frac{a^2}{2u} - \frac{b^2}{2(1-u)}\right\} E^{a,0,+}_{[0,u]} \otimes E^{b,0,+}_{[0,1-u]}[F(w_1 \bullet w_2)], \tag{1.6}$$

where $E^{x,y,+}_{[0,t]}$ denotes the expectation with respect to $P^{x,y,+}_{[0,t]}$, the law of pinned 3-dimensional Bessel process over $[0,t]$ starting at x and ending at y, and for $w_1 \in C([0,u];\mathbb{R}), w_2 \in C([0,1-u;\mathbb{R})$ with $w_1(u) = w_2(1-u)$, the notation $w_1 \bullet w_2$ stands for the path given as (2.15) below. See [10, formula (3)]. Note that in this case, $\partial\Omega = \{0\}$. We recall the fact that, for $a > 0$,

$$P_a(\tau_\Omega(B) \in du) = \frac{a}{\sqrt{2\pi u^3}} \exp\left(-\frac{a^2}{2u}\right) du,$$

and that, conditionally on $\tau_\Omega(B) = u$, $\{B_t, 0 \le t \le \tau_\Omega(B)\}$ has the same law as $P^{a,0,+}_{[0,u]}$. From these, we see that (1.6) is rewritten as, in our notation,

$$-2(a+b)e^{-2ab}\mathbb{E}^{a,0,b}_{[0,1]}[h(S_0(w))F(w)]. \tag{1.7}$$

We may find that the formula (1.5) gives a generalization of (1.7).

In [10], the formula (1.6) was proved in the case $a = b$; the proof given there relied upon an explicit formula relating a pinned 3-dimensional Bessel process to a pinned Brownian motion, known as Biane's theorem. Recently in [2], a generalization of Zambotti's result has been obtained in the case where the (1-dimensional) Wiener measure is restricted to a space of paths staying between two curves. Their method is based on polygonal approximations of Brownian motions.

Integration by parts formulae (in fact, divergence formulae) in infinite dimensional spaces have been studied by Goodman [3], Shigekawa [9] and other researchers in a framework of an abstract Wiener space.

The rest of the paper is organized as follows: In the next section, we prove Theorem 1.1 by using Proposition 2.1, which will be proved in Subsection 2.2; in Subsection 2.3, we prove Proposition 2.3 that is used in the proof of Proposition 2.1. In Section 3, we give concluding remarks.

Throughout this paper, we write $\nabla = (\partial/\partial x_k)_{1\le k\le d}$ and $\Delta = \nabla \cdot \nabla$, the Laplacian. To indicate a variable, we sometimes write ∇_x and Δ_x with $x = (x_k)_{1\le k\le d}$.

2 Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

2.1 Proof of Theorem 1.1

We consider the family of functionals F of the form:

$$F(w) = \prod_{i=1}^{n-1} f_i(w(t_i)) \tag{2.1}$$

for $n \in \mathbb{N}$, $f_i \in C_b^1(\mathbb{R}^d)$, and a partition $\{0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1\}$ of the interval $[0, 1]$.

Proposition 2.1. *Assume* (A0)–(A2). *Then, for every F of the form* (2.1)*, and for every $h \in C_0^2((0, 1))$,* (IbP) *holds with* (BC) *given by* (1.5).

We give a proof of this proposition in the next subsection. Once this proposition is shown, then Theorem 1.1 is proved by approximations.

Proof of Theorem 1.1. We prove Theorem 1.1 in two steps.
Step 1. Approximation of elements in $\mathcal{F}C_b^1$. In this step, by using Proposition 2.1, we prove the assertion of Theorem 1.1 for $F \in \mathcal{F}C_b^1$ and $h \in C_0^2((0, 1))$.

Note that, by the Riesz-Markov theorem, $l : W \to \mathbb{R}$ belongs to W^* if and only if there exists $\lambda = (\lambda_k)_{1 \le k \le d}$ with $\lambda_k : [0, 1] \to \mathbb{R}$ of finite variation, such that $l(w) = \int_0^1 w(s) \cdot d\lambda(s)$. Therefore, by approximating $l(w) \equiv \langle l, w \rangle$ by Riemann-Stieltjes sums, it suffices to prove the assertion for F given by:

$$F(w) = \tilde{f}(w(t_1), \cdots, w(t_{n-1})), \ \tilde{f} \in C_b^1((\mathbb{R}^d)^{n-1}).$$

Moreover, for any $\tilde{f} = \tilde{f}(x^1, \cdots, x^{n-1}) \in C_b^1((\mathbb{R}^d)^{n-1})$, we may find a family $f_{i,j} = f_{i,j}(x^i) \in C_b^1(\mathbb{R}^d), 1 \le i \le n-1, j \in \mathbb{N}$, such that

$$\lim_{M \to \infty} \sum_{j=1}^{M} \left(\prod_{i=1}^{n-1} f_{i,j} \right) = \tilde{f} \quad \text{uniformly on } (\overline{\Omega})^{n-1}.$$

Since (IbP) is linear in F, it thus suffices to prove the assertion for F of the form (2.1), and for $h \in C_0^2((0, 1))$, which is nothing but the assertion of Proposition 2.1.
Step 2. Approximation of elements in H. In this step, using the conclusion obtained in the previous step, we complete the proof of Theorem 1.1.

