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Suzuki, Atsushi
Faculty of Mathematics, Kyushu University

Tabata, Masahisa
Faculty of Mathematics, Kyushu University

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A. Suzuki & M. Tabata

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Faculty of Mathematics
Kyushu University
Fukuoka, JAPAN

Finite Element Matrices in Congruent Subdomains and their Effective Use for Large-Scale Computations

Atsushi Suzuki* and Masahisa Tabata

Department of Mathematical Sciences, Faculty of Mathematics, Kyushu University, Fukuoka 812-8581, Japan

Abstract

The structure of finite element matrices in congruent subdomains is studied. When a domain has a form of symmetries and/or periodicities, it is decomposed into a union of congruent subdomains, each of which is an image of a reference subdomain by an affine transformation with an orthogonal matrix whose components consist of -1, 0, and 1. Stiffness matrices in subdomains are expressed by one in the reference subdomain with renumbering indices and changing signs corresponding to the orthogonal matrices. The memory requirements for a finite element solver are reduced by the domain decomposition, which is useful in large-scale computations. Reducing rates of memory requirements to store matrices are reported with examples of domains.

Keywords: finite element matrices; congruent subdomains; domain decomposition; orthogonal transformation; memory reduction

1 INTRODUCTION

As calculation speed of computers becomes faster, finite element computations of three dimensional problems with over one million unknowns are carried out. While finite element methods have an advantage in dealing with complex geometries of domain, large memory is required to store stiffness and mass matrices. The limitation of computer resources forces us to use efficient algorithms that can save memory storage. For domains having some form of symmetries or periodicities, several computational techniques have been developed to reduce the memory requirements. When a domain has a form of periodicity, by assuming a periodic solution, Zienkiewicz [12] reduced a problem to one in a unit subdomain. When a domain has a form of symmetry, Bossavit [2] presented a strategy to reduce the size of a finite element problem by group representation theory. Bonnet [3] extended this to the boundary integral equation method. In their approaches, no symmetry is assumed for the solution, although there is a restriction in the domain decomposition. Every subdomain should have the same type of boundary conditions imposed. Owing to this restriction, the maximum decomposition number of a cubic domain is 8 in case of the Poisson equation subject to Dirichlet boundary conditions.

We assume some symmetries and/or periodicities for domains, but not for solutions. Precisely, we suppose that a domain is decomposed into congruent subdomains that are images of a reference domain by affine transformations with orthogonal matrices. We decompose the finite element matrices corresponding to subdomains independently of the boundary conditions. This strategy of the decomposition can be adopted by virtue of the framework of the variational formulation of the finite element method. In the finite element method, while natural boundary conditions are included in a functional of a variational form, essential boundary conditions

*E-mail: asuzuki@math.kyushu-u.ac.jp

are treated as restrictions to function spaces. The restrictions are realized by projections to a solution space from the space with the total degrees of freedom. Finite element matrices, e.g., stiffness matrices and mass matrices, are defined in the whole domain independently of the boundary conditions. Therefore, more flexible decompositions can be adopted. For example, a cubic domain is decomposed into a union of cuboids with the decomposition number more than 8. Memory requirements to store finite element matrices are reduced by the decomposition into congruent subdomains. In the case of vector-valued basis, however, extra costs are required to transform stiffness matrices. The costs are caused by procedures of multiplications of the orthogonal matrix at each nodal point. If a domain has a form of symmetry, it can be decomposed into congruent subdomains that are transformed by orthogonal matrices whose components consist of -1, 0, and 1. Using this domain decomposition, the finite element matrices in the transformed subdomains are expressed by a sub-matrix in the reference subdomain by renumbering indices and changing signs. This property drastically reduces memory requirements in an iterative solver, whose main operation consists of matrix-vector multiplication. In this paper, we omit details on an iterative solver with projections to treat boundary conditions.

The contents of this paper are as follows. In Section 2 we review properties of finite element bases and matrices obtained by affine transformations with general orthogonal matrices. In Section 3 we introduce a kind of restriction to the orthogonal transformations, and study properties of transformation of matrices with vector-valued basis under the restriction. In Section 4 we consider a domain decomposition into congruent subdomains, which are transformed by a class of orthogonal matrices. We give a representation of total matrix by sub-matrix in a reference subdomain. In Section 5 we show reducing rates of memory requirements for matrices with examples of domains and their decompositions. Although the results in this paper hold for any dimensional domains having some symmetries and/or periodicities, the effect of reducing memory is remarkable in three dimensional problems. We, therefore, describe the study in three dimensional cases.

In this paper, the kth component of the vector u is denoted by $[u]_k$, and the component at the (k,l) entry of matrix A by $[A]_{k\,l}$.

2 FINITE ELEMENT MATRICES UNDER ORTHOGONAL TRANSFORMATIONS

In this section, we review fundamental properties of finite element bases and matrices obtained by affine transformations with orthogonal matrices and parallel displacements.

2.1 An affine transformation with an orthogonal matrix

Let F be an affine transformation in $\mathbb{R}^3$ defined by

$$F(x) := Rx + c\,, \tag{1}$$

where $R \in \mathbb{R}^{3\times 3}$ is an orthogonal matrix, and $c \in \mathbb{R}^3$ is a vector. Let Ω be a domain in $\mathbb{R}^3$ and Ω' be the image of Ω by F. Let p and u be scalar- and vector-valued functions in $x \in \Omega$. We define transformed scalar- and vector-valued functions in $x' := F(x) \in \Omega'$ by

$$p' := p \circ F^{-1}\,,$$
$$u' := R(u \circ F^{-1})\,.$$

Let ∇ be the nabla operator, $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)^T$ and $D(u)$ be a tensor,

$$[D(u)]_{k\,l} := \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) \qquad (1 \le k, l \le 3)\,.$$

Let ∇' be the nabla operator in x' and $D'(u')$ be the tensor corresponding to $D(u)$. We have the following relations.

Lemma 1 *We have*

$$\nabla' p' = R\,\nabla p\,, \tag{2}$$
$$\nabla' \cdot u' = \nabla \cdot u\,, \tag{3}$$
$$\nabla' \times u' = (\det R) R(\nabla \times u)\,, \tag{4}$$
$$D'(u') = R\,D(u) R^{-1}\,, \tag{5}$$
$$D'(u') : D'(v') = D(u) : D(v)\,, \tag{6}$$
$$\nabla' \times u' \cdot \nabla' \times v' = \nabla \times u \cdot \nabla \times v\,. \tag{7}$$

2.2 Finite element bases and matrices

Suppose that Ω is a polyhedral domain for the simplicity of the treatment. Let $\mathcal{T}_h$ be a partition of $\bar{\Omega}$ by tetrahedra, where h is the maximum diameter of tetrahedral elements. Let $\mathcal{L}^1_k(\Omega) \subset H^1(\Omega) \cap C^0(\bar{\Omega})$ be the finite element space of polynomials of degree k [4, 5]. Let n_G be the number of nodal points in $\bar{\Omega}$. We define an index set $\Lambda_G := \{1, 2, \ldots, n_G\}$ and denote a nodal point by P_μ for $\mu \in \Lambda_G$. Let $\{\psi_\mu\}_{\mu \in \Lambda_G}$ be a finite element basis of $\mathcal{L}^1_k(\Omega)$. The finite element basis satisfies

$$\psi_\mu(P_\nu) = \delta_{\mu\,\nu} \qquad (\mu,\, \nu \in \Lambda_G)\,, \tag{8}$$

where P_ν is a nodal point and $\delta_{\mu\,\nu}$ is the Kronecker delta.

