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https://hdl.handle.net/2324/11832

出版情報：International Mathematics Research Notices. 2004, pp.2497-2521, 2004. Hindawi Publishing Corporation
バージョン：

MHF Preprint Series

Kyushu University
21st Century COE Program
Development of Dynamic Mathematics with High Functionality

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MHF 2004-9

(Received March 18, 2004)

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Hypergeometric solutions to the q-Painlevé equations

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Abstract

Hypergeometric solutions to seven q-Painlevé equations in Sakai's classification are constructed. Geometry of plane curves is used to reduce the q-Painlevé equations to the three-term recurrence relations for q-hypergeometric functions.

1 Introduction

It is well-known that the continuous Painlevé equations $\mathrm{P_J}$ (J=II,...,VI) admit particular solutions expressible in terms of various hypergeometric functions. The coalescence cascade of hypergeometric functions, from the Gauss hypergeometric function to the Airy function, corresponds to that of Painlevé equations, from $\mathrm{P_{VI}}$ to $\mathrm{P_{II}}$ [1]. The similar situation is expected for the discrete Painlevé equations.

The discrete Painlevé equations and their solutions have been studied for many years from various view points. In particular, Sakai [2] gave a natural framework of discrete Painlevé equations by means of geometry of rational surfaces. Among the 22 types in Sakai's classification, there are ten types of q-Painlevé equations corresponding to the following degeneration diagram of type of affine Weyl groups:

$$\begin{array}{l} E_8^{(1)} \to E_7^{(1)} \to E_6^{(1)} \to D_5^{(1)} \to A_4^{(1)} \to (A_2{+}A_1)^{(1)} \to (A_1{+}A_1')^{(1)} \to A_1^{(1)} \to A_0^{(1)} \\ \hspace{24.5em} \searrow \\ \hspace{25.5em} A_1^{(1)\prime} \end{array}$$

In this paper, we construct hypergeometric solutions to the first seven q-Painlevé equations. (The last three cases are irrelevant to the hypergeometric solutions.)

Usually, in order to construct hypergeometric solutions we first look for special situations in which the discrete Painlevé equation is reducible to the discrete Riccati equation, and linearize it into second order linear difference equations. Then we identify the linear equations with the three-term relations of appropriate hypergeometric functions. Reduction to the discrete Riccati equations has been found for all of the q-Painlevé equations [3, 4]. But so far full step of the above procedure has been carried out only for q-$\mathrm{P_{VI}}$ ($D_5^{(1)}$) [5], q-$\mathrm{P_{IV}}$ [6] and q-$\mathrm{P_{III}}$ [7] ($(A_2+A_1)^{(1)}$) due to technical difficulty.

In the previous paper [8], we studied the elliptic Painlevé equation, the master equation of all the Painlevé and discrete Painlevé equations. There we gave an algebraic formulation of τ functions and a geometric description of the equation in terms of plane curves. We also showed that it admits elliptic hypergeometric function ${}_{10}E_9$ as a hypergeometric solution. The q-Painlevé equation with affine Weyl group symmetry of type $E_8^{(1)}$ can be regarded naturally as a limiting case of the elliptic Painlevé equation. The construction of this paper provides explicit hypergeometric solutions for all the seven types of q-Painlevé equations.

In Section 2 we give a geometric method to construct hypergeometric solutions for discrete Painlevé equations. By using this method, we identify the hypergeometric functions appearing for each q-Painlevé equation in Section 3. In Section 4 we give the list of q-Painlevé equations and their explicit hypergeometric solutions.

2 Decoupling in terms of invariants

Consider the discrete Riccati equation

$$\overline{x} = \frac{ax+b}{cx+d}. \tag{1}$$

Here $x = x(t)$ is the unknown variable and $\overline{x} = x(qt)$. We will also use the notation $\underline{x} = x(t/q)$. In general, the coefficients a, b, c, d also depend on t. Putting $x = F/G$, the Riccati equation is decoupled into two linear equations for F and G (the contiguity relations):

$$h\overline{F} = aF + bG, \quad h\overline{G} = cF + dG, \tag{2}$$

where h is a decoupling factor. We then have the following three-term recurrence relations:

$$\begin{aligned} h\underline{h}\underline{b}\overline{F} - \underline{h}(a\underline{b} + b\underline{d})F + b(\underline{a}\underline{d} - \underline{b}\underline{c})\underline{F} = 0, \\ h\underline{h}\underline{c}\overline{G} - \underline{h}(d\underline{c} + c\underline{a})G + c(\underline{a}\underline{d} - \underline{b}\underline{c})\underline{G} = 0. \end{aligned} \tag{3}$$

Our task is to solve the Riccati equation (1) through eqs. (2) and (3). For the Riccati equation (1) arising from q-Painlevé equations with higher symmetries, the coefficients of eqs. (3) are polynomials depending on many parameters. By suitable choice of decoupling factor h and gauge factor g : $F = g\Phi$, eqs. (3) are expected to reduce to some q-hypergeometric equations which typically take the form

$$A\overline{\Phi} + (B - A - C)\Phi + C\underline{\Phi} = 0. \tag{4}$$

Here the coefficients A, B, C are of compact factorized form, but, $B - A - C$ is not. Accordingly, the second coefficients in eqs.(3) consist of huge number of terms (more than hundred for $E_8^{(1)}$). This is the main technical difficulty to manipulate these equations. Our basic strategy to overcome this difficulty is to express these coefficients in terms of invariants such as determinants. To do this, the geometric formulation of the discrete Painlevé equations developed in the previous paper [8] is useful.

Consider a configuration of nine points $P_1, \ldots, P_9$ in $\mathbb{P}^2$. We denote by C_0 the unique cubic curve passing through them. In the context of discrete Painlevé equations, the nine points $P_1, \ldots, P_9$ play the role of parameters for the difference equations; some of them may be regarded as independent variables. An additional generic point P_{10} is regarded as the dependent variable. The commuting family of the time evolutions T_{ij}, the translation associated with a pair of points (P_i, P_j) $(i, j = 1, \ldots, 9; i \neq j)$, is described as follows. Let us take T_{89} as an example. Under the translation T_{89}, the points P_i $(i \neq 8, 9, 10)$ are invariant and the new points $\overline{P}_8$ and $\overline{P}_9$ are determined so that

$$P_8 + P_9 = \overline{P}_8 + \overline{P}_9, \quad P_1 + \cdots + P_7 + P_8 + \overline{P}_9 = 0, \tag{5}$$

with respect to the addition on the cubic C_0, where $\overline{P}_j = T_{89}(P_j)$. This means that $\overline{P}_9$ is the additional intersection point of the pencil (one parameter family) of cubic curves defined by the eight points P_i $(i \neq 9, 10)$. Using this pencil of cubics, the transformation $T_{89}(P_{10})$ is geometrically described as follows. Consider a cubic curve C passing through the nine points P_i $(i \neq 9)$. The new point $\overline{P}_{10}$ is determined by

$$\overline{P}_{10} + \overline{P}_9 = P_{10} + P_8, \tag{6}$$

with respect to the addition on the curve C.

In view of configuration of nine points $P_1, \ldots, P_9$, there are two typical situations where the dynamical system admits reduction to discrete Riccati equations.

(1) The case where three points are collinear.
(2) The case where a point is infinitely near to another.

For each case we construct below the corresponding Riccati equation and its linearization.

