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Ishikawa, Masao
Department of Mathematics, Faculty of Education, Tottori University

Wakayama, Masato
Faculty of Mathematics, Kyushu University

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M. Ishikawa & M. Wakayama

Faculty of Mathematics

Kyushu University
Fukuoka, JAPAN

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Masao ISHIKAWA and Masato WAKAYAMA

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*Department of Mathematics, Faculty of Education, Tottori University

†Faculty of Mathematics, Kyushu Unversity

Abstract

The initial purpose of the present paper is to provide a combinatorial proof of the minor summation formula of Pfaffians in [8] based on the lattice path method. The second aim is to study the applications of the minor summation formula to obtaining several identities such as a variant of the Sundquist formula established in [31] about some extension of Pfaffian's version of the Cauchy determinant, a simple proof of Kawanaka's formula concerning a q-series identity involving Schur functions [15] (also, of the identity in [16] which is regarded as a determinant veriosn of the previous one from our point of view). We also establish a certain identity similar to the Kawanaka formula.

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Contents

1 Introduction

Recently, some applications of the minor summation formulas of Pfaffians presented in [8] have been made in several directions, e.g., to study a certain limit law for shifted Schur measures in [32], to find an explicit description of the skew-Capelli identity in [17] and to make further generalizations of the so-called Littlewood formulas for instance in [11] (see also [21], [13], [14]), etc. Moreover, the formula has been generalized to the case of hyperPfaffians in [23] (see also [22]).

In this paper we hence treat again the minor summation formulas of Pfaffians and derive several basic formulas concerning Pfaffians from certain combinatorial theoretical points of view. In order to develop the study of these formulas nicely, we present also a Pfaffian version of the Lewis-Caroll formula (Dodgeson's identity) and of the Plücker relation. The proof and the discussion concerning these latter formulas have not been developed sufficiently in our previous papers since they are not directly related to the actual proofs in our several applications of the minor summation formulas to obtaining various generating functions of the Schur functions etc, whereas they have been studied from the early stage of the research.

One of the main purpose of the present paper is to prove the minor summation formulas using the lattice path methods (see [30]) combined with the Lewis-Caroll formula for Pfaffians. The proof thus obtained enables us to provide a combinatorial interpretation of the minor summation formula through lattice paths. The point is that the Lewis-Caroll formula for Pfaffians plays efficiently to reduce the proof of the minor summation formulas, for instance, comparing with a similar combinatorial discussion given in [30], and consequently the present proof may give much clear meaning of the formulas.

The paper is organized as follows. In Section 2 we present two Pfaffian identities which may be called a Pfaffian version of the Lewis-Caroll formula and the Plücker relations respectively , and in Section 3 we rewrite the various type of minor summation formulas, where the Pfaffian analogue of the Lewis-Caroll formula has a key role in rewriting. In fact, we define the notion of the matrix formed by coPfaffians which may sound abuse of langueges, but we need to define a Pfaffian counterpart of the matrix of cofactors in the determinant theory, and then we express the minor summation formulas with these matrices of coPfaffians. Without the notion of matrices for coPfaffians, it seems very hard to discover a Gessel-Viennot type combinatorial proofs developed in Section 4, which simplfy the lattice paths' proof of the minor summation formulas. As another aim of the paper, we shall give in Section 5 some variant of the Sundquist formula obtained in [31] which is considered as a two variable pfafffian identities. The proof of this formula is elementary but involves several interesting identities. In Section 6 and Section 7 we give certain applications of the minor summation formulas to q-series. Actually, in Section 6 we show that Kawanaka's q-Littlewood identity is easily derived from the minor summation formulas, and in Section 7 we show that Kawanaka's q-Cauchy identity is proved by the Binet-Cauchy formula with some combinatorics. We also establish a certain identity similar to Kawanaka's q-Littlewood identity in the same line.

Though the notion of Pfaffians is less familiar than that of determinants it is also well-known that the Pfaffian (of a skew-symmetric matrix) is expressed as a square root of the determinant of the corresponding

matrix. We recall then first a more combinatorial definition of Pfaffians presented in [30]. Let $\mathfrak{S}_n$ be the symmetric group for the set of the letters $1, 2, \ldots, n$, and for each permutation $\sigma \in \mathfrak{S}_n$ let $\operatorname{sgn} \sigma$ stand for $(-1)^{\ell(\sigma)}$, where $\ell(\sigma)$ denotes the number of inversions in σ.

Let $n = 2r$ be an even integer and let $A = (a_{ij})_{1 \leq i,j \leq n}$, (i.e. $a_{ji} = -a_{ij}$) be an n by n skew symmetric matrix whose entries a_{ij} are in a commutative ring. The *Pfaffian* $\operatorname{Pf}(A)$ of A is defined by

$$\operatorname{Pf}(A) = \sum \epsilon(\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \sigma_n) \, a_{\sigma_1 \sigma_2} \ldots a_{\sigma_{n-1} \sigma_n}. \tag{1}$$

Here the summation is over all partitions $\{\{\sigma_1, \sigma_2\}_<, \ldots, \{\sigma_{n-1}, \sigma_n\}_<\}$ of $[n]$ into 2-elements blocks, and $\epsilon(\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \sigma_n)$ denotes the sign of the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{n-1} & \sigma_n \end{pmatrix}.$$

For instance, when $n = 4$, the equation above reads:

$$\operatorname{Pf} \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{14} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

Note that a skew symmetric matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ is determined by its upper triangular entries a_{ij} for $1 \leq i < j \leq n$.

A permutation $(\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \sigma_n)$ which arises from a partition of $[n]$ into 2-elements blocks is called a *perfect* maching or a 1-*factor*. For each $\pi \in \mathfrak{S}_n$, put $A^\pi = (a_{\pi(i)\pi(j)})$. From the definition above it is easy to see that

$$\operatorname{Pf}(A^\pi) = \operatorname{sgn} \pi \, \operatorname{Pf}(A). \tag{2}$$

2 The Lewis-Caroll formula and the Plücker relations

We provide a pfaffian version of Lewis-Caroll's formula (Dodgson's identity). We first recall the so-called Lewis-Caroll formula, or known as Jacobi's formula which is an identity among the minor determinants. The reader can find a restricted version of this identity and related topics in [1] and [28]. Furthermore, we present the Plücker's relations. The latter relations are also treated in [3], and in [18] they are called the (generalized) basic identity. We give a brief proof of ordinary Lewis-Caroll's formula, which is needed for establishing the Pfaffian version, to make also this paper self-contained. We only use Cramer's formula to prove it.

Let us denote by $\mathbb{N}$ the set of non-negative integers, and by $\mathbb{Z}$ the set of integers. Let $[n]$ denote the set $\{1, 2, \ldots, n\}$, for a positive integer n. Let n, M and N be positive integers such that $n \leq M, N$ and let T be any M by N matrix. For any multi-indices $I = \{i_1, \ldots, , i_n\}_< \subseteq [M]$ and $J = \{j_1, \ldots, j_n\}_< \subseteq [N]$ of cardinality n, let $T_J^I = T_{j_1 \ldots j_n}^{i_1 \ldots i_n}$ be the submatrix of T obtained by picking up the rows indexed by I and the columns indexed by J, i.e.,

$$T_J^I = \begin{pmatrix} t_{i_1 j_1} & \cdots & t_{i_1 j_n} \\ \vdots & \ddots & \vdots \\ t_{i_n j_1} & \cdots & t_{i_n j_n} \end{pmatrix}.$$

Let a_{ij} be a fixed element of a given square matrix A, and denote by $(A; i, j)$ a square submatrix obtained by removing the ith row and jth column of A. The determinant of $(A; i, j)$ is called a *minor* corresponding to a_{ij}, and the number $(-1)^{i+j} \det(A; i, j)$ is called a (i, j)-*cofactor* of A. The cofactor matrix $\widetilde{A}$ of A is the matrix whose (i, j)-entry is the (i, j)-cofactor of A.

Theorem 2.1 *Let A be an n by n matrix and $\widetilde{A}$ be its cofactor matrix. Let $r \leq n$ and $I, J \subseteq [n]$, $\sharp I = \sharp J = r$. Then*

$$\det \widetilde{A}^I_J = (-1)^{|I|+|J|}(\det A)^{r-1} \det A^{\overline{I}}_{\overline{J}}, \tag{3}$$

where $\overline{I}, \overline{J} \subseteq [n]$ stand for the complements of I, J, respectively in $\subseteq [n]$. Here we denote $|I| = \sum_{i \in I} i$.

Proof. Let $\Delta(i, j) = (-1)^{i+j} \det(A; i, j)$ denote the (i, j)-cofactor of A. Then, by definition, the matrix of cofactors is

$$\widetilde{A} = \left(\Delta(i, j) \right) = \left((-1)^{i+j} \det(A; i, j) \right).$$

Put $\overline{I} = \{p_1, \cdots, p_{n-r}\}$, $\overline{J} = \{q_1, \cdots, q_{n-r}\}$, and let $M = (M_{ij})$ be the matrix defined by

$$M_{ij} = \begin{cases} \Delta(i, j) & \text{if } i \in I, \\ 1 & \text{if } i = p_k \in \overline{I} \text{ and } j = q_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is a direct simple algebra to see that the (i, j)-entry of the matrix $A^t M = B = (b_{ij})$ is given by

$$B_{ij} = \begin{cases} \delta_{ij} \det A & \text{if } j \in I, \\ a_{iq_k} & \text{if } j = p_k \in \overline{I}. \end{cases}$$

Thus we have $\det B = (\det A)^r \det A^{\overline{I}}_{\overline{J}}$. Meanwhile, it is easy to see that

$$\det M = (-1)^{|\overline{I}|+|\overline{J}|} \det \widetilde{A}^I_J = (-1)^{|I|+|J|} \det \widetilde{A}^I_J$$

This proves the theorem. □

Example 2.2 *We put $I = J = \{1, n\} \subset [n]$ in the formula above and obtain the Desnannot-Jacobi adjoint matrix theorem:*

$$|M||M^{2,\ldots,n-1}_{2,\ldots,n-1}| = |M^{1,\ldots,n-1}_{1,\ldots,n-1}||M^{2,\ldots,n}_{2,\ldots,n}| - |M^{1,\ldots,n-1}_{2,\ldots,n}||M^{2,\ldots,n}_{1,\ldots,n-1}|,$$

which is also called Dodgeson's formula (or the Lewis-Caroll formula). For the details and the interesting story of the relations with the alternating sign matrices, see [1].

Let n be an even integer, and let A be a skew symmetric matrix of size n. Assume that $\mathrm{Pf}(A)$ is nonzero, which implies A is non-singular. For $1 \leq i < j \leq n$, let $(A; \{i, j\}, \{i, j\})$ denotes an $(n-2)$ by $(n-2)$ skew symmetric submatrix obtained by removing both the ith and jth rows and both the ith and jth columns of A. Let us define $\gamma(i, j)$ by

$$\gamma(i, j) = (-1)^{i+j-1} \mathrm{Pf}(A; \{i, j\}, \{i, j\}) \tag{4}$$

for $1 \leq i < j \leq n$. We define the values of $\gamma(i, j)$ for $1 \leq j \leq i \leq n$ as $\gamma(j, i) = -\gamma(i, j)$ always holds. Then the following expansion formula of Pfaffians along any row (resp. column) holds:

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Proposition 2.3 *Let n be an even integer and $A = (a_{ij})$ be an n by n skew symmetric matrix. For any i, j we have*

$$\delta_{ij} \operatorname{Pf}(A) = \sum_{k=1}^{n} a_{kj} \gamma(k, i), \tag{5}$$

$$\delta_{ij} \operatorname{Pf}(A) = \sum_{k=1}^{n} a_{ik} \gamma(j, k). \tag{6}$$

Since a_{ij} and $\gamma(i, j)$ are skew symmetric, the reader sees immediately that the identities (5) and (6) are equivalent. Moreover, to prove the general case it is sufficient to show the case where $i = j = 1$ in view of the formula (2). This case can be proved combinatorially from the definition (1) of Pfaffian. If we multiply the both sides of (5) by $\operatorname{Pf}(A)$ and use the basic relation between determinants and Pfaffians; $\det A = [\operatorname{Pf}(A)]^2$ (for a combinatorial proof, see for e.g. [30]), we obtain

$$\sum_{k=1}^{n} a_{ki} \gamma(k, j) \operatorname{Pf}(A) = \delta_{ij} \left[\operatorname{Pf}(A)\right]^2 = \delta_{ij} \det A.$$

Comparing this identity with the ordinary expansion of $\det A$, we obtain the following relation between $\Delta(i, j)$ and $\gamma(i, j)$:

$$\Delta(i, j) = \gamma(i, j) \operatorname{Pf}(A). \tag{7}$$

Definition 2.4 *Given a skew symmetric matrix A, let us call $\gamma(i, j)$ a copfaffian corresponding to a_{ij} (or (i, j)-copfaffian), and let $\widehat{A}$ denote the skew matrix whose (i, j)-entry is $\gamma(i, j)$, which we call the copfaffian matrix of A.*

Example 2.5 *Let $P_n(s, t)$ denotes the skew symmetric matrix, whose (i, j)-entry is given by $s^{(i-1) \bmod 2 + j \bmod 2} t^{j-i-1}$ for $1 \leq i < j \leq n$, where $x \bmod 2$ stands for the remainder of x divided by 2. In Lemma 7 of [8], we proved the formula*

$$\operatorname{Pf}(x_i y_j)_{1 \leq i < j \leq n} = \prod_{i=1}^{[n/2]} x_{2i-1} \prod_{j=1}^{[n/2]} y_{2j} \tag{8}$$

for an even integer n. From this formula, it is easy to see that the (i, j)-copfaffian of $P_n(s, t)$ is $(-1)^{j-i-1} s^{j-i-1} t^{(i-1) \bmod 2 + j \bmod 2}$. If $I = \{i_1, i_2, \ldots, i_{2r-1}, i_{2r}\}_<$, then the formula (8) also implies

$$\operatorname{Pf}\left[P_n(s, t)_I^I\right] = s^{\sum_{k=1}^{2r}(i_k - k) \bmod 2} t^{\sum_{k=1}^{2r}(-1)^k i_k - r}. \tag{9}$$

The following result is considered as a Pfaffian version of Lewis-Caroll's formula.

Theorem 2.6 *Let n be an even integer, and let A be an n by n skew symmetric matrix. Then, for any $I \subseteq [n]$ such that $\sharp I = 2r$, we have*

$$\operatorname{Pf}\left[(\widehat{A})_I^I\right] = (-1)^{|I|-r} \left[\operatorname{Pf}(A)\right]^{r-1} \operatorname{Pf}(A_{\overline{I}}^{\overline{I}}). \tag{10}$$

In particular, we have $\widehat{\widehat{A}} = (\operatorname{Pf} A)^{m-2} A$ with $n = 2m$.

Proof. Let $\widetilde{A} = (\Delta(i,j))$ denote the matrix of the cofactors of A. From (7) we have $\widetilde{A} = \mathrm{Pf}(A)\,\widehat{A}$, thus $(\widetilde{A})_I^I = \mathrm{Pf}(A)\,(\widehat{A})_I^I$. It follows that

$$\det(\widetilde{A})_I^I = [\mathrm{Pf}(A)]^{2r}\,\det(\widehat{A})_I^I = (\det A)^r\,\det(\widehat{A})_I^I.$$

On the other hand, Theorem 2.1 implies that $\det(\widetilde{A}_I^I) = (\det A)^{2r-1}\det A_{\overline{I}}^{\overline{I}}$. Comparing these two identities, we obtain

$$\det(\widehat{A})_I^I = (\det A)^{r-1}\det A_{\overline{I}}^{\overline{I}}.$$

By taking the square root of both sides of this identity, we obtain

$$\mathrm{Pf}\left(\widehat{A}_I^I\right) = \pm\,[\mathrm{Pf}(A)]^{r-1}\,\mathrm{Pf}\left(A_{\overline{I}}^{\overline{I}}\right). \tag{11}$$

To finish the proof we have to determine the sign. We substitute

$$S_n = P_n(1,1) = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ -1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 0 \end{pmatrix} \tag{12}$$

in the both sides of (11). Since $\widehat{S_n} = (\widehat{S}_{ij})$ with $\widehat{S}_{ij} = (-1)^{i+j-1}$ for $i < j$, Applying (9), we obtain $\mathrm{Pf}\,\widehat{S}_I = (-1)^{|I|-r}$ and $\mathrm{Pf}\,S_{\overline{I}} = 1$, which shows the desired sign. This proves the theorem. $\square$

Example 2.7 *Given a skew symmetric matrix A of size n, take $I = \{1,2,3,4\}$ in the theorem, then we obtain a formula which reads*

$$\mathrm{Pf}(A)\,\mathrm{Pf}(A_{5,\ldots,n}^{5,\ldots,n}) = \gamma(1,2)\gamma(3,4) - \gamma(1,3)\gamma(2,4) + \gamma(1,4)\gamma(2,3).$$

This may be regarded as a Pfaffian version of Dodgson's identity given in Example 2.2.

We give some examples of the copfaffian matrices. Let $n = 2r$. If $S_n = (S_{ij})$ with $S_{ij} = 1$ for $i < j$, then we have $\widehat{S_n} = (\widehat{S}_{ij})$ with $\widehat{S}_{ij} = (-1)^{i+j-1}$ for $i < j$ as obtained in the above proof. Put $T_n = (T_{ij}) = P_n(0,1)$ and $\widehat{T_n} = \left(\widehat{T}_{ij}\right)$, then

$$T_{ij} = \begin{cases} 1 & \text{if } 1 \leq i < j, \text{ and } i \text{ and } j-i \text{ are both odd}, \\ 0 & \text{otherwise}. \end{cases}$$

$$\widehat{T}_{ij} = \begin{cases} 1 & \text{if } j = i+1, \\ 0 & \text{otherwise}, \end{cases}$$

For example,

$$T_4 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}, \qquad \widehat{T}_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Let E_m denote the identity matrix of size m, and let O_m denote the zero matrix of size m. Let J_m denotes the symmetric matrix of size m defined by

$$J_m = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$

We let $n = 2m$ and put $K_n = \begin{pmatrix} O_m & J_m \\ -J_m & O_m \end{pmatrix}$ and $L_n = \begin{pmatrix} O_m & E_m \\ -E_m & O_m \end{pmatrix}$. Then it is easy to see that ${}^tK_n = -K_n$, ${}^tL_n = -L_n$, $K_n^2 = L_n^2 = -I_n$, $\mathrm{Pf}(K_n) = 1$ and $\mathrm{Pf}(L_n) = (-1)^{\frac{m(m-1)}{2}}$. From Cramer's formula and (7), we have $\widehat{A} = \mathrm{Pf}(A)\,{}^tA^{-1}$ for a non-singular matrix A, which immediately implies $\widehat{K_n} = K_n$ and $\widehat{L_n} = (-1)^{m(m-1)/2}L_n$.

We next state a Pfaffian analogue of the Plücker relations (or known as the Grassmann-Plücker relations for determinants) and make a remark on a relation with the Lewis-Caroll formula. It is an algebraic identity of degree two describing the relations among several subpfaffians. We point out that this identity has been proved in [6] and in [3] in the framework of an exterior algebra.

Theorem 2.8 *Suppose m, n are odd integers. Let A be an $(m+n) \times (m+n)$ skew symmetric matrix. Fix a sequence of integers $I = \{i_1 < i_2 < \cdots < i_m\} \subseteq [m+n]$ such that $\sharp I = m$. Denote the complement of I by $\overline{I} = \{k_1, k_2, \ldots, k_n\} \subseteq [m+n]$ which has the cardinality n. Then the following relation holds.*

$$\sum_{j=1}^{m}(-1)^{j-1}\mathrm{Pf}\left(A_{I\backslash\{i_j\}}^{I\backslash\{i_j\}}\right)\mathrm{Pf}\left(A_{\{i_j\}\cup\overline{I}}^{\{i_j\}\cup\overline{I}}\right) = \sum_{j=1}^{n}(-1)^{j-1}\mathrm{Pf}\left(A_{I\cup\{k_j\}}^{I\cup\{k_j\}}\right)\mathrm{Pf}\left(A_{\overline{I}\backslash\{k_j\}}^{\overline{I}\backslash\{k_j\}}\right).$$
$$(13)$$

Proof. We only use the expansion formula for Pfaffian given in Proposition 2.3. In fact, if we expand $\mathrm{Pf}(A_{\{i_j\}\cup\overline{I}})$ along the i_jth row/column on the left-hand side and also expand $\mathrm{Pf}(A_{I\cup\{k_j\}})$ along the k_jth row/column on the right-hand side, and finally compare it, then it is immediate to see the desired equality. This identity is also proved directly from the definition (1) of Pfaffian by usining the notion of matching and related combinatorics. □

Given a skew symmetric matrix A, we use the notation $A(i_1, i_2, \ldots, i_{2k})$ for $A_{i_1, i_2, \ldots, i_{2k}}^{i_1, i_2, \ldots, i_{2k}}$. Then the formula in the following assertion, which is called by the basic identity in [18], is regarded as a special case of the Plücker relations above.

