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Error Estimates of Finite Element Methods for Nonstationary Thermal Convection Problems with Temperature-Dependent Coefficients

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Summary General error estimates are proved for a class of finite element schemes for nonstationary thermal convection problems with temperature-dependent coefficients. These variable coefficients turn the diffusion and the buoyancy terms to be nonlinear, which increases the nonlinearity of the problems. An argument based on the energy method leads to optimal error estimates for the velocity and the temperature without any stability conditions. Error estimates are also provided for schemes modified by approximate coefficients, which are used conveniently in practical computations.

Key words thermal convection problem, temperature-dependent coefficients, error estimates, finite element method

1 Introduction

We are concerned here with finite element methods for thermal convection problems with temperature-dependent coefficients. In several cases of constant coefficients, error estimates of finite element approximations have already been developed. Some stationary problems have been studied by Bernardi et al. [1] and Boland and Layton [3]; a nonstationary one in a semi-discrete form by Boland and Layton [2]; some simplified nonstationary ones with infinite Prandtl number by Itoh and Tabata [12], Tabata and Suzuki [17], and Tagami and Itoh [20]. On the other hand, as pointed out by Getling [6], we often need to take into account the variation of material coefficients depending on the temperature in physical and engineering problems. Especially, variable coefficients play an important role in the formation of convection patterns. Our aim of this paper is to establish general error estimates of finite element methods for nonstationary thermal convection problems with temperature-dependent coefficients.

As far as we know, there are few researches on full-discrete finite element methods for nonstationary thermal convection problems with variable coefficients, and even with constant coefficients, from the mathematical point of view. For a simplified thermal convection problem error estimates have been established in the case of temperature-dependent coefficients by Tabata [15] and Tabata and Suzuki [18]. Their mathematical model does not include the inertia terms in the motion of fluid, because the slow velocity case is considered and the Prandtl number is set to be infinity. In this paper the original thermal convection problem without such reduction is treated. More precisely, we consider the following nonstationary thermal convection problem with temperature-dependent coefficients: Find

the velocity u, the pressure p, and the temperature θ

$$(u, p, \theta) : \ \Omega \times (0, T) \to \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$$

such that

$$\begin{cases} \partial_t u + (u \cdot \nabla)\, u - \nabla \cdot \big[\nu(\theta) D(u)\big] + \nabla p - \beta(\theta)\, \theta = f & \text{in } \Omega \times (0,T), \quad (1a) \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0,T), \quad (1b) \\ \partial_t \theta + (u \cdot \nabla)\, \theta - \nabla \cdot \big(\kappa(\theta) \nabla \theta\big) = g & \text{in } \Omega \times (0,T), \quad (1c) \\ u = u_D & \text{on } \Gamma \times (0,T), \quad (1d) \\ \theta = \theta_D & \text{on } \Gamma \times (0,T), \quad (1e) \\ u = u^0 & \text{in } \Omega \text{ at } t = 0, \quad (1f) \\ \theta = \theta^0 & \text{in } \Omega \text{ at } t = 0, \quad (1g) \end{cases}$$

where T (> 0) denotes a time; Ω a bounded domain in $\mathbb{R}^d$ ($d = 2$, 3) with Lipschitz-continuous boundary Γ;

$$(f,\, g) : \ \Omega \times (0,T) \to \mathbb{R}^d \times \mathbb{R}$$

a set of external force and heat source;

$$(u_D, \theta_D) : \ \Gamma \times (0,T) \to \mathbb{R}^d \times \mathbb{R}$$

a set of boundary velocity and temperature;

$$(u^0,\, \theta^0) : \ \Omega \to \mathbb{R}^d \times \mathbb{R}$$

a set of initial velocity and temperature;

$$(\nu,\, \kappa,\, \beta) : \ \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^d$$

a set of generalized viscosity, thermal conductivity, and thermal expansion coefficients; $D(u)$ the strain-rate tensor defined by

$$D(u) \equiv \frac{1}{2}(\nabla u + \nabla u^T).$$

The existence, uniqueness, and regularity result for (1) was obtained by Lorca and Boldrini [13], though their framework is slightly different with ours.

