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Abstract

In this paper, the investigation into stochastic calculus related with the KdV equation, which was initiated by S. Kotani [4] and made in succession by N. Ikeda and the author [2, 11], is continued. Reflectionless potentials give important examples in the scattering theory and the study of the KdV equation; they are expressed concretely by their corresponding scattering data, and give a rise of solitons of the KdV equation. N. Ikeda and the author [2] established a mapping ψ of a family $\mathcal{G}_0$ of probability measures on the 1-dimensional Wiener space to the space Ξ_0 of reflectionless potentials. The mapping gives a probabilistic expression of reflectionless potential. In this paper, it will be shown that ψ is bijective, and hence $\mathcal{G}_0$ and Ξ_0 can be identified. The space Ξ_0 was extended to the one Ξ of generalized reflectionless potentials, and was used by V. Marchenko to investigate the Cauchy problem for the KdV equation and by S. Kotani to construct KdV-flows. As an application of the identification of $\mathcal{G}_0$ and Ξ_0 via ψ, taking advantage of the Brownian sheet, it will be seen that convergences of elements in $\mathcal{G}_0$ realizes the extension of Ξ_0 to Ξ.

Key words: Brownian sheet; reflectionless potential; Ornstein-Uhlenbeck process,
AMS 2000 subject classifications: 60H30; 60B10; 34L25

1. Introduction

Let $\mathcal{W}$ be the space of all $\mathbf{R}$-valued continuous functions w on $[0,\infty)$ with $w(0) = 0$, and $\mathcal{B}$ be its Borel σ-field, $\mathcal{W}$ being equipped with the topology of uniform convergence on compacts. The coordinate mapping on $\mathcal{W}$ is denoted by $X(x)$; $X(x,w) = w(x)$, $w \in \mathcal{W}$, $x \in [0,\infty)$. Let Σ_0 be the set of measures on $\mathbf{R}$ of the form $\sum_{j=1}^n c_j^2 \delta_{p_j}$ for some $n \in \mathbf{N}$ and $p_j \in \mathbf{R}$, $c_j > 0$, $1 \le j \le n$ with $p_i \ne p_j$ if $i \ne j$, where δ_p is the Dirac measure concentrated at p. For $\sigma \in \Sigma_0$, set

$$R_\sigma(x,y) = \int_{\mathbf{R}} \frac{e^{\zeta(x+y)} - e^{\zeta|x-y|}}{2\zeta}\sigma(d\zeta),$$

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and let P^σ be the probability measure on $(\mathcal{W}, \mathcal{B})$ so that $\{X(x)\}_{x\geq 0}$ is a centered Gaussian process with covariance function R_σ (a construction of P^σ will be given in Sect. 2). Put

$$\mathcal{G}_0 = \{P^\sigma | \sigma \in \Sigma_0\}.$$

N. Ikeda and the author ([2]) showed that, for each P^σ, the function

$$(1) \qquad \psi(P^\sigma)(x) = 4\Big(\frac{d}{dx}\Big)^2 \log\bigg(\int_{\mathcal{W}} \exp\Big(-\frac{1}{2}\int_0^x X(y)^2 dy\Big) dP^\sigma\bigg), \quad x \geq 0,$$

is well defined and coincides with the restriction of a reflectionless potential to $[0, \infty)$, and the associated scattering data was specified in terms of σ. Since reflectionless potentials are real analytic, we may and will think of ψ as a mapping of $\mathcal{G}_0$ to the space Ξ_0 of reflectionless potentials. It should be recalled that reflectionless potentials give a rise of solitons of the KdV equation ([6, 8]). A review on these results and the definition of reflectionless potential will be given in Sect. 2. The first aim of this paper is to show that the mapping ψ of $\mathcal{G}_0$ to Ξ_0 is bijective. See Theorem 1. In this sense, the set Ξ_0 of analytic future and the set $\mathcal{G}_0$ of probabilistic future are identified. Moreover, we shall establish a probabilistic expression of $u \in \Xi_0$ through ψ. See Corollaries 1 and 2.

A generalized reflectionless potential u is a limit of a sequence $\{u_n\}$ of reflectionless potentials u_n such that $\mathrm{Spec}(-(d/dx)^2 + u_n) \subset [-\lambda_0, \infty)$, $n = 1, 2, \dots$, for some $\lambda_0 > 0$ in the topology of uniform convergence on compacts, where $\mathrm{Spec}(-(d/dx)^2 + u_n)$ denotes the spectrum of $-(d/dx)^2 + u_n$. The space Ξ of generalized potentials was used by V. Marchenko ([7]) to study the Cauchy problem for the KdV equation, and by S. Kotani ([4]) to construct KdV-flows. Let Σ be the space of all finite measures on $\mathbf{R}$ with compact support, and

$$\mathcal{G} = \{P^\sigma | \sigma \in \Sigma\},$$

where we have naturally extended the notation P^σ to Σ. On account of the identification of Ξ_0 and $\mathcal{G}_0$ stated in the above paragraph, arises a natural question if one can describe the relation between convergences of reflectionless potentials to generalized ones and convergences of probability measures in $\mathcal{G}_0$ to those in $\mathcal{G}$. The second aim of this paper is to answer affirmatively to this question. Namely, we shall study the convergence of $\psi(P^\sigma)$'s with P^σ not only in $\mathcal{G}_0$ but also in $\mathcal{G}$. In particular, the convergence of elements in Ξ_0 defining those in Ξ will be realized through the convergence of elements in $\mathcal{G}_0$ to those in $\mathcal{G}$. Moreover, we shall show that the surjectivity of ψ on $\mathcal{G}_0$ extends to $\mathcal{G}$; every $u \in \Xi$ admits $P^\sigma \in \mathcal{G}$ so that $\psi(P^\sigma) = u$ on $[0, \infty)$. The expression of such u on $(-\infty, 0]$ by ψ and σ will be also given. For these, see Theorem 3 and Remark 1. A key ingredient for the investigation is to realize the above P^σ by using the Brownian sheet and reduce every estimations to the ones for Wiener integrals associated with the Brownian sheet.

The organization of the paper is as follows. In Sect. 2, we shall show the bijectivity of ψ in (1) after reviewing the result in [2]. In the section, a construction of P^σ for $\sigma \in \Sigma_0$ is given. Sect. 3 is devoted to introducing compound Ornstein-Uhlenbeck processes which are indispensable to discuss the convergence of P^σ's.

The Brownian sheet plays a key role to construct such processes. Another realization of P^σ with the Brownian sheet will be also given there. In the last section, we shall observe the uniform convergence on compacts of reflectionless potentials via the convergence of P^σ's. The surjectivity of $\psi : \mathcal{G} \to \Xi$ will be seen there.

2. Reflectionless potentials

We start this section by reviewing the result in [2]. In what follows, every element in $\mathbf{R}^n$ is regarded as a column vector, and ${}^tA$ stands for the transpose of matrix A.

Let

$$\Sigma_0 = \Big\{ \sum_{j=1}^n c_j^2 \delta_{p_j} \Big| c_j > 0,\, p_j \in \mathbf{R},\, p_i \neq p_j\, (i \neq j),\, n = 1, 2, \dots \Big\},$$

where δ_p denotes the Dirac measure concentrated at p. For $\sigma = \sum_{j=1}^n c_j^2 \delta_{p_j} \in \Sigma_0$, we define the n-dimensional Ornstein-Uhlenbeck process $\{\xi_\sigma(y)\}_{y \geq 0}$ and the 1-dimensional Gaussian process $\{X_\sigma(y)\}_{y\geq 0}$ by

$$\begin{aligned} \xi_\sigma(y) &= e^{yD_\sigma} \int_0^y e^{-zD_\sigma} dB(z) = {}^t\Big(e^{yp_j} \int_0^y e^{-zp_j} dB^j(z)\Big)_{1\leq j \leq n}, \\ X_\sigma(y) &= \langle \mathbf{c}, \xi_\sigma(y) \rangle, \end{aligned} \tag{2}$$

where $\{B(y) = (B^1(y), \dots, B^n(y))\}_{y\geq 0}$ is an n-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, $dB(z)$ stands for the Itô integral with respect to $B(z)$, D_σ denotes the $n \times n$ diagonal matrix with $p_1, \dots, p_n$ as diagonal entries, $e^A = \sum_{j=0}^\infty A^j/j!$ for $n \times n$ matrix A, $\mathbf{c} = {}^t(c_1, \dots, c_n)$, and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbf{R}^n$. It should be mentioned that the law of X_σ does not depend on the order of pairs (p_j, c_j)'s, while ξ_σ does. It is easily seen that

$$\begin{aligned} \int_\Omega X_\sigma(x) X_\sigma(y) dP &= \sum_{j=1}^n \frac{c_j^2}{2p_j} \{ e^{p_j(x+y)} - e^{p_j|x-y|} \} \\ &= \int_{\mathbf{R}} \frac{e^{\zeta(x+y)} - e^{\zeta|x-y|}}{2\zeta} \sigma(d\zeta) = R_\sigma(x, y). \end{aligned}$$

Hence P^σ is realized as the induced measure of X_σ on $\mathcal{W}$; $P^\sigma = P \circ X_\sigma^{-1}$. Note that

$$\frac{d}{dx} \int_{\mathcal{W}} X(x)^2 dP^\sigma = \int_{\mathbf{R}} e^{2\zeta x} \sigma(d\zeta).$$

Hence $\sigma = \mu$ if $P^\sigma = P^\mu$. Thus Σ_0 is identified with $\mathcal{G}_0$.

