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Stochastic oscillatory integrals — asymptotics and exact expressions for quadratic phase function — *

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Introduction

After R. Feynman introduced the path integral named after him, a lot of mathematical investigations to make this formally defined integral mathematically rigorous are made. Apart form these mathematical legacies of Feynman's path integrals, many probabilistic studies influenced by his path integrals have been done. Such probabilistic works were pioneered by M. Kac, who participated at the lecture presented by R. Feynman at the Cornell University in 1947, and instantaneously had an insight into similarity between Feynman's path integrals and the Wiener integrals. He introduced a new direction of studies in the probability theory; exploring in the forest of Wiener integrals, the mathematically rigorous counterparts to Feynman's path integrals, keeping a point of view of the path integrals in mind. He established the celebrated Feynman-Kac formula. The theory of large deviation is one of fruits born by studies in this direction. The Feynman-Kac formula corresponds to Schrödinger operators with scalar potential, and hence to Laplace transform type Wiener integrals. So does the the theory of large deviation.

In the asymptotic theory, two methods are widely known; one is the Laplace method for Laplace transform type integrals, and the other is the method of stationary phase for Fourier transform type integrals. As for Wiener integrals, the Laplace method is what is dealt with in the theory of large deviation. A Fourier transform type Wiener integral arises naturally

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in the probability theory. Needless to say, a characteristic function is among typical examples. A Fourier transform type Wiener integral interesting us more appears in a probabilistic investigation on Schrödinger operators with vector potential (magnetic field). Namely the heat kernel $p(t,x,y)$ associated with the Schrödinger operator with magnetic field $d\theta$ on $\mathbb{R}^n$, θ being a C^∞ 1-form, is represented as

$$p(t,x,y) = \int_{\mathcal{W}} \exp\left[\sqrt{-1}\int_{w_x[0,t]}\theta\right]\delta_y(w_x(t))\mu(dw), \quad t>0,\ x,y\in\mathbb{R}^n,$$

where $\mathcal{W}$ is the classical n-dimensional Wiener space, $w_x(t) = x + w(t)$ ($w\in\mathcal{W}$), $\int_{w_x[0,t]}\theta$ is a stochastic line integral of θ along $\{w_x(s) : s\in[0,t]\}$, and $\delta_y(w_x(t))\mu(dw)$ stands for the integration with respect to Watanabe's pull-back of Dirac's delta function concentrated at y via the nondegenerate Wiener functional $w_x(t)$. For the definition of the pull-back, see [14, 6].

A Fourier transform type Wiener integral is in general of the form

$$I(\lambda;q,\psi) = \int_{\mathcal{W}} \exp[\sqrt{-1}\,\lambda q(w)]\psi(w)\mu(dw),$$

where $q,\psi : \mathcal{W}\to\mathbb{R}$ are $\mathbb{R}$-valued Wiener functionals and $\lambda\in\mathbb{R}$. We call $I(\lambda;q,\psi)$ a stochastic oscillatory integral with phase function q and amplitude function ψ. What we are interested in is the asymptotic behavior of $I(\lambda;q,\psi)$ as $\lambda\to\infty$, which corresponds to, so called, the semiclassical limits in the theory of Feynman's path integrals (set $\lambda = 1/\hbar$, the reciprocal of the Planck constant). Recalling the well-developed method of stationary phase on the Euclidean spaces of finite dimension, the investigations of the asymptotics of stochastic oscillatory integrals may be carried out in three steps;

(1) to establish a exact expression when the phase function q is quadratic,

(2) to localize the integral around the stationary points of q, i.e. the points where the gradient of q vanishes,

(3) to introduce a local coordinate system around a stationary point under which the phase function is quadratic.

In this paper, we discuss about the first two steps.

1 Analyticity

In this section, we review analytic functions on the classical N-dimensional Wiener space $\mathcal{W}$, the space of $\mathbb{R}^N$-valued continuous functions on $[0,T]$ starting at the origin at time 0. For a separable Hilbert space E, we denote by

$\mathbb{D}^{\infty,\infty-}(E)$ the space of infinitely differentiable E-valued Wiener functionals in the sense of the Malliavin calculus, whose derivatives of all orders are p-th integrable with respect to μ for any $p \in (1,\infty)$. For details see [6, 14]. We say $\psi \in \mathbb{D}^{\infty,\infty-}(\mathbb{R})$ is analytic ($\psi \in C^\omega$ in notation) if there is $p \in (1,\infty)$ such that

$$\sum_{n=0}^{\infty} \frac{r^n}{n!} \|\nabla^n \psi\|_{L^p(H^{\otimes n})} < \infty \quad \text{for any } r > 0,$$

where H is the Cameron-Martin subspace of $\mathcal{W}$, $H^{\otimes n}$ is the Hilbert space of Hilbert-Schmidt n-linear mappings on H, $L^p(H^{\otimes n})$ is the $H^{\otimes n}$-valued p-th integrable functions with respect to μ, and ∇ stands for the Malliavin gradient. Choosing appropriate version of Malliavin gradients $\nabla^n\psi$'s, we have an expansion

$$\psi(w+h) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \nabla^n \psi(w), h^{\otimes n} \rangle_{H^{\otimes n}} \quad \text{for every } w \in \mathcal{W}, h \in H, \tag{1}$$

where $\langle \cdot, \cdot \rangle_{H^{\otimes n}}$ stands for the inner product on $H^{\otimes n}$. See [7, 11]. In what follows, we always consider such nice versions of ψ and $\nabla^n\psi$'s as above, and these versions will be used to evaluate ψ and so on.