The problem (1) is discretized by the backward Euler method in time and by the conforming finite elements in space. For the discrete problem error estimates of the velocity and the temperature are established without any stability conditions. They are optimal in the sense that they have the same orders as the interpolation errors of the finite elements. Moreover, an estimate of the pressure is obtained, which is optimal in the space-dependent coefficient case, though not so in general. The derivation of the error estimates is based on the energy method, which is an extension of that in the Navier–Stokes equations with constant coefficients. We can refer to Tabata and Tagami [19], whose sources are found in some previous works, Girault and Raviart [7], Guermond and Quartapelle [9], and Heywood and Rannacher [11]. In the proofs, caused by the variable coefficients, we need additional estimates of some nonlinear remainders arising from the diffusion and the buoyancy terms. Throughout this paper we assume some regularity of the exact solution. This assumption is known to cause nonlocal compatibility conditions on the given data as discussed by Heywood and Rannacher [10] in the case of the Navier–Stokes equations. We do not, however, touch on the behavior of the solution near the initial time but confine ourselves to such an ideal case.

The problem (1) with the variable coefficients ν, κ, and β is often considered in practical applications, for example, the process of glass production by Ungan and Viskanta [21] and the mantle convection inside the Earth by Ratcliff et al. [14]. In practical finite element

codes of these examples some approximation may be introduced to variable coefficients, that is, they are replaced by interpolants. We show conditions where the same error estimates are maintained in the case when the approximate order in space is equal to 1 or 2.

This paper is organized as follows. In Section 2 we recall the maximum principle for (1c) as well as the existence and uniqueness result for (1). In Section 3, after presenting a finite element approximation to (1), we state the main results on error estimates. In Section 4 we prove the error estimates. In Section 5 we give practical schemes that maintain the convergence order. Finally we give concluding remarks in Section 6.

Throughout this paper we use c, c^*, c_ε, and c_ε^* as generic positive constants. The constant c^* may depend on the exact solution and the given data for the continuous problem, and c_ε and c_ε^* may depend on a positive constant ε. We note that these constants are independent of the time increment and the space discretization parameter and may be different at each occurrence.

2 Existence and uniqueness of the solution

Let us recall the d-dimensional bounded domain Ω with boundary Γ and consider $\mathbb{R}$-valued functions defined in Ω. For real number $p \geq 1$ let $L^p(\Omega)$ be the space of functions pth power summable over Ω. Likewise, let $L^\infty(\Omega)$ be the space of functions essentially bounded in Ω. The norms of $L^p(\Omega)$ and $L^\infty(\Omega)$ are denoted by $\|\,.\,\|_{0,p}$ and $\|\,.\,\|_{0,\infty}$, respectively. The space $L^2(\Omega)$ is equipped with the inner product

$$(\theta, \psi) \equiv \int_\Omega \theta\, \psi\, dx \qquad \text{for } \theta,\ \psi \in L^2(\Omega)$$

and we drop the subscript p $(= 2)$ in referring to the norm of $L^2(\Omega)$. For integer $k \geq 1$ let $W^{k,p}(\Omega)$ be the space of functions in $L^p(\Omega)$ with derivatives up to the kth order. The norm of $W^{k,p}(\Omega)$ is denoted by $\|\,.\,\|_{k,p}$. When $p = 2$, we denote by $H^k(\Omega)$ the space $W^{k,2}(\Omega)$ and drop the subscript p $(= 2)$ in referring to the norm of $H^k(\Omega)$. For integer $m \geq 0$ we denote by $\mathscr{C}^m(\Omega)$ the space of functions m times continuously differentiable in Ω. The space $\mathscr{C}^m(\overline{\Omega})$ consists of functions in $\mathscr{C}^m(\Omega)$ bounded and uniformly continuous in Ω with derivatives up to the mth order, and the space $\mathscr{C}^{m,1}(\overline{\Omega})$ consists of functions in $\mathscr{C}^m(\overline{\Omega})$ that are Lipschitz-continuous in $\overline{\Omega}$ with derivatives up to the mth order. When we take Ω having a $\mathscr{C}^{k-1,1}$-class boundary for integer $k \geq 1$, let $H^{k-1/2}(\Gamma)$ be the space of traces of the functions in $H^k(\Omega)$ to Γ.

We denote by $L^p(\Omega)^d$ the d-product space of $L^p(\Omega)$. As for functions, norms, seminorms, and inner products, we use the same notation in $\mathbb{R}$- and in $\mathbb{R}^d$-valued function spaces mentioned above. Moreover, if there is no ambiguity, we use the abbreviate notation of the spaces, for example, L^p instead of $L^p(\Omega)$.