Let $\mathcal{S}$ be the set of all sequence $\{\eta_j, m_j\}_{1\leq j\leq n}$ of length $2n$, $n = 1, 2, \dots$, of positive real numbers such that $\eta_1 < \cdots < \eta_n$. The reflectionless potential $u_{\mathbf{s}}$ with scattering data $\mathbf{s} = \{\eta_j, m_j\}_{1\leq j\leq n} \in \mathcal{S}$ is by definition the function

$$u_{\mathbf{s}}(x) = -2\Big(\frac{d}{dx}\Big)^2 \log \det(I + G_{\mathbf{s}}(x)), \quad x \in \mathbf{R},$$

where $G_{\mathbf{s}}(x)$ is the $n \times n$ matrix given by

$$G_{\mathbf{s}}(x) = \left(\frac{\sqrt{m_i m_j} e^{-(\eta_i+\eta_j)x}}{\eta_i + \eta_j} \right)_{1 \le i,j \le n}.$$

Set

$$\Xi_0 = \{u_{\mathbf{s}} | \mathbf{s} \in \mathcal{S}\}.$$

Solving the scattering problem for the Sturm-Liouville operator $-(d/dx)^2 + u_{\mathbf{s}}$, one obtains scattering data $\mathbf{s} \in \mathcal{S}$ from $u_{\mathbf{s}}$ ([3, 6, 7]). Thus, Ξ_0 and $\mathcal{S}$ are identified. It may be interesting to recall ([6, 8]) that if we set

$$\mathbf{s}(t) = \{\eta_j, m_j \exp(-2\eta_j^3 t)\}_{1 \le j \le n},$$

then the function $v(x,t) = -u_{\mathbf{s}(t)}(x)$ solves the KdV equation

$$\frac{\partial v}{\partial t} = \frac{3}{2} v \frac{\partial v}{\partial x} + \frac{1}{4} \frac{\partial^3 v}{\partial x^3}.$$

For $\sigma \in \Sigma_0$, without loss of generality, we may and will assume that there exist $m \le n$ and $1 \le j(1) < \cdots < j(m) \le n$ such that
(H) $|p_k| \le |p_{k+1}|$, $p_{j(\ell)} > 0$, $p_{j(\ell)+1} = -p_{j(\ell)}$, $\#\{|p_1|, \ldots, |p_n|\} = n - m$,
where $1 \le k \le n-1$ and $1 \le \ell \le m$. Then, the equation $\sum_{j=1}^n c_j^2/(r - p_j^2) = 1$ admits $n-m$ roots $0 < r_1 < \cdots < r_{n-m}$. Define the mapping $\overline{\psi} : \Sigma_0 \to \mathcal{S}$ so that $\overline{\psi}(\sigma) = \{\eta_j, m_j\}_{1 \le j \le n} \in \mathcal{S}$ is given by

$$\{\eta_1 < \cdots < \eta_n\} = \{p_{j(1)}, \ldots, p_{j(m)}, r_1^{1/2}, \ldots, r_{n-m}^{1/2}\},$$

$$m_i = \begin{cases} 2\eta_{j(\ell)} \dfrac{c_{j(\ell)+1}^2}{c_{j(\ell)}^2} \displaystyle\prod_{k \ne j(\ell)} \frac{\eta_k + \eta_{j(\ell)}}{\eta_k - \eta_{j(\ell)}} \prod_{k \ne j(\ell), j(\ell)+1} \frac{p_k + \eta_{j(\ell)}}{p_k - \eta_{j(\ell)}}, & \text{if } i = j(\ell), \\ -2\eta_i \displaystyle\prod_{k \ne i} \frac{\eta_k + \eta_i}{\eta_k - \eta_i} \prod_{k=1}^{n} \frac{p_k + \eta_i}{p_k - \eta_i}, & \text{otherwise.} \end{cases} \tag{3}$$

It was seen in [2] that

$$\begin{aligned} \log \int_{\mathcal{W}} \exp\left(-\frac{1}{2} \int_0^x X(y)^2 dy \right) dP^\sigma &= -\frac{1}{2} \log \det(I + G_{\overline{\psi}(\sigma)}(x)) + \\ &+ \frac{1}{2} \log \det(I + G_{\overline{\psi}(\sigma)}(0)) - \frac{x}{2} \sum_{j=1}^n (p_j + \eta_j), \quad x \ge 0. \end{aligned} \tag{4}$$

In particular, $\psi(P^\sigma)$ in (1) satisfies that

$$\psi(P^\sigma) = u_{\overline{\psi}(\sigma)} \quad \text{on } [0, \infty) \quad \text{for any } P^\sigma \in \mathcal{G}_0. \tag{5}$$

If $u, v \in \Xi_0$ coincide on $[0, \infty)$, then so on $\mathbf{R}$, since they are real analytic. Thus, we may and will think of $\psi(P^\sigma)$, $P^\sigma \in \mathcal{G}_0$, as functions on $\mathbf{R}$, and hence ψ as a mapping of $\mathcal{G}_0$ to Ξ_0.

We are now ready to state our first main result.

Theorem 1. *(i) $\psi : \mathcal{G}_0 \to \Xi_0$ is bijective.*
(ii) Let $P^\sigma \in \mathcal{G}_0$ and $u = \psi(P^\sigma)$. Represent as $\sigma = \sum_{j=1}^n c_j^2 \delta_{p_j}$ and define $\tilde{\sigma} = \sum_{j=1}^n c_j^2 \delta_{-p_j}$. Then it holds that

$$u(x) = \psi(P^{\tilde{\sigma}})(-x), \quad x \leq 0.$$

Due to this theorem, $\mathcal{G}_0$ and Ξ_0 can be identified. The theorem immediately implies that

Corollary 1. *Let $\tilde{\mathcal{G}}_0 = \{Q^\sigma = (P^\sigma, P^{\tilde{\sigma}}) | \sigma \in \Sigma_0\}$, where $\tilde{\sigma}$ is defined as in Theorem 1. Then the mapping $\tilde{\psi}$ defined by*

$$\tilde{\psi}(Q^\sigma)(x) = \begin{cases} \psi(P^\sigma)(x), & \text{if } x \geq 0, \\ \psi(P^{\tilde{\sigma}})(-x), & \text{if } x < 0, \end{cases}$$

is a bijection from $\tilde{\mathcal{G}}_0$ to Ξ_0.

Furthermore, we have that

Corollary 2. *Let $P^\sigma \in \mathcal{G}_0$ and $u = \psi(P^\sigma)$. Extend the Brownian motion $\{B(y)\}_{y \geq 0}$ used in (2) to $y \leq 0$ so that $B(y) = B(-y)$, and define $\xi_\sigma(y)$ and $X_\sigma(y)$ by (2) for $y \leq 0$:*

$$\xi_\sigma(y) = e^{yD_\sigma} \int_0^y e^{-zD_\sigma} dB(z) = -e^{yD_\sigma} \int_y^0 e^{-zD_\sigma} dB(z), \quad X_\sigma(y) = \langle \mathbf{c}, \xi_\sigma(y) \rangle.$$

Then it holds that

$$u(x) = 4\Big(\frac{d}{dx}\Big)^2 \log\bigg(\int_\Omega \exp\bigg(-\frac{1}{2}\int_{\min\{0,x\}}^{\max\{0,x\}} X_\sigma(y)^2 dy\bigg) dP\bigg), \quad x \in \mathbf{R}.$$

Proof. Let $\sigma = \sum_{j=1}^n c_j^2 \delta_{p_j}$ and $\tilde{\sigma} = \sum_{j=1}^n c_j^2 \delta_{-p_j}$. Since $D_{\tilde{\sigma}} = -D_\sigma$, it is easily seen that

$$\xi_\sigma(y) = \xi_{\tilde{\sigma}}(-y), \quad y \leq 0.$$

Hence $X_\sigma(y) = X_{\tilde{\sigma}}(-y)$, $y \leq 0$, and

$$\int_x^0 X_\sigma(y)^2 dy = \int_0^{-x} X_{\tilde{\sigma}}(y)^2 dy, \quad x \leq 0.$$