We consider a partition of $\mathcal{T}'_h$ of $\bar{\Omega}'$ by tetrahedra $\{F(K)\}_{K \in \mathcal{T}_h}$. We denote by $\{P'_{\mu'}\}_{\mu' \in \Lambda'_G}$ the set of nodal points in Ω', where Λ'_G is an index set. We define a bijection f from Λ_G onto Λ'_G,

$$f(\mu) = \mu' \qquad (\mu \in \Lambda_G,\ \mu' \in \Lambda'_G) \tag{9}$$

by $x'(P'_{\mu'}) = F\,x(P_\mu)$.

2.2.1 Transformation of scalar-valued finite element bases and matrices

We define a scalar-valued finite element space $X_h := \mathcal{L}^1_k(\Omega)$. Setting $\Lambda_X = \Lambda_G$, the basis is nothing but $\{\psi_\mu\}_{\mu \in \Lambda_X}$ and the dimension n_X is equal to n_G. Let $\{\psi'_{\mu'}\}_{\mu' \in \Lambda'_X}$ be the finite element basis in Ω', where $\Lambda'_X = \Lambda'_G$.

Lemma 2 *Let $\mu \in \Lambda_X$. We have*

$$\psi'_{\mu'} = \psi_\mu \circ F^{-1}\,,$$

where $\mu' = f(\mu)$.

We define the following bilinear form and inner product,

$$a_0(p, q\,;\, \Omega) := \int_\Omega \nabla p \cdot \nabla q \, dx\,,$$
$$(p, q\,;\, \Omega) := \int_\Omega p\, q\, dx\,,$$

for $p, q \in X_h$. We omit Ω if there is no confusion, e.g., we write $a_0(\cdot,\cdot)$ in place of $a_0(\cdot,\cdot\,;\,\Omega)$. We note that the bilinear form a_0 is used, for example, in a weak formulation of the Poisson equation.

Let A_0 and M_0 be stiffness and mass matrices,

$$[A_0]_{\mu\,\nu} := a_0(\psi_\nu, \psi_\mu\,;\,\Omega) \qquad (\mu, \nu \in \Lambda_X)\,, \tag{10a}$$
$$[M_0]_{\mu\,\nu} := (\psi_\nu, \psi_\mu\,;\,\Omega) \qquad (\mu, \nu \in \Lambda_X)\,. \tag{10b}$$

Let A'_0 and M'_0 be stiffness and mass matrices in Ω'.

Proposition 1 *Let $\mu, \nu \in \Lambda_X$. We have*

$$[A'_0]_{\mu'\,\nu'} = [A_0]_{\mu\,\nu}\,,$$
$$[M'_0]_{\mu'\,\nu'} = [M_0]_{\mu\,\nu}\,,$$

where $\mu' = f(\mu)$ and $\nu' = f(\nu)$.

These are easily proved from Lemma 2 and (2).

2.2.2 Transformation of vector-valued finite element bases and matrices

We define a vector-valued finite element space $Y_h := \mathcal{L}^1_k(\Omega)^3$. We note that the dimension $n_Y = 3\,n_G$. Let $\Lambda_Y := \{1, 2, \ldots, n_Y\}$ be an index set and $\{\varphi_\alpha\}_{\alpha\in\Lambda_Y}$ be a finite element basis of Y_h. We associate a pair $[\alpha_0,\, k]$ of a nodal point number and a component number with an index $\alpha \in \Lambda_Y$ and identify them

$$\alpha = [\alpha_0,\, k] \qquad (\alpha_0 \in \Lambda_G,\ k \in \{1, 2, 3\})\,.$$

We assume that the association of $[\alpha_0,\, k]$ with $\alpha \in \Lambda_Y$ is put as

$$\alpha_0 = \lfloor(\alpha - 1)/3\rfloor + 1, \qquad k = ((\alpha - 1) \bmod 3) + 1\,, \tag{11}$$

where $\lfloor\cdot\rfloor$ denotes the greatest integer less than or equal to the argument. This means the numbering of the degrees of freedom is done as $(3(\alpha_0 - 1) + 1, 3(\alpha_0 - 1) + 2, 3(\alpha_0 - 1) + 3)$ at a nodal point P_{α_0}. We do not lose generality on the numbering of the indices by this assumption. The basis of Y_h satisfies that

$$[\varphi_\alpha(P_{\beta_0})]_l = \delta_{\alpha\,\beta} \qquad (\alpha,\, \beta = [\beta_0,\, l] \in \Lambda_Y)\,,$$

where P_{β_0} is a nodal point. We note that the vector-valued basis is expressed by the scalar-valued basis as

$$[\varphi_\alpha]_l = \delta_{k\,l}\psi_{\alpha_0} \quad (\alpha = [\alpha_0,\, k] \in \Lambda_Y, l = 1, 2, 3)\,. \tag{12}$$

Let $\{\varphi'_{\alpha'}\}_{\alpha'\in\Lambda'_Y}$ be the finite element basis in Ω', where Λ'_Y is the index set defined by $\{\alpha' = 3(\alpha'_0 - 1) + k'\ ;\ \alpha'_0 \in \Lambda'_G,\ k' = 1, 2, 3\}$. We consider a bijection f_Y from Λ_Y onto Λ'_Y where a nodal point α_0 is transformed to α'_0 by (9) for $\alpha = [\alpha_0, k]$ and $\alpha' = [\alpha'_0, k']$.

Remark 1 As for the correspondence k to k' we have ambiguity. We can choose any permutation of $\{1, 2, 3\}$ at every nodal point. In the next section, however, f_Y is uniquely defined by (18).

Lemma 3 *Let $\alpha = [\alpha_0, k] \in \Lambda_Y$. We have*

$$\varphi'_{\alpha'} = \sum_{1\le l\le 3} [R]_{k'\,l} R(\varphi_{[\alpha_0,\, l]} \circ F^{-1})\,,$$

where $\alpha' = [\alpha'_0, k'] \in \Lambda'_Y = f_Y(\alpha)$.

