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Abstract

In this paper, we propose an estimator of the correlation dimension of the skelton for chaotic system with dynamic noise and prove the consistency of the estimator under some assumptions.

Key Words: Chaotic Dynamical System, Dynamic Noise, Correlation Dimension, U-statistics, Nadaraya-Watson Kernel Type Estimator.

1 Introduction

We consider trajectory $\{X_t\}_{t=1,2,\dots,N}$ generated from

$$X_t = F(X_{t-\tau}, X_{t-2\tau}, \dots, X_{t-d\tau}) + \varepsilon_t, \tag{1.1}$$

where $F : \mathbf{R}^d \to \mathbf{R}$ is unknown non-linear function such that $\{Y_t\}$ is ergodic where $Y_t = F(Y_{t-1}, Y_{t-2}, \dots, Y_{t-d})$, d and τ are unknown positive integers called embedding dimension and delay time, respectively. ε_t's are random variables on the probability space $(\Omega', \mathcal{F}', \mathcal{E})$, where $E[\varepsilon_t|\mathcal{A}_1^{t-1}(X)] = 0$, almost surely, and $E[\varepsilon_t^2|\mathcal{A}_1^{t-1}(X)] = \sigma^2$, $(\sigma > 0)$, almost surely, where $\mathcal{A}_s^t(X)$ denotes the sigma algebra generated by $(X_s, \dots, X_t)$. $\{\varepsilon_t\}$ is called dynamic noise. Model (1.1) may be represented as

$$\mathbf{X}_t = \mathbf{F}(\mathbf{X}_{t-\tau}) + \mathbf{e}_t, \tag{1.2}$$

where $\mathbf{F}(\mathbf{x}) = {}^t(F(\mathbf{x}), x_1, \dots, x_{d-1})$ for $\mathbf{x} = {}^t(x_1, x_2, \dots, x_d)$, $\mathbf{X}_t = {}^t(X_t,\ X_{t-\tau},\ \dots,\ X_{t-(d-1)\tau})$, and $\mathbf{e}_t = {}^t(\varepsilon_t, 0, \dots, 0)$. Then we consider

$$\mathbf{Y}_t \;=\; \mathbf{F}(\mathbf{Y}_{t-1}), \tag{1.3}$$

where $\mathbf{Y}_t = {}^t(Y_t, Y_{t-1}, \dots, Y_{t-d+1})$.

We refer to model (1.3) as the skeleton of model (1.2). Moreover, we consider a dynamical system $(\Omega, \mathcal{F}, \mu, \mathbf{F})$, where $\Omega \subseteq \mathbf{R}^d$ is closed, $\mathcal{F}$ is the completion of the Borel

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σ-field with respect to μ, and μ is an invariant measure, i.e. $\mu(\mathbf{F}^{-1}A) = \mu(A)$ for $A \in \mathcal{F}$. We propose to estimate the correlation dimension of its skeleton which is defined as follows.

Putting

$$C(r) = \iint_{\Omega\times\Omega} I(\|\mathbf{y}_1 - \mathbf{y}_2\| \leq r) d\mu(\mathbf{y}_1) d\mu(\mathbf{y}_2),$$

where I denotes an indicator function and $\|\cdot\|$ is a norm, Grassberger and Procaccia (1983a, b) defined the correlation dimension as

$$\nu = \lim_{r\to 0} \frac{\log C(r)}{\log r} \tag{1.4}$$

if the limit exists.

Many authors have discussed the effects of noise on dimension estimation. Wolff (1990) is concerned with the behavior of the correlation integral when applied to time series data from autoregressive and moving average processes. Wolff notes that, with such processes, estimates of the correlation dimension underestimate, an effect discussed by Dvořák and Klaschka (1990). Ramsey and Yuan (1989, 1990) looked at the statistical properties of estimates of the correlation dimension. They concluded that small numbers of observations and the presence of noise may result in the estimates being positively biased. The effect of noise on the correlation integral has been studied in a number of papers, Schreiber (1993), Diks (1996), Kugiumtzis (1997), and Oltmans and Verheijen (1997). These results are limited by the case of observation noise, i.e. they consider trajectory $\{X_t\}$ generated from $X_t = Y_t + \varepsilon_t$ where $Y_t = F(Y_{t-1}, Y_{t-2}, \ldots, Y_{t-d})$.

Smith (1992) proposed an estimator of the correlation dimension that seems to function well in the presence of observation noise. Unfortunately, his estimator does not work nearly so well in the case of dynamic noise. Further research on the effect of dynamic noise on the estimator of the correlation dimension seems desirable (Chan and Tong, 2001).

Our goal is to estimate the correlation dimension of F by observing $\{X_t\}$. In this paper, we propose an estimator of the correlation dimension based on the data filtered out noise by the Nadaraya-Watson kernel type estimator and show the consistency of the estimator under some assumptions and conditions mathematically. Kawaguchi (2003) showed the consistency of the estimator of the correlation dimension from the data generated from the deterministic system, taking the same approach as Serinko (1994). This is developed into the proof of the consistency. For filtering out the noise, we take the same approach as Yonemoto and Yanagawa (2003), who proposed an method for estimating Lyapunov exponent from nonlinear time series with dynamic noise and showed the consistency of the estimator.

This paper is organized as follows. We propose a method for estimating the correlation dimension in Section 2. In Section 3, the consistency of the estimator is proved.

2 The procedure for estimating the correlation dimension and theorem

We propose a procedure for estimating the correlation dimension from $\{X_t\}_{t=1,2,\ldots,N}$. Outline of the procedure is as follows.

Step 1. Estimate the embedding dimension d and delay time τ from $\{X_t\}_{t=1,2,\ldots,N}$ using the procedure proposed by Yonemoto and Yanagawa (2002). Denote the estimated embedding dimension and delay time by $\hat{d}$ and $\hat{\tau}$, respectively.

Step 2. Estimate the skeleton from $\{X_t\}_{t=1,2,\ldots,N}$ by the Nadaraya - Watson kernel type estimator (Nadaraya, 1964; Watson, 1964) using $\hat{d}$ and $\hat{\tau}$, and generating $\{\hat{Y}_{N,t}\}_{t=1,2,\ldots,K}$ from the estimated skeleton by giving an appropriate initial value.

