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Kajiwara, Kenji
Faculty of Mathematics, Kyushu University

Masuda, Tetsu
Department of Mathematics, Kobe University

Noumi, Masatoshi
Department of Mathematics, Kobe University

Ohta, Yasuhiro
Department of Mathematics, Kobe University

他

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K. Kajiwara, T. Masuda
M. Noumi, Y. Ohta
Y. Yamada

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Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

Cubic Pencils and Painlevé Hamiltonians

K. Kajiwara[1], T. Masuda[2], M. Noumi[2], Y. Ohta[2] and Y. Yamada[2]
[1] Graduate School of Mathematics, Kyushu University
[2] Department of Mathematics, Kobe University

Abstract We present a simple heuristic method to derive the Painlevé differential equations from the corresponding geometry of rational surafces.

1. Introduction

For each Painlevé equation, there exists an associated rational surface called the "space of initial conditions". This surface was introduced by Okamoto[1], and further studied by Takano and his collaborators. By the work of Sioda and Takano[2], the corresponding Painlevé equation was characterized as the unique Hamiltonian system satisfying certain holomorphy properties on the surface. Hence, in principle, one can recover the Painlevé equations from geometry.

This geometric approach to the Painlevé equations has been extended to the difference (or discrete) cases, from which the difference Painlevé equations (and their Bäcklund transformations) arise naturally as Cremona automorphisms of the surfaces[3]. Compared with the difference cases, however, the way how the differential Painlevé equations appear is rather indirect. The known method used so far to recover the differential Painlevé equations from geometry is either to take suitable continuous limit of discrete ones or to employ a deformation theory [4]. The aim of this note is to present yet another way, which is heuristic but much simpler.

The main idea of our method is to use cubic pencils. In our previous work[5], it is clarified that the cubic pencils play the essential role in the discrete Painlevé equation. It is natural to expect that they are also important in the differential Painlevé equations. Indeed, we find that the cubic pencils are directly related to the symplectic forms and Hamiltonians.

In Section 2, we explain our method in the case of the sixth Painlevé equation $P_{\rm VI}$. All the other degenerate cases are treated in Section 3. Finally, a relation of our cubic pencils and the Seiberg-Witten curves are discussed in Appendix A.

2. Procedure to obtain Hamiltonian

In this section, using the sixth Painlevé equation $P_{\rm VI}$ as an example, we explain a procedure to obtain the symplectic 2-form ω and the Hamiltonian H from the datum of the surface: the configuration of nine points on $\mathbb{P}^2$. The parameterization of the points is borrowed from [3].

Case $P_{\rm VI}$: (Fig.1, Add 4)

The configuration of the nine points for $P_{\rm VI}$ is given as follows,

$$\begin{array}{lll} P_1=(0:1:0), & P_2=(1:-a_2:1), & P_3=(1:-a_1-a_2:1),\\ P_4=(0:0:1), & P_5=(0:a_3:1), & P_6=(1:0:0),\\ P_7=(1:a_4:0), & P_8=((s-1)\epsilon:1:s\epsilon), & P_9=((s-1)\epsilon:1:s\epsilon-sa_0\epsilon^2). \end{array}$$

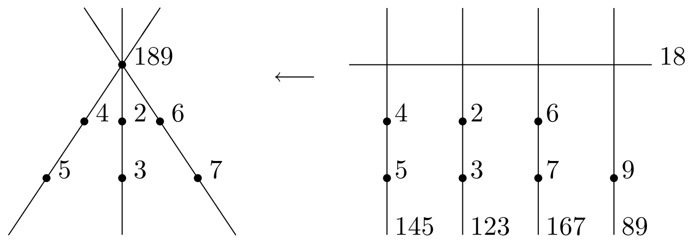

FIGURE 1. Configuration for $P_{\rm VI}$: In the right diagram, the labels i, ij and ijk represent the divisor classes $\mathcal{E}_i$, $\mathcal{E}_i - \mathcal{E}_j$ and $\mathcal{E}_0 - \mathcal{E}_i - \mathcal{E}_j - \mathcal{E}_k$ where $\mathcal{E}_0$ is the line in $\mathbb{P}^2$ and $\mathcal{E}_1, \dots, \mathcal{E}_9$ are the exceptional divisors.