Write $\mathcal{W} \oplus \sqrt{-1}\, H$, $H \oplus \sqrt{-1}\, H$, and $\mathcal{W} \oplus \sqrt{-1}\, \mathcal{W}$ for $\mathcal{W} \times H$, $H \times H$, and $\mathcal{W} \times \mathcal{W}$, respectively. Then $\mathcal{W} \oplus \sqrt{-1}\, \mathcal{W}$ is a real Banach space with norm

$$\|w + \sqrt{-1}\, w'\|_{\mathcal{W} \oplus \sqrt{-1}\mathcal{W}} = \|w\|_{\mathcal{W}} + \|w'\|_{\mathcal{W}},$$

where $\|\cdot\|_{\mathcal{W}}$ stands for the Banach norm on $\mathcal{W}$. For their elements $(w,h) \in \mathcal{W} \times H$, $(h,h') \in H \times H$, and $(w,w') \in \mathcal{W} \times \mathcal{W}$, we write $w + \sqrt{-1}\, h$, $h + \sqrt{-1}\, h'$, and $w + \sqrt{-1}\, w'$, respectively. Due to (1), ψ extends to a function $\widetilde{\psi}$ on $\mathcal{W} \oplus \sqrt{-1}\, H$, which we call a holomorphic prolongation of ψ, so that

$$\widetilde{\psi}(w + \sqrt{-1}\, h) = \sum_{n=0}^{\infty} \frac{\sqrt{-1}^{\,n}}{n!} \langle \nabla^n \psi(w), h^{\otimes n} \rangle_{H^{\otimes n}} \quad w \in \mathcal{W}, h \in H.$$

Let $H^{(2)}$ be the space of symmetric Hilbert-Schmidt operator of H to itself. If $S, T \in H^{(2)}$ possess a common normalized eigenfunctions, say $\{h_n\}_{n=1}^{\infty}$, and $S \geq -\delta I$ for some $\delta < 1$, then, expanding them as $S = \sum_{n=1}^{\infty} s_n h_n \otimes h_n$ and $T = \sum_{n=1}^{\infty} t_n h_n \otimes h_n$, we can define $\varphi_{S+\sqrt{-1}\,T} \in \mathbb{D}^{\infty,\infty-}(H \oplus \sqrt{-1}\, H)$ by

$$\varphi_{S+\sqrt{-1}\,T}(w) = \sum_{n=1}^{\infty} \big\{ (1 + s_n + \sqrt{-1}\, t_n)^{-1/2} - 1 \big\} (\nabla^* h_n)(w) h_n,$$

where ∇^* denotes the adjoint operator of ∇, and we have regarded $h_n \in H$ as a constant H-valued Wiener functional with value h_n, and used the branch of $z^{1/2}$ so that $1^{1/2} = 1$. We use $(I + S + \sqrt{-1}\,T)^{-1/2}$ to denote a $\mathcal{W} \oplus \sqrt{-1}\,H$-valued Wiener functional given by

$$(I + S + \sqrt{-1}\,T)^{-1/2} w = w + \varphi_{S+\sqrt{-1}\,T}(w).$$

2 Asymptotics and localization

For $A \in H^{(2)}$, let $Q_A = (\nabla^*)^2 A$ and $L_A = \nabla^* A$. We denote by $\mathcal{S}$ the space of complex valued rapidly decreasing function on $\mathbb{R}$. For $\varepsilon > 0$, put $G_\varepsilon(x) = (2\pi\varepsilon)^{-1/2} \exp[-x^2/(2\varepsilon)]$, $x \in \mathbb{R}$.

Theorem 2.1. *Let $A, B \in H^{(2)}$ and $\psi \in C^\omega$. Suppose that A is injective, and ψ and its holomorphic prolongation $\widetilde{\psi}$ satisfy that*

(ψ.1) $\displaystyle\sum_{n=0}^{\infty} \frac{r^n}{n!} \|\nabla^n \psi\|^2_{H^{\otimes n}} \in L^1(\mathbb{R})$ *for any $r > 0$,*

(ψ.2) *there is an $a \in \mathbb{R}$ such that*

$$\lim_{\lambda \to \infty} \int_{\mathcal{W}} \widetilde{\psi}\big(\{I - 2\sqrt{-1}\,(\lambda A + \xi B^2)\}^{-1/2} w\big) \mu(dw) = a \quad \text{for every } \xi \in \mathbb{R},$$

(ψ.3) *there exist $\lambda_0 \geq 0$, $C > 0$ and $n \in \mathbb{N}$ such that*

$$\left| \int_{\mathcal{W}} \widetilde{\psi}\big(\{I - 2\sqrt{-1}\,(\lambda A + \xi B^2)\}^{-1/2} w\big) \mu(dw) \right| \leq C(1 + |\xi|^n)$$
$$\text{for every } \lambda \geq \lambda_0,\ \xi \in \mathbb{R}.$$

If $f \equiv 1$ or $f : \mathbb{R} \to \mathbb{R}$ is of the form that $f = g * G_\varepsilon$ for some $g \in \mathcal{S}$ and $\varepsilon > 0$, where $*$ stands for the convolution product, then

(2) $\big\{\det_2(I - 2\sqrt{-1}\,\lambda A)\big\}^{1/2} I(\lambda; Q_A, \psi\, f(\|L_B\|^2_H)) \longrightarrow a f(0)$ *as $\lambda \to \infty$.*

In particular, if $f(0) = 1$ and $a \neq 0$ in addition, then

(3) $$\frac{I(\lambda; Q_A, \psi\, f(\|L_B\|^2_H))}{I(\lambda; Q_A, \psi)} \longrightarrow 1 \quad \text{as } \lambda \to \infty.$$

Corollary 2.1. *Let $A, B \in H^{(2)}$ be as in Theorem 2.1. Suppose that $\psi \in C^\omega$ enjoys (ψ.1) and*

(ψ.4) *there is an* $a \in \mathbb{R}$ *such that* $\lim_{\substack{w \in \mathcal{W},\, h \in H,\\ \|w+\sqrt{-1}\,h\|_{\mathcal{W}\oplus\sqrt{-1}\,\mathcal{W}}\to 0}} \widetilde{\psi}(w+\sqrt{-1}\,h) = a$,

(ψ.5) *there are* $\lambda_0 \geq 0$, $\delta > 0$, $C > 0$ *and* $n \in \mathbb{N}$ *such that*

$$\int_{\mathcal{W}} \big|\widetilde{\psi}\big(\{I - 2\sqrt{-1}\,(\lambda A + \xi B^2)\}^{-1/2} w\big)\big|^{1+\delta} \mu(dw) \leq C(1+|\xi|^n)$$

for every $\lambda \geq \lambda_0$, $\xi \in \mathbb{R}$.

