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Numerical Verification Methods of Solutions for the Free Boundary Problem

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Abstract

We propose herein a method by which to enclose the solution of an ordinary free boundary problem. The problem is reformulated as a nonlinear boundary value problem on a fixed interval including an unknown parameter, which corresponds to the free boundary point. By appropriately setting the function spaces that are dependent on the finite element approximations, the solution is represented as a fixed point on a compact map. Then, by using the finite element projection with constructive error estimates, a Newton-type verification procedure is derived via computer. In addition, numerical examples confirming the effectiveness of the present method are given.

1 Introduction

In the present paper, we consider a numerical method by which to verify the existence of solutions $(u(x), c)$ for the following free boundary problem:

$$\begin{array}{rcll} -u''(x) & = & \mathcal{F}(x, u(x), u'(x)) & \text{if} \quad x \in (0, c), \\ u(x) & > & 0 & \text{if} \quad x \in [0, c), \\ u(x) & = & 0 & \text{if} \quad x \in [c, \infty), \\ & & u(0) = u_0, \quad u'(c) = 0 & \end{array} \tag{1.1}$$

where $\mathcal{F}$ is a smooth function in three variables and $u_0 > 0$.

The problem (1.1) arises, for example, in the determining the shape of a rope fixed at $(0, u_0)$ on the $y - axis$ and touching the $x - axis$ for the first time at $(c, 0)$ [1][8]. Concerning this problem, Schäfer proposed an enclosure method that is formulated as linear complementarity problems based on a kind of the finite difference method [6]. However, this method requires a rather impracticable assumptions with respect to the existence and uniqueness of the solution to (1.1). In a present paper, we formulated the problem (1.1) as a fixed point equation on a certain function space together with the unknown parameter c, and applied our own numerical verification method of solutions for nonlinear boundary value problems. Our proposed method is based on the Newton-type formulation and the Banach fixed point theorem in [9]. Notice that no existential assumptions are required for solutions to the original problem. Rather, the existential assumptions follow from the verification results.

First, by setting

$$v(x) \equiv u(cx) - (1 - x)u_0, \qquad x \in [0, \infty),$$

problem (1.1) is reduced to the following problem of finding the function $v(x)$ and the real number c in $(0,1)$.

$$\begin{aligned} -v''(x) &= f(v,c) && \text{if} \quad x \in (0,1), \\ v(x) &> -(1-x)u_0 && \text{if} \quad x \in [0,1), \\ v(x) &= -(1-x)u_0 && \text{if} \quad x \in (1,\infty), \\ v(0) = v(1) &= 0, \quad v'(1) = u_0, \end{aligned} \tag{1.2}$$

where $f(v,c) \equiv c^2 \cdot \mathcal{F}(cx, v(x) + (1-x)u_0, (1/c)(v'(x) - u_0))$.

We denote the Sobolev spaces on $(0,1)$ by $H_0^1(0,1), W_\infty^2(0,1), W_{\infty,0}^1(0,1)$ and so on. In particular, we define the norm of a function in the first-order homogeneous L^p-Sobolev space by the L^p-norm of the first derivative, e.g., $||v||_{H_0^1} \equiv ||\nabla v||_{L^2}$, which is equivalent to the original norm in the same space. For arbitrary $g \in L^\infty(0,1)$, we refer to the (unique) solution of the following equation as $\phi = -\Delta^{-1} g \in W_{\infty,0}^1(0,1) \cap W_\infty^2(0,1)$

$$\begin{aligned} -\phi'' &= g \quad \text{in} \quad (0,1), \\ \phi(0) &= \phi(1) = 0. \end{aligned}$$

In the following section, we describe the formulation of verification conditions with residual form by introducing appropriate function spaces. In Section 3, the actual procedures to implement the verification algorithm will be discussed in detail. In addition, numerical examples confirming the effectiveness of the proposed method are presented in Section 4.

2 Formulation of the verification condition

In this section, we present the verification condition for the solution of (1.2) using the Newton-like operator with residual form. Rather than proving existence of a solution itself for (1.2), we enclose the difference between a solution (v,c) and an approximation $(\hat{v}_h, \hat{c}_h)$ as shown below(*cf.*[9]). In what follows, we assume that $f(\cdot,\cdot)$ in (1.2) is a bounded and continuous map from $W_{\infty,0}^1(0,1) \times \mathbf{R}$ into $L^2(0,1)$.

Let $(\hat{v}_h, \hat{c}_h)$ be an approximate solution of (1.2) which satisfies

$$\begin{aligned} (\hat{v}_h', \varphi_h') &= (f(\hat{v}_h, \hat{c}_h), \varphi_h), \quad \forall \varphi_h \in S_h, \\ \hat{v}_h'(1) &= u_0, \end{aligned}$$

where $(\cdot,\cdot)$ is the L^2 inner product on $(0,1)$ and S_h is the finite element subspace with mesh $0 = x_0 < \cdots < x_{N+1} = 1$. Let $h_i = x_{i+1} - x_i$, $(0 \le i \le N)$ and let $h = \max_{0 \le i \le N} h_i$.
We now take $\bar{v} \in W_{\infty,0}^1(0,1) \cap W_\infty^2(0,1)$ as a solution of the following Poisson equation:

$$\begin{aligned} -\bar{v}'' &= f(\hat{v}_h, \hat{c}_h) \quad \text{in} \quad (0,1), \\ \bar{v}(0) &= \bar{v}(1) = 0. \end{aligned} \tag{2.1}$$

The equation is then decomposed as $v - \hat{v}_h = (v - \bar{v}) + (\bar{v} - \hat{v}_h)$, and we define

$$w := v - \bar{v}, \quad v_0 := \bar{v} - \hat{v}_h, \quad d := c - \hat{c}_h.$$

Note that $P_h \bar{v} = \hat{v}_h$, where P_h is the standard H_0^1-projection defined in the next section.

Thus, if the existence of the approximate solution $(\hat{v}_h, \hat{c}_h)$ is numerically proven as a finite-dimensional problem, then our verification problem for the solutions of (1.2) can be written as the following residual equation:

$$\begin{aligned} -w'' &= f(\hat{v}_h + w + v_0, \hat{c}_h + d) - f(\hat{v}_h, \hat{c}_h) \quad \text{in} \quad (0,1), \\ w'(1) &= u_0 - \hat{v}_h'(1) - v_0'(1), \\ w(0) &= w(1) = 0. \end{aligned} \tag{2.2}$$

Moreover, the equation (2.2) is equivalent to the following Newton-like equation:

$$\begin{aligned} -w'' - f'(\hat{v}_h, \hat{c}_h)\cdot(w,d) &= g_1(w,d) \quad \text{in} \quad (0,1), \\ w'(1) &= g_2, \\ w(0) = w(1) &= 0, \end{aligned} \tag{2.3}$$

where $f'(\hat{v}_h, \hat{c}_h)$ denotes the Fréchet derivative of $f(v,c)$ at $(\hat{v}_h, \hat{c}_h)$ and

$$\begin{aligned} g_1(w,d) &\equiv f(w + v_0 + \hat{v}_h, d + \hat{c}_h) - f(\hat{v}_h, \hat{c}_h) - f'(\hat{v}_h, \hat{c}_h)\cdot(w,d), \\ g_2 &\equiv u_0 - \hat{v}_h'(1) - v_0'(1). \end{aligned}$$

Let

$$W^{\mathbf{R}} \equiv W \times \mathbf{R}, \qquad W \equiv W^1_{\infty,0}(0,1) \cap \left(\bigwedge_{i=0}^{N} C^1[x_i, x_{i+1}] \right).$$

Then, for $z = (v,c) \in W^{\mathbf{R}}$, we define the norm

$$|||z||| \equiv \max\left(\max_{0\le i\le N} \|v(x)\|_{C^1[x_i,x_{i+1}]}, |c| \right), \tag{2.4}$$

where $\|\cdot\|_{C^1[x_i,x_{i+1}]}$ is the usual C^1-norm on $[x_i, x_{i+1}]$. Note that W is a closed subspace of $W^1_{\infty,0}(0,1)$ and the derivative of a function in W has the usual boundary value at $x = 1$.

