Robustness of a Characteristic Finite Element Scheme of Second Order in Time Increment

Tabata, Masahisa  
Department of Mathematical Sciences, Faculty of Mathematics, Kyushu University

Fujima, Shoichi  
Department of Mathematical Sciences, Ibaraki University

http://hdl.handle.net/2324/11840
Robustness of a characteristic finite element scheme of second order in time increment

M. Tabata & S. Fujima

MHF 2004-23

(Received July 29, 2004)
Robustness of a Characteristic Finite Element Scheme of Second Order in Time Increment

Masahisa Tabata¹ and Shoichi Fujima²

¹ Department of Mathematical Sciences, Kyushu University Hakozaki, Fukuoka, 812-8581 Japan. tabata@math.kyushu-u.ac.jp
² Department of Mathematical Sciences, Ibaraki University Bunkyo, Mito, 310-8512 Japan. fujima@mx.ibaraki.ac.jp

1 Introduction

In constructing computational schemes for flow problems, a crucial point is how to approximate the convection terms. Mainly there are two ways of the approximation, one is of upwind type and the other is of characteristic type [1]. In this paper the latter is treated.

We consider the characteristic finite element method, which is a combination of characteristic method and the finite element method. As a solver of flow problems this method has such a nice property that the obtained linear system is symmetric. We can, therefore, solve large-scale problems using efficient iterative solvers, e.g., preconditioned CG method. Every difficulty of the scheme is concentrated in the integration of composite functions derived from the convection terms. Since those functions are not smooth in each element, numerical integration does not always work well.

We investigate the effect of errors caused by numerical integration in the convection-diffusion equation. Theoretically characteristic finite element schemes are proved to be unconditionally stable and the finite element solutions converge to the exact one when the discretization parameters tend to zero. Hence we are sure that the equation is solved correctly by those schemes if the integrations of the composite functions are done exactly or almost exactly. In real computation, however, we often encounter that the characteristic finite element scheme of first order in time increment, which is the most familiar in those methods, is unstable. This shows that the scheme is not robust with respect to errors introduced from numerical integration of the composite functions. Recently we have developed a characteristic finite element scheme of second order in time increment [2]. We show that this scheme is not only stable theoretically but also robust in practical computation.

2 Characteristic Finite Element Schemes with First and Second Orders in Time Increment

Let us consider a convection-diffusion problem in a bounded domain \( \Omega \) in \( \mathbb{R}^d \) (\( d = 2, 3 \)) and a time interval \([0, T]\),
\[
\frac{\partial \phi}{\partial t} + (u \cdot \nabla) \phi - \nu \Delta \phi = f, \quad (x, t) \in \Omega \times (0, T)
\]

\[
\phi = 0, \quad (x, t) \in \partial \Omega \times (0, T)
\]

\[
\phi = \phi^0, \quad x \in \Omega, \ t = 0
\]

where \( \phi \) is an unknown function, \( u \) is a given velocity, \( f \) is a given source, \( \nu \) is a viscosity, and \( \phi^0 \) is an initial function. Let \( V_h \) be the \( P_1 \) finite element space whose functions vanish on the boundary. Let \( \Delta t \) be a time increment and \( N_T = \lceil T/\Delta t \rceil \). The characteristic finite element method of first order in time increment is to find \( \phi^n_h \in V_h, \ n = 1, \ldots, N_T, \) such that

\[
\left( \frac{\phi^{n+1}_h - \phi^n_h \circ X^n_1}{\Delta t}, \psi_h \right) + \nu (\nabla \phi^{n+1}_h \circ X^n_1, \nabla \psi_h) = (\Pi_h f^{n+1}, \psi_h), \quad \forall \psi_h \in V_h \quad (1)
\]

\[
\phi^0_h = \Pi_h \phi^0
\]

where \( X^n_1(x) \equiv x - u^n(x) \Delta t \) and \( \Pi_h \) is the interpolation operator. Scheme (1) and (2) is proved to be unconditionally stable and convergent in \( O(\Delta t + h) \), where \( h \) is the representative element size. See [1]. In real computations, however, it often become unstable when numerical integration is used for the integration of the composite function term \( \phi^n_h \circ X^n_1 \).

