

Construction of hypergeometric solutions to the q -Painlevé equations

Kajiware, Kenji

Faculty of Mathematics, Kyushu University | School of Mathematics and Statistics F07,
University of Sydney

Masuda, Tetsu

Department of Mathematics, Kobe University

Noumi, Masatoshi

Department of Mathematics, Kobe University

Ohta, Yasuhiro

Department of Mathematics, Kobe University

他

<https://hdl.handle.net/2324/11848>

出版情報 : International Mathematics Research Notices. 2005, pp.1439–1463, 2005. Hindawi Publishing Corporation

バージョン :

権利関係 : The article has been accepted for publication in International Mathematics Research Notices (C): 2005 Published by Hindawi Publishing Corporation. All rights reserved.

MHF Preprint Series

Kyushu University
21st Century COE Program
Development of Dynamic Mathematics with
High Functionality

Construction of hypergeometric solutions to the q Painlevé equations

K. Kajiwara, T. Masuda,
M. Noumi, Y. Ohta,
Y. Yamada

MHF 2005-6

(Received February 1, 2005)

Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

Construction of Hypergeometric Solutions to the q -Painlevé Equations

K. Kajiwara^{1,3}, T. Masuda², M. Noumi², Y. Ohta² and Y. Yamada²

¹ Graduate School of Mathematics, Kyushu University, 6-10-1 Hakozaki, Fukuoka 812-8581, Japan

² Department of Mathematics, Kobe University, Rokko, Kobe 657-8501, Japan

³ School of Mathematics and Statistics F07, University of Sydney, Sydney, NSW 2006, Australia

January 30, 2005

Abstract

Hypergeometric solutions to the q -Painlevé equations are constructed by direct linearization of discrete Riccati equations. The decoupling factors are explicitly determined so that the linear systems give rise to q -hypergeometric equations.

1 Introduction

This article is a continuation of our previous work [1] on the hypergeometric solutions to the q -Painlevé equations in the following degeneration diagram of affine Weyl group symmetries [2, 3] :

$$E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_5^{(1)} \rightarrow A_4^{(1)} \rightarrow (A_2 + A_1)^{(1)} \rightarrow (A_1 + \frac{A_1}{|\alpha|^2=14})^{(1)} \quad (1)$$

The list of q -Painlevé equations we are going to investigate is given in section 3 below. We remark that these q -Painlevé equations were discovered through various approaches to discrete Painlevé equations, including singularity confinement analysis, compatibility conditions of linear difference equations, affine Weyl group symmetries and τ -functions on the lattices. Also, in Sakai's framework [2], each of these q -Painlevé equations is constructed in a unified manner as the birational action of a translation of the corresponding affine Weyl group on a certain family of rational surfaces.

In [4] we have introduced the formulation of discrete Painlevé equations based on the geometry of plane curves on \mathbb{P}^2 . On that basis we were able in the first part of [1] to find suitable coordinates for linearization of the q -Painlevé equations into three-term relations of hypergeometric functions. As a result we obtained the following degeneration diagram of basic hypergeometric functions corresponding to (1):

$$\begin{array}{ccccccc} \text{balanced} & & & \text{balanced} & & & \\ {}_{10}W_9 & \rightarrow & {}_8W_7 & \rightarrow & {}_3\varphi_2 & \rightarrow & {}_2\varphi_1 \rightarrow {}_1\varphi_1 \rightarrow \begin{array}{c} {}_1\varphi_1 \left(\begin{array}{c} a \\ 0 \end{array} ; q, z \right) \\ {}_1\varphi_1 \left(\begin{array}{c} 0 \\ b \end{array} ; q, z \right) \end{array} \rightarrow {}_1\varphi_1 \left(\begin{array}{c} 0 \\ -q \end{array} ; q, z \right) \end{array}$$

This shows the usefulness of geometric consideration in the study of particular solutions of discrete Painlevé equations. In order to determine explicit solutions to those equations written in the literature, further steps of precise computations are required since the variables to be solved are fixed in advance. We have shown only the results for them in the second part of [1]. The purpose of this article is to present these explicit solutions including subtle gauge factors and to show the calculation in detail based on the direct linearization of discrete Riccati equations.

This article is organized as follows: In section 2, we explain the procedure to construct hypergeometric solutions through the linearization of discrete Riccati equations. We demonstrate this procedure by taking the case of $E_7^{(1)}$ as an example. In section 3, we give the list of q -Painlevé equations and their hypergeometric solutions, with the data that are necessary for constructing solutions. Among the cases in the diagram (1) we have excluded the case of $D_5^{(1)}$, since a complete identification of hypergeometric solutions has already been given in [5].

2 Construction of Hypergeometric Solutions

2.1 Discrete Riccati Equation and Its Linearization

The q -Painlevé equations admit particular solutions characterized by discrete Riccati equations for special values of parameters. Reduction to discrete Riccati equation has been already done for all the q -Painlevé equations[3, 6, 7, 8]. We also note that such special situations have clear geometrical meaning, as was discussed in [1]. The basic idea for constructing hypergeometric solution is as follows: we linearize the discrete Riccati equations to yield second order linear q -difference equations. We then identify them with the three-term relation of an appropriate basic hypergeometric series.

Let us explain this procedure in detail. Suppose we have a discrete Riccati equation of the form

$$\bar{z} = \frac{Az + B}{Cz + D}, \quad (2)$$

where $z = z(t)$ and $\bar{z} = z(qt)$. We also use the notation $\underline{z} = z(t/q)$, and so forth. Moreover, the coefficients A, B, C and D are functions of t . First let us put an ansatz

$$z = \frac{F}{G}. \quad (3)$$

Then the discrete Riccati equation is linearized to

$$\frac{\bar{F}}{H} = AF + BG, \quad \frac{\bar{G}}{H} = CF + DG, \quad (4)$$

where H is an arbitrary decoupling factor. Eliminating G from eq.(4) we have for F the three-term relation

$$\bar{F} + c_1 F + c_2 \underline{F} = 0, \quad c_1 = -\frac{H}{B}(A\underline{B} + B\underline{D}), \quad c_2 = \frac{B}{\underline{B}} \frac{H\underline{H}(\underline{A}\underline{D} - \underline{B}\underline{C})}{B}. \quad (5)$$

The three-term relation for a basic hypergeometric series often takes the form

$$V_1(\bar{\Phi} - \Phi) + V_2\Phi + V_3(\underline{\Phi} - \Phi) = 0, \quad (6)$$

where the coefficients V_1, V_2 and V_3 are factorized into binomials involving the independent variable and parameters. Comparing eqs.(5) with (6), we have

$$\frac{V_2}{V_1} = 1 + c_1 + c_2, \quad \frac{V_3}{V_2} = c_2. \quad (7)$$

We look for the decoupling factor H so that these quantities factorize. We then identify the three-term relation with that for appropriate hypergeometric function. This is done by trial and error with the aid of computer algebra, but it is not practically difficult since we already know the hypergeometric function and its three-term relation to appear for each q -Painlevé equation.

Step 1. Find the decoupling factor H such that

$$\frac{V_2}{V_1} = 1 + c_1 + c_2, \quad \frac{V_3}{V_2} = c_2, \quad (8)$$

factorize. Then identify the three-term relation

$$V_1(\bar{F} - F) + V_2F + V_3(\underline{F} - F) = 0, \quad (9)$$

with that for an appropriate hypergeometric function.

Similarly, we have for G the three-term relation

$$\bar{G} + \tilde{d}_1 G + \tilde{d}_2 \underline{G} = 0, \quad \tilde{d}_1 = -\frac{H}{\underline{C}}(D\underline{C} + C\underline{A}), \quad \tilde{d}_2 = \frac{C}{\underline{C}} \frac{H\underline{H}(\underline{A}\underline{D} - \underline{B}\underline{C})}{\underline{C}}. \quad (10)$$

However, usually $1 + \tilde{d}_1 + \tilde{d}_2$ does not factorize for H obtained above. Replacing G with κG ($\bar{\kappa} = k\kappa$), we have

$$z = \frac{1}{\kappa} \frac{F}{G}, \quad (11)$$

$$\bar{G} + d_1 G + d_2 \underline{G} = 0, \quad d_1 = -\frac{H}{k\underline{C}}(\underline{DC} + \underline{CA}), \quad d_2 = \frac{C}{\underline{C}} \frac{HH}{kk}(\underline{AD} - \underline{BC}). \quad (12)$$

We then look for k so that $1 + d_1 + d_2$ factorizes, and identify eq.(12) with the three-term relation of the same hypergeometric function as F with different parameters. Putting $H/k = K$, this is equivalent to the following procedure:

Step 2. *In the three-term relation*

$$\bar{G} + d_1 G + d_2 \underline{G} = 0, \quad d_1 = -\frac{K}{\underline{C}}(\underline{DC} + \underline{CA}), \quad d_2 = \frac{C}{\underline{C}} \frac{KK}{kk}(\underline{AD} - \underline{BC}), \quad (13)$$

find decoupling factor K so that

$$\frac{U_2}{U_1} = 1 + d_1 + d_2, \quad \frac{U_3}{U_2} = d_2, \quad (14)$$

factorize. Then identify the three-term relation with that for an appropriate hypergeometric function. Now we have

$$z = \frac{1}{\kappa} \frac{\theta_1 \Phi}{\theta_2 \Psi}, \quad \frac{H}{K} = k, \quad \frac{\bar{\kappa}}{\kappa} = k, \quad (15)$$

where Φ and Ψ are some hypergeometric functions, and θ_i ($i = 1, 2$) are constants(gauge factors).