We pick $h^* \in H$ and fix it. Note that there exists a sequence $\{h_m\}_{m \in \mathbb{N}} \subset C_0^2((0, 1))$ such that $\|h_m - h^*\|_H \to 0, m \to \infty$. Since H is continuously embedded in W, $|h_m - h^*|_W \to 0$, from which we see that, as $m \to \infty$, (BC) for h_m given by (1.5) converges to that for h^*. Therefore, in order to complete the proof, it now suffices to show: for every

$F \in \mathcal{F}C_b^1$,

$$\lim_{m\to\infty}\int_{W_\Omega}\partial_{h_m}F(w)\,d\mathcal{W}^{a,b}_{[0,1]}(w)=\int_{W_\Omega}\partial_{h^*}F(w)\,d\mathcal{W}^{a,b}_{[0,1]}(w), \tag{2.2}$$

$$\lim_{m\to\infty}\int_{W_\Omega}F(w)[h_m](w)\,d\mathcal{W}^{a,b}_{[0,1]}(w)=\int_{W_\Omega}F(w)[h^*](w)\,d\mathcal{W}^{a,b}_{[0,1]}(w). \tag{2.3}$$

Let $F \in \mathcal{F}C_b^1$ be expressed as $F(w)=f(\langle l_1,w\rangle,\cdots,\langle l_N,w\rangle)$. Then we have

$$\partial_h F(w)\equiv \frac{d}{d\varepsilon}F(w+\varepsilon h)\Big|_{\varepsilon=0}=\sum_{i=1}^N\frac{\partial f}{\partial x_i}(\langle l_1,w\rangle,\cdots,\langle l_N,w\rangle)\langle l_i,h\rangle,$$

hence

$$|\partial_{h_m}F(w)-\partial_{h^*}F(w)|\le |h_m-h^*|_W\times\max_{1\le i\le N}\sup_{x\in\mathbb{R}^N}\left|\frac{\partial f(x)}{\partial x_i}\right|\max_{1\le i\le N}|l_i|_{W^*},$$

which implies (2.2). Since h_m converges to h^* in H, $[h_m](w)$ converges to $[h^*](w)$ in $L^2(W;\mathcal{W}^{a,b}_{[0,1]})$, from which (2.3) follows. So the proof is complete. □

2.2 Proof of Proposition 2.1

In this subsection, we prove Proposition 2.1.

For notational simplicity, we denote by $(\mathrm{IbP})_1$ and $(\mathrm{IbP})_2$, the LHS of (IbP) and the first term on the RHS of (IbP), respectively. Then (IbP) is restated as:

$$(\mathrm{IbP})_1=(\mathrm{IbP})_2+(\mathrm{BC}). \tag{IbP$'$}$$

Let F be of the form (2.1). For $\underline{x}=(x^0,x^1,\cdots,x^{n-1},x^n)\in(\mathbb{R}^d)^{n+1}$, we set

$$\mathfrak{F}(\underline{x})=\prod_{i=0}^n f_i(x^i)\quad (f_0=f_n\equiv 1),$$
$$\nu_{a,b}(d\underline{x})=\delta_a(dx^0)dx^1\cdots dx^{n-1}\delta_b(dx^n),\ a,b\in\Omega,$$

$\delta_x, x\in\mathbb{R}^d$, being the Dirac measure at x. We begin with the following lemma:

Lemma 2.2. *For F given by (2.1), and for $h\in C_0^2((0,1))$, we have*

$$(\mathrm{IbP})_k=-\frac{1}{p(1;a,b)}\int_{\Omega^{n+1}}\nu_{a,b}(d\underline{x})\,\mathfrak{F}(\underline{x})\mathfrak{I}_k(\underline{x}),\quad k=1,2, \tag{2.4}$$

where

$$\mathfrak{I}_1(\underline{x})=\sum_{i=1}^{n-1}h(t_i)\cdot\nabla_{x^i}\{\prod_{j=0}^{n-1}p_\Omega(t_{j+1}-t_j;x^j,x^{j+1})\},$$

$$\mathfrak{I}_2(\underline{x})=\sum_{i=0}^{n-1}\{\prod_{j\ne i}p_\Omega(t_{j+1}-t_j;x^j,x^{j+1})\}$$
$$\times\int_{t_i}^{t_{i+1}}du\int_\Omega dx\,(x\cdot h''(u))p_\Omega(u-t_i;x^i,x)p_\Omega(t_{i+1}-u;x^{i+1},x).$$

Proof. Note that, for F of the form (2.1),

$$\partial_h F(w) = \sum_{i=1}^{n-1} (h(t_i) \cdot \nabla f_i(w(t_i))) \prod_{j \neq i} f_j(w(t_j)).$$

Recall that the joint distribution $\mathcal{W}^{a,b}_{[0,1]}(w(t_1) \in dx^1, \cdots, w(t_{n-1}) \in dx^{n-1})$ is given by

$$\frac{1}{p(1;a,b)} \left\{ \prod_{i=1}^{n} p(t_i - t_{i-1}; x^{i-1}, x^i) \right\} dx^1 \cdots dx^{n-1}, \quad x^0 = a, x^n = b. \tag{2.5}$$

The expression for $(\mathrm{IbP})_1$ follows from these, the relation (1.1) and the divergence theorem. For $h \in C_0^2((0,1))$, note that the Wiener integral $[h](w)$ can be defined pathwisely via: $[h](w) = -\int_0^1 h''(s) \cdot w(s)\, ds$. The expression for $(\mathrm{IbP})_2$ follows from this, and again from (2.5) and the relation (1.1). □

For (BC), we have the following expression:

Proposition 2.3. *For F given by* (2.1)*, we have*

$$(\mathrm{BC}) = -\frac{1}{p(1;a,b)} \int_{\Omega^{n+1}} \nu_{a,b}(d\underline{x})\, \mathfrak{F}(\underline{x}) \mathfrak{I}_3(\underline{x}), \tag{2.6}$$

where

$$\begin{aligned}\mathfrak{I}_3(\underline{x}) =& \frac{1}{2} \sum_{i=0}^{n-1} \{ \prod_{j \neq i} p_\Omega(t_{j+1} - t_j; x^j, x^{j+1}) \} \\ & \times \int_{t_i}^{t_{i+1}} du \int_{\partial\Omega} \sigma(dx)\, (n_x \cdot h(u)) \frac{\partial}{\partial n_x} p_\Omega(u - t_i; x^i, x) \frac{\partial}{\partial n_x} p_\Omega(t_{i+1} - u; x^{i+1}, x).\end{aligned}$$

The proof is given in the next subsection. Comparing (2.4) and (2.6), we see that the proof of Proposition 2.1 is reduced to showing the following proposition:

Proposition 2.4. *It holds that, for all $\underline{x} \in (\mathbb{R}^d)^{n+1}$,*

$$\mathfrak{I}_1(\underline{x}) = \mathfrak{I}_2(\underline{x}) + \mathfrak{I}_3(\underline{x}).$$

This follows from:

Proposition 2.5. *For $\kappa = (\kappa_k)_{1 \le k \le d}$, $\kappa_k \in C_b^2((0,\infty)) \cap C^1([0,\infty))$, it holds that, for all $t > 0$ and $y, z \in \Omega$,*

$$\begin{aligned}&\int_0^t du \int_\Omega dx\, (x \cdot \kappa''(u)) p_\Omega(u; y, x) p_\Omega(t - u; z, x) \\ &\qquad + \frac{1}{2} \int_0^t du \int_{\partial\Omega} \sigma(dx)\, (n_x \cdot \kappa(u)) \frac{\partial}{\partial n_x} p_\Omega(u; y, x) \frac{\partial}{\partial n_x} p_\Omega(t - u; z, x) \\ &= (z \cdot \kappa'(t) - y \cdot \kappa'(0)) p_\Omega(t; y, z) + \kappa(0) \cdot \nabla_y p_\Omega(t; y, z) + \kappa(t) \cdot \nabla_z p_\Omega(t; y, z).\end{aligned} \tag{2.7}$$

Using this proposition, we prove Proposition 2.4:

Proof of Proposition 2.4. By Proposition 2.5,

$$
\begin{aligned}
&\mathfrak{I}_2(\underline{x}) + \mathfrak{I}_3(\underline{x}) \\
&= \left\{ \prod_{i=0}^{n-1} p_\Omega(t_{i+1} - t_i; x^i, x^{i+1}) \right\} \sum_{i=0}^{n-1} (x^{i+1} \cdot h'(t_{i+1}) - x^i \cdot h'(t_i)) \\
&\quad + \sum_{i=0}^{n-1} \left\{ \prod_{j \neq i} p_\Omega(t_{j+1} - t_j; x^j, x^{j+1}) \right\} \\
&\qquad \times \left\{ h(t_i) \cdot \nabla_{x^i} p_\Omega(t_{i+1} - t_i; x^i, x^{i+1}) + h(t_{i+1}) \cdot \nabla_{x^{i+1}} p_\Omega(t_{i+1} - t_i; x^i, x^{i+1}) \right\}
\end{aligned}
$$

Since $h'(0) = h'(1) = 0$ by assumption, the first term on the RHS vanishes. Rearranging the second term, we see that this is equal to $\mathfrak{I}_1(\underline{x})$. □

Now we are prepared to prove Proposition 2.1:

Proof of Proposition 2.1. Combining Lemma 2.2 and Propositions 2.3 and 2.4 leads to (IbP′), which shows the proposition. □

For the rest of this subsection, we prove Proposition 2.5. The key is the following lemma:

Lemma 2.6. *Let f, g be functions of class $C^2(\Omega)$ which satisfy $f|_{\partial\Omega} = g|_{\partial\Omega} = 0$ and are C^1 up to the boundary. Then we have, for all $v \in \mathbb{R}^d$,*

$$\int_\Omega dx\, (v \cdot \nabla f)\Delta g + \int_\Omega dx\, (v \cdot \nabla g)\Delta f + \int_{\partial\Omega} \sigma(dx)\, (v \cdot n_x) \frac{\partial f}{\partial n_x} \frac{\partial g}{\partial n_x} = 0.$$

Proof. Noting the identity:

$$(v \cdot \nabla f)\Delta g + (v \cdot \nabla g)\Delta f = \operatorname{div}\{(v \cdot \nabla f)\nabla g + (v \cdot \nabla g)\nabla f - (\nabla f \cdot \nabla g)v\},$$

we see that the assertion follows immediately from the divergence theorem, and from the fact that $\nabla f = (\partial f/\partial n_x)n_x$ and $\nabla g = (\partial g/\partial n_x)n_x$ at $x \in \partial\Omega$ by assumption. □

We proceed to the proof of Proposition 2.5. For $f \in C_0(\Omega)$, we write

$$U_f(t, x) = \int_\Omega dy\, f(y) p_\Omega(t; x, y), \quad t > 0,\ x \in \overline{\Omega}.$$

By definition,

$$\frac{\partial}{\partial t} U_f(t, x) = \frac{1}{2} \Delta U_f(t, x) \tag{2.8}$$

and $\lim_{t\downarrow 0} U_f(t,x) = f(x)$ for every $x \in \overline{\Omega}$.

For arbitrarily fixed $f, g \in C_0(\Omega)$ and $\gamma > 0$, we multiply the first term on the LHS of (2.7) by $e^{-\gamma t} f(y)g(z)$, and integrate it over $(0,\infty) \times \Omega^2$ with respect to $dtdydz$. Then it becomes

$$\int_0^\infty \int_0^\infty ds dt\, e^{-\gamma s} e^{-\gamma t} \int_\Omega dx\, (x \cdot \kappa''(s)) U_f(s,x) U_g(t,x). \tag{2.9}$$

Lemma 2.7. (2.9) *is equal to* $I_1 + I_2$. *Here*

$$\begin{aligned} I_1 &= \int_0^\infty dt\, e^{-\gamma t} \int_\Omega dx\, \{(x \cdot \kappa'(t)) g(x) U_f(t,x) - (x \cdot \kappa'(0)) f(x) U_g(t,x)\}, \\ I_2 &= \int_0^\infty dt\, e^{-\gamma t} \int_\Omega dx\, \{(\kappa(0) \cdot \nabla U_g(t,x)) f(x) + (\kappa(t) \cdot \nabla U_f(t,x)) g(x)\} \\ &\quad + \frac{1}{2} \int_0^\infty \int_0^\infty ds dt\, e^{-\gamma s} e^{-\gamma t} \int_\Omega dx\, \{(\kappa(s) \cdot \nabla U_g(t,x)) \Delta U_f(s,x) \\ &\qquad\qquad + (\kappa(s) \cdot \nabla U_f(s,x)) \Delta U_g(t,x)\}. \end{aligned}$$