Case (1): This case is further divided into two types depending on the choice of the translation T_{ab} and the three points P_l, P_m, P_n lying on a line ℓ. Namely, (1a) $\{a,b\} \cap \{l,m,n\} = \phi$ and (1b) $\{a,b\} \subset \{l,m,n\}$. In both cases, if $P_{10} \in \ell$ then $\overline{P}_{10} = T_{ab}(P_{10}) \in \ell$ and the motion of P_{10} is described by a discrete Riccati equation on the line ℓ.

(1a): Consider the case when the three points P_5, P_6, P_7 are on a line ℓ and the translation is T_{89} as an example. In such a case, $P_{10} \in \ell \Rightarrow \overline{P}_{10} \in \ell$ follows from the fact that the curve C is decomposed into the line ℓ (passing through $P_5, P_6, P_7, \overline{P}_{10}$) and the conic C_2 (passing through $P_1, P_2, P_3, P_4, P_8, \overline{P}_9$).

Proposition 2.1 *The time evolution of the point P_{10} under T_{89} is determined by the linear equation*

$$\begin{aligned} &\frac{d_{jk9}d_{ki8}d_{ij\overline{9}}d_{568}}{d_{i8\overline{9}}}\left(\lambda_{\{ijk\}}\frac{d_{ik\overline{9}}}{d_{ik8}}d_{jk\overline{10}} - d_{jk10}\right) \\ &+\frac{d_{jk8}d_{ki9}d_{ij\underline{8}}d_{569}}{d_{i9\underline{8}}}\left(\mu_{\{ijk\}}\frac{d_{ik\underline{8}}}{d_{ik9}}d_{jk\underline{10}} - d_{jk10}\right) = d_{ijk}d_{k89}d_{56j}d_{jk10}, \end{aligned} \tag{7}$$

where $\{ijk\} \subset \{1234\}$ and $d_{abc} = \det[P_a, P_b, P_c]$. The gauge factors $\lambda_{\{ijk\}}$ and $\mu_{\{ijk\}}$ can be chosen as follows:

$$\begin{aligned} &\lambda_{\{123\}} = 1, && \lambda_{\{124\}} = \frac{(14)}{(13)} = \frac{(24)}{(23)}, \\ &\lambda_{\{134\}} = \frac{(14)}{(12)} = \frac{(34)}{(32)}, && \lambda_{\{234\}} = \frac{(24)}{(21)} = \frac{(34)}{(31)}, \\ &(ij) = \frac{d_{ij8}}{d_{ij\overline{9}}}, && \mu_I = \lambda_I\big|_{8\to 9,\overline{9}\to\underline{8}}. \end{aligned} \tag{8}$$

[Proof] When $\{ijk\} = \{123\}$, eq.(7) is reformulation of [8], Proposition 4.2 in terms of determinants. The same proof can be applied for other choice of $\{ijk\} \subset \{1234\}$. The only point we should take care is the gauge factors λ_I, μ_I that are symmetric with respect to the indices $I = \{ijk\}$. By choosing the relative normalization of homogeneous coordinates of $\overline{P}_{10}$, P_{10} and $\underline{P}_{10}$, one can put $\lambda_{\{123\}} = \mu_{\{123\}} = 1$. Then we should determine the other factors λ_I and μ_I, $I = \{124\}, \{134\}, \{234\}$ consistently. To do this, let us consider eqs.(7) for $(i,j,k) = (1,2,3)$ and $(i,j,k) = (4,2,3)$, both with the same unknown variable $d_{2,3,10}$. Comparing the corresponding coefficients, we have

$$c_{\{123\}}\lambda_{\{123\}}\frac{d_{13\overline{9}}}{d_{138}} = c_{\{423\}}\lambda_{\{423\}}\frac{d_{43\overline{9}}}{d_{438}}, \quad c_{ijk} = \frac{d_{jk9}d_{ki8}d_{ij\overline{9}}d_{568}}{d_{i8\overline{9}}d_{ijk}d_{k89}d_{56j}}. \tag{9}$$

Using the fact that P_1, P_2, P_3, P_4, P_8 and $\overline{P}_9$ are on the conic C_2, we have

$$\frac{c_{123}}{c_{423}} = \frac{d_{381}d_{2\overline{9}1}d_{8\overline{9}4}d_{234}}{d_{384}d_{2\overline{9}4}d_{8\overline{9}1}d_{231}} = 1. \tag{10}$$

Hence,

$$\lambda_{\{423\}} = \lambda_{\{123\}}\frac{d_{13\overline{9}}}{d_{138}}\frac{d_{438}}{d_{43\overline{9}}} = \frac{(34)}{(31)} = \frac{(24)}{(21)}, \tag{11}$$

where the last equality also follows from the conic condition like eq.(10). Other factors λ_I, μ_I can be determined similarly. □

(1b): Consider next the case when the three points P_7, P_8, P_9 are on a line ℓ and translation is T_{89} for instance. Then $\overline{P}_{10} \in \ell$ whenever $P_{10} \in \ell$. This fact follows from

$$\begin{aligned} P_8 + P_9 &= -P_7 = \overline{P}_8 + \overline{P}_9, \\ P_8 + P_{10} &= -P_7 = \overline{P}_9 + \overline{P}_{10}. \end{aligned} \tag{12}$$

For a point P with homogeneous coordinates $(x:y:z)$, we set

$$m(P) = [x^3, x^2y, x^2z, xy^2, xyz, xz^2, y^3, y^2z, yz^2, z^3]^t. \tag{13}$$

Then $\overline{P}_{10}$ is determined as a point on ℓ such that

$$X = \det[m(P_1), \ldots, m(P_8), m(P_{10}), m(\overline{P}_{10})] = 0. \tag{14}$$

We parameterize the points $P_{10}, \overline{P}_{10} \in \ell$ by setting

$$\begin{aligned} P_{10} &= d_{179} P_8\, v - d_{178} P_9 = d_{189} P_7 + d_{179}(v-1) P_8, \\ \overline{P}_{10} &= d_{17\overline{9}} \overline{P}_8\, \overline{v} - d_{17\overline{8}} \overline{P}_9 = d_{1\overline{89}} \overline{P}_7 + d_{17\overline{9}}(\overline{v}-1) \overline{P}_8. \end{aligned} \tag{15}$$

The last equality follows from the identity $d_{ijk}P_l - d_{jkl}P_i + d_{kli}P_j - d_{lij}P_k = 0$ and $d_{789} = d_{7\overline{89}} = 0$. The coordinate v is chosen so that the three points $P_{10} = P_9, P_7, P_8$ correspond to $v = 0, 1, \infty$, respectively (and similar for $\overline{v}$).

Proposition 2.2 *The coordinate $\overline{v}$ of $\overline{P}_{10}$ is determined by the Riccati equation*

$$\overline{v} = -\frac{A_1 + A_2\, v}{A_4\, v}. \tag{16}$$

In terms of the variable F such that $v = F/\underline{F}$, we obtain the three-term recurrence relation

$$A_4(\overline{F} - F) + A_1(\underline{F} - F) + A_5 F = 0. \tag{17}$$

The explicit forms of the coefficients A_i are given in the proof.