Then (2) *holds provided that* $f \equiv 1$ *and* $f = g * G_\varepsilon$ *for some* $g \in \mathcal{S}$ *and* $\varepsilon > 0$. *Moreover, if* $f(0) = 1$ *and* $a \neq 0$ *in addition, then* (3) *holds.*

Proof of Corollary 2.1. (ψ.3) follows from (ψ.5). Since the operator $H \oplus \sqrt{-1}\,H \ni h + \sqrt{-1}\,h' \mapsto (I - \sqrt{-1}\,\lambda A - \sqrt{-1}\,\xi B^2)^{-1/2} h \in H \oplus \sqrt{-1}\,H$ converges to 0 strongly as $\lambda \to \infty$, by virtue of [4, Corollary 5.1], we see that $\|\{I - 2\sqrt{-1}\,(\lambda A + \xi B^2)\}^{-1/2} \cdot\|_{\mathcal{W}\oplus\sqrt{-1}\,\mathcal{W}}$ does to 0 in probability. Hence, due to (ψ.4), $\widetilde{\psi}\big(\{I - 2\sqrt{-1}\,(\lambda A + \xi B^2)\}^{-1/2}\cdot\big)$ converges to a in probability. Thus, in conjunction with the uniform integrability due to (ψ.5), this yields the convergence as described in (ψ.2). □

Remark 2.1. (i) It should be recalled (cf. [7]) that

$$I(\lambda; Q_A, \mathbf{1}) = \big\{\det{}_2(I - 2\sqrt{-1}\,\lambda A)\big\}^{-1/2},$$

and hence the quantity appearing in (2) is equal to

$$\frac{I(\lambda; Q_A, \psi\, f(\|L_B\|_H^2))}{I(\lambda; Q_A, \mathbf{1})}.$$

(ii) A stationary point of Q_A is a point where ∇Q_A vanishes. Mention ([7]) that $\nabla Q_A = 2L_A$. Since A is injective, the origin is the only one stationary point of Q_A. Moreover $\|L_B(w)\|_{\mathcal{W}} \leq C\|L_B(w)\|_H$ for some $C > 0$, and $\|L_B(w)\|_{\mathcal{W}}$ determines a measurable norm in the sense of Gross [4]. Hence $\{\|L_B\|_H^2 < b\}$ is a neighborhood of the origin. If g is of compact support, then $g * G_\varepsilon(x)$ decays as fast as $\exp[-x^2/(2\varepsilon)]$ as $|x| \to \infty$. Due to the factor $f(\|L_B\|_H^2)$, the main contribution to the the numerator of the fractional expression in (3) is made by the integration around the origin. Thus, (3) asserts that the main contribution to the asymptotic behavior of $\int_{\mathcal{W}} \exp\big[\sqrt{-1}\,\lambda Q_A\big]\psi\, d\mu$ as $\lambda \to \infty$ comes from the integration around the origin, the stationary point of Q_A.
(iii) A sufficient condition for (ψ.3) holds is that there are $C \geq 0$ and $0 < \delta < \delta_0/2$ such that

$$|\widetilde{\psi}(w+\sqrt{-1}\,h)| \leq C\big(1 + \exp\big[\delta\|w+\sqrt{-1}\,h\|_{\mathcal{W}\oplus\sqrt{-1}\,\mathcal{W}}^2\big]\big), \quad w \in \mathcal{W}, h \in H,$$

where $\delta_0 > 0$ was chosen so that $\int_{\mathcal{W}} \exp[\delta_0 \|w\|_{\mathcal{W}}^2] \mu(dw) < \infty$. See [13].
(iv) If $\phi : \mathcal{W} \to \mathbb{R}$ is continuous, multilinear, and symmetric, then $\psi(w) := \phi(w, \dots, w)$ satisfies the conditions (ψ.1), (ψ.4), and (ψ.5). Namely, it holds that

$$\widetilde{\psi}(w + \sqrt{-1}\, h) = \sum_{k=0}^{n} \binom{n}{k} \sqrt{-1}^{\,k} \phi(\underbrace{w, \dots, w}_{n-k}, \underbrace{h, \dots, h}_{k}) \quad w \in \mathcal{W}, h \in H,$$

and hence the condition (ψ.5) follows from [4, Theorem 5].

Before proceeding to the proof of Theorem 2.1, we recall a fact about $\det_2(I + A)$. For the proof, see [2, XI.9].

Lemma 2.1. *Let S, T be Hilbert-Schmidt operators of $H \oplus \sqrt{-1}\, H$ to itself. Then $\det_2(I+S)\mathrm{tr}[\{(I+S)^{-1} - I\}T]$ is well defined, irrespective to whether $I + S$ is invertible or not, and it holds that*

$$\frac{d}{du} \det_2(I + S + uT) = \det_2(I + S + uT)\mathrm{tr}[\{(I + S + uT)^{-1} - I\}T].$$

In particular, if $\det_2(I + S + uT) \neq 0$ for every $u \in [0, 1]$ and T is of trace class, then it holds that

$$\det_2(I + S + T) = \det_2(I + S) \exp\left[\int_0^1 \mathrm{tr}[\{(I + S + uT)^{-1} - I\}T] du \right].$$

Proof of Theorem 2.1. Put

$$q(\lambda, \xi) = \int_{\mathcal{W}} \widetilde{\psi}\big(\{I - 2\sqrt{-1}\,(\lambda A - \xi B^2)\}^{-1/2} w\big)\, \mu(dw).$$

By Assumptions (ψ.2) and (ψ.3),

$$(4) \quad \sup_{\lambda \geq \lambda_0} |q(\lambda, \xi)| \leq C(1 + |\xi|^n) \quad \text{and} \quad \lim_{\lambda \to \infty} q(\lambda, \xi) = a \quad \text{for every } \xi \in \mathbb{R}.$$

Due to [7, Theorem 7.8], we have that

$$I(\lambda; Q_A, \psi) = \big\{\det_2(I - 2\sqrt{-1}\,\lambda A)\big\}^{-1/2} q(\lambda, 0) \quad \text{for any } \lambda \in \mathbb{R}.$$

Thus, by (4), the convergence in (2) takes place for $f \equiv 1$.

Let $f = g * G_\varepsilon$ for some $g \in \mathcal{S}$ and $\varepsilon > 0$. There exists $\widetilde{g} \in \mathcal{S}$ such that $g(x) = \int_{\mathbb{R}} \widetilde{g}(\xi) e^{-\sqrt{-1}\, x\xi} d\xi$, and then

$$(5) \qquad f(x) = \int_{\mathbb{R}} \widetilde{f}(\xi) e^{-\sqrt{-1}\, x\xi} d\xi, \quad \text{where } \widetilde{f}(\xi) = \widetilde{g}(\xi) e^{-\varepsilon \xi^2/2}.$$

Since $\|L_B\|_H^2 = Q_{B^2} + \operatorname{tr} B^2$ (cf.[7]), by virtue of [7, Theorem 7.8] again, we see that

$$(6)\quad I(\lambda; Q_A, \psi\, f(\|L_B\|_H^2)) = \int_{\mathbb{R}} d\xi\, \widetilde{f}(\xi)\big\{\det{}_2(I - 2\sqrt{-1}\,\lambda A + 2\sqrt{-1}\,\xi B^2)\big\}^{-1/2} e^{-\sqrt{-1}\,\xi \operatorname{tr} B^2} q(\lambda, \xi).$$