Proof. Using integration by parts in the variable s, we see that (2.9) is rewritten as

$$\begin{aligned} &- \int_0^\infty dt\, e^{-\gamma t} \int_\Omega dx\, (x \cdot \kappa'(0)) f(x) U_g(t,x) \\ &\quad + \gamma \int_0^\infty \int_0^\infty ds dt\, e^{-\gamma s} e^{-\gamma t} \int_\Omega dx\, (x \cdot \kappa'(s)) U_f(s,x) U_g(t,x) \\ &\qquad - \int_0^\infty \int_0^\infty ds dt\, e^{-\gamma s} e^{-\gamma t} \int_\Omega dx\, (x \cdot \kappa'(s)) U_g(t,x) \frac{\partial}{\partial s} U_f(s,x). \end{aligned} \tag{2.10}$$

For the second term, note that, by integration by parts,

$$\gamma \int_0^\infty dt\, e^{-\gamma t} U_g(t,x) = g(x) + \int_0^\infty dt\, e^{-\gamma t} \frac{\partial}{\partial t} U_g(t,x). \tag{2.11}$$

Inserting this into the second term of (2.10), we see that (2.10) is rewritten as $I_1 + II$ with

$$II = \int_0^\infty \int_0^\infty ds dt\, e^{-\gamma s} e^{-\gamma t} \int_\Omega dx\, (x \cdot \kappa'(s)) \left\{ U_f(s,x) \frac{\partial}{\partial t} U_g(t,x) - U_g(t,x) \frac{\partial}{\partial s} U_f(s,x) \right\}.$$

So, in order to prove the lemma, it suffices to show

$$II = I_2. \tag{2.12}$$

Using the relation (2.8) for $U_f(s,x)$ and $U_g(t,x)$, and the divergence theorem in the definition of II, we see that

$$II = - \int_0^\infty \int_0^\infty ds dt\, e^{-\gamma s} e^{-\gamma t} \int_\Omega dx\, (\kappa'(s) \cdot \nabla U_g(t,x)) U_f(s,x). \tag{2.13}$$

Here we used the fact that $U_f(s,x) = U_g(t,x) = 0$ for $x \in \partial\Omega$. Using integration by parts in the variable s, we see that (2.13) is rewritten further as:

$$\begin{aligned} II = &\int_0^\infty dt\, e^{-\gamma t} \int_\Omega dx\, (\kappa(0) \cdot \nabla U_g(t,x)) f(x) \\ &\qquad - \gamma \int_0^\infty \int_0^\infty ds dt\, e^{-\gamma s} e^{-\gamma t} \int_\Omega dx (\kappa(s) \cdot \nabla U_g(t,x)) U_f(s,x) \\ &\qquad\quad + \int_0^\infty \int_0^\infty ds dt\, e^{-\gamma s} e^{-\gamma t} \int_\Omega dx\, (\kappa(s) \cdot \nabla U_g(t,x)) \frac{\partial}{\partial s} U_f(s,x). \end{aligned} \tag{2.14}$$

By the divergence theorem, the second term is equal to:

$$\begin{aligned} &\gamma \int_0^\infty \int_0^\infty ds dt\, e^{-\gamma s} e^{-\gamma t} \int_\Omega dx\, (\kappa(s) \cdot \nabla U_f(s,x)) U_g(t,x) \\ &= \int_0^\infty ds\ e^{-\gamma s} \int_\Omega dx\, (\kappa(s) \cdot \nabla U_f(s,x)) g(x) \\ &\quad + \int_0^\infty \int_0^\infty ds dt\, e^{-\gamma s} e^{-\gamma t} \int_\Omega dx\, (\kappa(s) \cdot \nabla U_f(s,x)) \frac{\partial}{\partial t} U_g(t,x), \end{aligned}$$

where we used (2.11) for the equality. Replacing by this the second term on the RHS of (2.14), and using the relation (2.8) for $U_f(s,x)$ and $U_g(t,x)$, we get (2.12). This ends the proof. □

Using Lemma 2.7, we prove Proposition 2.5:

Proof of Proposition 2.5. By Lemma 2.7, and by taking the inverse Laplace transform, we have, for all $t > 0$,

$$\begin{aligned} &\int_0^t du \int_\Omega dx\, (x \cdot \kappa''(u)) U_f(u,x) U_g(t-u,x) \\ &\qquad = \int_\Omega dx\ \{(x \cdot \kappa'(t)) g(x) U_f(t,x) - (x \cdot \kappa'(0)) f(x) U_g(t,x)\} \\ &\qquad + \int_\Omega dx\ \{(\kappa(0) \cdot \nabla U_g(t,x)) f(x) + (\kappa(t) \cdot \nabla U_f(t,x)) g(x)\} \\ &\qquad\quad + \frac{1}{2} \int_0^t du \int_\Omega dx\, \{(\kappa(u) \cdot \nabla U_g(t-u,x)) \Delta U_f(u,x) \\ &\qquad\qquad\qquad + (\kappa(u) \cdot \nabla U_f(u,x)) \Delta U_g(t-u,x)\}. \end{aligned}$$

Since $f, g \in C_0(\Omega)$ are arbitrary, we obtain, for a.e. $y, z \in \Omega$,

$$\begin{aligned} &\int_0^t du \int_\Omega dx\, (x \cdot \kappa''(u)) p_\Omega(u; y, x) p_\Omega(t-u; z, x) \\ &\quad = (z \cdot \kappa'(t) - y \cdot \kappa'(0)) p_\Omega(t; y, z) + \kappa(0) \cdot \nabla_y p_\Omega(t; y, z) + \kappa(t) \cdot \nabla_z p_\Omega(t; y, z) \\ &\qquad + \frac{1}{2} \int_0^t du \int_\Omega dx\, \{(\kappa(u) \cdot \nabla_x p_\Omega(t-u; z, x)) \Delta_x p_\Omega(u, y, x) \\ &\qquad\qquad + (\kappa(u) \cdot \nabla_x p_\Omega(u; y, x)) \Delta_x p_\Omega(t-u; z, x)\}. \end{aligned}$$

Here we used the commutative relations (1.2) and (1.3). By continuity, this relation holds for all $y, z \in \Omega$. Note that, by Lemma 2.6, the second term on the RHS is equal to

$$-\frac{1}{2}\int_0^t du \int_{\partial\Omega} \sigma(dx)\,(n_x \cdot \kappa(u))\frac{\partial}{\partial n_x}p_\Omega(u; y, x)\frac{\partial}{\partial n_x}p_\Omega(t-u; z, x).$$

This shows the proposition. $\square$

2.3 Proof of Proposition 2.3

In this section, we prove Proposition 2.3 that was used in the proof of Proposition 2.1.