Step 3. Estimate the correlation dimension by using the generated data.

We give the details of Step 2 and 3.

(Step 2 of the procedure)

Yonemoto and Yanagawa (2003) proposed a following method for estimating the skeleton. First, for $\mathbf{x} = {}^t(x_1, x_2, \ldots, x_d)$, $F(\mathbf{x})$ is estimated by

$$\hat{F}_N(\mathbf{x}) = \frac{\sum_{i=(\hat{d}-1)\hat{\tau}+1}^{N-\hat{\tau}} K_{h_N}(\mathbf{x}-\mathbf{X}_i)X_{i+\hat{\tau}}}{\sum_{i=(\hat{d}-1)\hat{\tau}+1}^{N-\hat{\tau}} K_{h_N}(\mathbf{x}-\mathbf{X}_i)},$$

where

$$K_{h_N}(\mathbf{x}-\mathbf{X}_i) = \frac{1}{h_N^{\hat{d}}} K\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_N}\right),$$

$K(\mathbf{x}) = \prod_{i=1}^{\hat{d}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x_i^2}{2}\right)$ which is called kernel function, and $h_N \in \mathbf{R}_{>0} \to 0$ as $N \to \infty$ which is called bandwidth. Next, put $\hat{\mathbf{F}}_N(\mathbf{x}) = {}^{\mathbf{t}}(\hat{F}_N(\mathbf{x}), x_1, \ldots, x_{\hat{d}-1})$. Giving an appropriate initial vector $\hat{\mathbf{Y}}_{N,0}$, we generate $\{\hat{Y}_{N,t}\}_{t=1,2,\ldots,K}$ by

$$\hat{\mathbf{Y}}_{N,t} = \hat{\mathbf{F}}_N(\hat{\mathbf{Y}}_{N,t-1}), \quad t = 1, 2, \ldots, K, \tag{2.1}$$

where $\hat{\mathbf{Y}}_{N,t} = {}^t(\hat{Y}_{N,t}, \hat{Y}_{N,t-1}, \ldots, \hat{Y}_{N,t-\hat{d}+1})$.

(Step 3 of the procedure)

Let
$$C_K(r, \hat{\mathbf{Y}}_N) = \binom{K}{2}^{-1} \sum_{i<j}^{K} I(\|\hat{\mathbf{Y}}_{N,i} - \hat{\mathbf{Y}}_{N,j}\| \le r),$$
$$C_{K2}(r, \hat{\mathbf{Y}}_N) = \binom{K}{3}^{-1} \sum_{i\ne j, i\ne k, j\ne k}^{K} I(\|\hat{\mathbf{Y}}_{N,i} - \hat{\mathbf{Y}}_{N,j}\| \le r, \|\hat{\mathbf{Y}}_{N,i} - \hat{\mathbf{Y}}_{N,k}\| \le r),$$

and $r_j^{(M_{N,K})} = r_0 s^{M_{N,K} - \left[\frac{M_{N,K}}{2}\right] + j}$, $(j = 0, 1, \ldots, L_{N,K} = [\frac{M_{N,K}}{2}])$ where some given $0 < s < 1$, $r_0 > 1$, and

$$M_{N,K} = \max\{m \in \mathbf{Z}_{>0};\ C_{K2}(r_m, \hat{\mathbf{Y}}_N) \neq 0,\ a.e. \text{ for } r_m = r_0 s^m\}. \tag{2.2}$$

We propose an estimator of the correlation dimension as follows.

$$\hat{\nu}_{N,K} = \sum_{j=0}^{L_{N,K}} (u_j - \bar{u}) \log C_K(r_j^{(M_{N,K})}, \hat{\mathbf{Y}}_N) \Big/ \sum_{j=0}^{L_{N,K}} (u_j - \bar{u})^2 \tag{2.3}$$

where, $u_j = \log r_j^{(M_{N,K})}$ and $\bar{u} = (L_{N,K} + 1)^{-1} \sum_{j=0}^{L_{N,K}} u_j$.

In this section, our goal is to prove the consistency of this estimator. For the estimator of the embedding dimension and delay time, the consistency was proved in Fueda and Yanagawa (2001). For convenience, denote $\hat{\nu}(\hat{F}_N(\hat{d}, \hat{\tau}), \hat{d})$ by the estimator (2.3) and $\nu(F(d, \tau), d)$ by the correlation dimension (1.4). Let $\mathcal{E} \times \mu$ denote a product measure of an invariant measure for dynamical system and a measure for dynamic noise. Then, for any $\varepsilon > 0$, there exists $N_0 \in \mathbf{N}$ such that for any $N > N_0$

$$\begin{aligned}
&\mathcal{E} \times \mu(|\hat{\nu}(\hat{F}_N(\hat{d}, \hat{\tau}), \hat{d}) - \nu(F(d, \tau), d)| > \varepsilon) \\
&\quad\leq\ \mathcal{E} \times \mu(|\hat{\nu}(\hat{F}_N(\hat{d}, \hat{\tau}), \hat{d}) - \nu(F(d, \tau), d)| > \varepsilon,\ \hat{d} = d) + \mathcal{E} \times \mu(\hat{d} \neq d) \\
&\quad=\ \mathcal{E} \times \mu(|\hat{\nu}(\hat{F}_N(d, \hat{\tau}), d) - \nu(F(d, \tau), d)| > \varepsilon) \\
&\quad\leq\ \mathcal{E} \times \mu(|\hat{\nu}(\hat{F}_N(d, \hat{\tau}), d) - \nu(F(d, \tau), d)| > \varepsilon,\ \hat{\tau} = \tau) + \mathcal{E} \times \mu(\hat{\tau} \neq \tau) \\
&\quad=\ \mathcal{E} \times \mu(|\hat{\nu}(\hat{F}_N(d, \tau), d) - \nu(F(d, \tau), d)| > \varepsilon).
\end{aligned}$$

Hence in this section we assume that the embedding dimension and delay time are known.

