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Hypergeometric solutions to the \(q\)-Painlevé equations

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Abstract

Hypergeometric solutions to seven \(q\)-Painlevé equations in Sakai’s classification are constructed. Geometry of plane curves is used to reduce the \(q\)-Painlevé equations to the three-term recurrence relations for \(q\)-hypergeometric functions.

1 Introduction

It is well-known that the continuous Painlevé equations \(P_J\) (\(J=\text{II}, \ldots, \text{VI}\)) admit particular solutions expressible in terms of various hypergeometric functions. The coalescence cascade of hypergeometric functions, from the Gauss hypergeometric function to the Airy function, corresponds to that of Painlevé equations, from \(P_{\text{VI}}\) to \(P_{\text{II}}\) [1]. The similar situation is expected for the discrete Painlevé equations.

The discrete Painlevé equations and their solutions have been studied for many years from various viewpoints. In particular, Sakai [2] gave a natural framework of discrete Painlevé equations by means of geometry of rational surfaces. Among the 22 types in Sakai’s classification, there are ten types of \(q\)-Painlevé equations corresponding to the following degeneration diagram of type of affine Weyl groups:

\[
E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_5^{(1)} \rightarrow (A_2 + A_1)^{(1)} \rightarrow (A_1 + A_1')^{(1)} \rightarrow A_1^{(1)} \rightarrow A_0^{(1)}
\]

In this paper, we construct hypergeometric solutions to the first seven \(q\)-Painlevé equations. (The last three cases are irrelevant to the hypergeometric solutions.)

Usually, in order to construct hypergeometric solutions we first look for special situations in which the discrete Painlevé equation is reducible to the discrete Riccati equation, and linearize it into second order linear difference equations. Then we identify the linear equations with the three-term relations of appropriate hypergeometric functions. Reduction to the discrete Riccati equations has been found for all of the \(q\)-Painlevé equations [3, 4]. But so far full step of the above procedure has been carried out only for \(q\)-\(P_{\text{V1}}\) \(D_5^{(1)}\) [5], \(q\)-\(P_{\text{IV}}\) [6] and \(q\)-\(P_{\text{III}}\) [7] \((A_2 + A_1)^{(1)}\) due to technical difficulty.

In the previous paper [8], we studied the elliptic Painlevé equation, the master equation of all the Painlevé and discrete Painlevé equations. There we gave an algebraic formulation of \(\tau\) functions and a geometric description of the equation in terms of plane curves. We also showed that it admits elliptic hypergeometric function \(E_{10}^{(1)}\) as a hypergeometric solution. The \(q\)-Painlevé equation with affine Weyl group symmetry of type \(E_8^{(1)}\) can be regarded naturally as a limiting case of the elliptic Painlevé equation. The construction of this paper provides explicit hypergeometric solutions for all the seven types of \(q\)-Painlevé equations.

In Section 2 we give a geometric method to construct hypergeometric solutions for discrete Painlevé equations. By using this method, we identify the hypergeometric functions appearing for each \(q\)-Painlevé equation in Section 3. In Section 4 we give the list of \(q\)-Painlevé equations and their explicit hypergeometric solutions.
Consider the discrete Riccati equation
\[ \overline{x} = \frac{ax + b}{cx + d}. \]  
Here \( x = x(t) \) is the unknown variable and \( \overline{x} = x(qt) \). We will also use the notation \( \overline{x} = x(t/q) \). In general, the coefficients \( a, b, c, d \) also depend on \( t \). Putting \( x = F/G \), the Riccati equation is decoupled into two linear equations for \( F \) and \( G \) (the contiguity relations):
\[ hF = aF + bG, \quad hG = cF + dG, \]  
where \( h \) is a decoupling factor. We then have the following three-term recurrence relations:
\[ hhF - h(ab + bd)F + b(ad - be)F = 0, \]
\[ hhG - h(dc + ca)G + c(ad - be)G = 0. \]  

Our task is to solve the Riccati equation (1) through eqs. (2) and (3). For the Riccati equation (1) arising from \( q \)-Painlevé equations with higher symmetries, the coefficients of eqs. (3) are polynomials depending on many parameters. By suitable choice of decoupling factor \( h \) and gauge factor \( g : F = g\Phi \), eqs. (3) are expected to reduce to some \( q \)-hypergeometric equations which typically take the form
\[ A\Phi + (B - A - C)\Phi + C\Phi = 0. \]  
Here the coefficients \( A, B, C \) are of compact factorized form, but, \( B - A - C \) is not. Accordingly, the second coefficients in eqs.(3) consist of huge number of terms (more than hundred for \( E_8^{(1)} \)). This is the main technical difficulty to manipulate these equations. Our basic strategy to overcome this difficulty is to express these coefficients in terms of invariants such as determinants. To do this, the geometric formulation of the discrete Painlevé equations developed in the previous paper [8] is useful.

Consider a configuration of nine points \( P_1, \ldots, P_9 \) in \( \mathbb{P}^2 \). We denote by \( C_0 \) the unique cubic curve passing through them. In the context of discrete Painlevé equations, the nine points \( P_1, \ldots, P_9 \) play the role of parameters for the difference equations; some of them may be regarded as independent variables. An additional generic point \( P_{10} \) is regarded as the dependent variable. The commuting family of the time evolutions \( T_{ij} \), the translation associated with a pair of points \((P_i, P_j)\) \((i,j = 1, \ldots, 9; i \neq j)\), is described as follows. Let us take \( T_{89} \) as an example. Under the translation \( T_{89} \), the points \( P_i \) \((i \neq 8, 9, 10)\) are invariant and the new points \( P_8 \) and \( P_9 \) are determined so that
\[ P_8 + P_9 = P_8 + P_9, \quad P_1 + \cdots + P_7 + P_8 + P_9 = 0, \]  
with respect to the addition on the cubic \( C_0 \), where \( P_j = T_{89}(P_j) \). This means that \( P_9 \) is the additional intersection point (one parameter family) of cubic curves defined by the eight points \( P_i \) \((i \neq 9, 10)\). Using this pencil of cubics, the transformation \( T_{89}(P_{10}) \) is geometrically described as follows. Consider a cubic curve \( C \) passing through the nine points \( P_i \) \((i \neq 9)\). The new point \( P_{10} \) is determined by
\[ P_{10} + P_9 = P_{10} + P_8, \]  
with respect to the addition on the curve \( C \).

In view of configuration of nine points \( P_1, \ldots, P_9 \), there are two typical situations where the dynamical system admits reduction to discrete Riccati equations.