[Proof] The determinant X in eq.(14) is at most cubic with respect to both the variables v and $\overline{v}$. It has trivial zeros at $v = 1, \infty$ and $\overline{v} = 0, 1$ corresponding to $P_{10} = P_7, P_8$ and $\overline{P}_{10} = \overline{P}_9, P_7$. Therefore, X is factorized in the form

$$X = (v-1)(\overline{v}-1)\overline{v}(A_1 + A_2\, v + A_3\, \overline{v} + A_4\, v\, \overline{v}). \tag{18}$$

First consider A_3. Comparing the coefficient $v^0 \overline{v}^3$ of X, we have

$$A_3 = \det[m(P_1), \ldots, m(P_8), -{d_{178}}^3 m(P_9), {d_{17\overline{9}}}^3 m(\overline{P}_8)] = 0. \tag{19}$$

Next, the coefficient A_1 is determined as

$$A_1 = -\frac{\partial X}{\partial \overline{v}}\Big|_{v=0, \overline{v}=1} = {d_{178}}^3 {d_{1\overline{89}}}^2 d_{17\overline{9}} \det[m(P_1), \ldots, m(P_9), dm(P_7, \overline{P}_8)]. \tag{20}$$

Here, we have used the relations

$$\begin{aligned} &m(P_{10}) = -{d_{178}}^3 m(P_9) + O(v), \\ &m(\overline{P}_{10}) = {d_{1\overline{89}}}^3 m(P_7) + (\overline{v}-1) {d_{1\overline{89}}}^2 d_{17\overline{9}} dm(P_7, \overline{P}_8) + O((\overline{v}-1)^2), \\ &dm(P, Q) = \frac{d}{d\epsilon} m(P + \epsilon\, Q)\Big|_{\epsilon=0}. \end{aligned} \tag{21}$$

Similarly, the coefficient A_4 is given by

$$\begin{aligned} A_4 \quad &= \frac{\partial}{\partial v}\left(\text{coefficient of } \overline{v}^3 \text{ in } X\right)\Big|_{v=1} \\ &= {d_{17\overline{9}}}^3 {d_{189}}^2 d_{179} \det[m(P_1), \ldots, m(P_8), dm(P_7, P_8), m(\overline{P}_8)]. \end{aligned} \tag{22}$$

Finally, for the coefficient A_2, it is rather convenient to consider the combination $A_5 = A_1 + A_2 + A_4$, which is determined by

$$\begin{aligned} A_5 \quad &= \frac{\partial^2 X}{\partial v \partial \overline{v}}\Big|_{v=\overline{v}=1} \\ &= {d_{1\overline{89}}}^2\, {d_{189}}^2\, d_{17\overline{9}}\, d_{179} \det[m(P_1), \ldots, m(P_8), dm(P_7, P_8), dm(P_7, \overline{P}_8)]. \end{aligned} \tag{23}$$

The Riccati equation (16) is derived from $X = 0$. Since $A_3 = 0$, it is easily decoupled into eq.(17). □

Case (2) Consider the case where P_9 is an infinitely near point of P_8 ($P_9 \to P_8$). If P_{10} is also infinitely near to P_8, then so is the translation $\overline{P}_{10} = T_{ab}(P_{10})$ $(a, b \neq 8, 9)$. The point $\overline{P}_{10}$ is determined by solving the following system of algebraic equations

$$\begin{aligned}
&\det[P_a, P_{10}, Q] = 0, \qquad \det[P_{\overline{b}}, P_{\overline{10}}, Q] = 0,\\
&\det[m(P_1), \ldots, \widehat{m(P_b)}, \ldots, m(P_8), dm(P_8, P_9), m(P_{10}), m(Q)]\big|_{\epsilon^3} = 0,\\
&\det[m(P_1), \ldots, \widehat{m(P_b)}, \ldots, m(P_8), dm(P_8, P_9), m(\overline{P}_{10}), m(Q)]\big|_{\epsilon^3} = 0,
\end{aligned} \tag{24}$$

including an intermediate infinitely near point Q, where

$$P_{10} = P_8 + \epsilon \begin{bmatrix} 0 \\ u \\ v \end{bmatrix}, \ \overline{P}_{10} = P_8 + \epsilon \begin{bmatrix} 0 \\ \overline{u} \\ \overline{v} \end{bmatrix}, \ Q = P_8 + \epsilon \begin{bmatrix} 0 \\ r \\ s \end{bmatrix}, \tag{25}$$

with an infinitely small parameter ϵ, and $f\big|_{\epsilon^n}$ stands for the coefficient of ϵ^n in the Taylor expansion of f at $\epsilon = 0$. Eq.(24) can be reduced to a linear relation for the homogeneous variables $(u : v)$ and $(\overline{u} : \overline{v})$. More precisely, there are four solutions to eq.(24), three of them are trivial ones: $P_{10} = Q$, $\overline{P}_{10} = Q$ or $P_{10} = \overline{P}_{10}$, and remaining one gives a linear relation between P_{10} and $\overline{P}_{10}$ which is in fact the Riccati equation. The variables $(u : v)$ etc. in eq.(25) represent the slope of the line P_8P_{10} etc, in a (temporary) coordinate of $\mathbb{P}^1$ (the exceptional curve which is the blown up of P_8). It is convenient to make a change of coordinates $(u : v) \to (U : V) = (au + bv : cu + dv)$ in such a way that the lines P_8P_a, P_8P_b, P_8P_9 correspond to $(0 : 1), (1 : 0), (1 : 1)$, respectively. Then, in these coordinates, we obtain a three-term recurrence relation for $F = U/V$ in the form of eq.(17) with factorized coefficients.

3 Identification of the Hypergeometric Functions

We apply the results in the preceding section to special configurations of points, and derive the corresponding hypergeometric special solutions to each of the q-Painlevé equations in the degeneration diagram [2]

$$E_8^{(1)} \to E_7^{(1)} \to E_6^{(1)} \to D_5^{(1)} \to A_4^{(1)} \to (A_2{+}A_1)^{(1)} \to (A_1{+}A_1')^{(1)}. \tag{26}$$

Let us recall the definition and terminology of the q-hypergeometric series [9]. The q-hypergeometric series ${}_r\varphi_s$ is given by,

$$\begin{gathered}
{}_r\varphi_s \left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_r; q)_n}{(b_1; q)_n \cdots (b_s; q)_n (q; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n,\\
(a; q)_n = (1-a)(1-qa)\cdots(1-q^{n-1}a).
\end{gathered} \tag{27}$$

The q-hypergeometric series ${}_{r+1}\varphi_r$ is called *balanced* [1] if the condition

$$qa_1a_2\cdots a_{r+1} = b_1b_2\cdots b_r, \quad z = q, \tag{28}$$

is satisfied, and is called *well-poised* if the condition

$$qa_1 = a_2b_1 = \cdots = a_{r+1}b_r, \tag{29}$$

is satisfied. Moreover, it is called *very-well-poised* if it satisfies

$$a_2 = qa_1^{\frac{1}{2}}, \quad a_3 = -qa_1^{\frac{1}{2}}, \tag{30}$$

[1]For ${}_3\varphi_2$ series, it appears that two different conventions are used in the literature. This convention is due to [9], while the series ${}_3\varphi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix} ; q, z \right)$ with $z = de/abc$ is also called "balanced ${}_3\varphi_2$ series" in [12]. For ${}_3\varphi_2$ series, we use latter convention without notice.

in addition to eq.(29), and denoted as ${}_{r+1}W_r$:

$$
{}_{r+1}W_r(a_1;a_4,\ldots,a_{r+1};q,z) = {}_r\varphi_s\left(\begin{array}{c} a_1, qa_1^{\frac{1}{2}}, -qa_1^{\frac{1}{2}}, a_4\ldots, a_{r+1} \\ a_1^{\frac{1}{2}}, -a_1^{\frac{1}{2}}, qa_1/a_4, \ldots, qa_1/a_{r+1} \end{array} ;q,z\right). \tag{31}
$$