Corollary 2.9 *Let A be a skew symmetric matrix of size N. Let $I = \{i_1, i_2, \ldots, i_{2k}\}$ be a subset of $[N]$. Take an integer l which satisfies $2k + 2l \le N$. Then we have*

$$\mathrm{Pf}(A(1, 2, \ldots, 2l))\,\mathrm{Pf}(A(i_1, i_2, \ldots, i_{2k}, 1, \ldots, 2l))$$
$$= \sum_{j=2}^{2k}(-1)^j\,\mathrm{Pf}(A(1, 2, \ldots, 2l, i_1, i_j))\,\mathrm{Pf}(A(i_2, \ldots, \widehat{i}_j, \ldots, i_{2k}, 1, \ldots, 2l)).$$
$$(14)$$

Proof. Given a skew symmetric matrix $A = (a_{ij})_{1\le i,j\le N}$ of size N and a subset $I = \{i_1, i_2, \ldots, i_{2k}\}$, we consider the skew symmetric matrix $B = (b_{ij})_{1\le i,j\le 2k+4l}$ of size $2k+4l$, whose (i, j)-entry b_{ij} is equal to $a_{p_i p_j}$, where the sequence $\{p_i\}_{i=1}^{2k+4l}$ is determined by

$$\begin{cases} p_\nu = \nu & \text{if } 1 \le \nu \le 2l, \\ p_{2l+\nu} = i_\nu & \text{if } 1 \le \nu \le 2k, \\ p_{2k+2l+\nu} = \nu & \text{if } 1 \le \nu \le 2l. \end{cases}$$

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Now, apply Theorem 2.8 to B with $m = 2l + 1$, $n = 2k + 2l - 1$, $I = \{1, 2, \ldots, 2l+1\}$ and $\overline{I} = \{2l+2, 2l+3, \ldots, 2k+4l\}$. Then, since each summand on the left-hand side vanish except for the case $j = 2l - 1$ ($i_j = i_1$), the desired identity immediately follows from the identity of Theorem 2.8.
□

Remark 2.10 *If we put $k = 2$ in this corollary, then the identity is nothing but the identity in Example 2.7. This implies the basic identity partially covers the Lewis-Caroll formula.*

3   Summation formulas of Pfaffians

In this section we review the summation formulas of Pfaffians. We restate the theorems in [8] and, in the next section, we shal give certain combinatorial proofs of Theorem 3.2, 3.6 and 3.7. In [8] we gave algebraic proofs of these thorems, instead in this paper, we use the lattice paths combined with the Lewis-Caroll formula (i.e. Theorem 2.6) to prove these theorems. Especially the Lewis-Caroll formula is useful to simplify our lattice path proof, comparing with the combinatorial discussion given in [30], and give more insights to explain these formulas. Our proofs of the theorems described here will be postponed until the next section. We restate our theorems in [8] in the form in which the copfaffian matrix $\widehat{A}$ appears instead of the original weight matrix A. This enable us to interpret these Pfaffians more directly and our proofs are not merely a weighted version of Stembridge's lattice path proofs, but more simplifyed and explain how copfaffian $\widehat{A}$ appears in Okada's summation formula [26].

Lemma 3.1 *Let m and $N = 2N'$ be even integers such that $m \leq N$. Let A (resp. B) be a skew symmetric matrix of size m (resp. N). Let T be an m by N rectangular matrix. We assume that $\mathrm{Pf}(B) \neq 0$. Then*

$$\mathrm{Pf}(B)^{-1} \mathrm{Pf}\left(A + TB\,{}^tT\right) = \mathrm{Pf}\begin{pmatrix} A & TJ_N \\ -J_N\,{}^tT & \frac{1}{\mathrm{Pf}(B)}J_N\,{}^t\widehat{B}J_N \end{pmatrix}$$

$$= \mathrm{Pf}\begin{pmatrix} J_m\,{}^tAJ_m & J_mT \\ -{}^tTJ_m & \frac{1}{\mathrm{Pf}(B)}{}^t\widehat{B} \end{pmatrix} = (-1)^{N'} \mathrm{Pf}\begin{pmatrix} A & T \\ -{}^tT & \frac{1}{\mathrm{Pf}(B)}{}^t\widehat{B} \end{pmatrix} \quad (15)$$

Proof. From the formula (5), we have $B\widehat{B} = -E_N$, and since B is skew-symmetric, we also have ${}^tB = J_N B J_N = -B$, which means $J_N\,{}^tBJ_N = B$.

$$\begin{pmatrix} A & TJ_N \\ -J_N\,{}^tT & \frac{1}{\mathrm{Pf}(B)}J_N\,{}^t\widehat{B}J_N \end{pmatrix}\begin{pmatrix} E_n & O \\ J_N B\,{}^tT & J_N B J_N \end{pmatrix} = \begin{pmatrix} A + TB\,{}^tT & TBJ_N \\ O & E_N \end{pmatrix}.$$

By taking the determinants of the both sides, we obatin

$$\det\begin{pmatrix} A & TJ_N \\ -J_N\,{}^tT & \frac{1}{\mathrm{Pf}(B)}J_N\,{}^t\widehat{B}J_N \end{pmatrix}\det B = \det\left(A + TB\,{}^tT\right)$$

By taking the square root of this identity, we have

$$\pm\mathrm{Pf}(B)\,\mathrm{Pf}\begin{pmatrix} A & TJ_N \\ -J_N\,{}^tT & \frac{1}{\mathrm{Pf}(B)}J_N\,{}^t\widehat{B}J_N \end{pmatrix} = \mathrm{Pf}\left(A + TB\,{}^tT\right).$$

To determine the sign, we compare the coefficient of $\mathrm{Pf}(A)$ of both sides as a polynomial of the entries of A and B. This coefficient is clearly equal to

+1 in $\operatorname{Pf}\left(A + TB^tT\right)$. Meanwhile, from the definition (1) of Pfaffian, this term only appears in the form $\operatorname{Pf}(A)\operatorname{Pf}(\frac{1}{\operatorname{Pf}(B)}J_N{}^t\widehat{B}J_N) = \frac{\operatorname{Pf}(A)}{\operatorname{Pf}(B)^{N'}}\operatorname{Pf}(\widehat{B})$ on the left-hand side. By (10) we obtain $\operatorname{Pf}(\widehat{B}) = \operatorname{Pf}(B)^{N'-1}$. This shows the sign is +1 and we obtain the desired result. All the other identies are derived by the same method by multiplying an appropriate matrix to make it into a triangular matrix. The last identity is obtained from the first by the identity (2), which shows how a permutation of rows/columns changes the sign of a Pfaffian. This completes the proof of the lemma. $\square$

We restate Theorem 1 of [8] in the following form:

Theorem 3.2 *Let m and $N = 2N'$ be even integers such that $m \leq N$. Let $T = (t_{ik})_{1 \leq i \leq m, 1 \leq k \leq N}$ be an m by N rectangular matrix. Let $A = (a_{ij})_{1 \leq i,j \leq N}$ be a skew-symmetric matrix of size N and let $\widehat{A}$ denote its copfaffian matrix. Then*

$$\sum_{\substack{I \subseteq [N] \\ \sharp I = m}} \operatorname{Pf}(A_I^I)\det(T_I) = \operatorname{Pf}(Q) = \operatorname{Pf}(A)\operatorname{Pf}\begin{pmatrix} O_m & TJ_N \\ -J_N{}^tT & \frac{1}{\operatorname{Pf}(A)}J_N{}^t\widehat{A}J_N \end{pmatrix}$$

$$= \operatorname{Pf}(A)\operatorname{Pf}\begin{pmatrix} O_m & J_mT \\ -{}^tTJ_m & \frac{1}{\operatorname{Pf}(A)}{}^t\widehat{A} \end{pmatrix} = (-1)^{N'}\operatorname{Pf}(A)\operatorname{Pf}\begin{pmatrix} O_n & T \\ {}^tT & \frac{1}{\operatorname{Pf}(A)}{}^t\widehat{A} \end{pmatrix}.$$

$$(16)$$

Here $Q = (Q_{ij}) = TA\,{}^tT$, and its entries are given by

$$Q_{ij} = \sum_{1 \leq k < l \leq N} a_{kl}\det(T_{kl}^{ij}), \qquad (1 \leq i,j \leq m). \tag{17}$$

The first identity, i.e. $\sum \operatorname{Pf}(A_I^I)\det(T_I) = \operatorname{Pf}(Q)$, holds even if N is not even. When m is odd, we can immediately derive a similar formula from the case when m is even. So we only treat even cases in this paper. If we take $m = N = 2r$ and $A = K_m$ in (16), then $\det(T) = \operatorname{Pf}(K_m)\det(T) = \operatorname{Pf}(T K_m{}^tT)$. This means that every determinant of even degree can be represented by a pfaffian of the *same* degree.

When m and N are even integers such that $0 \leq m \leq N$, and X and Y are m by N rectangular matrices, taking $A = {}^tXK_mX$ and $T = Y$ in Theorem 3.2, we obtain the following corollay, which is the so-called Cauchy-Binet formula. For another proof which use Theorem 3.2, see [7].

Corollary 3.3 *Assume $m \leq N$, and let $X = (x_{ij})_{1 \leq i \leq m, 1 \leq j \leq N}$ and $Y = (y_{ij})_{1 \leq i \leq m, 1 \leq j \leq N}$ be any m by N matrices. Then*

$$\sum_{\substack{K \subseteq [N] \\ \sharp K = m}} \det(X_K)\det(Y_K) = \det\left(X{}^tY\right). \tag{18}$$

We also list one formula concerning the skew symmetric part of a general square matrix as a simple reduction from the minor summation formula. Actually the following type of pfaffians may arise naturally when one considers the imaginary part of Hermitian forms. For the proof, see [12]

Corollary 3.4 *Fix positive integers m, n such that $m \leq 2n$. Let A and B be arbitrary $n \times m$ matrices, and X be an $n \times n$ symmetric matrix.*

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(i.e. $^tX = X$). Let P be the skew symmetric matrix defined by $P = {}^tAXB - {}^tBXA$. Then we have

$$\mathrm{Pf}\,(P) = \sum_{\substack{K \subseteq [2n] \\ \sharp K = m}} \mathrm{Pf}\left[\begin{pmatrix} O_n & X \\ -X & O_n \end{pmatrix}^K_K\right] \det\left[\begin{pmatrix} A \\ B \end{pmatrix}^K\right].$$

In particular, when $m = 2n$ we have

$$\mathrm{Pf}\,(P) = \det(X)\det\begin{pmatrix} A \\ B \end{pmatrix}.$$

Proof. Apply the above theorem to the $2n \times 2n$ skew symmetric matrix $\begin{pmatrix} O_n & X \\ -X & O_n \end{pmatrix}$ and the $2n \times m$ matrix $\begin{pmatrix} A \\ B \end{pmatrix}$. Then the elementary identity

$$^t\begin{pmatrix} A \\ B \end{pmatrix}\begin{pmatrix} O_n & X \\ -X & O_n \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = {}^tAXB - {}^tBXA$$

immediately asserts the corollary. $\square$

Before we state the next theorem, we prepare a lemma:

Lemma 3.5 *Let m, n be nonnegative integers, and let $N = 2N'$ be an even integer such that $m \le N$. Let A (resp. B) be a nonsingular skew-symmetric matrix of size m (resp. N). Let G (resp. H) be an m by n (resp. m by N) rectangular matix. Assume that $\mathrm{Pf}(B) \ne 0$. Then*

$$\mathrm{Pf}(B)^{-1}\,\mathrm{Pf}\begin{pmatrix} A + HB\,{}^tH & GJ_n \\ -J_n\,{}^tG & O_n \end{pmatrix} = \mathrm{Pf}\begin{pmatrix} A & HJ_N & GJ_n \\ -J_N\,{}^tH & \frac{1}{\mathrm{Pf}(B)}J_N\,{}^t\widehat{B}J_N & O_{N,n} \\ -J_n\,{}^tG & O_{n,N} & O_n \end{pmatrix}$$

$$= \mathrm{Pf}\begin{pmatrix} -J_mAJ_m & J_mG & J_mH \\ -{}^tGJ_m & O_n & O_{n,N} \\ -{}^tHJ_m & O_{N,n} & \frac{1}{\mathrm{Pf}(B)}\,{}^t\widehat{B} \end{pmatrix} = (-1)^{N'}\,\mathrm{Pf}\begin{pmatrix} A & G & H \\ -{}^tG & O_n & O_{n,N} \\ -{}^tH & O_{N,n} & \frac{1}{\mathrm{Pf}(B)}\,{}^t\widehat{B} \end{pmatrix}$$

$$\tag{19}$$

Proof. The identity (5) implies

$$\begin{pmatrix} A & HJ_N & GJ_n \\ -J_N\,{}^tH & \frac{1}{\mathrm{Pf}(B)}J_N\,{}^t\widehat{B}J_N & O_{N,n} \\ -J_n\,{}^tG & O_{n,N} & O_n \end{pmatrix}\begin{pmatrix} E_m & O_{m,N} & O_{m,n} \\ J_N B\,{}^tH & J_N BJ_N & O_{N,n} \\ O_{n,m} & O_{n,N} & E_n \end{pmatrix}$$

$$= \begin{pmatrix} A + HB\,{}^tH & HBJ_N & GJ_n \\ O_{N,n} & E_N & O_{N,n} \\ -J_n\,{}^tG & O_{n,N} & O_n \end{pmatrix}.$$

By taking the determinant and its square root of the both sides, we obatin

$$\pm\mathrm{Pf}(B)\,\mathrm{Pf}\begin{pmatrix} A & HJ_N & GJ_n \\ -J_N\,{}^tH & \frac{1}{\mathrm{Pf}(B)}J_N\,{}^t\widehat{B}J_N & O_{N,n} \\ -J_n\,{}^tG & O_{n,N} & O_n \end{pmatrix} = \mathrm{Pf}\begin{pmatrix} A + HB\,{}^tH & GJ_n \\ -J_n\,{}^tG & O_n \end{pmatrix}.$$

We can determine the sign by comparing the coefficient of $g_{11}\cdots g_{mm}a_{m+1,m+2}\cdots a_{n-1,n}$ of the both sides using the definition (1) and Theorem 2.6. The other identies can be proved in the same way. $\square$

The following theorem gives a minor summation formula, in which the index set I of a minor in the sum always include some fixed column index set, say $\{1, 2, \ldots, n\}$. (See [8] and [30].)

Theorem 3.6 *Let m, n and N be positive integers such that $m - n$ and $N = 2N'$ are even and $0 \leq m - n \leq N$. Let $A = (a_{ij})_{1 \leq i,j \leq N}$ be a nonsingular skew-symmetric matrix of size N. Let $T = (t_{ij})_{1 \leq i \leq m, 1 \leq j \leq n+N}$ be an m by $(n+N)$ rectangular matrix. We put the sets of column indices $R^0 = \{1, \ldots, n\}$ and $R = \{n+1, \ldots, n+N\}$. Then*

$$\sum_{\substack{I \subseteq R \\ \sharp I = m-n}} \mathrm{Pf}(A_I^I) \det(T_{R^0 \uplus I}) = \mathrm{Pf} \begin{pmatrix} Q & T_{R^0} J_n \\ -J_n {}^t T_{R^0} & O_n \end{pmatrix},$$

$$= \mathrm{Pf} \begin{pmatrix} O_m & T_R J_N & T_{R^0} J_n \\ -J_N {}^t T_R & \frac{1}{\mathrm{Pf}(A)} J_N {}^t \widehat{A} J_N & O_{N,n} \\ -J_n {}^t T_{R^0} & O_{n,N} & O_n \end{pmatrix} = \mathrm{Pf} \begin{pmatrix} O_m & J_m T_{R^0} & J_m T_R \\ -{}^t T_{R^0} J_m & O_n & O_{n,N} \\ -{}^t T_R J_m & O_{N,n} & \frac{1}{\mathrm{Pf}(A)} {}^t \widehat{A} \end{pmatrix}$$

$$= (-1)^{N'} \mathrm{Pf} \begin{pmatrix} O_m & T_{R^0} & T_R \\ -{}^t T_{R^0} & O_n & O_{n,N} \\ -{}^t T_R & O_{N,n} & \frac{1}{\mathrm{Pf}(A)} {}^t \widehat{A} \end{pmatrix} \tag{20}$$

where Q is the m by m skew-symmetric matrix defined by $Q = T_R A \, {}^t T_R$, i.e.

$$Q_{ij} = \sum_{1 \leq k < l \leq N} a_{kl} \det(T_{kl}^{ij}), \qquad (1 \leq i, j \leq m). \tag{21}$$

We shall later restate a combinatrial description (and a proof) of this theorem as Theorem 4.4.

The following theorem shows a minor summation formula for both the rows and columns.

Theorem 3.7 *Let M and $N = 2N'$ be even integers such that $M \leq N$. Let $T = (t_{ij})_{1 \leq i \leq M, 1 \leq j \leq N}$ be any M by N rectangular matrix, and let $A = (a_{ij})_{1 \leq i,j \leq M}$ (resp. $B = (b_{ij})_{1 \leq i,j \leq N}$) be a nonsingular skew-symmetric matrix of size M (resp. size N). Then*

$$\sum_{\substack{r=0 \\ r \,:\, even}}^{\min(M,N)} z^r \sum_{\substack{I \subseteq [M] \\ \sharp I = r}} \sum_{\substack{J \subseteq [N] \\ \sharp J = r}} \mathrm{Pf}(A_I^I) \mathrm{Pf}(B_J^J) \det(T_J^I) = \mathrm{Pf}(A) \mathrm{Pf} \left[\frac{1}{\mathrm{Pf}(A)} \widehat{A} + z^2 T B \, {}^t T \right]$$

$$= \mathrm{Pf}(A) \mathrm{Pf}(B) \mathrm{Pf} \begin{bmatrix} \frac{1}{\mathrm{Pf}(A)} \widehat{A} & z T J_N \\ -z J_N {}^t T & \frac{1}{\mathrm{Pf}(B)} \widehat{B} \end{bmatrix} = \mathrm{Pf}(A) \mathrm{Pf}(B) \mathrm{Pf} \begin{bmatrix} \frac{1}{\mathrm{Pf}(A)} J_n \widehat{A} J_n & z J_n T \\ -z {}^t T J_n & \frac{1}{\mathrm{Pf}(B)} \widehat{B} \end{bmatrix}$$

$$= (-1)^{N'} \mathrm{Pf}(A) \mathrm{Pf}(B) \mathrm{Pf} \begin{bmatrix} \frac{1}{\mathrm{Pf}(A)} \widehat{A} & z T \\ -z {}^t T & \frac{1}{\mathrm{Pf}(B)} \widehat{B} \end{bmatrix}. \tag{22}$$

We also have the

Corollary 3.8 *Let M and N be nonnegative integers such that $M \leq N$. Let $T = (t_{ij})$ be an M by N rectangular matrix. Let $B = (b_{ij})_{0 \leq i,j \leq N}$ be a skew-symmetric matrix of size $(N + 1)$,*

1. *If M is odd and $A = (a_{ij})_{0 \leq i,j \leq M}$ is a nonsingular skew-symmetric matrix of size $(M + 1)$, then*

$$\sum_{\substack{0 \leq r \leq M \\ r \,even}} z^r \sum_{\substack{I \subseteq [M] \\ \sharp I = r}} \sum_{\substack{J \subseteq [N] \\ \sharp J = r}} \mathrm{Pf}(A_I^I) \mathrm{Pf}(B_J^J) \det(T_J^I)$$

$$+ \sum_{\substack{0 \leq r \leq M \\ r \, odd}} z^r \sum_{\substack{I \subseteq [M] \\ \sharp I = r}} \sum_{\substack{J \subseteq [N] \\ \sharp J = r}} \mathrm{Pf}(A_{\{0\} \cup I}^{\{0\} \cup I}) \mathrm{Pf}(B_{\{0\} \cup J}^{\{0\} \cup J}) \det(T_J^I)$$

$$= \mathrm{Pf}(A) \mathrm{Pf} \left(\frac{1}{\mathrm{Pf}(A)} \widehat{A} + Q \right), \tag{23}$$

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where $Q = (Q_{ij})_{0 \le i,j \le M}$ is given by

$$Q_{ij} = \begin{cases} 0, & \text{if } i = j = 0, \\ z \sum_{1 \le k \le N} b_{0k} t_{jk}, & \text{if } i = 0 \text{ and } 1 \le j \le M, \\ z \sum_{1 \le k \le N} b_{k0} t_{jk}, & \text{if } j = 0 \text{ and } 1 \le i \le M, \\ z^2 \sum_{1 \le k < l \le N} b_{kl} \det(T_{kl}^{ij}), & \text{if } 1 \le i,j \le M. \end{cases}$$

$$(24)$$

2. *If M is even and $A = (a_{ij})_{0 \le i,j \le M+1}$ is a nonsingular skew-symmetric matrix of size $(M+2)$, then*

$$\sum_{\substack{0 \le r \le M \\ r \text{ even}}} z^r \sum_{\substack{I \subseteq [M] \\ \sharp I = r}} \sum_{\substack{J \subseteq [N] \\ \sharp J = r}} \mathrm{Pf}(A_I^I)\,\mathrm{Pf}(B_J^J)\det(T_J^I)$$

$$+ \sum_{\substack{0 \le r \le M \\ r \text{ odd}}} z^r \sum_{\substack{I \subseteq [M] \\ \sharp I = r}} \sum_{\substack{J \subseteq [N] \\ \sharp J = r}} \mathrm{Pf}(A_{\{0\} \cup I}^{\{0\} \cup I})\,\mathrm{Pf}(B_{\{0\} \cup J}^{\{0\} \cup J})\det(T_J^I)$$

$$= \mathrm{Pf}(A)\,\mathrm{Pf}\left(\frac{1}{\mathrm{Pf}(A)}\widehat{A} + Q\right), \qquad (25)$$

where $Q = (Q_{ij})_{0 \le i,j \le M+1}$ is given by

$$Q_{ij} = \begin{cases} 0 & \text{if } i = j = 0, \\ z \sum_{1 \le k \le N} b_{0k} t_{jk}, & \text{if } i = 0 \text{ and } 1 \le j \le M, \\ z \sum_{1 \le k \le N} b_{k0} t_{jk}, & \text{if } j = 0 \text{ and } 1 \le i \le M, \\ z^2 \sum_{1 \le k < l \le N} b_{kl} \det(T_{kl}^{ij}), & \text{if } 1 \le i,j \le M, \\ 0 & \text{if } i = M+1 \text{ or } j = M+1. \end{cases}$$

$$(26)$$

The proof of the following lemma is parallel to the proof of Lemma 3.5 and we omit the proof.