1. The case where three points are collinear.
2. The case where a point is infinitely near to another.

For each case we construct below the corresponding Riccati equation and its linearization.
In such a case, the same proof can be applied for other choice of $H$. Hence, the line $ℓ$ where the last equality also follows from the conic condition like eq. (10). Other factors $P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9$.

**Proposition 2.1** The time evolution of the point $P_{10}$ under $T_{89}$ is determined by the linear equation

$$
\frac{d_{ijk}d_{kis}d_{ijσ}d_{568}}{d_{ijσ}} \left( \lambda_{(ijk)} \frac{d_{ijs}}{d_{iks}} d_{kσm} - d_{jk10} \right) + \frac{d_{jks}d_{kis}d_{ijσ}d_{569}}{d_{ijkσ}} \left( \mu_{(ijk)} \frac{d_{ijs}}{d_{iks}} d_{kσ10} - d_{jσ10} \right) = d_{ijk}d_{kσ9}d_{569}d_{jk10},
$$

where $\{ijk\} \subset \{1234\}$ and $d_{abc} = \det[P_a, P_b, P_c]$. The gauge factors $\lambda_{(ijk)}$ and $\mu_{(ijk)}$ can be chosen as follows:

$$
\begin{align*}
\lambda_{(123)} &= 1, & \lambda_{(124)} &= \frac{(14)}{(13)} = \frac{24}{23}, \\
\lambda_{(134)} &= \frac{(14)}{(12)} = \frac{34}{32}, & \lambda_{(234)} &= \frac{(24)}{(21)} = \frac{34}{31},
\end{align*}
$$

$$
\begin{align*}
(ij) &= \frac{d_{ijk}}{d_{ijσ}}, & \mu_I &= \lambda_I \big|_{8 \rightarrow 9}, \frac{9}{σ}. \tag{8}
\end{align*}
$$

**[Proof]** When $\{ijk\} = \{123\}$, eq.(7) is reformulation of [8], Proposition 4.2 in terms of determinants. The same proof can be applied for other choice of $\{ijk\} \subset \{1234\}$. The only point we should take care is the gauge factors $\lambda_I, \mu_I$ that are symmetric with respect to the indices $I = \{ijk\}$. By choosing the relative normalization of homogeneous coordinates of $P_{10}, P_{10}$ and $P_{10}$, one can put $\lambda_{(123)} = \mu_{(123)} = 1$. Then we should determine the other factors $\lambda_1$ and $\mu_1, I = \{124\}, \{134\}, \{234\}$ consistently. To do this, let us consider eqs.(7) for $(i, j, k) = (1, 2, 3)$ and $(i, j, k) = (4, 2, 3)$, both with the same unknown variable $d_{2,3,10}$. Comparing the corresponding coefficients, we have

$$
\begin{align*}
(c_{123})\lambda_{(123)} \frac{d_{ijs}}{d_{ijs}} &= (c_{423})\lambda_{(423)} \frac{d_{ijs}}{d_{ijs}}, & (c_{ijk}) = \frac{d_{jks}d_{kis}d_{ijσ}d_{568}}{d_{569}d_{jks}d_{kis}d_{569}}.
\end{align*}
$$

Using the fact that $P_1, P_2, P_3, P_4, P_5$ and $P_9$ are on the conic $C_2$, we have

$$
\frac{c_{123}}{c_{423}} = \frac{d_{381}d_{3σ1}d_{3σ4}d_{234}}{d_{384}d_{2σ4}d_{5σ1}d_{231}} = 1.
$$

Hence,

$$
\lambda_{(423)} = \lambda_{(123)} \frac{d_{ijs}}{d_{ijs}} \frac{d_{ijσ}}{d_{ijσ}} = \frac{(34)}{(31)} = \frac{(24)}{(21)},
$$

where the last equality also follows from the conic condition like eq.(10). Other factors $\lambda_I, \mu_I$ can be determined similarly.

**Case (1):** This case is further divided into two types depending on the choice of the translation $T_{ab}$ and the three points $P_1, P_m, P_n$ lying on a line $ℓ$. Namely, (1a) $\{a, b\} \cap \{l, m, n\} = \emptyset$ and (1b) $\{a, b\} \subset \{l, m, n\}$. In both cases, if $P_{10} = ℓ$ then $\overline{P_{10}} = T_{ab}(P_{10}) = ℓ$ and the motion of $P_{10}$ is described by a discrete Riccati equation on the line $ℓ$.

(1a): Consider the case when the three points $P_5, P_6, P_7$ are on a line $ℓ$ and the translation is $T_{89}$ as an example. In such a case, $P_{10} \in ℓ \Rightarrow \overline{P_{10}} \in ℓ$ follows from the fact that the curve $C$ is decomposed into the line $ℓ$ (passing through $P_5, P_6, P_7, P_{10}$) and the conic $C_2$ (passing through $P_1, P_2, P_3, P_4, P_8, P_9$).

(1b): Consider next the case when the three points $P_7, P_8, P_9$ are on a line $ℓ$ and translation is $T_{89}$ for instance. Then $\overline{P_{10}} \in ℓ$ whenever $P_{10} \in ℓ$. This fact follows from

$$
\begin{align*}
P_8 + P_9 &= -P_7 = \overline{P_7} + \overline{P_9}, & P_9 + P_9 &= -P_7 = \overline{P_7} + \overline{P_9},
\end{align*}
$$

For a point $P$ with homogeneous coordinates $(x : y : z)$, we set

$$
m(P) = [x^3, x^2y, x^2z, xy^2, xyz, xz^2, y^3, y^2z, yz^2, z^3].
$$

3
Then \( \overline{P}_{10} \) is determined as a point on \( \ell \) such that

\[
X = \det[m(P_1), \ldots, m(P_8), m(P_{10})] = 0. \tag{14}
\]

We parameterize the points \( P_{10}, \overline{P}_{10} \in \ell \) by setting

\[
P_{10} = d_{178}P_8 v - d_{178}P_3 = d_{189}P_7 + d_{179}(v - 1)P_8,
\]

\[
\overline{P}_{10} = d_{178}\overline{P}_8 \overline{v} - d_{178}\overline{P}_3 = d_{189}\overline{P}_7 + d_{179}(\overline{v} - 1)\overline{P}_8. \tag{15}
\]

The last equality follows from the identity \( d_{ijk}P_i - d_{jkl}P_i + d_{kli}P_j - d_{lij}P_k = 0 \) and \( d_{780} = d_{789} = 0 \). The coordinate \( v \) is chosen so that the three points \( P_{10} = P_9, P_7, P_8 \) correspond to \( v = 0, 1, \infty \), respectively (and similar for \( \overline{v} \)).