Here variables $a_0, a_1, \dots, a_4$ and s are parameters parameterizing the configuration. The additional variable ϵ is an infinitesimal parameter introduced in order to handle some infinitesimally near points.

The configuration for $P_{\rm VI}$ contains a sequence of infinitely near points $P_{189} = (P_1 \leftarrow P_8 \leftarrow P_9)$. Where $P_i \leftarrow P_j$ means that the point P_j belongs to the exceptional curve $\mathcal{E}_i \simeq \mathbb{P}^1$ which is the total transform of P_i. Here, we represent such configuration by using an infinitesimal parameter ϵ. For instance, the condition that a curve $F(x, y, z) = 0$ pass through $P_{18} = (P_1 \leftarrow P_8)$ can be written as

$$F = (s-1)F_x + sF_z = 0, \quad (\text{at } P_1) \tag{1}$$

or equivalently

$$F(P_8) = F\big((s-1)\epsilon, 1, s\epsilon\big) = O(\epsilon^2). \tag{2}$$

Similarly, $F(x, y, z) = 0$ passes through P_1, P_8 and P_9 if and only if

$$F(P_9) = F\big((s-1)\epsilon : 1 : s\epsilon - sa_0\epsilon^2\big) = O(\epsilon^3). \tag{3}$$

Our basic object is a cubic curve passing through the nine points $P_1, \dots, P_9$. When the parameters a_i are generic, the cubic curve C_0 passing through the nine points is uniquely determined as

$$G = xz(z-x) = 0. \tag{4}$$

This cubic determines the symplectic form ω:

$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{G}, \tag{5}$$

which can be written as $\omega = df \wedge dg$, with canonical coordinates

$$f = \frac{z}{z-x}, \quad g = \frac{y(z-x)}{xz}. \tag{6}$$

When the parameters a_i satisfy the condition $\delta = a_0 + a_1 + 2a_2 + a_3 + a_4 = 0$, the cubic curve passing through the nine points forms a pencil (one parameter family) Fig. 2:

$$\lambda F(x, y, z) + \mu G(x, y, z) = 0, \tag{7}$$

where

$$\begin{aligned} F = &-(s-1)y^2z + a_3(s-1)yz^2 - a_4sx^2y + a_2(a_1+a_2)x^2z \\ &+ sxy^2 + (a_1 + 2a_2 + a_3 - a_3s + a_4s)xyz. \end{aligned} \tag{8}$$

In terms of the canonical variables f, g, the pencil equation $\lambda F + \mu G = 0$ can be written as $\lambda H + \mu = 0$ where

$$\begin{aligned} H = &f(f-1)(f-s)g^2 + \big[(a_1+2a_2)(f-1)f + a_3(s-1)f + a_4s(f-1)\big]g \\ &+ a_2(a_1+a_2)(f-1). \end{aligned} \tag{9}$$

Note that the choice of F involves the ambiguity such as $F \to c_1F + c_2G$ where c_1, c_2 are constants. This ambiguity, however, results only in changing H as $H \to c_1H + c_2$.