The degeneration diagram of q-Painlevé equations (26) corresponds to that of q-hypergeometric series:

$$
\begin{array}{c}\text{balanced}\\ {}_{10}W_9\end{array} \to {}_8W_7 \to \begin{array}{c}\text{balanced}\\ {}_3\varphi_2\end{array} \to {}_2\varphi_1 \to {}_1\varphi_1 \to \begin{array}{c} {}_1\varphi_1\left(\begin{array}{c} a \\ 0 \end{array};q,z\right) \\ {}_1\varphi_1\left(\begin{array}{c} 0 \\ b \end{array};q,z\right)\end{array} \to {}_1\varphi_1\left(\begin{array}{c} 0 \\ -q \end{array};q,z\right). \tag{32}
$$

3.1 Case $E_8^{(1)}$

In this case we take the configuration of nine points lying on a nodal cubic curve C_0. We can parameterize the nine points $P_i = P(u_i)$ as follows:

$$
P(u) = \Big((1-ubc)(1-u/b)(1-u/c) : (1-uca)(1-u/c)(1-u/a) : (1-uab)(1-u/a)(1-u/b)\Big). \tag{33}
$$

The function $P(u)$ parameterize a nodal cubic C_0 passing through $(1:0:0), (0:1:0), (0:0:1)$ with a node at $(1:1:1)$. Then the determinant d_{ijk} is given by

$$
d_{ijk} = [ij][ik][jk][ijk]d_0, \tag{34}
$$

where $[ij] = u_i - u_j$, $[ijk] = 1 - u_iu_ju_k$ and d_0 is a constant independent of u.

We apply Case (1a) where P_5, P_6, P_7 are collinear ($u_5u_6u_7 = 1$) and $T = T_{89}$. Putting $(i,j,k) = (1,2,3)$ and substituting the determinants (34) in the three-term recurrence relation (7), we obtain the linear equation for $F = d_{2,3,10}$

$$
\begin{aligned}
& A(l\overline{F} - F) + BF + C(m\underline{F} - F) = 0, \\
& \frac{A}{B} = \frac{[58][68][138][568][29][2\overline{9}][12\overline{9}][239]}{[25][26][123][256][89][8\overline{9}][18\overline{9}][389]}, \quad l = \frac{[1\overline{9}][3\overline{9}][13\overline{9}]}{[18][38][138]}, \\
& \left(\frac{C}{B}, m\right) = \left(\frac{A}{B}, l\right)\Big|_{8\leftrightarrow 9, \underline{8}\leftrightarrow\overline{9}},
\end{aligned} \tag{35}
$$

where $u_{\overline{9}} = qu_9$,$u_{\underline{8}} = qu_8$ and $qu_1u_2u_3u_4u_8u_9 = 1$. The parameters u_i are transformed by T_{89} as $u_9 \mapsto qu_9$, $u_8 \mapsto q^{-1}u_8$ while the other u_i are invariant. After the gauge transformation $F = g\,\Phi$ with $\overline{g} = l^{-1}g$, eq.(35) is solved by the balanced ${}_{10}W_9$ series [10]:

$$
\begin{aligned}
\Phi \quad &= \Phi(a_0; a_1, \ldots, a_7; q) \\
&= {}_{10}W_9(a_0; a_1, \ldots, a_7; q, q) \\
&\quad + \frac{(qa_0, a_7/a_0; q)_\infty}{(a_0/a_7, qa_7^2/a_0; q)_\infty} \prod_{k=1}^{6} \frac{(a_k, qa_7/a_k; q)_\infty}{(qa_0/a_k, a_ka_7/a_0; q)_\infty} \\
&\qquad \times {}_{10}W_9(a_7^2/a_0; a_1a_7/a_0, \ldots, a_6a_7/a_0, a_7; q, q),
\end{aligned} \tag{36}
$$

with the balancing condition $q^2a_0^3 = a_1a_2\cdots a_7$. The parameters a_i $(i = 0, 1, \ldots, 7)$ are given by

$$
\begin{aligned}
& a_0 = u_2^2u_3/q, \quad a_1 = u_1u_2u_3, \quad a_2 = u_2u_3u_4, \quad a_3 = u_2/u_5, \\
& a_4 = u_2/u_6, \quad a_5 = u_2u_5u_6, \quad a_6 = u_2u_3u_8, \quad a_7 = u_2u_3u_9.
\end{aligned} \tag{37}
$$

When one of the parameters $a_1, a_2, \ldots, a_6$ is q^{-N} $(N \in \mathbb{Z}_{\geq 0})$ the second term of eq.(36) vanishes, which can also be derived as the trigonometric limit of our previous result [8].

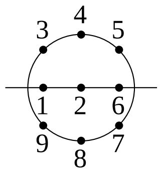

Figure 1: $E_7^{(1)}$

3.2 Case $E_7^{(1)}$

In this case we consider the configuration of nine points P_i in $\mathbb{P}^2$ among which three ($i = 1, 2, 6$) are on a line and six ($i = 3, 4, 5, 7, 8, 9$) are on a conic (Fig.1). We parameterize those nine points as follows:

$$P_i = \begin{cases} (-u_i : 0 : 1) & (i = 1, 2, 6) \\ (1 : u_i : u_i^2) & (i = 3, 4, 5, 7, 8, 9) \end{cases}. \tag{38}$$

Putting P_5, P_6, P_7 to be collinear ($u_5u_6u_7 = 1$), we again apply Case (1) with $(i, j, k) = (1, 2, 3)$. Then we obtain the three-term recurrence relation (7) with respect to T_{89}:

$$\begin{aligned} &A(l\overline{F} - F) - BF + C(m\underline{F} - F) = 0, \\ &\frac{A}{B} = \frac{qu_9[58][138][568][239]}{u_3u_5[26][89][8\overline{9}][18\overline{9}]}, \quad l = \frac{[3\overline{9}][13\overline{9}]}{[38][138]}, \\ &\left(\frac{C}{B}, m\right) = \left(\frac{A}{B}, l\right)\Big|_{8\leftrightarrow 9, \underline{8}\leftrightarrow\overline{9}}, \end{aligned} \tag{39}$$

where $F = d_{2,3,10}$ and $qu_1u_2u_3u_4u_8u_9 = 1$. The action of $T = T_{89}$ on the parameters u_i are the same as $E_8^{(1)}$ case.

By the gauge transformation $F = g\,\Phi$ with $\overline{g} = l^{-1}\dfrac{[7\overline{9}][258]}{[78][25\overline{9}]}\,g$, eq.(39) is solved by the ${}_8W_7$ series [11],

$$\Phi \quad = {}_8W_7(a_0; a_1, \ldots, a_5; q, z), \quad z = \frac{q^2a_0^2}{a_1a_2a_3a_4a_5}, \tag{40}$$

where the parameters a_i $(i = 0, 1, \ldots, 5)$ are given by

$$\begin{aligned} &a_0 = u_2u_5^2, \quad a_1 = u_5/u_8, \quad a_2 = u_5/u_9, \\ &a_3 = qu_5/u_3, \quad a_4 = u_5/u_4, \quad a_5 = u_2/u_6. \end{aligned} \tag{41}$$

3.3 Case $E_6^{(1)}$

In this case the nine points are divided into three groups of three points that are collinear (Fig.2). The parameterization is given as follows:

$$P_i = \begin{cases} (1 : -u_i : 0) & (i = 1, 2, 6) \\ (0 : 1 : -u_i) & (i = 3, 4, 7) \\ (-u_i : 0 : 1) & (i = 5, 8, 9) \end{cases}. \tag{42}$$