**Proposition 2.2** The coordinate \( \overline{v} \) of \( \overline{P}_{10} \) is determined by the Riccati equation

\[
\overline{v} = -\frac{A_1 + A_2 v}{A_4 v}. \tag{16}
\]

In terms of the variable \( F \) such that \( v = F/\overline{F} \), we obtain the three-term recurrence relation

\[
A_4(\overline{F} - F) + A_1(F - F) + A_3F = 0. \tag{17}
\]

The explicit forms of the coefficients \( A_i \) are given in the proof.

[Proof] The determinant \( X \) in eq. (14) is at most cubic with respect to both the variables \( v \) and \( \overline{v} \). It has trivial zeros at \( v = 1, \infty \) and \( \overline{v} = 0, 1 \) corresponding to \( P_{10} = P_7, P_8 \) and \( \overline{P}_{10} = \overline{P}_9, \overline{P}_7 \). Therefore, \( X \) is factorized in the form

\[
X = (v - 1)(\overline{v} - 1)\overline{v}(A_1 + A_2 v + A_3 \overline{v} + A_4 v \overline{v}). \tag{18}
\]

First consider \( A_3 \). Comparing the coefficient \( v^0 \overline{v}^3 \) of \( X \), we have

\[
A_3 = \det[m(P_1), \ldots, m(P_8), -d_{178}m(P_3), d_{178}^3m(\overline{P}_8)] = 0. \tag{19}
\]

Next, the coefficient \( A_1 \) is determined as

\[
A_1 = -\left. \frac{\partial X}{\partial v} \right|_{v=0, \overline{v}=1} = d_{178}^3 d_{178}^2 d_{178} \det[m(P_1), \ldots, m(P_8), dm(P_7, \overline{P}_8)]. \tag{20}
\]

Here, we have used the relations

\[
m(P_{10}) = -d_{178}^3m(P_3) + O(v),
\]

\[
m(\overline{P}_{10}) = d_{178}^3m(P_7) + (\overline{v} - 1)d_{178}^2d_{178}dm(P_7, \overline{P}_8) + O((\overline{v} - 1)^2),
\]

\[
dm(P, Q) = \left. \frac{\partial}{\partial \epsilon} m(P + \epsilon Q) \right|_{\epsilon=0}.
\]

Similarly, the coefficient \( A_4 \) is given by

\[
A_4 = \left. \frac{\partial}{\partial v} \left( \text{coefficient of } \overline{v}^3 \text{ in } X \right) \right|_{v=1}
= d_{178}^3 d_{189}^2 d_{178} \det[m(P_1), \ldots, m(P_8), dm(P_7, P_8), m(\overline{P}_8)]. \tag{22}
\]

Finally, for the coefficient \( A_2 \), it is rather convenient to consider the combination \( A_3 = A_1 + A_2 + A_4 \), which is determined by

\[
A_3 = \left. \frac{\partial^2 X}{\partial v \partial \overline{v}} \right|_{v=1, \overline{v}=1}
= d_{178}^2 d_{189}^2 d_{178} d_{178} \det[m(P_1), \ldots, m(P_8), dm(P_7, P_8), dm(P_7, \overline{P}_8)]. \tag{23}
\]

The Riccati equation (16) is derived from \( X = 0 \). Since \( A_3 = 0 \), it is easily decoupled into eq.(17). \( \square \)
Consider the case where $P_9$ is an infinitely near point of $P_8$ ($P_9 \to P_9$). If $P_{10}$ is also infinitely near to $P_9$, then so is the translation $P_{10} = T_{a b}(P_{10})$ ($a, b \neq 8, 9$). The point $P_{10}$ is determined by solving the following system of algebraic equations

\[
\begin{align*}
\det[P_a, P_{10}, Q] &= 0, \\
\det[m(P_1), \ldots, m(P_8), m(P_{10}), m(Q)]|_{e_3} &= 0, \\
\det[m(P_1), \ldots, m(P_8), m(P_{10}), m(Q)]|_{e_3} &= 0,
\end{align*}
\]

including an intermediate infinitely near point $Q$, where

\[
P_{10} = P_8 + \epsilon \begin{bmatrix} 0 \\ u \\ \pi \end{bmatrix}, \quad P_{10} = P_8 + \epsilon \begin{bmatrix} 0 \\ \pi \\ r \end{bmatrix}, \quad Q = P_8 + \epsilon \begin{bmatrix} 0 \\ r \\ s \end{bmatrix},
\]

with an infinitely small parameter $\epsilon$, and $f|_{e_3}$ stands for the coefficient of $\epsilon^n$ in the Taylor expansion of $f$ at $\epsilon = 0$. Eq.(24) can be reduced to a linear relation for the homogeneous variables $(u : v)$ and $(\pi : \tau)$. More precisely, there are four solutions to eq.(24), three of them are trivial ones: $P_{10} = Q$, $P_{10} = Q$ or $P_{10} = T_{10}$, and remaining one gives a linear relation between $P_{10}$ and $T_{10}$ which is in fact the Riccati equation. The variables $(u : v)$ etc. in eq.(25) represent the slope of the line $P_8 P_{10}$ etc. in a (temporary) coordinate of $\mathbb{P}^1$ (the exceptional curve which is the blown up of $P_8$). It is convenient to make a change of coordinates $(u : v) \mapsto (U : V) = (au + bv : cu + dv)$ in such a way that the lines $P_8 P_9$, $P_9 P_{10}$, $P_8 P_9$ correspond to $(0 : 1), (1 : 0), (1 : 1)$, respectively. Then, in these coordinates, we obtain a three-term recurrence relation for $F = U/V$ in the form of eq.(17) with factorized coefficients.