Since $P^{\tilde{\sigma}} = P \circ X_{\tilde{\sigma}}^{-1}$, in conjunction with Theorem 1(ii), this yields that

$$u(x) = \psi(P^{\tilde{\sigma}})(-x) = 4\Big(\frac{d}{dx}\Big)^2 \log\bigg(\int_\Omega \exp\bigg(-\frac{1}{2}\int_x^0 X_\sigma(y)^2 dy\bigg) dP\bigg)$$

for $x \leq 0$, which completes the proof. $\square$

Proof of Theorem 1. (i) Let $\mathbf{s} = \{\kappa_j, q_j\}_{1 \leq j \leq n} \in \mathcal{S}$. For $\lambda \in \mathbf{C}$ with $\Im\lambda \geq 0$, denote by $e^+(x; \lambda)$ and $e^-(x; -\lambda)$ the right and left Jost solutions of

$$\{-(d/dx)^2 + u_{\mathbf{s}}\}\phi = \lambda^2 \phi,$$

respectively, i.e. $e^+(x;\lambda)$ and $e^-(x;-\lambda)$ satisfy the above ordinary differential equation and $e^{\pm}(x;\pm\lambda) \sim e^{\pm\sqrt{-1}\lambda x}$ as $x \to \pm\infty$, where and in the sequel the symbol $\pm$ takes the same sign $+$ or $-$ simultaneously. It was shown in [5, 7] that there exist $\lambda_j \in C^\infty(\mathbf{R};\mathbf{R})$, $1 \le j \le n$, such that $\lambda_i(x) \ne \lambda_j(x)$ if $i \ne j$ for each $x \in \mathbf{R}$, and

$$e^{\pm}(x;\pm\lambda) = e^{\pm\sqrt{-1}\lambda x} \prod_{j=1}^{n} \frac{\lambda - (\pm\sqrt{-1}\lambda_j(x))}{\lambda + \sqrt{-1}\kappa_j}. \tag{6}$$

Define $k(\alpha)$, $1 \le \alpha \le n$, so that $|\lambda_{k(\alpha)}(0)| \le |\lambda_{k(\alpha+1)}(0)|$, $1 \le \alpha \le n-1$ and $\lambda_{k(\alpha)}(0) = -\lambda_{k(\alpha+1)}(0) > 0$ if $|\lambda_{k(\alpha)}(0)| = |\lambda_{k(\alpha+1)}(0)|$. Note that, in the latter condition, $\lambda_{k(\alpha)}(0)$ and $\lambda_{k(\alpha+1)}(0)$ have signs opposite to the ones in [7]. The following properties were seen in [7]; (A) $\lambda_j'(0) < 0$, $1 \le j \le n$, (B) for $1 \le \alpha \le n$, either of the following two cases occurs; (a) $\kappa_{\alpha-1} < |\lambda_{k(\alpha)}(0)| < \kappa_\alpha$, or (b) $\lambda_{k(\alpha)}(0) = -\lambda_{k(\alpha+1)}(0) = \kappa_\alpha$, where $\kappa_0 = 0$, (C) it holds that

$$\frac{1}{q_\alpha} = \frac{\kappa_\alpha^2 - \lambda_{k(\alpha)}(0)^2}{2\kappa_\alpha(\kappa_\alpha + \lambda_{k(\alpha)}(0))^2} \prod_{s\ne\alpha} \left(\frac{\kappa_\alpha - \lambda_{k(s)}(0)^2}{\kappa_\alpha^2 - \kappa_s^2}\right)\left(\frac{\kappa_\alpha - \kappa_s}{\kappa_\alpha + \lambda_{k(s)}(0)}\right)^2 \tag{7}$$

if $\kappa_\alpha \ne |\lambda_j(0)|$ for any $j = 1, \dots, n$, and

$$\frac{1}{q_\alpha} = \frac{\lambda'_{k(\alpha)}(0)}{2\kappa_\alpha \lambda'_{k(\alpha+1)}(0)} \frac{\kappa_{\alpha+1} - \kappa_\alpha}{\kappa_{\alpha+1} + \kappa_\alpha} \prod_{s\ne\alpha} \left(\frac{\kappa_\alpha - \lambda_{k(s)}(0)^2}{\kappa_\alpha^2 - \kappa_s^2}\right)\left(\frac{\kappa_\alpha - \kappa_s}{\kappa_\alpha + \lambda_{k(s)}(0)}\right)^2 \tag{8}$$

if $\kappa_\alpha = \lambda_{k(\alpha)}(0)$, and

$$\prod_{j=1}^{n}(z - \kappa_j^2) = \left\{\prod_{j=1}^{n}(z - \lambda_j(0)^2)\right\}\left\{1 - \sum_{j=1}^{n} \frac{-\lambda_j'(0)}{z - \lambda_j(0)^2}\right\}. \tag{9}$$

Let $u = u_{\mathbf{s}} \in \Xi_0$ with $\mathbf{s} = \{\kappa_j, q_j\}_{1\le j\le n} \in \mathcal{S}$. Define

$$p_\alpha(\mathbf{s}) = \lambda_{k(\alpha)}(0), \quad c_\alpha(\mathbf{s}) = \sqrt{-\lambda'_{k(\alpha)}(0)}, \quad \sigma(\mathbf{s}) = \sum_{j=1}^{n} c_j(\mathbf{s})^2 \delta_{p_j(\mathbf{s})}.$$

Set $\overline{\psi}(\sigma(\mathbf{s})) = \{\eta_j, m_j\}_{1\le j\le n}$. Since $p_j(\mathbf{s})$'s satisfy the condition (H), by (9), we see that $\eta_j = \kappa_j$, $1 \le j \le n$. Substituting these into (7) and (8), and then comparing with (3), we obtain that $m_j = q_j$, $1 \le j \le n$. Hence $\overline{\psi}(\sigma(\mathbf{s})) = \mathbf{s}$. Due to (5), $\psi(P^{\sigma(\mathbf{s})}) = u_{\mathbf{s}}$, which means that ψ is surjective.

Let $\sigma = \sum_{j=0}^{\infty} c_j^2 \delta_{p_j} \in \Sigma_0$, and assume that (H) is satisfied. Let $\mathbf{s} = \overline{\psi}(\sigma)$. It was shown in the proof of [7, Lemma 1.4] that $p_j(\mathbf{s}) = p_j$ and $c_j(\mathbf{s}) = c_j$, $1 \le j \le n$. Hence, if we define the mapping $\phi : \Xi_0 \to \mathcal{G}_0$ by $\phi(u_{\mathbf{s}}) = P^{\sigma(\mathbf{s})}$, then by (5), $\phi(\psi(P^\sigma)) = P^\sigma$. Thus ψ is injective.

(ii) Let $\sigma = \sum_{j=1}^{n} c_j^2 \delta_{p_j}$ and $u = \psi(P^\sigma)$. If we set $\overline{\psi}(\sigma) = \mathbf{s} = \{\kappa_j, q_j\}$, as was seen in the proof of (i), $u = u_{\mathbf{s}}$, $p_j(\mathbf{s}) = p_j$, and $c_j(\mathbf{s}) = c_j$, $j = 1, \dots, n$.

Put $\tilde{u}(x) = u(-x)$, $x \in \mathbf{R}$. Denote by $\tilde{e}^+(x;\lambda)$ and $\tilde{e}^-(x;-\lambda)$ the right and left Jost solutions associated with $\tilde{u}$, respectively. It is straightforward to see that

$\tilde{e}^+(x;\lambda) = e^-(-x;-\lambda)$ and $\tilde{e}^-(x;-\lambda) = e^+(-x;\lambda)$, $e^+(x;\lambda)$ and $e^-(x;-\lambda)$ being the right and left Jost solutions related with u, respectively. This implies that

$$W[\tilde{e}^+(*;\lambda), \tilde{e}^-(*;-\lambda)] = W[e^+(*;\lambda), e^-(*;-\lambda)],$$
$$W[\tilde{e}^-(*;-\xi), \tilde{e}^+(*;-\xi)] = W[e^-(*;\xi), e^+(*;\xi)]$$

for any $\lambda \in \mathbf{C}$ with $\Im\lambda \geq 0$ and $\xi \in \mathbf{R}$, where $W[f,g]$ denotes the Wronskian of f and g: $W[f,g] = f'g - fg'$. Hence, by virtue of the direct and inverse scattering theory (cf. [6]), $\tilde{u} \in \Xi_0$ and there exist $\tilde{q}_1, \ldots, \tilde{q}_n > 0$ so that, if we set $\tilde{\mathbf{s}} = \{\kappa_j, \tilde{q}_j\}$ then $\tilde{u} = u_{\tilde{\mathbf{s}}}$. Due to (6), we have that

$$\tilde{e}^{\pm}(x;\pm\lambda) = e^{\pm\sqrt{-1}\lambda x} \prod_{j=1}^{n} \frac{\lambda - (\pm\sqrt{-1}(-\lambda_j(-x)))}{\lambda + \sqrt{-1}\kappa_j}.$$

