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Abstract

We construct generating functions for the entries of Hankel determinant formula for the Okamoto polynomials which characterize a class of rational solutions to the Painlevé IV equation. Generating functions are characterized as asymptotic expansions of log derivative of Ai and Bi, which are solutions of the Airy equation.

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1 Introduction

In the theory of integrable systems, structure of exact solutions are considered to reflect essential mathematical structures behind the equations. In the soliton theory, as clarified by the Sato theory[12], the dependent variables which are introduced intuitively in the Hirota's method (τ functions) are regarded as the most fundamental object, and determinant or Pfaffian structure of the τ functions is reflection of the solution space and the transformation group acting on it. For the Painlevé equations, it is considered that the τ functions and their determinant structure are of the same importance as the soliton equations, although the Painlevé equations are ordinary differential equations while the soliton equations are partial differential equations. In the series of works[15, 16, 17, 18], Okamoto has introduced the τ functions through the Hamiltonians, and studied their properties. He has shown that the τ functions satisfy the Toda equation in general, and that some class of transcendental classical solutions (hypergeometric solutions, or special function solutions) can be expressible in terms of determinants whose entries are given in terms of special functions of hypergeometric type. Moreover, for algebraic (or rational) solutions, which are another class of classical solutions to the Painlevé equations, it has been revealed that the special polynomials that characterize the solutions are specialization of the Schur functions or their generalization (universal characters) by studying the determinant formula of Jacobi-Trudi type [4, 7, 8, 10, 11, 14].

For example, the Painlevé II equation(P_{II})

$$\frac{d^2u}{dx^2} = 2u^3 - 4xu + 4\left(\alpha + \frac{1}{2}\right),\tag{1}$$

admits a one-parameter family of transcendental classical solution expressible in terms of the solution of the Airy equation when α is an integer, and rational solutions when α is a half-integer. As for the rational solutions, the following fact is known: let T_N ($N \in \mathbb{Z}$) be the polynomials which are generated by the Toda equation

$$T_N'' T_N - (T_N')^2 = T_{N+1}T_{N-1} - x\,T_N^2, \quad T_0 = 1, \quad T_1 = x, \quad ' = \frac{d}{dx}.\tag{2}$$

Then we have $T_N = T_{-N-1}$, and it is shown that the function

$$u = \frac{d}{dx}\log\frac{T_{N+1}}{T_N},\tag{3}$$

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satisfy P_{II} (1) with $\alpha = N + \frac{1}{2}$. The polynomials T_N is called the Yablonski-Vorob'ev (Y-V) polynomials. It is known that the Y-V polynomials admit the following determinant formula of Jacobi-Trudi type[7]:

$$T_N = c_N \begin{vmatrix} p_N & p_{N+1} & \cdots & p_{2N-1} \\ p_{N-2} & p_{N-1} & \cdots & p_{2N-3} \\ \vdots & \vdots & \ddots & \vdots \\ p_{-N+1} & p_{-N+2} & \cdots & p_1 \end{vmatrix}, \quad \sum_{k=0}^{\infty} p_k \lambda^k = \exp\left[x\lambda + \frac{\lambda^3}{3}\right], \quad p_n = 0 \ (n < 0), \tag{4}$$

$$c_N = (2N-1)^{!!} = (2N-1)!!(2N-3)!! \cdots 3!!. \tag{5}$$

Since the Schur function $S_Y(t_1, t_2, t_3, \ldots)$ associated with the 2-core Young diagram $Y = (N, N-1, \cdots, 1)$ is given by the determinant in (4) with p_k replaced by $\sum_{k=0}^{\infty} p_k \lambda^k = \exp \sum_{n=1}^{\infty} t_n \lambda^n$ [12], the above formula immediately implies that T_N are nothing but the specialization of 2-core Schur functions. Furthermore, since the 2-core Schur functions give rational solutions to KdV or modified KdV equation[12], this coincides with the fact that P_{II} can be derived from similarity reduction of the modified KdV equation(see, for example, [1]). In this manner, the determinant formula is useful to clarify the essential property of the object.