Set $X \equiv H^1(\Omega)$ and $Y \equiv X^d$. Let $\Psi \equiv H_0^1(\Omega)$ be a space of functions in X that vanish on Γ, and $V \equiv \Psi^d$. As usual we denote by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$. Set $M \equiv L^2(\Omega)$. Let Q be a function space defined by

$$Q \equiv \Big\{ q \in M;\ \int_\Omega q\, dx = 0 \Big\}.$$

For each $\omega \in H^{1/2}(\Gamma)$ let $\Psi(\omega)$ be an affine space of X defined by

$$\Psi(\omega) \equiv \{\psi \in X;\ \psi - \theta_\omega \in \Psi\},$$

where $\theta_\omega \in X$ is an extension of ω. Likewise, for each $w \in H^{1/2}(\Gamma)^d$ let $V(w)$ be an affine space of Y defined by

$$V(w) \equiv \{v \in Y;\ v - u_w \in V\},$$

where $u_w \in Y$ is an extension of w.

For real numbers s_0 and s_1 $(0 \le s_0 < s_1)$ and a Banach space Z we define in the usual way the Z-valued function spaces $L^p(s_0, s_1; Z)$, $W^{k,p}(s_0, s_1; Z)$, and $\mathscr{C}^m([s_0, s_1]; Z)$. We denote by $\|\,.\,\|_{L^p(s_0,s_1;Z)}$ the norm of $L^p(s_0, s_1; Z)$. Similarly, we define the Z-valued distribution space $\mathscr{D}'(s_0, s_1; Z)$.

Now we give the following propositions on the maximum principle for (1c) as well as the existence and uniqueness result for (1). They were obtained by [13] in a slightly different setting.

Proposition 1 *Suppose that the boundary Γ consists of connected components Γ_i, $1 \le i \le I$. Assume that the given functions f, g, u_D, θ_D, u^0, and θ^0 satisfy*

$$f \in L^2\big(0,T; H^{-1}(\Omega)^d\big), \quad g \in L^\infty\big(0,T; L^\infty(\Omega)\big), \tag{2a}$$

$$u_D \in H^1\big(0,T; H^{1/2}(\Gamma)^d\big), \quad \int_{\Gamma_i} u_D \cdot n \, ds = 0 \ \text{ for } \ 1 \le i \le I, \tag{2b}$$

$$\theta_D \in H^1\big(0,T; H^{1/2}(\Gamma)\big) \cap L^\infty\big(0,T; L^\infty(\Gamma)\big), \tag{2c}$$

$$u^0 \in V\big(u_D(\,.\,,0)\big), \quad \theta^0 \in \Psi\big(\theta_D(\,.\,,0)\big) \cap L^\infty(\Omega). \tag{2d}$$

Furthermore, assume that the given functions ν, κ, and β satisfy

$$\nu,\ \kappa \in \mathscr{C}(\overline{\Omega} \times [0,T] \times \mathbb{R};\, \mathbb{R}^+), \tag{3a}$$

$$\beta \in \mathscr{C}(\overline{\Omega} \times [0,T] \times \mathbb{R};\, \mathbb{R}^d). \tag{3b}$$

Then, the problem (1) has at least one solution (u, p, θ) such that

$$u \in L^\infty\big(0,T; L^2(\Omega)^d\big) \cap L^2\big(0,T; V(u_D)\big), \quad p \in \mathscr{D}'(0,T; M), \tag{4a}$$

$$\theta \in L^\infty\big(0,T; L^\infty(\Omega)\big) \cap L^2\big(0,T; \Psi(\theta_D)\big). \tag{4b}$$

Proposition 2 *Under the assumptions of Proposition 1 the temperature θ obtained from (1) satisfies for all $t \in (0,T]$*

$$\|\theta(t)\|_{L^\infty(\Omega)} \le t\|g\|_{L^\infty(0,\,t;\,L^\infty(\Omega))} + \max\big\{\|\theta_D\|_{L^\infty(0,\,t;\,L^\infty(\Gamma))}, \|\theta^0\|_{L^\infty(\Omega)}\big\}. \tag{5}$$

Proposition 3 *Replacing (3) by*

$$\nu,\ \kappa \in \mathscr{C}^{0,1}(\overline{\Omega} \times [0,T] \times \mathbb{R};\, \mathbb{R}^+), \tag{6a}$$

$$\beta \in \mathscr{C}^{0,1}(\overline{\Omega} \times [0,T] \times \mathbb{R};\, \mathbb{R}^d), \tag{6b}$$

we keep the other assumptions in Proposition 1. In addition, we suppose that the velocity u and the temperature θ obtained from (1) satisfy

$$u \in L^2\big(0,T; W^{1,\infty}(\Omega)^d\big), \qquad \theta \in L^2\big(0,T; W^{1,\infty}(\Omega)\big). \tag{7}$$

Then, the solution of (1) is unique.