For a Brownian motion B starting from $a \in \Omega$, we denote by $\mathcal{P}^{a,x}_{[0,u]}$ the law of $\{B_s, 0 \le s \le \tau_\Omega(B)\}$ given $\tau_\Omega(B) = u$ and $B_{\tau_\Omega(B)} = x$, and by $\mathcal{E}^{a,u,x}_{[0,u]}$ the expectation with respect to $\mathcal{P}^{a,u,x}_{[0,u]}$.

Lemma 2.8. (i) *It holds that, for $A \in \mathcal{B}(W)$,*

$$\mathbb{P}^{a,x,b}_{[0,1]}(A|S_x = u) = \mathcal{P}^{a,x}_{[0,u]} \otimes \mathcal{P}^{b,x}_{[0,1-u]}(w_1 \bullet w_2 \in A),$$

where, for $w_1 \in C([0,u];\mathbb{R}^d)$ and $w_2 \in C([0,1-u];\mathbb{R}^d)$ with $w_1(u) = w_2(1-u)$, $w_1 \bullet w_2$ denotes the path defined by

$$(w_1 \bullet w_2)(s) = \begin{cases} w_1(s), & 0 \le s \le u, \\ w_2(1-s), & u \le s \le 1. \end{cases} \tag{2.15}$$

(ii) *The distribution of S_x under $\mathbb{P}^{a,x,b}_{[0,1]}$ is given by*

$$\mathbb{P}^{a,x,b}_{[0,1]}(S_x \in du) = (Z_x)^{-1}\frac{\partial}{\partial n_x}p_\Omega(u; a, x)\frac{\partial}{\partial n_x}p_\Omega(1-u; b, x)\,du,$$

where Z_x is the normalization; i.e.,

$$Z_x = \int_0^1 du\,\frac{\partial}{\partial n_x}p_\Omega(u; a, x)\frac{\partial}{\partial n_x}p_\Omega(1-u; b, x).$$

Proof. The assertion (i) is obvious from the definition of $\mathbb{P}^{a,x,b}_{[0,1]}$. (ii) follows from (1.4). $\square$

By (1.5) and the definition of Z_x, we may write (BC) as:

$$\text{(BC)} = -\frac{1}{2p(1; a, b)}\int_{\partial\Omega} \sigma(dx)\,\mathbb{E}^{a,x,b}_{[0,1]}[(n_x \cdot h(S_x(w)))F(w)]Z_x. \tag{2.16}$$

Note that, by Lemma 2.8, the integrand in (2.16) relative to $\sigma(dx)$ may be written as:

$$\int_0^1 du\,(n_x \cdot h(u))\mathcal{E}^{a,x}_{[0,u]} \otimes \mathcal{E}^{b,x}_{[0,1-u]}[F(w_1 \bullet w_2)]\frac{\partial}{\partial n_x}p_\Omega(u; a, x)\frac{\partial}{\partial n_x}p_\Omega(1-u; b, x). \tag{2.17}$$

Lemma 2.9. *For $0 < r < u$, let $G(w) = G(w(s), s \leq r)$ be a bounded, measurable functional on $C([0,u];\mathbb{R}^d)$. Then it holds that*

$$\begin{aligned}&\mathcal{E}^{a,x}_{[0,u]}[G(w)]\frac{\partial}{\partial n_x}p_\Omega(u;a,x)\\&=\int_\Omega dy\,p(r;a,y)\mathcal{W}^{a,y}_{[0,r]}[G(w)\mathbf{1}_{\{w(s)\in\Omega,0\leq s\leq r\}}]\frac{\partial}{\partial n_x}p_\Omega(u-r;y,x).\end{aligned}$$

Here $\mathcal{W}^{a,y}_{[0,r]}[\cdot]$ denotes the expectation relative to $\mathcal{W}^{a,y}_{[0,r]}$, and the notation $\mathbf{1}_A$ stands for the indicator function of an event A.

An intuition to Lemma 2.9 is that we may identify $\mathcal{E}^{a,x}_{[0,u]}[G(w)]$ with:

$$\begin{aligned}&\lim_{\substack{z\to x\\ z\in\Omega}}\frac{\mathcal{W}^{a,z}_{[0,u]}[G(w)\mathbf{1}_{\{w(s)\in\Omega,0\leq s\leq u\}}]}{\mathcal{W}^{a,z}_{[0,u]}(w(s)\in\Omega,0\leq s\leq u)}\\&=\lim_{\substack{z\to x\\ z\in\Omega}}\frac{p(u;a,z)\mathcal{W}^{a,z}_{[0,u]}[G(w)\mathbf{1}_{\{w(s)\in\Omega,0\leq s\leq u\}}]}{p_\Omega(u;a,z)},\end{aligned}$$

where the equalitiy follows from the relation (1.1). Roughly saying, Lemma 2.9 follows from this, conditioning on $w(r)$, and L'Hospital's rule.