Finally we determine the gauge factors θ_1 and θ_2 :

Step 3. *Compare the linear relations,*

$$\frac{\theta_1 \bar{\Phi}}{H} = A\theta_1 \Phi + \kappa B\theta_2 \Psi, \quad \frac{\bar{\kappa}\theta_2 \bar{\Psi}}{H} = C\theta_1 \Phi + \kappa D\theta_2 \Psi, \quad (16)$$

with contiguity relations of the hypergeometric functions to determine θ_1 and θ_2 .

2.2 An Example: Case of $E_7^{(1)}$

In this section we demonstrate the construction of the hypergeometric solution to the q -Painlevé equation of type $E_7^{(1)}$ as an example, following the procedure in the previous section.

Before proceeding, let us first summarize the definition and terminology of the basic hypergeometric series [9]. The basic hypergeometric series ${}_r\varphi_s$ is given by

$${}_r\varphi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_r; q)_n}{(b_1; q)_n \cdots (b_s; q)_n (q; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n, \quad (17)$$

$$(a; q)_n = (1-a)(1-qa) \cdots (1-q^{n-1}a).$$

The basic hypergeometric series ${}_{r+1}\varphi_r$ is called *balanced*¹ if the condition

$$qa_1 a_2 \cdots a_{r+1} = b_1 b_2 \cdots b_r, \quad z = q, \quad (18)$$

is satisfied, and is called *very-well-poised* if the condition

$$qa_1 = a_2 b_1 = \cdots = a_{r+1} b_r, \quad a_2 = qa_1^{\frac{1}{2}}, \quad a_3 = -qa_1^{\frac{1}{2}}, \quad (19)$$

¹The “balanced ${}_3\varphi_2$ ” in the diagram (1) is due to the convention that was used in [10].

is satisfied. A very-well-poised hypergeometric series ${}_{r+1}\varphi_r$ is denoted as ${}_{r+1}W_r$:

$${}_{r+1}W_r(a_1; a_4, \dots, a_{r+1}; q, z) = {}_r\varphi_s \left(\begin{matrix} a_1, qa_1^{\frac{1}{2}}, -qa_1^{\frac{1}{2}}, a_4, \dots, a_{r+1} \\ a_1^{\frac{1}{2}}, -a_1^{\frac{1}{2}}, qa_1/a_4, \dots, qa_1/a_{r+1} \end{matrix} ; q, z \right). \quad (20)$$

Now the q -Painlevé equation of type $E_7^{(1)}$ is given by [2, 3]

$$\begin{aligned} \frac{(f\bar{g} - \bar{t}t)(fg - t^2)}{(f\bar{g} - 1)(fg - 1)} &= \frac{(f - b_1t)(f - b_2t)(f - b_3t)(f - b_4t)}{(f - b_5)(f - b_6)(f - b_7)(f - b_8)}, \\ \frac{(fg - t^2)(\bar{f}\bar{g} - \bar{t}\bar{t})}{(fg - 1)(\bar{f}\bar{g} - 1)} &= \frac{\left(g - \frac{t}{b_1}\right)\left(g - \frac{t}{b_2}\right)\left(g - \frac{t}{b_3}\right)\left(g - \frac{t}{b_4}\right)}{\left(g - \frac{1}{b_5}\right)\left(g - \frac{1}{b_6}\right)\left(g - \frac{1}{b_7}\right)\left(g - \frac{1}{b_8}\right)}, \end{aligned} \quad (21)$$

where t is the independent variable and b_i ($i = 1, \dots, 8$) are parameters satisfying

$$\bar{t} = qt, \quad b_1b_2b_3b_4 = q, \quad b_5b_6b_7b_8 = 1. \quad (22)$$

Proposition 2.1 [3, 6] *In case of $b_1b_3 = b_5b_7$, eq.(21) admits a specialization to the discrete Riccati equation,*

$$\bar{g} = \frac{(\bar{t} - 1)f + t\{-(b_6 + b_8)\bar{t} + (b_2 + b_4)\}}{\{-(b_6 + b_8) + (b_2 + b_4)t\}f + b_6b_8(1 - \bar{t})}, \quad (23)$$

$$f = \frac{(t^2 - 1)b_5b_7g + t\{(b_1 + b_3) - (b_5 + b_7)t\}}{\{t(b_1 + b_3) - (b_5 + b_7)\}g + (1 - t^2)}. \quad (24)$$

As was pointed out in [1], in the cases of $E_{6,7,8}^{(1)}$ the variables f and g are not expressed by ratio of hypergeometric functions. We choose the variable as²

$$z = \frac{g - t/b_1}{g - 1/b_5}. \quad (25)$$

Then the discrete Riccati equation (23) and (24) is rewritten as

$$\bar{z} = \frac{Az + B}{Cz + D},$$

with

$$\begin{aligned} A &= b_1b_5(-b_3 + b_5t) \\ &\times \left[b_4b_6b_8q^2t^3 + (b_1b_4b_5 - b_4^2b_5 - b_1b_4b_6 - b_1b_4b_8 - b_5b_6b_8q)qt^2 \right. \\ &\quad \left. + (b_1b_4^2 + b_4b_5b_6q + b_4b_5b_8q + b_1b_6b_8q - b_4b_6b_8q)t - b_1b_4b_5 \right], \end{aligned} \quad (26)$$

$$B = -(b_1 - b_4)b_5^2(b_1b_4 - qb_6b_8)t(b_5 - b_3t)(-1 + qt^2), \quad (27)$$

$$C = -b_1^2b_4(b_5 - b_6)(b_5 - b_8)(b_3 - b_5t)(-1 + qt^2), \quad (28)$$

$$\begin{aligned} D &= b_1b_5(b_5 - b_3t) \\ &\times \left[-b_1b_4b_5qt^3 + (b_1b_4^2 + b_4b_5b_6q + b_4b_5b_8q + b_1b_6b_8q - b_4b_6b_8q)t^2 \right. \\ &\quad \left. + (b_1b_4b_5 - b_4^2b_5 - b_1b_4b_6 - b_1b_4b_8 - b_5b_6b_8q)t + b_4b_6b_8 \right]. \end{aligned} \quad (29)$$

²This variable z should be understood as a ratio of τ functions.

Step 1. We choose the decoupling factor H as

$$H = \frac{1}{qb_1b_5(b_5t - b_3)(b_1t - b_5)(b_4t - b_6)(b_4t - b_8)}. \quad (30)$$

Then we have for F the three-term relation

$$\begin{aligned} V_1(\bar{F} - F) + V_2F + V_3(\underline{F} - F) &= 0, \\ \frac{V_2}{V_1} &= \frac{b_5(b_3 - b_4)(b_1b_4 - qb_6b_8)(-1 + t)(1 + t)(-1 + qt^2)}{q(b_5 - b_1t)(-b_6 + b_4t)(-b_8 + b_4t)(-b_3 + b_5t)}, \\ \frac{V_3}{V_1} &= \frac{(b_5 - b_3t)(-b_1 + b_5t)(-b_4 + b_6t)(-b_4 + b_8t)(-1 + qt^2)}{(b_5 - b_1t)(-b_6 + b_4t)(-b_8 + b_4t)(-b_3 + b_5t)(-q + t^2)}. \end{aligned} \quad (31)$$

The three-term relation for the very-well-poised basic hypergeometric series

$$\Phi = {}_8W_7 \left(a_0; a_1, a_2, a_3, a_4, a_5; q, \frac{q^2 a_0^2}{a_1 a_2 a_3 a_4 a_5} \right), \quad (32)$$

is given by [11]

$$\begin{aligned} U_1(\bar{\Phi} - \Phi) + U_2\Phi + U_3(\underline{\Phi} - \Phi) &= 0, \quad \bar{\Phi} = \Phi|_{a_2 \rightarrow qa_2, a_3 \rightarrow a_3/q}, \quad \underline{\Phi} = \Phi|_{a_2 \rightarrow a_2/q, a_3 \rightarrow qa_3}, \\ U_1 &= \frac{(1 - a_2) \left(1 - \frac{a_0}{a_2}\right) \left(1 - \frac{qa_0}{a_2}\right) \left(1 - \frac{qa_0}{a_1 a_3}\right) \left(1 - \frac{qa_0}{a_3 a_4}\right) \left(1 - \frac{qa_0}{a_3 a_5}\right)}{a_3 \left(1 - \frac{a_2}{a_3}\right) \left(1 - \frac{qa_2}{a_3}\right)}, \quad U_3 = U_1|_{a_2 \leftrightarrow a_3}, \\ U_2 &= \frac{qa_0^2}{a_1 a_2 a_3 a_4 a_5} \left(1 - \frac{qa_0}{a_2 a_3}\right) (1 - a_1)(1 - a_4)(1 - a_5). \end{aligned} \quad (33)$$