Lemma 3.9 *Let n, m and M be nonnegatuve integers, and let $N = 2N'$ be an even integer. Let A (resp. B) be a skew symmetric matrix of size n (resp. N) such that B is nonsingular. Let T_{11}, T_{12}, T_{21} and T_{22} be an m by n, m by N, M by n and M by N rectangular matix, respectively. Then*

$$\mathrm{Pf}(B)^{-1}\,\mathrm{Pf}\begin{pmatrix} T_{12}B^t T_{12} & T_{12}B^t T_{22} & T_{11}J_n \\ T_{22}B^t T_{12} & A + T_{22}B\,^t T_{22} & T_{21}J_n \\ -J_n\,^t T_{11} & -J_n\,^t T_{21} & O_n \end{pmatrix}$$

$$= \mathrm{Pf}\begin{pmatrix} O_m & O_{m,M} & T_{12}J_N & T_{11}J_n \\ O_{M,m} & A & T_{22}J_N & T_{21}J_n \\ -J_N\,^t T_{12} & -J_N\,^t T_{22} & \frac{1}{\mathrm{Pf}(B)}J_N\,^t \widehat{B}J_N & O_{N,n} \\ -J_n\,^t T_{11} & -J_n\,^t T_{21} & O_{n,N} & O_n \end{pmatrix} \qquad (27)$$

Theorem 3.10 *Let m, n, M, N be nonnegative integers such that $M \equiv N \equiv m - n \equiv 0 \pmod 2$. We put $R^0 = \{1, \ldots, m\}$, $S^0 = \{1, \ldots, n\}$, $R = \{m+1, \ldots, m+M\}$ and $S = \{n+1, \ldots, n+N\}$, Let $T = (t_{ij})_{1 \le i \le m+M, 1 \le j \le n+N}$ be any $(m+M)$ by $(n+N)$ rectangular matrix, and let $A = (a_{ij})_{1 \le i,j \le M}$ (resp. $B = (b_{ij})_{1 \le i,j \le N}$) be a nonsigular skew-symmetric matrix of size M (resp. size N).*

1. If $0 \leq m - n \leq N$, then

$$\sum_{\substack{0 \leq r \leq \min(M, N-m+n) \\ r \text{ even}}} z^{r+m} \sum_{\substack{I \subseteq R \\ \sharp I = r}} \sum_{\substack{J \subseteq S \\ \sharp J = r+m-n}} \mathrm{Pf}(A_I^I) \, \mathrm{Pf}(B_J^J) \det(T_{S^0 \uplus J}^{R^0 \uplus I})$$

$$= \mathrm{Pf}(A) \, \mathrm{Pf}(B) \, \mathrm{Pf} \begin{pmatrix} O_m & O_{m,M} & z T_S^{R^0} J_N & z T_{S^0}^{R^0} J_n \\ O_{M,m} & \frac{1}{\mathrm{Pf}(A)} \widehat{A} & z T_S^R J_N & z T_{S^0}^R J_n \\ -z J_N {}^t T_S^{R^0} & -z J_N {}^t T_S^R & \frac{1}{\mathrm{Pf}(B)} J_N {}^t \widehat{B} J_N & O_{N,n} \\ -z J_n {}^t T_{S^0}^{R^0} & -z J_n {}^t T_{S^0}^R & O_{n,N} & O_n \end{pmatrix}$$

$$(28)$$

2. If $0 \leq n - m \leq M$, then

$$\sum_{\substack{0 \leq r \leq \min(M-n+m, N) \\ r \text{ even}}} z^{r+n} \sum_{\substack{I \subseteq R \\ \sharp I = r+n-m}} \sum_{\substack{J \subseteq S \\ \sharp J = r}} \mathrm{Pf}(A_I^I) \, \mathrm{Pf}(B_J^J) \det(T_{S^0 \uplus J}^{R^0 \uplus I})$$

$$= \mathrm{Pf}(A) \, \mathrm{Pf}(B) \, \mathrm{Pf} \begin{pmatrix} O_m & O_{m,M} & z T_S^{R^0} J_N & z T_{S^0}^{R^0} J_n \\ O_{M,m} & \frac{1}{\mathrm{Pf}(A)} \widehat{A} & z T_S^R J_N & z T_{S^0}^R J_n \\ -z J_N {}^t T_S^{R^0} & -z J_N {}^t T_S^R & \frac{1}{\mathrm{Pf}(B)} J_N {}^t \widehat{B} J_N & O_{N,n} \\ -z J_n {}^t T_{S^0}^{R^0} & -z J_n {}^t T_{S^0}^R & O_{n,N} & O_n \end{pmatrix}$$

$$(29)$$

Corollary 3.11 *Let m, n, M and N be nonnegatie integers such that $m \equiv n \pmod 2$. We put $R^0 = \{1, \ldots, m\}$, $S^0 = \{1, \ldots, n\}$, $R = \{m+1, \ldots, m+M\}$ and $S = \{n+1, \ldots, n+N\}$, Let $T = (T_j^i)_{1 \leq i \leq m+M, 1 \leq j \leq n+N}$ be any $(m+M)$ by $(n+N)$ rectangular matrix, and let $B = (b_{ij})_{0 \leq i,j \leq N}$ be any skew-symmetric matrix of size $(N+1)$.*

1. *Assume $0 \leq m - n \leq N$.*

   (a) *If M is odd and $A = (a_{ij})_{1 \leq i,j \leq M+1}$ is a nonsingular skew-symmetric matrix of size $(M+1)$, then*

$$\sum_{\substack{0 \leq r \leq \min(M, N-m+n) \\ r \text{ even}}} z^{r+m} \sum_{\substack{I \subseteq R \\ \sharp I = r}} \sum_{\substack{J \subseteq S \\ \sharp J = r+m-n}} \mathrm{Pf}(A_I^I) \, \mathrm{Pf}(B_J^J) \det(T_{S^0 \uplus J}^{R^0 \uplus I})$$

$$+ \sum_{\substack{0 \leq r \leq \min(M, N-m+n) \\ r \text{ odd}}} z^{r+m} \sum_{\substack{I \subseteq R \\ \sharp I = r}} \sum_{\substack{J \subseteq S \\ \sharp J = r+m-n}} \mathrm{Pf}(A_{I \uplus \{M+1\}}^{I \uplus \{M+1\}}) \, \mathrm{Pf}(B_{J \uplus \{N+1\}}^{J \uplus \{N+1\}}) \det(T_{S^0 \uplus J}^{R^0 \uplus I})$$

$$= \mathrm{Pf}(A) \, \mathrm{Pf} \begin{pmatrix} Q^{11} & Q^{12} & z T_{S^0}^{R^0} J_n \\ -{}^t Q^{12} & \frac{1}{\mathrm{Pf}(A)} \widehat{A} + Q^{22} & z \overline{T}_{S^0}^R J_n \\ -z J_n {}^t T_{S^0}^{R^0} & -z J_n {}^t \overline{T}_{S^0}^R & O_n \end{pmatrix} \qquad (30)$$

   where $Q^{11} = (Q_{ij}^{11})_{1 \leq i,j \leq m}$, $Q^{12} = (Q_{ij}^{12})_{1 \leq i \leq m, \, 1 \leq j \leq M+1}$, $Q^{22} = (Q_{ij}^{22})_{1 \leq i,j \leq M+1}$ is given by

$$Q_{ij}^{11} = z^2 \sum_{1 \leq k < l \leq N} b_{kl} \det(T_{n+k, n+l}^{ij})$$

$$Q_{ij}^{12} = \begin{cases} z^2 \sum_{1 \leq k < l \leq N} b_{kl} \det(T_{n+k, n+l}^{i, j+m}) & \text{if } 1 \leq j \leq M, \\ 0 & \text{if } j = M+1, \end{cases}$$

$$Q_{ij}^{22} = \begin{cases} z^2 \sum_{1 \leq k < l \leq N} b_{kl} \det(T_{n+k, n+l}^{m+i, m+j}) & \text{if } 1 \leq i, j \leq M, \\ z \sum_{1 \leq k \leq N} b_{N+1, k} T_{n+k}^{m+j} & \text{if } i = M+1 \text{ and } 1 \leq j \leq M, \\ z \sum_{1 \leq k \leq N} b_{k, N+1} T_{n+k}^{m+i} & \text{if } j = M+1 \text{ and } 1 \leq i \leq M, \\ 0 & \text{if } i = j = M+1, \end{cases}$$

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and $\overline{T}^R_{S0}$ is the $(M+1)$ by n matrix in which its first M rows are the same as T^R_{S0} and the entries of its bottom row are all zero.

(b) If M is even and $A = (a_{ij})_{1 \le i,j \le M+2}$ is a nonsingular skew-symmetric matrix of size $(M+2)$, then

$$\sum_{\substack{0 \le r \le \min(M, N-m+n) \\ r \ even}} z^{r+m} \sum_{\substack{I \subseteq R \\ \sharp I = r}} \sum_{\substack{J \subseteq S \\ \sharp J = r+m-n}} \mathrm{Pf}(A^I_I) \, \mathrm{Pf}(B^J_J) \det(T^{R^0 \uplus I}_{S0 \uplus J})$$

$$+ \sum_{\substack{0 \le r \le \min(M, N-m+n) \\ r \ odd}} z^{r+m} \sum_{\substack{I \subseteq R \\ \sharp I = r}} \sum_{\substack{J \subseteq S \\ \sharp J = r+m-n}} \mathrm{Pf}(A^{I \uplus \{M+1\}}_{I \uplus \{M+1\}}) \, \mathrm{Pf}(B^{J \uplus \{N+1\}}_{J \uplus \{N+1\}}) \det(T^{R^0 \uplus I}_{S0 \uplus J})$$

$$= \mathrm{Pf}(A) \, \mathrm{Pf} \begin{pmatrix} Q^{11} & Q^{12} & z T^{R^0}_{S0} J_n \\ -{}^t Q^{12} & \frac{1}{\mathrm{Pf}(A)} \widehat{A} + Q^{22} & z \overline{T}^R_{S0} J_n \\ -z J_n \, {}^t T^{R^0}_{S0} & -z J_n \, \overline{T}^R_{S0} & O_n \end{pmatrix} \qquad (31)$$

where $Q^{11} = (Q^{11}_{ij})_{1 \le i,j \le m}$, $Q^{12} = (Q^{12}_{ij})_{1 \le i \le m, \, 1 \le j \le M+2}$, $Q^{22} = (Q^{22}_{ij})_{1 \le i,j \le M+2}$ is given by

$$Q^{11}_{ij} = z^2 \sum_{1 \le k < l \le N} b_{kl} \det(T^{ij}_{n+k, n+l})$$

$$Q^{12}_{ij} = \begin{cases} z^2 \sum_{1 \le k < l \le N} b_{kl} \det(T^{i, j+m}_{n+k, n+l}) & \text{if } 1 \le j \le M, \\ 0 & \text{if } M+1 \le j \le M+2, \end{cases}$$

$$Q^{22}_{ij} = \begin{cases} z^2 \sum_{1 \le k < l \le N} b_{kl} \det(T^{m+i, m+j}_{n+k, n+l}) & \text{if } 1 \le i,j \le M, \\ z \sum_{1 \le k \le N} b_{N+1,k} T^{m+j}_{n+k} & \text{if } i = M+1 \text{ and } 1 \le j \le M, \\ z \sum_{1 \le k \le N} b_{k,N+1} T^{m+i}_{n+k} & \text{if } j = M+1 \text{ and } 1 \le i \le M, \\ 0 & \text{if } i = j = M+1, \\ 0 & \text{if } i = M+2 \text{ or } j = M+2, \end{cases}$$

and T^{*R}_{S0} is the $(M+2)$ by n matrix in which its first M rows are the same as T^R_{S0} and the entries of the last two rows are all zero.

4 Proofs by Lattice Paths

In this section we give combinatorial proofs of the summation formulas of Pfaffians, i.e. Theorem 3.2, 3.6, 3.7, which are stated in Section 3. In [26] Okada gave this type of the formula related to a certain plane partition enumeration problem, but his formula was a very special case, i.e. $A = S_N$, of ours. In [30] J.Stembridge gave a lattice path interpretation of this special summation formula, and gave proofs from this point of view. We follow his line in part and give a lattice path interpretations of our formulas and proofs from this viewpoint. Moreover, the Pfaffian analogue of the Lewis-Caroll formula (Theorem 2.6) makes possible the story clear. Thus, in this section, we obtain an improved and a much simplified version of Stembridge's proof. We may say our proofs are closer to Gessel-Viennot's original proofs in [5] than those given in [30]. We note that Stembrige's proof [30] can be also generalized almost parallelly to proving Theorem 3.2, 3.6, 3.7, but we do not develop the proofs in this direction.

Now we review the basic terminology of lattice paths and fix notation. We follow the basic terminology in [5] and [30]. Let $D = (V, E)$ be an

acyclic digraph without multiple edges. Further we assume that there are only finitely many paths between any two vertices. If u and v are any pair of vertices in D, let $\mathcal{P}(u, v)$ denote the set of all directed paths from u to v in D. Fix a positive integer n. An *n-vertex* is an n-tuple $\boldsymbol{v} = (v_1, \ldots, v_n)$ of n vertices of D. Given any pair of n-vertices $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, \ldots, v_n)$, an *n-path* from $\boldsymbol{u}$ to $\boldsymbol{v}$ is an n-tuple $\boldsymbol{P} = (P_1, \ldots, P_n)$ of n paths such that $P_i \in \mathcal{P}(u_i, v_i)$. Let $\mathcal{P}(\boldsymbol{u}, \boldsymbol{v})$ denote the set of all n-paths from $\boldsymbol{u}$ to $\boldsymbol{v}$. Two directed paths P and Q will be said to be nonintersecting if they share no common vertex. An n-path $\boldsymbol{P}$ is said to be nonintersecting if P_i and P_j are nonintersecting for any $i \neq j$. Let $\mathcal{P}^0(\boldsymbol{u}, \boldsymbol{v})$ denote the subset of $\mathcal{P}(\boldsymbol{u}, \boldsymbol{v})$ which consists of all nonintersecting n-paths.

We assign a commutative indeterminate x_e to each edge e of D and call it the *weight* of the edge. Set the weight of a path P to be the product of the weights of its edges and denote it by $\mathrm{wt}(P)$. If u and v are any pair of vertices in D, define

$$h(u, v) = \sum_{P \in \mathcal{P}(u,v)} \mathrm{wt}(P).$$

The weight of an n-path is defined to be the product of the weights of its components. The sum of the weights of n-paths in $\mathcal{P}(\boldsymbol{u}, \boldsymbol{v})$ (resp. $\mathcal{P}^0(\boldsymbol{u}, \boldsymbol{v})$) is denoted by $F(\boldsymbol{u}, \boldsymbol{v})$ (resp. $F^0(\boldsymbol{u}, \boldsymbol{v})$).

Definition 4.1 *If I and J are ordered sets of vertices of D, then I is said to be D-compatible with J if, whenever $u < u'$ in I and $v > v'$ in J, every path $P \in \mathcal{P}(u, v)$ intersects every path $Q \in \mathcal{P}(u', v')$.*

The following famous lemma is from [5], and we recall its proof here again to make this paper self-contained.

Lemma 4.2 *(Lindström-Gessel-Viennot) Let $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, \ldots, v_n)$ be two n-vertices in an acyclic digraph D. Then*

$$\sum_{\pi \in S_n} \mathrm{sgn}\,\pi \; F^0(\boldsymbol{u}^\pi, \boldsymbol{v}) = \det[h(u_i, v_j)]_{1 \leq i, j \leq n}. \tag{32}$$

Here, for any permutation $\pi \in S_n$, let $\boldsymbol{u}^\pi$ denote $(u_{\pi(1)}, \ldots, u_{\pi(n)})$. In particular, if $\boldsymbol{u}$ is D-compatible with $\boldsymbol{v}$, then

$$F^0(\boldsymbol{u}, \boldsymbol{v}) = \det[h(u_i, v_j)]_{1 \leq i, j \leq n}. \tag{33}$$

Proof. From the definition of determinants we have

$$\det[h(u_i, v_j)]_{1 \leq i, j \leq n} = \sum_{\pi \in \mathfrak{S}_n} \mathrm{sgn}(\pi) h(u_1, v_{\pi(1)}) h(u_2, v_{\pi(2)}) \ldots h(u_r, v_{\pi(n)}). \tag{34}$$

For $\pi \in \mathfrak{S}_n$, let $P(I, J; \pi)$ denote the set of all the n-paths $\boldsymbol{P} = \{P_1, \ldots, P_n\}$ such that each path P_i connects u_i with $v_{\pi(i)}$ for $i = 1, \ldots, n$. Let $P^0(I, J; \pi)$ denote the subset of $P(I, J; \pi)$ which consists of all noninter-secting paths $\boldsymbol{P} \in P(I, J; \pi)$. Let us define sets Π and Π^0 of configurations by

$$\Pi = \left\{ (\pi, \boldsymbol{P}) : \pi \in \mathfrak{S}_n \text{ and } \boldsymbol{P} \in \mathcal{P}(I, J; \pi)) \right\},$$
$$\Pi^0 = \left\{ (\pi, \boldsymbol{P}) : \pi \in \mathfrak{S}_n \text{ and } \boldsymbol{P} \in \mathcal{P}^0(I, J; \pi) \right\}.$$

Then the right-hand side of (34) is the generating function of config-urations $(\pi, \boldsymbol{P}) \in \Pi$ with the weight $\mathrm{wt}(\pi, \boldsymbol{P}) = \mathrm{sgn}(\pi) \mathrm{wt}(\boldsymbol{P})$. Now

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we describe an involution on the set $\Pi \setminus \Pi^0$ which reverse the sign of the associated weight. First fix an arbitrary total order on V. Let $C = (\pi, \boldsymbol{P}) \in \Pi \setminus \Pi^0$. Among all vertices that occurs as intersecting points, let v denote the least vertex with respect to the fixed order. Among paths that pass through v, assume that P_i and P_j are the two whose indices i and j are smallest. Let $P_i(\to v)$ (resp. $P_i(v \to)$) denote the subpath of P_i from u_i to v (resp. from v to $v_{\pi(i)}$). Set $C' = (\pi', \boldsymbol{P}')$ to be the configuration in which $P'_k = P_k$ for $k \neq i, j$,

$$P'_i = P_i(\to v)P_j(v \to), \qquad P'_j = P_j(\to v)P_i(v \to),$$

and $\pi' = \pi \circ (i, j)$. It is easy to see that $C' \in \Pi$ and $\mathrm{wt}(C') = - \mathrm{wt}(C)$. Thus $C \mapsto C'$ defines a sign reversing involution and, by this involution, one may cancel all of the terms $\{\mathrm{wt}(C) : C \in \Pi \setminus \Pi^0\}$ and only the terms $\{\mathrm{wt}(C) : C \in \Pi^0\}$ remains. Since $h(\boldsymbol{u}^\pi, \boldsymbol{v}) = h(\boldsymbol{u}, \boldsymbol{v}^{\pi^{-1}})$, we obtain the resulting identity. In particular, if $\boldsymbol{u}$ is D-compatible with $\boldsymbol{v}$, the configurations $C \in \Pi^0$ occur only when $\pi = \mathrm{id}$, and are counted with the weight $\mathrm{wt}(P)$. This proves the lemma. $\square$

Here we briefly review the definition of Pfaffians. Let $v = (v_1, \ldots, v_n)$ be an ordered list of n objects, and assume that n is even. A permutation $\tau = (v_{\tau(1)}, v_{\tau(2)}, \ldots, v_{\tau(n-1)}, v_{\tau(n)})$ of vertices is said to be a *perfect matching* if it satisfies $\tau(2i - 1) < \tau(2i)$ and $\tau(2i - 1) < \tau(2i + 1)$ for all i such that the both sides are defined. We say that the vertices $v_{\tau(2i-1)}$ and $v_{\tau(2i)}$ are *connected* to each other in this perfect matching τ for each $1 \leq i \leq n/2$. We will write $\mathcal{F}(v)$ for the set of perfect matchings of v, and $\mathcal{F}_n$ for the set of perfect matchings of $[n]$. We may identify the perfect matiching $\tau = (v_{\tau(1)}, v_{\tau(2)}, \ldots, v_{\tau(n-1)}, v_{\tau(n)})$ with $\tau = \{(v_{\tau(1)}, v_{\tau(2)}), \ldots, (v_{\tau(n-1)}, v_{\tau(n)})\}$ which lists the set of pairs of connected vetices, and we call a pair (v_i, v_j) in τ an edge connecting between v_i and v_j. We can express a pefect matching graphically by embedding the vertices along the x-axis in the plane and representing the edges by curves in the upper half plane. Two edges (v_i, v_j) and (v_k, v_l) in τ will be said to be *closed* if the corresponding edges always intersect in such an embedding. Define the *sign* of τ, denoted by $\mathrm{sgn}\, \tau$, to be $(-1)^k$, where k denotes the number of crossed pairs of edges in τ. For an example of such a pefect matching, see Figure 1 bellow, in which $\tau = \{(v_1, v_4), (v_2, v_5), (v_3, v_6)\}$ and the sign of the perfect matching becomes (-1).

Let $S = \{v_1 < \cdots < v_N\}$ be a finite totally ordered subset of V. We assume a commutative indeterminate $a_{v_i v_j}$ is assigned to each pair (v_i, v_j) $(i < j)$ of vertices in S. We write the assembly of the indeterminates as $A = (a_{v_i v_j})_{i < j}$. This upper triangular array uniquely defines a skew-symmetric matrix of size N, and we use the same symbol A to express this skew-symmetric matrix. Suppose m is even and $\boldsymbol{u} = (u_1, \ldots, u_m)$ is an m-vertex. We will consider a generating function of the set of nonintersecting
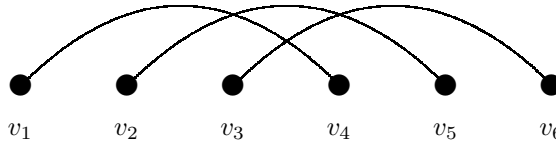

Figure 1: A perfect matching

m-paths from $\boldsymbol{u}$ to S weighted by subpfaffians of A:

$$Q(\boldsymbol{u}, S; A) = \sum_{\substack{I \subseteq S \\ \sharp I = m}} \mathrm{pf}(A_I^I) F^0(\boldsymbol{u}, I)$$

The following theorem gives an answer to this problem, and is interpreted as a lattice path version of Theorem 3.2.