We again consider the Case (1a) with P_5, P_6, P_7 being collinear and $T = T_{89}$. Then the three-term relation (7) with $(i, j, k) = (1, 2, 3)$ implies

$$\frac{[58][138][239]}{u_1u_3[8\overline{9}]}\left(\frac{[13\overline{9}]}{[138]}\overline{F} - F\right) + \frac{[59][238][139]}{u_1u_3[\underline{8}9]}\left(\frac{[13\underline{8}]}{[139]}\underline{F} - F\right) = -[257][89]F, \tag{43}$$

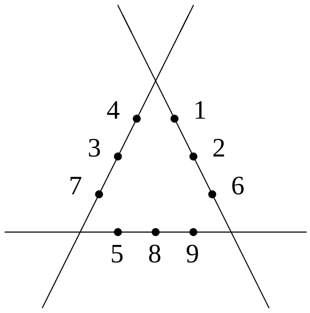

Figure 2: $E_6^{(1)}$

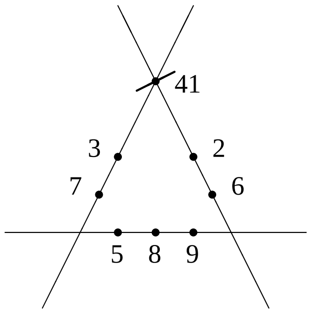

Figure 3: $D_5^{(1)}$

where $F = d_{2,3,10}$. This equation is solved by the balanced ${}_3\varphi_2$ series [12]

$$F = {}_3\varphi_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix} ; q, \frac{b_1 b_2}{a_1 a_2 a_3}\right), \tag{44}$$

where the parameters are given by

$$\begin{gathered} a_1 = u_3/u_7, \quad a_2 = u_2 u_3 u_8, \quad a_3 = u_2 u_3 u_9, \\ b_1 = u_2 u_3 u_5, \quad b_2 = q u_1 u_2 u_3^2 u_8 u_9. \end{gathered} \tag{45}$$

3.4 Case $D_5^{(1)}$

This is a limiting case where $P_1 = (\epsilon : 1 : -u_1\epsilon)|_{\epsilon\to 0}$ becomes infinitely near to $P_4 = (0 : 1 : 0)$, while the other P_i ($i \neq 1, 4$) are the same as the Case $E_6^{(1)}$ (Fig.3). Accordingly, the corresponding linear difference equation is

$$\frac{u_8[58][239]}{[8\overline{9}]}\left(\frac{qu_9}{u_8}\overline{F} - F\right) + \frac{u_9[59][238]}{[\underline{8}9]}\left(\frac{qu_8}{u_9}\underline{F} - F\right) = [257][89]F, \tag{46}$$

with $u_5 u_6 u_7 = 1$ and $q u_1 u_2 u_3 u_8 u_9 = 1$. This equation is solved by the ${}_2\varphi_1$ series [5]:

$$F = g\, {}_2\varphi_1\left(\begin{matrix} a_1, a_2 \\ b_1 \end{matrix} ; q, z\right), \quad \overline{g} = \frac{1 - u_5/qu_9}{1 - u_5/u_8}\, g, \tag{47}$$

where the parameters are given by

$$a_1 = u_2 u_3 u_8, \quad a_2 = u_2 u_3 u_9, \quad b_1 = q u_2 u_3 u_8 u_9/u_5, \quad z = q u_7/u_3. \tag{48}$$

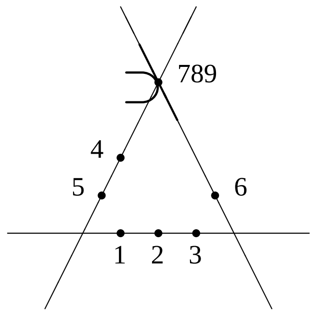

Figure 4: $A_4^{(1)}$

3.5 Case $A_4^{(1)}$

This case is a further degeneration of Case $D_5^{(1)}$:

$$\begin{array}{lll} P_1 = (1:0:1) & P_2 = (a_2:0:1) & P_3 = (a_1a_2:0:1) \\ P_4 = (0:1:1) & P_5 = (0:1:a_4) & P_6 = (1:-a_3:0) \\ P_7 = (0:1:0) & P_8 = (\epsilon:1:0) & P_9 = (\epsilon:1:\dfrac{a_0}{a_2}\epsilon^2) \end{array}, \tag{49}$$

where ϵ is an infinitesimal parameter and $a_0a_1\cdots a_4 = q$. This configuration contains a sequence of infinitely near points $P_9 \to P_8 \to P_7$, while (P_1, P_2, P_3), (P_4, P_5, P_7) and (P_6, P_7, P_8) are collinear (Fig.4).

Consider the case where P_1, P_5, P_6 are collinear ($a_3a_4 = 1$) and the time evolution is $T = T_{56}$ ($a_3 \mapsto a_3/q,\ a_4 \mapsto a_4q$). This situation corresponds to the Case (1b). Applying the Proposition 2.2, we obtain the linear equation,

$$\frac{a_2}{a_0}(a_3 - q)(\overline{F} - F) = (\underline{F} - F) + (1 - a_2)(1 - a_1a_2)F. \tag{50}$$

This equation is solved by

$$F = g\,{}_2\varphi_1\left(\begin{array}{c} a_0, a_0a_1 \\ 0 \end{array}; q, qa_4\right), \quad \overline{g} = \frac{a_0/a_2}{1 - a_3/q}\,g. \tag{51}$$

We note that the above solution can be rewritten in terms of ${}_1\varphi_1$ series by using the relation [13]

$${}_2\varphi_1\left(\begin{array}{c} a, b \\ 0 \end{array}; q, z\right) = \frac{(bz;q)_\infty}{(z;q)_\infty}\,{}_1\varphi_1\left(\begin{array}{c} b \\ bz \end{array}; q, az\right). \tag{52}$$

3.6 Case $(A_2+A_1)^{(1)}$

This case is a further degeneration of Case $A_4^{(1)}$ such that $P_5 \to P_3$ (Fig.5):

$$\begin{array}{lll} P_1 = (1:0:1) & P_2 = (a_1:0:1) & P_3 = (0:0:1) \\ P_4 = (0:1:1) & P_5 = (-\dfrac{b_1}{a_2}\epsilon:\epsilon:1) & P_6 = (1:-a_2:0) \\ P_7 = (0:1:0) & P_8 = (\epsilon:1:0) & P_9 = (\epsilon:1:\dfrac{b_0}{a_1}\epsilon^2) \end{array}, \tag{53}$$

where $a_0a_1a_2 = b_0b_1 = q$.