Comparing eqs.(31) with (33), we identify F with ${}_8W_7$ as

$$F \propto {}_8W_7 \left(\frac{b_1 b_8}{b_3 b_5}; \frac{qb_8}{b_5}, \frac{b_1 t}{b_5}, \frac{b_1}{b_5 t}, \frac{b_2}{b_3}, \frac{b_4}{b_3}; q, \frac{b_5}{b_6} \right). \quad (34)$$

Step 2. We choose the decoupling factor K as

$$K = \frac{1}{b_1 b_5 (qb_5 t - b_3)(b_1 t - b_5)(b_4 t - b_6)(b_4 t - b_8)}. \quad (35)$$

Then we have for G the three-term relation

$$\begin{aligned} X_1(\bar{G} - G) + X_2F + X_3(\underline{G} - G) &= 0, \\ \frac{X_2}{X_1} &= \frac{b_5(b_3 - b_4)(b_1b_4 - b_6b_8)(-1 + t)(1 + t)(-1 + qt^2)}{(b_5 - b_1t)(-b_6 + b_4t)(-b_8 + b_4t)(-b_3 + qb_5t)}, \\ \frac{X_3}{X_1} &= \frac{(qb_5 - b_3t)(-b_1 + b_5t)(-b_4 + b_6t)(-b_4 + b_8t)(-1 + qt^2)}{(b_5 - b_1t)(-b_6 + b_4t)(-b_8 + b_4t)(-b_3 + qb_5t)(-q + t^2)}. \end{aligned} \quad (36)$$

Comparing eqs.(36) with (33), we have

$$G \propto {}_8W_7 \left(\frac{b_1 b_8}{b_3 b_5}; \frac{b_8}{b_5}, \frac{b_1 t}{b_5}, \frac{b_1}{b_5 t}, \frac{b_2}{b_3}, \frac{b_4}{b_3}; q, \frac{qb_5}{b_6} \right). \quad (37)$$

Moreover, from $k = H/K = (1 - b_3/qb_5t)/(1 - b_3/b_5t)$, we have $\kappa = 1 - b_3/b_5t$. Therefore we obtain

$$z \propto \frac{1}{1 - \frac{b_3}{b_5t}} \frac{{}_8W_7 \left(a_0; qa_1, a_2, a_3, a_4, a_5; q, \frac{qa_0^2}{a_1 a_2 a_3 a_4 a_5} \right)}{{}_8W_7 \left(a_0, a_1, a_2, a_3, a_4, a_5; q, \frac{q^2 a_0^2}{a_1 a_2 a_3 a_4 a_5} \right)}, \quad (38)$$

with

$$a_0 = \frac{b_1 b_8}{b_3 b_5}, \quad a_1 = \frac{b_8}{b_5}, \quad a_2 = \frac{b_1 t}{b_5}, \quad a_3 = \frac{b_1}{b_5 t}, \quad a_4 = \frac{b_2}{b_3}, \quad a_5 = \frac{b_4}{b_3}, \quad \frac{q^2 a_0^2}{a_1 a_2 a_3 a_4 a_5} = \frac{q b_5}{b_6}. \quad (39)$$

Step 3. Let us put

$$F = \theta(qa_1, a_2, a_3) \Phi(qa_1, a_2, a_3), \quad G = \theta(a_1, a_2, a_3) \Phi(a_1, a_2, a_3), \quad (40)$$

where $\theta(a_1, a_2, a_3)$ is a gauge factor to be determined. Here, we have omitted the dependence of a_0, a_4 and a_5 , since they are not relevant to the calculation. Then linear equations (16) yield

$$\frac{1}{\kappa H B} \frac{\theta(qa_1, qa_2, a_3/q)}{\theta(a_1, a_2, a_3)} \Phi(qa_1, qa_2, a_3/q) = \frac{A}{\kappa B} \frac{\theta(qa_1, a_2, a_3)}{\theta(a_1, a_2, a_3)} \Phi(qa_1, a_2, a_3) + \Phi(a_1, a_2, a_3), \quad (41)$$

and

$$\frac{\bar{\kappa}}{H D} \frac{\theta(a_1, qa_2, a_3/q)}{\theta(a_1, a_2, a_3)} \Phi(a_1, qa_2, a_3/q) = \frac{C}{\kappa D} \frac{\theta(qa_1, a_2, a_3)}{\theta(a_1, a_2, a_3)} \Phi(qa_1, a_2, a_3) + \Phi(a_1, a_2, a_3), \quad (42)$$

respectively. Now, we have the contiguity relations for $\Phi = {}_8W_7$ [11]

$$\frac{a_1 \left(1 - \frac{a_0 q}{a_1 a_3}\right) \left(1 - \frac{a_0 q}{a_1 a_4}\right) \left(1 - \frac{a_0 q}{a_1 a_5}\right)}{1 - \frac{a_0 q}{a_1}} \Phi(a_1/q, a_2, a_3) - \left(a_1 \leftrightarrow a_2\right) \quad (43)$$

$$= (a_1 - a_2) \left(1 - \frac{a_0^2 q^2}{a_1 a_2 a_3 a_4 a_5}\right) \Phi(a_1, a_2, a_3),$$

$$(a_2 - 1) \left(1 - \frac{a_0}{a_2}\right) \Phi(a_1/q, qa_2, a_3) + \left(1 - \frac{a_1}{q}\right) \left(1 - \frac{a_0 q}{a_1}\right) \Phi(a_1, a_2, a_3) \quad (44)$$

$$= \left(a_2 - \frac{a_1}{q}\right) \left(1 - \frac{a_0 q}{a_1 a_2}\right) \Phi(a_1/q, a_2, a_3).$$

We denote eqs.(43) and (44) as CR1[a_1, a_2, a_3] and CR2[a_1, a_2, a_3], respectively. Moreover, note that the relations CR1[a_1, a_2, a_3] and CR2[a_1, a_2, a_3] hold for any permutation of a_1, a_2 and a_3 , since these parameters are on equal footing in $\Phi = {}_8W_7$.

Now we eliminate $\Phi(a_1, a_2, a_3/q)$ from CR1[a_1, a_3, a_2] and CR2[a_3, a_2, a_1]. Shifting a_1 to qa_1 , we have a linear relation among $\Phi(a_1, qa_2, a_3/q)$, $\Phi(a_1, a_2, a_3)$ and $\Phi(a_1/q, a_2, a_3)$, which should coincide with eq. (41). Similarly, eliminating $\Phi(qa_1, a_2, a_3/q)$ from CR1[qa_1, a_3, a_2] and CR2[a_3, a_1, a_2], we have a linear relation among $\Phi(qa_1, a_2, a_3)$, $\Phi(a_1, a_2, a_3)$ and $\Phi(a_1, a_2, a_3/q)$. Elimination further $\Phi(a_1, a_2, a_3/q)$ from this relation and CR2[a_3, a_2, a_1] yields a linear relation among $\Phi(a_1, qa_2, a_3/q)$, $\Phi(qa_1, a_2, a_3)$ and $\Phi(a_1, a_2, a_3)$, which should coincide with eq.(42). From these calculations, we find that $\theta(a_1, a_2, a_3)$ should satisfy

$$\frac{\theta(a_1, qa_2, a_3/q)}{\theta(a_1, a_2, a_3)} = \frac{\left(1 - \frac{a_2}{a_0}\right) \left(1 - \frac{a_3 a_5}{a_0 q}\right)}{\left(1 - \frac{a_2 a_5}{a_0}\right) \left(1 - \frac{a_3}{a_0 q}\right)} = \frac{\left(1 - \frac{b_3 t}{b_8}\right) \left(1 - \frac{b_4}{b_8 q t}\right)}{\left(1 - \frac{b_4 t}{b_8}\right) \left(1 - \frac{b_3}{b_8 q t}\right)}, \quad (45)$$

$$\frac{\theta(qa_1, a_2, a_3)}{\theta(a_1, a_2, a_3)} = 1 - \frac{a_1}{a_0} = 1 - \frac{b_3}{b_1},$$

$$\frac{\theta(qa_1, qa_2, a_3/q)}{\theta(a_1, a_2, a_3)} = \frac{\theta(a_1, qa_2, a_3/q)}{\theta(a_1, a_2, a_3)} \times \frac{\theta(qa_1, a_2, a_3)}{\theta(a_1, a_2, a_3)},$$

which yield

$$\theta(a_1, a_2, a_3) = \frac{\left(\frac{a_2}{a_0 q}, \frac{a_3 a_5}{a_0 q}, \frac{a_1}{a_0 q}\right)_\infty}{\left(\frac{a_2 a_5}{a q}, \frac{a_3}{a_0 q}\right)_\infty} = \frac{\left(\frac{b_3 t}{q b_8}, \frac{b_4}{q b_8 t}, \frac{b_3}{q b_1}\right)_\infty}{\left(\frac{b_4 t}{q b_8}, \frac{b_3}{q b_8 t}\right)_\infty}. \quad (46)$$