Theorem 4.3 *Let m and N be even integers such that $0 \le m \le N$. Let $\boldsymbol{u} = (u_1, \ldots, u_m)$ be an m-vertex and $S = \{V_1 < \cdots < V_N\}$ be a totally ordered set of vertices in an acyclic digraph D. Let $A = (a_{V_i V_j})_{1 \le i < j \le N}$ be an skew-symmetric matrix indexed by S, and let $\widehat{A}$ denote its copfaffian matrix. Then*

$$\sum_{\substack{I \subseteq S \\ \sharp I = m}} \mathrm{Pf}(A_I^I) \sum_{\pi \in S_m} \mathrm{sgn}\, \pi \, F^0(\boldsymbol{u}^\pi, I) = \mathrm{Pf}(A) \, \mathrm{Pf}\begin{pmatrix} O_m & H(\boldsymbol{u}, S) J_N \\ -J_N \, {}^t H(\boldsymbol{u}, S) & \frac{1}{\mathrm{Pf}(A)} J_N {}^t\widehat{A} J_N \end{pmatrix}.$$

$$(35)$$

In particular, if $\boldsymbol{u}$ is D-compatible with S, then we have

$$\sum_{\substack{I \subseteq S \\ \sharp I = m}} \mathrm{Pf}(A_I^I) F^0(\boldsymbol{u}^\pi, I) = \mathrm{Pf}(A) \, \mathrm{Pf}\begin{pmatrix} O_m & H(\boldsymbol{u}, S) J_N \\ -J_N \, {}^t H(\boldsymbol{u}, S) & \frac{1}{\mathrm{Pf}(A)} J_N {}^t\widehat{A} J_N \end{pmatrix}.$$

$$(36)$$

Proof. First we put $\widehat{A} = (\widehat{a}_{V_i V_j})$ and let $\alpha_{V_i V_j} = \frac{1}{\mathrm{Pf}(A)} \widehat{a}_{V_i V_j}$ for $1 \le i, j \le N$. Since multiplying J_N from right reverse the order of columns of $H(\boldsymbol{u}, S)$, we have

$$\mathrm{Pf}\begin{pmatrix} O_n & H(\boldsymbol{u}, S) J_N \\ -J_N \, {}^t H(\boldsymbol{u}, S) & \frac{1}{\mathrm{Pf}(A)} J_N {}^t\widehat{A} J_N \end{pmatrix} = \sum_\tau \mathrm{sgn}\, \tau \prod_{(u_i, V_j) \in \tau} h(u_i, V_j) \prod_{(V_k, V_l) \in \tau} \alpha_{V_k V_l}$$

summed over all perfect matchings τ on $(u_1, \ldots, u_m, V_N, \ldots, V_1)$ in which there are no edges connecting any two vertices of u. For an example of such a perfect matching, see Figure 2 bellow. We may interpret this Pfaffian as a generating function for all $(m+1)$-tuples $C = (\tau, P_1, \ldots, P_m)$ such that $P_i \in \mathcal{P}(u_i, V_j)$ if there is an edge $(u_i, V_j) \in \tau$. This implies that each vertex, say u_i, in $\boldsymbol{u}$ is always connected to a vertex, say V_j, in S, and remaining $(N - n)$ vertices of S are connected each other by edges which we write $(V_k, V_l) \in \tau$. The weight assigned to $C = (\tau, P_1, \ldots, P_m)$ shall be $\mathrm{sgn}\, \tau \left(\prod_{(V_k, V_l) \in \tau} \alpha_{V_k V_l} \right) w(P_1) \cdots w(P_m)$. Let Σ denote the set of configulations $C = (\tau, P_1, \ldots, P_m)$ satisfying the above condition, and let Σ^0 denote the subset consisting of all configulations $C = (\tau, P_1, \ldots, P_m)$ such that $(P_1, \ldots, P_m)$ is non-itersecting. We will show that there is
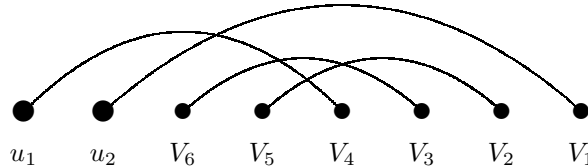

Figure 2: Proof of Theorem 4.3

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a sign-reversing involution on $\Sigma \setminus \Sigma^0$, i.e. the set of the configulations $C = (\tau, P_1, \ldots, P_m)$ with at least one pair of intersecting paths. Our proof here is essentially the same as that in Lemma 4.2. To describe the involution, first choose a fixed total order of the vertices, and consider an arbitrary configuration $C = (\tau, P_1, \ldots, P_m) \in \Sigma \setminus \Sigma^0$. Among all vertices that occurs as intersecting points, let v denote the vertex which precedes all other points of intersections with respect to the fixed order. Among paths that pass through v, assume that P_i and P_j are the two whose indices i and j are smallest. We define $C' = (\tau', P_1', \ldots, P_m')$ to be the configuration where $P_k' = P_k$ for $k \neq i, j$,

$$P_i' = P_i(\to v)P_j(v \to), \qquad P_j' = P_j(\to v)P_i(v \to),$$

and, if (u_i, V_k) and (u_j, V_l) are the edges of τ, then (u_i, V_l) and (u_j, V_k) are in τ' and all the other edges of τ' are the same as τ. Note that the multisets of edges appearing in $(P_1, \ldots, P_m)$ and $(P_1', \ldots, P_m')$ are identical, which means $\mathrm{wt}(C') = -\mathrm{wt}(C)$. Since this involution changes the sign of the associated weight, one may cancel all of the terms appearing in $\Sigma \setminus \Sigma^0$, aside from those with non-intersecting paths. For $C = (\tau, P_1, \ldots, P_m) \in \Sigma^0$, let I denote the set of vertices of S connected to a vertex in $\boldsymbol{u}$, and let $\overline{I}$ denote the complementary set of I in S. Put $I = \{V_{i_1}, \ldots, V_{i_m}\}_<$, then we can find a unique permutation $\pi \in S_m$ such that each $u_{\pi(k)}$ is connected to V_{i_k} in τ for $k = 1, \ldots, m$. The remaining edges in τ which does not contribute to this permutation perform a perfect matching on $\overline{I}$ which we denote by σ. The $\mathrm{sgn}\,\tau$ is eqaul to $(-1)^{s(I, \overline{I})} \mathrm{sgn}\,\pi\,\mathrm{sgn}\,\sigma$, where $s(I, \overline{I})$ denote the shuffle number to merge I with $\overline{I}$ into S. Thus, if we put $m = 2m'$ and $N = 2N'$ for nonnegative integers m' and N', the sum of weights is equal to

$$\sum_I (-1)^{s(I, \overline{I})} \sum_{\pi \in S_m} \mathrm{sgn}\,\pi\, F^0(\boldsymbol{u}^\pi, I) \frac{1}{(\mathrm{Pf}\,A)^{N'-m'}} \mathrm{Pf}\left(\widehat{A_{\overline{I}}^{\overline{I}}}\right),$$

where I runs over all subsets of S of cardinality m. Theorem 2.6 implies

$$\mathrm{Pf}\left(\widehat{A_{\overline{I}}^{\overline{I}}}\right) = (-1)^{|\overline{I}| + N' - m'} \mathrm{Pf}(A)^{N'-m'-1} \mathrm{Pf}(A_I^I).$$

Since $I \cup \overline{I} = S$, we have $|I| + |\overline{I}| = \binom{N+1}{2} \equiv N' \pmod 2$. Meanwhile, it is easy to see $(-1)^{s(I, \overline{I})} = (-1)^{|I| - m'}$. This immediately implies (35). This completes the proof. $\square$

In fact Theorem 4.3 is equivalent to Theorem 3.2. In [8] we gave an algebraic proof of Theorem 3.2 using the exterior algrbra. One easily sees that Theorem 4.3 is an easy consequence of Theorem 3.2 and Lemma 4.2. Here we give a proof that derives Theorem 3.2 from Theorem 4.3. Similarly one can derive Theorem 3.6 from Theorem 4.4, and also Theorem 3.7 from Theorem 4.5, but we will not give the details here and leave it to the reader.

Proof of Theorem 3.2. First we define a digraph D with vertex set $\mathbb{Z}^2$ and edges directed from u to v whenever $v - u = (1, 0)$ or $(0, 1)$. For $u = (i, j)$, we assign the weight x_j (resp. 1) to the edge with $v - u = (1, 0)$ (resp. $(0, 1)$). If $u = (i, 1)$ and $v = (j, r)$, then $\lim_{r \to \infty} h(u, v) = h_{j-i}(x)$ is well-known as a complete symmetric function $h_{j-i}(x)$, which is defined by $\sum_{k \geq 0} h_k(x)t^k = \prod_{i \geq 1} \frac{1}{1 - x_i t}$ (See [24]). Thus, if we fix constants $(\lambda_1, \cdots, \lambda_m)$ and $(\mu_1, \cdots, \mu_m)$ which satisfy $\lambda_1 < \cdots < \lambda_m$ and $\mu_1 < \cdots < \mu_m$, and take the vertices $u_i = (\lambda_i, 1)$ and $v_i = (\mu_i, r)$ for

$i = 1, \cdots, m$, then, $\boldsymbol{u}$ and $\boldsymbol{v}$ are D-compatible, and from Lemma 4.2, we deduce

$$\lim_{r \to \infty} F^0(\boldsymbol{u}, \boldsymbol{v}) = \det \left(h_{\mu_j - \lambda_i}(x) \right)_{1 \leq i, j \leq m}.$$

Let N be a positive interger such that $N \geq m \geq 0$ and $A = (a_{ij})$ be an N by N skew-symmetric matrix. We let $\boldsymbol{u} = (u_1, \cdots, u_m)$ with $u_i = (Ni, 1)$ for $i = 1 \cdots m$ and $S = \{v_1, \cdots, v_N\}$ with $v_j = (j + Nm, r)$ for $j = 1 \cdots N$. Then, from Theorem 4.3 and by taking the limit $r \to \infty$, we obtain

$$\sum_{I = \{i_1 < \cdots < i_m\} \subseteq [N]} \mathrm{Pf}(A_I) \det \left(h_{i_j + N(m-i)}(x) \right)_{1 \leq i, k \leq m} = \mathrm{Pf}\,(Q)$$

where $Q = (Q_{ij})_{1 \leq i, j \leq m}$ is given by

$$Q_{ij} = \sum_{1 \leq k < l \leq N} a_{kl} \begin{vmatrix} h_{k+N(n-i)}(x) & h_{l+N(n-i)}(x) \\ h_{k+N(n-j)}(x) & h_{l+N(n-j)}(x) \end{vmatrix}$$

We use the fact that the $h_1, \cdots, h_{Nm}$ are algebraically independent over $\mathbb{Q}$ (See [24]). Thus we can replace each $h_{j+N(m-i)}$ with any commutative indeterminate t_{ij}, and we obtain Theorem 3.2. $\square$

We next consider the lattice path version of Theorem 3.6. Let m, n and N be positive integers such that $m - n$ is even and $0 \leq m - n \leq N$. Suppose that $S^0 = \{v_1 < \cdots < v_n\}$ is a fixed ordered list of vertices, and let $S = \{V_1 < \cdots < V_N\}$ be a totally ordered set disjoint with S^0. For a subset I of S let $S^0 \uplus I$ denote the union of S^0 and I, ordered so that each v_i precedes each $w \in I$. Let $A = (a_{V_i V_j})_{1 \leq i, j \leq N}$ be a skew-symmetric matrix indexed by the totally ordered set S as before. We will obtain a formula of the genarating function weighted by subpfaffians of A as follows:

$$Q(\boldsymbol{u}; S^0, S; A) = \sum_{\substack{I \subseteq S \\ \sharp I = m - n}} \mathrm{Pf}(A_I^I) F^0(\boldsymbol{u}, S^0 \uplus I)$$

Theorem 4.4 *Let m, n and N be positive integers such that $m - n$ and N are even integers and $0 \leq m - n \leq N$. Let $\boldsymbol{u} = (u_1, \cdots, u_m)$ be an m-vertex and $S^0 = \{v_1 < \cdots < v_n\}$ be an n-vertex in an acyclic digraph D. Let $S = \{V_1 < \cdots < V_N\}$ be a finite totally ordered set of vertices which is disjoint with S_0. Let $A = (a_{V_i V_j})$ be an N by N skew-symmetric matrix indexed by S. Then*

$$\sum_{\substack{I \subseteq S \\ \sharp I = m - n}} \mathrm{Pf}(A_I^I) \sum_{\pi \in S_m} \mathrm{sgn}\,\pi\, \boldsymbol{F}^0(u^\pi, S^0 \uplus I)$$

$$= \mathrm{Pf}(A)\,\mathrm{Pf} \begin{pmatrix} O_m & H(\boldsymbol{u}; S) J_N & H(\boldsymbol{u}; S^0) J_n \\ -J_N\,{}^t H(\boldsymbol{u}; S) & \frac{1}{\mathrm{Pf}(A)} J_N\,{}^t \widehat{A} J_N & O_{N,n} \\ -J_n\,{}^t H(\boldsymbol{u}; S^0) & O_{n,N} & O_n \end{pmatrix} \qquad (37)$$

where

$$H(\boldsymbol{u}; S^0) = (h(u_i, v_j))_{1 \leq i \leq m, 1 \leq j \leq n},$$
$$H(\boldsymbol{u}; S) = (h(u_i, V_j))_{1 \leq i \leq m, 1 \leq j \leq N}.$$

In particular, if $\boldsymbol{u}$ is D-compatible with $I^0 \uplus S$, then we have

$$\sum_{\substack{I \subseteq S \\ \sharp I = m-n}} \mathrm{Pf}(A_I^I)\boldsymbol{F}^0(u, S^0 \uplus I)$$

$$= \mathrm{Pf}(A)\,\mathrm{Pf}\begin{pmatrix} O_m & H(\boldsymbol{u};S)J_N & H(\boldsymbol{u};S^0)J_n \\ -J_N{}^t H(\boldsymbol{u};S) & \frac{1}{\mathrm{Pf}(A)}J_N{}^t\widehat{A}J_N & O_{N,n} \\ -J_n{}^t H(\boldsymbol{u};S^0) & O_{n,N} & O_n \end{pmatrix} \qquad (38)$$

Proof. First we put $\alpha_{kl} = \frac{1}{\mathrm{Pf}(A)}\widehat{a}_{kl}$ as before. We have

$$\mathrm{Pf}\begin{pmatrix} O_m & H(\boldsymbol{u};S)J_N & H(\boldsymbol{u};S^0)J_n \\ -J_N{}^t H(\boldsymbol{u};S) & \frac{1}{\mathrm{Pf}(A)}J_N{}^t\widehat{A}J_N & O_{N,n} \\ -J_n{}^t H(\boldsymbol{u};S^0) & O_{n,N} & O_n \end{pmatrix}$$

$$= \sum_\tau \mathrm{sgn}\,\tau \prod_{(u_i,V_j)\in\tau} h(u_i,V_j) \prod_{(V_i,V_j)\in\tau} \alpha_{V_iV_j} \prod_{(u_i,V_j)\in\tau} h(u_i,V_j) \qquad (39)$$

summed over all perfect matchings τ of $(u_1,\ldots,u_m,V_N,\ldots,V_1,v_n,\ldots,v_1)$ in which there are no edges connetcting any two vertices of $\boldsymbol{u}$, and each vertex in S^0 must be connected to a vertex in $\boldsymbol{u}$. An example of such a perfect matching is given below. This may be interpreted as the generating function for all $(m+1)$-tuples $C = (\tau, P_1,\ldots,P_m)$ such that $P_i \in \mathcal{P}(u_i,v_j)$ if there is an edge $(u_i,v_j)\in\tau$, and $P_i \in \mathcal{P}(u_i,V_j)$ if there is an edge $(u_i,V_j)\in\tau$. The weight assigned to $C = (\tau,P_1,\ldots,P_m)$ shall be $\mathrm{sgn}\,\tau \prod_{(V_k,V_l)\in\tau} \alpha_{V_kV_l} w(P_1)\ldots w(P_m)$. We claim that the sign-reversing involution used in the previous proofs can be applied to this situation as well. In fact, quite the same arguments show that one may cancel all of the terms appearing in (39), aside from those with non-intersecting paths. In τ associated with these configuration $C = (\tau,P_1,\ldots,P_m)$, each v_k $(k = 1,\ldots,n)$ is always connected to a vetex in $\boldsymbol{u}$. This means that exactly n vertices of $\boldsymbol{u}$ are connected to vertices in S^0, and the remaining $(m-n)$ vertices are connected to certain vertices in S. Let I denote the set of vertices in S connected to vertices in $\boldsymbol{u}$, and let $\overline{I}$ denote its complementary set in S. Note that $\sharp I = (m-n)$ and $\sharp\overline{I} = (N-m+n)$. Let $S^0 \uplus I$ denote the juxtaposition of vertices from I^0 and I arranged in this order, and, if we put $I^0 \uplus I = (u_1^*,\ldots,u_m^*)$ then there is a unique permutation $\pi \in S_m$ such that each $u_{\pi(k)}$ is connected to u_k^* for $k = 1,\ldots,m$. The remaining edges of τ whose both endpoints are included $\overline{I}$ define a unique perfect matching on $\overline{I}$. In this situation $\mathrm{sgn}\,\tau$ is equal to $s(I,\overline{I})\,\mathrm{sgn}\,\pi\,\mathrm{sgn}\,\sigma$. Thus, if we put $m - n = 2m'$ and $N = 2N'$ for nonnegative integers m' and N', then the sum of weights is equal to

$$\sum_I (-1)^{s(I,\overline{I})}\,\mathrm{sgn}\,\pi\,\boldsymbol{F}^0(u^\pi, I^0 \uplus I)\frac{1}{(\mathrm{Pf}\,A)^{N'-m'}}\,\mathrm{Pf}(\widehat{A}_{\overline{I}}^{\overline{I}}),$$
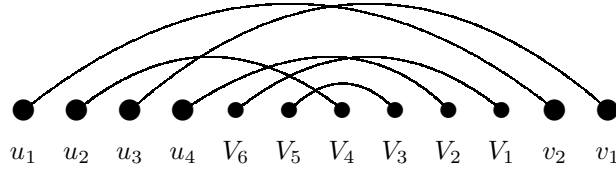

Figure 3: Proof of Thoerem 4.4

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where I runs over all subsets of S of cardinality $(m-n)$. From Theorem 2.6 we have

$$\mathrm{Pf}\left(\widehat{A^{\overline{I}}_{\overline{I}}}\right) = (-1)^{|\overline{I}|+N'-m'}(\mathrm{Pf}\,A)^{N'-m'-1}\mathrm{Pf}(A^I_I).$$

Since $I \cup \overline{I} = S$, we have $|I| + |\overline{I}| = \binom{N+1}{2} \equiv N' \pmod 2$. Meanwhile, it is easy to see $(-1)^{s(I,\overline{I})} = (-1)^{|I|-m'}$. This immediately implies (37). Lastly, if $\boldsymbol{u}$ is D-compatible with $I^0 \uplus S$, then there is no non-intersecting path unless $\pi = id$, which immediately implies (38). This completes the proof. $\square$

Now we consider a theorem to prove Theorem 3.7. For the purpose we consider the more general problem of enumerating sets of non-intersecting paths in which both

Theorem 4.5 *Let M and N be even integers. Let $R = \{u_1 < \cdots < u_M\}$ and $S = \{v_1 < \cdots < v_N\}$ be totally ordered subsets of vertices in an acyclic digraph D. Let $A = (a_{u_i u_j})$ (resp. $B = (b_{v_i v_j})$ be a non-sigular skew-symmetric matrix indexed by the vertices of R (resp. S). Then*

$$\sum_{\substack{0 \leq r \leq \min(M,N) \\ r\ even}} z^r \sum_{\substack{I \subseteq R \\ \sharp\, I = r}} \sum_{\substack{J \subseteq S \\ \sharp\, J = r}} \mathrm{Pf}(A^I_I)\,\mathrm{Pf}(B^J_J) \sum_{\pi \in S_n} \mathrm{sgn}\,\pi\, F^0(I^\pi, J)$$

$$= \mathrm{Pf}(A)\,\mathrm{Pf}(B)\,\mathrm{Pf}\begin{pmatrix} \frac{1}{\mathrm{Pf}(A)}\widehat{A} & zH(R,S)J_N \\ -zJ_N\,{}^tH(R,S) & \frac{1}{\mathrm{Pf}(B)}J_N{}^t\widehat{B}J_N \end{pmatrix}. \tag{40}$$

In particular, if R is compatible with S, then

$$\sum_{\substack{0 \leq r \leq \min(M,N) \\ r\ even}} z^r \sum_{\substack{I \subseteq R \\ \sharp\, I = r}} \sum_{\substack{J \subseteq S \\ \sharp\, J = r}} \mathrm{Pf}(A^I_I)\,\mathrm{Pf}(B^J_J)F^0(I, J)$$

$$= \mathrm{Pf}(A)\,\mathrm{Pf}(B)\,\mathrm{Pf}\begin{pmatrix} \frac{1}{\mathrm{Pf}(A)}\widehat{A} & zH(R,S)J_N \\ -zJ_N\,{}^tH(R,S) & \frac{1}{\mathrm{Pf}(B)}J_N{}^t\widehat{B}J_N \end{pmatrix}. \tag{41}$$

Proof. First we put $\alpha_{ij} = \frac{1}{\mathrm{Pf}(A)}\widehat{a}_{ij}$ and $\beta_{ij} = \frac{1}{\mathrm{Pf}(B)}\widehat{b}_{ij}$. Then, we have

$$\mathrm{Pf}\begin{pmatrix} \frac{1}{\mathrm{Pf}(A)}\widehat{A} & zH(R,S)J_N \\ -zJ_N\,{}^tH(R,S) & \frac{1}{\mathrm{Pf}(B)}J_N{}^t\widehat{B}J_N \end{pmatrix}$$

$$= \sum_\tau \mathrm{sgn}\,\tau \prod_{(u_i,u_j)\in\tau} \alpha_{u_i u_j} \prod_{(u_i,v_j)\in\tau} zh(u_i,v_j) \prod_{(v_i,v_j)\in\tau} \beta_{v_i v_j}$$

summed over all perfect matchings τ on $(u_1,\ldots,u_M,v_N,\ldots,v_1)$. An example of such a perfect matching is Figure 4 bellow. As before we may interpret this Pfaffian as a generating function for all $(r+1)$-tuples $C = (\tau, P_1, \ldots, P_r)$ which satifies (i) r is an even integer such that $0 \leq$
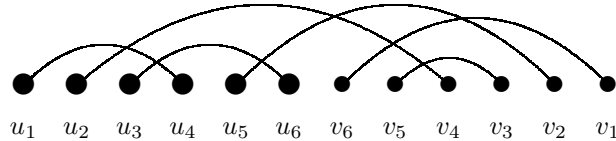