By the definition of $p_j(\mathbf{s})$ and $c_j(\mathbf{s})$, this implies that $p_j(\tilde{\mathbf{s}}) = -p_j(\mathbf{s}) = -p_j$ and $c_j(\tilde{\mathbf{s}}) = c_j(\mathbf{s}) = c_j$, $j = 1, \ldots, n$. In particular, $\sigma(\tilde{\mathbf{s}}) = \tilde{\sigma}$. Thus $\psi(\tilde{\sigma}) = \tilde{\mathbf{s}}$, and hence $\tilde{u} = \psi(P^{\tilde{\sigma}})$ on $[0,\infty)$, which completes the proof. □

3. The Brownian sheet

3.1. Wiener integral with respect to the Brownian sheet

Let $\{W(p,x)\}_{(p,x)\in\mathbf{R}_+^2}$ be the Brownian sheet on a probability space $(\Omega, \mathcal{F}, P)$, where $\mathbf{R}_+^2 = [0,\infty)^2$, i.e. $\{W(p,x)\}_{(p,x)\in\mathbf{R}_+^2}$ is a centered Gaussian system with covariance function $\int_\Omega W(p,x)W(q,y)dP = \min\{p,q\}\min\{x,y\}$. Denote by $L^2(\mathbf{R}_+^2)$ and $L^2(P)$ the spaces of square integrable functions with respect to the Lebesgue measure on $\mathbf{R}_+^2$ and P, respectively. There exists a linear isometry $\mathcal{I} : L^2(\mathbf{R}_+^2) \to L^2(P)$ such that

$$\mathcal{I}(\chi_{[a,b)\times[c,d)}) = W(b,d) - W(a,d) - W(b,c) + W(a,c),$$

for any $0 \leq a < b < \infty$ and $0 \leq c < d < \infty$, where χ_A is the indicator function of A. In the sequel, we shall write

$$\int_{\mathbf{R}_+^2} h(q,z)W(dq,dz)$$

for $\mathcal{I}(h)$, and call it the Wiener integral of h.

We shall see the dependence of the Wiener integrals on parameters. To do this, let $T > 0$ and take a family $\phi = \{\phi(\cdot,\cdot;t) \,|\, t \in [0,T]\} \subset L^2(\mathbf{R}_+^2)$ such that

$$K_\phi \equiv \sup_{0\leq s<t\leq T} \frac{1}{|t-s|} \int_{\mathbf{R}_+^2} |\phi(q,z;t) - \phi(q,z;s)|^2 dqdz < \infty, \tag{10}$$

and put $Z_\phi(y) = \int_{\mathbf{R}_+^2} \phi(q,z;y)W(dq,dz)$, $y \in [0,T]$. It then holds that

$$\int_\Omega |Z_\phi(t) - Z_\phi(s)|^{2m} dP \leq \frac{(2m)!}{2^m m!} K_\phi^m |t-s|^m \quad \text{for any } t,s \in [0,T], \tag{11}$$

because, for any $h \in L^2(\mathbf{R}^2_+)$, its Wiener integral is a centered Gaussian random variable with variance $\|h\|^2_{L^2(\mathbf{R}^2_+)}$ and hence

$$\int_\Omega \left(\int_{\mathbf{R}^2_+} h(q,z) W(dq,dz) \right)^{2m} dP = \frac{(2m)!}{2^m m!} \|h\|^{2m}_{L^2(\mathbf{R}^2_+)}, \quad m \in \mathbf{N}.$$

By Kolmogorov's continuity theorem, $\{Z_\phi(y)\}_{y\in[0,T]}$ admits a continuous version, say $\{Z_\phi(y)\}_{y\in[0,T]}$ again. We moreover have that

Theorem 2. *Let $T > 0$ and $m \in \mathbf{N}, \geq 2$. Then, there exists a constant $C_{m,T} > 0$ such that, for any family $\phi = \{\phi(\cdot,\cdot;t) \,|\, t \in [0,T]\} \subset L^2(\mathbf{R}^2_+)$ with $K_\phi < \infty$, where K_ϕ is defined by (10), the Wiener integral*

$$Z_\phi(y) = \int_{\mathbf{R}^2_+} \phi(q,z;y) W(dq,dz)$$

satisfies that

$$\int_\Omega \sup_{0\leq s<t\leq T} \frac{|Z_\phi(t) - Z_\phi(s)|^{2m}}{|t-s|^{m-(3/2)}} dP \leq C_{m,T} K_\phi^m. \tag{12}$$

Moreover, if $Z_\phi(0) = 0$ in addition, then it holds that

$$\int_\Omega \sup_{y\in[0,T]} |Z_\phi(y)|^{2m} dP \leq C_{m,T} K_\phi^m T^{m-(3/2)}.$$

Proof. To see the assertion, we apply the following inequality, which can be concluded easily from [10, Theorem 2.1.3]; for each $\alpha > 0$, $\beta > 2$, $T > 0$, and continuous function $f : [0,T] \to \mathbf{R}$, it holds that

$$\sup_{0\leq s<t\leq T} \frac{|f(t)-f(s)|^\alpha}{|t-s|^{\beta-2}} \leq 2^{3\alpha+2} \left(\frac{\beta}{\beta-2} \right)^\alpha \int_0^T \int_0^T \frac{|f(t)-f(s)|^\alpha}{|t-s|^\beta} dtds.$$

Plugging (11) into this estimation with $\alpha = 2m$ and $\beta = m + (1/2)$, we have that

$$\begin{aligned}\int_\Omega \sup_{0\leq s<t\leq T} & \frac{|Z_\phi(t) - Z_\phi(s)|^{2m}}{|t-s|^{m-(3/2)}} dP \\ &\leq 2^{6m+2} \left(\frac{2m+1}{2m-3} \right)^{2m} \frac{(2m)!}{2^m m!} K_\phi^m \int_0^T \int_0^T |t-s|^{-1/2} dtds.\end{aligned}$$

Thus we obtain (12). The last inequality is an immediate consequence of (12). $\square$

3.2. Representation with the Brownian sheet

We first reconstruct $P^\sigma \in \mathcal{G}_0$ by using the Brownian sheet. For this purpose, let $\mathcal{Q}$ be the set of all sequence $\alpha = \{(p_j, d_j)\}_{1\leq j\leq n}$ of points in $\mathbf{R}^2$ with $p_i \neq p_j$ if $i \neq j$, $n = 1, 2, \ldots$ Every $\sigma = \sum_{j=1}^n c_j^2 \delta_{p_j} \in \Sigma_0$ determines the element $\{(p_j, c_j)\}_{1\leq j\leq n} \in \mathcal{Q}$, denoted by σ again, if we order p_j's so that the condition (H) is fulfilled.

For $\alpha = \{(p_j, d_j)\}_{1\le j\le n} \in \mathcal{Q}$, $a \ge 0$ and $b \in \mathbf{R}$ with $-a \le b < p_1$, define $0 \le q_0 < q_1 < \cdots < q_n$ by

$$q_0 = b + a, \quad q_k = q_0 + \sum_{j=1}^{k} |p_j - p_{j-1}|, \quad k = 1, \ldots, n \quad (p_0 = b). \tag{13}$$

The $\mathbf{R}^n$-valued process

$$W_\alpha(y) = \left(\frac{W(q_j, y) - W(q_{j-1}, y)}{\sqrt{q_j - q_{j-1}}} \right)_{1\le j\le n}$$

is an n-dimensional Brownian motion, and then using this for $\{B(z)\}$ in (2), we define

$$\xi_{a,b,\alpha}(y) = e^{yD_\alpha} \int_0^y e^{-zD_\alpha} dW_\alpha(z) \quad \text{and} \quad X_{a,b,\alpha}(y) = \langle \mathbf{d}, \xi_{a,b,\alpha}(y) \rangle,$$

where D_α denotes the diagonal matrix with p_j's as diagonal elements and $\mathbf{d} = {}^t(d_1, \ldots, d_n)$. Then it is easily seen that

$$X_{a,b,\alpha}(y) = \int_{\mathbf{R}_+^2} h_{a,b,\alpha}(q, z; y) W(dq, dz), \tag{14}$$

where

$$h_{a,b,\alpha}(q, z; y) = \sum_{j=1}^{n} \frac{e^{(y-z)p_j} d_j}{\sqrt{q_j - q_{j-1}}} \chi_{[q_{j-1}, q_j)\times[0,y)}(q, z).$$

Moreover, if $\sigma \in \Sigma_0$, then, by virtue of the observation made in Sect. 2, it holds that

$$P^\sigma = P \circ X_{a,b,\sigma}^{-1}.$$

We next introduce another compound Ornstein-Uhlenbeck process. For $a \ge 0$ and a piecewise continuous function $g : [0, \infty) \to \mathbf{R}$ with compact support, we define $h_{a,g}(\cdot, \cdot; y) \in L^2(\mathbf{R}_+^2)$, $y \in [0, \infty)$, by

$$h_{a,g}(q, z; y) = e^{(y-z)(q-a)} g(q) \chi_{[0,y)}(z), \quad (q, z) \in \mathbf{R}_+^2,$$

and then put

$$X_{a,g}(y) = \int_{\mathbf{R}_+^2} h_{a,g}(q, z; y) W(dq, dz), \quad y \in [0, \infty).$$

We shall give some remarks on $X_{a,b,\alpha}$ and $X_{a,g}$. Firstly notice that $X_{a,b,\alpha}$ and $X_{a,g}$ are both continuous Gaussian processes starting at 0 at time 0. Namely, being Gaussian processes follows from their definition by Wiener integrals. The continuity is a consequence of the observation made before Theorem 2 and the next lemma.