Therefore we arrive at the final result

$$\begin{aligned}
z &= \frac{1}{1 - \frac{b_3}{b_5 t}} \frac{\theta(qa_1, a_2, a_3)}{\theta(a_1, a_2, a_3)} \frac{\Phi(qa_1, a_2, a_3)}{\Phi(a_1, a_2, a_3)} \\
&= \frac{1 - \frac{b_3}{b_1}}{1 - \frac{b_3}{b_5 t}} \frac{{}_8W_7\left(a_0; qa_1, a_2, a_3, a_4, a_5; q, \frac{qa_0^2}{a_1 a_2 a_3 a_4 a_5}\right)}{{}_8W_7\left(a_0, a_1, a_2, a_3, a_4, a_5; q, \frac{q^2 a_0^2}{a_1 a_2 a_3 a_4 a_5}\right)}.
\end{aligned} \tag{47}$$

3 Hypergeometric Solutions

Hypergeometric solutions to other q -Painlevé equations can be constructed by the same procedure as that was demonstrated in the previous section. Instead of describing full procedure, we give a list of equations, solutions and the other data that are necessary for construction of solutions. We note that the case of $D_5^{(1)}$ is omitted as mentioned in the introduction.

3.1 Case of $E_8^{(1)}$

3.1.1 Equation and Solution

(1) q -Painlevé equation [3, 6, 12]

$$\begin{aligned}
\frac{(\bar{g}\bar{s}t - f)(gst - f) - (\bar{s}^2 t^2 - 1)(s^2 t^2 - 1)}{\left(\frac{\bar{g}}{\bar{s}t} - f\right)\left(\frac{g}{st} - f\right) - \left(1 - \frac{1}{\bar{s}^2 t^2}\right)\left(1 - \frac{1}{s^2 t^2}\right)} &= \frac{P(f, t, m_1, \dots, m_7)}{P(f, t^{-1}, m_7, \dots, m_1)}, \\
\frac{(fst - g)(fst - g) - (s^2 \bar{t}^2 - 1)(s^2 t^2 - 1)}{\left(\frac{f}{s\bar{t}} - g\right)\left(\frac{f}{st} - g\right) - \left(1 - \frac{1}{s^2 \bar{t}^2}\right)\left(1 - \frac{1}{s^2 t^2}\right)} &= \frac{P(g, s, m_7, \dots, m_1)}{P(g, s^{-1}, m_1, \dots, m_7)},
\end{aligned} \tag{48}$$

where

$$\begin{aligned}
P(f, t, m_1, \dots, m_7) &= f^4 - m_1 t f^3 + (m_2 t^2 - 3 - t^8) f^2 \\
&\quad + (m_7 t^7 - m_3 t^3 + 2m_1 t) f + (t^8 - m_6 t^6 + m_4 t^4 - m_2 t^2 + 1),
\end{aligned} \tag{49}$$

and m_k ($k = 1, 2, \dots, 7$) are the elementary symmetric functions of k -th degree in b_i ($i = 1, 2, \dots, 8$) with

$$b_1 b_2 \cdots b_8 = 1. \tag{50}$$

Moreover,

$$\bar{t} = qt, \quad t = q^{\frac{1}{2}} s. \tag{51}$$

(2) Constraint on parameters [6]

$$qb_1 b_3 b_5 b_7 = 1, \quad b_2 b_4 b_6 b_8 = q. \tag{52}$$

(3) Hypergeometric solution

$$z = \frac{g - \left(\frac{s}{b_1} + \frac{b_1}{s}\right)}{g - \left(\frac{s}{b_8} + \frac{b_8}{s}\right)} = \lambda \frac{\Phi(q^4 a_0; a_1, q^2 a_2, \dots, q^2 a_7)}{\Phi(a_0; a_1, \dots, a_7)}, \tag{53}$$

where Φ is defined in terms of the balanced $_{10}W_9$ series by

$$\begin{aligned} \Phi(a_0; a_1, \dots, a_7) &= {}_{10}W_9(a_0; a_1, \dots, a_7; q^2, q^2) \\ &+ \frac{\left(q^2 a_0, \frac{a_7}{a_0}; q^2\right)_\infty}{\left(\frac{a_0}{a_7}, \frac{q^2 a_7^2}{a_0}; q^2\right)_\infty} \prod_{k=1}^6 \frac{\left(a_k, \frac{q^2 a_7}{a_k}; q^2\right)_\infty}{\left(\frac{q^2 a_0}{a_k}, \frac{a_k a_7}{a_0}; q^2\right)_\infty} {}_{10}W_9\left(\frac{a_7^2}{a_0}; \frac{a_1 a_7}{a_0}, \dots, \frac{a_6 a_7}{a_0}, a_7; q^2, q^2\right). \end{aligned} \quad (54)$$

Here, a_i ($i = 0, 1, \dots, 7$) and λ are given by

$$\begin{aligned} a_0 &= \frac{1}{qb_1 b_2 b_8^2}, \quad a_1 = \frac{q^2}{b_2 b_8 t^2}, \quad a_2 = \frac{s^2}{b_2 b_8}, \\ a_i &= \frac{b_i}{b_8} \quad (i = 3, 5, 7), \quad a_i = \frac{b_i}{b_1} \quad (i = 4, 6), \end{aligned} \quad (55)$$

and

$$\lambda = \frac{b_1 b_4 b_6}{b_8 s^2} \frac{\left(1 - \frac{b_4 b_6}{b_1 b_8}\right) \left(1 - q^2 \frac{b_4 b_6}{b_1 b_8}\right) (1 - b_3 b_5 t^2) (1 - b_3 b_7 t^2) (1 - b_5 b_7 t^2) \prod_{i=2,4,6} \left(1 - \frac{b_i}{b_1}\right)}{\left(1 - \frac{s^2}{b_1 b_8}\right) \left(1 - \frac{q^2 s^2}{b_1 b_8}\right) \left(1 - \frac{b_4}{b_8}\right) \left(1 - \frac{b_6}{b_8}\right) \left(1 - \frac{q}{b_1 b_8 s^2}\right) \prod_{i=3,5,7} \left(1 - \frac{b_4 b_6}{b_1 b_i}\right)}, \quad (56)$$

respectively.

3.1.2 Data

(1) Riccati equation

$$\begin{vmatrix} 1 & f & g & fg \\ 1 & f_1 & g_1 & f_1 g_1 \\ 1 & f_3 & g_3 & f_3 g_3 \\ 1 & f_5 & g_5 & f_5 g_5 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & f & \bar{g} & f\bar{g} \\ 1 & f_8 & \bar{g}_8 & f_8 \bar{g}_8 \\ 1 & f_6 & \bar{g}_6 & f_6 \bar{g}_6 \\ 1 & f_4 & \bar{g}_4 & f_4 \bar{g}_4 \end{vmatrix} = 0, \quad (57)$$

where

$$f_i = b_i t + \frac{1}{b_i t}, \quad g_i = \frac{s}{b_i} + \frac{b_i}{s}. \quad (58)$$

The Riccati equation for $z = \frac{g - g_1}{g - g_8}$ is given by

$$\bar{z} = \frac{Az + B}{Cz + D}, \quad (59)$$

$$\begin{aligned} B &= -f_{35} g_{13} g_{15} d'_{1468}, \\ C &= f_{46} \bar{g}_{48} \bar{g}_{68} d_{1358}, \\ D &= -f_{35} f_{46} f_{18} g_{13} g_{15} \bar{g}_{48} \bar{g}_{68}, \\ \Delta &= AD - BC = f_{13} f_{35} f_{15} f_{46} f_{48} f_{68} g_{13} g_{15} g_{35} g_{18} \bar{g}_{46} \bar{g}_{48} \bar{g}_{68} \bar{g}_{18}, \end{aligned} \quad (60)$$

where $f_{ij} = f_i - f_j$ and

$$d_{1358} = \begin{vmatrix} 1 & f_1 & g_1 & f_1 g_1 \\ 1 & f_3 & g_3 & f_3 g_3 \\ 1 & f_5 & g_5 & f_5 g_5 \\ 1 & f_8 & g_8 & f_8 g_8 \end{vmatrix}, \quad d'_{1468} = \begin{vmatrix} 1 & f_1 & \bar{g}_1 & f_1 \bar{g}_1 \\ 1 & f_4 & \bar{g}_4 & f_4 \bar{g}_4 \\ 1 & f_6 & \bar{g}_6 & f_6 \bar{g}_6 \\ 1 & f_8 & \bar{g}_8 & f_8 \bar{g}_8 \end{vmatrix}, \quad (61)$$

respectively.