Figure 4: Proof of Thoerem 4.5

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$r \leq \min(M, N)$, (ii) there are exactly r edges whose one endpoint is in R and the other endpoint is S, and (iii) τ is a perfect matching on $(u_1, \ldots, u_M, v_N, \ldots, v_1)$ such that $P_i \in \mathcal{P}(u_i, v_j)$ if and only if there is an edge $(u_i, v_j) \in \tau$. The weight assigned to $C = (\tau, P_1, \ldots, P_r)$ shall be

$$\operatorname{sgn} \tau \, z^r \prod_{(u_i, u_j) \in \tau} \alpha_{u_i u_j} \, w(P_1) \cdots w(P_r) \prod_{(v_i, v_j) \in \tau} \beta_{v_i v_j}.$$

The same argument as in the proof of Theorem 4.3 shows us that we can define a sign-reversing inolution on the set of the configulations $C = (\tau, P_1, \ldots, P_r)$ with at least one pair of intersecting paths, and this involution cancell all of the terms involving intersecting configulations of paths. Thus we need to sum over only non-intersecting configulations. Given a perect matching τ on $(u_1, \ldots, u_M, v_N, \ldots, v_1)$ such that there are exactly r edges conecting a vetex in R and a vertex in S. Let I (resp. J) denote the subset of R (resp. S) which is composed of such endpoints of τ. Thus $\sharp I = \sharp J = r$, and r must be even. Let $\overline{I}$ (resp. $\overline{J}$) denote the complementary set of I (resp. J) in R (resp. S). Put $I = \{u_{i_1}, \ldots, u_{i_r}\}_<$ and $J = \{v_{j_1}, \ldots, v_{j_r}\}_<$, then there is a unique permutation π such that $u_{i_{\pi(\nu)}}$ is connected to v_{j_ν} in τ for $\nu = 1, \ldots, r$. If we put $M = 2M'$, $N = 2N'$ and $r = 2r'$, then the sum of weights becomes

$$\sum_{\substack{r=0 \\ r=2r'}}^{M} \sum_{\substack{I \subseteq R \\ \sharp I = r}} \sum_{\substack{J \subseteq S \\ \sharp J = r}} (-1)^{s(I, \overline{I}) + s(J, \overline{J})} \sum_{\pi \in S_n} \operatorname{sgn} \pi \, z^r F^0(I^\pi, J) \frac{1}{\operatorname{Pf}(A)^{M'-r'}} \operatorname{Pf}(\widehat{A_{\overline{I}}^{\overline{I}}}) \frac{1}{\operatorname{Pf}(B)^{N'-r'}} \operatorname{Pf}(\widehat{B_{\overline{J}}^{\overline{J}}}).$$

By (10) we have $\operatorname{Pf}(\widehat{A_{\overline{I}}^{\overline{I}}}) = (-1)^{|\overline{I}|-M'+r'} \operatorname{Pf}(A)^{M'-r'-1} \operatorname{Pf}(A_I^I)$ and $\operatorname{Pf}(\widehat{B_{\overline{J}}^{\overline{J}}}) = (-1)^{|\overline{J}|-N'+r'} \operatorname{Pf}(B)^{N'-r'-1} \operatorname{Pf}(B_J^J)$. Further it is easy to see that $s(I, \overline{I}) \equiv |I| - r' \pmod 2$ and $s(J, \overline{J}) \equiv |J| - r' \pmod 2$ Since $I \cup \overline{I} = R$ and $J \cup \overline{J} = S$, we have $|I| + |\overline{I}| = \binom{M+1}{2} \equiv M' \pmod 2$ and $|J| + |\overline{J}| = \binom{N+1}{2} \equiv N' \pmod 2$. These identities immediately implies (40). $\square$

In the following theorem we assume D-compatiblity of two regions to make our notation simple, but the more general theorem is also possible.

Theorem 4.6 *Let m, n, M and N be nonnegative integers such that $M \equiv N \equiv m - n \equiv 0 \pmod 2$. Let $R^0 = (u_1, \ldots, u_m)$ (resp. $S^0 = (v_1, \ldots, v_n)$) be an m-vertex (resp. an n-vertex) in an acyclic digraph D. Let $R = \{U_1 < \cdots < U_M\}$ (resp. $S = \{V_1 < \cdots < V_N\}$) be totally ordered subsets of vertices in D which is disjoint with R^0 (resp. S^0). Assume that $R^0 \uplus R$ is D-compatible with $S^0 \uplus S$. Let $A = (a_{U_i U_j})$ (resp. $B = (b_{V_i V_j})$) be a skew-symmetric matrix indexed by the vertices of R (resp. S).*

  1. If $0 \leq m - n \leq N$, then

$$\sum_{\substack{0 \leq r \leq \min(M, N-m+n) \\ r \text{ even}}} z^r \sum_{\substack{I \subseteq R \\ \sharp I = r}} \sum_{\substack{J \subseteq S \\ \sharp J = r+m-n}} \operatorname{Pf}(A_I^I) \operatorname{Pf}(B_J^J) F^0(R^0 \uplus I, S^0 \uplus J)$$

$$= \operatorname{Pf}(A) \operatorname{Pf}(B) \operatorname{Pf} \begin{pmatrix} O_m & O_{m,M} & H(R^0, S)J_N & H(R^0, S^0)J_n \\ O_{M,m} & \frac{1}{\operatorname{Pf}(A)}\widehat{A} & H(R; S)J_N & H(R; S^0)J_n \\ -J_N{}^t H(R^0, S) & -J_N{}^t H(R; S) & \frac{1}{\operatorname{Pf}(B)}J_N{}^t \widehat{B}J_N & O_{N,n} \\ -J_n{}^t H(R^0, S^0) & -J_n{}^t H(R; S^0) & O_{n,N} & O_n \end{pmatrix}$$

$$\tag{42}$$

2. If $0 \le n - m \le M$, then

$$\sum_{\substack{0 \le r \le \min(M-n+m,N) \\ r \text{ even}}} z^r \sum_{\substack{I \subseteq R \\ \sharp I = r+n-m}} \sum_{\substack{J \subseteq S \\ \sharp J = r}} \operatorname{Pf}(A_I^I) \operatorname{Pf}(B_J^J) F^0(R^0 \uplus I, S^0 \uplus J)$$

$$= \operatorname{Pf}(A) \operatorname{Pf}(B) \operatorname{Pf} \begin{pmatrix} O_m & O_{m,M} & H(R^0,S)J_N & H(R^0,S^0)J_n \\ O_{M,m} & \frac{1}{\operatorname{Pf}(A)}\widehat{A} & H(R;S)J_N & H(R;S^0)J_n \\ -J_N{}^tH(R^0,S) & -J_N{}^tH(R;S) & \frac{1}{\operatorname{Pf}(B)}J_N{}^t\widehat{B}J_N & O_{N,n} \\ -J_n{}^tH(R^0,S^0) & -J_n{}^tH(R;S^0) & O_{n,N} & O_n \end{pmatrix}$$

$$(43)$$

Proof. We can assume $0 \le m - n \le N$ since for the other case we can prove them almost parallelly. First we put $\alpha_{xy} = \frac{1}{\operatorname{Pf}(A)}\widehat{a}_{xy}$ for $x,y \in R$ and $\beta_{xy} = \frac{1}{\operatorname{Pf}(B)}\widehat{b}_{xy}$ for $x,y \in S$ as before. Further we put $R^0 \uplus R = (u_1,\ldots,u_m,U_1,\ldots,U_M) = (u_1^*,\ldots,u_{m+M}^*)$ and $S^0 \uplus S = (v_1,\ldots,v_n,V_1,\ldots,V_N) = (v_1^*,\ldots,v_{n+N}^*)$ for convenience. Then we have

$$\operatorname{Pf} \begin{pmatrix} O_m & O_{m,M} & H(R^0,S)J_N & H(R^0,S^0)J_n \\ O_{M,m} & \frac{1}{\operatorname{Pf}(A)}\widehat{A} & H(R;S)J_N & H(R;S^0)J_n \\ -J_N{}^tH(R^0,S) & -J_N{}^tH(R;S) & \frac{1}{\operatorname{Pf}(B)}J_N{}^t\widehat{B}J_N & O_{N,n} \\ -J_n{}^tH(R^0,S^0) & -J_n{}^tH(R;S^0) & O_{n,N} & O_n \end{pmatrix}$$

$$= \sum_{\tau} \operatorname{sgn}\tau \prod_{(U_k,U_l)\in\tau} \alpha_{U_kU_l} \prod_{(V_k,V_l)\in\tau} \beta_{V_kV_l} \prod_{(u_k^*,v_l^*)\in\tau} h(u_k^*,v_l^*) \qquad (44)$$

summed over all perfect matching on $(u_1^*,\ldots,u_{m+M}^*,v_{n+N}^*,\ldots,v_1^*)$. We can interpret this Pfaffian as a generating function for all $(r+m+1)$-tuples $C = (\tau, P_1, \ldots, P_{r+m})$ which satisfies (i) r is an even integer such that $0 \le r \le \min(M, N-m+n)$, (ii) τ is a perfect matching on $(u_1^*,\ldots,u_{m+M}^*,v_{n+N}^*,\ldots,v_1^*) = (u_1,\ldots,u_m,U_1,\ldots,U_M,V_N,\ldots,V_1,v_n,\ldots,v_1)$ such that $P_i \in \mathcal{P}(u_k^*,v_l^*)$ if and only if there is an edge $(u_k^*,v_l^*) \in \tau$ which is connecting a vertex in $R^0 \uplus R$ and a vertex in $S^0 \uplus S$. (iii) each vertex in R^0 must be connected to a vertex in $S^0 \uplus S$, (iv) each vertex in S^0 must be connected to a vertex in $R^0 \uplus R$, (v) and there are exactly r edges connecting a vertex in R with a vertex in $S^0 \uplus S$. The weight asigned to $C = (\tau, P_1, \ldots, P_{r+m})$ shall be $\operatorname{sgn}\tau \prod_{(U_k,U_l)\in\tau} \alpha_{U_kU_l} \prod_{(V_k,V_l)\in\tau} \beta_{V_kV_l} \operatorname{wt}(P_1)\cdots\operatorname{wt}(P_{r+m})$. It is easy to see that the sign-reversing involution used in the previous proofs is applicable exactly as before, and we may cancell all the terms appearing in (44), aside from those with non-intersecting paths. Thus we only need to consider configulations $C = (\tau, P_1, \ldots, P_{r+m})$ with non-intersecting paths. From the assumption that $R^0 \uplus R$ is D-compatible with $S^0 \uplus S$, (a) each v_i must be connected to u_i in τ and $P_i \in \mathcal{P}(u_i,v_i)$ for $i = 1,\ldots,n$, (b) each u_i must be connected to V_{i-n} in τ and $P_i \in \mathcal{P}(u_i,V_{i-n})$ for $i = (n+1),\ldots,m$, (c) there are r vertices $I = \{U_{i_1},\ldots,U_{i_r}\}_{1 \le i_1 < \cdots < i_r \le M}$ in R and r vertices $J = \{V_{j_1},\ldots,V_{j_r}\}_{m-n < j_1 < \cdots < j_r \le N}$ such that each U_{i_k} is connected to V_{j_k} in τ and $P_k \in \mathcal{P}(U_{i_k},V_{j_k})$ for $k = 1,\ldots,r$, (d) and the remaining $(M-r)$ vertices in R are connected each other in τ and the remaining $(N-m+n-r)$ vertices in S are connected each other in τ. Thus r must be even. We set $\overline{I}$ (resp. $\overline{J}$) to be the complementary set of I (resp. J) in R (resp. $S \setminus \{V_1,\ldots,V_{m-n}\}$). If we put $M = 2M'$, $N-m+n = 2N'$ and $r = 2r'$ for nonnegative integers M', N' and r', then the sum of the weights becomes

$$\sum_{\substack{r=0 \\ r=2r' \text{ even}}}^{\min(M,N-m+n)} \sum_{\substack{I \subseteq R \\ \sharp I = r}} \sum_{\substack{J \subseteq S \\ \sharp J = r}} (-1)^{s(I,\overline{I})+s(J,\overline{J})} \frac{1}{\operatorname{Pf}(A)^{M'-r'}\operatorname{Pf}(B)^{N'-r'}} \operatorname{Pf}(\widehat{A}_{\overline{I}}^{\overline{I}}) \operatorname{Pf}(\widehat{B}_{\overline{J}}^{\overline{J}}) F^0(R^0 \uplus I, S^0 \uplus J).$$

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By (10) we have $\mathrm{Pf}(\widehat{A_{\overline{I}}^{\overline{I}}}) = (-1)^{|\overline{I}|-M'+r'}\,\mathrm{Pf}(A)^{M'-r'-1}\,\mathrm{Pf}(A_I^I)$ and $\mathrm{Pf}(\widehat{B_{\overline{J}}^{\overline{J}}}) = (-1)^{|J|-N'+r'}\,\mathrm{Pf}(B)^{N'-r'-1}\,\mathrm{Pf}(B_J^J)$. Further it is easy to see that $s(I,\overline{I}) \equiv |I| - r' \pmod 2$ and $s(J,\overline{J}) \equiv |J| - r' \pmod 2$ Since $I \cup \overline{I} = R$ and $J \cup \overline{J} = S$, we have $|I| + |\overline{I}| = \binom{M+1}{2} \equiv M' \pmod 2$ and $|J| + |\overline{J}| = \binom{N+1}{2} - \binom{m-n+1}{2} \equiv N' \pmod 2$. These identities immediately implies (40). $\square$

5 A variant of the Sundquist formula

We give here some variant (both a statement and a proof) of the Sundquist formula [31]. Indeed, we prove the

Theorem 5.1

$$\mathrm{pf}\left(\frac{y_i - y_j}{a + b(x_i + x_j) + cx_ix_j}\right)_{1 \le i,j \le 2n} \times \prod_{1 \le i < j \le 2n} \{a + b(x_i + x_j) + cx_ix_j\}$$

$$= (ac - b^2)^{\binom{n}{2}} \sum_{\substack{I \subseteq [2n] \\ \sharp I = n}} (-1)^{|I|-\binom{n+1}{2}} y_I \Delta_I(x) \Delta_{I^c}(x) J_I(x) J_{I^c}(x)$$

$$= (ac - b^2)^{\binom{n}{2}} \sum_{\substack{I \subseteq [2n] \\ \sharp I = n \\ i_1 < j_1}} (-1)^{|I|-\binom{n+1}{2}} (y_I + (-1)^n y_{I^c}) \Delta_I(x) \Delta_{I^c}(x) J_I(x) J_{I^c}(x),$$

$$(45)$$

where the sum runs over all n-elements subset $I = \{i_1 < \cdots < i_n\}$ of $[2n] = \{1, 2, \ldots, 2n\}$ such that $i_1 < j_1$ and $|I| = i_1 + \cdots + i_n$. Moreover $I^c = \{j_1 < \cdots < j_n\}$ is the complementary subset of I in $[2n]$ and

$$\Delta_I(x) = \prod_{\substack{i,j \in I \\ i < j}} (x_i - x_j),$$

$$J_I(x) = J_I(x; a, b, c) = \prod_{\substack{i,j \in I \\ i < j}} \{a + b(x_i + x_j) + cx_ix_j\},$$

$$y_I = \prod_{i \in I} y_i.$$

In particular, if the relation $ac = b^2$ holds then

$$\mathrm{pf}\left(\frac{y_i - y_j}{a + b(x_i + x_j) + cx_ix_j}\right)_{1 \le i,j \le 2n} = 0.$$

Example 5.2 *In the case of $n = 2$, if we put $a = c = 1$ and $b = 0$ then the theorem above reads*

$$\mathrm{pf}\left[\frac{y_i - y_j}{1 + x_ix_j}\right]_{1 \le i < j \le 4} \times \prod_{1 \le i < j \le 4} (1 + x_ix_j)$$

$$= (y_1y_2 + y_3y_4)(x_1 - x_2)(x_3 - x_4)(1 + x_1x_2)(1 + x_3x_4)$$

$$- (y_1y_3 + y_2y_4)(x_1 - x_3)(x_2 - x_4)(1 + x_1x_3)(1 + x_2x_4)$$

$$+ (y_1y_4 + y_2y_3)(x_1 - x_4)(x_2 - x_3)(1 + x_1x_4)(1 + x_2x_3)$$

Proof of the theorem. Since

$$a + b(x_i + x_j) + cx_ix_j = (\sqrt{c}x_i + \frac{b}{\sqrt{c}})(\sqrt{c}x_j + \frac{b}{\sqrt{c}}) + a - \frac{b^2}{c}$$

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it is enough to show the theorem for the case $a = c = 1$ and $b = 0$.

Moreover, since the second equality follows immediately from the first one if one notes the fact that $|I| + |I^c| \equiv n \mod 2$, we give a proof of the first equality. We prove it by induction on a size of the matrix in question using Corollary 2.3. Assume the formula holds for n. Consider the $n+1$ case. If we exapand first the pfaffian of the left-hand side along the first row, then the induction hypothesis yeilds

$$\text{LHS} = \prod_{0 \leq i < j \leq 2n+1} (1 + x_i x_j) \sum_{\alpha=1}^{2n+1} (-1)^{\alpha-1} \frac{y_0 - y_\alpha}{1 + x_0 x_\alpha} \, \text{pf} \left(\frac{y_i - y_j}{a + b(x_i + x_j) + c x_i x_j} \right)_{\substack{1 \leq i, j \leq 2n+1 \\ i, j \neq \alpha}}$$

$$= \prod_{0 \leq i < j \leq 2n+1} (1 + x_i x_j) \sum_{\alpha=1}^{2n+1} \frac{y_0 - y_\alpha}{1 + x_0 x_\alpha} \frac{(-1)^{\alpha-1}}{\prod_{\substack{i < j \\ i, j \neq 0, \alpha}} (1 + x_i x_j)}$$

$$\times \sum_{I(\check{\alpha})} (-1)^{|I(\check{\alpha})| - \frac{n(n+1)}{2} - j_\alpha} y_{I(\check{\alpha})} a_{I(\check{\alpha})}(x),$$

where we put

$$a_{I(\check{\alpha})}(x) = \Delta_{I(\check{\alpha})}(x) \Delta_{I(\check{\alpha})^c}(x) J_{I(\check{\alpha})}(x) J_{I(\check{\alpha})^c}(x) (= a_{I(\check{\alpha})^c}(x)).$$

Here $I(\check{\alpha})$ in the interior sum indicates the sum over the subset $I(\check{\alpha}) = \{i_1, \dots, i_n\} \subseteq \{1, 2, \dots, \alpha - 1, \alpha + 1, \dots, 2n + 1\} (\subseteq [2n + 1])$ satisfying $\#I(\check{\alpha}) = n$ and one put $j_\alpha = \#\{i \in I(\check{\alpha}) ; i > \alpha\}$, and $I(\check{\alpha})^c$ being the complement of the subset $I(\check{\alpha})$ in $\{1, 2, \dots, \alpha - 1, \alpha + 1, \dots, 2n + 1\}$.

It is hence easy to see that

$$\text{LHS} = \sum_{\alpha=1}^{2n+1} \sum_{I(\check{\alpha})} (-1)^{|I(\check{\alpha})| - \frac{n(n+1)}{2} - j_\alpha + \alpha - 1} (y_0 - y_\alpha) y_{I(\check{\alpha})} a_{I(\check{\alpha})}(x) \prod_{\substack{k=1 \\ k \neq \alpha}}^{2n+1} (1 + x_0 x_k) \prod_{\substack{j=1 \\ j \neq \alpha}}^{2n+1} (1 + x_\alpha x_j).$$

We now arrange the sum above in the following way; Take $K = \{k_1 < \dots < k_n\} \subseteq [2n + 1]$ such that $\#K = n$. Note that the complement K^c of K in $[2n + 1]$ is equal to the complement $(\{0\} \cup K)^c$ of $\{0\} \cup K$ in $\{0, 1, 2, \dots, 2n + 1\}$ and $\#K^c = n + 1$. Thus if $K = I(\check{\alpha})$, then since $I(\check{\alpha})^c \cup K = \{1, 2, \dots, \alpha - 1, \alpha + 1, \dots, 2n + 1\}$ we observe that $I(\check{\alpha})^c = (K^c \text{ in } [2n + 1]) \backslash \{\alpha\}$. This indicates

$$\sum_{\alpha=1}^{2n+1} \sum_{\substack{K = I(\check{\alpha}) \\ \subseteq \{1, 2, \dots, \alpha-1, \alpha+1, \dots, 2n+1\}}} * = \sum_{\substack{K \subseteq [2n+1] \\ \sharp K = n}} \sum_{\substack{\alpha \\ I(\check{\alpha}) = K}} *.$$

One hence rewrites LHS as follows.

$$\text{LHS} = \sum_{\substack{K \subseteq [2n+1] \\ \sharp K = n}} \sum_{\alpha \in K^c} (-1)^{|K| - \frac{n(n+1)}{2} - j_\alpha(K) + \alpha - 1}$$

$$\times (y_{K \cup \{0\}} - y_{K \cup \{\alpha\}}) a_K(x) \prod_{\substack{k=1 \\ k \neq \alpha}}^{2n+1} (1 + x_0 x_k) \prod_{\substack{j=1 \\ j \neq \alpha}}^{2n+1} (1 + x_\alpha x_j),$$

where $j_\alpha(K) = \#\{i \in K ; i > \alpha\}$. Note that any subset I in $\{0, 1, \dots, 2n+1\}$ such that $\#I = n + 1$ can be represented by either $I = K \cup \{0\}$ or $I = K \cup \{\alpha\}$ for some $K \subseteq [2n + 1]$ with $\#K = n$ and $\alpha \in K^c$. Hence it

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is easy to see the RHS of the target formula is written as

$$\text{RHS} = \sum_{\substack{I \subseteq \{0,1,\dots,2n+1\} \\ \sharp I = n+1}} (-1)^{|I|+n+1-\frac{(n+1)(n+2)}{2}} y_I a_I(x)$$

$$= \sum_{\substack{K \subseteq [2n+1] \\ \sharp K = n}} (-1)^{|K\cup\{0\}|-\frac{n(n+1)}{2}} y_{K\cup\{0\}} a_{K\cup\{0\}}(x)$$

$$+ \sum_{\substack{0 \notin I \subseteq \{0,1,\dots,2n+1\} \\ \sharp I = n+1}} (-1)^{|I|-\frac{n(n+1)}{2}} y_I a_I(x).$$

Hence the first formula of the theorem follows immediately from the following lemma. $\square$