Remark 1 *$W^{\mathbf{R}}$ is a Banach space with the norm $|||\cdot|||$.*

We also introduce the following norms:

$$\begin{aligned} |||z|||_{W^1_{\infty,0}} &\equiv \max\left(\|v\|_{W^1_{\infty,0}}, |c| \right), \\ |||z|||_{L^\infty} &\equiv \max\left(\|v\|_{L^\infty}, |c| \right). \end{aligned} \tag{2.5}$$

We then have the following:

Lemma 1 *The norm $|||\cdot|||_{W^1_{\infty,0}}$ defined in (2.5) coincides with $|||\cdot|||$ in (2.4).*

Proof : For arbitrary $v \in W$, it follows that $|||\cdot|||_{W^1_{\infty,0}} \le |||\cdot|||$ because

$$\max_{0\le i\le N} \|v(x)\|_{C^1[x_i,x_{i+1}]} = \max\left(\|v\|_{W^1_{\infty,0}}, \|v\|_{L^\infty} \right).$$

On the other hand, from $v(0) = v(1) = 0$ we can write $v(x) = \int_0^x v'(t)\,dt$. Hence, we obtain

$$|v(x)| = \left| \int_0^x v'(t)\,dt \right| \le x\|v\|_{W^1_{\infty,0}} \le \|v\|_{W^1_{\infty,0}}.$$

Thus, the proof is given. ∎

For $\mathbf{w} = (w,d) \in W^{\mathbf{R}}$, we define the linear operator $\mathcal{L}$ as

$$\mathcal{L}\mathbf{w} \equiv \begin{pmatrix} -w''(x) + f'(\hat{v}_h, \hat{c}_h)\cdot(w,d) \\ w'(1) \end{pmatrix}, \tag{2.6}$$

where the second derivative is interpreted in the weak sense. If $\mathcal{L}$ is invertible, then, by setting $g(\mathbf{w}) := (g_1(\mathbf{w}), g_2)$, the equation (2.3) can be written in the following fixed point form:

$$\mathbf{w} = F(\mathbf{w}) \left(\equiv \mathcal{L}^{-1} g(\mathbf{w}) \right).$$

Here, notice that the nonlinear operator F becomes compact on $W^{\mathbf{R}}$ from the compact imbedding $W^2_\infty(0,1) \hookrightarrow C^1(0,1)$.

Thus, for some positive number α, setting $Y \equiv \{\mathbf{w} \in W^{\mathbf{R}} : |||\mathbf{w}|||_{W^1_{\infty,0}} \le \alpha\}$, if

$$F(Y) \quad \subset \quad Y, \tag{2.7}$$

then there exists $\hat{\mathbf{w}} \in Y$ such that, from Schauder's fixed point theorem, we have $F(\hat{\mathbf{w}}) = \hat{\mathbf{w}}$ where Y is the *candidate set.* Since the center of Y is zero, a sufficient condition of this inclusion is written in the form

$$|||F(Y)|||_{W^1_{\infty,0}} \equiv \sup_{\mathbf{w}\in Y} |||F(\mathbf{w})|||_{W^1_{\infty,0}} \quad < \quad \alpha. \tag{2.8}$$

The detailed procedure for estimating the right-hand side of (2.8) via computer is presented in the following two sections.

3 Computable verification condition for the inverse of the linearized operator

In this section, we describe numerical methods by which to show the invertibility of the following linear operator and estimate the norm of the inverse. First, (2.6) can generally be written in the form

$$\mathcal{L}z \quad \equiv \quad \begin{pmatrix} -v''(x) + p(x)v'(x) + q(x)v(x) + c \cdot r(x) \\ v'(1) \end{pmatrix}, \tag{3.1}$$

where $z = (v(x), c) \in \left(W^1_{\infty,0}(0,1) \cap W^2_\infty(0,1)\right) \times \mathbf{R}$, and we assume that $p(x) \in W^1_\infty(0,1)$, $q(x), r(x) \in L^\infty(0,1)$.

3.1 Invertibility condition of the operator $\mathcal{L}$

For arbitrary $\phi \in H^1_0(0,1)$, we define the H^1_0-projection $P_h : H^1_0 \to S_h$ of ϕ by

$$((\phi - P_h\phi)', \varphi_h') = 0 \quad \forall \varphi_h \in S_h. \tag{3.2}$$

The following are assumed to hold for this H^1_0-projection:

Assumption 1 *For arbitrary $\phi \in W^1_{\infty,0}(0,1) \cap W^2_\infty(0,1)$, there exists a constant $C_0 > 0$ with independent of h such that*

$$\|(\phi - P_h\phi)'\|_{L^\infty} \le C_0 h \|\phi''\|_{L^\infty}.$$

Assumption 2 *For arbitrary $\phi \in W^1_{\infty,0}(0,1)$, there exist positive constants C and C_p independent of h such that*

$$\|\phi - P_h\phi\|_{L^\infty} \le Ch\|(\phi - P_h\phi)'\|_{L^\infty}$$

and

$$\|P_h\phi\|_{W^1_{\infty,0}} \le C_p\|\phi\|_{W^1_{\infty,0}},$$

respectively.

In addition, by letting $\{\varphi_i\}_{i=1}^N$ be a basis of S_h, we have the following.

Assumption 3 *We define the vector for any* $\phi_h = \sum_{i=1}^{N} a_i\varphi_i \in S_h$, *by* $\vec{\phi}_h \equiv (a_1, \cdots, a_N)^T$. *Then, there exists a matrix* $\mathbf{P} \in \mathbf{R}^{M\times N}$ *with* $\mathrm{rank}(\mathbf{P}) = N$ $(M \geq N)$ *such that*

$$\|\phi_h'\|_{L^\infty} = \|\mathbf{P}\vec{\phi}_h\|_\infty$$

where $\|\cdot\|_\infty$ *is the matrix maximum norm.*

In particular, we define the following finite element subspace:

$$PL(0,1) \ \equiv \ \mathrm{span}\{\varphi_1, \cdots, \varphi_N\}$$

where the base functions $\{\varphi_i\}_{i=1}^N$ are defined as

$$\varphi_i \equiv \varphi_i(x) = \begin{cases} (x - x_{i-1})/h_{i-1} & \text{if} \quad x \in [x_{i-1}, x_i], \\ (x_{i+1} - x)/h_i & \text{if} \quad x \in [x_i, x_{i+1}], \\ 0 & \quad\text{otherwise.} \end{cases}$$

We then consider a number of properties in the case that $S_h = PL(0,1)$. First, by [7], $P_h\phi(x_i) = \phi(x_i)$, $0 \leq i \leq N+1$. The constant in Assumption 1 can be taken as $C_0 = \dfrac{1}{2}$. We also have the following lemma:

Lemma 2 *Let* $S_h = PL(0,1)$. *Then the constants in Assumption 2 can be taken as follows:*

$$C = \frac{1}{2} \qquad \text{and} \qquad C_p = 1,$$

respectively.