Lemma 1. *Let g, α be as above and $T > 0$. Set*

$$K_{\alpha,T} = e^{2TM(\alpha)}\{1 + T^2 M(\alpha)^2\} S(\alpha),$$
$$K_{a,g,T} = \{1 + (T_0 + a)^2 T^2\} e^{2T(T_0+a)} \int_0^\infty g(q)^2 dq,$$

where $M(\alpha) = \sup_{1\le j\le n} |p_j|$, $S(\alpha) = \sum_{j=1}^n d_j^2$, and T_0 is chosen so that $g(q) = 0$ if $q \ge T_0$. Then it holds that

$$K_{h_{a,b,\alpha}} \le K_{\alpha,T} \quad \text{and} \quad K_{h_{a,g}} \le K_{a,g,T},$$

where $K_{h_{a,b,\alpha}}$ and $K_{h_{a,g}}$ are defined by (10) with $\phi = h_{a,b,\alpha}$ and $h_{a,g}$, respectively.

Proof. For any $0 \le s < t \le T$, it holds that

$$\begin{aligned}
|h_{a,b,\alpha}(q,z;t) - h_{a,b,\alpha}(q,z;s)| &\le \sum_{j=1}^n \frac{e^{TM(\alpha)}|d_j|}{\sqrt{q_j - q_{j-1}}} \chi_{[q_{j-1},q_j)\times[s,t)}(q,z) \\
&\quad + \sum_{j=1}^\infty \frac{M(\alpha)e^{TM(\alpha)}(t-s)|d_j|}{\sqrt{q_j - q_{j-1}}} \chi_{[q_{j-1},q_j)\times[0,s)}(q,z), \\
|h_{a,g}(q,z;t) - h_{a,g}(q,z;s)|& \\
\le e^{T(T_0+a)}|g(q)|\{\chi_{[s,t)}(z) + (t-s)(T_0+a)\chi_{[0,s)}(z)\}.&
\end{aligned}$$

These imply the desired conclusion. □

Secondly, observe that for $\sigma, \mu \in \Sigma_0$, if A and B are chosen so that $A + B$ is sufficiently large, then

$$P^{\sigma+\mu} = P \circ \{X_{a,b,\sigma} + X_{A,B,\mu}\}^{-1}. \tag{15}$$

Namely, note that $h_{A,B,\mu}(q,z;y) = 0$ if $q \le A + B$. Hence, if $A + B$ is so large that $q_n \le A + B$, where q_n is defined by (13) for σ, then $h_{a,b,\sigma}h_{A,B,\mu} = 0$, and which implies the independence of $X_{a,b,\sigma}$ and $X_{A,B,\mu}$. Then

$$\begin{aligned}
\int_\Omega \{X_{a,b,\sigma}(x) + X_{A,B,\mu}(x)\}\{X_{a,b,\sigma}(y) + X_{A,B,\mu}(y)\}dP& \\
= R_\sigma(x,y) + R_\mu(x,y) = R_{\sigma+\mu}(x,y).&
\end{aligned}$$

Thus $P^{\sigma+\mu}$ is realized as the law of $X_{a,b,\sigma} + X_{A,B,\mu}$.

Thirdly, if $\sigma \in \Sigma$ is of the form

$$\sigma(d\xi) = f(\xi)d\xi + \mu(d\xi),$$

where $f : \mathbf{R} \to [0,\infty)$ is a piecewise continuous function with compact support and $\mu \in \Sigma_0$, then, choosing $a > 0$ so that $\operatorname{supp} f \subset [-a,a]$, and setting $g(\xi) = \sqrt{f(\xi - a)}$, we have that

$$P^\sigma = P \circ \{X_{a,g} + X_{A,B,\mu}\}^{-1} \tag{16}$$

for A and B with sufficiently large $A+B$. In fact, it holds that

$$\sigma(d\xi) = g(\xi + a)^2 \chi_{[-a,\infty)}(\xi) d\xi + \mu(d\xi),$$

and we may and will think of g as a piecewise continuous function on $[0,\infty)$ with compact support. It is easily seen that the covariance function of $X_{a,g}$ is

$$\int_\Omega X_{a,g}(x) X_{a,g}(y) dP = \int_{\mathbf{R}} \frac{e^{\xi(x+y)} - e^{\xi|x-y|}}{2\xi} g(\xi + a)^2 \chi_{[-a,\infty)}(\xi) d\xi.$$

Take $\gamma > 0$ so that $\operatorname{supp} \mu \subset [-\gamma, \gamma]$. Since $\operatorname{supp} g \subset [0, 2a]$, for $A \geq 0$ and $B \leq 0$ such that $-A \leq B \leq -\gamma$ and $2a < A + B$, we have that $h_{a,g} h_{A,B,\mu} = 0$. Then $X_{a,g}$ and $X_{A,B,\mu}$ are independent, and hence the Gaussian process $X_{a,g} + X_{A,B,\mu}$ possesses the covariance function $R_\sigma(x,y)$. Thus P^σ coincides with the law of $X_{a,g} + X_{A,B,\mu}$.

Finally, Theorem 2 and Lemma 1 yields that

Proposition 1. *Let g, α, a, b be as above. Then, for any $T > 0$ and $m \in \mathbf{N}$, there exists a constant $C_{m,T}$, depending only on T and m, such that the following estimations hold with $(Z, K) = (X_{a,b,\alpha}, K_{\alpha,T})$ or $(Z,K) = (X_{a,g}, K_{a,g,T})$.*

$$\int_\Omega \sup_{0 \leq s < t \leq T} \frac{|Z(t) - Z(s)|^{2m}}{|t-s|^{m-(3/2)}} dP \leq C_{m,T} K^m,$$
$$\int_\Omega \sup_{y \in [0,T]} |Z(y)|^{2m} dP \leq C_{m,T} K^m T^{m-(3/2)}.$$

4. Generalized reflectionless potentials

In this section, we shall show that the convergence of $P^\sigma \in \mathcal{G}_0$ implies that of reflectionless potentials to generalized one in the topology of uniform convergence on compacts.

For $T > 0$, let $\mathcal{W}_T$ be the space of all continuous $w : [0,T] \to \mathbf{R}$ with $w(0) = 0$. Naturally $\mathcal{W}_T \subset \mathcal{W}$, and every probability measure P can be restricted to $\mathcal{W}_T$. The restriction will be denoted by $P|_{\mathcal{W}_T}$. For $\sigma \in \Sigma$, put

$$\Phi_\sigma(x) = \int_{\mathcal{W}} \exp\Big(-\frac{1}{2}\int_0^x X(y)^2 dy\Big) dP^\sigma.$$

As will be seen in the next theorem, Φ_σ is C^2, and then one can define

$$\psi(P^\sigma) = 4\Big(\frac{d}{dx}\Big)^2 \log \Phi_\sigma.$$

Our goal of this section is

Theorem 3. *(i) For $\sigma \in \Sigma$, Φ_σ is C^2.*
(ii) Let $\sigma_n \in \Sigma_0$ and $\sigma \in \Sigma$. Suppose that $\bigcup_{n \in \mathbf{N}} \operatorname{supp} \sigma_n \subset [-\beta, \beta]$ for some $\beta > 0$, and σ_n tends to σ vaguely. Then Φ_{σ_n} and its first and second derivatives

Φ'_{σ_n} *and* Φ''_{σ_n} *converge to* Φ_σ, Φ'_σ, *and* Φ''_σ *uniformly on every bounded interval in* $[0,\infty)$, *respectively. In particular,* $\psi(P^{\sigma_n})$ *tends to* $\psi(P^\sigma)$ *uniformly on every bounded interval in* $[0,\infty)$. *Moreover, for every* $\varepsilon > 0$, *there exists* $n_0 \in \mathbf{N}$ *such that*

$$\mathrm{Spec}(-(d/dx)^2 + \psi(P^{\sigma_n})) \subset [-\beta^2 - \sigma(\mathbf{R}) - \varepsilon, \infty), \quad n \geq n_0. \tag{17}$$