On the other hand, it is known that T_N admits the following Hankel determinant formula[7]:

$$T_N = \begin{vmatrix} a_0 & a_1 & \cdots & a_{N-1} \\ a_1 & a_2 & \cdots & a_N \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1} & a_N & \cdots & a_{2N-2} \end{vmatrix}, \quad a_0 = x, \quad a_n = a'_{n-1} + \sum_{k=0}^{n-2} a_k a_{n-2-k}. \tag{6}$$

If a determinant formula reflects some mathematical structure, what does this formula imply? In particular, what are the entries a_n? In order to investigate this problem, a generating function for a_n was considered in [2], and the following result was obtained:

Theorem 1.1 *[2] We define the generating function $F(x,t)$ of a_n $(n \geq 0)$ by*

$$F(x,t) = \sum_{n=0}^{\infty} a_n \, (-2t)^{-n}. \tag{7}$$

Then, the following asymptotic expansion around $t \to \infty$ holds for any subsector of $|\arg t| < \frac{\pi}{2}$:

$$F(x,t) \sim \frac{\partial}{\partial t} \log \theta(x,t), \quad \theta(x,t) = e^{\frac{2t^3}{3}} \mathrm{Ai}(t^2 - x), \tag{8}$$

where Ai *is the Airy function.*

This result is quite suggestive, since log derivative of the Airy function is another particular solution to P_{II}. In fact,

$$u = \frac{d}{dx} \log \mathrm{Ai}(2^{1/3} x), \tag{9}$$

satisfies P_{II} (1) with $\alpha = 0$. Is this phenomenon caused by an accident? In order to answer this question, it may be an important problem to investigate whether similar phenomena take place for other Painlevé equations or not.

In this article, we consider the Painlevé IV equation (P_{IV})

$$\frac{d^2 y}{dx^2} = \frac{1}{2y}\left(\frac{dy}{dx}\right)^2 + \frac{3}{2}y^3 + 6xy + \frac{9}{2}\left(x^2 - \frac{4}{3}\alpha\right)y - \frac{\beta}{2y}, \tag{10}$$

where α and β are parameters, and the Okamoto polynomials which characterize a class of rational solutions for P_{IV}. We construct the Hankel determinant formula for the Okamoto polynomials and discuss the generating functions for the entries of determinants.

2 Okamoto polynomials and Hankel determinant formula

Let Q_n ($n \in \mathbb{Z}$) be polynomials generated by the following Toda equation:

$$Q_n'' Q_n - (Q_n')^2 = Q_{n+1} Q_{n-1} - (x^2 + 2n + 1) Q_n^2, \tag{11}$$

$$Q_0 = 1, \ Q_1 = x^2 + 1. \tag{12}$$

Then, it is known that [17, 13]

$$y = \frac{d}{dx} \log \frac{Q_{n+1}}{Q_n} - x, \tag{13}$$

satisfies P$_{\mathrm{IV}}$ (10) with $\alpha = n + 1$, $\beta = 1$. The polynomials Q_n are called the Okamoto polynomials. A Jacobi-Trudi type determinant formula for Q_n is discussed in [8, 14, 13], and it is shown that Q_n are specialization of 3-core Schur functions.

Let us construct the Hankel determinant formula for Q_n. A Hankel determinant formula for the general solution of the Toda equation has been presented in [6] as follows:

Theorem 2.1 *[6] Let $\{\tau_n\}_{n \in \mathbb{Z}}$ functions satisfying the Toda equation:*

$$\tau_n'' \tau_n - \tau_n'^2 = \tau_{n+1} \tau_{n-1} - \psi \varphi \, \tau_n^2, \quad \tau_{-1} = \psi, \quad \tau_0 = 1, \quad \tau_1 = \varphi. \tag{14}$$

Let $\{a_n\}_{n \in \mathbb{N}}$, $\{b_n\}_{n \in \mathbb{N}}$ be sequences defined by the recursion relation

$$a_n = a_{n-1}' + \psi \sum_{k=0}^{n-2} a_k a_{n-k}, \quad b_n = b_{n-1}' + \varphi \sum_{k=0}^{n-k} b_k b_{n-k}, \quad a_0 = \varphi, \quad b_0 = \psi, \tag{15}$$

respectively. Then τ_n is expressed as

$$\tau_n = \begin{cases} \det(a_{i+j-2})_{i,j \leq n} & n > 0, \\ 1 & n = 0, \\ \det(b_{i+j-2})_{i,j \leq |n|} & n < 0. \end{cases} \tag{16}$$

Applying Theorem 2.1 to the Toda equation for Q_n (11), we have the following formula:

Proposition 2.2

$$Q_n = \begin{cases} \det\left(a_{i+j-2}\right)_{1 \leq i,j \leq n}, & n \geq 1 \\ 1 & n = 0, -1 \\ \det\left(b_{i+j-2}\right)_{1 \leq i,j \leq |n|-1}, & n \leq -1 \end{cases} \tag{17}$$

$$a_n = a_{n-1}' + 2x a_{n-1} + \sum_{k=0}^{n-2} a_k a_{n-2-k}, \quad a_0 = x^2 + 1, \quad a_1 = 2x^3 + 4x, \tag{18}$$

$$b_n = b_{n-1}' - 2x b_{n-1} + \sum_{k=0}^{n-2} b_k b_{n-2-k}, \quad b_0 = x^2 - 1, \quad b_1 = 2x^3 - 4x. \tag{19}$$