Proof : Letting $e = \phi - P_h\phi$, from the property described above, we have $e(x_i) = 0, \quad i = 0, \cdots, N+1$. Then, for any $x \in [x_i, x_{i+1}]$, $i = 0, \cdots, N$, we have

$$e(x) = \int_{x_i}^{x} e'(t)dt, \qquad e(x) = -\int_{x}^{x_{i+1}} e'(t)dt.$$

Hence, we obtain

$$\begin{aligned} |e(x)| &= \left|\int_{x_i}^{x} e'(t)dt\right| \leq \int_{x_i}^{x} |e'(t)|dt, \\ |e(x)| &= \left|-\int_{x}^{x_{i+1}} e'(t)dt\right| \leq \int_{x}^{x_{i+1}} |e'(t)|dt. \end{aligned}$$

and

$$2|e(x)| \leq \int_{x_i}^{x_{i+1}} |e'(t)|dt, \tag{3.3}$$

which yields the desired estimates for the constant C.
Next, notice that $(P_h\phi)'(x) = (\phi(x_i) - \phi(x_{i-1}))/h_i$ for all $x \in [x_i, x_{i+1}]$, $i = 0, \cdots, N$. Hence, we obtain

$$\begin{aligned} |(P_h\phi)'(x)| &= \frac{1}{h_i}\left|\int_{x_{i-1}}^{x_i} \phi'(x)\,dx\right| \\ &\leq \frac{1}{h_i}\|\phi'\|_{L^\infty[x_i,x_{i+1}]}h_i \\ &= \|\phi\|_{W^1_{\infty,0}}. \end{aligned}$$

Thus, the proof of the lemma is given. ∎

Next, we present the following property for use later in the present paper:

Lemma 3 *Let $\psi \in W^1_{\infty,0}(0,1)$. Then we have*

$$\|\psi\|_{W^1_{\infty,0}} = \sup_{\xi \in L^1} \frac{(\psi', \xi)}{\|\xi\|_{L^1}} \quad \leq \quad \sup_{\zeta \in W^1_{1,0}} \frac{2|(\psi', \zeta')|}{\|\zeta'\|_{L^1}}$$

where $W^1_{1,0}(0,1) \equiv \{\zeta \in W^1_1(0,1) : \zeta(0) = \zeta(1) = 0\}$.

Proof : The first equality follows from the definition of the norm in $W^1_{\infty,0}(0,1)$ and the usual dual property. Let

$$\zeta(x) \equiv \int_0^x \xi(t)\, dt - x \int_0^1 \xi(t)\, dt$$

for arbitrary $\xi(x) \in L^1(0,1)$. Then, from $\zeta(0) = \zeta(1) = 0$, it follows that $\zeta(x) \in W^1_{1,0}(0,1)$. Since $\psi \in W^1_{\infty,0}(0,1)$, we have

$$\begin{aligned}
(\psi', \zeta') &= (\psi', \xi - \int_0^1 \xi(t)\, dt) \\
&= (\psi', \xi) - \int_0^1 \psi'(t)\, dt \cdot \int_0^1 \xi(t)\, dt \\
&= (\psi', \xi) - [\psi(1) - \psi(0)] \int_0^1 \xi(t)\, dt \\
&= (\psi', \xi).
\end{aligned}$$

We therefore obtain

$$\begin{aligned}
\|\zeta'\|_{L^1} \equiv \int_0^1 |\zeta'(x)|\, dx &= \int_0^1 \left| \xi(x) - \int_0^1 \xi(t)\, dt \right| dx \\
&\leq \int_0^1 |\xi(x)|\, dx + \int_0^1 |\xi(t)|\, dt \\
&= 2\|\xi\|_{L^1}.
\end{aligned}$$

Thus, the proof is completed. ∎

The following lemma is required for the actual verification procedures.

Lemma 4 *For the case in which $S_h = PL(0,1)$, the matrix $\mathbf{P}$ in Assumption 3 is given by $\mathbf{P} = \mathbf{H}^{-1}\mathbf{Q}$, where*

$$\mathbf{Q} = \begin{bmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & 1 \\ -1 & 0 & \cdots & 0 \end{bmatrix}, \qquad \mathbf{H} = diag(h_1, \cdots, h_N, h_0).$$

The proof of this lemma is obtained by direct computation.
We also define the matrices $\mathbf{G} = (\mathbf{G}_{i,j})$, $\mathbf{r} = (\mathbf{r}_i)$, $\mathbf{v} = (\mathbf{v}_j)$, $\mathbf{D} = (\mathbf{D}_{i,j})$ as

$$\begin{aligned}
\mathbf{G}_{i,j} &= (\varphi_j', \varphi_i') + (p \cdot \varphi_j', \varphi_i) + (q \cdot \varphi_j, \varphi_i), \\
\mathbf{r}_i &= (r(x), \varphi_i), \\
\mathbf{v}_j &= \varphi_j'(1), \\
\mathbf{D}_{i,j} &= (\varphi_j', \varphi_i').
\end{aligned}$$

In addition, we define the following matrix maximum norm, which is the norm of the inverse operator for $\mathcal{L}$ in the finite-dimensional sense:

$$M \equiv \left\| \begin{bmatrix} \mathbf{P} & \\ & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P}^T\mathbf{P}\mathbf{D}^{-1}\mathbf{G} & \mathbf{P}^T\mathbf{P}\mathbf{D}^{-1}\mathbf{r} \\ \mathbf{v} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{P}^T & \\ & 1 \end{bmatrix} \right\|_{\infty}. \tag{3.4}$$

Note that, for the case in which $S_h = PL(0,1)$, we have $\mathbf{P}^T\mathbf{P} = \mathbf{H}^{-1/2}\mathbf{D}\mathbf{H}^{-1/2}$ in Lemma 4, and if $h = h_i$ for $i = 0, \cdots, N$, i.e., the uniform mesh, the matrix norm M in (3.4) is reduced to

$$M = \left\| \begin{bmatrix} \mathbf{Q} & \\ & 1 \end{bmatrix} \begin{bmatrix} \mathbf{G}h & \mathbf{r} \\ \mathbf{v}h & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{Q}^T & \\ & 1 \end{bmatrix} \right\|_{\infty}. \tag{3.5}$$

$$\tag{3.6}$$

Next, for $z = (v, c) \in W^{\mathbf{R}}$, we define the operator A on $W^{\mathbf{R}}$ as

$$Az \equiv \begin{bmatrix} \Delta^{-1} & \\ & 1 \end{bmatrix} \begin{pmatrix} p \cdot v' + q \cdot v + c \cdot r \\ c - v'(1) \end{pmatrix}.$$