Finally, there exists $u \in \Xi$ *such that* $\psi(P^\sigma) = u$ *on* $[0,\infty)$.
(iii) Let $g_n : \mathbf{R} \to [0,\infty)$ *be piecewise continuous, and* $\mu \in \Sigma_0$. *Assume that*

$$\bigcup_{n\in\mathbf{N}} \mathrm{supp}\, g_n \subset [-\beta,\beta] \quad \textit{for some } \beta > 0, \quad \sup_{n\in\mathbf{N}} \int_{\mathbf{R}} g_n(\xi)^2 d\xi < \infty,$$

and $\sigma_n \in \Sigma$ *defined by* $\sigma_n(d\xi) = g_n(\xi)^2 d\xi + \mu(d\xi)$ *converges to some* $\sigma \in \Sigma$ *vaguely. Then* Φ_{σ_n}, Φ'_{σ_n}, *and* Φ''_{σ_n} *converge to* Φ_σ, Φ'_σ, *and* Φ''_σ *uniformly on every bounded interval in* $[0,\infty)$, *respectively. In particular,* $\psi(P^{\sigma_n})$ *tends to* $\psi(P^\sigma)$ *uniformly on every bounded interval in* $[0,\infty)$.
(iv) For every $u \in \Xi$, *there exists* $P^\sigma \in \mathcal{G}$ *such that* $\psi(P^\sigma) = u$ *on* $[0,\infty)$.

We shall give several remarks on the theorem before getting into the proof.

Remark 1. (a) Repeating the arguments in Lemmas 3, 4, and 5 below, one can show that Φ_σ is C^∞.
(b) Let $\sigma \in \Sigma$. Fix $\beta > 0$ so that $\mathrm{supp}\,\sigma \subset [-\beta,\beta]$, and define $\sigma_n \in \Sigma_0$ by $\sigma_n(d\xi) = \sum_{j=-n}^{n} \sigma([j\beta/n,(j+1)\beta/n))\delta_{j\beta/n}$. Then σ_n's satisfy the assumption in (ii).
(c) The identification of Ξ_0 and $\mathcal{G}_0$ extends to that of Ξ and $\mathcal{G}$ as follows. First let $P^\sigma \in \mathcal{G}$. Define $\sigma_n \in \Sigma_0$ as in (b). By Theorem 1, $\psi(P^{\sigma_n})(x) = \psi(P^{\tilde{\sigma}_n})(-x)$, $x \leq 0$. Define $\tilde{\sigma} \in \Sigma$ by $\tilde{\sigma}(A) = \sigma(-A)$, $A \in \mathcal{B}(\mathbf{R})$, where $-A = \{-x | x \in A\}$. Since $\mathrm{supp}\,\tilde{\sigma}_n \subset [-\beta,\beta]$ and $\tilde{\sigma}_n$ tends to $\tilde{\sigma}$ vaguely, by (ii), we see that $\psi(P^{\tilde{\sigma}_n})$ converges to $\psi(P^{\tilde{\sigma}})$ uniformly on compacts in $[0,\infty)$. As will be seen in the proof of Lemma 4 below, there exist $u \in \Xi$ and a subsequence $\{\sigma_{n_j}\}$ of $\{\sigma_n\}$ such that $\psi(P^{\sigma_{n_j}})$ converges to $u \in \Xi$ uniformly on compacts in $\mathbf{R}$. Hence we have that $u = \psi(P^\sigma)$ on $[0,\infty)$ and $= \psi(P^{\tilde{\sigma}})$ on $(-\infty,0]$.

Conversely, let $u \in \Xi$. As will be seen in the proof of (iv) (Lemma 8 below), there exist $P^{\sigma_n} \in \mathcal{G}_0$, $n \in \mathbf{N}$, such that $\psi(P^{\sigma_n})$ converges to u uniformly on compacts in $\mathbf{R}$, $\bigcup_{n\in\mathbf{N}} \mathrm{supp}\,\sigma_n \subset [-\beta,\beta]$ for some $\beta > 0$, and σ_n tends to some $\sigma \in \Sigma$ vaguely. Then, in repetition of the above argument, we see that u coincides with $\psi(P^\sigma)$ on $[0,\infty)$ and $\psi(P^{\tilde{\sigma}})$ on $(-\infty,0]$.
(d) A correspondence between Ξ and Σ was studied by Marchenko [7] and Kotani [4] in an analytical manner. The relation between Ξ and $\mathcal{G}$ investigated above is a probabilistic counterpart to their observation.
(e) Every $\psi(P^\sigma)$, $P^\sigma \in \mathcal{G}$, can be approximated by $\psi(P^{\sigma_n})$'s with σ_n of the form as described in (iii). Namely, let $P^\sigma \in \mathcal{G}$. Take a nonnegative C^∞ function $\phi : \mathbf{R} \to \mathbf{R}$ with compact support such that $\int_{\mathbf{R}} \phi(x)dx = 1$. Define $g_n : \mathbf{R} \to [0,\infty)$ by $g_n(x)^2 = \int_{\mathbf{R}} n\phi(n(x-\xi))\sigma(d\xi)$, and set $\sigma_n(d\xi) = g_n(\xi)^2 d\xi$. Then $\bigcup_{n\in\mathbf{N}} g_n \subset [-\beta,\beta]$ for some $\beta > 0$, $\int_{\mathbf{R}} g_n(\xi)^2 d\xi = \sigma(\mathbf{R})$, and σ_n converges to σ vaguely.

(f) The convergence discussed in (iii) relates to the convergence of finite-zone potentials to reflectionless ones discussed in [1, 9]. Namely, for $u \in \Xi$ of finite-zone, the σ appearing in (iv) was computed by Kotani [4] to be represented as $\sigma(d\xi) = g(\xi)^2 d\xi + \mu(d\xi)$ for some piecewise continuous g with compact support and $\mu \in \Sigma_0$. As finite-zone potentials tends to a reflectionless potential, the support of g shrinks to a discrete point set ([9]). This is the situation investigated in (iii).

We now proceed to the proof of Theorem 3. It is broken into several steps, each step being a lemma. In the sequel, let $\{W(p,x)\}_{(p,x)\in\mathbf{R}_+^2}$ be the Brownian sheet on $(\Omega, \mathcal{F}, P)$ as in Sect. 3.

Lemma 2. *Let $T > 0$ and $\{\{Z_\beta(y)\}_{y\in[0,T]} \mid \beta \in \Lambda\}$ be a family of continuous processes Z_β defined on $(\Omega, \mathcal{F}, P)$ with $Z_\beta(0) = 0$. Suppose that*

$$A_m = \sup_{\beta\in\Lambda} \int_\Omega \sup_{0\le s<t\le T} \frac{|Z_\beta(t) - Z_\beta(s)|^{2m}}{|t-s|^{m-(3/2)}} dP < \infty, \quad m = 2, 3, \ldots$$

Then the family $\{P \circ Z_\beta^{-1}\}_{\beta\in\Lambda}$ of the laws of Z_β's on $\mathcal{W}_T$ is tight.

Let $Q : \mathbf{R}^2 \to \mathbf{R}$ be a polynomial, and put

$$\Phi_{\beta,\gamma}(x) = \int_\Omega Q(Z_\beta(x), Z_\gamma(x)) \exp\Big(-\frac{1}{2}\int_0^x Z_\beta(y)^2 dy\Big) dP, \quad x \in [0,T].$$

Then $\Phi_{\beta,\gamma}$, $\beta, \gamma \in \Lambda$, are equi-continuous and uniformly bounded on $[0,T]$.

Finally, if $Q \equiv 1$, then $\Phi_{\beta,\gamma}$'s are all C^1, and $\Phi'_{\beta,\gamma}$'s are also equi-continuous and uniformly bounded on $[0,T]$.

Proof. The finiteness of A_m implies the tightness. It also yields that

$$B_m = \sup_{\beta\in\Lambda} \int_\Omega \sup_{t\in[0,T]} |Z_\beta(t)|^{2m} dP < \infty, \quad m = 2, 3, \ldots$$

Then, as an application of the dominated convergence theorem and the second assertion, we obtain the third assertion.