(2) Three-term and contiguity relations for hypergeometric function [13]

(a) Three-term relation

$$U_1(\bar{\Phi} - \Phi) + U_2\Phi + U_3(\underline{\Phi} - \Phi) = 0, \quad (62)$$

where

$$U_1 = \frac{a_1(1-a_2)\left(1 - \frac{a_0}{a_2}\right)\left(1 - \frac{q^2 a_0}{a_2}\right)}{\left(1 - \frac{q^2 a_2}{a_1}\right) \prod_{j=3}^7 \left(1 - \frac{q^2 a_0}{a_1 a_j}\right)}, \quad (63)$$

$$U_2 = -(a_1 - a_2) \left(1 - \frac{q^2 a_0}{a_1 a_2}\right) \prod_{j=3}^7 (1 - a_j), \quad U_3 = U_1|_{a_1 \leftrightarrow a_2},$$

$$\bar{\Phi} = \Phi(a_0; a_1/q^2, q^2 a_2, a_3, \dots, a_7), \quad \underline{\Phi} = \Phi(a_0; q^2 a_1, a_2/q^2, a_3, \dots, a_7), \quad (64)$$

and Φ is defined by eq.(54).

(b) Contiguity relations

$$\begin{aligned} & \Phi(a_0; a_1/q^2, q^2 a_2, a_3, \dots, a_7) - \Phi(a_0; a_1, a_2, a_3, \dots, a_7) \\ &= V_1 \Phi(q^4 a_0^2; a_1, q^2 a_2, \dots, q^2 a_7), \end{aligned} \quad (65)$$

$$\begin{aligned} & V_2 \Phi(q^4 a_0^2; a_1, q^2 a_2, q^2 a_3, \dots, q^2 a_7) - V_3 \Phi(q^4 a_0^2; q^2 a_1, a_2, q^2 a_3, \dots, q^2 a_7) \\ &= V_4 \Phi(a_0; a_1, a_2, a_3, \dots, a_7), \end{aligned} \quad (66)$$

where

$$\begin{aligned} V_1 &= \frac{\frac{q^2 a_0}{a_2} \left(1 - \frac{q^2 a_2}{a_1}\right) \left(1 - \frac{a_1 a_2}{q^2 a_0}\right) (1 - q^2 a_0) (1 - q^4 a_0) \prod_{j=3}^7 (1 - a_j)}{\left(1 - \frac{q^2 a_0}{a_1}\right) \left(1 - \frac{q^4 a_0}{a_1}\right) \left(1 - \frac{a_0}{a_2}\right) \left(1 - \frac{q^2 a_0}{a_2}\right) \prod_{j=3}^7 \left(1 - \frac{q^2 a_0}{a_j}\right)}, \\ V_2 &= \frac{a_1^2 (1 - a_2) \prod_{j=3}^7 \left(1 - \frac{q^2 a_0}{a_1 a_j}\right)}{\left(1 - \frac{q^2 a_0}{a_1}\right) \left(1 - \frac{q^4 a_0}{a_1}\right)}, \quad V_3 = V_2|_{a_1 \leftrightarrow a_2}, \\ V_4 &= \frac{a_1 \left(1 - \frac{a_2}{a_1}\right) \prod_{j=3}^7 \left(1 - \frac{q^2 a_0}{a_j}\right)}{(1 - q^2 a_0) (1 - q^4 a_0)}. \end{aligned} \quad (67)$$

(3) Decoupling factors

$$H = \frac{D}{\Delta} = -\frac{f_{18}}{f_{13} f_{15} f_{48} f_{68} g_{35} g_{18} \bar{g}_{46} \bar{g}_{18}}, \quad K = \frac{1}{D} = -\frac{1}{f_{35} f_{46} f_{18} g_{13} g_{15} \bar{g}_{48} \bar{g}_{68}}, \quad (68)$$

so that

$$k = \frac{H}{K} = \frac{D^2}{\Delta} = \frac{f_{35} f_{46} f_{18}^2 g_{13} g_{15} \bar{g}_{48} \bar{g}_{68}}{f_{13} f_{15} f_{48} f_{68} g_{35} \bar{g}_{46} \bar{g}_{18}}. \quad (69)$$

(4) Identification

$$z = \frac{1}{\kappa} \frac{F}{G}, \quad F \propto \Phi(q^4 a_0; a_1, q^2 a_2, \dots, q^2 a_7), \quad G \propto \Phi(a_0; a_1, \dots, a_7), \quad \bar{\kappa} = k\kappa, \quad (70)$$

where a_i ($i = 0, \dots, 7$) are given by eq.(55).

- (5) Gauge factors
Putting

$$\begin{aligned} F &= \theta(q^4 a_0; a_1, q^2 a_2, \dots, q^2 a_7) \Phi(q^4 a_0; a_1, q^2 a_2, \dots, q^2 a_7), \\ G &= \theta(a_0; a_1 \dots, a_7) \Phi(a_0; a_1 \dots, a_7), \end{aligned} \quad (71)$$

we have:

$$\frac{\theta(a_0; a_1/q^2, a_2 q^2, \dots, a_7)}{\theta(a_0; a_1 \dots, a_7)} = 1, \quad \frac{1}{\kappa} \frac{\theta(q^4 a_0; a_1, q^2 a_2, \dots, q^2 a_7)}{\theta(a_0; a_1 \dots, a_7)} = \lambda. \quad (72)$$

3.2 Case of $E_6^{(1)}$

3.2.1 Equation and Solution

- (1) q -Painlevé equation [3, 14, 15]

$$\begin{aligned} (f\bar{g} - 1)(fg - 1) &= \bar{t}\bar{t} \frac{(f - b_1 t)(f - b_2 t)(f - b_3 t)(f - b_4 t)}{(f - b_5 t)\left(f - \frac{t}{b_5}\right)}, \\ (f\bar{g} - 1)(fg - 1) &= t^2 \frac{\left(g - \frac{1}{b_1}\right)\left(g - \frac{1}{b_2}\right)\left(g - \frac{1}{b_3}\right)\left(g - \frac{1}{b_4}\right)}{(g - b_6 t)\left(g - \frac{t}{b_6}\right)}, \\ \bar{t} &= qt, \quad b_1 b_2 b_3 b_4 = 1. \end{aligned} \quad (73)$$

- (2) Constraint on parameters [3, 15]

$$b_1 b_2 = b_5 b_6. \quad (74)$$

- (3) Hypergeometric solution

$$z = \frac{g - \frac{1}{b_1}}{g - tb_6} = \frac{1 - \frac{b_3}{b_1}}{1 - \frac{b_1 b_2 b_3 t}{b_5}} \frac{\Phi(qa, b, c, d, e)}{\Phi(a, b, qc, d, e)}, \quad (75)$$

where Φ is the balanced ${}_3\varphi_2$ series defined by

$$\Phi(a, b, c, d, e) = {}_3\varphi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, \frac{de}{abc} \right), \quad (76)$$

with

$$a = \frac{b_3 b_5}{t}, \quad b = \frac{b_3}{b_2}, \quad c = b_1^2 b_2 b_3, \quad d = q \frac{b_3 b_5^2}{b_2}, \quad e = q b_1 b_2 b_3^2. \quad (77)$$

3.2.2 Data

- (1) Riccati equation [3, 15]

$$\bar{g} = \frac{1 + \frac{b_5 \bar{t}}{b_1 b_2} (f - b_1 - b_3)}{f - \bar{t} b_5}, \quad f = \frac{1 + b_6 t (b_3 b_4 g - b_3 - b_4)}{g - t b_6}. \quad (78)$$