Theorem 2.1. *Under the assumptions that are given in the following section, it follows that for any $\varepsilon > 0$,*

$$\lim_{K \to \infty} \lim_{N \to \infty} \mathcal{E} \times \mu(|\hat{\nu}_{N,K} - \nu| > \varepsilon) = 0.$$

3 Proof of Theorem 2.1

In this section, we give a proof of Theorem 2.1. At first, we decompose the estimator as follows.

Lemma 3.1. (Serinko, 1994)

$$\hat{\nu}_{N,K} = \nu + d_{N,K} + e_{N,K},$$

where

$$\begin{aligned}
d_{N,K} &= \frac{1}{S_{uu}} \sum_{j=0}^{L_{N,K}} \left\{ \log C(r_j^{(M_{N,K})}) - \nu \log r_j^{(M_{N,K})} \right\} (u_j - \bar{u}), \\
e_{N,K} &= \frac{1}{S_{uu}} \sum_{j=0}^{L_{N,K}} \left\{ \log C_K(r_j^{(M_{N,K})}, \hat{\mathbf{Y}}_N) - \log C(r_j^{(M_{N,K})}) \right\} (u_j - \bar{u}), \\
&\textit{and} \quad S_{uu} = \sum_{j=0}^{L_{N,K}} (u_j - \bar{u})^2.
\end{aligned}$$

Next, we prove the convergence of $d_{N,K}$ in probability. We assume the following assumptions.

Assumption 3.1. *There exists a compact set $G \subset \mathbf{R}^d$ such that for any $\mathbf{x} \in G$,*

$$\mathbf{F}(\mathbf{x}) + \mathbf{e} \in G \qquad a.e.,$$

where $\mathbf{e} = {}^t(\varepsilon, 0, \ldots, 0)$ and ε is identically distributed as $\{\varepsilon_t\}$.

Assumption 3.2. *F is C^1 class on G.*

Assumption 3.3. *For any $N \in \mathbf{N}$ such that $N \geq (d-1)\tau + 1$,*

$$\{\mathbf{X}_t\}_{t=(d-1)\tau+1,(d-1)\tau+2,\ldots,N} \subset G.$$

Assumption 3.4. *$\hat{F}_N$ is such that*

$$\lim_{N\to\infty} \mathcal{E} \times \mu \left(\sup_{\mathbf{x}\in G} |\hat{F}_N(\mathbf{x}) - F(\mathbf{x})| > \varepsilon \right) = 0.$$

Select an initial vector $\mathbf{Y}_1$ randomly from G° with uniform probability, and set $\hat{\mathbf{Y}}_{N,1} = \mathbf{Y}_1$, where G° denotes an interior of G. Then the following theorem holds.

Theorem 3.1. (Yonemoto and Yanagawa, 2003)
Under Assumption 3.1 through 3.4, for any $\varepsilon > 0$ and $t \in \{1, 2, \ldots, K\}$,

$$\lim_{N\to\infty} \mathcal{E} \times \mu(\|\hat{\mathbf{Y}}_{N,t} - \mathbf{Y}_t\| > \varepsilon) = 0.$$

Let $C_{K2}(r, \mathbf{Y}) = \binom{K}{3}^{-1} \sum_{i\neq j, i\neq k, j\neq k}^{K} I(\|\mathbf{Y}_i - \mathbf{Y}_j\| \leq r, \|\mathbf{Y}_i - \mathbf{Y}_k\| \leq r)$, we assum the following assumption.

Assumption 3.5. *For any $r > 0$ and $\varepsilon > 0$,*

$$\lim_{K\to\infty} \mu(C_{K2}(r, \mathbf{Y}) > \varepsilon) = 1.$$

Lemma 3.2. *Under Assumption 3.5 and the condition of Theorem 3.1, for $M_{N,K}$ in (2.2),*

$$\mathcal{E} \times \mu \left(\lim_{K\to\infty} \lim_{N\to\infty} M_{N,K} = \infty \right) = 1.$$

Proof. For any integers m, N_0 and K_0, let

$$\begin{aligned} &A(m, N_0, K_0) \\ &\quad = \Big\{ (\omega, \omega') \in \Omega \times \Omega'; C_{K2}(r_l, \hat{\mathbf{Y}}_N) > \varepsilon, \\ &\qquad\qquad \text{for all } N \geq N_0,\ K \geq K_0,\ l = 0, 1, \ldots, m, \text{ and } \varepsilon > 0 \\ &\qquad\qquad\qquad \textit{where } r_l = r_0 s^l \Big\}. \end{aligned}$$

Then, $A(m, N_0, K_0) \subset \{M_{N,K} \geq m$ for $N \geq N_0$ and $K \geq K_0\}$, and therefore for any m

$$\begin{aligned}\mathcal{E} \times \mu\left(\lim_{K\to\infty}\lim_{N\to\infty} M_{N,K} \geq m\right) &\geq \mathcal{E} \times \mu\left(\bigcap_{K_0=1}^{\infty}\bigcap_{N_0=1}^{\infty} A(m, N_0, K_0)\right) \\ &= \lim_{K_0\to\infty}\lim_{N_0\to\infty} \mathcal{E} \times \mu(A(m, N_0, K_0))\end{aligned}$$

Thus, in order to prove the lemma, it must be shown that for any m

$$\lim_{K_0\to\infty}\lim_{N_0\to\infty} \mathcal{E} \times \mu(A(m, N_0, K_0)) = 1.$$

For any $r > 0$ and $t \in \{1, 2, \ldots, K\}$,

$$\begin{aligned}&\mathcal{E} \times \mu(C_{K2}(r, \hat{\mathbf{Y}}_N) > \varepsilon) \\ &\quad= \mathcal{E} \times \mu(C_{K2}(r, \hat{\mathbf{Y}}_N) > \varepsilon \textit{ and } \|\hat{\mathbf{Y}}_{N,t} - \mathbf{Y}_t\| > r/3) \\ &\qquad\qquad + \mathcal{E} \times \mu(C_{K2}(r, \hat{\mathbf{Y}}_N) > \varepsilon \textit{ and } \|\hat{\mathbf{Y}}_{N,t} - \mathbf{Y}_t\| \leq r/3).\end{aligned}$$