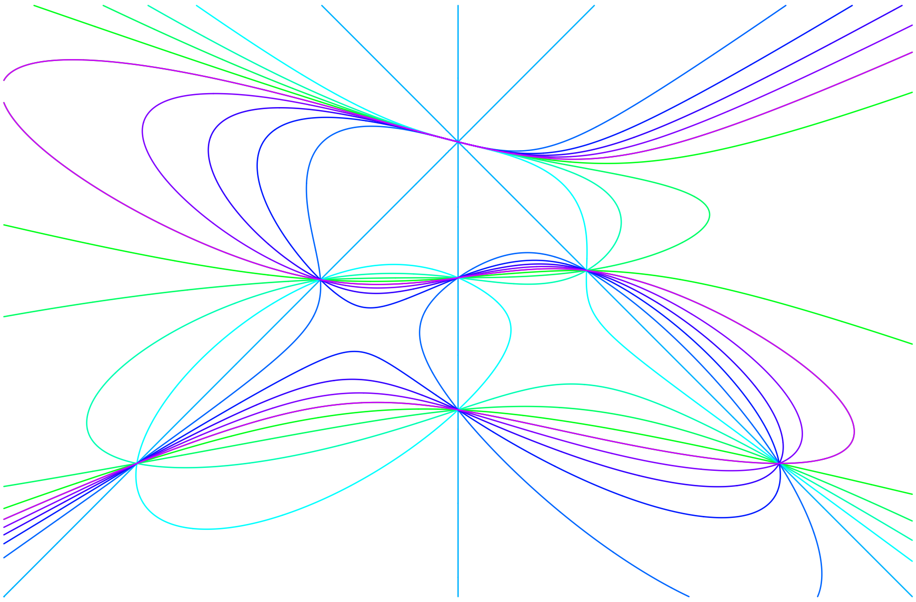

FIGURE 2. Cubic pencil for $P_{\rm VI}$ configuration

Painlevé eq.	Sakai's list [3]	configuration	symmetry
$P_{\rm VI}$	Add 4	$D_4^{(1)}$(Fig.1)	$D_4^{(1)}$
$P_{\rm V}$	Add 5	$D_5^{(1)}$(Fig.3)	$A_3^{(1)}$
$P_{\rm III}^{D_6^{(1)}}$	Add 6	$D_6^{(1)}$(Fig.4)	$(2A_1)^{(1)}$
$P_{\rm III}^{D_7^{(1)}}$	Add 7	$D_7^{(1)}$(Fig.5)	$A_1^{(1)}$
$P_{\rm III}^{D_8^{(1)}}$	Add 8	$D_8^{(1)}$(Fig.6)	$\mathfrak{S}_2$
$P_{\rm IV}$	Add 9	$E_6^{(1)}$(Fig.7)	$A_2^{(1)}$
$P_{\rm II}$	Add 10	$E_7^{(1)}$(Fig.8)	$A_1^{(1)}$
$P_{\rm I}$	Add 11	$E_8^{(1)}$(Fig.9)	–

TABLE 1. The Painlevé equations

At this stage, we drop the condition $\delta = 0$ by hand. We recognize then that H is a Hamiltonian for $P_{\rm VI}$[1], namely

Theorem 2.1. *With the above Hamiltonian H, the system of differential equation*

$$D_t f = \frac{\partial H}{\partial g}, \quad D_t g = -\frac{\partial H}{\partial f}, \quad D_t = s(s-1)\frac{d}{dt}, \quad \frac{ds}{dt} = \delta, \tag{10}$$

gives a Hamiltonian form of the sixth Painlevé equation $P_{\rm VI}$:

$$\begin{aligned}\frac{d^2 f}{dt^2} &= \frac{1}{2}\left(\frac{1}{f}+\frac{1}{f-1}+\frac{1}{f-s}\right)\left(\frac{df}{dt}\right)^2 - \delta\left(\frac{1}{s}+\frac{1}{s-1}+\frac{1}{f-s}\right)\frac{df}{dt}\\ &+ \frac{f(f-1)(f-s)}{s^2(s-1)^2}\left(\frac{a_1^2}{2}-\frac{a_4^2}{2}\frac{s}{f^2}+\frac{a_3^2}{2}\frac{s-1}{(f-1)^2}+\frac{(\delta^2-a_0^2)}{2}\frac{s(s-1)}{(f-s)^2}\right).\end{aligned} \tag{11}$$

In the next section, we will show similar results for all the cases in Table 1.

[1]For the autonomous case ($\delta = 0$) the pencil is invariant.

3. Degenerate cases

In this section, we consider the degenerate cases Add 5-11 in [3]. The constructions are essentially the same as the previous section (Add 4) and we give only the relevant data.