The Riccati equation for

$$z = \frac{g - 1/b_1}{g - t b_6}, \quad (79)$$

is given by,

$$\bar{z} = \frac{Az + B}{Cz + D},$$

$$\begin{aligned} A &= b_1 b_5 (b_3 b_5 - t)(b_5 - b_1 b_2 b_3 t)(-b_2 + b_5 q t), \\ B &= -b_5^2 t(b_1 - b_3)(-1 + b_1^2 b_2 b_3)(-b_2 + b_5 q t), \\ C &= q t b_1 (b_1 b_2 - b_5)(b_1 b_2 + b_5)(b_3 b_5 - t)(-b_5 + b_1 b_2 b_3 t), \\ D &= -b_5 \left[b_1 b_2 b_3 b_5^2 + (-b_1^3 b_2^2 b_3 b_5 - b_1^3 b_2^2 b_3 b_5 q - b_2 b_3 b_5^3 q) t \right. \\ &\quad \left. + (b_1^3 b_2^2 - b_1^2 b_2^2 b_3 + b_1^4 b_2^3 b_3^2 - b_1 b_5^2 + b_3 b_5^2 + b_1^3 b_2 b_3 b_5^2 + b_1^2 b_2^2 b_3 b_5^2 - b_1^2 b_2 b_3^2 b_5^2 + b_1^2 b_2^2 b_3 b_5^2 q) q t^2 \right. \\ &\quad \left. - b_1^4 b_2^3 b_3 b_5 q^2 t^3 \right]. \end{aligned} \quad (80)$$

(2) Three-term and contiguity relations for hypergeometric function [10]

(a) Three-term relation

$$V_1 (\bar{\Phi} - \Phi) + V_2 \Phi + V_3 (\Phi - \underline{\Phi}) = 0, \quad (81)$$

where

$$\begin{aligned} V_1 &= \left(1 - \frac{q}{z}\right)(1 - a), \quad V_2 = (1 - b)(1 - c), \quad V_3 = \frac{a}{z} \left(1 - \frac{d}{a}\right) \left(1 - \frac{e}{a}\right), \\ \Phi &= \Phi(a, b, c, d, e) = {}_3\varphi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q; z \right), \quad z = \frac{de}{abc}, \quad \bar{\Phi} = \Phi|_{a \rightarrow qa}, \quad \underline{\Phi} = \Phi|_{a \rightarrow a/q}. \end{aligned} \quad (82)$$

(b) Contiguity relations

$$(a - c)\Phi(a, b, c, d, e) + (1 - a)\Phi(qa, b, c, d, e) - (1 - c)\Phi(a, b, qc, d, e) = 0, \quad (83)$$

$$\begin{aligned} (a - c)(de - abc)\Phi(a, b, c, d, e) + bc(d - a)(e - a)\Phi(a/q, b, c, d, e) \\ - ab(d - c)(e - c)\Phi(a, b, c/q, d, e) = 0. \end{aligned} \quad (84)$$

(3) Decoupling factors

$$\begin{aligned} H &= \frac{1}{b_1 b_3 b_5 (-b_5 + b_1^2 b_2 t)(b_5 - b_1 b_2 b_3 t)(b_2 - q b_5 t)}, \\ K &= \frac{1}{b_1 b_3 b_5 (-b_5 + b_1^2 b_2 t)(b_5 - q b_1 b_2 b_3 t)(b_2 - q b_5 t)}, \\ k &= \frac{H}{K} = \frac{b_5 - q b_1 b_2 b_3 t}{b_5 - b_1 b_2 b_3 t}, \quad \kappa = 1 - \frac{b_1 b_2 b_3 t}{b_5}. \end{aligned} \quad (85)$$

(4) Identification

$$z = \frac{1}{\kappa} \frac{F}{G}, \quad F \propto \Phi(qa, b, c, d, e), \quad G \propto \Phi(a, b, qc, d, e), \quad (86)$$

where a, \dots, e are given by eq.(77).

(5) Gauge factors

Putting

$$F = \theta(qa, b, c, d, e)\Phi(qa, b, c, d, e), \quad G = \theta(a, b, qc, d, e)\Phi(a, b, qc, d, e), \quad (87)$$

we have:

$$\frac{\theta(a, b, c, d, e)}{\theta(a, b, qc, d, e)} = \frac{\theta(qa, b, c, d, e)}{\theta(a, b, qc, d, e)} = 1 - \frac{b_3}{b_1}, \quad \frac{\theta(a/q, b, qc, d, e)}{\theta(a, b, qc, d, e)} = 1. \quad (88)$$

3.3 Case of $A_4^{(1)}$

3.3.1 Equation and Solution

(1) q -Painlevé equation [3]

$$\begin{aligned}\bar{g}g &= \frac{\left(f + \frac{a_1}{t}\right)\left(f + \frac{1}{a_1 t}\right)}{1 + a_3 f}, \\ f\bar{f} &= \frac{\left(g + \frac{a_2}{s}\right)\left(g + \frac{1}{a_2 s}\right)}{1 + g/a_3}, \\ \bar{t} &= qt, \quad t = q^{\frac{1}{2}} s.\end{aligned}\tag{89}$$

(2) Constraint on parameters [3]

$$a_1 a_2 a_3^2 = q^{-\frac{1}{2}}.\tag{90}$$

(3) Hypergeometric solution

$$g = -\frac{1}{a_1 a_3^2 t} \frac{\Phi(\alpha_1, \alpha_2, z)}{\Phi(\alpha_1, q\alpha_2, z)}, \quad f = \frac{1}{a_3} \left(1 - \frac{1}{a_2^2}\right) \frac{\Phi(q\alpha_1, q\alpha_2, z)}{\Phi(\alpha_1, q\alpha_2, z)},\tag{91}$$

where Φ is the ${}_2\phi_1$ series defined by

$$\Phi(\alpha_1, \alpha_2, z) = {}_2\phi_1\left(\begin{matrix} \alpha_1, \alpha_2 \\ 0 \end{matrix}; q, z\right),\tag{92}$$

with

$$\alpha_1 = \frac{1}{a_2^2}, \quad \alpha_2 = a_1^2, \quad z = \frac{t}{a_1 a_3}.\tag{93}$$

Note that the solution is also expressible in terms of ${}_1\phi_1$ series by using the formula [16],

$${}_2\phi_1\left(\begin{matrix} a, b \\ 0 \end{matrix}; q, z\right) = \frac{(bz; q)_\infty}{(z; q)_\infty} {}_1\phi_1\left(\begin{matrix} b \\ bz \end{matrix}; q, az\right).\tag{94}$$

3.3.2 Data

(1) Riccati equation [3]

$$\bar{g} = \frac{a_3^2 g + \frac{1 - a_1^2}{a_1 t}}{-a_3 g + \left(\frac{1}{a_3^2} - \frac{1}{a_1 a_3 t}\right)}, \quad g = -\frac{f + \frac{1}{a_1 t}}{a_3^2}.\tag{95}$$

(2) Three-term and contiguity relations for hypergeometric function

(a) Three-term relation

$$\frac{\alpha_1 \alpha_2}{q} z (\bar{\Phi} - \Phi) + \frac{z}{q} (1 - \alpha_1)(1 - \alpha_2) \Phi + \left(\frac{z}{q} - 1\right) (\underline{\Phi} - \Phi) = 0,\tag{96}$$

where

$$\Phi(\alpha_1, \alpha_2, z) = {}_2\phi_1\left(\begin{matrix} \alpha_1, \alpha_2 \\ 0 \end{matrix}; q, z\right), \quad \bar{\Phi} = \Phi|_{z \rightarrow qz}, \quad \underline{\Phi} = \Phi|_{z \rightarrow z/q}.\tag{97}$$

(b) Contiguity relations

$$\Phi(\alpha_1, \alpha_2, z) - \alpha_2 \Phi(\alpha_1, \alpha_2, qz) = (1 - \alpha_2) \Phi(\alpha_1, q\alpha_2, z), \quad (98)$$

$$\Phi(\alpha_1, \alpha_2/q, z) - \Phi(\alpha_1, \alpha_2, z) = -\frac{\alpha_2 z}{q} (1 - \alpha_1) \Phi(q\alpha_1, \alpha_2, z), \quad (99)$$

$$\alpha_1 \alpha_2 z \Phi(\alpha_1, q\alpha_2, qz) = \Phi - (1 - \alpha_2 z) \Phi(\alpha_1, q\alpha_2, z). \quad (100)$$

(3) Decoupling factors

$$H = \frac{1}{a_1^2 a_3^2}, \quad K = \frac{1}{q a_1^2 a_3^2}, \quad k = \frac{H}{K} = q, \quad \kappa = t. \quad (101)$$

(4) Identification

$$f = \frac{1}{\kappa} \frac{F}{G}, \quad F \propto \Phi(\alpha_1, \alpha_2, z), \quad G \propto \Phi(\alpha_1, q\alpha_2, z), \quad (102)$$

with parameters given in eq.(93).