Proof For technical reason, we separate the cases $n > 0$ and $n < 0$. Putting $Q_n = \mathrm{e}^{-nx^2} R_n$ in (11), we have

$$R_n'' R_n - (R_n')^2 = R_{n+1} R_{n-1} - (x^2 + 1) R_n^2,$$
$$R_{-1} = \mathrm{e}^{-x^2}, \ R_0 = 1, \ R_1 = \mathrm{e}^{x^2}(x^2 + 1). \tag{20}$$

Applying Theorem 2.1 to (20) with $n > 0$, we have

$$R_n = \det(\tilde{a}_{i+j-2}), \quad \tilde{a}_n = \tilde{a}_{n-1}' + \mathrm{e}^{-x^2} \sum_{k=0}^{n-2} \tilde{a}_k \tilde{a}_{n-k}, \quad \tilde{a}_0 = \mathrm{e}^{x^2}(x^2 + 1). \tag{21}$$

Now putting $\tilde{a}_n = \mathrm{e}^{x^2} a_n$ and noticing that $R_n = \mathrm{e}^{nx^2} Q_n$, we obtain the formula for $n > 0$ as

$$Q_n = \det(a_{i+j-2}), \quad a_n = a_{n-1}' + 2x a_{n-1} + \sum_{k=0}^{n-2} a_k a_{n-k}, \quad a_0 = x^2 + 1. \tag{22}$$

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For the case of $n < 0$, putting $Q_n = \mathrm{e}^{-(n+1)x^2} S_{n+1}$, we have

$$S_n'' S_n - (S_n')^2 = S_{n+1} S_{n-1} - (x^2 - 1)S_n^2,$$
$$S_{-1} = \mathrm{e}^{-x^2}(x^2 - 1), \ S_0 = 1, \ S_1 = \mathrm{e}^{x^2}. \tag{23}$$

Applying Theorem 2.1 to (23) with $n < 0$, we get the formula for $n < 0$ by similar calculations. $\square$

3 Generating functions

3.1 Main result

Let $I_\nu(z)$ be the modified Bessel function

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}, \quad |z| < \infty, \quad |\arg z| < \pi. \tag{24}$$

We define the functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ by

$$\mathrm{Ai}(z) = \frac{z^{1/2}}{3}\left[I_{-1/3}\left(\frac{2z^{3/2}}{3}\right) - I_{1/3}\left(\frac{2z^{3/2}}{3}\right)\right], \tag{25}$$

$$\mathrm{Bi}(z) = \frac{z^{1/2}}{3}\left[I_{-1/3}\left(\frac{2z^{3/2}}{3}\right) + I_{1/3}\left(\frac{2z^{3/2}}{3}\right)\right], \tag{26}$$

respectively. It is known that $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are independent solutions of the Airy equation

$$\frac{d^2 u}{dz^2} = zu, \tag{27}$$

and that they admit the following asymptotic expansion around $z \sim \infty$ [9]:

$$\mathrm{Ai}(z) \sim U_-(z) \quad |\arg z| < \pi,$$
$$\mathrm{Bi}(z) \sim U_+(z) \quad |\arg z| < \frac{\pi}{3}, \tag{28}$$

where

$$U_\pm(z) = \frac{1}{2\sqrt{\pi}} \exp\left(\pm\frac{2z^{3/2}}{3}\right) z^{-1/4} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{n!} \left(\pm\frac{4z^{3/2}}{3}\right)^{-n}. \tag{29}$$

Now we define the generating functions for the entries a_n and b_n of the Hankel determinant formula of Q_n obtained in Proposition 2.2 by

$$F(x,t) = \sum_{n=0}^{\infty} a_n t^{-3n}, \quad G(x,t) = \sum_{n=0}^{\infty} b_n t^{-3n}. \tag{30}$$

Then the main result of this article is stated as follows:

Theorem 3.1 *Define the functions $\theta_1(x,t)$, $\theta_2(x,t)$ by*

$$\begin{cases} \theta_1(x,t) = \mathrm{Bi}\left(\frac{t^4}{4} - xt\right)\exp\left(-\frac{t^6}{12} + \frac{xt^3}{2}\right), \\ \theta_2(x,t) = \mathrm{Ai}\left(\frac{t^4}{4} + xt\right)\exp\left(\frac{t^6}{12} + \frac{xt^3}{2}\right), \end{cases} \tag{31}$$

respectively. Then, the following asymptotic expansions hold around $t \sim \infty$:

$$F(x,t) \sim \frac{t^4}{x - t^3}\left[\frac{\partial}{\partial t}\log\theta_1(x,t) - \frac{x^2}{t}\right], \quad |\arg t| < \frac{\pi}{12}, \tag{32}$$

$$G(x,t) \sim \frac{t^4}{x + t^3}\left[\frac{\partial}{\partial t}\log\theta_2(x,t) + \frac{x^2}{t}\right], \quad |\arg t| < \frac{\pi}{4}. \tag{33}$$

4

3.2 Riccati equation

In the following we give the proof of Theorem 3.1. The strategy for proof is described as follows:

(1) We derive Riccati equations for F and G from the recursion relations (18) and (19), respectively.

(2) We then linearize the Riccati equations by a standard technique to yield the Airy equation.

(3) Finally we identify the functions by investigating asymptotic behaviours.

Proposition 3.2 *$F(x,t)$ and $G(x,t)$ satisfy the folloiwng Riccati equations,*

$$t^3 \frac{\partial F}{\partial x} = -F^2 + t^3(t^3 - 2x)F - t^6(x^2 + 1), \tag{34}$$

$$t^3 \frac{\partial G}{\partial x} = -G^2 + t^3(t^3 + 2x)G - t^6(x^2 - 1), \tag{35}$$

respectively.

proof: (34) can be derived as follows:

$$
\begin{aligned}
F^2 &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} a_k a_{n-k} \right) t^{-3n} = \sum_{n=0}^{\infty} \left(a_{n+2} - a'_{n+1} - 2x a_{n+1} \right) t^{-3n} \\
&= \left(t^6 F - t^6 a_0 - t^3 a_1 \right) - \left(t^3 \frac{\partial F}{\partial x} - t^3 a'_0 \right) - 2x \left(t^3 F - t^3 a_0 \right) \\
&= -t^3 \frac{\partial F}{\partial x} + t^3(t^3 - 2x)F - t^6(x^2 + 1),
\end{aligned}
$$

where we have used (18) to derive the second equality. We obtain (35) by the similar calculation.□

Since $F(x,t)$ and $G(x,t)$ are defined as formal series around $t \sim \infty$, it is useful to derive differential equations with respect to t. For this purpose we present the following lemma:

Lemma 3.3 *(1) a_n and b_n satisfy the following recursion relations,*

$$a'_n = xa'_{n-1} + (3n + 1)a_{n-1}, \tag{36}$$
$$b'_n = -xb'_{n-1} - (3n + 1)b_{n-1}, \tag{37}$$

respectively.

(2) $F(x,t)$ and $G(x,t)$ satisfy the following linear differential equations,

$$(x - t^3)\frac{\partial F}{\partial x} = t\frac{\partial F}{\partial t} - 4F - 2xt^3, \tag{38}$$
$$(x + t^3)\frac{\partial G}{\partial x} = t\frac{\partial G}{\partial t} - 4G + 2xt^3, \tag{39}$$

respectively.

The first statement can be proved by a simple induction. The second statement follows immediately from (36) and (37). □

From Lemma 3.3, we obtain the Riccati equations with respect to t.

Proposition 3.4 *$F(x,t)$ and $G(x,t)$ satisfy the following Riccati equations*

$$t \frac{\partial F}{\partial t} = -\frac{x - t^3}{t^3}F^2 + (-t^6 + 3t^3 x - 2x^2 + 4)F + t^3(t^3 x^2 + t^3 - x^3 + x), \tag{40}$$

$$t \frac{\partial G}{\partial t} = -\frac{x + t^3}{t^3}G^2 + (t^6 + 3xt^3 + 2x^2 + 4)G + t^3(-t^3 x^2 + t^3 - x^3 - x), \tag{41}$$

respectively.

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3.3 Linearization and identification of functions

We obtain the Airy equation by linearization of the Riccati equations by standard technique.