### 3 Identification of the Hypergeometric Functions

We apply the results in the preceding section to special configurations of points, and derive the corresponding hypergeometric special solutions to each of the $q$-Painlevé equations in the degeneration diagram [2]

\[
E^{(1)}_8 \rightarrow E^{(1)}_7 \rightarrow E^{(1)}_6 \rightarrow D^{(1)}_5 \rightarrow A^{(1)}_4 \rightarrow (A_2 + A_4)^{(1)} \rightarrow (A_1 + A_1)^{(1)}.
\]

Let us recall the definition and terminology of the $q$-hypergeometric series [9]. The $q$-hypergeometric series $r\varphi_s$ is given by,

\[
r\varphi_s \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} ; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_r; q)_n}{(b_1; q)_n \cdots (b_s; q)_n} \left[ (-1)^n q^{\frac{n(n-1)}{2}} \right] 1 + s - r z^n,
\]

The $q$-hypergeometric series $r^\varphi_r$ is called balanced \(^1\) if the condition

\[
qa_1 a_2 \cdots a_r + 1 = b_1 b_2 \cdots b_r, \quad z = q,
\]

is satisfied, and is called well-poised if the condition

\[
qa_1 = a_2 b_1 = \cdots = a_r b_r,
\]

is satisfied. Moreover, it is called very-well-poised if it satisfies

\[
a_2 = qa_1^2, \quad a_4 = -qa_1^2,
\]

\(^1\)For $3\varphi_2$ series, it appears that two different conventions are used in the literature. This convention is due to [9], while the series $3\varphi_2 \left( \begin{array}{c} a_1, b_1, c \\ d, e \end{array} ; q, z \right)$ with $z = de/abc$ is also called “balanced $3\varphi_2$ series” in [12]. For $3\varphi_2$ series, we use latter convention without notice.
in addition to eq.(29), and denoted as \( r+1 W_r \):

\[
r+1 W_r(a_1; a_4, \ldots, a_{r+1}; q, z) = r\varphi_r \left( \frac{a_1, qa_2, -qa_1^2, a_4, \ldots, a_{r+1}}{a_1^2, -a_1^2, qa_1/a_4, \ldots, qa_1/a_{r+1}} ; q, z \right). \tag{31}
\]

The degeneration diagram of \( q \)-Painlevé equations (26) corresponds to that of \( q \)-hypergeometric series:

\[
\text{balanced} \quad W_{10}^{1} \quad \rightarrow \quad 8 W_7 \quad \rightarrow \quad \text{balanced} \quad 3\varphi_2 \quad \rightarrow \quad 2\varphi_1 \quad \rightarrow \quad 1\varphi_1 \quad \rightarrow \quad 1\varphi_1 \left( \begin{array}{c} a \\ 0 \\ 0 \end{array} : q, z \right) \rightarrow \quad 1\varphi_1 \left( \begin{array}{c} 0 \\ -q \\ q, z \end{array} \right). \tag{32}
\]

### 3.1 Case \( E_8^{(1)} \)

In this case we take the configuration of nine points lying on a nodal cubic curve \( C_0 \). We can parameterize the nine points \( P_i = P(u_i) \) as follows:

\[
P(u) = \left( (1 - ubc)(1 - u/b)(1 - u/c) : (1 - uca)(1 - u/c)(1 - u/a) : (1 - uab)(1 - u/a)(1 - u/b) \right). \tag{33}
\]

The function \( P(u) \) parameterize a nodal cubic \( C_0 \) passing through \((1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\) with a node at \((1 : 1 : 1)\). Then the determinant \( d_{ijk} \) is given by

\[
d_{ijk} = [ij][ik][jk][ijk]d_0, \tag{34}
\]

where \([ij] = u_i - u_j, [ijk] = 1 - u_iu_ju_k \) and \( d_0 \) is a constant independent of \( u \).

We apply Case (1a) where \( P_0, P_8, P_7 \) are collinear \((u_5, u_6, u_7 = 1)\) and \( T = T_{90} \). Putting \((i, j, k) = (1, 2, 3)\) and substituting the determinants (34) in the three-term recurrence relation (7), we obtain the linear equation for \( F = d_{2,3,10} \)

\[
A(lF - F) + BF + C(mF - F) = 0,
\]

\[
\]

\[
\left( \frac{C}{B}, m \right) = \begin{pmatrix} A(lF - F) \\ A(lF - F) \end{pmatrix} \bigg|_{l = [8, 9, 8, 0]},
\]

where \( u_7 = qu_9, u_9 = qu_8, u_8 = q^{-1}u_8 \) while the other \( u_i \) are invariant. After the gauge transformation \( F = g \Phi \) with \( g = l^{-1} \), eq.(35) is solved by the balanced 10W9 series [10]:

\[
\Phi = \Phi(a_0; a_1, \ldots, a_7; q) = 10W_9(a_0; a_1, \ldots, a_7; q, q)\bigg|_{a_0 = a_1, \ldots, a_7} + \left( qa_0, a_7/0; q \right) \infty \prod_{k=1}^{6} \left( qa_0/ak, a_7/0; q \right) \infty \times 10W_9(a_7^2/a_0; a_1a_7/a_0, \ldots, a_6a_7/a_0, a_7; q, q), \tag{36}
\]

with the balancing condition \( q^2 a_3^3 = a_1 a_2 \cdots a_7 \). The parameters \( a_i \) \((i = 0, 1, \ldots, 7)\) are given by

\[
a_0 = u_2^2u_3/q, \quad a_1 = u_1u_2u_3, \quad a_2 = u_2u_3u_4, \quad a_3 = u_2/u_5, \quad a_4 = u_2/u_6, \quad a_5 = u_2u_3u_6, \quad a_6 = u_2u_3u_8, \quad a_7 = u_2u_3u_9. \tag{37}
\]

When one of the parameters \( a_1, a_2, \ldots, a_6 \) is \( q^{-N} (N \in \mathbb{Z}_{\geq 0}) \) the second term of eq.(36) vanishes, which can also be derived as the trigonometric limit of our previous result [8].
3.2 Case $E_7^{(1)}$

In this case we consider the configuration of nine points $P_i$ in $\mathbb{P}^2$ among which three ($i = 1, 2, 6$) are on a line and six ($i = 3, 4, 5, 7, 8, 9$) are on a conic (Fig.1). We parameterize those nine points as follows:

$$P_i = \begin{cases} 
(-u_i : 0 : 1) & (i = 1, 2, 6) \\
(1 : u_i : u_i^2) & (i = 3, 4, 5, 7, 8, 9)
\end{cases} \quad (38)$$

Putting $P_5, P_6, P_7$ to be collinear ($u_5, u_6, u_7 = 1$), we again apply Case (1) with $(i, j, k) = (1, 2, 3)$. Then we obtain the three-term recurrence relation (7) with respect to $T_{89}$:

$$A(lF - F) - BF + C(mF - F) = 0, \quad (43)$$

where $F = d_{2,3,10}$ and $qu_1u_2u_3u_4u_8u_9 = 1$. The action of $T = T_{89}$ on the parameters $u_i$ are the same as $E_8^{(1)}$ case.

