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1 Introduction

In constructing computational scheemes for flow problems, a crucial point is how to approximate the convection terms. Mainly there are two ways of the approximation, one is of upwind type and the other is of characteristic type [1]. In this paper the latter is treated.

We consider the characteristic finite element method, which is a combination of characteristic method and the finite element method. As a solver of flow problems this method has such a nice property that the obtained linear system is symmetric. We can, therefore, solve large-scale problems using efficient iterative solvers, e.g., preconditioned CG method. Every difficulty of the scheme is concentrated in the integration of composite functions derived from the convection terms. Since those functions are not smooth in each element, numeirical integration does not always work well.

We investigate the effect of errors caused by numerical integration in the convection-diffusion equation. Theoretically characteristic finite element schemes are proved to be unconditionally stable and the finite element solutions converge to the exact one when the discretization paremeters tend to zero. Hence we are sure that the equation is solved correctly by those shcemes if the integrations of the composite functions are done exactly or alomost exactly. In real computation, however, we often encounter that the characteristic finite element scheme of first order in time increment, which is the most familiar in those methods, is unstable. This shows that the scheme is not robust with respect to errors introduced from numerical integration of the composite functions. Recently we have developed a characteristic finite element scheme of second order in time increment [2]. We show that this scheme is not only stable theoretically but also robust in practical computation.

2 Characteristic Finite Element Schemes with First and Second Orders in Time Increment

Let us consider a convection-diffusion problem in a bounded domain Ω in $\mathbf{R}^d$ $(d = 2, 3)$ and a time interval $[0, T]$,

$$\frac{\partial \phi}{\partial t} + (u \cdot \nabla)\phi - \nu \Delta \phi = f, \qquad (x,t) \in \Omega \times (0,T)$$
$$\phi = 0, \qquad (x,t) \in \partial\Omega \times (0,T)$$
$$\phi = \phi^0, \qquad x \in \Omega,\ t = 0$$

where ϕ is an unknown function, u is a given velocity, f is a given source, ν is a viscosity, and ϕ^0 is an initial function. Let V_h be the P_1 finite element space whose functions vanish on the boundary. Let Δt be a time increment and $N_T = \lfloor T/\Delta t \rfloor$. The characteristic finite element method of first order in time increment is to find $\phi_h^n \in V_h$, $n = 1, ..., N_T$, such that

$$(\frac{\phi_h^{n+1} - \phi_h^n \circ X_1^n}{\Delta t}, \psi_h) + \nu(\nabla \phi_h^{n+1}, \nabla \psi_h) = (\Pi_h f^{n+1}, \psi_h), \quad \forall \psi_h \in V_h \tag{1}$$
$$\phi_h^0 = \Pi_h \phi^0 \tag{2}$$

where $X_1^n(x) \equiv x - u^{n+1}(x)\Delta t$ and Π_h is the interpolation operator. Scheme (1) and (2) is proved to be unconditionally stable and convergent in $O(\Delta t{+}h)$, where h is the representative element size. See [1]. In real computations, however, it often become unstable when numerical integration is used for the integration of the composite function term $\phi_h^n \circ X_1^n$.

In a recent paper [2] we have developed a characteristic finite element scheme of second order in time increment by using the second-order Runge-Kutta method

$$X_2^n(x) \equiv x - u^{n+1/2}(x - u^n x \Delta t/2)\Delta t$$

in place of X_1^n. We have also pointed out that a correction term is necessary for the real second order scheme in Δt. The scheme is

$$\begin{aligned} &\left(\frac{\phi_h^{n+1} - \phi_h^n \circ X_2^n}{\Delta t}, \psi_h\right) + \frac{\nu}{2}(\nabla \phi_h^{n+1} + \nabla \phi_h^n \circ X_1^n, \nabla \psi_h) \\ &+ \frac{\nu \Delta t}{2}(J^n \nabla \phi_h^n \circ X_1^n, \nabla \psi_h) \\ = \ &\frac{1}{2}\left(\Pi_h f^{n+1} + \Pi_h f^n \circ X_1^n, \psi_h\right), \qquad \forall \psi_h \in V_h \end{aligned} \tag{3}$$

where $J \equiv \partial u/\partial x$ is the Jacobian matrix. Scheme (3) and (2) is proved to be unconditionally stable and the error estimate

$$\max\{||\phi_h^n - \phi^n||_{L^2(\Omega)};\ n = 0, ..., N_T\} \le c(\Delta t^2 + h) \tag{4}$$

can be obtained, where ϕ is the exact solution and ϕ_h is the finite element solution found in the P_1 space. When the P_k finite element space is used, the right-hand side of (4) is replaced by $c(\Delta^2 + h^k)$.