Proposition 3.5   *(1) Putting*

$$F = t^3 \frac{v_x}{v} - xt^3 + \frac{t^6}{2},$$

(42)

the Riccati equation (34) reduces to the following linear equation:

$$\frac{\partial^2 v}{\partial x^2} = t^3 \left(-x + \frac{t^3}{4}\right) v.$$

(43)

(2) Similarly, putting

$$F = \frac{t^4}{x - t^3} \frac{v_t}{v} - xt^3 + \frac{t^6}{2},$$

(44)

the Riccati equation (40) reduces to the following linear equation:

$$\frac{\partial^2 v}{\partial t^2} = -\frac{3t^2}{x - t^3} \frac{\partial v}{\partial t} + t(x - t^3)^2 \left(-x + \frac{t^3}{4}\right) v.$$

(45)

(3) (43) and (45) are transformed into the Airy equation

$$\frac{d^2 v}{dz^2} = zv,$$

(46)

by the change of variable

$$z = t \left(-x + \frac{t^3}{4}\right).$$

(47)

A similar proposition holds for $G(x, t)$:

Proposition 3.6   *(1) Putting*

$$G = t^3 \frac{w_x}{w} + xt^3 + \frac{t^6}{2},$$

(48)

the Riccati equation (35) reduces to the following linear equation:

$$\frac{\partial^2 w}{\partial x^2} = t^3 \left(x + \frac{t^3}{4}\right) w.$$

(49)

(2) Similarly, putting

$$G = \frac{t^4}{x + t^3} \frac{w_t}{w} + xt^3 + \frac{t^6}{2},$$

(50)

the Riccati equation (41) reduces to the following linear equation:

$$\frac{\partial^2 w}{\partial t^2} = \frac{3t^2}{x + t^3} \frac{\partial w}{\partial t} + t(x + t^3)^2 \left(x + \frac{t^3}{4}\right) w.$$

(51)

(3) (49) and (51) are transformed into the Airy equation

$$\frac{d^2 w}{dz^2} = zw,$$

(52)

by the change of variable,

$$z = t \left(x + \frac{t^3}{4}\right).$$

(53)

We omit the proof since it is done by elementary calculations.

Remark 3.7 *Assertions (1) and (2) in Proposition 3.5 are consistent. Namely, (1) and (2) can be transformed to each other under the assumption that (x, t) dependence of u is given by $u = u(z)$. A similar remark also holds for Proposition 3.6.*

Let us finally identify the function by considering its asymptotic behaviour around $t \sim \infty$. Solving (44) in terms of v, we have from (30)

$$v(x, t) = \text{const.} \times \exp\left[\frac{1}{12}t^6 - \frac{1}{2}xt^3\right] \times \frac{1}{t} \times \exp\left[-\sum_{n=0}^{\infty} \frac{xa_{n-1} - a_n}{3n} t^{-3n}\right]. \tag{54}$$

The function v is given as linear combination of Ai(z) and Bi(z). We compare the above expression with (28) and (29). Noticing that we have from (47)

$$\frac{2}{3}z^{\frac{3}{2}} \overset{t \to \infty}{\sim} \frac{t^6}{12} - \frac{1}{2}xt^3 + \cdots, \quad z^{-\frac{1}{4}} \overset{t \to \infty}{\sim} \frac{\sqrt{2}}{t}\left(1 + \frac{x}{t^3} + \cdots\right),$$

we find that u should be

$$u(x, t) \sim U_+(z), \tag{55}$$

by comparing the exponential factors. This proves the statement regarding F in Theorem 3.1. The statement for G can be proved in a similar manner by using Proposition 3.6. $\square$

4   Concluding remarks

In this article, we have shown that the generating functions associated with the Hankel determinant formula for Okamoto polynomials can be characterized by the log derivative of solutions of the Airy equation. It is an interesting observation that Bi and Ai, the solutions of Airy equation with different asymptotic behaviours, appear in opposite directions with respect to n.

We have posed the question as to whether the appearance of the log derivative of the Airy function in the generating function associated with Y-V polynomials is an accident or not. The result of this article appears to imply that the answer is "No". It may be an interesting problem to characterize generating functions associated with special polynomials for other Painlevé equations in terms of the log derivative of special functions of hypergeometric type.

If this phenomena is not caused by accident, we should clarify the reason and mechanism underlying it. Unfortunately we cannot give an answer to this problem yet. Recently, Hankel determinant formula for generic solutions to P$_{\text{II}}$ and the generating functions of its entries have been discussed in [3]. It is shown that the generating functions are also characterized by log derivative of solutions of certain linear differential equations, which are nothing but the auxiliary linear problem for P$_{\text{II}}$. This result is not a conclusive answer to the problem, but it may imply some (unknown) correspondence between the Painlevé equations and their auxiliary linear problems. At least, it may be true that there is a nontrivial mathematical structure behind Hankel determinant formulae, which deserves further study.

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