By the gauge transformation $F = g \Phi$ with $\tilde{g} = l^{-1} \frac{[79][258]}{[78][259]} g$, eq.(39) is solved by the $sW_7$ series [11].

$$\Phi = sW_7(a_0; a_1, \ldots, a_5; q, z), \quad z = \frac{q^2 a_0^2}{a_1 a_2 a_3 a_4 a_5}, \quad (40)$$

where the parameters $a_i$ ($i = 0, 1, \ldots, 5$) are given by

$$a_0 = u_2 u_3^2, \quad a_1 = u_3 / u_8, \quad a_2 = u_5 / u_9, \quad a_3 = qu_5 / u_3, \quad a_4 = u_5 / u_4, \quad a_5 = u_2 / u_6. \quad (41)$$

3.3 Case $E_6^{(1)}$

In this case the nine points are divided into three groups of three points that are collinear (Fig.2). The parameterization is given as follows:

$$P_i = \begin{cases} 
(1 : -u_i : 0) & (i = 1, 2, 6) \\
(0 : 1 : -u_i) & (i = 3, 4, 7) \\
(-u_i : 0 : 1) & (i = 5, 8, 9)
\end{cases} \quad (42)$$

We again consider the Case (1a) with $P_5, P_6, P_7$ being collinear and $T = T_{89}$. Then the three-term relation (7) with $(i, j, k) = (1, 2, 3)$ implies

$$\frac{[58][138][239]}{u_1 u_2 [89]} \left( \frac{[139]}{[138]} F - F \right) + \frac{[59][238][139]}{u_1 u_3 [89]} \left( \frac{[138]}{[139]} F - F \right) = -[257][89]F, \quad (43)$$
where $F = d_{2,3,10}$. This equation is solved by the balanced $3\varphi_2$ series \[12\]
\begin{align*}
F &= 3\varphi_2 \left( a_1, a_2, a_3 ; q, \frac{b_1 b_2}{a_1 a_2 a_3} \right), \tag{44}
\end{align*}
where the parameters are given by
\begin{align*}
a_1 &= u_3/u_7, \quad a_2 = u_2 u_3 u_8, \quad a_3 = u_2^2 u_9, \\
b_1 &= u_2 u_3 u_5, \quad b_2 = q u_1 u_2^2 u_8 u_9. \tag{45}
\end{align*}

### 3.4 Case $D_5^{(1)}$

This is a limiting case where $P_1 = (\epsilon : 1 : -u_1 \epsilon)|_{\epsilon \to 0}$ becomes infinitely near to $P_4 = (0 : 1 : 0)$, while the other $P_i$ ($i \neq 1, 4$) are the same as the Case $E_6^{(1)}$ (Fig.3). Accordingly, the corresponding linear difference equation is
\begin{align*}
\frac{u_5[58][239]}{[89]} \left( \frac{q u_9}{u_8} F - F \right) + \frac{u_9[59][238]}{[89]} \left( \frac{q u_8}{u_9} F - F \right) = [257][89] F, \tag{46}
\end{align*}
with $u_5 u_6 u_7 = 1$ and $q u_1 u_2 u_3 u_8 u_9 = 1$. This equation is solved by the $2\varphi_1$ series \[5\]:
\begin{align*}
F &= g_2 \varphi_1 \left( a_1, a_2, b_1, q, z \right), \quad g = \frac{1 - u_5/q u_9}{1 - u_5/u_8}, \tag{47}
\end{align*}
where the parameters are given by
\begin{align*}
a_1 &= u_2 u_3 u_8, \quad a_2 = u_2 u_3 u_9, \quad b_1 = q u_2 u_3 u_8 u_9 / u_5, \quad z = q u_7 / u_3. \tag{48}
\end{align*}
3.5 Case $A_4^{(1)}$

This case is a further degeneration of Case $D_5^{(1)}$:

\[
P_1 = (1 : 0 : 1) \quad P_2 = (a_2 : 0 : 1) \quad P_3 = (a_1 a_2 : 0 : 1) \\
P_4 = (0 : 1 : 1) \quad P_5 = (0 : 1 : a_4) \quad P_6 = (1 : -a_3 : 0) ,
\]

\[
P_7 = (0 : 1 : 0) \quad P_8 = (\epsilon : 1 : 0) \quad P_9 = (\epsilon : 1 : \frac{a_0}{a_2} \epsilon^2),
\]

where $\epsilon$ is an infinitesimal parameter and $a_0 a_1 \cdots a_4 = q$. This configuration contains a sequence of infinitely near points $P_9 \to P_8 \to P_7$, while $(P_1, P_2, P_3), (P_4, P_5, P_7)$ and $(P_6, P_7, P_8)$ are collinear (Fig.4).

Consider the case where $P_1, P_5, P_6$ are collinear ($a_3 a_4 = 1$) and the time evolution is $T = T_{56} (a_3 \mapsto a_3/q, a_4 \mapsto a_4 q)$. This situation corresponds to the Case (1b). Applying the Proposition 2.2, we obtain the linear equation,

\[
a_0 \left( a_3 q - q \right) (F - F') = (F - F') + (1 - a_2)(1 - a_1 a_2) F.
\]

This equation is solved by

\[
F = g_2 \varphi_1 \left( \frac{a_0}{a_2} a_3 : q, q a_4 \right), \quad g = \frac{a_0/a_2}{1 - a_3/q} g.
\]

We note that the above solution can be rewritten in terms of $1 \varphi_1$ series by using the relation [13]

\[
2 \varphi_1 \left( \frac{a, b}{0} : q, z \right) = \left( \frac{b z; q}{z; q} \right)_{\infty} - \varphi_1 \left( \frac{b}{z} : q, a z \right).
\]

3.6 Case $(A_2 + A_1)^{(1)}$

This case is a further degeneration of Case $A_4^{(1)}$ such that $P_5 \to P_3$ (Fig.5):

\[
P_1 = (1 : 0 : 1) \quad P_2 = (a_1 : 0 : 1) \quad P_3 = (0 : 0 : 1) \\
P_4 = (0 : 1 : 1) \quad P_5 = (\frac{b_1}{a_2} \epsilon : \epsilon : 1) \quad P_6 = (1 : -a_2 : 0) ,
\]

\[
P_7 = (0 : 1 : 0) \quad P_8 = (\epsilon : 1 : 0) \quad P_9 = (\epsilon : 1 : \frac{b_0}{a_1} \epsilon^2),
\]

where $a_0 a_1 a_2 = b_0 b_1 = q$. 
In this case there are two different kinds of time evolutions, which are referred to as \(q\text{-}P_{III}\) and \(q\text{-}P_{IV}\), respectively. The actions on the parameters are given by