Case $P_{\rm V}$: (Fig.3, Add 5)

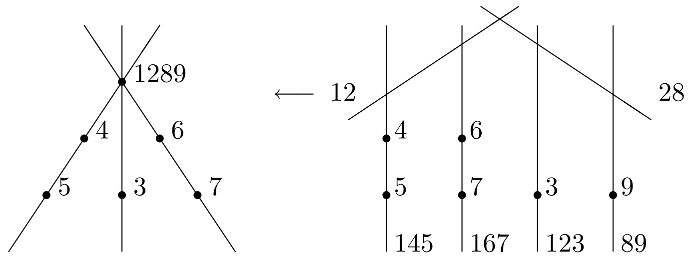

Figure 3. Configuration for $P_{\rm V}$

Condition for the cubic: $F(P_i) = 0$ $(i = 3, 4, 5, 6, 7)$ and $F(P_{1289}) = 0$,

$$\begin{aligned} &F(1, -a_2, 1) = F(0, 0, 1) = F(0, a_1, 1) = F(1, 0, 0) = F(1, a_3, 0) = 0, \\ &F(\epsilon, 1, \epsilon + s\epsilon^2 + s(s - a_0)\epsilon^3) = O(\epsilon^4). \end{aligned} \tag{12}$$

Pencil: $\lambda F + \mu G = 0$, $(\delta = a_0 + a_1 + a_2 + a_3 = 0)$

$$\begin{aligned} &F = a_3x^2y - xy^2 - a_2sx^2z + (a_1 - a_3 - s)xyz + y^2z - a_1yz^2, \\ &G = xz(z - x). \end{aligned} \tag{13}$$

Hamiltonian H and canonical variables f, g:

$$\begin{aligned} &H = f(f-1)g(g+s) - (a_1 + a_3)fg + a_1g + a_2sf, \\ &f = \frac{x}{x-z}, \quad g = \frac{y(x-z)}{xz}. \end{aligned} \tag{14}$$

Theorem 3.1. *With the above Hamiltonian H, the system of differential equation*

$$D_t f = \frac{\partial H}{\partial g}, \quad D_t g = -\frac{\partial H}{\partial f}, \quad D_t = s\frac{d}{dt}, \quad \frac{ds}{dt} = \delta, \tag{15}$$

gives a Hamiltonian form of the fifth Painlevé equation $P_{\rm V}$: $(y = 1 - 1/f)$

$$\begin{aligned} \frac{d^2y}{dt^2} &= \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{\delta}{s}\frac{df}{dt} \\ &+ \frac{(y-1)^2}{s^2}\left(\frac{a_1^2}{2}y - \frac{a_3^2}{2}\frac{1}{y}\right) + (a_0 - a_2)\frac{y}{s} - \frac{1}{2}\frac{y(y+1)}{(y-1)}. \end{aligned} \tag{16}$$

Case $P_{\rm III}^{D_6^{(1)}}$: (Fig.4, Add 6)

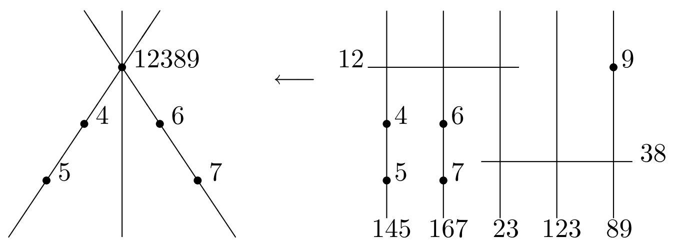

Figure 4. Configuration for $P_{\rm III}^{D_6^{(1)}}$

Condition for the cubic: $F(P_i) = 0$ $(i = 4, 5, 6, 7)$ and $F(P_{12389}) = 0$,

$$F(0,0,1) = F(0,a_1,1) = F(1,0,0) = F(1,b_1,0) = 0,$$
$$F(\epsilon, 1, \epsilon + s\epsilon^3 + s(b_1 - a_0)\epsilon^4) = O(\epsilon^5). \tag{17}$$

Pencil: $\lambda F + \mu G = 0$, $(\delta = a_0 + a_1 = 0)$

$$F = -b_1x^2y + xy^2 + sx^2z + (b_1 - a_1)xyz - y^2z + a_1yz^2, \quad G = xz(x - z). \tag{18}$$