(5) Gauge factors

Putting

$$F = \theta(\alpha_1, \alpha_2, z) \Phi(\alpha_1, \alpha_2, z), \quad G = \theta(\alpha_1, q\alpha_2, z) \Phi(\alpha_1, q\alpha_2, z), \quad (103)$$

we have:

$$\frac{\theta(\alpha_1, \alpha_2, qz)}{\theta(\alpha_1, \alpha_2, z)} = 1, \quad \frac{\theta(\alpha_1, q\alpha_2, z)}{\theta(\alpha_1, \alpha_2, z)} = -a_1 a_3^2. \quad (104)$$

3.4 Case of $(A_2 + A_1)^{(1)}$

3.4.1 Equation and Solution

(1) q -Painlevé equation [2, 3, 7, 8, 17]

$$\bar{g} g f = b_0 \frac{1 + a_0 t f}{a_0 t + f}, \quad g f \bar{f} = b_0 \frac{\frac{a_1}{t} + g}{1 + \frac{a_1}{t} g}, \quad \bar{t} = q t. \quad (105)$$

Eq. (105) admits two different specializations for hypergeometric solutions: (a) specialization of b_i (parameter of A_1), (b) specialization of a_i (parameter of A_2). See also [7, 8] for details.

(2) Constraint on parameters

(a)

$$b_0 = q. \quad (106)$$

(b)

$$a_0 a_1 = q. \quad (107)$$

(3) Hypergeometric solution

(a)

$$g = -\frac{a_1}{t} \left(1 - \frac{q^2}{a_0^2 a_1^2} \right) \frac{\Phi(b, z)}{\Phi(q^2 b, z)}, \quad f = \frac{q^2 t}{a_0 a_1^2} \frac{1}{1 - \frac{q^2}{a_0^2 a_1^2}} \frac{\Phi(q^2 b, q^2 z)}{\Phi(b, z)}, \quad (108)$$

where

$$\Phi = \Phi(b, z) = {}_1\varphi_1 \left(\begin{matrix} 0 \\ b \end{matrix}; q^2, z \right), \quad (109)$$

with

$$b = q^2 / a_0^2 a_1^2, \quad z = q^2 t^2 / a_1^2. \quad (110)$$

(b)

$$g = \frac{b_0}{a_0 t} \frac{\Psi(a, z)}{\Psi(a, q^2 z)}, \quad f = -a_0 t \frac{\Psi(a, q^2 z)}{\Psi(a, z)}, \quad (111)$$

where

$$\Psi = \Psi(a, z) = {}_1\varphi_1 \left(\begin{matrix} a \\ 0 \end{matrix} ; q^2, z \right), \quad (112)$$

with

$$a = a_0^2 t^2, \quad z = q/b_0. \quad (113)$$

3.4.2 Data

(1) Riccati equation

(a)

$$\bar{f} = \frac{(a_0^2 a_1^2 / q - a_0^2 q t^2) f - a_0 q t}{a_0 t f + 1}, \quad g = -a_0 a_1 \frac{1}{a_0 t + f}. \quad (114)$$

(b)

$$\bar{g} = -\frac{g - a_0 b_0 t}{a_0 t g - b_0}, \quad f g = -b_0. \quad (115)$$

(2) Three-term and contiguity relations for hypergeometric function

(a) (i) Three-term relation

$$\frac{b}{z} (\Phi(b, q^2 z) - \Phi(b, z)) + \Phi(b, z) + \frac{q^2}{z} (\Phi(b, z/q^2) - \Phi) = 0, \quad (116)$$

where

$$\Phi(b, z) = {}_1\varphi_1 \left(\begin{matrix} 0 \\ b \end{matrix} ; q^2, z \right). \quad (117)$$

(ii) Contiguity relations

$$\Phi(b, z) - \frac{b}{q^2} \Phi(b, q^2 z) = \left(1 - \frac{b}{q^2}\right) \Phi(b/q^2, z), \quad (118)$$

$$\Phi(b, z) - \Phi(b, z/q^2) = \frac{z/q^2}{1-b} \Phi(q^2 b, z). \quad (119)$$

(b) (i) Three-term relation

$$(1-a) \frac{z}{q^2} (\Psi(q^2 a, z) - \Psi(a, z)) - \frac{a z}{q^2} \Psi(a, z) + (\Psi(a/q^2, z) - \Psi(a, z)) = 0, \quad (120)$$

where

$$\Psi(a, z) = {}_1\varphi_1 \left(\begin{matrix} a \\ 0 \end{matrix} ; q^2, z \right). \quad (121)$$

(ii) Contiguity relations

$$\Psi(a, z) - a \Psi(a, q^2 z) = (1-a) \Psi(q^2 a, z), \quad (122)$$

$$\Psi(a, z) - \Psi(q, z/q^2) = (1-a) \frac{z}{q^2} \Psi(q^2 a, z). \quad (123)$$

(3) Decoupling factors

(a)

$$H = \frac{1}{q}, \quad K = 1, \quad k = \frac{H}{K} = \frac{1}{q}, \quad \kappa = \frac{a_1}{qt}. \quad (124)$$

(b)

$$H = -\frac{1}{1 - a_0^2 t^2}, \quad K = \frac{1}{q} \frac{1}{1 - a_0^2 t^2}, \quad k = \frac{H}{K} = q, \quad \kappa = a_0 t. \quad (125)$$

(4) Identification

(a)

$$f = \frac{1}{\kappa} \frac{F}{G}, \quad F \propto \Phi(q^2 b, q^2 z), \quad G \propto \Phi(b, z), \quad (126)$$

with parameters given in eq.(110).

(b)

$$g = \frac{1}{\kappa} \frac{F}{G}, \quad F \propto \Psi(a, z), \quad G \propto \Psi(a, q^2 z), \quad (127)$$

with parameters given in eq.(113).

(5) Gauge factors

(a) Putting

$$F = \theta(q^2 b, q^2 z) \Phi(q^2 b, q^2 z), \quad G = \theta(b, z) \Phi(b, z), \quad (128)$$

we have:

$$\frac{\theta(b, q^2 z)}{\theta(b, z)} = 1, \quad \frac{\theta(q^2 b, z)}{\theta(b, z)} = \frac{\theta(q^2 b, q^2 z)}{\theta(b, z)} = \frac{q}{a_0 a_1 \left(1 - \frac{q^2}{a_0^2 a_1^2}\right)}. \quad (129)$$

(b) Putting

$$F = \theta(a, z) \Psi(a, z), \quad G = \theta(a, q^2 z) \Psi(a, q^2 z), \quad (130)$$

we have:

$$\frac{\theta(q^2 a, z)}{\theta(a, z)} = 1, \quad \frac{\theta(a, q^2 z)}{\theta(a, z)} = \frac{\theta(q^2 a, q^2 z)}{\theta(a, z)} = \frac{1}{b_0}. \quad (131)$$

3.5 Case of $(A_1 + A'_1)^{(1)}$

3.5.1 Equation and Solution

(1) q -Painlevé equation [2, 3, 18]

$$(\bar{f}f - 1)(f\bar{f} - 1) = \frac{at^2 f}{f + t}, \quad \bar{t} = qt. \quad (132)$$

(2) Constraint on parameters

$$a = q. \quad (133)$$

(3) Hypergeometric solution

$$f = \frac{\Phi(qt)}{\Phi(t)}, \quad \Phi = {}_1\varphi_1 \left(\begin{matrix} 0 \\ -q \end{matrix}; q, -qt \right). \quad (134)$$

3.5.2 Data

(1) Riccati equation

$$\bar{f} = \frac{1}{f} - qt. \quad (135)$$

(2) Three-term relation

$$\Phi(qt) + t\Phi(t) = \Phi(t/q). \quad (136)$$

(3) Identification

$$f = \frac{F}{G}, \quad F = \Phi(qt), \quad G = \Phi(t). \quad (137)$$

We note that there is no need to introduce decoupling and gauge factors.

Acknowledgment One of the author(K.K.) would like to thank Professor Nalini Joshi for hospitality during his stay in University of Sydney.

References

- [1] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada, *Hypergeometric solutions to the q -Painlevé equations*, Int. Math. Res. Not. **2004** 2497-2521.
- [2] H. Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Commun. Math. Phys. **220** (2001) 165–229.
- [3] A. Ramani, B. Grammaticos, T. Tamizhmani and K.M. Tamizhmani, *Special function solutions of the discrete Painlevé equations*, Comp. Math. Appl. **42**(2001) 603–614.
- [4] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada, *$_{10}E_9$ solution to the elliptic Painlevé equation*, J. Phys. A: Math. Gen. **36** (2003) L263–L272.
- [5] H. Sakai, *Casorati determinant solutions for the q -difference sixth Painlevé equation*, Nonlinearity **11** (1998) 823–833.
- [6] M. Murata, H. Sakai and J. Yoneda, *Riccati solutions of discrete Painlevé equations with Weyl group symmetry of type $E_8^{(1)}$* , J. Math. Phys. **44** (2003) 1396–1414.
- [7] K. Kajiwara, M. Noumi and Y. Yamada, *A study on the fourth q -Painlevé equation*, J. Phys. A: Math. Gen. **34** (2001) 8563–8581.
- [8] K. Kajiwara and K. Kimura, *On a q -difference Painlevé III equation. I: Derivation, symmetry and Riccati type solutions*, J. Nonlin. Math. Phys. **10** (2003) 86–102.
- [9] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its Applications, **35**(1990) (Cambridge: Cambridge University Press).
- [10] D.P. Gupta, M.E.H. Ismail and D.R. Masson, *Contiguous relations, basic hypergeometric functions, and orthogonal polynomials. II. associated big q -Jacobi polynomials*, J. Math. Anal. Appl. **171**(1992) 477–497.
- [11] M. Ismail and M. Rahman, *The Associated Askey-Wilson Polynomials*, Trans. Amer. Math. Soc. **328**(1991) 201-237.
- [12] Y. Ohta, A. Ramani and B. Grammaticos, *An affine Weyl group approach to the eight-parameter discrete Painlevé equation*, J. Phys. A: Math. Gen. **34** (2001) 10523–10532.
- [13] D.P. Gupta and D.R. Masson, *Contiguous relations, continued fractions and orthogonality*, Trans. Amer. Math. Soc. **350** (1998) 769–808.