If $(I - A)$ is invertible, then $\mathcal{L}$ is also invertible, where I denotes the identity map on $W^{\mathbf{R}}$. As one of the main results, we obtain the following invertibility condition:

Theorem 1 *For the linear operator (3.1), if*

$$C_0 h(C_3 M K + C_4) < 1$$

then $\mathcal{L}$ is invertible. Here, $K = \max(2C_p(C_1 + C_2)h, 1)$, and the constants C_i $(i = 1, 2, 3, 4)$ are defined as

$$\begin{aligned} C_1 &= C(\|p\|_{L^\infty} + \|p'\|_{L^\infty}), & C_2 &= C\|q\|_{L^\infty}, \\ C_3 &= \|p\|_{L^\infty} + \|q\|_{L^\infty} + \|r\|_{L^\infty}, & C_4 &= \|p\|_{L^\infty} + Ch\|q\|_{L^\infty}. \end{aligned}$$

Proof : We consider the uniqueness of the solution for the fixed point equation $Az = z$, which implies the invertibility of $I - A$. First, as usual(e.g.,[4]etc.), we decompose the fixed point equation $z = Az$ as

$$\begin{aligned} \mathcal{P}_h z &= \mathcal{P}_h A z, \\ (I - \mathcal{P}_h) z &= (I - \mathcal{P}_h) A z \end{aligned}$$

where $\mathcal{P}_h \equiv P_h \times 1 : W \times \mathbf{R} \to S_h \times \mathbf{R}$. Here, P_h is the H_0^1-projection defined earlier.

Next, based on the same formulation as that in [4][9]etc., we define two operators by

$$N_h z \equiv \mathcal{P}_h z - [I - A]_h^{-1} \mathcal{P}_h (I - A) z$$

and

$$Tz \equiv N_h z + (I - \mathcal{P}_h) A z,$$

where $[I - A]_h^{-1}$ is the inverse of $\mathcal{P}_h(I - A)|_{S_h \times \mathbf{R}} : S_h \times \mathbf{R} \longrightarrow S_h \times \mathbf{R}$.
Next, for positive real numbers α and γ, we define the candidate set $Z \equiv Z_h + Z_\perp$ as

$$\begin{aligned} Z_h &= \left\{ z_h = (v_h, \hat{c}_h) \in S_h \times \mathbf{R} : |||z_h|||_{W^1_{\infty,0}} \le \gamma \right\}, \\ Z_\perp &= \left\{ z_\perp = (v_\perp, 0) \in W \times \{0\} : |||z_\perp|||_{W^1_{\infty,0}} \le \alpha, \quad |||z_\perp|||_{L^\infty} \le Ch\alpha \right\}. \end{aligned}$$

Then, based on the fact that $z = Az$ is equivalent to $z = Tz$, in order to prove the unique existence of a solution to $z = Az$ in the set Z, it is sufficient to show the inclusion $TZ \overset{\circ}{\subset} Z$, which implies that $\overline{TZ} \subset \overset{\circ}{Z}$, i.e., that the closure of TZ is included in the interior of Z. A sufficient condition of this inclusion can be written as

$$|||N_h Z|||_{W^1_{\infty,0}} \equiv \sup_{z\in Z} |||N_h z|||_{W^1_{\infty,0}} < \gamma, \tag{3.7}$$

$$|||(I-\mathcal{P}_h)AZ|||_{W^1_{\infty,0}} \equiv \sup_{z\in Z} |||(I-\mathcal{P}_h)Az|||_{W^1_{\infty,0}} < \alpha. \tag{3.8}$$

Therefore, in the following, by using the constants defined above, we try to estimate norms $|||N_h z|||_{W^1_{\infty,0}}$ and $|||(I-\mathcal{P}_h)Az|||_{W^1_{\infty,0}}$ in (3.7) and (3.8), respectively.

First, for arbitrary $z = z_h + z_\perp \in Z_h + Z_\perp$, by setting $\psi_h := N_h z = N_h(z_h + z_\perp)$, we have

$$\begin{aligned}\psi_h &= \mathcal{P}_h(z_h+z_\perp) - [I-A]_h^{-1}\mathcal{P}_h(I-A)(z_h+z_\perp)\\ &= [I-A]_h^{-1}\mathcal{P}_h A z_\perp.\end{aligned}$$

Next, setting $t_h := \mathcal{P}_h A z_\perp \in S_h \times \mathbf{R}$ and using the notation $\psi_h = (\psi_{S_h}, \psi_{\mathbf{R}})$ and $t_h = (t_{S_h}, t_{\mathbf{R}}) \in S_h \times \mathbf{R}$, ψ_h is given as the solution of the following weak form:

$$\begin{aligned}(\psi'_{S_h}, \varphi'_h) + (p\cdot\psi'_{S_h}, \varphi_h) + (q\cdot\psi_{S_h}, \varphi_h) + (\psi_{\mathbf{R}}\cdot r, \varphi_h) &= (t'_{S_h}, \varphi'_h),\\ \psi'_{S_h}(1) &= t_{\mathbf{R}}.\end{aligned} \tag{3.9}$$

Letting

$$\psi_{S_h} = \sum_{i=1}^{N} w_i\varphi_i, \qquad t_{S_h} = \sum_{i=1}^{N} t_i\varphi_i, \tag{3.10}$$

from (3.9) we have the matrix equation of the form

$$\begin{bmatrix}\mathbf{G} & \mathbf{r}\\ \mathbf{v} & 0\end{bmatrix}\begin{pmatrix}\vec{w}\\ \psi_{\mathbf{R}}\end{pmatrix} = \begin{bmatrix}\mathbf{D} & \\ & 1\end{bmatrix}\begin{pmatrix}\vec{t}\\ t_{\mathbf{R}}\end{pmatrix}, \tag{3.11}$$

where $\vec{w} = (w_1, \cdots, w_N)^T$ and $\vec{t} = (t_1, \cdots, t_N)^T$ are coefficient vectors of ψ_{S_h} and t_{S_h}, respectively. Therefore, from (3.10), (3.11) and Assumption 3, it follows that