Proof of Lemma 2.9. For every non-negative, measurable function f on $\partial\Omega$, and for every $t > r$, we have, by (1.4) and the definition of $\mathcal{P}^{a,x}_{[0,u]}$,

$$\begin{aligned}&E_a[G(B_s,s\leq r)\mathbf{1}_{\{\tau_\Omega(B)>t\}}f(B_{\tau_\Omega(B)})]\\&=\frac{1}{2}\int_t^\infty du\int_{\partial\Omega}\sigma(dx)f(x)\mathcal{E}^{a,x}_{[0,u]}[G(w(s),s\leq r)]\frac{\partial}{\partial n_x}p_\Omega(u;a,x).\end{aligned}\tag{2.18}$$

Since $\{\tau_\Omega(B) > t\} = \{B_s \in \Omega, 0 \leq s \leq t\}$, the LHS of (2.18) may also be written as, by the Markov property of Brownian motion,

$$E_a[G(B_s,s\leq r)\mathbf{1}_{\{B_s\in\Omega,0\leq s\leq r\}}V(t-r,B_r)].\tag{2.19}$$

Here, for $s > 0$ and $x \in \Omega$, $V(s,x) := E_x[f(B_{\tau_\Omega(B)})\mathbf{1}_{\{\tau_\Omega(B)>s\}}]$. Note that, by (1.4),

$$V(t-r,B_r)=\frac{1}{2}\int_t^\infty du\int_{\partial\Omega}\sigma(dx)f(x)\frac{\partial}{\partial n_x}p_\Omega(u-r;B_r,x).$$

Plugging this into (2.19) and using the law of B at time r, we see (2.19) is rewritten as

$$\begin{aligned}&\int_\Omega P_a(B_r\in dy)\,\mathcal{W}^{a,y}_{[0,r]}[G(w(s),s\leq r)\mathbf{1}_{\{w(s)\in\Omega,0\leq s\leq r\}}]\\&\qquad\times\frac{1}{2}\int_t^\infty du\int_{\partial\Omega}\sigma(dx)f(x)\frac{\partial}{\partial n_x}p_\Omega(u-r;y,x)\\&=\frac{1}{2}\int_t^\infty du\int_{\partial\Omega}\sigma(dx)f(x)\int_\Omega dy\,p(r;a,y)\\&\qquad\times\mathcal{W}^{a,y}_{[0,r]}[G(w(s),s\leq r)\mathbf{1}_{\{w(s)\in\Omega,0\leq s\leq r\}}]\frac{\partial}{\partial n_x}p_\Omega(u-r;y,x).\end{aligned}$$

Here we used Fubini's theorem for the equality (note that, by the non-negativity of p_Ω and the smoothness up to the boundary, for each $t > 0$ and $y \in \Omega$, $\frac{\partial}{\partial n_x} p_\Omega(t; y, x)$ is non-negative for every $x \in \partial\Omega$). Comparing this with the RHS of (2.18), we obtain the lemma. □

Now we are in a position to prove Proposition 2.3:

Proof of Proposition 2.3. We decompose (2.17) into the sum of $J_i, 0 \le i \le n-1$, defined by:

$$J_i = \int_{t_i}^{t_{i+1}} du\,(n_x \cdot h(u)) \mathcal{E}^{a,x}_{[0,u]} \otimes \mathcal{E}^{b,x}_{[0,1-u]}[F(w_1 \bullet w_2)] \frac{\partial}{\partial n_x} p_\Omega(u; a, x) \frac{\partial}{\partial n_x} p_\Omega(1-u; b, x).$$

Note that, by (2.16), and by this decomposition,

$$(\mathrm{BC}) = -\frac{1}{2p(1; a, b)} \int_{\partial\Omega} \sigma(dx) \sum_{i=0}^{n-1} J_i. \tag{2.20}$$

For F given by (2.1), we develop each J_i as

$$J_i = \int_{t_i}^{t_{i+1}} du\,(n_x \cdot h(u)) \times K_{i,1} \times K_{i,2},$$

where

$$K_{i,1} = \mathcal{E}^{a,x}_{[0,u]}[\prod_{j=0}^{i} f_j(w(t_j))] \frac{\partial}{\partial n_x} p_\Omega(u; a, x),$$

$$K_{i,2} = \mathcal{E}^{b,x}_{[0,1-u]}[\prod_{j=i+1}^{n} f_j(w(1-t_j))] \frac{\partial}{\partial n_x} p_\Omega(1-u; b, x).$$

By Lemma 2.9, $K_{i,1}$ is rewritten as:

$$\int_\Omega dx^i\, f_i(x^i) p(t_i; a, x^i) \mathcal{W}^{a,x^i}_{[0,t_i]}[\prod_{j=0}^{i-1} f_j(w(t_j)) \mathbf{1}_{\{w(s)\in\Omega, 0\le s\le t_i\}}] \frac{\partial}{\partial n_x} p_\Omega(u - t_i; x^i, x). \tag{2.21}$$

Note that, by using the joint distribution of $(w(t_0), \cdots, w(t_{i-1}))$ under $\mathcal{W}^{a,x^i}_{[0,t_i]}$, and the relation (1.1),

$$\begin{aligned} &p(t_i; a, x^i) \mathcal{W}^{a,x^i}_{[0,t_i]}[\prod_{j=0}^{i-1} f_j(w(t_j)) \mathbf{1}_{\{w(s)\in\Omega, 0\le s\le t_i\}}] \\ &= \int_{\Omega^i} \delta_a(dx^0) dx^1 \cdots dx^{i-1} \{\prod_{j=0}^{i-1} f_j(x^j) p_\Omega(t_{j+1} - t_j; x^j, x^{j+1})\}. \end{aligned}$$

Plugging this into (2.21), we have

$$K_{i,1} = \int_{\Omega^{i+1}} \delta_a(dx^0)dx^1 \cdots dx^i \, \{\prod_{j=0}^{i} f_j(x^j)\}$$
$$\times \{\prod_{j=0}^{i-1} p_\Omega(t_{j+1} - t_j; x^j, x^{j+1})\} \frac{\partial}{\partial n_x} p_\Omega(u - t_i; x^i, x).$$

Similarly we have

$$K_{i,2} = \int_{\Omega^{n-i}} dx^{i+1} \cdots dx^{n-1} \delta_b(dx^n)\{\prod_{j=i+1}^{n} f_j(x^j)\}$$
$$\times \{\prod_{j=i+1}^{n-1} p_\Omega(t_{j+1} - t_j; x^j, x^{j+1})\} \frac{\partial}{\partial n_x} p_\Omega(t_{i+1} - u; x^{i+1}, x).$$