Lemma 5.3 *Let K be the subset of $[2n+1]$ with $\#K = n$.*
(A) Suppose $0 \in I(\subseteq \{0,1,\dots,2n+1\})$. Then I is given by the form $I = K \cup \{0\}$. Note that K^c in $[2n+1] = I^c$ in $\{0,1,\dots,2n+1\}$. Then

$$\sum_{\alpha \in K^c} \prod_{\substack{k=1 \\ k\neq\alpha}}^{2n+1} (1+x_0 x_k) \prod_{\substack{j=1 \\ j\neq\alpha}}^{2n+1} (1+x_\alpha x_j)(-1)^{\alpha-1-j_\alpha(K)} \Delta_K(x) J_K(x) \Delta_{K^c\backslash\{\alpha\}}(x) J_{K^c\backslash\{\alpha\}}(x)$$

$$= \Delta_{K\cup\{0\}}(x) \Delta_{K^c}(x) J_{K\cup\{0\}}(x) J_{K^c}(x).$$

(B) Suppose $I \notin (\subseteq \{0,1,\dots,2n+1\})$. Thus I is written as the form $I = K \cup \{\alpha\}$ for some $K \subseteq [2n+1]$ and $\alpha \in K^c$. Note that in this case $|I| = |K \cup \{\alpha\}| = |K| + \alpha$. Then

$$\sum_{\substack{K\cup\{\alpha\}=I \\ K\subseteq[2n+1], \alpha\notin K \\ \sharp K = n,}} \prod_{\substack{k=1 \\ k\neq\alpha}}^{2n+1} (1+x_0 x_k) \prod_{\substack{j=1 \\ j\neq\alpha}}^{2n+1} (1+x_\alpha x_j)(-1)^{j_\alpha(K)} \Delta_K(x) J_K(x) \Delta_{K^c\backslash\{\alpha\}}(x) J_{K^c\backslash\{\alpha\}}(x)$$

$$= \Delta_I(x) \Delta_{I^c}(x) J_I(x) J_{I^c}(x).$$

Proof. We first prove the case (A). Since $\alpha \notin K$ one has

$$\prod_{\substack{j=1 \\ j\neq\alpha}}^{2n+1} (1+x_\alpha x_j) J_{K^c\backslash\{\alpha\}}(x) = \prod_{j\in K} (1+x_\alpha x_j) J_{K^c}(x), \qquad (46)$$

$$\prod_{\substack{k=1 \\ j\neq\alpha}}^{2n+1} (1+x_0 x_k) J_K(x) = \prod_{\substack{k\in K^c \\ k\neq\alpha}} (1+x_\alpha x_k) J_{K\cup\{0\}}(x). \qquad (47)$$

Put

$$\mathbb{J}_\alpha^K(x) = \prod_{\substack{k\in K^c \\ k\neq\alpha}} (1+x_0 x_k) \prod_{j\in K} (1+x_\alpha x_j)(-1)^{\alpha-1-j_\alpha(K)} \Delta_{K^c\backslash\{\alpha\}}(x). \quad (48)$$

Then, by (5.1, 2), to prove (A) it is enough to show

$$\sum_{\alpha \in K^c} \mathbb{J}_\alpha^K(x) = \prod_{k\in K} (x_0 - x_k) \Delta_{K^c}(x). \qquad (49)$$

We shall prove (5.4) by two steps.

the 1st step: Note that

$$\Delta_{K^c}(x) = \prod_{\substack{j\neq\alpha \\ j\in K^c}} (x_\alpha - x_j)(-1)^{\alpha-n-1-j_\alpha(K)} \Delta_{K^c\backslash\{\alpha\}}(x). \qquad (50)$$

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Hence we rewrite $\mathbb{J}_\alpha(x)$ as

$$\mathbb{J}_\alpha^K(x) = \frac{(-1)^n}{(1+x_0 x_\alpha)\prod_{\substack{j\neq\alpha \\ j\in K^c}}(x_\alpha - x_j)}\prod_{k\in K^c}(1+x_0 x_k)\prod_{j\in K}(1+x_\alpha x_j)\Delta_{K^c}(x).$$

Since $\mathbb{J}_\alpha^K(x)$ is a polynomial in x_i's, the expression of $\mathbb{J}_\alpha(x)$ above combined with the simple relation

$$\prod_{j\neq\alpha}(x_\alpha - x_j) = \prod_{j\neq\alpha,\beta}(x_\alpha - x_j)\times(x_\alpha - x_\beta)\quad(\beta\neq\alpha,\ \beta\notin K)$$

implies that $\mathbb{J}_\alpha^K(x) + \mathbb{J}_\beta^K(x)$ is divided by $x_\alpha - x_\beta$. Moreover, if $\gamma\neq\alpha,\beta$ then $x_\alpha - x_\beta|\Delta_{K^c\setminus\{\gamma\}}(x)$, whence $x_\alpha - x_\beta|\mathbb{J}_\gamma^K(x)$. It follows that $\sum_{\alpha\in K^c}\mathbb{J}_\alpha^K(x)$ can be divided by $x_\beta - x_\gamma$ for all $\beta,\gamma\in K^c$. Hence we see that $\Delta_{K^c}(x)|\sum_{\alpha\in K^c}\mathbb{J}_\alpha^K(x)$.

the 2nd step: We shall next prove the relation:

$$\prod_{k\in K}(x_0 - x_k)|\sum_{\alpha\in K^c}\mathbb{J}_\alpha^K(x). \tag{51}$$

To prove (51) we may assume $K = \{1,\dots,n\}$. In this case, $K^c = \{n+1,\dots,2n+1\}$. Suppose now $x_1 = x_0$. Then one writes

$$\sum_{\alpha=n+1}^{2n+1}\mathbb{J}_\alpha^K(x) = \prod_{k=n+1}^{2n+1}(1+x_1 x_k)\sum_{\alpha=n+1}^{2n+1}\prod_{j=2}^{n}(1+x_\alpha x_j)(-1)^\alpha\Delta_{K^c\setminus\{\alpha\}}(x).$$

By this expression, in order to show (51) it is sufficient to show the

Lemma 5.4 *As a polynomial in $x_{n+1},\dots,x_{2n+1}$ we have*

$$\sum_{\alpha=n+1}^{2n+1}\prod_{j=2}^{n}(1+x_\alpha x_j)(-1)^\alpha\Delta_{K^c\setminus\{\alpha\}}(x)\equiv 0.$$

Proof. Denote the left hand side's polynomial in question by $P(x)$. Put also

$$\mathbb{K}_\alpha(x) = \prod_{j=2}^{n}(1+x_\alpha x_j)(-1)^\alpha\Delta_{K^c\setminus\{\alpha\}}(x) = \frac{\Delta_{K^c}(x)(-1)^{n+1}\prod_{j=2}^{n}(1+x_\alpha x_j)}{\prod_{j\neq\alpha}(x_\alpha - x_j)}.$$

Note that $P(x) = \sum_{\alpha=n+1}^{2n+1}\mathbb{K}_\alpha(x)$. From the second expression of $\mathbb{K}_\alpha(x)$ it is easy to see that if $\alpha < \beta$ then

$$\mathbb{K}_\alpha(x_{n+1},\dots,x_\beta,\dots,x_\alpha,\dots,x_{2n+1}) = -\mathbb{K}_\beta(x_{n+1},\dots,x_\alpha,\dots,x_\beta,\dots,x_{2n+1}),$$

whence $\mathbb{K}_\alpha(x) + \mathbb{K}_\beta(x) = 0$ when $x_\alpha = x_\beta$. Also for $\gamma\neq\alpha,\beta$, $x_\alpha - x_\beta$ is factored out in $\Delta_{K^c\setminus\{\gamma\}}(x)$. It follows then immediately to see that $x_\alpha - x_\beta|P(x)$ for any $\alpha,\beta\in K^c$. Thus $P(x)$ can be divided by $\Delta_{K^c}(x)$. On the other hand we have $\deg(P(x)) = n - 1 + \binom{n}{2} < \binom{n+1}{2} = \deg(\Delta_{K^c}(x))$. Hence we conclude that $P(x) = 0$. This proves the lemma. $\square$

Combining the results obtained in the 1st and the 2nd steps above, one concludes that

$$\prod_{k \in K} (x_0 - x_k) \Delta_{K^c}(x) \Big| \sum_{\alpha \in K^c} \mathbb{J}_\alpha(x).$$

Let us now compair the degree of both sides: As polynomials in x_j ($j \in K^c$), we see that

$$\deg(\prod_{k \in K} (x_0 - x_k) \Delta_{K^c}(x)) = \deg(\Delta_{K^c}(x)) = \binom{n+1}{2} = n + \binom{n}{2} = \deg(\sum_{\alpha \in K^c} \mathbb{J}_\alpha^K(x)).$$

As polynomials in x_0, the degrees of both sides are equal to n. Thus there is a constant $C = C_K(x_{k_1}, \dots, x_{k_n})$ such that

$$\sum_{\alpha \in K^c} \mathbb{J}_\alpha^K(x) = C \prod_{k \in K} (x_0 - x_k) \Delta_{K^c}(x).$$

We now prove $C = 1$. It is obvious to show this when $K = [n]$. Since the coefficient of x_0^n which does not contain x_{n+1} in the left hand side is obviously given by the one of $\mathbb{J}_{n+1}^K(x)$, and hence it is equal to $(-1)^n x_{n+2} \cdots x_{2n+1} \Delta_{K^c \setminus \{n+1\}}(x)$. Thus if we look at the relation $\Delta_{K^c}(x) = \prod_{j=n+2}^{2n+1} (x_{n+1} - x_j) \Delta_{K^c \setminus \{n+1\}}(x)$, it follows that $C = 1$. This completes the proof of the case (A).

We next prove the case (B). For each K, α such that $I = K \cup \{\alpha\}$ note that

$$\Delta_I(x) = (-1)^{j_\alpha(K)+n} \prod_{j \in K} (x_\alpha - x_j) \Delta_K(x). \tag{52}$$

Since I^c in $\{0, 1, \dots, 2n+1\} = K^c$ in $\{1, \dots, \alpha-1, \alpha+1, \dots, 2n+1\} \cup \{0\} = (K^c \setminus \{\alpha\}) \cup \{0\}$ one has

$$\Delta_{I^c}(x) = \prod_{j \in K^c \setminus \{\alpha\}} (x_0 - x_j) \Delta_{K^c \setminus \{\alpha\}}(x). \tag{53}$$

Similarly we have

$$\prod_{\substack{j=1 \\ j \neq \alpha}}^{2n+1} (1 + x_\alpha x_j) J_K(x) = \prod_{j \in K^c \setminus \{\alpha\}} (1 + x_\alpha x_j) J_I(x), \tag{54}$$

$$\prod_{\substack{k=1 \\ k \neq \alpha}}^{2n+1} (1 + x_0 x_k) J_{K^c \setminus \{\alpha\}}(x) = \prod_{k \in K} (1 + x_0 x_k) J_{I^c}(x), \tag{55}$$

whence we find the identity in (B) is equivalent to

$$\sum_{\substack{K \cup \{\alpha\} = I \\ K \subseteq [2n+1], \alpha \notin K \\ \sharp K = n,}} \prod_{k \in K}^{2n+1} (1 + x_0 x_k) \prod_{j \in K^c \setminus \{\alpha\}}^{2n+1} (1 + x_\alpha x_j) (-1)^{j_\alpha(K)} \Delta_K(x) \Delta_{K^c \setminus \{\alpha\}}(x) = \Delta_I(x) \Delta_{I^c}(x). \tag{56}$$

Thus we prove (56). It is enough to prove (56) for the case $I = \{1, \dots, n+1\}$. Note the fact $K^c \setminus \{\alpha\} = \{n+2, \dots, 2n+2\}$. Since $K = I \setminus \{\alpha\}$ and $j_\alpha(K) = n + 1 - \alpha$, the idenitity we should prove is the following.

$$\sum_{\alpha=1}^{n+1} \mathbb{L}_\alpha(x) = \prod_{j=n+2}^{2n+1} (x_0 - x_j) \Delta_I(x), \tag{57}$$

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where we put

$$\mathbb{L}_\alpha(x) = \prod_{\substack{k=1 \\ k \neq \alpha}}^{n+1}(1 + x_0 x_k) \prod_{j=n+2}^{2n+1}(1 + x_\alpha x_j)(-1)^{n+1-\alpha}\Delta_K(x). \qquad (58)$$

It is clear that $\mathbb{L}_\alpha(x)$ can be divided by $x_\beta - x_\gamma$ for any $\beta, \gamma \neq \alpha\,(1 \leq \beta, \gamma \leq n+1)$. Suppose $K = \{1, \ldots, \alpha-1, \alpha+1, \ldots, n+1\}$. For any $\beta \neq \alpha\,(1 \leq \beta \leq n+1)$ since

$$\Delta_K(x) = \Delta_{K \setminus \{\beta\}}(x) \times \prod_{\substack{i=1 \\ i \neq \alpha, \beta}}^{n+1}(x_i - x_\beta)(-1)^{n+1-\beta},$$

we have for $\alpha < \beta$

$$\mathbb{L}_\alpha(x_1, \ldots, x_\beta, \ldots, x_\alpha, \ldots, x_{2n+1}) = -\mathbb{L}_\beta(x_1, \ldots, x_\alpha, \ldots, x_\beta, \ldots, x_{2n+1}).$$

This shows $x_\beta - x_\alpha | \mathbb{L}_\alpha(x) + \mathbb{L}_\beta(x)$. Hence one has $\Delta_I(x) | \sum_{\alpha=1}^{n+1} \mathbb{L}_\alpha(x)$.

Our next task is to show that $\sum_{\alpha=1}^{n+1} \mathbb{L}_\alpha(x)$ is divided by $\prod_{j=n+2}^{2n+1}(x_0 - x_j)$. Thus, suppose $x_0 = x_{2n+1}$. Then we have

$$\mathbb{L}_\alpha(x) = \prod_{k=1}^{n+1}(1 + x_0 x_k) \prod_{j=n+2}^{2n}(1 + x_\alpha x_j)(-1)^{n+1-\alpha}\Delta_K(x).$$

Therefore what we have to show is

$$\sum_{\alpha=1}^{n+1} \mathbb{L}_\alpha(x) = \prod_{k=1}^{n+1}(1 + x_0 x_k)(-1)^{n+1} \sum_{\alpha=1}^{n+1} \prod_{j=n+2}^{2n}(1 + x_\alpha x_j)(-1)^\alpha \Delta_{I \setminus \{\alpha\}}(x) \equiv 0,$$

but this is indeed true by Lemma 5.2. Hence we see that

$$\prod_{j=n+2}^{2n+1}(x_0 - x_j)\Delta_I(x) \Big| \sum_{\alpha=1}^{n+1} \mathbb{L}_\alpha(x).$$

Compairing the degree, and further the coefficients of x_0^n which does not contain for instance x_{n+1} of the both sides, by the quite similar way as we have done in the proof of (A), we see that (5.12) holds true. This completes the proof of the case (B), whence Lemma 5.1 follows. $\square$

As a corollary of this theorem we obtain the identity established in [31]. It is considered as a two variables generalization of the pfaffian version of Cauchy's determinant formula (see e.g., Lemma 8 in [8].)

$$\mathrm{pf}(\frac{x_j - x_i}{1 - tx_i x_j})_{1 \leq i < j \leq 2n} = t^{n(n-1)} \frac{\prod_{1 \leq i < j \leq 2n}(x_j - x_i)}{\prod_{1 \leq i < j \leq 2n}(1 - tx_i x_j)}.$$

We have already given a way of reduction of the following corollary from Theorem 5.1 in [12] by using

$$\prod_{1 \leq i < j \leq n}(1 + x_i x_j) = \sum_{\lambda = (\alpha_1, \cdots, \alpha_p | \alpha_1 + 1, \cdots, \alpha_p + 1)} s_\lambda(x_1, \cdots, x_n)$$

where $s_\lambda = s_\lambda(x_1, \cdots, x_n)$ are the Schur functions, so we omit it.

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Corollary 5.5

$$\mathrm{pf}\left(\frac{y_i - y_j}{1 + x_i x_j}\right)_{1 \leq i,j \leq 2n} \times \prod_{1 \leq i < j \leq 2n} (1 + x_i x_j) = \sum_{\lambda,\mu} a_{\lambda+\delta_n, \mu+\delta_n}(x,y),$$

where the sums runs over pairs of partitions

$$\lambda = (\alpha_1, \cdots, \alpha_p | \alpha_1 + 1, \cdots, \alpha_p + 1), \mu = (\beta_1, \cdots, \beta_p | \beta_1 + 1, \cdots, \beta_p + 1)$$

in Frobenius notation with $\alpha_1, \beta_1 < n - 1$. Also, for α and β partitions (compositions, in general) of length n, we put

$$a_{\alpha,\beta}(x,y) = \sum_{\sigma \in \mathfrak{S}_{2n}} \epsilon(\sigma) \sigma(x_1^{\alpha_1} y_1 \cdots x_n^{\alpha_n} y_n x_{n+1}^{\beta_1} \cdots x_{2n}^{\beta_n}),$$

where $\sigma \in \mathfrak{S}_{2n}$ acts on each of two sets of variables $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_n\}$ by permuting indices, and $\delta_n = (n - 1, n - 2, \cdots, 0)$. □

Suppose n is even. If we put $y_i = 1$ for all $1 \leq i \leq 2n$ in the theorem then it is immediate to see the

Corollary 5.6

$$\sum_{\substack{I \subseteq [2n] \\ \sharp I = n \\ i_1 < j_1}} (-1)^{|I|} \Delta_I(x) \Delta_{I^c}(x) J_I(x) J_{I^c}(x) = 0$$

holds for even n. □

Remark 5.7 *It is naturally thought the theorem as the identity of two variables related as a A_n-type root system. It would be intersting to establish the B_n, C_n, D_n etc analogues of Theorem 5.1 and its corollaries.*

6 Kawanaka's q-Littlewood formula

In [15], Kawanaka gave a certain q-series identity which is a generalization of the classical Littlewood identity:

$$\sum_{\lambda} s_\lambda(x) = \prod_i \frac{1}{1 - x_i} \prod_{i < j} \frac{1}{1 - x_i x_j}.$$

First of all, we recall his generalization of the Littlewood formula. Let $(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n)$ and $(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$ for complex numbers a, q such that $|q| < 1$. We write $(a)_n$ (resp. $(a)_\infty$) for $(a;q)_n$ (resp. $(a;q)_\infty$) in short when there is no fear of confusions.

Theorem 6.1 *(Kawanaka)*

$$\sum_{\lambda} \prod_{\alpha \in \lambda} \frac{1 + q^{h(\alpha)}}{1 - q^{h(\alpha)}} s_\lambda(x) = \prod_{i=1}^{n} \frac{(-x_i q; q)_\infty}{(x_i; q)_\infty} \prod_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j}. \tag{59}$$

In this section we give a short proof of this identity as an application of the minor summation formula and then use this method to obtain a similar formula as follows.

Theorem 6.2

$$\sum_\lambda \frac{q^{n(\lambda)}}{\prod_{\alpha\in\lambda}(1-q^{h(\alpha)})} s_\lambda(x) = \frac{1}{\prod_{i=1}^n (x_i;q)_\infty}. \tag{60}$$

Here $n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i$ for a partition λ. More generally,

$$\sum_\lambda q^{n(\lambda)} \frac{\prod_{i=1}^n (a;q)_{\lambda_i+n-i}}{\prod_{\alpha\in\lambda}(1-q^{h(\alpha)})} s_\lambda(x) = \prod_{i=1}^{n-1}(a;q)_i \prod_{i=1}^n \frac{(aq^{n-1}x_i;q)_\infty}{(x_i;q)_\infty}. \tag{61}$$

In order to prove the theorems, we recall here a well-known formula called the q-binomial formula as a lemma.

Lemma 6.3

$$\sum_{n=0}^\infty \frac{(a)_n}{(q)_n} x^n = \frac{(ax)_\infty}{(x)_\infty}. \square \tag{62}$$

The following lemma is a generalization of the q-binomial formula and becomes the key to the proof of Kawanaka's identity.

Lemma 6.4

$$\sum_{k,l\geq 0} \frac{(-q)_k(-q)_l}{(q)_k(q)_l} \frac{q^l-q^k}{q^k+q^l} x^k y^l = \frac{(qx)_\infty}{(x)_\infty} \frac{(qy)_\infty}{(y)_\infty} \frac{x-y}{1-xy}$$

Proof. Put

$$F(x,y) = \sum_{k,l\geq 0} \frac{(-q)_k(-q)_l}{(q)_k(q)_l} \frac{q^l-q^k}{q^k+q^l} x^k y^l = \sum_{k,l\geq 0} a_{kl} x^k y^l,$$

$$G(x,y) = \frac{x-y}{1-xy} \prod_{r=0}^\infty \frac{1+xq^{r+1}}{1-xq^r} \frac{1+yq^{r+1}}{1-yq^r} = \sum_{k,l\geq 0} b_{kl} x^k y^l.$$

Then

$$F(x,0) = \sum_{k\geq 0} \frac{(-q)_{k-1}}{(q)_{k-1}} x^k = x\frac{(-qx)_\infty}{(x)_\infty} = G(x,0).$$

In the same way we have $F(0,y) = -y\frac{(-qy)_\infty}{(y)_\infty} = G(0,y)$, and this shows $a_{k,0} = b_{k,0}$ and $a_{0,l} = b_{0,l}$ for $k,l \geq 0$. We show the coefficent of $x^k y^l$ of $(1-xy)F(x,y)$ is equal to the coefficent of $x^k y^l$ of $(1-xy)G(x,y)$ for $k,l \geq 1$. An easy calculation shows that

$$a_{kl} - a_{k-1,l-1} = 2(q^l - q^k)\frac{(-q)_{k-1}(-q)_{l-1}}{(q)_k(q)_l}.$$

On the other hand, the coefficent of $x^k y^l$ in $(1-xy)G(x,y) = (x-y)\frac{(-qx)_\infty(-qy)_\infty}{(x)_\infty(y)_\infty}$ is

$$\frac{(-q)_{k-1}}{(q)_{k-1}} \frac{(-q)_l}{(q)_l} - \frac{(-q)_k}{(q)_k} \frac{(-q)_{l-1}}{(q)_{l-1}} = 2(q^l - q^k)\frac{(-q)_{k-1}(-q)_{l-1}}{(q)_k(q)_l}.$$

This proves the lemma. $\square$

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The Schur functions are well-known symmetric functions. The reader should consult [24] to see detailed explanations of these symmetric functions. Here we only use a well-known deteminantal expression for Schur functions. We use the notation of Macdonald's book [24]. For example, a *partition* is a non-increasing sequence of nonnegatie integers $\lambda = (\lambda_1, \lambda_2, \dots)$ with finite non-zero parts. The number of non-zero parts are called the *length* and denoted by $\ell(\lambda)$. We may assume that the number of variables are finite, say n, and $x = (x_1, \dots, x_n)$. Then the *Schur function* corresponding to a partition λ is defined to be

$$s_\lambda(x) = \frac{1}{\Delta(x)} \det(x_i^{\lambda_j + n - j}).$$

Here $\Delta(x) = \prod_{i<j}(x_i - x_j)$. In the discussion we identify a partition with its Ferrers graph. Given a partiton λ, the *hook-length* of λ at $\alpha = (i, j)$ is, by definition, given by $h(\alpha) = h(i, j) = \lambda_i + \lambda_j - i - j + 1$. A key observation to proving Kawanaka's formula is the following fact which the reader can veryfy directly from the diagram of a partition:

$$\prod_{\alpha \in \lambda} \frac{1 + q^{h(\alpha)}}{1 - q^{h(\alpha)}} = \prod_{i=1}^{n} \frac{(-q)_{\ell_i}}{(q)_{\ell_i}} \prod_{i<j} \frac{1 - q^{\ell_i - \ell_j}}{1 + q^{\ell_i - \ell_j}}$$

Here $\ell_i = \lambda_i + n - i$ with $\ell(\lambda) \leq n$. In fact, if one write a large Young diagram and fill each cell with its hook-length, then he will notice that it is easy to see that the numbers in the first row are $[\ell_1] - \{\ell_1 - \ell_j | 1 < j \leq n\}$, and the numbers in the second row are $[\ell_2] - \{\ell_2 - \ell_j | 2 < j \leq n\}$, etc. We also use the following famous identities. (For the proof, see [30].)