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\text{q-P}_{III} & a_1 & a_2 & a_0 & b_1 & b_0 \\
\hline
T_{92} & a_1/q & a_2 & a_0/q & b_1 & b_0 \\
T_{56} & a_1 & a_2q & a_0/q & b_1 & b_0 \\
\hline
\end{array}
\]

(54)

In the case of \(q\text{-}P_{III}\), when \((P_3, P_5, P_6)\) is collinear \((b_1 = 1)\), the linear equation with respect to \(T_{92}\) is derived by taking \((i, j, k) = (1, 4, 7)\) in eq.(7) of Proposition 2.1:

\[
a_1(T_{92}(F) - F) + a_1a_2(T_{92}^{-1}(F) - F) + F = 0, \quad F = d_{4,7,10}.
\]

(55)

This equation is solved by Jackson’s \(q\)-Bessel function [9]

\[
F = 1\varphi_1 \left( \frac{0}{q/a_2}; q, a_0 \right).
\]

(56)

For the case of \(q\text{-}P_{IV}\), taking \(P_2 \rightarrow P_1\) \((a_1 = 1)\), we get the linear equation with respect to \(T_{59}\),

\[
a_2T_{59}^{-1}(F) - b_1T_{59}(F) - (1 - b_1)F = 0.
\]

(57)

According to the argument of Case (2), this equation is obtained by taking \(F/T_{95}^{-1}(F)\) as the inhomogeneous coordinate of \(\mathbb{P}^1\) such that 0 and \(\infty\) correspond to the lines \(P_1P_5\) and \(P_1P_9\), respectively. The above equation is solved by

\[
F = 1\varphi_1 \left( \frac{a_2}{0}; q, b_0 \right).
\]

(58)

### 3.7 Case \((A_1 + A_1')^{(1)}\)

This case is obtained from Case \((A_2 + A_1)^{(1)}\) by taking \(P_6 \rightarrow P_2\) (Fig.6):

\[
\begin{align*}
P_1 &= (1 : 0 : 1) & P_2 &= (1 : 0 : 0) & P_3 &= (0 : 0 : 1) \\
P_4 &= (0 : 1 : 1) & P_5 &= \left( \frac{a_0}{b^2}; \epsilon : 1 \right) & P_6 &= (1 : a_1\epsilon : \epsilon) \\
P_7 &= (0 : 1 : 0) & P_8 &= (\epsilon : 1 : 0) & P_9 &= (\epsilon : 1 : -b\epsilon^2)
\end{align*}
\]

(59)

where \(a_0a_1 = q\).

When \((P_2, P_4, P_6)\) is collinear \((a_1 = 1)\), we obtain the Riccati equation

\[
T_{95}(y) = \frac{b(1 - y)}{y}, \quad P_{10} = (1 : y : y),
\]

(60)
Figure 6: \((A_1 + A'_1)^{(1)}\)

with respect to \(T_{95} (b \mapsto bq)\), which is linearized as

\[
T_{95}(F) - T_{95}^{-1}(F) + q^{-\frac{1}{2}} b^\frac{1}{2} F = 0,
\]

through

\[
y = q^{-\frac{1}{2}} b^\frac{1}{2} \frac{F}{T_{95}^{-1}(F)}.
\]

This is solved by a \(q\)-analogue of the Airy function

\[
F = \psi_1 \left( \begin{array}{c} 0 \\ -q^\frac{1}{2} : q^\frac{1}{2}, -q^\frac{1}{2} b^\frac{1}{2} \end{array} \right).
\]

4 Hypergeometric Solutions for \(q\)-Painlevé Equations

In the previous section, the relevant hypergeometric functions are identified for each \(q\)-Painlevé equation. However, our choice of the dependent variables and parameters is not always the same as in the literature. In this section we give a list of hypergeometric solutions for the \(q\)-Painlevé equations in the forms appearing in preceding works. Full details of construction of the solutions will be given in a forthcoming paper.

4.1 Case \(E_8^{(1)}\)

\(q\)-Painlevé Equation [3, 4, 18]

\[
\begin{align*}
\frac{(gst - f)(gst - g) - (s^2 t^2 - 1)(s^2 t^2 - 1)}{(gst - f) - \left(1 - \frac{1}{s^2 t^2}\right)} &= P(f, t, m_1, \ldots, m_7),
\end{align*}
\]

\[
\begin{align*}
\frac{(gst - g)(gst - g) - (s^2 t^2 - 1)(s^2 t^2 - 1)}{\frac{f}{sl} - \frac{g}{s} - \left(1 - \frac{1}{s^2 t^2}\right)} &= \frac{P(g, s, m_7, \ldots, m_1)}{P(g, s^{-1}, m_1, \ldots, m_7)},
\end{align*}
\]

where

\[
P(f, t, m_1, \ldots, m_7) = f^4 - m_1 t f^3 + (m_2 t^2 - 3 - t^8) f^2 + (m_3 t^7 - m_3 t^3 + 2 m_1 t) f + (t^8 - m_6 t^6 + m_4 t^4 - m_4 t^2 + 1),
\]

and \(m_k (k = 1, 2, \ldots, 7)\) are the elementary symmetric functions of \(k\)-th degree in \(b_i (i = 1, 2, \ldots, 8)\) with \(b_1 b_2 \cdots b_8 = 1\).

Moreover,

\[
\overline{t} = qt, \; t = q^\frac{1}{2} s.
\]
Constraint on Parameters \[3\]
\[qb_1b_2b_3b_7 = 1, \quad b_2b_4b_8 = q. \quad (68)\]

Hypergeometric Solution A hypergeometric solution is given by
\[
g - \left( \frac{s}{b_1} + \frac{b_1}{s} \right) = \lambda \frac{\Phi(q^4a_0; a_1, q^2a_2, \ldots, q^2a_7; q^2)}{\Phi(a_0; a_1, \ldots, a_7; q^2)} ,
\]
where \(\Phi\) is the balanced 10\(W_9\) series defined in eq. (36), and \(a_i (i = 0, 1, \ldots, 7)\) and \(\lambda\) are given by
\[
a_0 = \frac{1}{qb_1b_2b_8^2}, \quad a_1 = \frac{q^2}{b_2b_8^2}, \quad a_2 = \frac{s^2}{b_2b_8},
\]
\[
a_i = \frac{b_i}{b_8} (i = 3, 5, 7), \quad a_i = \frac{b_i}{b_4} (i = 4, 6),
\]
and
\[
\lambda = \frac{b_1b_2b_9}{bs^2} \left( 1 - \frac{b_4b_6}{b_1b_8} \right) \left( 1 - q^2 \frac{b_4b_6}{b_1b_8} \right) \left( 1 - b_3b_5t^2 \right) \left( 1 - b_3b_7t^2 \right) \left( 1 - b_5b_7t^2 \right) \prod_{s=2, 4, 6} \left( 1 - \frac{b_1}{b_1} \right),
\]
respectively.