Hamiltonian H and canonical variables f, g:

$$H = f^2g^2 + \left[f^2 - (a_1 + b_1)f - s\right]g - a_1f,$$
$$f = \frac{y(z - x)}{xz}, \quad g = \frac{x}{z - x}. \tag{19}$$

Theorem 3.2. *With the above Hamiltonian H, the system of differential equation*

$$D_tf = \frac{\partial H}{\partial g}, \quad D_tg = -\frac{\partial H}{\partial f}, \quad D_t = s\frac{d}{dt}, \quad \frac{ds}{dt} = \delta, \tag{20}$$

gives a Hamiltonian form of the third Painlevé equation $P_{\mathrm{III}}^{D_6^{(1)}}$:

$$\frac{d^2f}{dt^2} = \frac{1}{f}\left(\frac{df}{dt}\right)^2 - \frac{\delta}{s}\frac{df}{dt} + \frac{f^2}{s^2}(f + a_1 - b_1) - \frac{1}{f} - \frac{a_0 + 2a_1 + b_1}{s}. \tag{21}$$

Case $P_{\mathrm{III}}^{D_7^{(1)}}$: (Fig.5, Add 7)

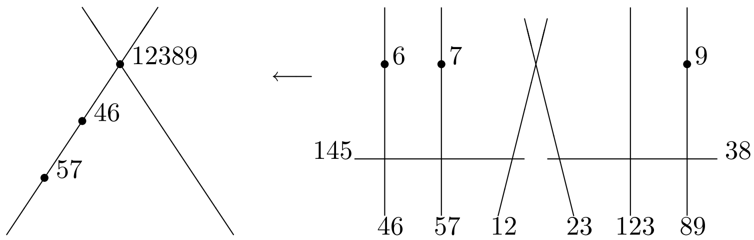

FIGURE 5. Configuration for $P_{\mathrm{III}}^{D_7^{(1)}}$

Condition for the cubic: $F(P_{46}) = F(P_{57}) = F(P_{12389}) = 0$,

$$F(\epsilon, 0, 1) = O(\epsilon^2), \quad F(\epsilon, 1 + a_1\epsilon, 1) = O(\epsilon^2), \quad F(\epsilon, 1, s\epsilon^3 - a_0s\epsilon^4) = O(\epsilon^5). \tag{22}$$

Pencil: $\lambda F + \mu G = 0$, $(\delta = a_0 + a_1 = 0)$

$$F = -sx^3 - a_1xyz + y^2z - yz^2, \quad G = x^2z. \tag{23}$$

Hamiltonian H and canonical variables f, g:

$$H = f^2g^2 + (a_1f + s)g - f,$$
$$f = \frac{yz}{x^2}, \quad g = -\frac{x}{z}. \tag{24}$$

Theorem 3.3. *With the above Hamiltonian H, the system of differential equation*

$$D_tf = \frac{\partial H}{\partial g}, \quad D_tg = -\frac{\partial H}{\partial f}, \quad D_t = s\frac{d}{dt}, \quad \frac{ds}{dt} = \delta, \tag{25}$$

gives a Hamiltonian form of the third Painlevé equation $P_{\mathrm{III}}^{D_7^{(1)}}$:

$$\frac{d^2f}{dt^2} = \frac{1}{f}\left(\frac{df}{dt}\right)^2 - \frac{\delta}{s}\frac{df}{dt} + 2\frac{f^2}{s^2} - \frac{1}{f} + \frac{a_0}{s}. \tag{26}$$