Lemma 6.5 *Let n be an even integer. Let $x_1, \dots, x_n$ be indeterminates. Then*

$$\mathrm{Pf} \left[\frac{x_i - x_j}{x_i + x_j} \right]_{1 \leq i < j \leq n} = \prod_{1 \leq i < j \leq n} \frac{x_i - x_j}{x_i + x_j}, \tag{63}$$

$$\mathrm{Pf} \left[\frac{x_i - x_j}{1 - x_i x_j} \right]_{1 \leq i < j \leq n} = \prod_{1 \leq i < j \leq n} \frac{x_i - x_j}{1 - x_i x_j}. \quad \square \tag{64}$$

Let $A = (\alpha_{ij})_{i,j \geq 0}$ denote a skew symmetric matrix. If $\lambda = (\lambda_1, \lambda_2, \cdots)$ is any partition, then we put $\ell_i = \lambda_i + n - i$ for $i = 1, \dots, n$ with a fixed even integer $n \geq \ell(\lambda)$. As an application of Theorem 4.3 we obtain the following formula from the definition of Schur functions.

Lemma 6.6 *Let n be an even integer. We denote by $s_\lambda(x)$ the Schur functions of n variables corresponding to a partition λ. Then*

$$\sum_\lambda \mathrm{Pf}(\alpha_{\ell_p \ell_q})_{1 \leq p, q \leq n} s_\lambda(x) = \frac{1}{\Delta(x)} \mathrm{Pf}(\beta_{ij})_{1 \leq i, j \leq n} \tag{65}$$

where λ runs all the partition satisfying $\ell(\lambda) \leq n$ and

$$\beta_{ij} = \sum_{k,l \geq 0} \alpha_{kl} x_i^k x_j^l = \sum_{0 \leq k < l} \alpha_{kl} \begin{vmatrix} x_i^k & x_i^l \\ x_j^k & x_j^l \end{vmatrix},$$

and $\Delta(x) = \prod_{1 \leq i < j \leq n}(x_i - x_j)$.

Now we are in the position to prove the Kawanaka formula above.

Proof of Theorem 6.1. It is enough to prove the case where n is even. For a partition $\lambda = (\lambda_1, \cdots, \lambda_n)$ $(\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, we put $\ell_i = \lambda_i + n - i$ $(i = 1, 2, \cdots, n)$ as above. If we put $\alpha_{kl} = \frac{(-q)_k (-q)_l}{(q)_k (q)_l} \frac{q^l - q^k}{q^k + q^l}$ in (65), then (63) and the remark above show

$$\mathrm{Pf} \left[\alpha_{\ell_i \ell_j} \right]_{1 \leq i, j \leq n} = \prod_{i=1}^n \frac{(-q)_{l_i}}{(q)_{l_i}} \prod_{1 \leq i < j \leq n} \frac{q^{l_j} - q^{l_i}}{q^{l_i} + q^{l_j}} = \prod_{\alpha \in \lambda} \frac{1 + q^{h(x)}}{1 - q^{h(x)}}$$

Thus, Lemma 6.4 and (64) show

$$\Delta(x) \sum_{\lambda} \prod_{\alpha \in \lambda} \frac{1 + q^{h(x)}}{1 - q^{h(x)}} s_\lambda(x) = \mathrm{Pf} \left[\frac{(-qx_i)_\infty}{(x_i)_\infty} \frac{(-qx_j)_\infty}{(x_j)_\infty} \frac{x_i - x_j}{1 - x_i x_j} \right]_{1 \leq i < j \leq n}$$

$$= \prod_{i=1}^n \frac{(-qx_i)_\infty}{(x_i)_\infty} \mathrm{Pf} \left[\frac{x_i - x_j}{1 - x_i x_j} \right]_{1 \leq i < j \leq n}$$

$$= \Delta(x) \prod_{i=1}^n \frac{(-qx_i)_\infty}{(x_i)_\infty} \prod_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j}$$

This proves the theorem. $\square$

As a formula similar to (63) and (64), the following is a Pfaffian version of the Vendermonde determinant.

Lemma 6.7 *Let $n = 2r$ be an even integer. Then*

$$\mathrm{Pf} \left[\frac{(x_i^r - x_j^r)^2}{x_i - x_j} \right]_{1 \leq i < j \leq n} = \prod_{1 \leq i < j \leq n} (x_i - x_j). \qquad (66)$$

Proof. To prove this identity we can adopt exactly the same method to prove the Vandermonde determinant. The reader can find the fundamental ideas to evaluate determinants in [20], and we will see these ideas are also applicable to evaluate Pfaffians. First of all, the left-hand side is a polynomial of x_i's of degree at most $\frac{n(n-1)}{2}$ since each entry of the Pfaffian is a polynomial of degree $(n-1)$. If we put $x_i = x_j$ with $i \neq j$, then the Pfaffian vanishes since all the entries of a row/column become zero. This shows that $(x_i - x_j)$ divides the Pfaffian as a polynomial, which implies that the complete product $\prod_{1 \leq i < j \leq n} (x_i - x_j)$ must divide the Pfaffian. Thus we conclude that

$$\mathrm{Pf} \left[\frac{(x_i^r - x_j^r)^2}{x_i - x_j} \right]_{1 \leq i < j \leq n} = c \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

To determine the constant c, we check the coefficient of $x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ in the both sides. In the Pfaffian in question this monomial appears only when the corresponding perfect matching becomes $((1, n), (2, n - 1), \ldots, (r, r + 1))$, and the reader can easily confirm $c = 1$. This proves the lemma. $\square$

This proof shows that replacing the entries of the Pfaffian by appropriate polynomials will be an interesting problem.

Proof of Theorem 6.2. Now we consider a skew symmetric matrix $A = (\alpha_{k,l})$ of size $n = 2r$ whose (k, l)-entry is defined by

$$\alpha_{k,l} = \frac{(a)_k (a)_l}{(q)_k (q)_l} \frac{(q^{rl} - q^{rk})^2}{q^l - q^k}.$$

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For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ we put $\ell_i = \lambda + n - i$. Then, from Lemma 6.7 we obtain

$$
\begin{aligned}
\mathrm{Pf}[\alpha_{\ell_i,\ell_j}]_{1 \le i < j \le n} &= \prod_{i=1}^{n} \frac{(a)_i}{(q)_i} \prod_{1 \le i < j \le n} (q^{\ell_j} - q^{\ell_i}) \\
&= q^{\sum_{i=1}^{n}(i-1)(\lambda_i+n-i)} \prod_{i=1}^{n} \frac{(a)_i}{(q)_i} \prod_{1 \le i < j \le n} (1 - q^{\ell_i - \ell_j}) \\
&= \frac{q^{n(\lambda) + \frac{1}{6}n(n-1)(n-2)} \prod_{i=1}^{n}(a)_{\lambda_i+n-i}}{\prod_{c \in \lambda}(1 - q^{h(c)})}.
\end{aligned}
$$

Now, to apply Lemma 6.6, we need to study the sum:

$$
\begin{aligned}
f_n(x,y) &= \sum_{k,l \ge 0} \frac{(a)_k (a)_l}{(q)_k (q)_l} \frac{(q^{rl} - q^{rk})^2}{q^l - q^k} x^k y^l, \\
&= \sum_{k,l \ge 0} \frac{(a)_k (a)_l}{(q)_k (q)_l} \left\{ \sum_{\nu=1}^{r} q^{(\nu-1)k} q^{(n-\nu)l} - \sum_{\nu=1}^{r} q^{(n-\nu)k} q^{(\nu-1)l} \right\} x^k y^l, \\
&= \sum_{\nu=1}^{r} \frac{(aq^{\nu-1}x)_\infty}{(q^{\nu-1}x)_\infty} \frac{(aq^{n-\nu}y)_\infty}{(q^{n-\nu}y)_\infty} - \sum_{\nu=1}^{r} \frac{(aq^{n-\nu}x)_\infty}{(q^{n-\nu}x)_\infty} \frac{(aq^{\nu-1}y)_\infty}{(q^{\nu-1}y)_\infty}.
\end{aligned}
$$

Thus we have

$$
f_n(x,y) = \frac{(aq^{n-1}x)_\infty (aq^{n-1}y)_\infty}{(x)_\infty (y)_\infty} g_n(x,y),
$$

where $g_n(x,y)$ is a polynomial of x and y defined by

$$
\begin{aligned}
g_n(x,y) = &\sum_{\nu=1}^{r} \prod_{k=1}^{\nu-1}(1 - q^{k-1}x) \prod_{k=\nu}^{n-1}(1 - aq^{k-1}x) \prod_{k=1}^{n-\nu}(1 - q^{k-1}y) \prod_{k=n-\nu+1}^{n-1}(1 - aq^{k-1}y) \\
&- \sum_{\nu=1}^{r} \prod_{k=1}^{n-\nu}(1 - q^{k-1}x) \prod_{k=n-\nu+1}^{n-1}(1 - aq^{k-1}x) \prod_{k=1}^{\nu-1}(1 - q^{k-1}y) \prod_{k=\nu}^{n-1}(1 - aq^{k-1}y).
\end{aligned}
$$

By applying Lemma 6.6, it is not hard to see that in order to prove (61), it suffices to prove the following lemma, Lemma 6.8. Note also that the formula (60) is obtained by putting $a = 0$ in (61). Hence the theorem follows. $\square$

Lemma 6.8 *Let n be even integer and let $g_n(x,y)$ be as above. Then we have*

$$
\mathrm{Pf}[g_n(x_i, x_j)]_{1 \le i < j \le n} = q^{\frac{1}{6}n(n-1)(n-2)} \prod_{k=1}^{n-1}(a)_k \prod_{1 \le i < j \le n}(x_i - x_j).
$$

Proof. The method we use here is quite similar to that of we used in the proof of Lemma 6.7. First of all the reader should notice that $g(x,y)$ is of degree $(n-1)$ as a polynomial in the variable x. This shows that $\mathrm{Pf}[g(x_i, x_j)]$ is a polynomial of degree at most $(n-1)$ if we see it as a polynomial of a fixed variable x_i. Since $g(x,y)$ is skew symmetric, i.e. $g(y,x) = -g(x,y)$, and this show that $(x_i - x_j)$ divides the Pfaffian, and, as before, the complete produce $\prod_{i<j}(x_i - x_j)$ must divide the Pfaffian. Thus we conclude that

$$
\mathrm{Pf}[g(x_i, x_j)]_{1 \le i < j \le n} = c \prod_{1 \le i < j \le n}(x_i - x_j).
$$

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If we see the right-hand side as a polynomial of a fixed variable x_i, then it is of degree $(n-1)$, and this shows the constant c must not include x_i. Now, to determine the constant c, which is independent of x_i, we compare the coefficient of the monomial $\prod_{i=1}^{n} x_i^{n-i}$ of the both sides. First we consider the left-hand side. The Pfaffian is the sum of polynomials $\operatorname{sgn}\sigma\, g_n(x_{i_1}, x_{j_1}) \cdots g_n(x_{i_r}, x_{j_r})$ for all perfect matching $\sigma = ((i_1, j_1), \ldots, (i_r, j_r))$ of $[n]$, The monomial which contributes to the monimial $\prod_{i=1}^{n} x_i^{n-i}$ in the polynomial $g_n(x_{i_k}, x_{j_k})$ is $x_{i_k}^{n-i_k} x_{j_k}^{n-j_k}$. This shows hence that the coefficent of $\prod_{i=1}^{n} x_i^{n-i}$ in the left-hand side Pfaffian is equal to $\operatorname{Pf}\left[[x^{n-i}y^{n-j}]g_n(x,y)\right]$. Thus, to prove the desired identity it suffices to prove the identity

$$\operatorname{Pf}\left[[x^{n-i}y^{n-j}]g_n(x,y)\right]_{1 \le i < j \le n} = q^{\frac{1}{6}n(n-1)(n-2)}\prod_{k=1}^{n-1}(a)_k.$$

Here $[x^a y^b]f(x,y)$ stands for the coefficent of the $x^a y^b$ in the polynomial $f(x,y)$. Since the determinination of the sign of the Pfaffain is easy, to prove the identity it suffices to show the following lemma. $\square$

Lemma 6.9 *Let $n = 2r$ be an even integer and let $h_n(x,y) = h_n(a,b,q,t;x,y)$ be the polynomial of x and y defined by*

$$\sum_{\nu=1}^{r}\prod_{k=1}^{\nu-1}(1-q^{k-1}x)\prod_{k=\nu}^{n-1}(1-aq^{k-1}x)\prod_{k=1}^{n-\nu}(1-t^{k-1}y)\prod_{k=n-\nu+1}^{n-1}(1-bt^{k-1}y)$$

$$-\sum_{\nu=1}^{r}\prod_{k=1}^{n-\nu}(1-q^{k-1}x)\prod_{k=n-\nu+1}^{n-1}(1-aq^{k-1}x)\prod_{k=1}^{\nu-1}(1-t^{k-1}y)\prod_{k=\nu}^{n-1}(1-bt^{k-1}y).$$

Then we have

$$\det\left[[x^{n-i}y^{n-j}]h_n(x,y)\right]_{1 \le i,j \le n} = (qt)^{\frac{1}{6}n(n-1)(n-2)}\prod_{i=1}^{n-1}(a;q)_i\prod_{j=1}^{n-1}(b;t)_j.$$

Proof. First of all, we can easily observe that $[x^{n-i}y^{n-j}]h_n(x,y)$ is a polynomial of degree $(n-1)$ with respect to a, and also of degree $(n-1)$ with respect to b. This means the determinant is a polynomial of degreee at most $n(n-1)$ with respect to each variable a or b. Now we show that $\prod_{i=1}^{n-1}(a;q)_i\prod_{j=1}^{n-1}(b;t)_j$ divides the determinant. For this purpose we want to show that $(1 - aq^{i-1})$ divides the determinant $(n-i)$ times for each $i = 1, \ldots, (n-1)$. For example, if we substitute $a = 1$, then it is easy to see from $h_n(x,y)$ that all the columns of the matrix become proportinal and the rank of matrix become at most 1. This shows $(1-a)$ divides the matrix at least $(n-1)$ times. Next, if we substitute $a = q^{-1}$, then the dimension of the vector space spanned by n polynomials $\prod_{k=1}^{\nu-1}(1-q^{k-1}x)\prod_{k=\nu}^{n-1}(1-aq^{k-1}x)$ for $\nu = 1, \ldots, n$ is at most 2 since all these polynomials are multiples of $\prod_{k=1}^{n-2}(1-q^{k-1}x)$. This shows that at most two column vectors of the matrix $\left[[x^{n-i}y^{n-j}]h_n(x,y)\right]$ is linearly independent, which means $(1-aq)$ divides the matrix at least $(n-2)$ times. By the similar arguments, it is easy to see that

$$\det\left[[x^{n-i}y^{n-j}]h_n(x,y)\right]_{1 \le i,j \le n} = c\prod_{i=1}^{n-1}(a;q)_i\prod_{j=1}^{n-1}(b;t)_j,$$

where c is a constant independent of a and b. To find c, we compare the constant term of the both sides regarding them as polynomials of a and b. If we extract only the constant from the matrix $\left[[x^{n-i}y^{n-j}]h_n(x,y)\right]$ and set the resulting matrix to be $G = (g_{ij})_{1 \leq i,j \leq n}$, then $g_{ij} = 0$ if $i + j < n$. Further, by checking the coefficient of $x^{n-\nu}y^{\nu}$ in $h_n(x,y)$, we have

$$g_{\nu,n-\nu} = -(-1)^{n-\nu}q^{\sum_{k=1}^{n-\nu}(k-1)}(-1)^{\nu-1}t^{\sum_{k=1}^{\nu-1}(k-1)} = q^{\frac{(n-\nu-1)(n-\nu)}{2}}t^{\frac{(\nu-2)(\nu-1)}{2}}$$

for $1 \leq \nu \leq r$. By the same way we have $g_{n-\nu,\nu} = -q^{\frac{(\nu-2)(\nu-1)}{2}}t^{\frac{(n-\nu-1)(n-\nu)}{2}}$ for $1 \leq \nu \leq r$. This shows

$$c = q^{\sum_{\nu=1}^{n}\frac{(\nu-2)(\nu-1)}{2}}t^{\sum_{\nu=1}^{n}\frac{(\nu-2)(\nu-1)}{2}} = (qt)^{\frac{1}{6}n(n-1)(n-2)},$$

and this completes the proof of the lemma. $\square$

7 Kawanaka's q-Cauchy identity

In [16] Kawanaka gave a q-Cauchy formula, which is considered as a deteminant version of the formula in the previous section.

Theorem 7.1 *(Kawanaka) Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ be two independent sequences of variables. For a partition λ let $s_\lambda(x)$ and $s_\lambda(y)$ be the corresponding Schur functions in x and y respectively. Then we have the following identity:*

$$\sum_{\lambda,\mu} q^{|\lambda-\mu|+|\mu-\lambda|}J_{\lambda\mu}(q^2)s_\lambda(x)s_\mu(y)$$

$$=\prod_{i\geq 1}\frac{(-qx_i;q^2)_\infty}{(qx_i;q^2)_\infty}\prod_{j\geq 1}\frac{(-qy_j;q^2)_\infty}{(qy_j;q^2)_\infty}\prod_{i,j\geq 1}\frac{1}{1-x_iy_j}. \qquad (67)$$

Here $J_{\lambda\mu}(t)$ is a rational function defined below.

Let λ and μ be partitions, and let $c = (i,j)$ be any cell in the plane. As a natural generaliztion of the hook length $h_\lambda(c) = \lambda_i + \lambda_j' - i - j + 1$,

$$h_{\lambda\mu}(c) = \lambda_i + \mu_j' - i - j + 1$$

is defined in [16]. For example, let $\lambda = (4,3,1,1)$ and $\mu = (3,3)$. If we fill each cell c of λ with the numbers $h_{\lambda\mu}(c)$, then it looks as follows:

5	4	3	0
3	2	1	
0			
−1			

In [16],

$$n(\lambda,\mu) = \sum_{(i,j)\in\lambda-\mu}(\lambda_j' - i) = \sum_{(i,j)\in\lambda-\mu}(i - \mu_j' - 1)$$

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is also defined as a generalization of $n(\lambda) = \sum_{i \geq 1}(i-1)\lambda_i$ (see [24]). Let t be an indeterminate. For partitions λ and μ, we define a rational function $J_{\lambda\mu}(t)$ in t by

$$J_{\lambda\mu}(t) = t^{n(\lambda,\mu)} \prod_{c\in\lambda} \frac{1+t^{h_{\lambda\mu}(c)}}{1-t^{h_\lambda(c)}} \cdot t^{n(\mu,\lambda)} \prod_{c\in\mu} \frac{1+t^{h_{\mu\lambda}(c)}}{1-t^{h_\mu(c)}}.$$

Lemma 7.2

$$\sum_{k,l\geq 0} \frac{(-q^2;q^2)_k}{(q^2;q^2)_k} \frac{(-q^2;q^2)_l}{(q^2;q^2)_l} \frac{2x^k y^l}{q^{k-l}+q^{l-k}} = \frac{(-qx;q^2)_\infty}{(qx;q^2)_\infty} \frac{(-qy;q^2)_\infty}{(qy;q^2)_\infty} \frac{1}{1-xy}$$

(68)

Proof. Put

$$F(x,y) = \sum_{k,l\geq 0} \frac{(-q^2;q^2)_k}{(q^2;q^2)_k} \frac{(-q^2;q^2)_l}{(q^2;q^2)_l} \frac{2x^k y^l}{q^{k-l}+q^{l-k}} = \sum_{k,l\geq 0} a_{kl} x^k y^l,$$

$$G(x,y) = \frac{(-qx;q^2)_\infty}{(qx;q^2)_\infty} \frac{(-qy;q^2)_\infty}{(qy;q^2)_\infty} \frac{1}{1-xy}.$$

We will compare the coefficients of $x^k y^l$ of $(1-xy)F(x,y)$ and $(1-xy)G(x,y)$. By Lemma 6.3 we have

$$(1-xy)G(x,y) = \sum_{k,l\geq 0} \frac{(-1;q^2)_k(-1;q^2)_l}{(q^2;q^2)_k(q^2;q^2)_l} q^{k+l} x^k y^l$$

Meanwhile, the coefficient of $x^k y^l$ of $(1-xy)F(x,y)$ for $k,l \geq 0$ equals

$a_{kl} - a_{k-1,l-1}$

$$= \frac{(-q^2;q^2)_{k-1}(-q^2;q^2)_{l-1}}{(q^2;q^2)_k(q^2;q^2)_l}\{(1+q^{2k})(1+q^{2l}) - (1-q^{2k})(1-q^{2l})\}\frac{2q^k q^l}{q^{2k}+q^{2l}}$$

$$= 4\frac{(-q^2;q^2)_{k-1}(-q^2;q^2)_{l-1}}{(q^2;q^2)_k(q^2;q^2)_l} q^{k+l} = \frac{(-1;q^2)_k(-1;q^2)_l}{(q^2;q^2)_k(q^2;q^2)_l} q^{k+l}.$$

The coefficient of x^k of $(1-xy)F(x,y)$ is equal to

$$\frac{(-q^2;q^2)_k}{(q^2;q^2)_k} \frac{2q^k}{1+q^{2k}} = \frac{(-1;q^2)_k}{(q^2;q^2)_k} q^k.$$

In the same way it is easy to see that the coefficient of y^l of $(1-xy)F(x,y)$ is equal to $\frac{(-1;q^2)_l}{(q^2;q^2)_l}q^l$. Thus we have $(1-xy)F(x,y) = (1-xy)G(x,y)$, and this completes the proof. $\square$

The following identities are known as the Cauchy determinants (see [24]).