Combining these, obtain

$$J_i = \int_{t_i}^{t_{i+1}} du \, (n_x \cdot h(u)) \int_{\Omega^{n+1}} \nu_{a,b}(d\underline{x})\mathfrak{F}(\underline{x})\{\prod_{j \neq i} p_\Omega(t_{j+1} - t_j; x^j, x^{j+1})\}$$
$$\times \frac{\partial}{\partial n_x} p_\Omega(u - t_i; x^i, x) \frac{\partial}{\partial n_x} p_\Omega(t_{i+1} - u; x^{i+1}, x).$$

Summing up these for $0 \le i \le n-1$ and noting (2.20), we obtain (2.6). So the proposition is proved. □

3 Concluding remarks

(i) The formula (1.5) asserts that (BC) is given by the integral of

$$\mathbb{E}^{a,x,b}_{[0,1]}[F(w)h(S_x(w))] \tag{3.1}$$

with respect to the vector-valued measure $m(dx)$ on $\partial\Omega$ given by:

$$m(dx) = -\frac{1}{2p(1; a, b)} \left\{ \int_0^1 du \frac{\partial}{\partial n_x} p_\Omega(u; a, x) \frac{\partial}{\partial n_x} p_\Omega(1 - u; b, x) \right\} n_x \, \sigma(dx).$$

As we see, the normal derivative of p_Ω (in fact, p_Ω itself) is not involved in (3.1). So, if we obtain another expression of the measure $m(dx)$ in which, at least the normal derivative of p_Ω is not involved, then we may extend our result to more general Ω's such as, say, Caccioppoli sets.

(ii) In [10], integration by parts formulae were applied to the study of stochastic partial differential equations with reflection introduced by Nualart-Pardoux [8]; in particular, he gave a characterization of local times in equations in terms of Revuz measures, to which boundary measures in (BC) were associated. Our result might have a similar application.

References

[1] Aizenman, M., Simon, B.: Brownian motion and Harnack inequality for Schrödinger operators. Comm. Pure and Appl. Math. **35**, 209–273 (1982)

[2] Funaki, T., Ishitani, K.: Integration by parts formulae for Wiener measures on a path space between two curves, preprint (2005)

[3] Goodman, V.: A divergence theorem for Hilbert space. Trans. Amer. Math. Soc. **164**, 411–426 (1972)

[4] Itô, S.: The fundamental solution of the parabolic equation in a differentiable manifold, II. Osaka Math. J. **6**, 167–185 (1954)

[5] Itô, S.: A boundary value problem of partial differential equations of parabolic type. Duke Math. J. **24**, 299–312 (1957)

[6] Itô, S.: Diffusion Equations (in Japanese), Kinokuniya-shoten, Tokyo (1979)

[7] Jost, J.: Partial Differential Equations, Springer, New York (2002)

[8] Nualart, D., Pardoux, E.: White noise driven quasilinear SPDEs with reflection. Probab. Theory Relat. Fields **93**, 77–89 (1992)

[9] Shigekawa, I.: Vanishing theorem of the Hodge-Kodaira operator for differential forms on a convex domain of the Wiener space. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **6**, 53–63 (2003)

[10] Zambotti, L.: Integration by parts formulae on convex sets of paths and applications to SPDEs with reflection. Probab. Theory Relat. Fields **123**, 579–600 (2002)

List of MHF Preprint Series, Kyushu University

21st Century COE Program
Development of Dynamic Mathematics with High Functionality

MHF2003-1 Mitsuhiro T. NAKAO, Kouji HASHIMOTO & Yoshitaka WATANABE
A numerical method to verify the invertibility of linear elliptic operators with applications to nonlinear problems

MHF2003-2 Masahisa TABATA & Daisuke TAGAMI
Error estimates of finite element methods for nonstationary thermal convection problems with temperature-dependent coefficients

MHF2003-3 Tomohiro ANDO, Sadanori KONISHI & Seiya IMOTO
Adaptive learning machines for nonlinear classification and Bayesian information criteria

MHF2003-4 Kazuhiro YOKOYAMA
On systems of algebraic equations with parametric exponents

MHF2003-5 Masao ISHIKAWA & Masato WAKAYAMA
Applications of Minor Summation Formulas III, Plücker relations, Lattice paths and Pfaffian identities

MHF2003-6 Atsushi SUZUKI & Masahisa TABATA
Finite element matrices in congruent subdomains and their effective use for large-scale computations

MHF2003-7 Setsuo TANIGUCHI
Stochastic oscillatory integrals - asymptotic and exact expressions for quadratic phase functions -

MHF2003-8 Shoki MIYAMOTO & Atsushi YOSHIKAWA
Computable sequences in the Sobolev spaces

MHF2003-9 Toru FUJII & Takashi YANAGAWA
Wavelet based estimate for non-linear and non-stationary auto-regressive model

MHF2003-10 Atsushi YOSHIKAWA
Maple and wave-front tracking — an experiment

MHF2003-11 Masanobu KANEKO
On the local factor of the zeta function of quadratic orders

MHF2003-12 Hidefumi KAWASAKI
Conjugate-set game for a nonlinear programming problem

MHF2004-1 Koji YONEMOTO & Takashi YANAGAWA
Estimating the Lyapunov exponent from chaotic time series with dynamic noise

MHF2004-2 Rui YAMAGUCHI, Eiko TSUCHIYA & Tomoyuki HIGUCHI
State space modeling approach to decompose daily sales of a restaurant into time-dependent multi-factors

MHF2004-3 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Cubic pencils and Painlevé Hamiltonians

MHF2004-4 Atsushi KAWAGUCHI, Koji YONEMOTO & Takashi YANAGAWA
Estimating the correlation dimension from a chaotic system with dynamic noise

MHF2004-5 Atsushi KAWAGUCHI, Kentarou KITAMURA, Koji YONEMOTO, Takashi YANAGAWA & Kiyofumi YUMOTO
Detection of auroral breakups using the correlation dimension