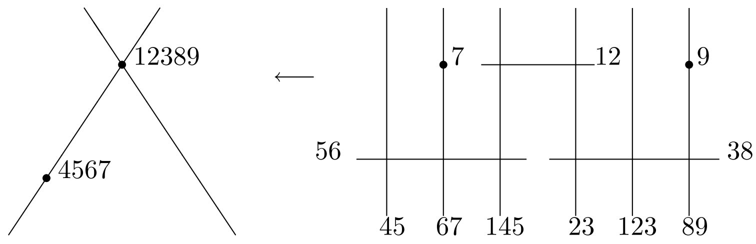

Figure 6. Configuration for $P_{\mathrm{III}}^{D_8^{(1)}}$

Case $P_{\mathrm{III}}^{D_8^{(1)}}$: (Fig.6, Add 8)
Condition for the cubic: $F(P_{4567}) = F(P_{12389}) = 0$,

$$F(\epsilon^2, \epsilon, 1) = O(\epsilon^4), \quad F(\epsilon, 1, s\epsilon^3 - as\epsilon^4) = O(\epsilon^5). \tag{27}$$

Pencil: $\lambda F + \mu G = 0$, $(\delta = a = 0)$

$$F = -sx^3 + y^2 z - xz^2, \quad G = x^2 z. \tag{28}$$

Hamiltonian H and canonical variables f, g:

$$\begin{aligned} H &= f^2 g^2 - f - \frac{s}{f}, \\ f &= \frac{z}{x}, \quad g = -\frac{y}{z}. \end{aligned} \tag{29}$$

Theorem 3.4. *With the above Hamiltonian H, the system of differential equation*

$$D_t f = \frac{\partial H}{\partial g}, \quad D_t g = -\frac{\partial H}{\partial f}, \quad D_t = s\frac{d}{dt}, \quad \frac{ds}{dt} = \delta, \tag{30}$$

gives a Hamiltonian form of the third Painlevé equation $P_{\mathrm{III}}^{D_8^{(1)}}$:

$$\frac{d^2 f}{dt^2} = \frac{1}{f}\left(\frac{df}{dt}\right)^2 - \frac{\delta}{s}\frac{df}{dt} + 2\frac{f^2}{s^2} - \frac{2}{s}. \tag{31}$$

Case P_{IV}: (Fig.7, Add 9)

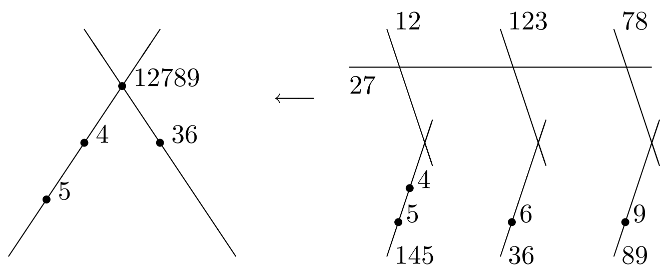

Figure 7. Configuration for P_{IV}

Condition for the cubic: $F(P_4) = F(P_5) = F(P_{36}) = F(P_{12789}) = 0$,

$$\begin{aligned} &F(0,0,1) = F(0,a_1,1) = 0, \\ &F(1, -a_2\epsilon, \epsilon) = O(\epsilon^2), \quad F(\epsilon, 1, \epsilon^2 + s\epsilon^3 + (s^2 - a_0)\epsilon^4) = O(\epsilon^5). \end{aligned} \tag{32}$$

Pencil: $\lambda F + \mu G = 0$, $(\delta = a_0 + a_1 + a_2 = 0)$

$$F = -x^2 y - a_2 x^2 z - sxyz + y^2 z - a_1 yz^2, \quad G = xz^2. \tag{33}$$

Hamiltonian H and canonical variables f, g:

(34)
$$H = fg(g - f - s) - a_2 f - a_1 g,$$
$$f = \frac{x}{z}, \quad g = \frac{y}{x}.$$

Theorem 3.5. *With the above Hamiltonian H, the system of differential equation*

(35)
$$D_t f = \frac{\partial H}{\partial g}, \quad D_t g = -\frac{\partial H}{\partial f}, \quad D_t = \frac{d}{dt}, \quad \frac{ds}{dt} = \delta,$$

gives a Hamiltonian form of the fourth Painlevé equation $P_{\rm IV}$:

(36)
$$\frac{d^2 f}{dt^2} = \frac{1}{2f}\left(\frac{df}{dt}\right)^2 + \frac{3}{2}f^3 + 2sf^2 + \frac{1}{2}\left[s^2 + 2(a_2 - a_0)\right]f - \frac{a_1^2}{2f}.$$