Proposition 7.3

$$\det\left(\frac{1}{x_i+y_j}\right)_{1\leq i,j\leq n} = \frac{\prod_{1\leq i<j\leq n}(x_i-x_j)\prod_{1\leq i<j\leq n}(y_i-y_j)}{\prod_{1\leq i,j\leq n}(x_i+y_j)}, \quad (69)$$

$$\det\left(\frac{1}{1-x_iy_j}\right)_{1\leq i,j\leq n} = \frac{\prod_{1\leq i<j\leq n}(x_i-x_j)\prod_{1\leq i<j\leq n}(y_i-y_j)}{\prod_{1\leq i,j\leq n}(1-x_iy_j)}. \quad (70)$$

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Lemma 7.4 *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ be partitions such that $\ell(\lambda), \ell(\mu) \leq n$. We put $k_i = \lambda_i + n - i$ and $\ell_i = \mu_i + n - i$ for $1 \leq i \leq n$. If we put*

$$a_{kl} = \frac{(-q^2; q^2)_k}{(q^2; q^2)_k} \frac{(-q^2; q^2)_l}{(q^2; q^2)_l} \frac{2q^{k+l}}{q^{2k} + q^{2l}},$$

then

$$\det(a_{k_i \ell_j})_{1 \leq i,j \leq n} = q^{|\lambda - \mu| + |\mu - \lambda|} J_{\lambda\mu}(q^2). \tag{71}$$

Proof. By (69) we have

$$\det(a_{k_i \ell_j})_{1 \leq i,j \leq n}$$

$$= 2^n q^{\sum_i k_i + \sum_j \ell_j} \prod_i \frac{(-q^2; q^2)_{k_i}}{(q^2; q^2)_{k_i}} \prod_j \frac{(-q^2; q^2)_{\ell_j}}{(q^2; q^2)_{\ell_j}} \cdot \frac{\prod_{i<j}(q^{2k_j} - q^{2k_i}) \prod_{i<j}(q^{2\ell_j} - q^{2\ell_i})}{\prod_{i,j}(q^{2k_i} + q^{2\ell_j})}$$

$$= 2^n q^{\sum_i (2i-1)k_i + \sum_j (2j-1)\ell_j} \frac{\prod_{i<j}(1 - q^{2(k_i - k_j)})}{\prod_i (q^2; q^2)_{k_i}} \cdot \frac{\prod_{i<j}(1 - q^{2(\ell_i - \ell_j)})}{\prod_j (q^2; q^2)_{\ell_j}} \cdot \frac{\prod_i (-q^2; q^2)_{k_i} \prod_j (-q^2; q^2)_{\ell_j}}{\prod_{i,j}(q^{2k_i} + q^{2\ell_j})}$$

Since $\prod_{c \in \lambda} \frac{1}{1 - q^{2h_\lambda(c)}} = \frac{\prod_{1 \leq i < j \leq n} 1 - q^{2(k_i - k_j)}}{\prod_{i=1}^n (q^2; q^2)_{k_i}}$ and $\prod_{c \in \lambda} \frac{1}{1 - q^{2h_\mu(c)}} = \frac{\prod_{1 \leq i < j \leq n} 1 - q^{2(\ell_i - \ell_j)}}{\prod_{j=1}^n (q^2; q^2)_{\ell_j}}$,
it is enough to show that

$$2^n q^{\sum_{i=1}^n (2i-1)k_i + \sum_{j=1}^n (2j-1)\ell_j} \frac{\prod_{i=1}^n (-q^2; q^2)_{k_i} \prod_{j=1}^n (-q^2; q^2)_{\ell_j}}{\prod_{i,j=1}^n (q^{2k_i} + q^{2\ell_j})}$$

$$= q^{|\lambda - \mu| + |\mu - \lambda| + 2n(\lambda, \mu) + 2n(\mu, \lambda)} \prod_{c \in \lambda} \left(1 + q^{2h_{\lambda\mu}(c)}\right) \prod_{c \in \mu} \left(1 + q^{2h_{\mu\lambda}(c)}\right).$$

Note that $h_{\lambda\mu}(c) = \lambda_i - j + \mu'_j - i + 1 = \lambda_i - j + \sharp\{r : \mu_r \geq j\} - i + 1$.
This implies that, if we draw the Young diagram of λ and fill each cell c with $h_{\lambda\mu}(c)$, then we find that the numbers in the ith row of λ are

$$[\sharp\{r : \mu_r > \lambda_i\} - i + 1, k_i] - \{k_i - \ell_r : \mu_r \leq \lambda_i\}.$$

Here we write $[a, b] = \{a, a+1, a+2, \ldots, b\}$ for integers $a, b \in \mathbb{Z}$. It is also easy to see that, if we write the Young diagram of μ and fill each cell c with $h_{\mu\lambda}(c)$, then the numbers in the jth row of μ are

$$[\sharp\{r : \lambda_r \geq \mu_j\} - j + 1, \ell_j] - \{\ell_j - k_r : \lambda_j < \mu_r\}.$$

Thus it is enough to show that

$$2^n q^{\sum_{i=1}^n (2i-1)k_i + \sum_{j=1}^n (2j-1)\ell_j} \frac{\prod_{i=1}^n (-q^2; q^2)_{k_i} \prod_{j=1}^n (-q^2; q^2)_{\ell_j}}{\prod_{i,j=1}^n (q^{2k_i} + q^{2\ell_j})}$$

$$= q^{|\lambda - \mu| + |\mu - \lambda| + 2n(\lambda, \mu) + 2n(\mu, \lambda)} \frac{\prod_{i=1}^n \prod_{m=\sharp\{r:\mu_r > \lambda_i\} - i + 1}^{k_i} \left(1 + q^{2m}\right) \prod_{j=1}^n \prod_{m=\sharp\{r:\lambda_r \geq \mu_j\} - j + 1}^{\ell_j} \left(1 + q^{2m}\right)}{\prod_{\lambda_i \geq \mu_j} \left(1 + q^{2(k_i - \ell_j)}\right) \prod_{\lambda_i < \mu_j} \left(1 + q^{2(\ell_i - k_j)}\right)}$$

It is easy to see that

$$\prod_{\mu_j \leq \lambda_i} \left(1 + q^{2(k_i - \ell_j)}\right) \prod_{\lambda_i < \mu_j} \left(1 + q^{2(\ell_i - k_j)}\right) = q^{-2P(\lambda, \mu)} \prod_{i,j=1}^n (q^{2k_i} + q^{2\ell_j}),$$

where

$$P(\lambda, \mu) = \sum_{\lambda_i \geq \mu_j} \ell_j + \sum_{\lambda_i < \mu_j} k_i = \sum_{i=1}^n \sharp\{r : \mu_r > \lambda_i\} k_i + \sum_{j=1}^n \sharp\{r : \lambda_r \geq \mu_j\} \ell_j$$

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Let A and B be the sets of lattice points defined by

$$A = \bigcup_{i=1}^{n} \{(i-1,y) : \sharp\{1 \leq r \leq n : \mu_r > \lambda_i\} \leq y \leq \lambda_i + n - 1\},$$

$$B = \bigcup_{j=1}^{n} \{(x,j-1) : \sharp\{1 \leq r \leq n : \lambda_r \geq \mu_j\} \leq x \leq \mu_j + n - 1\}.$$

Then we have

$$\prod_{i=1}^{n} \prod_{m=\sharp\{r:\mu_r>\lambda_i\}-i+1}^{k_i} \left(1+q^{2m}\right) = \prod_{(x,y)\in A} \left(1+q^{2(y-x)}\right),$$

$$\prod_{j=1}^{n} \prod_{m=\sharp\{r:\lambda_r\geq\mu_j\}-j+1}^{\ell_j} \left(1+q^{2m}\right) = \prod_{(x,y)\in B} \left(1+q^{2(x-y)}\right).$$

For example, if $n = 4$, $\lambda = (4,3,1,1)$ and $\mu = (3,3)$, then the big circles in Figure 5 are in A and the small circles are in B. The numbers asigned to big circles are $y - x$ and the numbers asigned to small circles are $x - y$. Let

$$A_1 = \bigcup_{i=1}^{n} \{(i-1,y) : \sharp\{1 \leq r \leq n : \mu_r > \lambda_i\} \leq y \leq n - 1\},$$

$$B_1 = \bigcup_{j=1}^{n} \{(x,j-1) : \sharp\{1 \leq r \leq n : \lambda_r \geq \mu_j\} \leq x \leq n - 1\},$$

$$A_2 = \bigcup_{i=1}^{n} \{(i-1,y) : \sharp\{n \leq y \leq \lambda_i + n - 1\},$$

$$B_2 = \bigcup_{j=1}^{n} \{(x,j-1) : \sharp\{n \leq x \leq \mu_j + n - 1\}.$$

Then we have $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. It is also easy to see that $A_1 \cup B_1 = [0, n-1] \times [0, n-1]$, which implies that, as a multiset,

$$\bigcup_{i=1}^{n} \{|y-i+1| : \sharp\{r : \mu_r > \lambda_i\} \leq y \leq n - 1\} \cup \bigcup_{j=1}^{n} \{|x-j+1| : \sharp\{r : \lambda_r \geq \mu_j\} \leq x \leq n - 1\}$$
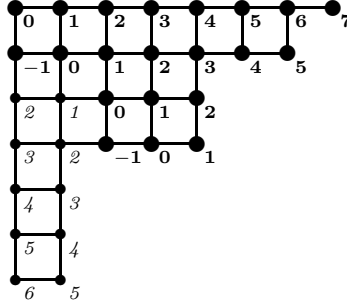

Figure 5: Lattice points

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is equal to $\{|x - y|; (x, y) \in [0, n - 1] \times [0, n - 1]\}$, and is composed of n 0's, $2(n-1)$ 1's, $2(n-2)$ 2's, ... , 2 $(n-1)$'s. This shows that

$$\prod_{(x,y)\in A} \left(1 + q^{2(y-x)}\right) \prod_{(x,y)\in B}^{n} \left(1 + q^{2(x-y)}\right)$$

$$= q^{-2Q(\lambda,\mu)} 2^n \prod_{i=1}^{n} (-q^2; q^2)_{k_i} \prod_{j=1}^{n} (-q^2; q^2)_{\ell_j},$$

where

$$Q(\lambda, \mu) = \sum_{\substack{(x,y)\in A \\ x<y}} (y - x) + \sum_{\substack{(x,y)\in B \\ x>y}} (x - y)$$

$$= \sum_{\substack{i \\ \lambda_i \geq \mu_i}} \binom{i - 1 - \sharp\{r : \mu_r > \lambda_i\}}{2} + \sum_{\substack{i \\ \lambda_i < \mu_i}} \binom{i - 1 - \sharp\{r : \lambda_r \geq \mu_i\}}{2}.$$

Thus the proof will be done if we show the identity

$$|\lambda - \mu| + |\mu - \lambda| + 2n(\lambda, \mu) + 2n(\mu, \lambda) + 2P(\lambda, \mu) - 2Q(\lambda, \mu)$$

$$= \sum_{i=1}^{n} (2i - 1)k_i + \sum_{j=1}^{n} (2j - 1)\ell_j.$$

In the example above, we have $|\lambda - \mu| = 3$, $|\mu - \lambda| = n(\mu, \lambda) = 0$, $n(\lambda, \mu) = 1$, $P(\lambda, \mu) = 64$, $Q(\lambda, \mu) = 4$, and $3 + 2 + 64 - 4 = 65 = \sum_{i=1}^{4}(2i - 1)k_i + \sum_{j=1}^{4}(2j - 1)\ell_j$. In the following lemma we prove the identity above. $\square$

Lemma 7.5 *Let n be a nonnegative integer, and let λ and μ be partitions which satisfies $\ell(\lambda), \ell(\mu) \leq n$. Then the identity*

$$|\lambda - \mu| + |\mu - \lambda| + 2n(\lambda, \mu) + 2n(\mu, \lambda) + 2P(\lambda, \mu) - 2Q(\lambda, \mu)$$

$$= \sum_{i=1}^{n} (2i - 1)k_i + \sum_{j=1}^{n} (2j - 1)\ell_j. \tag{72}$$

holds.

Proof. We proceed by induction on n. When $n = 1$, assume $\lambda_1 \geq \mu_1$. Then it is easy to see that $n(\lambda, \mu) = n(\mu, \lambda) = Q(\lambda, \mu) = 0$ and $P(\lambda, \mu) = 2\mu$. This shows that the left-hand side equals $\lambda_1 + \mu_1$ and it coincides with the right-hand sides. In the case $\lambda_1 < \mu_1$, we can prove it similarly. Asuume $n \geq 2$ and (72) holds up to $(n-1)$. Given partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$, we put $\widetilde{\lambda} = (\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_{n-1}) = (\lambda_1, \ldots, \lambda_{n-1})$ and $\widetilde{\mu} = (\widetilde{\mu}_1, \ldots, \widetilde{\mu}_{n-1}) = (\mu_1, \ldots, \mu_{n-1})$. Further we set $\widetilde{k}_i = \widetilde{\lambda}_i + n - 1 - i$ and $\widetilde{\ell}_i = \widetilde{\mu}_i + n - 1 - i$ for $1 \leq i \leq n-1$. Then, by the induction hypothesis, we may assume that (72) holds for $(n-1)$, $\widetilde{\lambda}$, $\widetilde{\mu}$, $\widetilde{k}$ and $\widetilde{\ell}$. First we assume that $\lambda_n \geq \mu_n$. Thus we have $|\lambda - \mu| + |\mu - \lambda| = |\widetilde{\lambda} - \widetilde{\mu}| + |\widetilde{\mu} - \widetilde{\lambda}| + \lambda_n - \mu_n$. From the condition $\lambda_n \geq \mu_n$, we can find an integer s such that $0 \leq s < n$ and $\mu_s > \lambda_n \geq \mu_{s+1}$ holds. Here we use the convention that $\lambda_0 = \mu_0 = \infty$. Using this s, we can express the statistics on λ and μ with the statistics on $\widetilde{\lambda}$ and $\widetilde{\mu}$. For example, if we write the Young diagram of λ and μ and fill the cell $(i, j) \in \lambda - \mu$ with the number $\mu'_j - i - 1$, then we easily see

that

$$n(\lambda, \mu) = n(\widetilde{\lambda}, \widetilde{\mu}) + (n - s - 1)(\lambda_n - \mu_{s+1}) + \sum_{i=s+1}^{n-1} (n - i - 1)(\mu_i - \mu_{i+1}),$$

$$= n(\widetilde{\lambda}, \widetilde{\mu}) + (n - s - 1)\lambda_n - \sum_{i=s+1}^{n-1} \mu_i$$

$$n(\mu, \lambda) = n(\widetilde{\mu}, \widetilde{\lambda}).$$

For a fixed i such that $1 \le i < n$, from the fact that $\mu_s > \lambda_n \ge \mu_{s+1}$, it is easy to see that

$$\sharp\{r : \mu_r > \lambda_i\} = \sharp\{r : \widetilde{\mu}_r > \widetilde{\lambda}_i\},$$

$$\sharp\{r : \lambda_r \ge \mu_i\} = \begin{cases} \sharp\{r : \widetilde{\lambda}_r \ge \widetilde{\mu}_i\} & \text{if } 1 \le i \le s, \\ \sharp\{r : \widetilde{\lambda}_r \ge \widetilde{\mu}_i\} + 1 & \text{if } s + 1 \le i < n. \end{cases}$$

From these facts we have

$$P(\lambda, \mu) = P(\widetilde{\lambda}, \widetilde{\mu}) + (n - 1)^2 + \sum_{j=s+1}^{n-1} \widetilde{\ell}_j + s\lambda_n + n\mu_n,$$

$$Q(\lambda, \mu) = Q(\widetilde{\lambda}, \widetilde{\mu}) + \binom{n - 1 - s}{2}.$$

Here we used the fact $\sum_{i=1}^{n-1} \sharp\{r : \mu_r > \lambda_i\} + \sum_{j=1}^{n-1} \sharp\{r : \lambda_r \ge \mu_j\} = (n-1)^2$, which is easy to confirm. From these identities, we obtain

$$|\lambda - \mu| + |\mu - \lambda| + 2n(\lambda, \mu) + 2n(\mu, \lambda) + 2P(\lambda, \mu) - 2Q(\lambda, \mu)$$

$$= |\widetilde{\lambda} - \widetilde{\mu}| + |\widetilde{\mu} - \widetilde{\lambda}| + 2n(\widetilde{\lambda}, \widetilde{\mu}) + 2n(\widetilde{\mu}, \widetilde{\lambda}) + 2P(\widetilde{\lambda}, \widetilde{\mu}) - 2Q(\widetilde{\lambda}, \widetilde{\mu})$$

$$+ (2n - 1)\lambda_n + (2n - 1)\mu_n + 2(n - 1)^2.$$

By the induction hypothesis we have $|\widetilde{\lambda} - \widetilde{\mu}| + |\widetilde{\mu} - \widetilde{\lambda}| + 2n(\widetilde{\lambda}, \widetilde{\mu}) + 2n(\widetilde{\mu}, \widetilde{\lambda}) + 2P(\widetilde{\lambda}, \widetilde{\mu}) - 2Q(\widetilde{\lambda}, \widetilde{\mu}) = \sum_{i=1}^{n-1}(2i - 1)\widetilde{k}_i + \sum_{i=1}^{n-1}(2i - 1)\widetilde{\ell}_i = \sum_{i=1}^{n-1}(2i - 1)k_i + \sum_{i=1}^{n-1}(2i - 1)\ell_i - 2(n - 1)^2$, and this proves the desired identity. In the case of $\lambda_n < \mu_n$, we may find an interger s which satisfes $0 \le s < n$ and $\lambda_s \ge \mu_n > \lambda_{s+1}$. A similar arguments will lead to the desired identity again. $\square$

Proof of Theorem 7.1.
We may assume that the number of variables are finite, i.e., $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Assume $N \ge n$ is a positive integer. Let T and S be two n by N rectangular matirces defined by

$$T = \left(x_i^{N-j}\right)_{j=1,\ldots,N}^{i=1,\ldots,n}, \qquad S = \left(y_i^{N-j}\right)_{j=1,\ldots,N}^{i=1,\ldots,n}.$$

Let A be an N by N square matrix defined by

$$A = \left(\frac{(-q^2; q^2)_{N-i}}{(q^2; q^2)_{N-i}} \frac{(-q^2; q^2)_{N-j}}{(q^2; q^2)_{N-j}} \frac{2}{q^{i-j} + q^{j-i}}\right)_{i,j=1,\ldots,N}.$$

Now we compute $\lim_{N \to \infty} \det {}^t TAS$ in two different ways. By the Cauchy-Binet formula (18), we have

$$\det {}^t TAS = \sum_{\substack{I \subseteq [N] \\ \sharp I = n}} \sum_{\substack{J \subseteq [N] \\ \sharp J = n}} \det T_I \det A_J^I \det S_J.$$

Put $I = \{i_1, \ldots, i_n\}$ and $J = \{j_1, \ldots, j_n\}$. Then there exist partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ such that $\lambda_1, \mu_1 \leq N - n$ and we can write $N - i_r = \lambda_r + n - r$ and $N - j_r = \mu_r + n - r$ for $r = 1, \ldots, n$. Then it is easy to see that $\det T_I = \det \left(x_i^{\lambda_j + n - j} \right) = \Delta(x) s_\lambda(x)$ and $\det S_J = \det \left(y_i^{\mu_j + n - j} \right) = \Delta(y) s_\mu(y)$. As before we put $k_r = \lambda_r + n - r$ and $\ell_r = \mu_r + n - r$ for $r = 1, \ldots, n$. Then, by Lemma 7.4, we obtain

$$\det A_J^I = \det \left[\frac{(-q^2; q^2)_{k_i}}{(q^2; q^2)_{k_i}} \frac{(-q^2; q^2)_{\ell_j}}{(q^2; q^2)_{\ell_j}} \frac{2}{q^{k_i - \ell_j} + q^{\ell_j - k_i}} \right] = q^{|\lambda - \mu| + |\mu - \lambda|} J_{\lambda\mu}(q^2).$$

On the other hand, by Lemma 7.2, we have

$$\lim_{N \to \infty} \det {}^t T A S = \det \left[\frac{(-qx_i; q^2)_\infty}{(qx_i; q^2)_\infty} \frac{(-qy_j; q^2)_\infty}{(qy_j; q^2)_\infty} \frac{1}{1 - x_i y_j} \right]_{i,j=1,\ldots,n}.$$

Thus (67) is an immediate consequence of (70). This proves the theorem.
$\square$

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Masao ISHIKAWA, Department of Mathematics, Faculty of Education, Tottori University, Tottori 680-8551, Japan

E-mail address: ishikawa@fed.tottori-u.ac.jp

Masato Wakayama, Faculty of Mathematics, Kyushu Unversity, Hakozaki, Fukuoka 812-8581, Japan

E-mail address: wakayama@math.kyushu-u.ac.jp

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