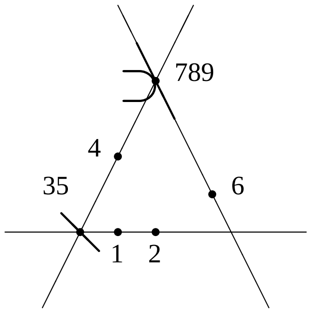

Figure 5: $(A_2{+}A_1)^{(1)}$

In this case there are two different kinds of time evolutions, which are referred to as q-P$_{\rm III}$ and q-P$_{\rm IV}$, respectively. The actions on the parameters are given by

		a_1	a_2	a_0	b_1	b_0
q-P$_{\rm III}$	T_{92}	a_1/q	a_2	$a_0 q$	b_1	b_0
	T_{56}	a_1	$a_2 q$	a_0/q	b_1	b_0
q-P$_{\rm IV}$	T_{59}	a_1	a_2	a_0	qb_1	b_0/q

. (54)

In the case of q-P$_{\rm III}$, when (P_3, P_5, P_6) is collinear ($b_1 = 1$), the linear equation with respect to T_{92} is derived by taking $(i, j, k) = (1, 4, 7)$ in eq.(7) of Proposition 2.1:

$$a_1(T_{92}(F) - F) + a_1 a_2(T_{92}^{-1}(F) - F) + F = 0, \quad F = d_{4,7,10}. \tag{55}$$

This equation is solved by Jackson's q-Bessel function [9]

$$F = {}_1\varphi_1\left(\begin{array}{c} 0 \\ q/a_2 \end{array} ; q, a_0\right). \tag{56}$$

For the case of q-P$_{\rm IV}$, taking $P_2 \to P_1$ ($a_1 = 1$), we get the linear equation with respect to T_{59},

$$a_2 T_{59}^{-1}(F) - b_1 T_{59}(F) - (1 - b_1)F = 0. \tag{57}$$

According to the argument of Case (2), this equation is obtained by taking $F/T_{95}^{-1}(F)$ as the inhomogeneous coordinate of $\mathbb{P}^1$ such that 0 and ∞ correspond to the lines P_1P_5 and P_1P_9, respectively. The above equation is solved by

$$F = {}_1\varphi_1\left(\begin{array}{c} a_2 \\ 0 \end{array} ; q, b_0\right). \tag{58}$$

3.7 Case $(A_1{+}A_1')^{(1)}$

This case is obtained from Case $(A_2{+}A_1)^{(1)}$ by taking $P_6 \to P_2$ (Fig.6):

$$\begin{array}{lll} P_1 = (1:0:1) & P_2 = (1:0:0) & P_3 = (0:0:1) \\ P_4 = (0:1:1) & P_5 = (\frac{a_0}{b}\epsilon : \epsilon : 1) & P_6 = (1 : a_1\epsilon : \epsilon) \\ P_7 = (0:1:0) & P_8 = (\epsilon : 1 : 0) & P_9 = (\epsilon : 1 : -b\epsilon^2) \end{array}, \tag{59}$$

where $a_0 a_1 = q$.

When (P_2, P_4, P_6) is collinear ($a_1 = 1$), we obtain the Riccati equation

$$T_{95}(y) = \frac{b(1-y)}{y}, \quad P_{10} = (1 : y : y), \tag{60}$$

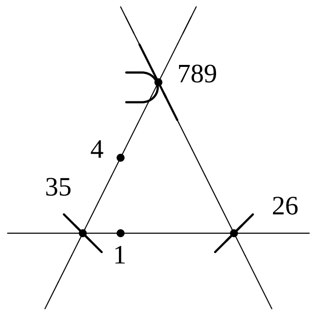

Figure 6: $(A_1+A_1')^{(1)}$

with respect to T_{95} $(b \mapsto bq)$, which is linearized as

$$T_{95}(F) - T_{95}^{-1}(F) + q^{-\frac{1}{4}} b^{\frac{1}{2}} F = 0, \tag{61}$$

through

$$y = q^{-\frac{1}{4}} b^{\frac{1}{2}} \frac{F}{T_{95}^{-1}(F)}. \tag{62}$$

This is solved by a q-analogue of the Airy function

$$F = {}_1\varphi_1 \left(\begin{matrix} 0 \\ -q^{\frac{1}{2}} \end{matrix} ; q^{\frac{1}{2}}, -q^{\frac{1}{4}} b^{\frac{1}{2}} \right). \tag{63}$$

4 Hypergeometric Solutions for q-Painlevé Equations

In the previous section, the relevant hypergeometric functions are identified for each q-Painlevé equation. However, our choice of the dependent variables and parameters is not always the same as in the literature. In this section we give a list of hypergemetric solutions for the q-Painlevé equations in the forms appearing in preceding works. Full details of construction of the solutions will be given in a forthcoming paper.

4.1 Case $E_8^{(1)}$

q-Painlevé Equation [3, 4, 18]

$$\begin{aligned}
\frac{(\overline{g}\overline{s}t - f)(gst - f) - (\overline{s}^2t^2 - 1)(s^2t^2 - 1)}{\left(\dfrac{\overline{g}}{\overline{s}t} - f\right)\left(\dfrac{g}{st} - f\right) - \left(1 - \dfrac{1}{\overline{s}^2t^2}\right)\left(1 - \dfrac{1}{s^2t^2}\right)} &= \frac{P(f, t, m_1, \ldots, m_7)}{P(f, t^{-1}, m_7, \ldots, m_1)}, \\
\frac{(\underline{f}s\underline{t} - g)(fst - g) - (s^2\underline{t}^2 - 1)(s^2t^2 - 1)}{\left(\dfrac{\underline{f}}{s\underline{t}} - g\right)\left(\dfrac{f}{st} - g\right) - \left(1 - \dfrac{1}{s^2\underline{t}^2}\right)\left(1 - \dfrac{1}{s^2t^2}\right)} &= \frac{P(g, s, m_7, \ldots, m_1)}{P(g, s^{-1}, m_1, \ldots, m_7)},
\end{aligned} \tag{64}$$

where

$$\begin{aligned}
P(f, t, m_1, \ldots, m_7) = f^4 - m_1 t f^3 + (m_2 t^2 - 3 - t^8) f^2 \\
+(m_7 t^7 - m_3 t^3 + 2m_1 t) f + (t^8 - m_6 t^6 + m_4 t^4 - m_2 t^2 + 1),
\end{aligned} \tag{65}$$

and m_k $(k = 1, 2, \ldots 7)$ are the elementary symmetric functions of k-th degree in b_i $(i = 1, 2, \ldots, 8)$ with

$$b_1 b_2 \cdots b_8 = 1. \tag{66}$$

Moreover,

$$\overline{t} = qt, \ t = q^{\frac{1}{2}} s. \tag{67}$$

Constraint on Parameters [3]

$$qb_1b_3b_5b_7 = 1, \quad b_2b_4b_6b_8 = q. \tag{68}$$

Hypergeometric Solution A hypergeometric solution is given by

$$\frac{g - \left(\dfrac{s}{b_1} + \dfrac{b_1}{s}\right)}{g - \left(\dfrac{s}{b_8} + \dfrac{b_8}{s}\right)} = \lambda\, \frac{\Phi(q^4a_0; a_1, q^2a_2, \ldots, q^2a_7; q^2)}{\Phi(a_0; a_1 \ldots, a_7; q^2)}, \tag{69}$$

where Φ is the balanced ${}_{10}W_9$ series defined in eq. (36), and a_i $(i = 0, 1, \ldots, 7)$ and λ are given by

$$\begin{aligned} &a_0 = \frac{1}{qb_1b_2b_8^2}, \quad a_1 = \frac{q^2}{b_2b_8t^2}, \quad a_2 = \frac{s^2}{b_2b_8}, \\ &a_i = \frac{b_i}{b_8}\ (i = 3, 5, 7), \quad a_i = \frac{b_i}{b_1}\ (i = 4, 6), \end{aligned} \tag{70}$$

and

$$\lambda = \frac{b_1b_4b_6}{b_8s^2}\, \frac{\left(1 - \dfrac{b_4b_6}{b_1b_8}\right)\left(1 - q^2\dfrac{b_4b_6}{b_1b_8}\right)(1 - b_3b_5t^2)(1 - b_3b_7t^2)(1 - b_5b_7t^2)\displaystyle\prod_{i=2,4,6}\left(1 - \frac{b_i}{b_1}\right)}{\left(1 - \dfrac{s^2}{b_1b_8}\right)\left(1 - \dfrac{q^2s^2}{b_1b_8}\right)\left(1 - \dfrac{b_4}{b_8}\right)\left(1 - \dfrac{b_6}{b_8}\right)\left(1 - \dfrac{q}{b_1b_8s^2}\right)\displaystyle\prod_{i=3,5,7}\left(1 - \frac{b_4b_6}{b_1b_i}\right)}, \tag{71}$$

respectively.