3 Finite element approximation

Let h be a space discretization parameter tending to 0 and $\{\Omega_h\}_{h\downarrow 0}$ a sequence of approximate domains to Ω. We introduce finite dimensional spaces X_h and M_h approximating X and M, respectively. Let Ψ_h be a subspace of X_h approximating Ψ. Let V_h be a subspace of $Y_h \equiv X_h^d$ approximating V. Let Q_h be a subspace of M_h consisting of functions whose mean values vanish over Ω_h. We employ $H^1(\Omega_h)$-norm for X_h, $H^1(\Omega_h)^d$-norm for Y_h, and $L^2(\Omega_h)$-norm for M_h, respectively. Since we use conforming finite elements, the Korn inequality

$$\inf_{v_h \in V_h} \frac{1}{\|v_h\|_1^2} \int_{\Omega_h} |D(v_h)|^2 \, dx \ge \alpha^* \tag{8}$$

and the Poincarè inequality

$$\inf_{\psi_h \in \Psi_h} \frac{1}{\|\psi_h\|_1^2} \int_{\Omega_h} |\nabla \psi_h|^2 \, dx \geq \gamma^* \tag{9}$$

hold for all h, where α^* and γ^* are positive constants independent of h.

The spaces V_h, Q_h, and X_h are supposed to satisfy "inf-sup condition", "kth order approximability", and "inverse inequality" as follows:

Hypothesis 1 *There exist positive constants h_0 and β^* such that*

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{1}{\|v_h\|_1 \, \|q_h\|_0} \int_{\Omega_h} q_h \nabla \cdot v_h \, dx \geq \beta^* \quad \text{for all } h \in (0, h_0].$$

Hypothesis 2 *There exist a positive integer k and two operators $\Pi_{X_h} \in \mathscr{L}(H^1(\Omega); X_h)$ and $\Pi_{Q_h} \in \mathscr{L}(Q; Q_h)$ such that for any integer $\ell \in [0, k]$*

$$\| \psi - \Pi_{X_h} \psi \|_0 + h \| \psi - \Pi_{X_h} \psi \|_1 \leq c \, h^{\ell+1} \, \|\psi\|_{\ell+1} \quad \text{for all } \psi \in H^{\ell+1}(\Omega),$$
$$\| q - \Pi_{Q_h} q \|_0 \leq c \, h^{\ell} \, \|q\|_{\ell} \quad \text{for all } q \in Q \cap H^{\ell}(\Omega).$$

Hypothesis 3 *For any integers ℓ and m $(0 \leq \ell \leq m \leq 1)$ and any real numbers p and q $(1 \leq p \leq q \leq +\infty)$ it holds that*

$$\|\psi_h\|_{m,\,q} \leq c \, h^{\ell - m + d(1/q - 1/p)} \, \|\psi_h\|_{\ell,\,p} \qquad \text{for all } \psi_h \in X_h.$$

Remark 1 In the finite element method every integral over Ω is replaced by that over Ω_h. In this paper we use the same notation for these two integrals, for example, a_0 that appears later is used for a trilinear form over Ω as well as Ω_h. Furthermore, precisely speaking, ψ in the left-hand side of the inequality in Hypothesis 2 should be replaced by an extension $\widetilde{\psi} \in H^{\ell+1}(\Omega \cup \Omega_h)$ of ψ. Such an extension is not written explicitly but we understand that a suitable extension is done. Errors caused by this difference of the domains can be proved to be less than approximation errors by finite element spaces under an appropriate condition.

Remark 2 We give an example of finite elements satisfying Hypotheses 1–3. We consider a uniformly regular triangulation of a polygonal or polyhedral region. If the $P2/P1/P2$ elements are taken for the velocity, the pressure, and the temperature, then Hypotheses 1–3 with $k = 2$ hold. It still holds for general domains with piecewise regular boundaries if we use the isoparametric transformation and if h is less than a positive constant h_0. We refer to Ciarlet [4] and Tabata [16] for the isoparametric transformation and the uniform inf-sup condition in approximate domains, respectively. For other choices of elements see Girault and Raviart [8], and references their in.