Proof. From (8) and Lemma 2 we obtain

$$\begin{aligned}
\left[\sum_{1\le l\le 3} [R]_{k'\,l} R(\varphi_{[\alpha_0,\, l]} \circ F^{-1})\right]_n &= \sum_{1\le l\le 3} [R]_{k'\,l} \sum_{1\le m\le 3} [R]_{n\,m} [\varphi_{[\alpha_0,\, l]} \circ F^{-1}]_m \\
&= \sum_{1\le l\le 3} [R]_{k'\,l} \sum_{1\le m\le 3} [R]_{n\,m} \delta_{l\,m} \psi_{\alpha_0} \circ F^{-1} \\
&= \sum_{1\le l\le 3} [R]_{k'\,l} [R]_{n\,l}\, \psi_{\alpha_0} \circ F^{-1} \\
&= \delta_{k'\,n} \psi'_{\alpha'_0} = [\varphi'_{\alpha'}]_n\,,
\end{aligned}$$

for $n = 1, 2, 3$. □

We define the following bilinear forms and inner product,

$$a_1(u,v\,;\Omega) := 2\int_\Omega D(u) : D(v)\,dx \qquad (u,\,v\in Y_h)\,,$$
$$a_2(u,v\,;\Omega) := \int_\Omega \nabla\cdot u\,\nabla\cdot v\,dx \qquad (u,\,v\in Y_h)\,,$$
$$a_3(u,v\,;\Omega) := \int_\Omega \nabla\times u\cdot\nabla\times v\,dx \qquad (u,\,v\in Y_h)\,,$$
$$b(v,q\,;\Omega) := -\int_\Omega \nabla\cdot v\,q\,dx \qquad (v\in Y_h \text{ and } q\in X_h)\,,$$
$$(u,v\,;\Omega) := \int_\Omega u\cdot v\,dx \qquad (u,\,v\in Y_h)\,.$$

Here we use the same notation $(\cdot,\cdot\,;\Omega)$ to represent the L^2-inner products in the scalar- and vector-valued function spaces.

Remark 2 The bilinear forms a_1 and a_2 are used, for example, in a weak formulation of the Navier equations [5]. The bilinear form b concerns with the incompressibility of the fluid. A combination of a_1 and b is used in a weak formulation of the Stokes equations [4, 6]. The bilinear form a_3 is used in another weak formulation of the Stokes equations [8].

Let $A_m(m=1,2,3)$, B, and M_1 be stiffness and mass matrices defined by

$$[A_m]_{\alpha\,\beta} := a_m(\varphi_\beta,\varphi_\alpha\,;\Omega) \qquad (m=1,2,3,\ \alpha,\beta\in\Lambda_Y)\,, \tag{13a}$$
$$[B]_{\mu\,\beta} := b(\varphi_\beta,\psi_\mu\,;\Omega) \qquad (\mu\in\Lambda_X,\ \beta\in\Lambda_Y)\,, \tag{13b}$$
$$[M_1]_{\alpha\,\beta} := (\varphi_\beta,\varphi_\alpha\,;\Omega) \qquad (\alpha,\beta\in\Lambda_Y)\,. \tag{13c}$$

We note that $[A_m]_{[\alpha_0,\,\cdot\,][\beta_0,\,\cdot\,]}(m=1,2,3)$ and $[M_1]_{[\alpha_0,\,\cdot\,][\beta_0,\,\cdot\,]}$ are 3×3 matrices corresponding to two nodal points with indices $\alpha_0,\beta_0\in\Lambda_G$, and $[B]_{\mu\,[\beta_0,\,\cdot\,]}$ is a 1×3 matrix. Let $A'_m(m=1,2,3)$, B', and M'_1 be stiffness and mass matrices in Ω'.

Proposition 2 *Let $\mu,\alpha_0,\beta_0\in\Lambda_G$. We have*

$$[A'_m]_{[\alpha'_0,\,\cdot\,][\beta'_0,\,\cdot\,]} = R[A_m]_{[\alpha_0,\,\cdot\,][\beta_0,\,\cdot\,]}R^{-1} \quad (m=1,2,3)\,, \tag{14a}$$
$$[B']_{\mu'\,[\beta'_0,\,\cdot\,]} = [B]_{\mu\,[\beta_0,\,\cdot\,]}R^{-1}\,, \tag{14b}$$
$$[M'_1]_{[\alpha'_0,\,\cdot\,][\beta'_0,\,\cdot\,]} = R[M_1]_{[\alpha_0,\,\cdot\,][\beta_0,\,\cdot\,]}R^{-1}\,, \tag{14c}$$

where $\alpha'_0=f(\alpha_0)$, $\beta'_0=f(\beta_0)$, and $\mu'=f(\mu)$.

Proof. We prove (14a) with $m=1$. From Lemma 3 and (6) we obtain

$$\begin{aligned}
[A'_1]_{[\alpha'_0,\,k][\beta'_0,\,l]} &= 2\int_{\Omega'} D'(\varphi'_{[\beta'_0,\,l]}(x')) : D'(\varphi'_{[\alpha'_0,\,k]}(x'))dx' \\
&= 2\int_{\Omega'} D'\Big(\sum_{1\le t\le 3}[R]_{l\,t}R\varphi_{[\beta_0,\,t]}(F^{-1}x')\Big)\ :\ D'\Big(\sum_{1\le s\le 3}[R]_{k\,s}R\varphi_{[\alpha_0,\,s]}(F^{-1}x')\Big)dx' \\
&= \sum_{1\le s\le 3}\sum_{1\le t\le 3}[R]_{l\,t}[R]_{k\,s}\,2\int_\Omega D(\varphi_{[\beta_0,\,t]}(x)) : D(\varphi_{[\alpha_0,\,s]}(x))|\mathrm{det}R|dx \\
&= \sum_{1\le s\le 3}[R]_{k\,s}\sum_{1\le t\le 3}[A_1]_{[\alpha_0,\,s][\beta_0,\,t]}[R^T]_{t\,l}\,,
\end{aligned}$$

for $k,l=1,2,3$. Relations (14a) with $m=2,3$ and (14b) are shown in similar manners from (3) and (7). Relation (14c) is easily proved. □

3 FINITE ELEMENT MATRICES UNDER A CLASS OF ORTHOGONAL TRANSFORMATIONS

In Section 2.2, we studied relations between finite element matrices in the original domain with those in the domain that is mapped by an affine transformation with a general orthogonal matrix. In order to derive the latter from the former, in the vector-valued case, we need two multiplications of 3×3 matrices at each nodal point. Now we consider a special class of orthogonal transformations where these multiplications are not needed. Under this restriction to the orthogonal matrices, simple procedures to transform the matrices with vector-valued basis are obtained.

We consider the following restriction to the orthogonal matrix in the affine transformation.

Assumption 1 *The orthogonal matrix R is a product of a diagonal matrix $\Sigma = \mathrm{diag}\{\epsilon_k\}_{1\le k\le 3}$ and a permutation matrix P,*

$$R = \Sigma P\,, \tag{15}$$

where $\epsilon_k = 1$ or -1.

We call Σ a sign matrix.

Let σ be a permutation of $\{1,2,3\}$ defined by P,

$$\sigma(k) = k' \quad (k, k' \in \{1,2,3\})\,, \tag{16}$$

where $[P]_{k'\,k} = 1$. We note that

$$[R]_{\sigma(k)\,l} = \epsilon_{\sigma(k)}\delta_{k\,l} \quad (k, l \in \{1,2,3\})\,. \tag{17}$$

We define a bijection f_Y from Λ_Y onto Λ'_Y by

$$f_Y(\alpha) = [f(\alpha_0),\, \sigma(k)] \quad (\alpha = [\alpha_0,\, k] \in \Lambda_Y). \tag{18}$$

Under Assumption 1, we have a main result on relations of finite element matrices with vector-valued basis. Let $A_m (m = 1,2,3)$, B, and M_1 be the stiffness and mass matrices defined by (13a), (13b), and (13c) in Ω. Let $A'_m (m = 1,2,3)$, B', and M'_1 be the stiffness and mass matrices in Ω'. Let $\{\epsilon_k\}_{k=1,2,3}$ be components of the sign matrix.