In a recent paper [2] we have developed a characteristic finite element scheme of second order in time increment by using the second-order Runge-Kutta method

\[
X^n_2(x) \equiv x - u^{n+1/2}(x - u^n x \Delta t/2) \Delta t
\]

in place of \( X^n_1 \). We have also pointed out that a correction term is necessary for the real second order scheme in \( \Delta t \). The scheme is

\[
\left( \frac{\phi^{n+1}_h - \phi^n_h \circ X^n_2}{\Delta t}, \psi_h \right) + \frac{\nu}{2} (\nabla \phi^{n+1}_h \circ X^n_2, \nabla \psi_h)
\]

\[
+ \frac{\nu \Delta t}{2} (J^n \nabla \phi^n_h \circ X^n_1, \nabla \psi_h)
\]

\[
= \frac{1}{2} (\Pi_h f^{n+1} + \Pi_h f^n \circ X^n_1, \psi_h), \quad \forall \psi_h \in V_h \quad (3)
\]

where \( J \equiv \partial u/\partial x \) is the Jacobian matrix. Scheme (3) and (2) is proved to be unconditionally stable and the error estimate

\[
\max \{ ||\phi^n_h - \phi^n||_{L^2(\Omega)} ; \ n = 0, \ldots, N_T \} \leq c(\Delta t^2 + h) \quad (4)
\]

can be obtained, where \( \phi \) is the exact solution and \( \phi_h \) is the finite element solution found in the \( P_1 \) space. When the \( P_k \) finite element space is used, the right-hand side of (4) is replaced by \( c(\Delta t^2 + h^k) \).

### 3 Effect of Numerical Integrations

In order to get \( O(h) \) approximate solutions we take \( \Delta t = O(h) \) in (1) and \( \Delta t = O(\sqrt{h}) \) in (3). For both schemes we use numerical integration for the
composite function terms. We observe effect of numerical integrations in a model problem of the rotating Gaussian hill. Let \( \Omega \) be a square \([-1, 1] \times [-1, 1] \), \( T = 2\pi \), \( \nu = 5 \times 10^{-4} \), \( u = (-x_2, x_1) \), \( f = 0 \), and

\[
\phi^0(x) = \exp\left(-\frac{|x - x_c|^2}{\sigma}\right), \quad x_c = (0.25, 0), \quad \sigma = 0.01.
\]

Let us divide each side into \( N \) segments and get \( N^2 \) small squares. Connecting diagonals, we get \( 2N^2 \) triangles. The maximum side length \( h \) is equal to \( 2\sqrt{2}/N \). On this mesh we use the \( P_1 \) finite element. \( \Delta t \) is taken as mentioned above. We use the following numerical integration, which is denoted by \( I_h(k) \), \( k = 1, 2, 3 \), to the composite convection terms. We divide each element into congruent \( k^2 \) triangles and on the small triangles we use the numerical integration of the first order. For example, the numerical integration \( I_h(1; f) \) of a function \( f \) on element \( K \) is

\[
I_h(1; f) = \frac{\text{meas}(K)}{3} \{ f(A_1) + f(A_2) + f(A_3) \}
\]

where \( A_i, i = 1, 2, 3, \) are vertices of \( K \). Taking \( N = 64, 96, 128, 192 \), we calculate the relative errors

\[
\frac{||\phi_h - \Pi_h \phi||_{L^\infty(L^2)}}{||\Pi_h \phi||_{L^\infty(L^2)}}
\]

where

\[
||\phi_h||_{L^\infty(L^2)} = \max \{ ||\phi_h^n||_{L^2(\Omega)}; n = 0, \ldots, N_T \}.
\]

Table 1 shows the relative errors, where \( F(k) \) and \( S(k) \) means that the numerical integration \( I(k) \) is employed for the first and the second order characteristic finite element method, respectively. The symbol \( X \) shows no result is obtained by the instability.

**Table 1. Relative errors**

<table>
<thead>
<tr>
<th>N</th>
<th>F(1)</th>
<th>F(2)</th>
<th>F(3)</th>
<th>S(1)</th>
<th>S(2)</th>
<th>S(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>X</td>
<td>7.74e-01</td>
<td>2.40e-01</td>
<td>2.82e+00</td>
<td>6.91e-02</td>
<td>5.44e-02</td>
</tr>
<tr>
<td>96</td>
<td>X</td>
<td>2.00e-01</td>
<td>1.60e-01</td>
<td>1.36e-01</td>
<td>4.28e-02</td>
<td>3.59e-02</td>
</tr>
<tr>
<td>128</td>
<td>X</td>
<td>1.45e-01</td>
<td>1.19e-02</td>
<td>8.39e-02</td>
<td>3.07e-02</td>
<td>2.69e-02</td>
</tr>
<tr>
<td>192</td>
<td>X</td>
<td>9.36e-02</td>
<td>7.94e-02</td>
<td>4.44e-02</td>
<td>1.94e-02</td>
<td>1.77e-02</td>
</tr>
</tbody>
</table>