From Theorem 3.1,

$$\begin{aligned}\mathcal{E} \times \mu(C_{K2}(r, \hat{\mathbf{Y}}_N) > \varepsilon \textit{ and } \|\hat{\mathbf{Y}}_{N,t} - \mathbf{Y}_t\| > r/3) &\leq \mathcal{E} \times \mu(\|\hat{\mathbf{Y}}_{N,t} - \mathbf{Y}_t\| > r/3) \\ &\to 0 \qquad (N \to \infty).\end{aligned}$$

Further,

$$\begin{aligned}&\mathcal{E} \times \mu(C_{K2}(r, \hat{\mathbf{Y}}_N) > \varepsilon \textit{ and } \|\hat{\mathbf{Y}}_{N,t} - \mathbf{Y}_t\| \leq r/3) \\ &\geq \mathcal{E} \times \mu\Bigg(\binom{K}{3}^{-1} \sum_{i\neq j, i\neq k, j\neq k}^{K} I\Big(\|\hat{\mathbf{Y}}_{N,i} - \mathbf{Y}_i\| + \|\mathbf{Y}_i - \mathbf{Y}_j\| + \|\hat{\mathbf{Y}}_{N,j} - \mathbf{Y}_j\| \leq r, \\ &\qquad \|\hat{\mathbf{Y}}_{N,i} - \mathbf{Y}_i\| + \|\mathbf{Y}_i - \mathbf{Y}_k\| + \|\hat{\mathbf{Y}}_{N,k} - \mathbf{Y}_k\| \leq r\Big) > \varepsilon \\ &\qquad \textit{and } \|\hat{\mathbf{Y}}_{N,t} - \mathbf{Y}_t\| \leq r/3\Bigg) \\ &= \mathcal{E} \times \mu\Bigg(\binom{K}{3}^{-1} \sum_{i\neq j, i\neq k, j\neq k}^{K} I\Big(\|\mathbf{Y}_i - \mathbf{Y}_j\| \leq r - \|\hat{\mathbf{Y}}_{N,i} - \mathbf{Y}_i\| - \|\hat{\mathbf{Y}}_{N,j} - \mathbf{Y}_j\|, \\ &\qquad \|\mathbf{Y}_i - \mathbf{Y}_k\| \leq r - \|\hat{\mathbf{Y}}_{N,i} - \mathbf{Y}_i\| - \|\hat{\mathbf{Y}}_{N,k} - \mathbf{Y}_k\|\Big) > \varepsilon \\ &\qquad \textit{and } \|\hat{\mathbf{Y}}_{N,t} - \mathbf{Y}_t\| \leq r/3\Bigg) \\ &\geq \mathcal{E} \times \mu\Bigg(\binom{K}{3}^{-1} \sum_{i\neq j, i\neq k, j\neq k}^{K} I\Big(\|\mathbf{Y}_i - \mathbf{Y}_j\| \leq r/3, \|\mathbf{Y}_i - \mathbf{Y}_k\| \leq r/3\Big) > \varepsilon \\ &\qquad \textit{and } \|\hat{\mathbf{Y}}_{N,t} - \mathbf{Y}_t\| \leq r/3\Bigg) \\ &\geq \mathcal{E} \times \mu(C_{K2}(r/3, \mathbf{Y}) > \varepsilon) + \mathcal{E} \times \mu(\|\hat{\mathbf{Y}}_{N,t} - \mathbf{Y}_t\| \leq r/3) - 1.\end{aligned}$$

From Theorem 3.1 and Assumption 3.5,

$$\begin{aligned}
& \mathcal{E} \times \mu(C_{K2}(r/3, \mathbf{Y}) > \varepsilon) + \mathcal{E} \times \mu(\|\hat{\mathbf{Y}}_{N,t} - \mathbf{Y}_t\| \leq r/3) - 1 \\
\to \quad & \mathcal{E} \times \mu(C_{K2}(r/3, \mathbf{Y}) > \varepsilon) \qquad (N \to \infty) \\
\to \quad & 1 \qquad (K \to \infty).
\end{aligned}$$

Hence, for any m,

$$\lim_{K\to\infty} \lim_{N\to\infty} \mathcal{E} \times \mu(C_{K2}(r_m, \hat{\mathbf{Y}}_N) > \varepsilon) = 1.$$

From the definition of $A(m, N_0, K_0)$, the proof is completed. □

Lemma 3.3. *Under the condition of Lemma 3.2, for any* $\varepsilon > 0$

$$\lim_{K\to\infty} \lim_{N\to\infty} \mathcal{E} \times \mu(|d_{N,K}| > \varepsilon) = 0.$$

Proof. First, from the definition of the correlation dimension,

$$\frac{\log C(r) - \nu \log r}{\log r} \to 0 \qquad (r \to 0).$$

Thus, there exists a positive real $A(r) \to 0$ as $r \to 0$ such that for sufficiently small $r > 0$,

$$|\log C(r) - \nu \log r| \leq A(r)|\log r|.$$

Next, it follows by the Cauchy-Schwarz inequality that

$$\begin{aligned}
|d_{N,K}| \quad & \leq \quad \frac{1}{S_{uu}} \sqrt{\sum_{j=0}^{L_{N,K}} \left\{\log C(r_j^{(M_{N,K})}) - \nu \log r_j^{(M_{N,K})}\right\}^2} \sqrt{\sum_{j=0}^{L_{N,K}} (u_j - \bar{u})^2} \\
& \leq \quad \frac{1}{\sqrt{S_{uu}}} \sqrt{\sum_{j=0}^{L_{N,K}} \left\{A(r_j^{(M_{N,K})}) \log r_j^{(M_{N,K})}\right\}^2} \\
& \leq \quad \frac{1}{\sqrt{S_{uu}}} \times \max_{0\leq j\leq L_{N,K}} A(r_j^{(M_{N,K})}) \times \sqrt{\sum_{j=0}^{L_{N,K}} \left\{\log r_j^{(M_{N,K})}\right\}^2}.
\end{aligned}$$