MHF2004-6 Ryo IKOTA, Masayasu MIMURA & Tatsuyuki NAKAKI
A methodology for numerical simulations to a singular limit

MHF2004-7 Ryo IKOTA & Eiji YANAGIDA
Stability of stationary interfaces of binary-tree type

MHF2004-8 Yuko ARAKI, Sadanori KONISHI & Seiya IMOTO
Functional discriminant analysis for gene expression data via radial basis expansion

MHF2004-9 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Hypergeometric solutions to the q - Painlevé equations

MHF2004-10 Raimundas VIDŪNAS
Expressions for values of the gamma function

MHF2004-11 Raimundas VIDŪNAS
Transformations of Gauss hypergeometric functions

MHF2004-12 Koji NAKAGAWA & Masakazu SUZUKI
Mathematical knowledge browser

MHF2004-13 Ken-ichi MARUNO, Wen-Xiu MA & Masayuki OIKAWA
Generalized Casorati determinant and Positon-Negaton-Type solutions of the Toda lattice equation

MHF2004-14 Nalini JOSHI, Kenji KAJIWARA & Marta MAZZOCCO
Generating function associated with the determinant formula for the solutions of the Painlevé II equation

MHF2004-15 Kouji HASHIMOTO, Ryohei ABE, Mitsuhiro T. NAKAO & Yoshitaka WATANABE
Numerical verification methods of solutions for nonlinear singularly perturbed problem

MHF2004-16 Ken-ichi MARUNO & Gino BIONDINI
Resonance and web structure in discrete soliton systems: the two-dimensional Toda lattice and its fully discrete and ultra-discrete versions

MHF2004-17 Ryuei NISHII & Shinto EGUCHI
Supervised image classification in Markov random field models with Jeffreys divergence

MHF2004-18 Kouji HASHIMOTO, Kenta KOBAYASHI & Mitsuhiro T. NAKAO
Numerical verification methods of solutions for the free boundary problem

MHF2004-19 Hiroki MASUDA
Ergodicity and exponential β-mixing bounds for a strong solution of Lévy-driven stochastic differential equations

MHF2004-20 Setsuo TANIGUCHI
The Brownian sheet and the reflectionless potentials

MHF2004-21 Ryuei NISHII & Shinto EGUCHI
Supervised image classification based on AdaBoost with contextual weak classifiers

MHF2004-22 Hideki KOSAKI
On intersections of domains of unbounded positive operators

MHF2004-23 Masahisa TABATA & Shoichi FUJIMA
Robustness of a characteristic finite element scheme of second order in time increment

MHF2004-24 Ken-ichi MARUNO, Adrian ANKIEWICZ & Nail AKHMEDIEV
Dissipative solitons of the discrete complex cubic-quintic Ginzburg-Landau equation

MHF2004-25 Raimundas VIDŪNAS
Degenerate Gauss hypergeometric functions

MHF2004-26 Ryo IKOTA
The boundedness of propagation speeds of disturbances for reaction-diffusion systems

MHF2004-27 Ryusuke KON
Convex dominates concave: an exclusion principle in discrete-time Kolmogorov systems

MHF2004-28 Ryusuke KON
Multiple attractors in host-parasitoid interactions: coexistence and extinction

MHF2004-29 Kentaro IHARA, Masanobu KANEKO & Don ZAGIER
Derivation and double shuffle relations for multiple zeta values

MHF2004-30 Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Generalized partitioned quantum cellular automata and quantization of classical CA

MHF2005-1 Hideki KOSAKI
Matrix trace inequalities related to uncertainty principle

MHF2005-2 Masahisa TABATA
Discrepancy between theory and real computation on the stability of some finite element schemes

MHF2005-3 Yuko ARAKI & Sadanori KONISHI
Functional regression modeling via regularized basis expansions and model selection

MHF2005-4 Yuko ARAKI & Sadanori KONISHI
Functional discriminant analysis via regularized basis expansions

MHF2005-5 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Point configurations, Cremona transformations and the elliptic difference Painlevé equations

MHF2005-6 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Construction of hypergeometric solutions to the q - Painlevé equations

MHF2005-7 Hiroki MASUDA
Simple estimators for non-linear Markovian trend from sampled data:
I. ergodic cases

MHF2005-8 Hiroki MASUDA & Nakahiro YOSHIDA
Edgeworth expansion for a class of Ornstein-Uhlenbeck-based models

MHF2005-9 Masayuki UCHIDA
Approximate martingale estimating functions under small perturbations of dynamical systems

MHF2005-10 Ryo MATSUZAKI & Masayuki UCHIDA
One-step estimators for diffusion processes with small dispersion parameters from discrete observations

MHF2005-11 Junichi MATSUKUBO, Ryo MATSUZAKI & Masayuki UCHIDA
Estimation for a discretely observed small diffusion process with a linear drift

MHF2005-12 Masayuki UCHIDA & Nakahiro YOSHIDA
AIC for ergodic diffusion processes from discrete observations

MHF2005-13 Hiromichi GOTO & Kenji KAJIWARA
Generating function related to the Okamoto polynomials for the Painlevé IV equation

MHF2005-14 Masato KIMURA & Shin-ichi NAGATA
Precise asymptotic behaviour of the first eigenvalue of Sturm-Liouville problems with large drift

MHF2005-15 Daisuke TAGAMI & Masahisa TABATA
Numerical computations of a melting glass convection in the furnace

MHF2005-16 Raimundas VIDŪNAS
Normalized Leonard pairs and Askey-Wilson relations

MHF2005-17 Raimundas VIDŪNAS
Askey-Wilson relations and Leonard pairs

MHF2005-18 Kenji KAJIWARA & Atsushi MUKAIHIRA
Soliton solutions for the non-autonomous discrete-time Toda lattice equation

MHF2005-19 Yuu HARIYA
Construction of Gibbs measures for 1-dimensional continuum fields

MHF2005-20 Yuu HARIYA
Integration by parts formulae for the Wiener measure restricted to subsets in $\mathbb{R}^d$