Proposition 3 *Let $\mu \in \Lambda_X$ and $\alpha, \beta \in \Lambda_Y$. We have*

$$[A'_m]_{\alpha'\,\beta'} = \epsilon_{k'}\ [A_m]_{\alpha\,\beta}\ \epsilon_{l'} \quad (m = 1,2,3)\,, \tag{19a}$$

$$[B']_{\mu'\,\beta'} = [B]_{\mu\,\beta}\ \epsilon_{l'}\,, \tag{19b}$$

$$[M'_1]_{\alpha'\,\beta'} = [M_1]_{\alpha\,\beta}\,, \tag{19c}$$

where $\alpha' = [\alpha'_0,\, k'] = f_Y(\alpha)$, $\beta' = [\beta'_0,\, l'] = f_Y(\beta)$, and $\mu' = f(\mu)$.

Proof. Substituting (17) into (14a) and (14b) in Proposition 2, we obtain (19a) and (19b). Relation (19c) is shown from (8) and (14c) as follows,

$$\begin{aligned}
[M'_1]_{[\alpha'_0,\,k']\,[\beta'_0,\,l']} &= \epsilon_{k'}[M_1]_{[\alpha_0,\,k]\,[\beta_0,\,l]}\epsilon_{l'} \\
&= \epsilon_{k'}\epsilon_{l'}\int_\Omega \varphi_{[\beta_0,\,l]}(x)\cdot\varphi_{[\alpha_0,\,k]}(x)dx \\
&= \epsilon_{\sigma(k)}\epsilon_{\sigma(l)}\int_\Omega \sum_{1\le s\le 3}\delta_{l\,s}\psi_{\beta_0}(x)\delta_{k\,s}\psi_{\alpha_0}(x)dx \\
&= [M_1]_{[\alpha_0,\,k]\,[\beta_0,\,l]}\,. \qquad \Box
\end{aligned}$$

Remark 3 Multiplications of the matrices R and R^{-1} are unnecessary to get the stiffness and mass matrices in Ω' from those in Ω. For the stiffness matrices, operations of changing the signs corresponding to the orthogonal matrix are only required.

4 FINITE ELEMENT MATRICES OBTAINED FROM DOMAIN DECOMPOSITIONS

In this section, we introduce a domain decomposition into a union of congruent subdomains, each of which is an image of a reference subdomain by an orthogonal matrix described in Section 3. Representation of the total matrices by the sub-matrices in the reference subdomain is obtained as an application of Propositions 1 and 3.

4.1 A domain decomposition into a union of congruent subdomains

We decompose the domain Ω into the union of p non-overlapping subdomains,

$$\bar{\Omega} = \bigcup_{0 \le i < p} \bar{\Omega}^{(i)}, \quad \Omega^{(i)} \cap \Omega^{(j)} = \emptyset \quad (0 \le i < j < p)\,.$$

We call $\Omega^{(0)}$ reference subdomain.

Assumption 2 *Every $\Omega^{(i)}$ $(i = 1, 2, \ldots, p-1)$ is an image of the reference subdomain $\Omega^{(0)}$ by an affine transformation $F^{(i)}$ with an orthogonal matrix $R^{(i)} = \Sigma^{(i)} P^{(i)}$ satisfying Assumption 1.*

We note that $\Sigma^{(i)}$ is a sign matrix $\mathrm{diag}\{\epsilon_k^{(i)}\}_{1 \le k \le 3}$ and $P^{(i)}$ is a permutation matrix. We set the sign matrix $\Sigma^{(0)}$ and the permutation matrix $P^{(0)}$ to be the identity matrix.

Let $\mathcal{T}_h^{(0)}$ be a partition of $\bar{\Omega}^{(0)}$ by tetrahedra with the maximum element diameter h.

Assumption 3 *A union of partitions $\bigcup_{i=0}^{p-1} \mathcal{T}_h^{(i)}$ is a partition of $\bar{\Omega}$, where*

$$\mathcal{T}_h^{(i)} = \{K' \; ; \; K' = F^{(i)} K, \; \forall\, K \in \mathcal{T}_h^{(0)}\} \quad (1 \le i < p)\,.$$

We suppose that a domain Ω is decomposable into subdomains satisfying Assumption 2 and a mesh partition of the domain satisfying Assumption 3. Let n_G be the number of nodal points in $\bar{\Omega}$. We define an index set $\Lambda_G := \{1, 2, \ldots, n_G\}$ of nodal points $\{P_\mu\}_{\mu \in \Lambda_G}$. Let $\mathcal{L}_k^1(\Omega)$ be the finite element space of polynomials of degree k. We define scalar- and vector-valued finite element spaces $X_h := \mathcal{L}_k^1(\Omega)$ and $Y_h := \mathcal{L}_k^1(\Omega)^3$, respectively. Let $\Lambda_X := \{1, 2, \ldots, n_X (= n_G)\}$ and $\Lambda_Y := \{1, 2, \ldots, n_Y (= 3n_G)\}$ be index sets of finite element bases $\{\psi_\mu\}_{\mu \in \Lambda_X} \subset X_h$ and $\{\varphi_\alpha\}_{\alpha \in \Lambda_Y} \subset Y_h$, satisfying (8) and (12), respectively. Here, we associate a pair $[\alpha_0, k]$ with $\alpha \in \Lambda_Y$ as (11).

Let $\Lambda_G^{(i)} \subset \Lambda_G$ ($\#\Lambda_G^{(i)} = n_G^{(0)}$) be an index set of the nodal points in $\bar{\Omega}^{(i)}$ for $i = 0, 1, \ldots, p-1$. We assume that $\Lambda_G^{(0)} := \{1, 2, \ldots, n_G^{(0)}\}$. We define a bijection $f^{(i)}$ from $\Lambda_G^{(0)}$ onto $\Lambda_G^{(i)}$,

$$f^{(i)}(\mu) = \mu' \qquad (\mu \in \Lambda_G^{(0)}, \; \mu' \in \Lambda_G^{(i)})$$

by $x(P_{\mu'}) = F^{(i)} x(P_\mu)$. The bijection $f^{(0)}$ is identical on $\Lambda_X^{(0)}$. We define an index set corresponding to the ith subdomain,

$$\begin{aligned} &\Lambda_X^{(i)} := \Lambda_G^{(i)} \quad (0 \le i < p)\,, \\ &\Lambda_Y^{(i)} := \{\alpha \in \Lambda_Y \; ; \; \alpha = [\alpha_0, k], \; \alpha_0 \in \Lambda_G^{(i)}, \; k \in \{1, 2, 3\}\} \quad (0 \le i < p). \end{aligned}$$

We note that $\#\Lambda_X^{(i)} = n_G^{(0)}$ and $\#\Lambda_Y^{(i)} = 3\,n_G^{(0)}$ for $i = 0, 1, \ldots, p-1$, and $\Lambda_Y^{(0)} = \{1, 2, \ldots, n_Y^{(0)} (= 3\,n_G^{(0)})\}$. We define a bijection $f_Y^{(i)}$ $(i = 0, 1, \ldots, p-1)$ from $\Lambda_Y^{(0)}$ onto $\Lambda_Y^{(i)}$ by

$$f_Y^{(i)}(\alpha) = [f^{(i)}(\alpha_0), \, \sigma^{(i)}(k)] \quad (\alpha = [\alpha_0, \, k] \in \Lambda_Y^{(0)})\,.$$

Here $\sigma^{(i)}$ is a permutation defined from $P^{(i)}$ as (16).