Serinko(1994) gave

$$S_{uu} = \frac{1}{12} L_{N,K}(L_{N,K} + 1)(L_{N,K} + 2)(\log s)^2.$$

Thus, from Lemma 3.2 and the definition of $L_{N,K}$,

$$\sqrt{S_{uu}} = O_p({M_{N,K}}^{\frac{3}{2}}).$$

On the other hand, from Lemma 3.2,

$$\sqrt{\sum_{j=0}^{L_{N,K}} \left\{\log r_j^{(M_{N,K})}\right\}^2} = O_p({M_{N,K}}^{\frac{3}{2}}),$$

and for any $j \in \{0, 1, \ldots, L_{N,K}\}$,

$$\mathcal{E} \times \mu \left(\lim_{K\to\infty} \lim_{N\to\infty} A(r_j^{(M_{N,K})}) = 0 \right) = 1.$$

Therefore,

$$\mathcal{E} \times \mu \left(\lim_{K\to\infty} \lim_{N\to\infty} \left\{ \frac{1}{\sqrt{S_{uu}}} \max_{0\le j\le L_{N,K}} A(r_j^{(M_{N,K})}) \sqrt{\sum_{j=0}^{L_{N,K}} \left\{ \log r_j^{(M_{N,K})} \right\}^2} \right\} = 0 \right) = 1$$

and

$$\mathcal{E} \times \mu \left(\lim_{K\to\infty} \lim_{N\to\infty} |d_{N,K}| = 0 \right) = 1.$$

The convergence with probability 1 implies the convergence in probability, therefore the proof is completed. □

Next, we prove the convergence of $e_{N,K}$ in probability. In order to state the next theorem, we give some notations and assumptions. Let $\alpha = \{A_1, A_2, \ldots, A_m\}$ and $\beta = \{B_1, B_2, \ldots, B_n\}$ be finite measurable partitions of Ω. From these one may construct the following partitions:

1. $\alpha \vee \beta = \{A \cap B;\ A \in \alpha,\ B \in \beta\}$
2. $\mathbf{F}^{-1}\alpha = \{\mathbf{F}^{-1}A;\ A \in \alpha\}$
3. $\alpha_r^s = \mathbf{F}^{(-r)}\alpha \vee \mathbf{F}^{(-r-1)}\alpha \vee \cdots \vee \mathbf{F}^{(-s+1)}\alpha \vee \mathbf{F}^{(-s)}\alpha \quad (r, s \in \mathbf{N}, s.t.\ r < s)$,

where $\mathbf{F}^{(k)}$ denotes k times convolution of $\mathbf{F}$.

Let $\mathcal{F}_r^s$ denotes the σ-algebra generated by α_r^s $(r, s \in \mathbf{N},\ s.t.\ r < s)$ and $\mathcal{F}_0^\infty$ denotes the smallest σ-algebra which contains all of the $\mathcal{F}_r^s$ $(r, s \in \mathbf{N},\ s.t.\ r < s)$.

Definition 3.1. (Generator)

$$\alpha \textit{ is generator} \Longleftrightarrow \mathcal{F}_0^\infty = \mathcal{F}$$

Definition 3.2. (Weak Bernoulli)
A measurable partition α is said to be weak Bernoulli for dynamical system $(\Omega, \mathcal{F}, \mu, \mathbf{F})$ if

$$\beta_k = \sup_{r,s\in\mathbf{N}} \sum_{A\in\alpha_0^r} \sum_{B\in\alpha_{r+k}^{r+s+k}} |\mu(A \cap B) - \mu(A)\mu(B)|$$

goes to zero as $k \to \infty$. The β_k's are called the mixing coefficients.

We assume the following assumptions for the dynamical system $(\Omega, \mathcal{F}, \mu, \mathbf{F})$.

Assumption 3.6. $(\Omega, \mathcal{F}, \mu, \mathbf{F})$ *has the measurable partition α which is weak Bernoulli and generator.*

Assumption 3.7. $(\Omega, \mathcal{F}, \mu, \mathbf{F})$ *is such that the mixing coefficients satisfy*

$$\beta_k^{\frac{\delta}{2+\delta}} = O(k^{-(1+\varepsilon)})$$

for some $\delta > 0$ *and* $0 < \varepsilon < 1$.

Let $\eta^{(l)}(r) = \|\mu(\bar{B}_r(\mathbf{Y}_j)) - \mu(\bar{B}_r(\mathbf{Y}_j^{(l)}))\|_2^2$, where $\bar{B}_r(\mathbf{y}) = \{\mathbf{x}; \|\mathbf{x} - \mathbf{y}\| \leq r\}$ and $\mathbf{Y}_j^{(l)} = E[\mathbf{Y}_j|\alpha_j^{j+l}]$, $j = 1, 2, \cdots$. Let $\bar{\eta}^{(l)} = \sup_r \eta^{(l)}(r)$.

Assumption 3.8. $(\Omega, \mathcal{F}, \mu, \mathbf{F})$ *is such that*

$$\bar{\eta}^{(l)\frac{1}{2}} = o(l^{-(1+\gamma)})$$

for some $\gamma > 0$, $\gamma/(1+\gamma) > \varepsilon$.

Let $\psi_k^{(l)}(r) = \|I(\|\mathbf{Y}_i - \mathbf{Y}_j\| \leq r) - I(\|\mathbf{Y}_i^{(l)} - \mathbf{Y}_j^{(l)}\| \leq r)\|_2^2$ for $i, j = 1, 2, \cdots$ and $k = |i - j|$, and $\bar{\psi}_k^{(l)} = \sup_r \psi_k^{(l)}(r)$.

Assumption 3.9. $(\Omega, \mathcal{F}, \mu, \mathbf{F})$ *is such that for any sequence of reals* $\{c_n\}_{n=0}^{\infty}$ *satisfying* $\lim_{n\to\infty} c_n = \infty$ *and* $c_n = o(n^{\frac{1}{2}})$, *one has*

$$\sum_{k=0}^{n-1} \bar{\psi}_k^{(c_n)\frac{1}{2}} = o(n^{\frac{1}{2}}).$$

Let $b_K = \left(\frac{1}{K}\right)^{\frac{1}{2(\nu+\varepsilon_0)}}$ where $\varepsilon_0 > 0$, and

$$C_K(r, \mathbf{Y}) = \binom{K}{2}^{-1} \sum_{i<j}^{K} I(\|\mathbf{Y}_i - \mathbf{Y}_j\| \leq r).$$

Theorem 3.2. (Serinko, 1994)
If $(\Omega, \mathcal{F}, \mu, \mathbf{F})$ *satisfies Assumption.3.6 through 3.9, then whenever* ν *exists,*

$$\lim_{K\to\infty} r_K = 0 \quad \textit{and} \quad \limsup_{K\to\infty} \frac{b_K}{r_K} < \infty$$

imply

$$\lim_{K\to\infty} \mu\left(\left|\frac{C_K(r_K, \mathbf{Y}) - C(r_K)}{C(r_K)}\right| > \varepsilon\right) = 0.$$

Let $M_K = \lim_{N\to\infty} M_{N,K}$. We may prove that $M_K = \max\{m \in \mathbf{Z}_{>0};\ C_{K2}(r_m, \mathbf{Y}) \neq 0,\ \textit{a.e.}$ for $r_m = r_0 s^m\}$ similarly as the proof of Lemma 3.5.