$$\begin{aligned}|||\psi_h|||_{W^1_{\infty,0}} &\equiv \max(\|\psi_{S_h}\|_{W^1_{\infty,0}}, |\psi_{\mathbf{R}}|)\\ &= \max(\|\mathbf{P}\vec{w}\|_\infty, |\psi_{\mathbf{R}}|)\\ &= \left\|\begin{bmatrix}\mathbf{P} & \\ & 1\end{bmatrix}\begin{pmatrix}\vec{w}\\ \psi_{\mathbf{R}}\end{pmatrix}\right\|_\infty\\ &= \left\|\begin{bmatrix}\mathbf{P} & \\ & 1\end{bmatrix}\begin{bmatrix}\mathbf{G} & \mathbf{r}\\ \mathbf{v} & 0\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{D} & \\ & 1\end{bmatrix}\begin{pmatrix}\vec{t}\\ t_{\mathbf{R}}\end{pmatrix}\right\|_\infty\\ &= \left\|\begin{bmatrix}\mathbf{P} & \\ & 1\end{bmatrix}\begin{bmatrix}\mathbf{G} & \mathbf{r}\\ \mathbf{v} & 0\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{D} & \\ & 1\end{bmatrix}\begin{bmatrix}(\mathbf{P}^T\mathbf{P})^{-1}(\mathbf{P}^T\mathbf{P}) & \\ & 1\end{bmatrix}\begin{pmatrix}\vec{t}\\ t_{\mathbf{R}}\end{pmatrix}\right\|_\infty\\ &\le \left\|\begin{bmatrix}\mathbf{P} & \\ & 1\end{bmatrix}\begin{bmatrix}\mathbf{P}^T\mathbf{P}\mathbf{D}^{-1}\mathbf{G} & \mathbf{P}^T\mathbf{P}\mathbf{D}^{-1}\mathbf{r}\\ \mathbf{v} & 0\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{P}^T & \\ & 1\end{bmatrix}\right\|_\infty \left\|\begin{pmatrix}\mathbf{P}\vec{t}\\ t_{\mathbf{R}}\end{pmatrix}\right\|_\infty\\ &= M\cdot\max(\|\mathbf{P}\vec{t}\|_\infty, |t_{\mathbf{R}}|)\\ &= M\cdot\max(\|t_{S_h}\|_{W^1_{\infty,0}}, |t_{\mathbf{R}}|)\\ &= M\cdot|||t_h|||_{W^1_{\infty,0}}.\end{aligned}$$

Thus from $t_h = \mathcal{P}_h A z_\perp$ and Lemma 2, and by setting $z_\perp = (v_\perp, 0)$, we obtain the estimation

$$\begin{aligned}
|||\psi_h|||_{W^1_{\infty,0}} &\leq M|||\mathcal{P}_h A z_\perp|||_{W^1_{\infty,0}} \\
&= M \cdot \max\left(\|P_h \Delta^{-1}(p \cdot v'_\perp + q \cdot v_\perp)\|_{W^1_{\infty,0}}, |v'_\perp(1)|\right) \\
&\leq M \cdot \max\left(C_p \|\Delta^{-1}(p \cdot v'_\perp + q \cdot v_\perp)\|_{W^1_{\infty,0}}, |v'_\perp(1)|\right). \qquad (3.12)
\end{aligned}$$

We now estimate $\|\Delta^{-1}(p \cdot v'_\perp)\|_{W^1_{\infty,0}}$ and $\|\Delta^{-1}(q \cdot v_\perp)\|_{W^1_{\infty,0}}$.
For the first term, setting $\psi_1 = \Delta^{-1}(p \cdot v'_\perp)$, by Lemma 3, we have

$$\begin{aligned}
\|\psi_1\|_{W^1_{\infty,0}} &\leq \sup_{\zeta \in W^1_{1,0}} \frac{2|(\psi'_1, \zeta')|}{\|\zeta'\|_{L^1}} \\
&= \sup_{\zeta \in W^1_{1,0}} \frac{2|(\psi''_1, \zeta)|}{\|\zeta'\|_{L^1}} \\
&= \sup_{\zeta \in W^1_{1,0}} \frac{2|(p \cdot v'_\perp, \zeta)|}{\|\zeta'\|_{L^1}} \qquad (3.13)\\
&= \sup_{\zeta \in W^1_{1,0}} \frac{2|(v_\perp, \operatorname{div}(p \cdot \zeta))|}{\|\zeta'\|_{L^1}} \\
&\leq \sup_{\zeta \in W^1_{1,0}} \frac{2\|v_\perp\|_{L^\infty}(\|p\|_{L^\infty} + \|p'\|_{L^\infty})\|\zeta'\|_{L^1}}{\|\zeta'\|_{L^1}} \\
&\leq 2Ch(\|p\|_{L^\infty} + \|p'\|_{L^\infty})\alpha,
\end{aligned}$$

where we have used the fact that $\|v_\perp\|_{L^\infty} \leq Ch\alpha$.
For the second term, setting $\psi_2 = \Delta^{-1} q \cdot v_\perp$ and by applying an argument similar to that given above, we have

$$\|\Delta^{-1}(q \cdot v_\perp)\|_{W^1_{\infty,0}} \leq 2Ch\|q\|_{L^\infty}. \qquad (3.14)$$

Thus, by (3.12) – (3.14) and taking into account that $|v'_\perp(1)| \leq \alpha$, we obtain the following estimate for the finite-dimensional part

$$|||N_h Z|||_{W^1_{\infty,0}} \leq M \cdot \max\left(2C_p(C_1 + C_2)h, 1\right)\alpha. \qquad (3.15)$$

Next, setting $z_h = (v_h, c_h)$, $z_\perp = (v_\perp, 0)$, we observe that

$$\begin{aligned}
\|p \cdot v'_h + q \cdot v_h + c_h \cdot r\|_{L^\infty} &\leq \|p\|_{L^\infty}\|v_h\|_{W^1_{\infty,0}} + \|q\|_{L\infty}\|v_h\|_{L^\infty} + \|r\|_{L^\infty}|c_h| \\
&\leq \|p\|_{L^\infty}\|v_h\|_{W^1_{\infty,0}} + \|q\|_{L\infty}\|v_h\|_{W^1_{\infty,0}} + \|r\|_{L^\infty}|c_h| \\
&\leq (\|p\|_{L^\infty} + \|q\|_{L\infty} + \|r\|_{L^\infty})\gamma, \\
\|p \cdot v'_\perp + q \cdot v_\perp\|_{L^\infty} &\leq \|p\|_{L^\infty}\|v_\perp\|_{W^1_{\infty,0}} + \|q\|_{L^\infty}\|v_\perp\|_{L^\infty} \\
&\leq \|p\|_{L^\infty}\|v_\perp\|_{W^1_{\infty,0}} + Ch\|q\|_{L^\infty}\|v_\perp\|_{W^1_{\infty,0}} \\
&\leq (\|p\|_{L^\infty} + Ch\|q\|_{L^\infty})\alpha.
\end{aligned}$$

In addition, from Assumption 1, we have

$$\begin{aligned}
|||(I - \mathcal{P}_h)AZ|||_{W^1_{\infty,0}} &\leq C_0 h \sup_{z \in Z} \|p \cdot (v_h + v_\perp)' + q \cdot (v_h + v_\perp) + c_h \cdot r\|_{L^\infty} \\
&\leq C_0 h \sup_{z \in Z} \left(\|p \cdot v'_h + q \cdot v_h + c_h \cdot r\|_{L^\infty} + \|p \cdot v'_\perp + q \cdot v_\perp\|_{L^\infty}\right).
\end{aligned}$$

Thus, we obtain

$$|||(I - \mathcal{P}_h)AZ|||_{W^1_{\infty,0}} \leq C_0 h(C_3\gamma + C_4\alpha). \qquad (3.16)$$

Next, from (3.15) and (3.16), we rewrite the invertibility condition (3.7) and (3.8) as

$$M \cdot \max(2C_p(C_1+C_2)h, 1)\,\alpha \quad < \quad \gamma, \tag{3.17}$$
$$C_0 h(C_3\gamma + C_4\alpha) \quad < \quad \alpha. \tag{3.18}$$

Setting $K = \max(2C_p(C_1+C_2)h, 1)$, for arbitrary small $\varepsilon > 0$, if we set $\gamma := MK\alpha + \varepsilon$, then the condition (3.17) holds. Therefore, by substituting this expression for γ into (3.18) we have

$$C_0 h(C_3(MK\alpha + \varepsilon) + C_4\alpha) < \alpha$$

which is equivalent to

$$1 - C_0 h(C_3 MK + C_4) > 0$$

Thus the proof is given. ∎

3.2 Norm Estimation of the Inverse Operator

Our purpose in this section is to estimate the operator norm $|||(I-A)^{-1}|||_{W^1_{\infty,0}}$.