Fig. 1 depicts the results of Table 1 in the logarithmic scale, where the dotted line shows the theoretical convergence order \( h \). In both cases \( O(h) \) convergence is obtained when the solutions are calculated without overflow. These results shows that the second order scheme is more robust than the
Fig. 1. Relative errors vs. element size $h$

Fig. 2. Stereographs of F1 and F2(2$\pi$)
first order scheme with respect to the numerical integration error and produce better numerical results.

In Fig. 2 the stereograph of F1 before the overflow and that of S2 at $t = 2\pi$ are shown, and in Fig. 3 those of S1 and S2 at $t = 2\pi$. Those of the other convergent cases are almost same as that of S2.

We can explain the numerical results as follows. To obtain the $L^2$-stability we substitute $\phi_h^{n+1}$ into $\psi_h$ in (3). We denote by $(\cdot, \cdot)_h$ the value obtained by numerical integration of $(\cdot, \cdot)$. We have

$$\left| \frac{(\phi_h^n \circ X_h^n, \phi_h^{n+1}) - (\phi_h^n \circ X_h^n, \phi_h^{n+1})_h}{\Delta t} \right| \leq c \frac{h}{\Delta t} \| \nabla \phi_h^n \|_{L^2(\Omega)} \| \phi_h^{n+1} \|_{L^2(\Omega)}. \tag{5}$$

In the second order scheme the right-hand side is evaluated by

$$c\Delta t \| \nabla \phi_h^n \|_{L^2(\Omega)} \| \phi_h^{n+1} \|_{L^2(\Omega)}$$

by virtue of $\Delta t = O(\sqrt{h})$. In the first order scheme, however, $\Delta t = O(h)$ and such an estimate does not hold. A simple estimate of the right-hand side of (5) is

$$c \| \nabla \phi_h^n \|_{L^2(\Omega)} \| \phi_h^{n+1} \|_{L^2(\Omega)}$$

which implies that the instability may occur as the time-step number increases.

4 Concluding Remarks

In using the characteristic finite element method we have to pay much attention to the errors caused by numerical integration. Compared to the first
order scheme, the second order scheme is robust with respect to the numerical integration error, which has been recognized by numerical results. We have also given a rough explanation for the robustness. Under a mild condition on the numerical integration we can show that the second order scheme is stable with the numerical integration. A detail discussion and a complete proof of the stability will be presented in the forthcoming paper.

Acknowledgments

The first author wishes to express his thanks to support from the Japan Society for the Promotion of Science under Grant-in-Aid for Scientific Research (S), No. 16104001 and from the Ministry of Education, Culture, Sports, Science and Technology of Japan under Kyushu University 21st Century COE Program, Development of Dynamic Mathematics with High Functionality. The second author was supported by the Japan Society for the Promotion of Science under Grant-in-Aid for Scientific Research (C), No. 16540093.

References

List of MHF Preprint Series, Kyushu University
21st Century COE Program
Development of Dynamic Mathematics with High Functionality

MHF2003-1 Mitsuhiro T. NAKAO, Kouji HASHIMOTO & Yoshitaka WATANABE
A numerical method to verify the invertibility of linear elliptic operators with
applications to nonlinear problems

MHF2003-2 Masahisa TABATA & Daisuke TAGAMI
Error estimates of finite element methods for nonstationary thermal convection
problems with temperature-dependent coefficients

MHF2003-3 Tomohiro ANDO, Sadanori KONISHI & Seiya IMOTO
Adaptive learning machines for nonlinear classification and Bayesian information
criteria

MHF2003-4 Kazuhiro YOKOYAMA
On systems of algebraic equations with parametric exponents

MHF2003-5 Masao ISHIKAWA & Masato WAKAYAMA
Applications of Minor Summation Formulas III, Plücker relations, Lattice
paths and Pfaffian identities