Remark 3 The inequality (8) and Hypothesis 1 lead to the uniform solvability of the corresponding Stokes problem in approximate domains. So does the inequality (9) of the corresponding Poisson problem. It is well-known that Hypothesis 1, which concerns the choice of a pair of finite elements for the velocity and the pressure, plays an important role in the discretization. On the other hand, the choice of a pair of finite elements for the velocity and the temperature is less restrictive. In fact, it is not necessary to use finite element spaces derived from the same X_h. However, to avoid complicated notation and arguments, we choose the finite elements as above.

For each $\omega \in H^{1/2}(\Gamma)$ we define a finite element affine space $\Psi_h(\omega)$ by

$$\Psi_h(\omega) \equiv \{\psi_h \in X_h; \ \psi_h - \Pi_{X_h} \theta_\omega \in \Psi_h\}$$

and for each $w \in H^{1/2}(\Gamma)^d$ a finite element affine space $V_h(w)$ by

$$V_h(w) \equiv \{v_h \in Y_h; \ v_h - \Pi_{Y_h} u_w \in V_h\},$$

where $\Pi_{Y_h} \equiv \Pi^d_{X_h}$. We prepare bi- and tri-linear forms a_0, c_0, b, a_1, and c_1 defined by

$$a_0(\nu; u, v) \equiv 2\int_\Omega \nu\, D(u) : D(v)\, dx \qquad \text{for } (\nu, u, v) \in L^\infty(\Omega) \times Y \times Y,$$

$$c_0(\kappa; \theta, \psi) \equiv \int_\Omega \kappa\, \nabla\theta \cdot \nabla\psi\, dx \qquad \text{for } (\kappa, \theta, \psi) \in L^\infty(\Omega) \times X \times X,$$

$$b(v, q) \equiv -\int_\Omega q\, \nabla\cdot v\, dx \qquad \text{for } (v, q) \in Y \times M,$$

$$a_1(w, u, v) \equiv \frac{1}{2}\left\{\int_\Omega \big[(w\cdot\nabla)u\big]\cdot v\, dx - \int_\Omega \big[(w\cdot\nabla)v\big]\cdot u\, dx\right\} \quad \text{for } (w, u, v) \in Y \times Y \times Y,$$

$$c_1(w, \theta, \psi) \equiv \frac{1}{2}\left\{\int_\Omega (w\cdot\nabla\theta)\,\psi\, dx - \int_\Omega (w\cdot\nabla\psi)\,\theta\, dx\right\} \quad \text{for } (w, \theta, \psi) \in Y \times X \times X.$$

Let τ be a time increment and $N_T \equiv [T/\tau]$ a total step number. The time $n\tau$ is denoted by t_n. We denote by u^n the value $u(n\tau)$ at time step n and by $\overline{D}_\tau u^n$ the backward difference quotient $(u^n - u^{n-1})/\tau$. Set $\nu^n(\psi) \equiv \nu(x, t_n, \psi(x))$, and $\kappa^n(\psi)$ and $\beta^n(\psi)$ are defined similarly. We introduce the time discrete space $\ell^p(Z)$ associated with $L^p(0, T; Z)$; $\ell^p(Z)$ is the space of Z-valued sequences $w \equiv \{w_n;\ n = 1, \ldots, N_T\}$ with norm $\|\,.\,\|_{\ell^p(Z)}$ defined by

$$\|w\|_{\ell^p(Z)} \equiv \begin{cases} \displaystyle\left(\tau\sum_{n=1}^{N_T} \|w_n\|_Z^p\right)^{1/p} & \text{if } 1 \le p < +\infty, \\ \max\big\{\|w_n\|_Z;\ n = 1, \ldots, N_T\big\} & \text{if } p = +\infty. \end{cases}$$