Theorem 2 *Under the assumptions stated in Theorem 1, provided that*

$$\kappa \equiv C_0 h(C_3 MK + C_4) < 1$$

then we have the following estimation

$$|||(I-A)^{-1}|||_{W^1_{\infty,0}} \le (R+S) =: \mathcal{M}.$$

Here, R and S are defined by

$$R = (C_0 C_3 C_p hM + 1 + C_p)/(1-\kappa), \quad S = (KR + C_p)M.$$

Proof : Let ψ be an arbitrary element in $W^{\mathbf{R}}$. Then, by the invertibility of $(I-A)$, there exists a unique element $z \in W^{\mathbf{R}}$ satisfying $(I-A)z = \psi$. We next set

$$\begin{aligned} N_h z &= \mathcal{P}_h z - [I-A]_h^{-1}\mathcal{P}_h((I-A)z - \psi), \\ Tz &= N_h z + (I - \mathcal{P}_h)(Az + \psi). \end{aligned}$$

Here, $(I-A)z = \psi$ is equivalent to $Tz = z$. Using the unique decomposition $z = z_h + z_\perp$ ($z_h := \mathcal{P}_h z$, $z_\perp := z - \mathcal{P}_h z$), a few simple calculations can be performed to obtain the following

$$\begin{aligned} z_h &= [I-A]_h^{-1}(\mathcal{P}_h A z_\perp - \mathcal{P}_h \psi), \\ z_\perp &= (I-\mathcal{P}_h)A(z_h + z_\perp) + (I-\mathcal{P}_h)\psi. \end{aligned} \tag{3.19}$$

Hence, taking notice of $M = \|[I-A]_h^{-1}\|_{W^1_{\infty,0}}$ and the estimates in the proof of Theorem 1, we have by (3.19)

$$\begin{aligned} |||z_h|||_{W^1_{\infty,0}} &\le M|||\mathcal{P}_h A z_\perp - \mathcal{P}_h\psi|||_{W^1_{\infty,0}} \\ &\le MK|||z_\perp|||_{W^1_{\infty,0}} + M|||\mathcal{P}_h\psi|||_{W^1_{\infty,0}}, \end{aligned} \tag{3.20}$$

$$\begin{aligned} |||z_\perp|||_{W^1_{\infty,0}} &\le |||(I-\mathcal{P}_h)A(z_h+z_\perp)|||_{W^1_{\infty,0}} + |||(I-\mathcal{P}_h)\psi|||_{W^1_{\infty,0}} \\ &\le C_0 h(C_3|||z_h|||_{W^1_{\infty,0}} + C_4|||z_\perp|||_{W^1_{\infty,0}}) + |||(I-\mathcal{P}_h)\psi|||_{W^1_{\infty,0}}. \end{aligned} \tag{3.21}$$

Substituting the estimate of $|||z_h|||_{W^1_{\infty,0}}$ in (3.20) into the right-hand term of (3.21) and solving with respect to $|||z_\perp|||_{W^1_{\infty,0}}$, we obtain

$$\begin{aligned}|||z_\perp|||_{W^1_{\infty,0}} &\leq \left[C_0C_3hM|||\mathcal{P}_h\psi|||_{W^1_{\infty,0}} + |||(I-\mathcal{P}_h)\psi|||_{W^1_{\infty,0}}\right]/(1-\kappa) \\ &\leq \left[C_0C_3C_phM|||\psi|||_{W^1_{\infty,0}} + (1+C_p)|||\psi|||_{W^1_{\infty,0}}\right]/(1-\kappa) \\ &= \left[(C_0C_3C_phM + 1 + C_p)/(1-\kappa)\right]|||\psi|||_{W^1_{\infty,0}}. \qquad (3.22)\end{aligned}$$

By (3.20), we also have

$$\begin{aligned}|||z_h|||_{W^1_{\infty,0}} &\leq MKR|||\psi|||_{W^1_{\infty,0}} + M|||\mathcal{P}_h\psi|||_{W^1_{\infty,0}} \\ &\leq (KR + C_p)M|||\psi|||_{W^1_{\infty,0}}. \qquad (3.23)\end{aligned}$$

Thus the proof follows immediately from (3.22) and (3.23). ∎

In addition, we have the following a priori estimates for the solution of the linear problem.

Theorem 3 *Let* $z = (v,c) \in \left(W^1_{\infty,0}(0,1) \cap W^2_\infty(0,1)\right) \times \mathbf{R}$ *be a solution of the equation* $\mathcal{L}z = (g,s)$ *with* $(g,s) \in L^\infty(0,1) \times \mathbf{R}$. *Then, we have*

$$|||z|||_{W^1_{\infty,0}} \leq |||(I-A)^{-1}|||_{W^1_{\infty,0}} \cdot \max\left(\frac{1}{2}\|g\|_{L^\infty}, |s|\right),$$

if $\mathcal{L}$ *is invertible.*

Proof : Defining $\phi := -\Delta^{-1}g$ and $\psi := (\phi, s)$, and then taking into account that $(I-A)z = \psi$, it suffices to show

$$\|\phi\|_{W^1_{\infty,0}} \leq \frac{1}{2}\|g\|_{L^\infty}. \qquad (3.24)$$

From the definition of $-\phi$, a few elementary calculations show that

$$\phi'(x) = -\int_0^x tg(t)\,dt + \int_x^1 (1-t)g(t)\,dt.$$

Hence, we have

$$\begin{aligned}|\phi'(x)| &\leq \|g\|_{L^\infty}\left[\int_0^x t\,dt + \int_x^1 (1-t)\,dt\right] \\ &= \left(x^2 - x + \frac{1}{2}\right)\|g\|_{L^\infty} \\ &\leq \frac{1}{2}\|g\|_{L^\infty}.\end{aligned}$$

Therefore, the proof of this theorem is completed. ∎

In general, for verification of the solutions of the nonlinear problem (1.1), the following a priori estimates are more effective than Theorem 3.

Theorem 4 *Let* $z = (v,c) \in \left(W^1_{\infty,0}(0,1) \cap W^2_\infty(0,1)\right) \times \mathbf{R}$ *be a solution of the equation* $\mathcal{L}z = (g,s)$ *with* $(g,s) \in L^\infty(0,1) \times \mathbf{R}$. *Under the same assumptions in Theorem 1, provided that*

$$\kappa \equiv C_0h(C_3MK + C_4) < 1$$

then we have the following estimation:

$$|||z|||_{W^1_{\infty,0}} \le \mathcal{M}_1 \cdot \max\left(\frac{1}{2}\|g\|_{L^\infty}, |s|\right) + \mathcal{M}_2\|g\|_{L^\infty}.$$

Here, $\mathcal{M}_1$ and $\mathcal{M}_2$ are defined by

$$\mathcal{M}_1 = C_p M[\mathcal{M}_2 C_3 + 1] \quad and \quad \mathcal{M}_2 = S_0[MK + 1],$$

respectively, where $S_0 = \dfrac{C_0 h}{(1-\kappa)}$.