3 Effect of Numerical integrations

In order to get $O(h)$ approximate solutions we take $\Delta t = O(h)$ in (1) and $\Delta t = O(\sqrt{h})$ in (3). For both schemes we use numerical integration for the composite function terms. We observe effect of numerical integrations in a model problem of the rotating Gaussian hill. Let Ω be a square $[-1,1] \times [-1,1]$, $T = 2\pi$, $\nu = 5 \times 10^{-4}$, $u = (-x_2, x_1)$, $f = 0$, and

$$\phi^0(x) = \exp\left(-\frac{|x - x_c|^2}{\sigma}\right), \quad x_c = (0.25, 0), \quad \sigma = 0.01.$$

Let us divide each side into N segments and get N^2 small squares. Connecting diagonals, we get $2N^2$ triangles. The maximum side length h is equal to $2\sqrt{2}/N$. On this mesh we use the P_1 finite element. Δt is taken as mentioned above. We use the following numerical integration, which is denoted by $I_h(k)$, $k = 1, 2, 3$, to the conposite convection terms. We divide each element into congruent k^2 triangles and on the small triangles we use the numerical integration of the first order. For example, the numerical integration $I_h(1; f)$ of a function f on element K is

$$I_h(1; f) = \frac{\mathrm{meas}(K)}{3}\{f(A_1) + f(A_2) + f(A_3)\}$$

where A_i, $i = 1, 2, 3$, are vertices of K. Taking $N = 64, 96, 128, 192$, we calculate the relative errors

$$\frac{||\phi_h - \Pi_h\phi||_{\ell^\infty(L^2)}}{||\Pi_h\phi||_{\ell^\infty(L^2)}}$$

where

$$||\phi_h||_{\ell^\infty(L^2)} = \max\left\{||\phi_h^n||_{L^2(\Omega)}; n = 0, \cdots, N_T\right\}.$$

Table 1 shows the relative errors, where $F(k)$ and $S(k)$ means that the numerical integration $I(k)$ is employed for the first and the second order characteristic finite element method, respectively. The symbol X shows no result is obtained by the instability.

Table 1. Relative errors

N	F(1)	F(2)	F(3)	S(1)	S(2)	S(3)
64	X	7.74e-01	2.40e-01	2.82e+00	6.91e-02	5.44e-02
96	X	2.00e-01	1.60e-01	1.36e-01	4.28e-02	3.59e-02
128	X	1.45e-01	1.19e-02	8.39e-02	3.07e-02	2.69e-02
192	X	9.36e-02	7.94e-02	4.44e-02	1.94e-02	1.77e-02

Fig. 1 depicts the results of Table 1 in the logarithmic scale, where the dotted line shows the theoretical convergence order h. In both cases $O(h)$ convergence is obtained when the solutions are calculated without overflow. These results shows that the second order scheme is more robust than the