Hereafter, the boundary data u_D and θ_D are assumed to be independent of time. We introduce an approximate problem discretized by the backward Euler method in time and by the finite element method in space: Setting $u_h^0 \in V_h(u_D)$ and $\theta_h^0 \in \Psi_h(\theta_D)$ as approximations to u^0 and θ^0, respectively, we find $\big\{(u_h^n, p_h^n, \theta_h^n) \in V_h(u_D) \times Q_h \times \Psi_h(\theta_D);\ n = 1, \ldots, N_T\big\}$ such that for $n = 1, \ldots, N_T$

$$\begin{cases} (\overline{D}_\tau u_h^n, v_h) + a_0\big(\nu^n(\theta_h^{n-1}); u_h^n, v_h\big) + a_1(u_h^{n-1}, u_h^n, v_h) \\ \qquad\qquad + b(v_h, p_h^n) - \big(\beta^n(\theta_h^{n-1})\,\theta_h^{n-1}, v_h\big) = (f^n, v_h) & \text{for all } v_h \in V_h, \text{ (10a)} \\ b(u_h^n, q_h) = 0 & \text{for all } q_h \in Q_h, \text{(10b)} \\ (\overline{D}_\tau \theta_h^n, \psi_h) + c_0\big(\kappa^n(\theta_h^{n-1}); \theta_h^n, \psi_h\big) + c_1(u_h^{n-1}, \theta_h^n, \psi_h) = (g^n, \psi_h) & \text{for all } \psi_h \in \Psi_h. \text{ (10c)} \end{cases}$$

From the maximum principle (5) the temperature θ of (1) is uniformly bounded. Therefore, we can modify the functions ν, κ, and β so as to satisfy the inequalities

$$\nu_0 \le \nu(x, t, \xi) \le \nu_1, \quad \kappa_0 \le \kappa(x, t, \xi) \le \kappa_1, \quad \text{and} \quad |\beta(x, t, \xi)| \le \beta_1 \quad \text{for all } (x, t, \xi) \in \overline{\Omega} \times [0, T] \times \mathbb{R} \qquad (11)$$

with positive constants ν_0, ν_1, κ_0, κ_1, and β_1. Thus, we can suppose additional conditions (11) on ν, κ, and β without loss of generality.

Remark 4 The approximate problem (10) consists of two parts; one is a generalized Navier–Stokes part (10a) and (10b), and the other a generalized convection-diffusion part (10c). Since these are linear and completely separated at every time step n, we can solve them by virtue of (11) and obtain the solution $\{(u_h^n, p_h^n, \theta_h^n);\ n = 1, \ldots, N_T\}$ step by step from u_h^0 and θ_h^0.

Now we state the main results.

Theorem 1 *Suppose that Hypotheses* 1–3 *hold with a positive number* h_0 *and a positive integer* k*, that the solution* (u, p, θ) *of* (1) *satisfies*

$$u \in \mathscr{C}\big([0,T]; V(u_D) \cap W^{1,\infty}(\Omega)^d\big) \cap H^1\big(0,T; H^{k+1}(\Omega)^d\big) \cap H^2\big(0,T; L^2(\Omega)^d\big), \tag{12a}$$

$$p \in \mathscr{C}\big([0,T]; Q \cap H^k(\Omega)\big), \tag{12b}$$

$$\theta \in \mathscr{C}\big([0,T]; \Psi(\theta_D) \cap W^{1,\infty}(\Omega)\big) \cap H^1\big(0,T; H^{k+1}(\Omega)\big) \cap H^2\big(0,T; L^2(\Omega)\big), \tag{12c}$$

that the functions ν*,* κ*, and* β *satisfy* (6) *and* (11)*, and that the initial conditions* u_h^0 *and* θ_h^0 *satisfy*

$$\|u^0 - u_h^0\|_0, \quad \|\theta^0 - \theta_h^0\|_0 \le c^* h^k. \tag{13}$$

Then, for any $h \in (0, h_0]$ *the velocity* u_h *and the temperature* θ_h *of* (10) *satisfy*

$$\|u - u_h\|_{\ell^\infty(L^2)\cap\ell^2(H^1)}, \ \|\theta - \theta_h\|_{\ell^\infty(L^2)\cap\ell^2(H^1)} \le c^*(\tau + h^k). \tag{14}$$

Theorem 2 *Under the assumptions in Theorem* 1 *the pressure* p_h *of* (10) *satisfies*

$$\|p - p_h\|_{\ell^2(L^2)} \le c^* \tau^{-1/2}(\tau + h^k). \tag{15}$$

Taking $\tau = ch^k$*, we have*

$$\|p - p_h\|_{\ell^2(L^2)} \le c^* h^{k/2}. \tag{16}$$

If the functions ν and κ are independent of t and θ, then we can prove optimal estimates of the time derivatives of the velocity and the temperature, which improves the estimate of the pressure to the best possible.