Proof : The proof is similar to that for Theorem 2. We choose ψ in the proof of Theorem 2 such that $\psi := (\phi, s)$, where $\phi := -\Delta^{-1} g$. Then, the estimates $|||z_\perp|||_{W^1_{\infty,0}}$ in (3.21) and $|||z_h|||_{W^1_{\infty,0}}$ in (3.20) are bounded as follows:
First, from Assumption 1 and 2, we have

$$\begin{aligned} |||z_\perp|||_{W^1_{\infty,0}} &\le \left[C_0 C_3 h M |||\mathcal{P}_h\psi|||_{W^1_{\infty,0}} + |||(I-\mathcal{P}_h)\psi|||_{W^1_{\infty,0}}\right]/(1-\kappa) \\ &\le \left[C_0 C_3 C_p h M |||\psi|||_{W^1_{\infty,0}} + C_0 h\|g\|_{L^\infty}\right]/(1-\kappa). \end{aligned}$$

Thus, by (3.20) and Assumption 2, we also have

$$\begin{aligned} |||z_h|||_{W^1_{\infty,0}} &\le MKS_0\left[C_3 C_p M|||\psi|||_{W^1_{\infty,0}} + \|g\|_{L^\infty}\right] + M|||\mathcal{P}_h\psi|||_{W^1_{\infty,0}} \\ &\le MKS_0\left[C_3 C_p M|||\psi|||_{W^1_{\infty,0}} + \|g\|_{L^\infty}\right] + C_p M|||\psi|||_{W^1_{\infty,0}} \\ &\le C_p M\left[MKS_0 C_3 + 1\right]|||\psi|||_{W^1_{\infty,0}} + MKS_0\|g\|_{L^\infty}. \end{aligned}$$

Hence, from (3.24), we obtain

$$\begin{aligned} |||z|||_{W^1_{\infty,0}} &\le |||z_h|||_{W^1_{\infty,0}} + |||z_\perp|||_{W^1_{\infty,0}} \\ &\le C_p M[\mathcal{M}_2 C_3 + 1]|||\psi|||_{W^1_{\infty,0}} + S_0[MK+1]\|g\|_{L^\infty} \\ &\le \mathcal{M}_1 \cdot \max\left(\frac{1}{2}\|g\|_{L^\infty}, |s|\right) + \mathcal{M}_2\|g\|_{L^\infty}. \end{aligned}$$

Thus the proof of this theorem is completed. ∎

4 Numerical Examples

In this section, we present numerical examples for the original free boundary problem (1.1). The candidate set is taken as $Y \equiv \{\mathbf{w} = (w, d) \in W^{\mathbf{R}} : |||\mathbf{w}|||_{W^1_{\infty,0}} \le \alpha\}$ for a positive number α, and we use the finite element subspace $S_h \equiv PL(0,1)$. Since we enclose the solution for the residual equation (2.2) in Section 2, Y is the error for the approximate solution $(\hat{v}_h, \hat{c}_h)$. That is, the total error is estimated in the form $(\|v_0\|_{W^1_{\infty,0}} + \|w\|_{W^1_{\infty,0}}, |d|)$.

Example 1 *[8] In the following, we consider the rope problem:*

$$\begin{array}{rcll} u''(x) &=& \rho\sqrt{1+(u'(x))^2} & \text{if} \quad x \in (0,c), \\ u(x) &>& 0 & \text{if} \quad x \in [0,c), \\ u(x) &=& 0 & \text{if} \quad x \in [c,\infty), \\ && u(0) = u_0, \quad u'(c) = 0 & \end{array} \tag{4.25}$$

where $\rho > 0$ is a parameter, and we set $\rho = 1$ and $u_0 = 0.1$.

Numerical enclosure of solutions for Example 1 was presented by Schäfer [6]. It is known that this example has a globally unique solution for any u_0, and the free boundary point is verified by [6], for example, at $c = 0.44356825\cdots$ if $u_0 = 0.1$.

By (4.25) and by calculations using several kinds of norms, we obtain the existential condition (2.8) in the form

$$K_2\alpha^2 + K_1\alpha + K_0 \;<\; \alpha. \tag{4.26}$$

Here, the constants K_i, $0 \le i \le 2$ are given as

$$\begin{aligned}
K_2 &= 6\left(\frac{1}{2}\mathcal{M}_1 + \mathcal{M}_2\right),\\
K_1 &= 4\left(\frac{1}{2}\mathcal{M}_1 + \mathcal{M}_2\right)\|v_0\|_{W^1_{\infty,0}},\\
K_0 &= \left(\frac{1}{2}\mathcal{M}_1 + \mathcal{M}_2\right)\left(\|v_0\|^2_{W^1_{\infty,0}} + \min\left(\|\hat{v}_h' - u_0\|_{L^\infty}, |\hat{c}_h|\right)\|v_0\|_{W^1_{\infty,0}}\right)\\
&\quad + \mathcal{M}_1\left(|u_0 - \hat{v}_h'(1)| + \|v_0\|_{W^1_{\infty,0}}\right),
\end{aligned}$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are the constants defined in Theorem 4.
The verification results are shown in Table 1, where the constant $\mathcal{M}$ is defined in Theorem 2 and 'smallest α' indicates the smallest bound α that satisfies the verification condition (4.26). The exact free boundary point c is verified in the interval $[\hat{c}_h - \alpha, \hat{c}_h + \alpha]$.
Moreover, in this example, the condition $u(x) > 0$ in (1.1) is always satisfied, because $\rho\sqrt{1 + (u'(x))^2} > 0$ for any solution $u(x)$ of (1.1) at any $x \in [0, c)$.

Table 1: Verification Results for Example 1 ($c = 0.44356825\cdots$)

$1/h$	$\mathcal{M}$	$\mathcal{M}_1$	$\mathcal{M}_2$	$\|v_0\|_{W^1_{\infty,0}}$	smallest α	$\hat{c}_h$
100	23.1535	7.0462	4.0268e-2	1.0920e-3	0.0114928	0.4458024
200	22.7137	6.9021	1.9764e-2	5.4353e-4	0.0046142	0.4446812
300	22.5746	6.8566	1.3098e-2	3.6180e-4	0.0029257	0.4443093
400	22.5224	6.8396	9.8017e-3	2.7115e-4	0.0021470	0.4441237

In the numerical result given by Schäfer[6], the free boundary point is enclosed as $c \in [0.4412705, 0.4472135]$ when $h = 1/300$, which is a slightly finer enclosure. However, notice that our results require no assumption on the existence of solutions.

Example 2 *We next consider the following problem:*

$$\begin{aligned}
-u''(x) &= u(x) - 2 - \tfrac{1}{4}(u'(x))^2 && \text{if} \quad x \in (0, c),\\
u(x) &> 0 && \text{if} \quad x \in [0, c),\\
u(x) &= 0 && \text{if} \quad x \in [c, \infty),\\
&u(0) = u_0, \quad u'(c) = 0
\end{aligned}$$

where $u_0 = 1/2,\ 1/4,\ 1/9$.