To see the second assertion, let k be the degree of Q and take $C_0 < \infty$ such that

$$|Q(a,b) - Q(c,d)| \le C_0(1 + |a| + |b| + |c| + |d|)^{k-1}(|a-c| + |b-d|),$$
$$|Q(a,b)| \le C_0(1 + |a| + |b|)^k, \quad a, b, c, d \in \mathbf{R}.$$

Since $|e^{-\xi} - e^{-\eta}| \le |\xi - \eta|$ for $\xi, \eta \ge 0$, we have that

$$\begin{aligned}
&|\Phi_{\beta,\gamma}(x) - \Phi_{\beta,\gamma}(x')| \\
&\le C_0\Big(\int_\Omega \Big\{1 + 2\sup_{y\in[0,T]}|Z_\beta(y)| + 2\sup_{y\in[0,T]}|Z_\gamma(y)|\Big\}^{4(k-1)/3} dP\Big)^{3/4} \\
&\qquad \times\Big(\int_\omega \{|Z_\beta(x) - Z_\beta(x')| + |Z_\gamma(x) - Z_\gamma(x')|\}^4 dP\Big)^{1/4} \\
&\quad + \frac{C_0}{2}|x - x'| \int_\Omega \Big\{1 + \sup_{y\in[0,T]}|Z_\beta(y)| + \sup_{y\in[0,T]}|Z_\gamma(y)|\Big\}^{k+2} dP.
\end{aligned}$$

Hence there exists a constant $C < \infty$, depending only on A_m's and B_m's, such that

$$\sup_{\beta,\gamma\in\Lambda} |\Phi_{\beta,\gamma}(x) - \Phi_{\beta,\gamma}(x')| \le C\{|x-x'|^{1/8} + |x-x'|\}, \quad x, x' \in [0,T].$$

Thus $\Phi_{\beta,\gamma}$'s are equi-continuous on $[0,T]$. Since $\Phi_{\beta,\gamma}(0) = Q(0,0)$, $\Phi_{\beta,\gamma}$'s are then uniformly bounded on $[0,T]$. □

Lemma 3. *Let $\sigma = \sum_{j=1}^n c_j^2 \delta_{p_j} \in \Sigma_0$ and $-a \le b < p_1$. Φ_σ is C^∞ and its first and second derivatives are represented as*

$$\Phi'_\sigma(x) = -\frac{1}{2}\int_\Omega X_{a,b,\sigma}(x)^2 \exp\left(-\frac{1}{2}\int_0^x X_{a,b,\sigma}(y)^2 dy\right) dP,$$

$$\Phi''_\sigma(x) = -\frac{1}{4}\int_\Omega \left\{2\sigma(\mathbf{R}) + 4X_{a,b,\sigma}(x)X_{a,b,\alpha(\sigma)}(x) - X_{a,b,\sigma}(x)^4\right\} \times \exp\left(-\frac{1}{2}\int_0^x X_{a,b,\sigma}(y)^2 dy\right) dP,$$

where $\alpha(\sigma) = \{(p_j, p_j c_j)\}_{1\le j\le n}$.

Proof. By (4), Φ_σ is C^∞. Since $P^\sigma = P \circ X_{a,b,\sigma}^{-1}$,

$$\Phi_\sigma(x) = \int_\Omega \exp\left(-\frac{1}{2}\int_0^x X_{a,b,\sigma}(y)^2 dy\right) dP.$$

Moreover, by Proposition 1, we have that

$$\int_\Omega \sup_{y\in[0,T]} |X_{a,b,\sigma}(y)|^{2m} dP < \infty, \quad T > 0, m \ge 2.$$

By an application of the dominated convergence theorem, the desired expression of the first derivative is obtained.

Rewrite $\xi_{a,b,\sigma}$ used to define $X_{a,b,\sigma}$ as

$$\xi_{a,b,\sigma}(y) = W_\alpha(y) + \int_0^y D_\sigma \xi_{a,b,\sigma}(z) dz.$$

Then, as an application of Itô's formula, we have that

$$\begin{aligned} X_{a,b,\sigma}(x)^2 &= 2\int_0^x X_{a,b,\sigma}(z)\langle \mathbf{c}, dW_\alpha(z)\rangle \\ &\quad + \int_0^x \left\{\sum_{j=1}^n c_j^2 + 2X_{a,b,\sigma}(z)X_{a,b,\alpha(\sigma)}(z)\right\} dz, \end{aligned}$$

and hence that

$$\Phi'_\sigma(x) = -\frac{1}{4}\int_\Omega\int_0^x \left\{2\sum_{j=1}^n c_j^2 + 4X_{a,b,\sigma}(z)X_{a,b,\alpha(\sigma)}(z) - X_{a,b,\sigma}(z)^4\right\} \times \exp\left(-\frac{1}{2}\int_0^z X_{a,b,\sigma}(y)^2 dy\right) dz dP.$$

This implies that $\Phi'_{a,b,\sigma}$ is continuously differentiable and the second derivative of $\Phi_{a,b,\sigma}$ has the desired representation, because $\sigma(\mathbf{R}) = \sum_{j=1}^n c_j^2$. □

Lemma 4. *Let $\sigma_n \in \Sigma_0$ and $\sigma \in \Sigma$. Suppose that $\bigcup_{n\in\mathbf{N}} \operatorname{supp}\sigma_n \subset [-\beta, \beta]$ for some $\beta > 0$ and that σ_n tends to σ vaguely. Then Φ_σ is C^2, and Φ_{σ_n}, Φ'_{σ_n}, and Φ''_{σ_n} converge to Φ_σ, Φ'_σ, and Φ''_σ uniformly on every bounded interval in $[0,\infty)$, respectively. Moreover, the assertion (ii) in Theorem 3 holds.*

Proof. Let $a \geq 0$ and $b \in \mathbf{R}$ satisfy that $-a \leq b < -\beta$. Due to the assumption, it holds that

$$\sup_{n\in\mathbf{N}} M(\sigma_n) \leq \beta \quad \text{and} \quad \sup_{n\in\mathbf{N}} S(\sigma_n) < \infty. \tag{18}$$

By Proposition 1 we see that

$$\sup_{n\in\mathbf{N}} \int_\Omega \sup_{0\leq s<t\leq T} \frac{|X_{a,b,\sigma_n}(t) - X_{a,b,\sigma_n}(s)|^{2m}}{|t-s|^{m-(3/2)}} dP < \infty, \quad T > 0, m \geq 2. \tag{19}$$

Thus $\{P^{\sigma_n}|_{\mathcal{W}_T}\}_{n\in\mathbf{N}}$ is tight for any $T > 0$.

Since σ_n tends to σ vaguely and $\operatorname{supp}\sigma_n \subset [-\beta, \beta]$, $n \in \mathbf{N}$, we obtain the convergence of $R_{\sigma_n}(x,y)$ to $R_\sigma(x,y)$ for every $x, y \geq 0$. Hence every finite dimensional distribution of P^{σ_n} tends to that of P^σ. In conjunction with the tightness, this implies that $P^{\sigma_n}|_{\mathcal{W}_T}$ converges to $P^\sigma|_{\mathcal{W}_T}$ weakly for any $T > 0$. In particular, $\Phi_{\sigma_n} \to \Phi_\sigma$ point wise.

Since $M(\alpha(\sigma_n)) = M(\sigma_n)$ and $S(\alpha(\sigma_n)) \leq \beta^2 S(\sigma_n)$, by (18) and Proposition 1, we have that

$$\sup_{n\in\mathbf{N}} \int_\Omega \sup_{0\leq s<t\leq T} \frac{|X_{a,b,\alpha(\sigma_n)}(t) - X_{a,b,\alpha(\sigma_n)}(s)|^{2m}}{|t-s|^{m-(3/2)}} dP < \infty, \ T > 0, \ m \geq 2. \tag{20}$$

Then the equi-continuity and the uniform boundedness of Φ_{σ_n}, Φ'_{σ_n}, and Φ''_{σ_n} on any bounded interval in $[0,\infty)$ follow from (19), (20), Lemmas 2 and 3, and the fact that $\sigma_n(\mathbf{R}) \to \sigma(\mathbf{R})$ as $n \to \infty$. In conjunction with the point wise convergence of Φ_{σ_n} to Φ_σ, we see that Φ_σ and that Φ_{σ_n}, Φ'_{σ_n}, and Φ''_{σ_n} tend to Φ_σ, Φ'_σ, and Φ''_σ uniformly on any bounded interval in $[0,\infty)$, respectively. In particular, the first assertion of (ii) holds.

We shall show the second assertions of (ii). By [7, Lemma 1.4], it holds that

$$\operatorname{Spec}(-(d/dx)^2 + \psi(P^{\sigma_n})) \subset [-\lambda, \infty)$$

for some $\lambda > 0$ with $M(\sigma_n)^2 < \lambda \leq M(\sigma_n)^2 + \sigma_n(\mathbf{R})$. Since $M(\sigma_n)^2 \leq \beta^2$ and $\sigma_n(\mathbf{R}) \to \sigma(\mathbf{R})$ as $n \to \infty$, we obtain the second assertion of (ii).