By Lemma 2.1, setting

$$e(\lambda, \xi) = \exp\left[\int_0^{\xi} \operatorname{tr}\big(\big\{(I - 2\sqrt{-1}\,\lambda A + 2\sqrt{-1}\,\eta B^2)^{-1} - I\big\}(2\sqrt{-1}\,B^2)\big)\, d\eta\right]$$

we obtain

$$\det{}_2(I - 2\sqrt{-1}\,\lambda A + 2\sqrt{-1}\,\xi B^2) = \det{}_2(I - 2\sqrt{-1}\,\lambda A) e(\lambda, \xi).$$

Substituting this into (6), we arrive at

$$(7)\quad \big\{\det{}_2(I - 2\sqrt{-1}\,\lambda A)\big\}^{1/2} I(\lambda; Q_A, \psi\, f(\|L_B\|_H^2)) = \int_{\mathbb{R}} \widetilde{f}(\xi) e^{-\sqrt{-1}\,\xi \operatorname{tr} B^2} \frac{1}{e(\lambda, \xi)^{1/2}}\, q(\lambda, \xi) d\xi.$$

It is straightforward to show that

$$(8)\quad \big|\operatorname{tr}\big(\big\{(I - 2\sqrt{-1}\,\lambda A + 2\sqrt{-1}\,\eta B^2)^{-1} - I\big\}(2\sqrt{-1}\,B^2)\big)\big| \le 2\operatorname{tr} B^2.$$

Hence it holds that

$$|e(\lambda, \xi)| \ge \exp\big[-2|\xi| \operatorname{tr} B^2\big] \quad \text{for every } \lambda, \xi \in \mathbb{R},$$

which, combined with (4) and (5), implies that there is a constant $\widetilde{C}$ so that

$$(9)\quad \left|\widetilde{f}(\xi) e^{-\sqrt{-1}\,\xi \operatorname{tr} B^2} \frac{1}{e(\lambda, \xi)^{1/2}}\, q(\lambda, \xi)\right| \le \widetilde{C} |\widetilde{g}(\xi)| (1 + |\xi|^n) \quad \text{for every } \lambda \ge \lambda_0,\ \xi \in \mathbb{R}.$$

In conjunction with the bounded convergence theorem, (8) also implies that

$$(10)\qquad e(\lambda, \xi) \longrightarrow \exp[-2\sqrt{-1}\,\xi \operatorname{tr} B^2] \quad \text{as } \lambda \to \infty.$$

Plugging (9) and (10) into (7), and applying the dominated convergence theorem, we conclude that

$$\big\{\det{}_2(I - 2\sqrt{-1}\,\lambda A)\big\}^{1/2} I(\lambda; Q_A, \psi\, f(\|L_B\|_H^2)) \longrightarrow a \int_{\mathbb{R}} \widetilde{f}(\xi) d\xi = a f(0).$$

Thus the proof of (2) has been completed.

The convergence (3) follows from (2) immediately. □

In the above theorem, a key ingredient to see the localization is that $I(\lambda; Q_A, \psi)/I(\lambda; Q_A, 1)$ converges. In the remaining of this section, we shall give examples where a localization occurs while $I(\lambda; Q_A, \psi)/I(\lambda; Q_A, 1)$ diverges.

Example 2.1. Let $\{h_n\}_{n\in\mathbb{Z}^*}$ be an ONB of H, where $\mathbb{Z}^* = \mathbb{Z}\setminus\{0\}$. Fix $1/2 < \alpha < 1$, $\varepsilon > 0$ arbitrarily, and put $a_n = -a_{-n} = n^{-\alpha}$ $(n \in \mathbb{N})$, $A = \sum_{n\in\mathbb{Z}^*} a_n h_n \otimes h_n$, and $\psi_\varepsilon = \exp[-\sqrt{-1}\,\varepsilon Q_A]$. Suppose that $B \in H^{(2)}$ possesses $\{h_n\}_{n\in\mathbb{Z}_*}$ as eigenfunctions.

Since $I(\lambda; Q_A, \psi_\varepsilon) = I(\lambda - \varepsilon; Q_A, 1)$, by Lemma 2.1,

$$\frac{I(\lambda; Q_A, \psi_\varepsilon)}{I(\lambda; Q_A, 1)} = \exp\biggl[-\varepsilon\sqrt{-1}\int_0^1 \mathrm{tr}\Bigl[\bigl\{\bigl(I - 2\sqrt{-1}\,(\lambda-\varepsilon u)A\bigr)^{-1} - I\bigr\}A\Bigr]du\biggr].$$

It is straightforward to see

$$\begin{aligned}&-\varepsilon\sqrt{-1}\int_0^1 \mathrm{tr}\Bigl[\bigl\{\bigl(I - 2\sqrt{-1}\,(\lambda-\varepsilon u)A\bigr)^{-1} - I\bigr\}A\Bigr]du\\ &\quad= 4\varepsilon\int_0^1 (\lambda-\varepsilon u)\biggl(\sum_{n=1}^\infty \frac{1}{n^{2\alpha} + 4(\lambda-\varepsilon u)^2}\biggr)du.\end{aligned}$$

If we set

$$K_\alpha = \int_0^\infty \frac{dy}{y^{2\alpha}+4},$$

then

$$\sum_{n=1}^\infty \frac{1}{n^{2\alpha} + 4(\lambda-\varepsilon u)^2} = (\lambda-\varepsilon u)^{-2+(1/\alpha)}K_\alpha + O(\lambda^{-2}),$$

where $O(\lambda^{-2})$ is uniform in $u \in [0,1]$. Thus we can show that

$$\frac{I(\lambda; Q_A, \psi_\varepsilon)}{I(\lambda; Q_A, \mathbf{1})} = \exp\bigl[4\varepsilon K_\alpha \lambda^{(1/\alpha)-1} + O(1)\bigr].$$

Applying Lemma 2.2, which we shall give just after this example, we can also show that

$$\frac{I(\lambda; Q_A, \psi_\varepsilon \exp[-t\|L_B\|_H^2/2])}{I(\lambda; Q_A, \psi_\varepsilon)} = \frac{I(\lambda-\varepsilon; Q_A, \exp[-t\|L_B\|_H^2/2])}{I(\lambda-\varepsilon; Q_A, \mathbf{1})} \to 1$$

as $\lambda \to \infty$.