4.2 Representation of total matrices by sub-matrices

Let A_0, M_0, A_m $(m = 1,2,3)$, B, and M_1 be total stiffness and mass matrices in Ω defined by (10) and (13). Let $A_0^{(0)}$, $M_0^{(0)}$, $A_m^{(0)}$ $(m = 1,2,3)$, $B^{(0)}$, and $M_1^{(0)}$ be corresponding sub-matrices in the reference subdomain $\Omega^{(0)}$. By virtue of Propositions 1 and 3 we do not need to store the total stiffness and mass matrices in the whole domain. It is sufficient to construct and store these matrices only in the reference subdomain $\bar{\Omega}^{(0)}$. For scalar-valued finite element matrices, the transformations of stiffness and mass matrices in the reference subdomain $\Omega^{(0)}$ to the subdomains $\Omega^{(i)}$ are performed by the renumbering process $f^{(i)}$ of the indices of the bases. For vector-valued stiffness matrices, the renumbering process $f_Y^{(i)}$ and additional operations of changing the signs are needed. We obtain the following relations on total finite element matrices and sub-matrices in the reference subdomain.

For $\mu \in \Lambda_X$ and $\alpha \in \Lambda_Y$ we define $I(\mu)$ and $I(\alpha)$, subsets of $\{0, 1, \dots, p-1\}$, by

$$I(\mu) := \{j \in \{0,1,\dots,p-1\} \ ; \ \mu \in f^{(j)}(\Lambda_X^{(0)})\},$$
$$I(\alpha) := \{j \in \{0,1,\dots,p-1\} \ ; \ \alpha \in f_Y^{(j)}(\Lambda_Y^{(0)})\}.$$

For $\mu \in \Lambda_X$ and $j \in I(\mu)$ we define $\mu^{(j)} \in \Lambda_X^{(0)}$ by the unique element satisfying $f^{(j)}(\mu^{(j)}) = \mu$. Similarly for $\alpha \in \Lambda_Y$ and $j \in I(\alpha)$ we define $\alpha^{(j)} \in \Lambda_Y^{(0)}$ by $f_Y^{(j)}(\alpha^{(j)}) = \alpha$.

Theorem 1 *Let $\mu,\ \nu \in \Lambda_X$ and $\alpha = [\alpha_0,\, k],\ \beta = [\beta_0,\, l] \in \Lambda_Y$. We have*

$$[A_0]_{\mu\,\nu} = \sum_{j \in I(\mu)\cap I(\nu)} [A_0^{(0)}]_{\mu^{(j)}\,\nu^{(j)}}\,, \tag{20a}$$

$$[M_0]_{\mu\,\nu} = \sum_{j \in I(\mu)\cap I(\nu)} [M_0^{(0)}]_{\mu^{(j)}\,\nu^{(j)}}\,, \tag{20b}$$

$$[A_m]_{\alpha\,\beta} = \sum_{j \in I(\alpha)\cap I(\beta)} \epsilon_k^{(j)} [A_m^{(0)}]_{\alpha^{(j)}\,\beta^{(j)}} \epsilon_l^{(j)} \quad (m = 1,2,3), \tag{20c}$$

$$[B]_{\mu\,\beta} = \sum_{j \in I(\mu)\cap I(\beta)} [B^{(0)}]_{\mu^{(j)}\,\beta^{(j)}} \epsilon_l^{(j)}\,, \tag{20d}$$

$$[M_1]_{\alpha\,\beta} = \sum_{j \in I(\alpha)\cap I(\beta)} [M_1^{(0)}]_{\alpha^{(j)}\,\beta^{(j)}}\,. \tag{20e}$$

Proof. We prove relation (20c). By definition of the stiffness matrix (13a), we have

$$\begin{aligned}
[A_m]_{\alpha\,\beta} &= \sum_{0 \le j < p} a_m(\varphi_\beta,\, \varphi_\alpha\,;\, \Omega^{(j)}) \\
&= \sum_{j \in I(\alpha)\cap I(\beta)} a_m(\varphi_\beta,\, \varphi_\alpha\,;\, \Omega^{(j)}) \\
&= \sum_{j \in I(\alpha)\cap I(\beta)} \epsilon_k^{(j)} [A_m^{(0)}]_{\alpha^{(j)}\,\beta^{(j)}} \epsilon_l^{(j)}\,.
\end{aligned}$$

The last equality is obtained from Proposition 3 with $\Omega = \Omega^{(0)}$, $\Omega' = \Omega^{(j)}$ and $F = F^{(j)}$ for each j. The other relations are obtained similarly. □

Let $u \in \mathbb{R}^{n_Y}$ be a given vector. The product of A_m and u is calculated as follows. We prepare work vectors $\{v^{(i)}\}_{0 \le i < p}$, $\{w^{(i)}\}_{0 \le i < p} \subset \mathbb{R}^{n_Y^{(0)}}$.

Algorithm 1
Step 1. Decompose the vectors into sub-vectors with signs : for $j = 0,1,\dots,p-1$,

$$[v^{(j)}]_\delta := \epsilon_n^{(j)} [u]_{f_Y^{(j)}(\delta)} \quad (\delta = [\delta_0,\, n] \in \Lambda_Y^{(0)})\,,$$

where $f_Y^{(j)}(\delta) = \delta' \in \Lambda_Y$.
Step 2. Calculate matrix-vector products in subdomains : for $j = 0, 1, \ldots, p-1$,

$$w^{(j)} := A_m^{(0)}\, v^{(j)}\,.$$

Step 3. Change signs and sum up values on the interfaces of subdomains :

$$[A_m\, u]_\alpha = \sum_{j \in I(\alpha)} \epsilon_k^{(j)} [w^{(j)}]_{\alpha^{(j)}} \quad (\alpha = [\alpha_0,\, k] \in \Lambda_Y)\,.$$

In this algorithm we only need to store $A_m^{(0)}$ and $\{\epsilon_k^{(j)}\}_{k=1,2,3,j=0,\ldots,p-1}$. Therefore, memory requirement for a finite element solver is reduced drastically. The matrix-vector multiplication is a main operation in the execution of iterative solvers, for example, Krylov subspace methods [1, 7].