Assumption 3.10. *For some* $\varepsilon_0 > 0$ *and some* $\delta < \frac{1}{d+\varepsilon_0}$, *there exists* $K_0 \in \mathbf{N}$ *such that for any* $K > K_0$,

$$M_K > \frac{\delta \log K}{2|\log s|} \quad \textit{a.e.}.$$

Lemma 3.4. *Under Assumption 3.10, let* $r_K = r_0 s^{M_K}$*, then*

$$\lim_{K\to\infty} r_K = 0 \quad \textit{and} \quad \limsup_{K\to\infty} \frac{b_K}{r_K} < \infty \qquad \textit{a.e.}.$$

Proof. For the former part of lemma, we obtain immediately from Lemma 3.2. Setting $r'_K = K^{\frac{-1}{2(d+\varepsilon_0)}}$, Serinko(1994) proved

$$\limsup_{K\to\infty} \frac{b_K}{r'_K} < \infty.$$

Thus, the last part of lemma is proved if we prove $r_K > r'_K$ a.e. for any $K > K_0$.

From Assumption 3.10, we have

$$\begin{aligned}
\log \frac{r_K}{r'_K} &= \log \frac{r_0 s^{M_K}}{K^{\frac{-1}{2(d+\varepsilon_0)}}} \\
&= \log r_0 + M_K \log s + \frac{1}{2(d+\varepsilon_0)} \log K \\
&> \log r_0 + \left(-\delta + \frac{1}{d+\varepsilon_0}\right) \frac{1}{2} \log K > 0.
\end{aligned}$$

□

Lemma 3.5. *Under the condition of Theorem 3.1, for a fixed* K*, any* $\varepsilon > 0$ *and* $r > 0$

$$\lim_{N\to\infty} \mathcal{E} \times \mu \left(\left| C_K(r, \hat{\mathbf{Y}}_N) - C_K(r, \mathbf{Y}) \right| > \varepsilon \right) = 0.$$

Proof. It follows by the Markov inequality that

$$\begin{aligned}
\mathcal{E} \times \mu \left(\left| C_K(r, \hat{\mathbf{Y}}_N) - C_K(r, \mathbf{Y}) \right| > \varepsilon \right) &\le \frac{1}{\varepsilon} E[|C_K(r, \hat{\mathbf{Y}}_N) - C_K(r, \mathbf{Y})|] \\
= \frac{1}{\varepsilon} E \left[\left| \binom{K}{2}^{-1} \sum_{i<j}^{K} \{ I(\|\hat{\mathbf{Y}}_{N,i} - \hat{\mathbf{Y}}_{N,j}\| \le r) - I(\|\mathbf{Y}_i - \mathbf{Y}_j\| \le r) \} \right| \right] \\
\le \frac{1}{\varepsilon} \binom{K}{2}^{-1} \sum_{i<j}^{K} E[|I(\|\hat{\mathbf{Y}}_{N,i} - \hat{\mathbf{Y}}_{N,j}\| \le r) - I(\|\mathbf{Y}_i - \mathbf{Y}_j\| \le r)|]
\end{aligned}$$

It follows by the triangle inequality that for $i, j \in \{1, 2, \ldots, K\}$ $(i < j)$

$$\begin{aligned}
E[|I(\|\hat{\mathbf{Y}}_{N,i} - \hat{\mathbf{Y}}_{N,j}\| \le r) - I(\|\mathbf{Y}_i - \mathbf{Y}_j\| \le r)|] \\
\le E[|I(\|\hat{\mathbf{Y}}_{N,i} - \hat{\mathbf{Y}}_{N,j}\| \le r) - \phi_\delta(\hat{\mathbf{Y}}_{N,i} - \hat{\mathbf{Y}}_{N,j})|] \\
+E[|\phi_\delta(\mathbf{Y}_i - \mathbf{Y}_j) - I(\|\mathbf{Y}_i - \mathbf{Y}_j\| \le r)|] \\
+E[|\phi_\delta(\hat{\mathbf{Y}}_{N,i} - \hat{\mathbf{Y}}_{N,j}) - \phi_\delta(\mathbf{Y}_i - \mathbf{Y}_j)|]
\end{aligned} \tag{3.1}$$

where for any $\delta > 0$,

$$\phi_\delta(\mathbf{x}) = \begin{cases} 0 & \text{if } \|\mathbf{x}\| \ge r \\ 1 & \text{if } \|\mathbf{x}\| \le r - \delta \end{cases},$$

$0 \le \phi_\delta(\mathbf{x}) \le 1$ and $\phi_\delta \in C^1$.