4.2 Case $E_7^{(1)}$

q-Painlevé Equation [3, 4]

$$\left\{\begin{aligned} \frac{(\overline{g}f - t\overline{t})(gf - t^2)}{(\overline{g}f - 1)(gf - 1)} &= \frac{(f - b_1t)(f - b_2t)(f - b_3t)(f - b_4t)}{(f - b_5)(f - b_6)(f - b_7)(f - b_8)}, \\ \frac{(gf - t^2)(g\underline{f} - \underline{t}t)}{(gf - 1)(g\underline{f} - 1)} &= \frac{(g - t/b_1)(g - t/b_2)(g - t/b_3)(g - t/b_4)}{(g - 1/b_5)(g - 1/b_6)(g - 1/b_7)(g - 1/b_8)}, \end{aligned}\right. \tag{72}$$

where

$$\overline{t} = qt, \quad b_1b_2b_3b_4 = q, \quad b_5b_6b_7b_8 = 1. \tag{73}$$

Constraint on Parameters

$$b_1b_3 = b_5b_7. \tag{74}$$

Hypergeometric Solution A hypergeometric solution is given by the ${}_8W_7$ series [11],

$$z = \frac{g - t/b_1}{g - 1/b_5} = \frac{1 - b_3/b_1}{1 - b_3/b_5t}\, \frac{{}_8W_7\left(\dfrac{b_1b_8}{b_3b_5}; \dfrac{qb_8}{b_5}, \dfrac{b_2}{b_3}, \dfrac{b_1t}{b_5}, \dfrac{b_1}{b_5t}, \dfrac{b_4}{b_3}; q, \dfrac{b_5}{b_6}\right)}{{}_8W_7\left(\dfrac{b_1b_8}{b_3b_5}; \dfrac{b_8}{b_5}, \dfrac{b_2}{b_3}, \dfrac{b_1t}{b_5}, \dfrac{b_1}{b_5t}, \dfrac{b_4}{b_3}; q, \dfrac{qb_5}{b_6}\right)}. \tag{75}$$

In the terminating case, e.g. $b_4/b_3 = q^{-N}$ $(N \in \mathbb{Z}_{\geq 0})$, the solution is expressed in terms of the terminating balanced ${}_4\varphi_3$ series (Askey-Wilson polynomials) as

$$z = \frac{1 - b_3/b_1}{1 - b_3/b_5t}\, \frac{{}_4\varphi_3\left(\begin{matrix} b_1/b_2, b_1t/b_5, b_1/b_5t, b_4/b_3 \\ b_1/b_3, b_1b_4/b_5b_6, b_1b_4/b_5b_8 \end{matrix}; q, q\right)}{{}_4\varphi_3\left(\begin{matrix} qb_1/b_2, b_1t/b_5, b_1/b_5t, b_4/b_3 \\ qb_1/b_3, b_1b_4/b_5b_6, b_1b_4/b_5b_8 \end{matrix}; q, q\right)}, \tag{76}$$

by using Watson's transformation formula for the terminating ${}_8W_7$ series [9]

$$
\begin{aligned}
&{}_8W_7(a; b, c, d, e, f; q, q^2a^2/bcdef) \\
&\quad = \frac{(aq, aq/de, aq/df, aq/ef; q)_\infty}{(aq/d, aq/e, aq/f, aq/def; q)_\infty}\, {}_4\varphi_3\left(\begin{matrix} aq/bc, d, e, f \\ aq/b, aq/c, def/a \end{matrix}; q, q\right).
\end{aligned}
\tag{77}
$$

4.3 Case $E_6^{(1)}$

q-Painlevé Equation [3, 4]

$$
\begin{cases}
(\overline{g}f - 1)(gf - 1) = t\bar{t}\, \dfrac{(f - b_1)(f - b_2)(f - b_3)(f - b_4)}{(f - b_5t)(f - t/b_5)}, \\[2ex]
(gf - 1)(g\underline{f} - 1) = t^2\, \dfrac{(g - 1/b_1)(g - 1/b_2)(g - 1/b_3)(g - 1/b_4)}{(g - b_6t)(g - t/b_6)},
\end{cases}
\tag{78}
$$

where

$$
\bar{t} = qt, \quad b_1b_2b_3b_4 = 1. \tag{79}
$$

Constraint on Parameters

$$
b_5b_6 = b_1b_2. \tag{80}
$$

Hypergeometric Solution A hypergeometric solution is given in terms of the balanced ${}_3\varphi_2$ series [12] as

$$
z = \frac{g - 1/b_1}{g - tb_6} = \frac{1 - b_3/b_1}{1 - b_1b_2b_3t/b_5}\, \frac{{}_3\varphi_2\left(\begin{matrix} qb_3b_5/t, b_3/b_2, b_1^2b_2b_3 \\ qb_3b_5^2/b_2, qb_1b_2b_3^2 \end{matrix}; q, b_5t/b_1\right)}{{}_3\varphi_2\left(\begin{matrix} b_3b_5/t, b_3/b_2, qb_1^2b_2b_3 \\ qb_3b_5^2/b_2, qb_1b_2b_3^2 \end{matrix}; q, b_5t/b_1\right)}. \tag{81}
$$

In the terminating case, e.g. $b_3/b_2 = q^{-N}$ $(N \in \mathbb{Z}_{\geq 0})$, the solution can be rewritten in terms of the terminating ${}_3\varphi_2$ series (big q-Jacobi polynomials) as

$$
z = \frac{1 - b_2/b_1}{1 - b_1b_2b_3t/b_5}\, \frac{{}_3\varphi_2\left(\begin{matrix} b_5t/b_2, b_3/b_2, b_1^2b_2b_3 \\ qb_3b_5^2/b_2, b_1/b_2 \end{matrix}; q, q\right)}{{}_3\varphi_2\left(\begin{matrix} b_5t/b_2, b_3/b_2, qb_1^2b_2b_3 \\ qb_3b_5^2/b_2, qb_1/b_2 \end{matrix}; q, q\right)}, \tag{82}
$$

by using the formula [9]

$$
\begin{aligned}
{}_3\varphi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, de/abc\right) &= \frac{(e/b, e/c)_\infty}{(e, e/bc)_\infty}\, {}_3\varphi_2\left(\begin{matrix} d/a, b, c \\ d, qbc/e \end{matrix}; q, q\right) \\
&\quad + \frac{(d/a, b, c, de/bc)_\infty}{(d, e, bc/e, de/abc)_\infty}\, {}_3\varphi_2\left(\begin{matrix} e/b, e/c, de/abc \\ de/bc, qe/bc \end{matrix}; q, q\right).
\end{aligned}
\tag{83}
$$

4.4 Case $D_5^{(1)}$

q-Painlevé Equation (q-Painlevé VI equation) [5, 14]

$$
\begin{cases}
\overline{g}g = \dfrac{(f - a_1t)(f - a_2t)}{(f - a_3)(f - a_4)}, \\[2ex]
f\underline{f} = \dfrac{(g - b_1t/q)(g - b_2t/q)}{(g - b_3)(g - b_4)},
\end{cases}
\tag{84}
$$

where

$$
\frac{b_1b_2}{b_3b_4} = q\frac{a_1a_2}{a_3a_4}. \tag{85}
$$

Constraint on Parameters

$$\frac{b_1}{b_3} = q\frac{a_1}{a_3}, \quad \frac{b_2}{b_4} = \frac{a_2}{a_4}. \tag{86}$$