To see the last assertion of (ii), let $u_n = \psi(P^{\sigma_n}) \in \Xi_0$. By (17), $\{u_n\}_{n\in\mathbf{N}}$ is precompact in the topology of uniform convergence on bounded intervals in $\mathbf{R}$ ([7, Lemma 2.3]). Hence, by (17), there exists $u \in \Xi$ and a subsequence $\{u_{n_j}\}_{j\in\mathbf{N}}$ such that u_{n_j} converges to $u \in \Xi$ uniformly on any compact interval in $\mathbf{R}$. Combined with the convergence of $\psi(P^{\sigma_n})$ to $\psi(P^\sigma)$ on $[0,\infty)$, we see that $\psi(P^\sigma) = u$ on $[0,\infty)$. $\square$

Lemma 5. *The assertions (i) in Theorem 3 holds.*

Proof. Let $\sigma \in \Sigma$, and define $\sigma_n \in \Sigma_0$ by

$$\sigma_n(d\xi) = \sum_{j=-n}^{n} \sigma([j\beta/n, (j+1)\beta/n))\delta_{j\beta/n},$$

where $\beta > 0$ is chosen so that $\operatorname{supp}\sigma \subset [-\beta, \beta]$. Then $\bigcup_{n\in\mathbf{N}} \operatorname{supp}\sigma_n \subset [-\beta, \beta]$ and $\sigma_n \to \sigma$ vaguely. By Lemma 4, Φ_σ is C^2. □

Lemma 6. *Let $g : [0, \infty) \to [0, \infty)$ be piecewise continuous and $\mu \in \Sigma_0$. Assume that* $\operatorname{supp} g, \operatorname{supp}\mu \subset [-\beta, \beta]$ *for some $\beta > 0$. Define $\sigma \in \Sigma$ by*

$$\sigma(d\xi) = g(\xi)^2 d\xi + \mu(d\xi).$$

For $a > \beta$, $A > 0$, and $B < 0$ with $-A \le B \le -\beta$ and $a + \beta \le A + B$, define $X_\sigma = X_{a,g_a} + X_{A,B,\mu}$ and $\widetilde{X}_\sigma = X_{a,\tilde{g}_a} + X_{A,B,\alpha(\mu)}$, where $g_a(x) = g(x - a)$ and $\tilde{g}_a(x) = (x - a)g(x - a)$. Then it holds that

$$\Phi''_\sigma(x) = -\frac{1}{4}\int_\Omega \{2\sigma(\mathbf{R}) + 4X_\sigma(x)\widetilde{X}_\sigma(x) - X^4_\sigma(x)\} \exp\left(-\frac{1}{2}\int_0^x X_\sigma(y)^2 dy\right) dP.$$

Proof. Define $\sigma_n \in \Sigma_0$ by

$$\sigma_n(d\xi) = \sum_{j=1}^{n} g_a(j(a+\beta)/n)^2 \frac{a+\beta}{n} \delta_{(j(a+\beta)/n)-a}.$$

Then, $\operatorname{supp}(\sigma_n + \mu) \subset [-a, \beta] \cup \operatorname{supp}\mu$, $(\sigma_n + \mu)(\mathbf{R}) \le (a + \beta)\sup |g|^2 + \mu(\mathbf{R})$, and $\sigma_n + \mu$ tends to σ vaguely. By Lemma 4, $\Phi''_{\sigma_n+\mu}$ converges to Φ''_σ uniformly on any bounded interval in $[0, \infty)$.

Due to (15), we have that $P^{\sigma_n+\mu} = P \circ \{X_{a,-a,\sigma_n} + X_{A,B,\mu}\}^{-1}$. Moreover, in repetition of the argument used in the proof of Lemma 3, we see that

$$\begin{aligned} \Phi''_{\sigma_n+\mu}(x) \;=\; & -\frac{1}{4}\int_\Omega \Big\{ 2\{\sigma_n(\mathbf{R}) + \mu(\mathbf{R})\} \\ & +4\{X_{a,-a,\sigma_n}(x) + X_{A,B,\mu}(x)\}\{X_{a,-a,\alpha(\sigma_n)}(x) + X_{A,B,\alpha(\mu)}(x)\} \\ & -\{X_{a,-a,\sigma_n}(x) + X_{A,B,\mu}(x)\}^4 \Big\} \\ & \times \exp\left(-\frac{1}{2}\int_0^x \{X_{a,-a,\sigma_n}(y) + X_{A,B,\mu}(y)\}^2 dy\right) dP. \end{aligned} \tag{21}$$

Since $h_{a,-a,\sigma_n}(*; y)$ and $h_{a,-a,\alpha(\sigma_n)}(*; y)$ tend to h_{a,g_a} and $h_{a,\tilde{g}_a}$ in $L^2(\mathbf{R}^2_+)$ for every $y \in [0, \infty)$, respectively, $X_{a,-a,\sigma_n}(y)$ and $X_{a,-a,\alpha(\sigma_n)}(y)$ converge to X_{a,g_a} and $X_{a,\tilde{g}_a}$ in $L^2(P)$ for every $y \in [0, \infty)$, respectively. Moreover, by Proposition 1, we have that

$$\sup_{n\in\mathbf{N}} \int_\Omega \sup_{y\in[0,T]} \big\{|X_{a,-a,\sigma_n}(y)|^{2m} + |X_{a,-a,\alpha(\sigma_n)}(y)|^{2m}\big\} dP < \infty$$

for any $T > 0$ and $m \in \mathbf{N}$. Then, letting $n \to \infty$ in (21), we obtain the desired representation of Φ''_σ. □

Lemma 7. *The assertion (iii) of Theorem 3 holds.*

Proof. Take $a > \beta$ and $A > 0$, $B < 0$ so that $-A \le B < \inf(\operatorname{supp}\mu)$, $a + \beta < A + B$, and define $X_{\sigma_n} = X_{a,g_{n,a}} + X_{A,B,\mu}$ and $\widetilde{X}_{\sigma_n} = X_{a,\tilde{g}_{n,a}} + X_{A,B,\alpha(\mu)}$, where $g_{n,a}(\xi) = g_n(\xi - a)$ and $\tilde{g}_{n,a}(\xi) = (\xi - a)g_n(\xi - a)$. By (16), $P^{\sigma_n} = P \circ X_{\sigma_n}^{-1}$, and

$$\Phi_{\sigma_n}(x) = \int_\Omega \exp\Big(-\frac{1}{2}\int_0^x X_{\sigma_n}(y)^2 dy\Big) dP.$$

Since $\sup_{n\in\mathbf{N}} \int_{\mathbf{R}} \{(g_n)_a(\xi)\}^2 d\xi < \infty$ and $\sup_{n\in\mathbf{N}} \int_{\mathbf{R}} \{(\tilde{g}_n)_a(\xi)\}^2 d\xi < \infty$, it follows from Proposition 1 that

$$\sup_{n\in\mathbf{N}} \int_\Omega \sup_{0\le s<t\le T} \frac{|X_{\sigma_n}(t) - X_{\sigma_n}(s)|^{2m} + |\widetilde{X}_{\sigma_n}(t) - \widetilde{X}_{\sigma_n}(s)|^{2m}}{|t-s|^{m-(3/2)}} dP < \infty$$

for any $T > 0$ and $m \ge 2$. We then obtain the desired convergence in repetition of the proof of Lemma 4, only this time with X_{σ_n}, $\widetilde{X}_{\sigma_n}$, and Lemma 6 for X_{a,b,σ_n}, $X_{a,b,\alpha(\sigma_n)}$, and Lemma 3. We omit the details. $\square$

Lemma 8. *The assertion (iv) of Theorem 3 holds.*

Proof. Let $u \in \Xi$ and suppose that $\{u_n\}_{n\in\mathbf{N}} \subset \Xi_0$ satisfies that u_n converges to u uniformly on any bounded interval in $\mathbf{R}$ and $\bigcup_{n\in\mathbf{N}} \operatorname{Spec}(-(d/dx)^2 + u_n) \subset [-\lambda, \infty)$ for some $\lambda > 0$. By Theorem 1, for every $n \in \mathbf{N}$, there exists $P^{\sigma_n} \in \mathcal{G}_0$ such that $u_n = \psi(P^{\sigma_n})$. These σ_n's are in Σ_0, and it was seen in [7, Lemma 1.4 and Corollary after Lemma 2.1] that $\operatorname{supp}\sigma_n \subset [-\sqrt{\lambda}, \sqrt{\lambda}]$ and $\sigma_n(\mathbf{R}) \le \lambda$ for any $n \in \mathbf{N}$. [1] Then, choosing a subsequence if necessary, we may assume that σ_n converges to some $\sigma \in \Sigma$ vaguely. By Theorem 3 (ii), we see that $\psi(P^{\sigma_n})$ converges to $\psi(P^\sigma)$ uniformly on any bounded interval in $[0, \infty)$. Thus $u = \psi(P^\sigma)$ on $[0, \infty)$. $\square$

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