The first term of (3.1) is

$$\begin{aligned} &\iint (I(\|\mathbf{x}-\mathbf{y}\| \le r) - \phi_\delta(\mathbf{x}-\mathbf{y}))d(\mathcal{E}\times\mu)(\mathbf{x})d(\mathcal{E}\times\mu)(\mathbf{y}) \\ &\quad \le \iint_{r-\delta<\|\mathbf{x}-\mathbf{y}\|<r} d(\mathcal{E}\times\mu)(\mathbf{x})d(\mathcal{E}\times\mu)(\mathbf{y}) \\ &\quad = \mathcal{E}\times\mu(r-\delta < \|\hat{\mathbf{Y}}_{N,i} - \hat{\mathbf{Y}}_{N,j}\| < r). \end{aligned}$$

Thus, for any $\varepsilon_1 > 0$, there exists $\delta_1 \in \mathbf{N}$ such that for any $\delta < \delta_1$

$$E[|I(\|\hat{\mathbf{Y}}_{N,i} - \hat{\mathbf{Y}}_{N,j}\| \le r) - \phi_\delta(\hat{\mathbf{Y}}_{N,i} - \hat{\mathbf{Y}}_{N,j})|] < \varepsilon_1.$$

By the same way, for any $\varepsilon_2 > 0$, there exists $\delta_2 \in \mathbf{N}$ such that for any $\delta < \delta_2$

$$E[|\phi_\delta(\mathbf{Y}_i - \mathbf{Y}_j) - I(\|\mathbf{Y}_i - \mathbf{Y}_j\| \le r)|] < \varepsilon_2.$$

Therefore, from the continuity of ϕ_δ and Theorem 3.1, for any $\delta < \min\{\delta_1, \delta_2\}$,

$$\begin{aligned} &E[|I(\|\hat{\mathbf{Y}}_{N,i} - \hat{\mathbf{Y}}_{N,j}\| \le r) - I(\|\mathbf{Y}_i - \mathbf{Y}_j\| \le r)|] \\ &\quad < \varepsilon_1 + \varepsilon_2 + E[|\phi_\delta(\hat{\mathbf{Y}}_{N,i} - \hat{\mathbf{Y}}_{N,j}) - \phi_\delta(\mathbf{Y}_i - \mathbf{Y}_j)|] \\ &\quad \to \varepsilon_1 + \varepsilon_2 \qquad (N \to \infty). \end{aligned}$$

Consequently, $E[|I(\|\hat{\mathbf{Y}}_{N,i} - \hat{\mathbf{Y}}_{N,j}\| \le r) - I(\|\mathbf{Y}_i - \mathbf{Y}_j\| \le r)|] \to 0$ as $N \to \infty$ for $i, j \in \{1, 2, \ldots, K\}$ $(i < j)$. Hence, the proof is completed. □

Lemma 3.6. *Under the condition of Theorem 3.2 and Lemma 3.5, for any* $\varepsilon > 0$,

$$\lim_{K\to\infty}\lim_{N\to\infty} \mathcal{E}\times\mu(|\log C_K(r_K, \hat{\mathbf{Y}}_N) - \log C(r_K)| > \varepsilon) = 0.$$

Proof.

$$\begin{aligned} &\mathcal{E}\times\mu\left(\left|\log\frac{C_K(r_K,\hat{\mathbf{Y}}_N)}{C(r_K)}\right| > \varepsilon\right) \\ &= \mathcal{E}\times\mu\left(\log\frac{C_K(r_K,\hat{\mathbf{Y}}_N)}{C(r_K)} > \varepsilon\right) + \mathcal{E}\times\mu\left(\log\frac{C_K(r_K,\hat{\mathbf{Y}}_N)}{C(r_K)} < -\varepsilon\right) \\ &= \mathcal{E}\times\mu\left(\frac{C_K(r_K,\hat{\mathbf{Y}}_N)}{C(r_K)} > e^{\varepsilon}\right) + \mathcal{E}\times\mu\left(\frac{C_K(r_K,\hat{\mathbf{Y}}_N)}{C(r_K)} < e^{-\varepsilon}\right) \\ &= \mathcal{E}\times\mu\left(\frac{C_K(r_K,\hat{\mathbf{Y}}_N) - C(r_K)}{C(r_K)} > e^{\varepsilon} - 1\right) \\ &\qquad + \mathcal{E}\times\mu\left(\frac{-C_K(r_K,\hat{\mathbf{Y}}_N) + C(r_K)}{C(r_K)} > 1 - e^{-\varepsilon}\right). \end{aligned}$$

For any $\varepsilon' > 0$,

$$\begin{aligned}
&\mathcal{E} \times \mu\left(\left|\frac{C_K(r_K, \hat{\mathbf{Y}}_N) - C(r_K)}{C(r_K)}\right| > \varepsilon'\right) \\
&\quad\leq\ \mathcal{E} \times \mu\left(\left|\frac{C_K(r_K, \hat{\mathbf{Y}}_N) - C_K(r_K, \mathbf{Y})}{C(r_K)}\right| > \frac{\varepsilon'}{2}\right) \\
&\qquad\qquad + \mathcal{E} \times \mu\left(\left|\frac{C_K(r_K, \mathbf{Y}) - C(r_K)}{C(r_K)}\right| > \frac{\varepsilon'}{2}\right).
\end{aligned}$$

From Lemma 3.5, the first term goes to zero as $N \to \infty$ for a fixed K. From Theorem 3.2, the second term goes to zero as $K \to \infty$. □

Lemma 3.7. *Under the condition of Lemma 3.2, Lemma 3.6, and Assumption 3.10 for any* $\varepsilon > 0$,

$$\lim_{K\to\infty} \lim_{N\to\infty} \mathcal{E} \times \mu(|e_{N,K}| > \varepsilon) = 0.$$

Proof. It follows by the Cauchy-Schwarz inequality that

$$\begin{aligned}
|e_{N,K}| &\leq \frac{1}{S_{uu}} \sqrt{\sum_{j=0}^{L_{N,K}} \left\{\log C_K(r_j^{(M_{N,K})}, \hat{\mathbf{Y}}_N) - \log C(r_j^{(M_{N,K})})\right\}^2} \sqrt{\sum_{j=0}^{L_{N,K}} (u_j - \bar{u})^2} \\
&\leq \frac{1}{\sqrt{S_{uu}}} \sqrt{L_{N,K} + 1} \max_{0\leq j\leq L_{N,K}} \left|\log C_K(r_j^{(M_{N,K})}, \hat{\mathbf{Y}}_N) - \log C(r_j^{(M_{N,K})})\right| \\
&= \left\{\frac{1}{12}(\log s)^2 L_{N,K}(L_{N,K} + 2)\right\}^{-\frac{1}{2}} \\
&\qquad \times \max_{0\leq j\leq L_{N,K}} \left|\log C_K(r_j^{(M_{N,K})}, \hat{\mathbf{Y}}_N) - \log C(r_j^{(M_{N,K})})\right|
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\mathcal{E} \times \mu\Bigg(\left\{\frac{1}{12}(\log s)^2 L_{N,K}(L_{N,K} + 2)\right\}^{-\frac{1}{2}} \\
&\qquad\qquad \times \max_{0\leq j\leq L_{N,K}} \left|\log C_K(r_j^{(M_{N,K})}, \hat{\mathbf{Y}}_N) - \log C(r_j^{(M_{N,K})})\right| > \varepsilon\Bigg) \\
&\leq\ \mathcal{E} \times \mu\left(\left\{\frac{1}{12}(\log s)^2 L_{N,K}(L_{N,K} + 2)\right\}^{-\frac{1}{2}} > 1\right) \\
&\qquad + \mathcal{E} \times \mu\left(\max_{0\leq j\leq L_{N,K}} \left|\log C_K(r_j^{(M_{N,K})}, \hat{\mathbf{Y}}_N) - \log C(r_j^{(M_{N,K})})\right| > \varepsilon\right).
\end{aligned}$$