4.2 Case \(E_7^{(1)}\)

\(q\)-Painlevé Equation \[3, 4\]

\[
\begin{align*}
(gf - t^2)(gf - t^2) &= (f - b_1t)(f - b_2t)(f - b_3t)(f - b_4t), \\
(gf - t)(gf - t) &= (f - b_5t)(f - b_6t)(f - b_7t), \\
(gf - t)(gf - t) &= (g - t/b_5)(g - t/b_6)(g - t/b_7), \\
(gf - 1)(gf - 1) &= (g - 1/b_5)(g - 1/b_6)(g - 1/b_7),
\end{align*}
\]
where
\[
L = qt, \quad b_1b_2b_3b_4 = q, \quad b_5b_6b_7b_8 = 1.
\]

Constraint on Parameters
\[
b_2b_3 = b_5b_7.
\]

Hypergeometric Solution A hypergeometric solution is given by the 8\(W_7\) series \[11,\]
\[
z = g - t/b_1 = \frac{1 - b_3/b_1}{g - 1/b_5} \left( \frac{b_1b_8}{b_3b_5} \cdot \frac{b_2}{b_3} \cdot \frac{b_4}{b_5} \cdot \frac{b_7}{b_5} \cdot \frac{q}{b_7} \cdot \frac{b_5}{b_1}, \right) \left( \frac{b_1b_8}{b_3b_5} \cdot \frac{b_2}{b_3} \cdot \frac{b_4}{b_5} \cdot \frac{b_7}{b_5} \cdot \frac{q}{b_7} \cdot \frac{b_5}{b_1} \right). \quad (75)
\]

In the terminating case, e.g. \(b_4/b_3 = q^{-N} (N \in \mathbb{Z}_{\geq 0})\), the solution is expressed in terms of the terminating balanced 4\(\varphi\)3 series (Askey-Wilson polynomials) as
\[
z = \frac{1 - b_3/b_4}{1 - b_3/b_5} \left( \frac{b_1b_2}{b_3b_4} \cdot \frac{b_1t/b_5}{b_3b_4} \cdot \frac{b_1b_5t/b_4b_6}{b_3b_4} \cdot \frac{b_1b_4b_5b_6b_8}{b_3b_4} ; q, q \right), \quad (76)
\]
\[
\quad \frac{q}{b_7} \cdot \frac{b_5}{b_1}, \quad \frac{b_1b_8}{b_3b_5} \cdot \frac{b_2}{b_3} \cdot \frac{b_4}{b_5} \cdot \frac{b_7}{b_5} \cdot \frac{q}{b_7} \cdot \frac{b_5}{b_1} \right).
\]
by using Watson’s transformation formula for the terminating $sW_7$ series [9]

$$sW_7(a; b, c, d, e, f; q, q^2 a^2/bedf)$$

$$= \frac{(aq, aq/|de|, aq/df, aq/ef; q)_{\infty}}{(aq/d, aq/e, aq/f, aq/def; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(aq/b, aq/c, de/f/a : q, q)}{q^n}.$$

(77)

4.3 Case $E_6^{(1)}$

$q$-Painlevé Equation [3, 4]

$$\begin{cases}
3f = 1 - (g_1/g_2) = tT \left( \frac{(f - b_1)(f - b_2)(f - b_3)(f - b_4)}{(f - b_5)(f - t/b_5)} \right), \\
(gf - 1)(gf - 1) = \frac{(g - 1/b_1)(g - 1/b_2)(g - 1/b_3)(g - 1/b_4)}{(g - b_6)(g - t/b_6)}.
\end{cases}$$

(78)

where

$$T = qt, \quad b_1 b_2 b_3 b_4 = 1.$$ 

(79)

Constraint on Parameters

$$b_5 b_6 = b_1 b_2.$$ 

(80)

Hypergeometric Solution

A hypergeometric solution is given in terms of the balanced $3\varphi_2$ series [12] as

$$z = \frac{g - 1/b_1}{g - b_5} = \frac{1 - b_3/b_1}{1 - b_1 b_2 b_5 t/b_5} \sum_{n=0}^{\infty} \frac{qb_3 b_5 t/b_2, b_5 b_2 b_3 b_4/b_5; q, b_5 t/b_1}{qb_3 b_2^2/b_2, b_1 b_2^3/b_2; q, b_5 t/b_1}.$$ 

(81)

In the terminating case, e.g. $b_3/b_2 = q^{-N} (N \in \mathbb{Z}_{\geq 0})$, the solution can be rewritten in terms of the terminating $3\varphi_2$ series (big $q$-Jacobi polynomials) as

$$z = \frac{1 - b_2/b_1}{1 - b_1 b_2 b_5 t/b_5} \sum_{n=0}^{\infty} \frac{b_5 t/b_2, b_3/b_2, b_2 b_3 b_4/b_5; q, q}{b_3 b_2^2/b_2, b_1 b_2^3/b_2; q, q},$$ 

(82)

by using the formula [9]

$$3\varphi_2 \left( \frac{a, b, c}{d, e} ; q, de/abc \right) = \frac{(e/b, b/c \infty)}{(e, e/bc)_{\infty}} 3\varphi_2 \left( \frac{d/a, b, c}{d, qbc/e} ; q, q \right) + \frac{(d/a, b, c, de/abc)_{\infty}}{(d, e, bc/e, de/abc)_{\infty}} 3\varphi_2 \left( \frac{e/b, e/c, de/abc}{de/bc, qe/bc} ; q, q \right).$$

(83)

4.4 Case $D_5^{(1)}$

$q$-Painlevé Equation (q-Painlevé VI equation) [5, 14]

$$\begin{cases}
gg = \frac{(f - a_1 t)(f - a_2 t)}{(f - a_3)(f - a_4)}, \\
ff = \frac{(g - b_1 t/q)(g - b_2 t/q)}{(g - b_3)(g - b_4)}.
\end{cases}$$