MHF2003-6 Atsushi SUZUKI & Masahisa TABATA
Finite element matrices in congruent subdomains and their effective use for
large-scale computations

MHF2003-7 Setsuo TANIGUCHI
Stochastic oscillatory integrals - asymptotic and exact expressions for quadratic
phase functions -

MHF2003-8 Shoki MIYAMOTO & Atsushi YOSHIKAWA
Computable sequences in the Sobolev spaces

MHF2003-9 Toru FUJII & Takashi YANAGAWA
Wavelet based estimate for non-linear and non-stationary auto-regressive model

MHF2003-10 Atsushi YOSHIKAWA
Maple and wave-front tracking — an experiment

MHF2003-11 Masanobu KANEKO
On the local factor of the zeta function of quadratic orders

MHF2003-12 Hidefumi KAWASAKI
Conjugate-set game for a nonlinear programming problem
MHF2004-1 Koji YONEMOTO & Takashi YANAGAWA
Estimating the Lyapunov exponent from chaotic time series with dynamic noise

MHF2004-2 Rui YAMAGUCHI, Eiko TSUCHIYA & Tomoyuki HIGUCHI
State space modeling approach to decompose daily sales of a restaurant into time-dependent multi-factors

MHF2004-3 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Cubic pencils and Painlevé Hamiltonians

MHF2004-4 Atsushi KAWAGUCHI, Koji YONEMOTO & Takashi YANAGAWA
Estimating the correlation dimension from a chaotic system with dynamic noise

MHF2004-5 Atsushi KAWAGUCHI, Kentarou KITAMURA, Koji YONEMOTO, Takashi YANAGAWA & Kiyofumi YUMOTO
Detection of auroral breakups using the correlation dimension

MHF2004-6 Ryo IKOTA, Masayasu MIMURA & Tatsuyuki NAKAKI
A methodology for numerical simulations to a singular limit

MHF2004-7 Ryo IKOTA & Eiji YANAGIDA
Stability of stationary interfaces of binary-tree type

MHF2004-8 Yuko ARAKI, Sadanori KONISHI & Seiya IMOTO
Functional discriminant analysis for gene expression data via radial basis expansion

MHF2004-9 Kenji KAJIWARA, Tetsu MASUDA, Masatoshi NOUMI, Yasuhiro OHTA & Yasuhiko YAMADA
Hypergeometric solutions to the $q$ Painlevé equations

MHF2004-10 Raimundas VIDUNAS
Expressions for values of the gamma function

MHF2004-11 Raimundas VIDUNAS
Transformations of Gauss hypergeometric functions

MHF2004-12 Koji NAKAGAWA & Masakazu SUZUKI
Mathematical knowledge browser

MHF2004-13 Ken-ichi MARUNO, Wen-Xiu MA & Masayuki OIKAWA
Generalized Casorati determinant and Positon-Negaton-Type solutions of the Toda lattice equation

MHF2004-14 Nalini JOSHI, Kenji KAJIWARA & Marta MAZZOCCH
Generating function associated with the determinant formula for the solutions of the Painlevé II equation
MHF2004-15 Kouji HASHIMOTO, Ryohei ABE, Mitsuhiro T. NAKAO & Yoshitaka WATANABE
Numerical verification methods of solutions for nonlinear singularly perturbed problem

MHF2004-16 Ken-ichi MARUNO & Gino BIONDINI
Resonance and web structure in discrete soliton systems: the two-dimensional Toda lattice and its fully discrete and ultra-discrete versions

MHF2004-17 Ryuei NISHII & Shinto EGUCHI
Supervised image classification in Markov random field models with Jeffreys divergence

MHF2004-18 Kouji HASHIMOTO, Kenta KOBAYASHI & Mitsuhiro T. NAKAO
Numerical verification methods of solutions for the free boundary problem

MHF2004-19 Hiroki MASUDA
Ergodicity and exponential $\beta$-mixing bounds for a strong solution of Lévy-driven stochastic differential equations

MHF2004-20 Setsuo TANIGUCHI
The Brownian sheet and the reflectionless potentials

MHF2004-21 Ryuei NISHII & Shinto EGUCHI
Supervised image classification based on AdaBoost with contextual weak classifiers

MHF2004-22 Hideki KOSAKI
On intersections of domains of unbounded positive operators

MHF2004-23 Masahisa TABATA & Shoichi FUJIMA
Robustness of a characteristic finite element scheme of second order in time increment