Corollary 1 *Replacing* (6a) *by*

$$\nu, \ \kappa \in \mathscr{C}^{0,1}(\overline{\Omega}; \mathbb{R}^+) \tag{17}$$

and (13) *by*

$$\|u^0 - u_h^0\|_1, \quad \|\theta^0 - \theta_h^0\|_1 \le c^* h^k, \tag{18a}$$

$$b(u_h^0, q_h) = 0 \quad \textit{for all } q_h \in Q_h, \tag{18b}$$

we keep the other assumptions in Theorem 1*. Then, for any* $h \in (0, h_0]$ *the velocity* u_h *and the temperature* θ_h *of* (10) *satisfy*

$$\|\partial_t u - \overline{D}_\tau u_h\|_{\ell^2(L^2)}, \ \|u - u_h\|_{\ell^\infty(H^1)}, \\ \|\partial_t \theta - \overline{D}_\tau \theta_h\|_{\ell^2(L^2)}, \ \|\theta - \theta_h\|_{\ell^\infty(H^1)} \le c^*(\tau + h^k). \tag{19}$$

Corollary 2 *Under the assumptions in Corollary* 1 *the pressure* p_h *of* (10) *satisfies*

$$\|p - p_h\|_{\ell^2(L^2)} \le c^*(\tau + h^k). \tag{20}$$

Remark 5 If we choose the approximate initial value u_h^0 as the first component of the Stokes projection of $(u^0, 0)$ defined by (21), the condition (18) is satisfied by virtue of Lemma 1.

4 Proof of the error estimates

To begin with, we prepare some tools used frequently in the following.

Let $\overline{\nu}$ be a function in $\mathscr{C}^{0,1}(\overline{\Omega})$ such that the inequalities

$$\overline{\nu}_0 \le \overline{\nu}(x) \le \overline{\nu}_1 \quad \text{for all } x \in \overline{\Omega}$$

hold with positive constants $\overline{\nu}_0$ and $\overline{\nu}_1$. For each pair $(u,p) \in Y \times Q$ we define a Stokes projection $(w_h, r_h) \in V_h(u|_\Gamma) \times Q_h$ to be the finite element solution of the problem:

$$\begin{cases} a_0(\overline{\nu}; w_h, v_h) + b(v_h, r_h) = a_0(\overline{\nu}; u, v_h) + b(v_h, p) & \text{for all } v_h \in V_h, \quad (21a) \\ b(w_h, q_h) = b(u, q_h) & \text{for all } q_h \in Q_h. \quad (21b) \end{cases}$$

This projection has the convergence and boundedness result. The result is proved by a similar argument to that in the case of the constant coefficient; see for example [8] and [9].

Lemma 1 *Suppose that Hypotheses* 1 *and* 2 *hold with a positive number* h_0 *and a positive integer* k*, and that* (u,p) *belongs to* $H^{k+1}(\Omega)^d \times \big(Q \cap H^k(\Omega)\big)$*. Then, for any* $h \in (0, h_0]$ *the Stokes projection* (w_h, r_h) *of* (u,p) *satisfies*

$$\|u - w_h\|_1 + \|p - r_h\|_0 \le c\, h^k \big(\|u\|_{k+1} + \|p\|_k\big). \quad (22)$$

Moreover, suppose that Hypothesis 3 *holds. Then, the first component* w_h *satisfies*

$$\|w_h\|_{0,\infty} + \|w_h\|_{1,3} \le c\,\big(\|u\|_2 + \|p\|_1\big). \quad (23)$$

Remark 6 The boundedness (23) can be extended to the following, though not used in this paper: for $1 \le q < +\infty$ $(d = 2)$, or for $1 \le q \le 6$ $(d = 3)$ it holds

$$\|w_h\|_{1,q} \le c\,\big(\|u\|_2 + \|p\|_1\big).$$

In proving Theorem 1, we need estimates of the nonlinear terms arising from temperature-dependent coefficients. The following lemma is often used to deal with them.