Lemma 2.2. *Assume that $S, T \in H^{(2)}$ possess a common eigenvectors and no eigenvalue of S vanishes. Then, for $t > 0$, $\lambda \in \mathbb{R}$ and $\psi \in C^\omega$ satisfying (ψ.1) in Theorem 2.1, it holds*

$$I(\lambda; Q_S, \psi \exp[-t\|L_T\|_H^2/2]) = \big\{\det_2(I + tT^2 - 2\sqrt{-1}\,\lambda S)\exp[t\,\mathrm{tr}T^2]\big\}^{-1/2} \times \int_{\mathcal{W}} \widetilde{\psi}\big((I + tT^2 - 2\sqrt{-1}\,\lambda S)^{-1/2} w\big)\mu(dw).$$

Moreover,

$$\frac{I(\lambda; Q_S, \exp[-\|L_T\|_H^2/2])}{I(\lambda; Q_S, \mathbf{1})} = \left(\frac{\det_2(I - 2\sqrt{-1}\,\lambda S)}{\det_2(I + T^2 - 2\sqrt{-1}\,\lambda S)\exp[\mathrm{tr}T^2]}\right)^{1/2} \to 1$$

as $\lambda \to \infty$.

Proof. The first identity can be seen in repetition of the argument employed in the proof of [7, Theorem 7.8].

The first equality in the second assertion is an immediate consequence of the first identity. To see the convergence, let $\{s_n\}$ and $\{t_n\}$ be eigenvalues of S and T, respectively. Then

$$\mathrm{tr}\Big(\big\{(I + uT^2 - 2\sqrt{-1}\,\lambda S)^{-1} - I\big\}T^2\Big) = \sum_{n=1}^{\infty} \frac{t_n^2}{1 + ut_n^2 - 2\sqrt{-1}\,\lambda s_n} - \mathrm{tr}T^2,$$

which, in conjunction with Lemma 2.1 and the dominated convergence theorem, implies the desired convergence. □

Example 2.2. Let $\{h_n\}_{n\in\mathbb{Z}^*}$ be an ONB of H. Fix $\alpha > 1/2$, $1/2 < \gamma < 1$, and $b_n \in \mathbb{R}$ with $\sum_{n\in\mathbb{Z}^*} b_n^2 < \infty$. In this example, we consider the following two cases;

Case 1. $a_{-n} = a_n = n^{-\alpha}$, $b_n^2 = b_{-n}^2$, and $c_{-n} = c_n = n^{-\gamma}$, $n \in \mathbb{N}$,
Case 2. $a_{-n} = -a_n = n^{-\alpha}$, $b_n^2 = b_{-n}^2$, and $c_{-n} = c_n = n^{-\gamma}$, $n \in \mathbb{N}$.

Let $m \in \mathbb{N}$ and define

$$A = \sum_{n\in\mathbb{Z}^*} a_n h_n \otimes h_n, \quad B = \sum_{n\in\mathbb{Z}^*} b_n h_n \otimes h_n, \quad \psi = \sum_{n\in\mathbb{Z}^*} c_n\big\{(\nabla^* h_n)^{2m} - p_m\big\},$$

where $p_m = (2m)!/(2^m m!)$. Then we shall see that
(i) It holds that

$$\lim_{\lambda\to\infty} \lambda^{-(1-\gamma)/\alpha} \frac{I(\lambda; Q_A, \psi\exp[-t\|L_B\|_H^2/2])}{I(\lambda; Q_A, \mathbf{1})} = \begin{cases} K_1 & \text{in } \textbf{Case 1}, \\ K_2 & \text{in } \textbf{Case 2}, \end{cases}$$

where

$$K_1 = -2p_m \int_0^\infty \left(1 - \frac{1}{(1-2\sqrt{-1}\,y^{-\alpha})^m}\right) y^{-\gamma} dy \quad \text{and} \quad K_2 = \operatorname{Re} K_1.$$

(ii) In both **Case 1** and **Case 2**, it holds that

$$\lim_{\lambda\to\infty} \frac{I(\lambda; Q_A, \psi \exp[-t\|L_B\|_H^2/2])}{I(\lambda; Q_A, \psi)} = 1.$$

Proof. Set

$$J_t(\lambda) = \int_{\mathcal{W}} \widetilde{\psi}\big((I + tB^2 - 2\sqrt{-1}\,\lambda A)^{-1/2} w\big)\mu(dw),$$
$$E_t(\lambda) = \left(\frac{\det_2(I - 2\sqrt{-1}\,\lambda A)}{\det_2(I + tB^2 - 2\sqrt{-1}\,\lambda A)\exp[t\,\mathrm{tr} B^2]}\right)^{1/2}.$$

Due to Lemma 2.2, we have

$$\frac{I(\lambda; Q_A, \psi \exp[-t\|L_B\|_H^2/2])}{I(\lambda; Q_A, \mathbf{1})} = E_t(\lambda) J_t(\lambda), \tag{11}$$

$$\frac{I(\lambda; Q_A, \psi \exp[-t\|L_B\|_H^2/2])}{I(\lambda; Q_A, \psi)} = E_t(\lambda) \frac{J_t(\lambda)}{J_0(\lambda)}, \tag{12}$$

$$\lim_{\lambda\to\infty} E_t(\lambda) = 1. \tag{13}$$

Thus what we need to see is the behavior of $J_t(\lambda)$ as $\lambda \to \infty$.

By virtue of the splitting property of μ, it is easily seen that

$$J_t(\lambda) = -p_m \sum_{k=1}^m \sum_{j=0}^m \binom{m}{k}\binom{m}{j} (-1)^{(3k+j)/2} 2^{k+j} \lambda^{k+j} \times \left(\sum_{n\in\mathbb{Z}^*} \frac{c_n a_n^{k+j} (1+tb_n^2)^{m-j}}{\{(1+tb_n^2)^2 + 4\lambda^2 a_n^2\}^m}\right) + o(1) \quad \text{as } \lambda \to \infty.$$

Let

$$A_{t,k,j}(\lambda) = \sum_{n=1}^\infty \frac{c_n a_n^{k+j}(1+tb_n^2)^{m-j}}{\{(1+tb_n^2)^2 + 4\lambda^2 a_n^2\}^m} = \sum_{n=1}^\infty \frac{n^{-\gamma+(2m-k-j)\alpha}(1+tb_n^2)^{m-j}}{\{n^{2\alpha}(1+tb_n^2) + 4\lambda^2\}^m}.$$