From Lemma 3.2,

$$\lim_{K\to\infty} \lim_{N\to\infty} \mathcal{E} \times \mu\left(\left\{\frac{1}{12}(\log s)^2 L_{N,K}(L_{N,K} + 2)\right\}^{-\frac{1}{2}} > 1\right) = 0.$$

For any $j \in \{0, 1, \ldots, L_{N,K}\}$

$$\begin{aligned}
&\mathcal{E} \times \mu \left(\left| \log C_K(r_j^{(M_{N,K})}, \hat{\mathbf{Y}}_N) - \log C(r_j^{(M_{N,K})}) \right| > \varepsilon \right) \\
&\leq \mathcal{E} \times \mu \left(\left| \log C_K(r_j^{(M_{N,K})}, \hat{\mathbf{Y}}_N) - \log C_K(r_j^{(M_K)}, \hat{\mathbf{Y}}_N)) \right| > \varepsilon/3 \right) \\
&\qquad + \mathcal{E} \times \mu \left(\left| \log C(r_j^{(M_{N,K})} - \log C(r_j^{(M_K)}) \right| > \varepsilon/3 \right) \\
&\qquad\qquad + \mathcal{E} \times \mu \left(\left| \log C_K(r_j^{(M_K)}, \hat{\mathbf{Y}}_N) - \log C(r_j^{(M_K)}) \right| > \varepsilon/3 \right)
\end{aligned}$$

From right continuity of $C_K(r, Y)$, for any $\varepsilon > 0$, $\varepsilon_1 > 0$, and a fixed K

$$\mathcal{E} \times \mu \left(\exists N_1 \in \mathbf{N},\ \forall N > N_1,\ \sup_y \left| \log C_K(r_j^{(M_{N,K})}, y) - \log C_K(r_j^{(M_K)}, y) \right| > \varepsilon \right) < \varepsilon_1.$$

Thus,

$$\mathcal{E} \times \mu \left(\exists N_1 \in \mathbf{N},\ \forall N > N_1,\ \left| \log C_K(r_j^{(M_{N,K})}, \hat{\mathbf{Y}}_N) - \log C_K(r_j^{(M_K)}, \hat{\mathbf{Y}}_N) \right| > \varepsilon \right) < \varepsilon_1.$$

Therefore, there exists $N_1 \in \mathbf{N}$ such that for any $N > N_1$ and a fixed K,

$$\mathcal{E} \times \mu \left(\left| \log C_K(r_j^{(M_{N,K})}, \hat{\mathbf{Y}}_N) - \log C_K(r_j^{(M_K)}, \hat{\mathbf{Y}}_N) \right| > \varepsilon \right) < \varepsilon_1.$$

By the same way, from right continuity of $C(r)$, for any $\varepsilon > 0$ and $\varepsilon_2 > 0$, there exists $N_2 \in \mathbf{N}$ such that for any $N > N_2$ and a fixed K

$$\mathcal{E} \times \mu \left(\left| \log C(r_j^{(M_{N,K})}) - \log C(r_j^{(M_K)}) \right| > \varepsilon \right) < \varepsilon_2.$$

On the other hand, from Lemma 3.4 and Lemma 3.6, for any $\varepsilon > 0$ and $\varepsilon_3 > 0$, there exists $N_3 \in \mathbf{N}$ and $K_0 \in \mathbf{N}$ such that for any $N > N_3$ and $K > K_0$,

$$\mathcal{E} \times \mu \left(\left| \log C_K(r_j^{(M_K)}, \hat{\mathbf{Y}}_N) - \log C(r_j^{(M_K)}) \right| > \varepsilon \right) < \varepsilon_3.$$

Thus, for any $N > \max\{N_1, N_2, N_3\}$ and $K > K_0$

$$\mathcal{E} \times \mu \left(\left| \log C_K(r_j^{(M_{N,K})}, \hat{\mathbf{Y}}_N) - \log C(r_j^{(M_{N,K})}) \right| > \varepsilon \right) \leq \varepsilon_1 + \varepsilon_2 + \varepsilon_3.$$

Consequently,

$$\lim_{K\to\infty} \lim_{N\to\infty} \mathcal{E} \times \mu \left(\max_{0 \leq j \leq L_{N,K}} \left| \log C_K(r_j^{(M_{N,K})}, \hat{\mathbf{Y}}_N) - \log C(r_j^{(M_{N,K})}) \right| > \varepsilon \right) = 0.$$

Hence, the proof is completed. □

(Proof of Theorem 2.1)

From Lemma 3.1, for any $\varepsilon > 0$,

$$\mathcal{E} \times \mu(|\hat{\nu}_{N,K} - \nu| > \varepsilon) \leq \mathcal{E} \times \mu(|d_{N,K}| > \varepsilon/2) + \mathcal{E} \times \mu(|e_{N,K}| > \varepsilon/2).$$

Hence from Lemma 3.3 and Lemma 3.7,

$$\lim_{K\to\infty} \lim_{N\to\infty} \mathcal{E} \times \mu(|\hat{\nu}_{N,K} - \nu| > \varepsilon) = 0.$$

The proof of Theorem 2.1 is completed.

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