Case $P_{\rm II}$: (Fig.8, Add 10)

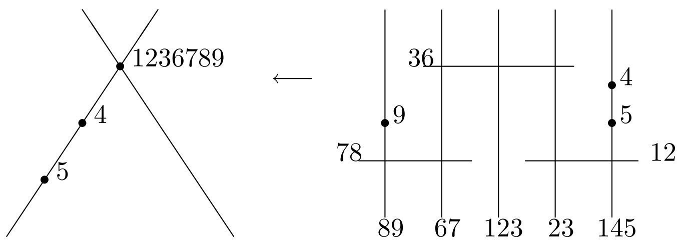

FIGURE 8. Configuration for $P_{\rm II}$

Condition for the cubic: $F(P_4) = F(P_5) = F(P_{1236789}) = 0$,

(37)
$$F(0, 0, 1) = F(0, a_1, 1) = 0, \quad F(\epsilon, 1, \epsilon^3 - s\epsilon^5 - a_0\epsilon^6) = O(\epsilon^7).$$

Pencil: $\lambda F + \mu G = 0$, $(\delta = a_0 + a_1 = 0)$

(38)
$$F = x^3 - sx^2 z - y^2 z + a_1 y z^2, \quad G = xz^2.$$

Hamiltonian H and canonical variables f, g:

(39)
$$H = g^2 + (f^2 + s)g + a_1 f, \quad f = \frac{y}{x}, \quad g = -\frac{x}{z}.$$

Theorem 3.6. *With the above Hamiltonian H, the system of differential equation*

(40)
$$D_t f = \frac{\partial H}{\partial g}, \quad D_t g = -\frac{\partial H}{\partial f}, \quad D_t = \frac{d}{dt}, \quad \frac{ds}{dt} = \delta,$$

gives a Hamiltonian form of the second Painlevé equation $P_{\rm II}$:

(41)
$$\frac{d^2 f}{dt^2} = 2f^3 + 2sf + (a_0 - a_1).$$

Case $P_{\rm I}$. (Fig.9, Add 11)

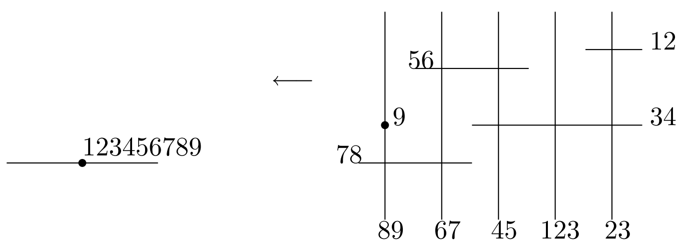

FIGURE 9. Configuration for $P_{\rm I}$

Condition for the cubic: $F(P_{123456789}) = 0$,

$$F(\epsilon, 1, \epsilon^3 + s\epsilon^7 + a\epsilon^8) = O(\epsilon^9). \tag{42}$$

Pencil: $\lambda F + \mu G = 0$: $(\delta = a = 0)$

$$F = -x^3 + y^2 z - sxz^2, \quad G = z^3. \tag{43}$$

Hamiltonian H and canonical variables f, g:

$$H = g^2 - f^3 - sf, \quad f = \frac{x}{z}, \quad g = \frac{y}{z}. \tag{44}$$

Theorem 3.7. *With the above Hamiltonian H, the system of differential equation*

$$D_t f = \frac{\partial H}{\partial g}, \quad D_t g = -\frac{\partial H}{\partial f}, \quad D_t = \frac{d}{dt}, \quad \frac{ds}{dt} = \delta, \tag{45}$$

gives a Hamiltonian form of the first Painlevé equation P_{I}:

$$\frac{d^2 f}{dt^2} = 6f^2 + 2s. \tag{46}$$

Appendix A. Relation to Seiberg-Witten curves

It may be interesting to note that the cubic pencils we considered in this paper are directly related with the Seiberg-Witten curves appearing in the $\mathcal{N} = 2$ supersymmetric gauge theory with SU(2) gauge group. The following is the Seiberg-Witten curves given in [6] and [7] (with some parameters rescaled).