Since $b_n \to 0$ as $n \to \infty$, we can show that

$$\lim_{\lambda\to\infty} \lambda^{k+j-(1-\gamma)/\alpha} A_{t,k,j}(\lambda) = \int_0^\infty \frac{y^{-\gamma+(2m-k-j)\alpha}}{\{y^{2\alpha}+4\}^m}\, dy \equiv \Gamma_{k,j}.$$

Thus, if we put

$$\widetilde{K}_i = -p_m \sum_{k=1}^{m} \sum_{j=0}^{m} \binom{m}{k} \binom{m}{j} (-1)^{(3k+j)/2} 2^{k+j} \{1 + (-1)^{(i-1)(k+j)}\} \Gamma_{k,j},$$

for $i = 1, 2$, then

$$\lim_{\lambda \to \infty} \lambda^{-(1-\gamma)/\alpha} J_t(\lambda) = \begin{cases} \widetilde{K}_1 & \text{in } \textbf{Case 1}, \\ \widetilde{K}_2 & \text{in } \textbf{Case 2}. \end{cases}$$

It is straightforward to see that $\widetilde{K}_i = K_i$, $i = 1, 2$. Plugging these into (11) and (12), we obtain the desired assertions (i) and (ii). □

3 Explicit expression and exponential decay

In the previous section, we have seen that the stochastic oscillatory integrals with phase function Q_A decays as fast as

$$I(\lambda; Q_A, \mathbf{1}) = \{\det_2(I - 2\lambda\sqrt{-1}\, A)\}^{-1/2}.$$

It is natural to ask how fast the determinant decays. In this section, we state results related to this question by giving an explicit expression of $I(\lambda; Q_A, \mathbf{1})$.

Given $A \in H^{(2)}$, decompose as

$$A = \sum_{n=1}^{\infty} a_n h_n \otimes h_n,$$

where $a_n \in \mathbb{R}$ satisfies that $\sum_{n=1}^{\infty} a_n^2 < \infty$ and $\{h_n\}_{n=1}^{\infty}$ is an ONB of H. For $\ell \in H$, define

$$f_{A,\ell}(x) = \frac{1}{2} \sum_{n:\, x a_n > 0} \left\{ \frac{1}{|x|} + \frac{\langle \ell, h_n \rangle_H^2}{|a_n|^3} \right\} e^{-x/a_n}, \quad x \neq 0,$$

and $f_{A,\ell}(0) = 0$. Then $(1 \wedge |x|^2) f_{A,\ell}(x)$ is integrable on $\mathbb{R}$, and so is $(e^{i\lambda x} - 1 - i\lambda x) f_{A,\ell}(x)$. See [8]. We have

Theorem 3.1 ([8]). *Let $A \in H^{(2)}$, $\ell \in H$, and $\gamma \in \mathbb{R}$.*
(i) It holds

$$I(\lambda; \tfrac{1}{2} Q_A + \nabla^* \ell + \gamma, \mathbf{1}) = \exp\left[-\frac{\|\ell_A\|_H^2}{2} \lambda^2 + \sqrt{-1}\, \lambda \gamma + \int_{\mathbb{R}} \left(e^{\sqrt{-1}\, \lambda x} - 1 - \sqrt{-1}\, \lambda x \right) f_{A,\ell}(x) dx \right]$$

for any $\lambda \in \mathbb{R}$, *where* $\ell_A = \sum_{n:a_n=0} \langle \ell, h_n \rangle_H h_n$.
(ii) If $\ell_A = 0$ *and there exists* $a > 0$ *such that*

$$\limsup_{\lambda \to \infty} \lambda^{-a} \int_{\mathbb{R}} \big(\cos(\lambda x) - 1\big) f_{A,\ell}(x) dx < 0,$$

then there exist $C, \lambda_0 > 0$ *such that*

$$\big| I(\lambda; \tfrac{1}{2} Q_A + \nabla^* \ell + \gamma, \mathbf{1}) \big| \le \exp\big[-C\lambda^a\big] \quad \textit{for every } \lambda \ge \lambda_0.$$

Moreover, in this case, for any $\delta > 1/a$, *the distribution on* $\mathbb{R}$ *of* $\frac{1}{2} Q_A + \nabla^* \ell + \gamma$ *under* μ *admits a density function in the Gevrey class of order* δ *with respect to the Lebesgue measure.*

The theorem asserts that the distribution of $\frac{1}{2} Q_A + \nabla^* \ell + \gamma$ is infinitely divisible, and the corresponding Lévy measure is $f_{A,\ell}(x)dx$. Moreover, the distribution of $\frac{1}{2} Q_A + \gamma$ is selfdecomposable. See [9, §8 and §15].

Put $N^+(y) = \#\{n : a_n > 1/y\}$. If $\lim_{y \to \infty} e^{-cy} N^+(y) = 0$ for any $c > 0$, and $\liminf_{y \to \infty} y^{-a} N^+(y) > 0$ for some $a > 0$, then this a satisfies the condition described in the theorem.

As an application of Theorem 3.1, we can show that a hypersurface in $\mathcal{W}$ determined by quadratic Wiener functional gets flatter at infinity. To state our result, we prepare a lemma on the nondegeneracy of Q_A.

Lemma 3.1. *Let* $A \in H^{(2)}$ *and suppose that the range* $R(A)$ *of* A *is of infinite dimension. Then* $Q_A/2$ *is smooth and nondegenerate in the sense of the Malliavin calculus.*

Proof. It suffices to show that $Q_A/2$ is nondegenerate. Decompose as $A = \sum_{n=1}^{\infty} a_n h_n \otimes h_n$, and take $1 \le n_1 < n_2 < \cdots < n_j < \dots$ such that $a_{n_j} \ne 0$, $j = 1, 2, \dots$, and $a_n = 0$ if $n \notin \{n_j; j \in \mathbb{N}\}$. Fix $m \in \mathbb{N}$ arbitrarily. Then, there exists $C_m > 0$ such that

$$\|\nabla Q_A\|_H^2 = \sum_{j=1}^{\infty} a_{n_j}^2 \big(\nabla^* h_{n_j}\big)^2 \ge C_m \sum_{j=1}^{m} \big(\nabla^* h_{n_j}\big)^2.$$

Recalling that

$$\int_{\mathbb{R}^m} |z|^{-p} e^{-|z|^2/2} dz = \begin{cases} < \infty, & \text{if } p < m, \\ = \infty, & \text{if } p \ge m, \end{cases} \tag{14}$$

we see that $\|\nabla Q_A\|_H^{-1}$ belongs to $L^{m-1}(\mu)$. Since m is arbitrary, $Q_A/2$ is nondegenerate in the sense of the Malliavin calculus. □