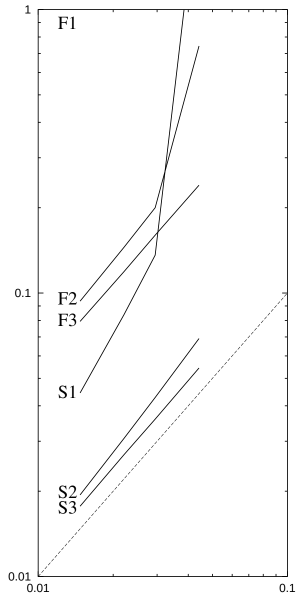

Fig. 1. Relative errors vs. element size h

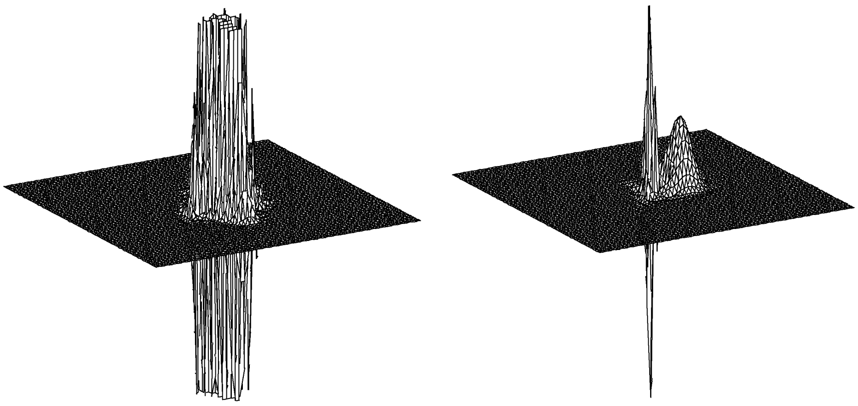

Fig. 2. Stereographs of F1 and F2(2π)

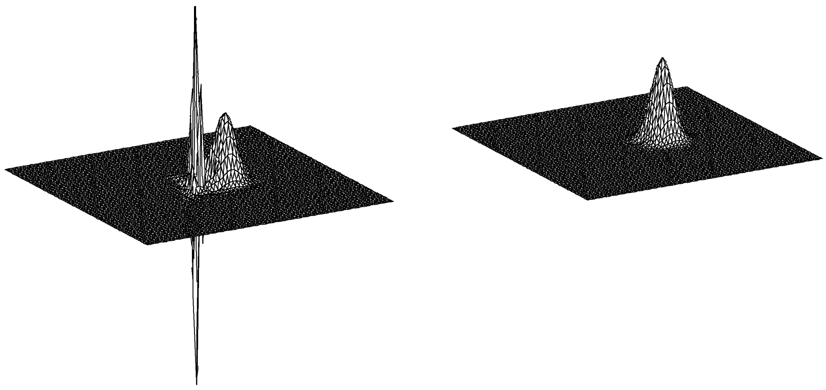

Fig. 3. Stereographs of S1(2π) and S2(2π)

first order scheme with respect to the numerical integration error and produce better numerical results.

In Fig. 2 the stereograph of F1 before the overflow and that of S2 at $t = 2\pi$ are shown, and in Fig. 3 those of S1 and S2 at $t = 2\pi$. Those of the other convergent cases are almost same as that of S2.

We can explain the numerical results as follows. To obtain the L^2-stability we substitute ϕ_h^{n+1} into ψ_h in (3). We denote by $(\cdot,\cdot)_h$ the value obtained by numerical integration of $(\cdot,\cdot)$. We have

$$\left|\frac{(\phi_h^n \circ X_2^n, \phi_h^{n+1}) - (\phi_h^n \circ X_2^n, \phi_h^{n+1})_h}{\Delta t}\right| \le c\frac{h}{\Delta t}||\nabla\phi_h^n||_{L^2(\Omega)}||\phi_h^{n+1}||_{L^2(\Omega)}. \quad (5)$$

In the second order scheme the right-hand side is evaluated by

$$c\Delta t||\nabla\phi_h^n||_{L^2(\Omega)}||\phi_h^{n+1}||_{L^2(\Omega)}$$

by virtue of $\Delta t = O(\sqrt{h})$. In the first order scheme, however, $\Delta t = O(h)$ and such an estimate does not hold. A simple estimate of the right-hand side of (5) is

$$c||\nabla\phi_h^n||_{L^2(\Omega)}||\phi_h^{n+1}||_{L^2(\Omega)}$$

which implies that the instability may occur as the time-step number increases.

4 Concluding Remarks

In using the characteristic finite element method we have to pay much attention to the errors caused by numerical integration. Compared to the first order scheme, the second order scheme is robust with respect to the numerical integration error, which has been recognized by numerical results. We have also given a rough explanation for the robustness. Under a mild condition on the numerical integration we can show that the second order scheme is stable with the numerical integration. A detail discussion and a complete proof of the stability will be presented in the forthcoming paper.

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