Remark 4 We have described the decomposition of the finite element matrices corresponding to the total nodal points. Here the matrices are independent of the type of the boundary conditions. In the framework of the finite element method, essential boundary conditions are treated as restrictions to function spaces. These restrictions can be realized by projections to a solution space from the space with total degrees of freedom.

5 REDUCTION OF MEMORY REQUIREMENTS BY DOMAIN DECOMPOSITIONS

We show two examples of domains which are decomposable into congruent subdomains satisfying Assumption 2. We discuss reducing rates of memory requirements to store matrices exploiting Theorem 1 by these examples.

5.1 Examples of domains decomposable into congruent subdomains

In the following examples, domains are approximated by polyhedra in finite element computations, though we neglect the differences of shapes caused by the approximation.

Example 1 An elliptic cylinder domain.

$$\Omega := \{x \in \mathbb{R}^3 \; ; \; \left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 < 1\,, \quad 0 < x_3 < c\}\,,$$

where $a, b, c > 0$, $a \neq b$.

Let p_0 be a decomposition number of the cylinder height. The elliptic cylinder domain is decomposed into $4\,p_0$ subdomains. A reference subdomain is defined by

$$\Omega_4^{(0)} := \{x \in \Omega \; ; \; 0 < x_1,\; 0 < x_2,\; 0 < x_3 < \frac{c}{p_0}\}\,,$$

and each subdomain is transformed by the combination of reflections to the x_1- and x_2-axes and a parallel displacement to the x_3-axis. Another setting of a reference subdomain

$$\Omega_2^{(0)} := \{x \in \Omega \; ; \; 0 < x_1,\; 0 < x_3 < \frac{c}{p_0}\}$$

leads to a decomposition into $2\,p_0$ subdomains. The other setting of a reference subdomain

$$\Omega_1^{(0)} := \{x \in \Omega \; ; \; 0 < x_3 < \frac{c}{p_0}\}$$

leads to a decomposition into p_0 subdomains. Table 1 shows several examples of decompositions of the elliptic cylinder domain. Figure 1 shows a decomposition into 24 subdomains when $p_0 = 6$, where the reference subdomain is cut off and every subdomain is shifted slightly to show the decomposition better. A mesh with the data listed in Table 3 is also shown there.

Example 2 A spherical shell domain.

$$\Omega := \{x \in \mathbb{R}^3 \ ; \ R_1 < |x| < R_2\} \, ,$$

where $|x|$ is the Euclidean norm of x and $0 < R_1 < R_2$.

Proposition 4 *Available subdomain numbers by transformations with the orthogonal matrices* (15) *are* $2, 3, 4, 6, 8, 12, 16, 24$*, and* 48.
Proof. The number of combination of three nonzero entries of such matrices is 6 and the number of combination of signs at three entries is 2^3, because each entry can take the value -1 or 1. Hence, the maximum number of the subdomains is 48. Examples of reference subdomains and generators of the orthogonal matrices are listed in Table 2. □

Figure 2 shows a decomposition when $p = 48$. The reference subdomain is cut off and every subdomain is shifted slightly. A mesh with the data listed in Table 3 is also shown there.

The present method to a spherical shell domain was applied in the numerical simulation of the Earth's mantle convection. For the details we refer to [9, 10, 11].

5.2 Reduction of memory requirements

We discuss reducing rates of memory requirements to store matrices exploiting Theorem 1. From the relation of total matrices and sub-matrices, the size of memory requirements in p subdomains would reduce to $1/p$ of that in a whole domain without using the decomposition. In reality, however, additional storages in renumbering the indices are necessary. We observe the practical reducing rates in Examples 1 and 2.

Since finite element matrices are sparse, it is efficient to store only nonzero components. We employ the Compressed Row Storage (CRS) format [1]. Let A be an $n_X \times n_X$ finite element matrix and n_A be the number of nonzero components of A. The CRS format uses three vectors $\vec{v}_A \in \mathbb{R}^{n_A}$, $\vec{c}_A \in \mathbb{N}^{n_A}$, and $\vec{r}_A \in \mathbb{N}^{n_X+1}$. Nonzero components of A are stored in $\vec{v}_A$, and information on rows and columns are in $\vec{r}_A$ and $\vec{c}_A$, respectively. In detail, $[\vec{r}_A]_i$ is equal to the total number of nonzero components in rows of A with indices from 1 to $i-1$. Let $[A]_{i\,j}$ be the kth nonzero component in the ith row. Then we set $[\vec{v}_A]_l = [A]_{i\,j}$ and $[\vec{c}_A]_l = j$, where $l = [\vec{r}_A]_{i+k}$. In usual computation, 8 and 4 bytes are necessary to store a real number with double precision for values of the matrix and an integer for an index, respectively. Therefore, to store the matrix A by the CRS format, $12n_A + 4(n_X + 1)$ bytes are required.

Since $A_0^{(0)}$, $M_0^{(0)}$, $A_m^{(0)}$ $(m = 1, 2, 3)$, and $M_1^{(0)}$ are symmetric matrices, the numbers of nonzero components to be stored are as follows,

$$\begin{aligned} n_{A_0^{(0)}} &:= \sum_{1 \le i \le n_X^{(0)}} \#\{j \in \Lambda_X^{(0)} \, ; \, [A_0^{(0)}]_{i\,j} \neq 0, \ j \le i\} \, , \\ n_{A_1^{(0)}} &:= \sum_{1 \le i \le n_Y^{(0)}} \#\{j \in \Lambda_Y^{(0)} \, ; \, [A_1^{(0)}]_{i\,j} \neq 0, \ j \le i\} \, , \\ n_{B^{(0)}} &:= \sum_{1 \le i \le n_X^{(0)}} \#\{j \in \Lambda_Y^{(0)} \, ; \, [B^{(0)}]_{i\,j} \neq 0\} \, . \end{aligned}$$

Similarly, $n_{M_0^{(0)}}$, $n_{A_2^{(0)}}$, $n_{A_3^{(0)}}$, and $n_{M_1^{(0)}}$ are defined. We note that $n_{A_0^{(0)}} = n_{M_0^{(0)}}$, $n_{A_1^{(0)}} = n_{A_2^{(0)}} = n_{A_3^{(0)}}$, and $n_{M_1^{(0)}} = 3n_{M_0^{(0)}}$. In order to practice Algorithm 1, we need integer vectors which describe the bijections $f^{(j)}$, $f_Y^{(j)}$ and sings $\{\epsilon_k^{(j)}\}_{1 \le k \le 3}$ for each j. Therefore, we need the

Table 1: Examples of decompositions of an elliptic cylinder domain into congruent subdomains (p : number of subdomains, p_0 : decomposition number of the cylinder height).

p	reference subdomain	p_0
2	$\Omega_2^{(0)}$	1
3	$\Omega_1^{(0)}$	3
4	$\Omega_4^{(0)}$	1
6	$\Omega_2^{(0)}$	3
8	$\Omega_4^{(0)}$	2
10	$\Omega_2^{(0)}$	5
$4n$	$\Omega_4^{(0)}$	$n \geq 3$