Hypergeometric Solution A hypergeometric solution is given by [5],

$$\begin{cases} f = a_3\,\dfrac{1 - a_4b_3/a_3b_4}{1 - b_3/b_4}\,\dfrac{{}_2\varphi_1\left(\begin{matrix} a_3/a_4, a_2b_4/a_4b_1 \\ a_3b_4/a_4b_3 \end{matrix};q, b_1t/b_3\right)}{{}_2\varphi_1\left(\begin{matrix} a_3/a_4, qa_2b_4/a_4b_1 \\ qa_3b_4/a_4b_3 \end{matrix};q, b_1t/b_3\right)}, \\ g = b_4\,\dfrac{1 - a_4b_3/a_3b_4}{1 - a_4/a_3}\,\dfrac{{}_2\varphi_1\left(\begin{matrix} a_3/a_4, qa_2b_4/a_4b_1 \\ a_3b_4/a_4b_3 \end{matrix};q, b_1t/qb_3\right)}{{}_2\varphi_1\left(\begin{matrix} qa_3/a_4, qa_2b_4/a_4b_1 \\ qa_3b_4/a_4b_3 \end{matrix};q, b_1t/qb_3\right)}. \end{cases} \tag{87}$$

In the terminating case, the above solution is expressible in terms of the little q-Jacobi polynomials.

Remark A class of hypergeometric solutions including the above solution has been constructed in terms of Casorati determinants in [5].

4.5 Case $A_4^{(1)}$

q-Painlevé Equation (q-Painlevé V equation) [2, 4, 15]

$$\begin{cases} \overline{g}g = \dfrac{(f + b_1/t)(f + 1/b_1t)}{1 + b_3f}, \\ f\underline{f} = \dfrac{(g + b_2/s)(g + 1/b_2s)}{1 + g/b_3}, \end{cases} \tag{88}$$

where

$$\overline{t} = qt, \quad t = q^{\frac{1}{2}}s. \tag{89}$$

Constraint on Parameters

$$b_1b_2b_3^2 = q^{-\frac{1}{2}}. \tag{90}$$

Hypergeometric Solution

$$\begin{cases} f = \dfrac{1 - b_2}{b_2^2b_3}\,\dfrac{{}_2\varphi_1\left(\begin{matrix} q/b_2^2, qb_1^2 \\ 0 \end{matrix};q, q^{\frac{1}{2}}b_2b_3t\right)}{{}_2\varphi_1\left(\begin{matrix} 1/b_2^2, qb_1^2 \\ 0 \end{matrix};q, q^{\frac{1}{2}}b_2b_3t\right)}, \\ g = -\dfrac{1}{b_1b_3^2t}\,\dfrac{{}_2\varphi_1\left(\begin{matrix} 1/b_2^2, b_1^2 \\ 0 \end{matrix};q, q^{\frac{1}{2}}b_2b_3t\right)}{{}_2\varphi_1\left(\begin{matrix} 1/b_2^2, qb_1^2 \\ 0 \end{matrix};q, q^{\frac{1}{2}}b_2b_3t\right)}. \end{cases} \tag{91}$$

In the terminating case, the solution is expressible in terms of the alternative q-Charlier polynomials or the q-Laguerre polynomials [13]. As we mentioned in the previous section, the above solution is rewritten in terms of ${}_1\varphi_1$ series.

4.6 Case $(A_2{+}A_1)^{(1)}$

q-Painlevé Equation (q-Painlevé III equation) [4, 6, 7, 16]

$$\left\{\begin{aligned} \overline{g}gf &= b_0\,\frac{1+a_0tf}{a_0t+f},\\ gf\underline{f} &= b_0\,\frac{a_1/t+g}{1+ga_1/t}, \end{aligned}\right. \tag{92}$$

where

$$\overline{t}=qt. \tag{93}$$

Constraint on Parameters Eq.(92) admits two kinds of specialization which yield different hypergeometric solutions:

(1) $b_0=q$.

(2) $a_0a_1=q$.

Hypergeometric Solution [4, 6, 7]

(1)

$$\left\{\begin{aligned} g &= -\frac{a_1}{t}\,(1-q^2/a_0^2a_1^2)\,\frac{{}_1\varphi_1\left(\begin{matrix}0\\ q^2a_0^2a_1^2\end{matrix};q^2,q^2t^2/a_1^2\right)}{{}_1\varphi_1\left(\begin{matrix}0\\ q^4a_0^2a_1^2\end{matrix};q^2,q^2t^2/a_1^2\right)},\\ f &= \frac{qt}{a_1}\,\frac{q/a_0a_1}{1-q^2/a_0^2a_1^2}\,\frac{{}_1\varphi_1\left(\begin{matrix}0\\ q^4a_0^2a_1^2\end{matrix};q^2,q^4t^2/a_1^2\right)}{{}_1\varphi_1\left(\begin{matrix}0\\ q^2a_0^2a_1^2\end{matrix};q^2,q^2t^2/a_1^2\right)}. \end{aligned}\right. \tag{94}$$

(2)

$$\left\{\begin{aligned} g &= \frac{b_0}{a_0t}\,\frac{{}_1\varphi_1\left(\begin{matrix}a_0^2t^2\\ 0\end{matrix};q^2,q/b_0\right)}{{}_1\varphi_1\left(\begin{matrix}a_0^2t^2\\ 0\end{matrix};q^2,q^3/b_0\right)},\\ f &= -a_0t\,\frac{{}_1\varphi_1\left(\begin{matrix}a_0^2t^2\\ 0\end{matrix};q^2,q^3/b_0\right)}{{}_1\varphi_1\left(\begin{matrix}a_0^2t^2\\ 0\end{matrix};q^2,q/b_0\right)}. \end{aligned}\right. \tag{95}$$

This solution is also expressible in terms of the ${}_1\varphi_1$ series or a specialization of the ${}_2\varphi_1$ series by using the formula [13]

$${}_1\varphi_1\left(\begin{matrix}z\\ 0\end{matrix};q,c\right) = (c;q)_\infty\,{}_0\varphi_1\left(\begin{matrix}-\\ c\end{matrix};q,cz\right) \tag{96}$$

$$= (c,z;q)_\infty\,{}_2\varphi_1\left(\begin{matrix}0,0\\ c\end{matrix};q,z\right). \tag{97}$$

In the terminating case, the solution is expressible in terms of the Stieltjes-Wigert polynomials [13].

Remark Two classes of Casorati determinant solutions which includes the above solutions as the simplest cases have been constructed in [6, 7].

4.7 Case $(A_1{+}A_1')^{(1)}$

q-Painlevé Equation (q-Painlevé II equation) [2, 4, 17]

$$(\overline{f}f-1)(f\underline{f}-1)=\frac{at^2f}{f+t},\quad \overline{t}=qt. \tag{98}$$

Constraint on Parameter

$$a=q. \tag{99}$$

Hypergeometric Solution

$$f=\frac{{}_1\varphi_1\left(\begin{matrix}0\\-q\end{matrix};q,-qt\right)}{{}_1\varphi_1\left(\begin{matrix}0\\-q\end{matrix};q,-t\right)}. \tag{100}$$

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