Theorem 3.2. *Let $A \in H^{(2)}$ and suppose that* $\dim R(A) = \infty$.
(i) Let $h \in H$ satisfy $h_A = 0$. Suppose that there exist $\alpha, \beta > 0$ such that

$$\liminf_{|\xi|\to\infty} |\xi|^{-\alpha} \int_{\mathbb{R}} (1 - \cos(\xi y)) f_A(y) dy > 0, \tag{15}$$

$$\liminf_{|\xi|\to\infty} |\xi|^{\beta} \langle h, (I + \xi^2 A^2)^{-1} h \rangle > 0, \tag{16}$$

where the limits may be infinity. Then, for each $0 < \alpha' < \alpha$, there exist $C_1, C_2 > 0$ such that

$$\left| \int_{\mathcal{W}} e^{\sqrt{-1}\lambda\nabla^* h} \delta_x(Q_A/2) d\mu \right| \le C_1 \exp\Big[-C_2 |\lambda|^{2\alpha'/(\alpha'+\beta)} \Big] \tag{17}$$

for any $\lambda \in \mathbb{R}$.
(ii) For any $\beta > 0$, there exists $h \in H$ such that $h_A = 0$ and (16) *holds.*

The second assertion of the the theorem reflects that $\{Q_A/2 = x\}$ gets flatter at infinity. To see this, first recall that $\int_{\mathcal{W}} e^{i\lambda\nabla^* h} d\mu = e^{-\lambda^2 \|h\|_H^2/2}$. We call this a flat case, since $\mathcal{W}$ is flat. Next suppose that A is of trace class. Due to the splitting property of μ, we think of $\{\nabla^* h_n\}$ as a coordinate system of $\mathcal{W}$. Then $\{Q_A/2 = x\}$ is regarded as an elliptic quadratic hypersurface given by $\sum_{n=1}^{\infty} a_n^2 (\nabla^* h_n)^2 = 2x + \operatorname{tr} A$, say S_A^∞. Let $S_{A,k}^\infty = S_A^\infty \cap \{\sum_{n=k}^{\infty} a_n^2 (\nabla^* h_n)^2 = 2x + \operatorname{tr} A\}$. The minimal radius of elliptic quadratic hypersurface $S_{A,k}^\infty$ is more than $1/\sup\{|a_n| : n \ge k\}$. Since $a_n \to 0$ as $n \to \infty$, the minimal radii of elliptic quadratic hypersurface $S_{A,k}^\infty \subset S_A^\infty$ tend to infinity as $k \to \infty$. This means $S_{A,k}^\infty$ gets flatter as $k \to \infty$. Choosing h associated with $S_{A,k}$, we can make the order of the exponential decay closer to the one for the flat case.

Proof of Theorem 3.2. (i) Due to the splitting property of μ, we may and will assume that A is not singular, i.e., $a_n \neq 0$ for any $n = 1, 2, \dots$ It holds that

$$\begin{aligned} \int_{\mathcal{W}} e^{\sqrt{-1}\lambda\nabla^* h} \delta_x(Q_A/2) d\mu & \\ = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\sqrt{-1}\xi x} \left(\int_{\mathcal{W}} e^{\sqrt{-1}\lambda\nabla^* h} e^{\sqrt{-1}\xi Q_A/2} d\mu \right) d\xi. & \end{aligned} \tag{18}$$

Thus what we need to estimate is the integral $\int_{\mathcal{W}} e^{\sqrt{-1}\lambda\nabla^* h} e^{\sqrt{-1}\xi Q_A/2} d\mu$.

By a complex change of variables formula established in [7] and Theorem 3.1, we obtain

$$\begin{aligned}&\int_{\mathcal{W}} e^{\sqrt{-1}\,\lambda\nabla^* h} e^{\sqrt{-1}\,\xi Q_A/2} d\mu \\ &\qquad = \exp\biggl[\int_{\mathbb{R}} \bigl(e^{\sqrt{-1}\,\xi y} - 1 - \sqrt{-1}\,\xi y\bigr) f_{A,0}(y) dy - \frac{\lambda^2}{2} n(\xi, h)\biggr],\end{aligned}$$

where $n(\xi, h) = \sum_{n=1}^{\infty} \langle h, h_n\rangle^2/(1 - \sqrt{-1}\,\xi a_n)$. Noting that $\operatorname{Re} n(\xi, h) = \langle h, (I + \xi^2 A^2)^{-1} h\rangle$, we obtain

$$\begin{aligned}&\biggl|\int_{\mathcal{W}} e^{\sqrt{-1}\,\nabla^* h} e^{\sqrt{-1}\,\xi Q_A/2} d\mu\biggr| \\ &\qquad = \exp\biggl[-\int_{\mathbb{R}} (1 - \cos(\xi y)) f_{A,0}(y) dy - \frac{\lambda^2}{2} \langle h, (I + \xi^2 A^2)^{-1} h\rangle\biggr].\end{aligned}$$

By (15) and the continuity of the mapping $\xi \mapsto \int_{\mathbb{R}} (1 - \cos(\xi y)) f_{A,0}(y) dy$, there exist $K_1, K_2, K_3 > 0$ and $M > 0$ such that

$$\begin{aligned}(19)\quad &\biggl|\int_{\mathcal{W}} e^{\sqrt{-1}\,\lambda\nabla^* h} e^{\sqrt{-1}\,\xi Q_A/2} d\mu\biggr| \\ &\qquad \le \begin{cases} K_1 \exp\bigl[-\lambda^2 \langle h, (I + M^2 A^2)^{-1} h\rangle/2\bigr], & |\xi| \le M, \\ -K_2 |\xi|^{\alpha} - K_3 \lambda^2 |\xi|^{-\beta}, & |\xi| \ge M. \end{cases}\end{aligned}$$