Table 2: Examples of decompositions of a spherical shell domain into congruent subdomains (p : number of subdomains).

p	reference subdomain	generator of $R^{(i)}$
2	$\Omega_2^{(0)} := \{x \in \Omega \ ; \ x_3 > 0\}$	A_3
3	$\Omega_3^{(0)} := \{x \in \Omega \ ; \ x_1 > x_2, \ x_1 > x_3\}$	B
4	$\Omega_4^{(0)} := \{x \in \Omega \ ; \ x_2 > 0, \ x_3 > 0\}$	A_2, A_3
6	$\Omega_6^{(0)} := \{x \in \Omega \ ; \ x_1 > \|x_2\|, \ x_1 > \|x_3\|\}$	A_0, B
8	$\Omega_8^{(0)} := \{x \in \Omega \ ; \ x_i > 0 \ (i = 1, 2, 3)\}$	A_1, A_2, A_3
12	$\Omega_{12}^{(0)} := \{x \in \Omega \ ; \ x_1 > \|x_2\|, \ x_1 > \|x_3\|, \ x_2 > x_3\}$	A_0, B, C
16	$\Omega_{16}^{(0)} := \{x \in \Omega \ ; \ x_i > 0 \ (i = 1, 2, 3), \ x_2 > x_3\}$	A_1, A_2, A_3, C
24	$\Omega_{24}^{(0)} := \{x \in \Omega \ ; \ x_i > 0 \ (i = 1, 2, 3), \ x_1 > x_2, \ x_1 > x_3\}$	A_1, A_2, A_3, B
48	$\Omega_{48}^{(0)} := \{x \in \Omega \ ; \ x_i > 0 \ (i = 1, 2, 3), \ x_1 > x_2, \ x_1 > x_3, \ x_2 > x_3\}$	A_1, A_2, A_3, B, C

Matrices A_0, A_1, A_2, A_3, B, and C are defined by

$$A_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Table 3: Mesh partitioning data (n_G and n_E : numbers of nodes and elements, n_X and n_Y : degrees of freedom of scalar- and vector-valued unknowns).

domain	n_G	n_E	n_X	n_Y
elliptic cylinder	226,981	1,296,000	226,981	680,943
spherical shell	324,532	1,868,544	324,532	973,596

following sizes of memories for $p(\geq 2)$,

$$m_{A_0}(p) := 12n_{A_0^{(0)}} + 4(n_X^{(0)} + 1) + 4p\, n_X^{(0)}, \tag{21a}$$

$$m_{A_1}(p) := 12n_{A_1^{(0)}} + 4(n_Y^{(0)} + 1) + 4p(n_Y^{(0)} + 3), \tag{21b}$$

$$m_B(p) := 12n_{B^{(0)}} + 4(n_X^{(0)} + 1) + 4p(n_X^{(0)} + n_Y^{(0)} + 3), \tag{21c}$$

$$m_{M_1}(p) := 12n_{M_1^{(0)}} + 4(n_Y^{(0)} + 1) + 4p\, n_Y^{(0)}. \tag{21d}$$

For $p = 1$, they are defined by

$$m_{A_0}(p) := 12n_{A_0} + 4(n_X + 1), \tag{22a}$$

$$m_{A_1}(p) := 12n_{A_1} + 4(n_Y + 1), \tag{22b}$$

$$m_B(p) := 12n_B + 4(n_X + 1), \tag{22c}$$

$$m_{M_1}(p) := 12n_{M_1} + 4(n_Y + 1). \tag{22d}$$

We define a reducing rate of memory requirement for a matrix C by $m_C(1)/m_C(p)$.

Now we employ the P1 element for Examples 1 and 2. In Table 3, we show mesh partitioning data. Figures 3 and 5 show the number of nodal points in the reference subdomain : $n_G^{(0)}$ by `NP`, and show the number of nonzero components of the stiffness and mass matrices in the reference subdomain : $n_{A_0^{(0)}}$, $n_{A_1^{(0)}}$, $n_{B^{(0)}}$ and $n_{M_1^{(0)}}$, by `A0`, `A1`, `B`, and `M1`, respectively. Figures 4 and 6 show the sizes of memory requirements for stiffness and mass matrices by the decompositions : $m_{A_0}(p)$, $m_{A_1}(p)$, $m_B(p)$ and $m_{M_1}(p)$, by `A0`, `A1`, `B`, and `M1`, respectively. The third terms of (21) show the memory requirements for processes of renumbering indices and changing signs. The sizes of index array $4p\, n_X^{(0)}$ and $4p\, n_Y^{(0)}$ increase slightly as the number of subdomains becomes large because there are overlaps of nodes among subdomains. Therefore reduction rate of memory requirements does not vary linearly, which are observed most clearly in case of the matrix B in Figures 4 and 6. In cases of the elliptic cylinder subdomain with 40 subdomains, reducing rates for the matrix B is 15.9, and in the spherical shell domain with 48 subdomains, it is 17.8.

6 CONCLUSION

We have studied properties of finite element matrices in a congruent subdomain obtained by an affine transformation with an orthogonal matrix whose components consist of −1, 0, and 1. We have obtained simple representations of total matrices by sub-matrices in a reference subdomain using a domain decomposition into congruent subdomains. We have discussed memory requirements with the CRS format to store finite element matrices using the representations. Reductions of memory requirements were observed with examples of an elliptic cylinder domain and a spherical shell domain. We have focused our attention on consumption of memory storage. Computational costs are also reduced by the domain decomposition, because integrations of finite element bases are done only in the reference subdomain.

In parallel computations, our approach is suitable to shared memory-type computers. Each processor is assigned to a group of subdomains for parallel execution of matrix-vector products operation in a Krylov subspace method. High parallel efficiency is obtained because the decomposition into congruent subdomains leads to almost optimal load balance.

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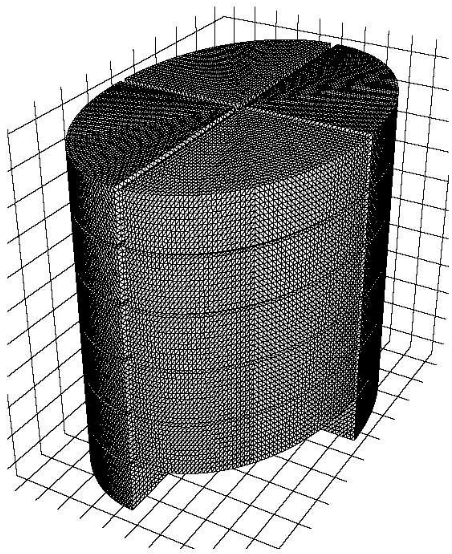

Figure 1: A domain decomposition of an elliptic cylinder into 24 subdomains.