The solution of Example 2 is given by $u(x) = (x - \sqrt{u_0})^2$, $c = \sqrt{u_0}$. As in the previous example, we find that the verification condition (2.8) leads to the following inequality with the error bound α.

$$T_3\alpha^3 + T_2\alpha^2 + T_1\alpha + T_0 \;<\; \alpha. \tag{4.27}$$

Here, the constants T_i, $0 \le i \le 3$ are given as follows:

$$
\begin{aligned}
T_3 &= \left(\frac{1}{2}\mathcal{M}_1 + \mathcal{M}_2\right), \\
T_2 &= \left(\frac{1}{2}\mathcal{M}_1 + \mathcal{M}_2\right)\left[\|\hat{v}_h + (1-x)u_0 - 2\|_{L^\infty} + Ch\|v_0\|_{W^1_{\infty,0}} + 2|\hat{c}_h| + 1\right], \\
T_1 &= \left(\frac{1}{2}\mathcal{M}_1 + \mathcal{M}_2\right)\left[2Ch|\hat{c}_h|\|v_0\|_{W^1_{\infty,0}} + \frac{1}{2}\|v_0\|_{W^1_{\infty,0}}\right], \\
T_0 &= \left(\frac{1}{2}\mathcal{M}_1 + \mathcal{M}_2\right)\left[Ch|\hat{c}_h|^2\|v_0\|_{W^1_{\infty,0}} + \frac{1}{2}\|\hat{v}_h' - u_0\|_{L^\infty}\|v_0\|_{W^1_{\infty,0}} + \frac{1}{4}\|v_0\|^2_{W^1_{\infty,0}}\right] \\
&\quad + \mathcal{M}_1\left(|u_0 - \hat{v}_h'(1)| + \|v_0\|_{W^1_{\infty,0}}\right).
\end{aligned}
$$

The verification results are listed in Tables 2-4 with the same symbols in Example 1. In this example, unlike in Example 1, it is not possible to assure *a priori* the positivity condition $u(x) > 0$, $x \in [0, c)$. However, using the verified data in Table 2-4, we find that $u(x) - 2 < 0$ for any $x \in [0, c)$, which implies $u(x) - 2 - \frac{1}{4}(u'(x))^2 < 0$. Therefore, we can also guarantee the positivity condition *a posteriori* from the computational results, which is considered to be an additional advantage of the present method.

Table 2: Verification Results for Example 2 with $u_0 = 1/2$ ($c = 0.70710678\cdots$)

$1/h$	$\mathcal{M}$	$\mathcal{M}_1$	$\mathcal{M}_2$	$\|v_0\|_{W^1_{\infty,0}}$	smallest α	$\hat{c}_h$
100	18.6727	5.5465	3.2815e-2	5.0631e-3	Fail	0.7106674
200	17.7984	5.2609	1.5671e-2	2.5157e-3	0.0227080	0.7088808
300	17.5243	5.1713	1.0294e-2	1.6736e-3	0.0127211	0.7082880
400	17.3905	5.1276	7.6643e-3	1.2539e-3	0.0089764	0.7079922

Table 3: Verification Results for Example 2 with $u_0 = 1/4$ ($c = 0.5$)

$1/h$	$\mathcal{M}$	$\mathcal{M}_1$	$\mathcal{M}_2$	$\|v_0\|_{W^1_{\infty,0}}$	smallest α	$\hat{c}_h$
100	17.9558	5.3133	3.1606e-2	2.5284e-3	0.0191939	0.5025183
200	17.4655	5.1525	1.5391e-2	1.2570e-3	0.0079673	0.5012545
300	17.3077	5.1008	1.0172e-2	8.3646e-4	0.0050757	0.5008353
400	17.2298	5.0753	7.5965e-3	6.2676e-4	0.0037295	0.5006261

Table 4: Verification Results for Example 2 with $u_0 = 1/9$ ($c = 0.33333333\cdots$)

$1/h$	$\mathcal{M}$	$\mathcal{M}_1$	$\mathcal{M}_2$	$\|v_0\|_{W^1_{\infty,0}}$	smallest α	$\hat{c}_h$
100	17.4811	5.1581	3.0807e-2	1.1229e-3	0.0065406	0.3350124
200	17.2377	5.0781	1.5199e-2	5.5850e-4	0.0030907	0.3341697
300	17.1581	5.0519	1.0088e-2	3.7167e-4	0.0020249	0.3338902
400	17.1186	5.0389	7.5497e-3	2.7851e-4	0.0015045	0.3337507

Remark 2 *In the present paper, in order to enclose the deterministic value of the derivatives of solutions at the free boundary point, as well as to use piecewise linear polynomials, i.e., C^0-element, we used the mesh dependent function space W, which may appear to be somewhat odd. Note however, that once the exact solution $u(x)$ of (1.1) is verified in that space, then from the regularity of solutions we have $u(x) \in W^2_\infty(0,1) \subset C^1[0,1]$, which implies that the solution is actually enclosed by the usual $W^1_{\infty,0}(0,1)$ norm.*

The numerical computations were carried out on a Dell Latitude C400 Intel Pentium Mobile 866-MHz CPU using INTLAB 4.1.2, a tool box in MATLAB 6.5.1 developed by Rump [5] for self-validating algorithms.

Approximate solutions of Example 1-2 are shown in the Figure 1 below.

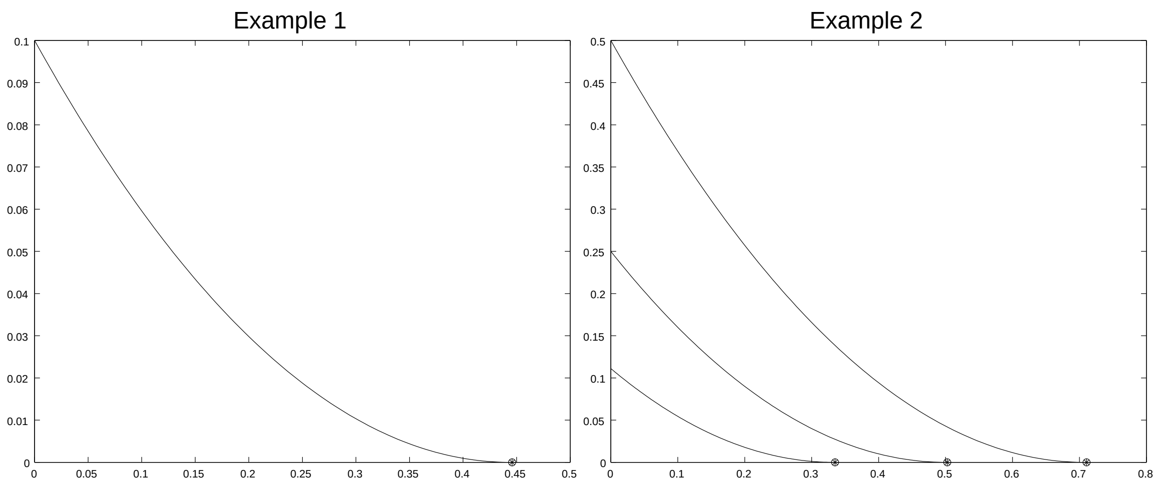

Figure 1: Approximate solutions of Examples 1-2

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