- [14] B. Grammaticos, A. Ramani and Y. Ohta, *A unified description of the asymmetric q - P_V and d - P_{IV} equations and their Schlesinger transformations*, J. Nonlin. Math. Phys. **2**(2003) 215-228.
- [15] K.M. Tamizhmani, A. Ramanim B. Grammaticos and Y. Ohta, *A study of the discrete P_V equation: Miura transformation and particular solutions*, Lett. Math. Phys. **38**(1996) 289–296.
- [16] R. Koekoek and R. F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q -analogue*, Delft University of Technology, Faculty of Information Technology and Systems, Department of Technical Mathematics and Informatics, Report no. 98-17 (1998).
- [17] K. Kajiwara, *On a q -difference Painlevé III equation. II: Rational solutions*, J. Nonlin. Math. Phys. **10** (2003) 282–303.
- [18] A. Ramani and B. Grammaticos, *Discrete Painlevé equations: coalescences, limits and degeneracies*, Physica A **228** (1996) 160–171.

List of MHF Preprint Series, Kyushu University

21st Century COE Program

Development of Dynamic Mathematics with High Functionality

- MHF2003-1 Mitsuhiro T. NAKAO, Kouji HASHIMOTO & Yoshitaka WATANABE
A numerical method to verify the invertibility of linear elliptic operators with applications to nonlinear problems
- MHF2003-2 Masahisa TABATA & Daisuke TAGAMI
Error estimates of finite element methods for nonstationary thermal convection problems with temperature-dependent coefficients
- MHF2003-3 Tomohiro ANDO, Sadanori KONISHI & Seiya IMOTO
Adaptive learning machines for nonlinear classification and Bayesian information criteria
- MHF2003-4 Kazuhiro YOKOYAMA
On systems of algebraic equations with parametric exponents
- MHF2003-5 Masao ISHIKAWA & Masato WAKAYAMA
Applications of Minor Summation Formulas III, Plücker relations, Lattice paths and Pfaffian identities
- MHF2003-6 Atsushi SUZUKI & Masahisa TABATA
Finite element matrices in congruent subdomains and their effective use for large-scale computations
- MHF2003-7 Setsuo TANIGUCHI
Stochastic oscillatory integrals - asymptotic and exact expressions for quadratic phase functions -
- MHF2003-8 Shoki MIYAMOTO & Atsushi YOSHIKAWA
Computable sequences in the Sobolev spaces
- MHF2003-9 Toru FUJII & Takashi YANAGAWA
Wavelet based estimate for non-linear and non-stationary auto-regressive model
- MHF2003-10 Atsushi YOSHIKAWA
Maple and wave-front tracking — an experiment
- MHF2003-11 Masanobu KANEKO
On the local factor of the zeta function of quadratic orders
- MHF2003-12 Hidefumi KAWASAKI
Conjugate-set game for a nonlinear programming problem

- MHF2004-1 Koji YONEMOTO & Takashi YANAGAWA
Estimating the Lyapunov exponent from chaotic time series with dynamic noise
- MHF2004-2 Rui YAMAGUCHI, Eiko TSUCHIYA & Tomoyuki HIGUCHI
State space modeling approach to decompose daily sales of a restaurant into time-dependent multi-factors
- MHF2004-3 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Cubic pencils and Painlevé Hamiltonians
- MHF2004-4 Atsushi KAWAGUCHI, Koji YONEMOTO & Takashi YANAGAWA
Estimating the correlation dimension from a chaotic system with dynamic noise
- MHF2004-5 Atsushi KAWAGUCHI, Kentarou KITAMURA, Koji YONEMOTO, Takashi YANAGAWA & Kiyofumi YUMOTO
Detection of auroral breakups using the correlation dimension
- MHF2004-6 Ryo IKOTA, Masayasu MIMURA & Tatsuyuki NAKAKI
A methodology for numerical simulations to a singular limit
- MHF2004-7 Ryo IKOTA & Eiji YANAGIDA
Stability of stationary interfaces of binary-tree type
- MHF2004-8 Yuko ARAKI, Sadanori KONISHI & Seiya IMOTO
Functional discriminant analysis for gene expression data via radial basis expansion
- MHF2004-9 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Hypergeometric solutions to the q -Painlevé equations
- MHF2004-10 Raimundas VIDŪNAS
Expressions for values of the gamma function
- MHF2004-11 Raimundas VIDŪNAS
Transformations of Gauss hypergeometric functions
- MHF2004-12 Koji NAKAGAWA & Masakazu SUZUKI
Mathematical knowledge browser
- MHF2004-13 Ken-ichi MARUNO, Wen-Xiu MA & Masayuki OIKAWA
Generalized Casorati determinant and Positon-Negaton-Type solutions of the Toda lattice equation
- MHF2004-14 Nalini JOSHI, Kenji KAJIWARA & Marta MAZZOCCO
Generating function associated with the determinant formula for the solutions of the Painlevé II equation

- MHF2004-15 Kouji HASHIMOTO, Ryohei ABE, Mitsuhiro T. NAKAO & Yoshitaka WATANABE
Numerical verification methods of solutions for nonlinear singularly perturbed problem
- MHF2004-16 Ken-ichi MARUNO & Gino BIONDINI
Resonance and web structure in discrete soliton systems: the two-dimensional Toda lattice and its fully discrete and ultra-discrete versions
- MHF2004-17 Ryuei NISHII & Shinto EGUCHI
Supervised image classification in Markov random field models with Jeffreys divergence
- MHF2004-18 Kouji HASHIMOTO, Kenta KOBAYASHI & Mitsuhiro T. NAKAO
Numerical verification methods of solutions for the free boundary problem
- MHF2004-19 Hiroki MASUDA
Ergodicity and exponential β -mixing bounds for a strong solution of Lévy-driven stochastic differential equations
- MHF2004-20 Setsuo TANIGUCHI
The Brownian sheet and the reflectionless potentials
- MHF2004-21 Ryuei NISHII & Shinto EGUCHI
Supervised image classification based on AdaBoost with contextual weak classifiers
- MHF2004-22 Hideki KOSAKI
On intersections of domains of unbounded positive operators
- MHF2004-23 Masahisa TABATA & Shoichi FUJIMA
Robustness of a characteristic finite element scheme of second order in time increment
- MHF2004-24 Ken-ichi MARUNO, Adrian ANKIEWICZ & Nail AKHMEDIEV
Dissipative solitons of the discrete complex cubic-quintic Ginzburg-Landau equation
- MHF2004-25 Raimundas VIDŪNAS
Degenerate Gauss hypergeometric functions
- MHF2004-26 Ryo IKOTA
The boundedness of propagation speeds of disturbances for reaction-diffusion systems
- MHF2004-27 Ryusuke KON
Convex dominates concave: an exclusion principle in discrete-time Kolmogorov systems

- MHF2004-28 Ryusuke KON
Multiple attractors in host-parasitoid interactions: coexistence and extinction
- MHF2004-29 Kentaro IHARA, Masanobu KANEKO & Don ZAGIER
Derivation and double shuffle relations for multiple zeta values
- MHF2004-30 Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Generalized partitioned quantum cellular automata and quantization of classical CA
- MHF2005-1 Hideki KOSAKI
Matrix trace inequalities related to uncertainty principle
- MHF2005-2 Masahisa TABATA
Discrepancy between theory and real computation on the stability of some finite element schemes
- MHF2005-3 Yuko ARAKI & Sadanori KONISHI
Functional regression modeling via regularized basis expansions and model selection
- MHF2005-4 Yuko ARAKI & Sadanori KONISHI
Functional discriminant analysis via regularized basis expansions
- MHF2005-5 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Point configurations, Cremona transformations and the elliptic difference Painlevé equations
- MHF2005-6 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Construction of hypergeometric solutions to the q -Painlevé equations