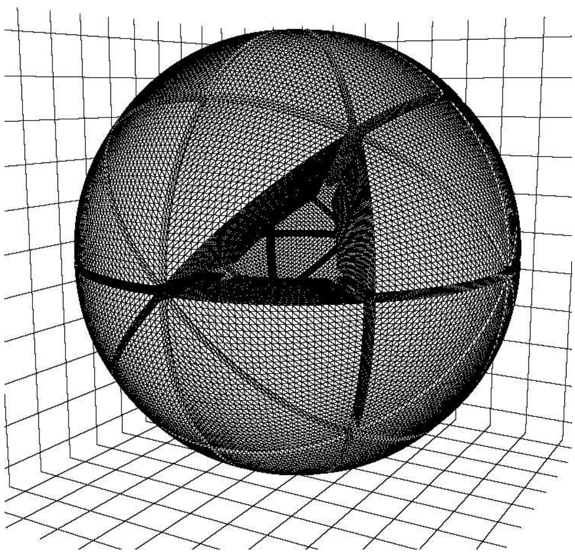

Figure 2: A domain decomposition of a spherical shell into 48 subdomains.

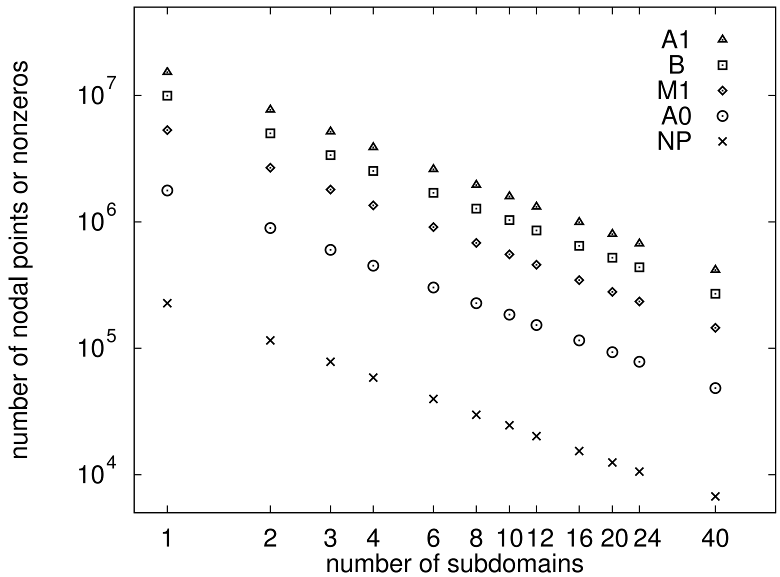

Figure 3: Numbers of nodal points and nonzero components of matrices in the reference subdomain in case of the elliptic cylinder domain.

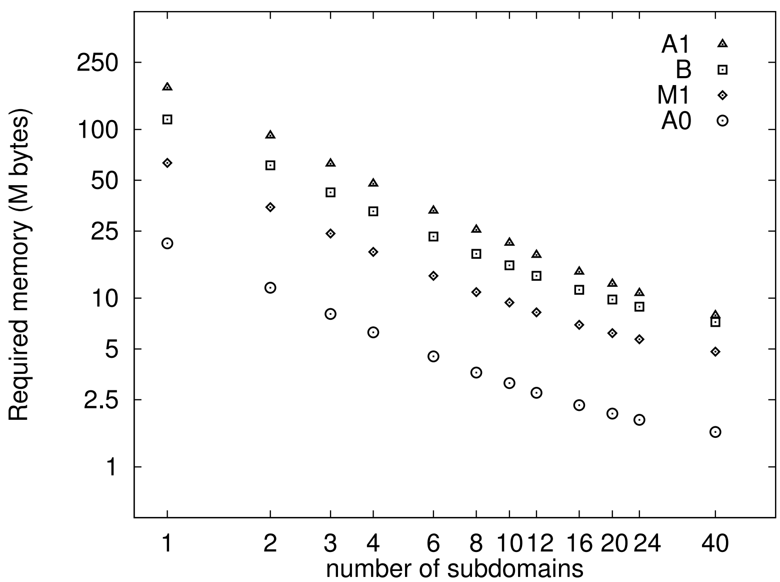

Figure 4: Memory requirements to store matrices in case of the elliptic cylinder domain.

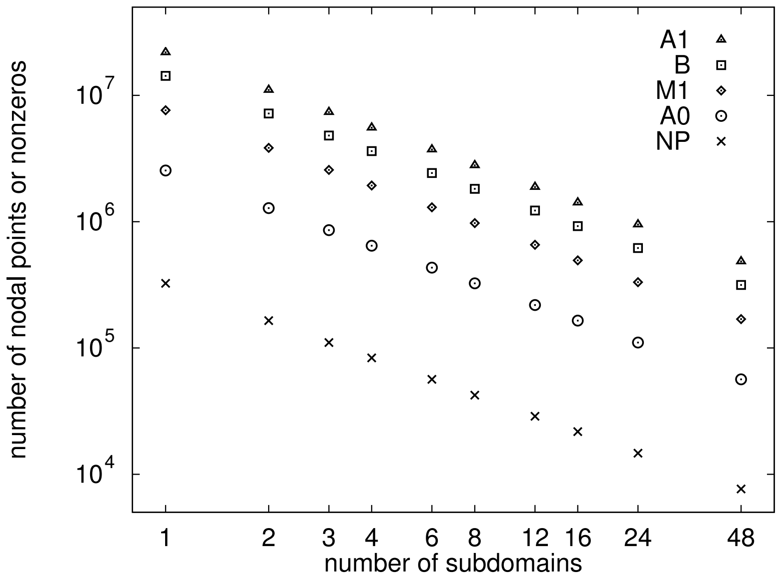

Figure 5: Numbers of nodal points and nonzero components of matrices in the reference subdomain in case of the spherical shell domain.

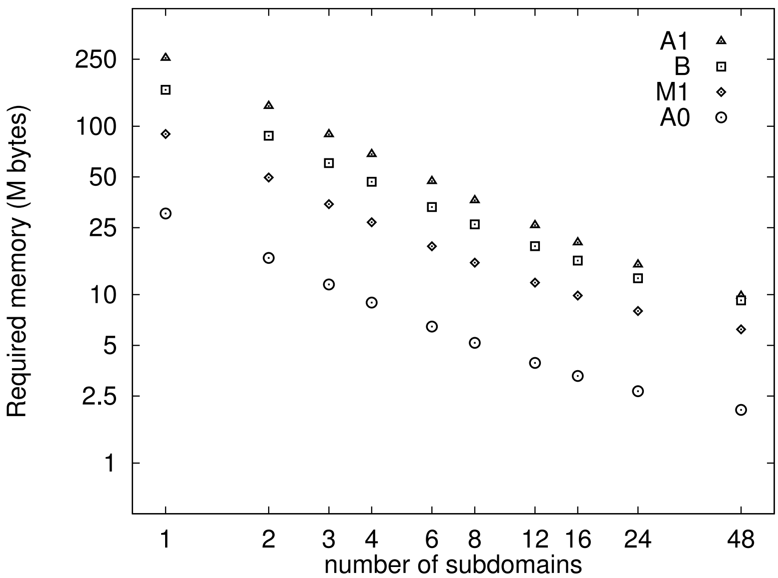

Figure 6: Memory requirements to store matrices in case of the spherical shell domain.

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