(84)

where

$$\frac{b_1 b_2}{b_3 b_4} = \frac{a_1 a_2}{a_3 a_4}.$$ 

(85)
Constraint on Parameters
\[ \frac{b_1}{b_3} = q^{-\frac{1}{2}}, \quad \frac{b_2}{b_4} = q. \tag{86} \]

Hypergeometric Solution
A hypergeometric solution is given by [5],
\[ \begin{aligned}
  f &= a_3 \frac{1 - a_4 b_3 / a_3 b_4}{1 - b_3 / b_4} 2 {}_{2} \phi_{1} \left( \begin{array}{c}
    a_3/a_4, a_2 b_1 / a_4 b_1 \\
    a_3 b_2 / a_4 b_3 
  \end{array} ; q, b_1 t / b_3 \right), \\
  g &= b_4 \frac{1 - a_4 b_3 / a_3 b_4}{1 - a_4 / a_3} 2 {}_{2} \phi_{1} \left( \begin{array}{c}
    a_3/a_4, q a_2 b_1 / a_4 b_1 \\
    a_3 b_4 / a_4 b_3 
  \end{array} ; q, b_1 t / q b_3 \right). 
\end{aligned} \tag{87} \]

In the terminating case, the above solution is expressible in terms of the little \( q \)-Jacobi polynomials.

Remark
A class of hypergeometric solutions including the above solution has been constructed in terms of Casorati determinants in [5].

4.5 Case \( A^{(1)}_1 \)

\( q \)-Painlevé Equation (\( q \)-Painlevé V equation) [2, 4, 15]
\[ \begin{aligned}
  \bar{f} g &= \frac{(f + b_1 / t)(f + 1 / b_1 t)}{1 + b_1 f}, \\
  \bar{f} \bar{f} &= \frac{(g + b_2 / s)(g + 1 / b_2 s)}{1 + g / b_3}, \tag{88} \end{aligned} \]

where
\[ \bar{t} = q t, \quad t = q^{\frac{1}{2}} s. \tag{89} \]

Constraint on Parameters
\[ b_1 b_2 b_3^2 = q^{-\frac{1}{2}}. \tag{90} \]

Hypergeometric Solution
\[ \begin{aligned}
  f &= \frac{1 - b_2}{b_2 b_3} \frac{1}{2} {}_{2} \phi_{1} \left( \begin{array}{c}
    q / b_2^2, q b_1^2 \\
    0 
  \end{array} ; q, q^{\frac{1}{2}} b_2 b_3 t \right), \\
  g &= - \frac{1}{b_1 b_2^3 t} 2 {}_{2} \phi_{1} \left( \begin{array}{c}
    1 / b_2^2, b_1^2 \\
    0 
  \end{array} ; q, q^{\frac{1}{2}} b_2 b_3 t \right) \tag{91} \end{aligned} \]

In the terminating case, the solution is expressible in terms of the alternative \( q \)-Charlier polynomials or the \( q \)-Laguerre polynomials [13]. As we mentioned in the previous section, the above solution is rewritten in terms of \( {}_{1} \phi_{1} \) series.
4.6 Case \((A_2 + A_1)^{(1)}\)

\(q\)-Painlevé Equation  \((q\)-Painlevé III equation\) \([4, 6, 7, 16]\)

\[
\begin{align*}
gf & = b_0 \frac{1 + a_0 f}{a_0 t + f}, \\
gff & = b_0 \frac{a_1/t + g}{1 + ga_1/t},
\end{align*}
\]

(92)

where

\(\tau = qt.\)  \hspace{1cm} (93)

**Constraint on Parameters**  Eq.(92) admits two kinds of specialization which yield different hypergeometric solutions:

(1)  \(b_0 = q.\)

(2)  \(a_0 a_1 = q.\)

**Hypergeometric Solution**  \([4, 6, 7]\)

(1)

\[
\begin{align*}
g & = -\frac{a_1}{t} (1 - q^2/a_0^2 a_1^2) \mathbf{1}_1 \varphi_1 \left( \begin{array}{c} 0 \\ q^2 a_0^2 a_1^2 \\ q^2, q^2 t^2/a_1^2 \end{array} \right), \\
f & = \frac{qt}{a_1} \frac{q/a_0 a_1}{1 - q^2/a_0^2 a_1^2} \mathbf{1}_1 \varphi_1 \left( \begin{array}{c} 0 \\ q^4 a_0^2 a_1^2 \\ q^2, q^4 t^2/a_1^2 \end{array} \right).
\end{align*}
\]

(94)

(2)

\[
\begin{align*}
g & = \frac{b_0}{a_0 t} \mathbf{1}_1 \varphi_1 \left( \begin{array}{c} a_0^2 t^2 \\ 0 \\ q^2, q/b_0 \end{array} \right), \\
f & = -\frac{a_0 t}{a_1} \mathbf{1}_1 \varphi_1 \left( \begin{array}{c} a_0^2 t^2 \\ 0 \\ q^2, q^3/b_0 \end{array} \right).
\end{align*}
\]

(95)

This solution is also expressible in terms of the \(\mathbf{1}_1 \varphi_1\) series or a specialization of the \(\mathbf{2}_1 \varphi_1\) series by using the formula \([13]\)

\[
\mathbf{1}_1 \varphi_1 \left( \begin{array}{c} z \\ 0 \\ q, c \end{array} \right) = (c; q) \infty_0 \varphi_1 \left( \begin{array}{c} - \\ c \\ q, c z \end{array} \right)
\]

(96)

\[
= \left( \begin{array}{c} c, z; q \infty_2 \varphi_1 \left( \begin{array}{c} 0 \\ 0 \\ c \end{array} ; q, z \right) \right).
\]

(97)

In the terminating case, the solution is expressible in terms of the Stieltjes-Wigert polynomials \([13]\).

**Remark**  Two classes of Casorati determinant solutions which includes the above solutions as the simplest cases have been constructed in \([6, 7]\).
4.7 Case $(A_1 + A_1')(1)$

$q$-Painlevé Equation  

$q$-Painlevé II equation [2, 4, 17]

\[
(\overline{f} f - 1)(f f - 1) = \frac{at^2 f}{f + t}, \quad \overline{t} = qt.
\]  

(98)

Constraint on Parameter

\[
a = q.
\]  

(99)

Hypergeometric Solution

\[
f = \frac{\phi_1 \left( \begin{array}{c} 0 \\ -q \\ q, -qt \end{array} \right)}{\phi_1 \left( \begin{array}{c} 0 \\ -q \\ q, -t \end{array} \right)}.
\]  

(100)

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