$$\begin{aligned}
D_8:\ & y^2 = x^3 - ux^2 + 2\Lambda_0^4 x.\\
D_7:\ & y^2 = x^2(x-u) + 2m_1\Lambda_1^3 x - \Lambda_1^6.\\
D_6:\ & y^2 = (x^2 - \Lambda_2^4)(x-u) + 2m_1 m_2 \Lambda_2^2 x - (m_1^2 + m_2^2)\Lambda_2^4.\\
D_5:\ & y^2 = x^2(x-u) - \Lambda_3^2(x-u)^2 - \sum_{i=1}^{3} m_i^2 \Lambda_3^2 (x-u)\\
& \quad + 2m_1 m_2 m_3 \Lambda_3 x - \sum_{1\le i<j\le 3} m_i^2 m_j^2 \Lambda_3^2.\\
D_4:\ & y^2 = x(x-\alpha u)(x-\beta u) - \frac{1}{4}(\alpha-\beta)^2 u_2 x^2\\
& \quad - \big(\frac{1}{4}(\alpha-\beta)^2 \alpha\beta u_4 - \frac{1}{2}\alpha\beta(\alpha^2-\beta^2)s_4\big)x\\
& \quad - (\alpha-\beta)\alpha^2\beta^2 s_4 u - \frac{1}{4}(\alpha-\beta)^2\alpha^2\beta^2 u_6,\\
& u_2 = \sum_{i=1}^{4} m_i^2, \quad u_4 = \sum_{1\le i<j\le 4} m_i^2 m_j^2, \quad u_6 = \sum_{1\le i<j<k\le 4} m_i^2 m_j^2 m_k^2,\\
& s_4 = \prod_{i=1}^{4} m_i, \quad \alpha = -\vartheta_3(\tau)^4, \quad \beta = -\vartheta_2(\tau)^4.\\
E_8:\ & y^2 = x^3 - 2Mx - u.\\
E_7:\ & y^2 = x^3 - 2ux - 2Mu + M^3 - 4m_1^2.\\
E_6:\ & y^2 = x^3 - 2(Mu + c_2)x - u^2 - \frac{M^3}{3}u + \frac{M^6}{108} - \frac{2M^2}{3}c_2 + \frac{8}{3}c_3,\\
& c_k = m_1^k + m_2^k + m_3^k \quad (c_1 = 0).
\end{aligned} \tag{47}$$

The correspondence between our cubic pencils and the above Seiberg-Witten curves is a direct consequence of their definition/construction[8] [9][10]. In fact, by comparing the Weierstrass canonical form of both curves, the relations of the parameters are explicitly determined as in Table 2.

Painlevé	SW curve	Relation of parameters
$P_{\rm I}$	E_8	$s=-2M$
$P_{\rm II}$	E_7	$a_1=4m_1, \quad s=-3M$
$P_{\rm IV}$	E_6	$a_1=2(m_1-m_2), \quad a_2=2(m_2-m_3), \quad s=2M$
${P_{\rm III}}^{D_8^{(1)}}$	D_8	$s=2\Lambda_0^4$
${P_{\rm III}}^{D_7^{(1)}}$	D_7	$a_1=2m_1, \quad s=2\Lambda_1^3$
${P_{\rm III}}^{D_6^{(1)}}$	D_6	$a_1=m_1-m_2, \quad b_1=m_1+m_2, \quad s=-2\Lambda_2^2$
$P_{\rm V}$	D_5	$a_1=-(m_1+m_3), \quad a_2=m_1+m_2, \quad a_3=m_3-m_1, \quad s=2\Lambda_3$
$P_{\rm VI}$	D_4	$a_1=m_3+m_4, \quad a_2=m_2-m_3, \quad a_3=m_1-m_2, \quad a_4=m_3-m_4, \quad s=\dfrac{\beta}{\alpha}$

Table 2. Painlevé equation and Seiberg-Witten curve

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