Plugging this into (18), we obtain

$$\begin{aligned}(20)\quad &\biggl|\int_{|\xi| \le M} e^{-\sqrt{-1}\,\xi x} \biggl(\int_{\mathcal{W}} e^{\sqrt{-1}\,\lambda\nabla^* h} e^{\sqrt{-1}\,\xi Q_A/2} d\mu\biggr) d\xi\biggr| \\ &\qquad \le 2K_1 M \exp\bigl[-\lambda^2 \langle h, (I + M^2 A^2)^{-1} h\rangle/2\bigr]\end{aligned}$$

for any $\lambda \in \mathbb{R}$. Moreover, (19) yields that, for any $L > M$ and $0 < \alpha' < \alpha$,

$$\begin{aligned}&\biggl|\int_{|\xi| \ge M} e^{-\sqrt{-1}\,\xi x} \biggl(\int_{X} e^{\sqrt{-1}\,\lambda\nabla^* h} e^{\sqrt{-1}\,\xi Q_A/2} d\mu\biggr) d\xi\biggr| \\ &\quad \le \int_{M \le |\xi| \le L} \exp[-K_2 |\xi|^{\alpha} - K_3 \lambda^2 |\xi|^{-\beta}] d\xi \\ &\qquad\quad + \int_{L \le |\xi|} \exp[-K_2 |\xi|^{\alpha} - K_3 \lambda^2 |\xi|^{-\beta}] d\xi \\ &\quad \le \exp[-\lambda^2 L^{-\beta}] \int_{\mathbb{R}} \exp[-K_2 |\xi|^{\alpha}] d\xi + \exp[-K_2 L^{\alpha'}] \int_{\mathbb{R}} \exp[-K_2 |\xi|^{\alpha - \alpha'}] d\xi.\end{aligned}$$

Substituting $L = \lambda^{2/(\alpha'+\beta)}$ into the above estimation, we find $C, C' > 0$ such that

$$\left| \int_{|\xi| \geq M} e^{-\sqrt{-1}\,\xi x} \left(\int_{\mathcal{W}} e^{\sqrt{-1}\,\lambda \nabla^* h} e^{\sqrt{-1}\,\xi Q_A/2} d\mu \right) d\xi \right| \leq C \exp\left[-C' |\lambda|^{2\alpha'/(\alpha'+\beta)} \right]$$

for any $\lambda \in \mathbb{R}$. In conjunction with (20), this implies (17).

(ii) Due to the splitting property of the Wiener measure μ, without loss of generality we may assume that A is nonsingular. Then obviously $h_A = 0$ for any $h \in H$.

Fix $\beta > 0$ and put $q = (\beta + 1)/2 > 1/2$. Since $\dim R(A) = \infty$ and $a_n \to 0$, we can define $n(k)$, $k = 0, 1, 2, \ldots$, by $n(0) = 0$ and $n(k) = \min\{n > n(k-1) : |a_n| < 1/k\}$, $k \geq 1$. Set

$$b_n = \begin{cases} k^{-q}, & \text{if } n = n(k),\ k = 1, 2, \ldots, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad h = \sum_{n=1}^{\infty} b_n h_n.$$

Then $h \in R(A)$ and

$$\langle h, (I + \xi^2 A^2)^{-1} h \rangle = \sum_{k=0}^{\infty} \frac{k^{-2q}}{1 + \xi^2 a_{n(k)}^2} \geq \sum_{k=0}^{\infty} \frac{k^{2-2q}}{k^2 + \xi^2}. \tag{21}$$

For $a, b \in \mathbb{R}$ with $b, c > 0$ and $b - a > 1$, we can easily see that

$$\liminf_{c \to \infty} c^{1-\{(1+a)/b\}} \sum_{n=1}^{\infty} \frac{n^a}{n^b + c} \geq \int_0^{\infty} \frac{y^a}{y^b + 1} dy > 0,$$

where the integral may be infinity. Hence, we can conclude from (21) that

$$\liminf_{|\xi| \to \infty} |\xi|^{\beta} \langle h, (I + \xi^2 A^2)^{-1} h \rangle = \liminf_{|\xi| \to \infty} |\xi|^{(2q-1)} \langle h, (I + \xi^2 A^2)^{-1} h \rangle > 0.$$

□

4 Cameron-Martin transform and Jacobi equation

In the preceding sections, the observations are based on the splitting property of the Wiener measure. Another approach to stochastic oscillatory integrals, which we show in this section, is based on the Itô calculus and the Cameron-Martin transform on the Wiener space. In this approach, Euler (Jacobi)

equations associated with Lagrange functions appear in the expression. Such an appearance of ODE's in classical mechanics are well known in the Feynman path integral theory. For example, see [3, 10].

Let $T > 0$ and $\alpha, \beta \in C^1([0,T]; \mathbb{R}^{n\times n})$, $\mathbb{R}^{n\times n}$ being the space of real $n \times n$ matrices. Define

$$q_{\alpha,\beta} = \frac{1}{2}\int_0^T \big\{\alpha(t)w(t), dw(t)\rangle_{\mathbb{R}^n} + \langle\beta(t)w(t), w(t)\rangle_{\mathbb{R}^n} dt\big\},$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}^n}$ denotes the standard inner product on $\mathbb{R}^n$ and $dw(t)$ does the Itô integral. For $z \in \mathbb{C}$, let $P_z(t)$ be a solution to a Jacobi equation

$$\begin{aligned} &P_z''(t) - z\alpha(t)P_z'(t) + z\big\{\beta(t) - (\alpha'(t)/2)\big\}P_z(t) = 0, \quad t \in [0,T], \\ &P_z(T) = I, \quad P_z'(T) = z\alpha(T)/2. \end{aligned}$$

Let Ω be the set of all $z \in \mathbb{C}$ such that $\det P_z(t) \neq 0$ for any $t \in [0,T]$. For $z \in \Omega$, we define a transform T_z on $\mathcal{W}$ by

$$T_z w(t) = -P_z(t)\int_0^t \big(P_z^{-1}\big)'(s)w(s)ds, \quad t \in [0,T].$$

Theorem 4.1 ([12]). *Suppose that ${}^t\alpha(s) = -\alpha(s)$ and ${}^t\beta(s) = \beta(s)$ for any $s \in [0,T]$. Let $\psi \in C^\omega(\mathcal{W})$ satisfy the conditions $(\psi.1, 2\&3)$ described in Theorem 2.1. Then it holds*

$$\int_{\mathcal{W}} e^{zq_{\alpha,\beta}}\psi d\mu = \frac{1}{\sqrt{\det P_z(0)}}\int_{\mathcal{W}} \psi(w + T_z w)\mu(dw)$$

for any $z \in \Omega_0$, the connected component of Ω containing the origin.

A similar expression of a Wiener integral

$$\int_{\mathcal{W}} \exp\left[-\frac{z}{2}\int_0^T X(t)^2 dt\right]\psi d\mu$$

via a Riccati equation is also available when $X(t)$ is a Gaussian process associated with n-solitons. Moreover, we can show a relation between covariance functions of Gaussian processes and scattering data for reflectionless potentials for n-solitons. The details will be discussed in the forthcoming paper [5].

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