Lemma 2 *Let* ϕ_i *and* θ_i*,* $i = 1, 2$*, and* ψ *be functions in* $L^2(\Omega)$*, and* λ *a function in* $\mathscr{C}^{0,1}(\overline{\Omega} \times \mathbb{R}; \mathbb{R})$*. Then, for any* $\overline{\phi} \in L^\infty(\Omega)$ *it holds that*

$$\begin{aligned} &\Big| \int_\Omega \lambda(\,.\,, \theta_1)\, \phi_1\, \psi\, dx \; - \int_\Omega \lambda(\,.\,, \theta_2)\, \phi_2\, \psi\, dx \Big| \\ &\qquad \le \max\big\{ \|\lambda\|_{\mathscr{C}(\overline{\Omega}\times\mathbb{R};\,\mathbb{R})},\ \|\overline{\phi}\|_{0,\infty} |\lambda|_{\mathscr{C}^{0,1}(\overline{\Omega}\times\mathbb{R};\,\mathbb{R})} \big\} \\ &\qquad\qquad \times \big(\|\theta_1 - \theta_2\|_0 + \|\phi_1 - \overline{\phi}\|_0 + \|\overline{\phi} - \phi_2\|_0 \big) \|\psi\|_0, \end{aligned} \quad (24)$$

where $|\,.\,|_{\mathscr{C}^{0,1}(\overline{\Omega}\times\mathbb{R};\,\mathbb{R})}$ *is defined by*

$$|\lambda|_{\mathscr{C}^{0,1}(\overline{\Omega}\times\mathbb{R};\,\mathbb{R})} \equiv \sup\Big\{ \frac{|\lambda(x,\theta) - \lambda(y,\psi)|}{|(x,\theta) - (y,\psi)|};\ (x,\theta), (y,\psi) \in \overline{\Omega} \times \mathbb{R} \Big\}.$$

Proof. From the mean value theorem and the Hölder inequality the left-hand side of (24) has the bound

$$\begin{aligned} &\int_\Omega \lambda(\,.\,, \theta_1)\phi_1\, \psi\, dx - \int_\Omega \lambda(\,.\,, \theta_2)\phi_2\, \psi\, dx \\ &\qquad = \int_\Omega \lambda(\,.\,, \theta_1)(\phi_1 - \overline{\phi})\, \psi\, dx \\ &\qquad\qquad + \int_\Omega \big\{ \lambda(\,.\,, \theta_1) - \lambda(\,.\,, \theta_2) \big\}\, \overline{\phi}\, \psi\, dx + \int_\Omega \lambda(\,.\,, \theta_2)(\overline{\phi} - \phi_2)\, \psi\, dx \\ &\qquad \le \Big\{ \|\lambda\|_{\mathscr{C}(\overline{\Omega}\times\mathbb{R};\,\mathbb{R})} \|\phi_1 - \overline{\phi}\|_0 + |\lambda|_{\mathscr{C}^{0,1}(\overline{\Omega}\times\mathbb{R};\,\mathbb{R})} \|\theta_1 - \theta_2\|_0 \|\overline{\phi}\|_{0,\infty} \\ &\qquad\qquad + \|\lambda\|_{\mathscr{C}(\overline{\Omega}\times\mathbb{R};\,\mathbb{R})} \|\overline{\phi} - \phi_2\|_0 \Big\} \|\psi\|_0, \end{aligned}$$

which leads to (24). □

The following lemma plays an important role to obtain optimal estimates for $\partial_t u$ and $\partial_t \theta$.

Lemma 3 *Let u_h, v_h, and w_h be functions in Y_h with $(w_h\cdot n)\,(u_h\cdot v_h) = 0$ on $\partial\Omega_h$. Then, for any positive number τ the inequality*

$$a_1(w_h, u_h, v_h) \le c\,\min(h^{-d/6},\, \tau^{-1/2})\|w_h\|_1\|u_h\|_1(\|v_h\|_0 + \sqrt{\tau}\,\|v_h\|_1) \tag{25}$$

holds.

Proof. As in [9] the inequality

$$a_1(w_h, u_h, v_h) \le c\,\big(\|w_h\|_{0,\infty} + \|w_h\|_{1,3}\big)\|u_h\|_1\|v_h\|_0$$

holds. From the inverse inequality in Hypothesis 3 and the Sobolev inequality, the estimate

$$\|w_h\|_{0,\infty} + \|w_h\|_{1,3} \le c\,h^{-d/6}\big(\|w_h\|_{0,6} + \|w_h\|_1\big) \le c\,h^{-d/6}\|w_h\|_1$$

holds. Therefore we have

$$a_1(w_h, u_h, v_h) \le c\,h^{-d/6}\|w_h\|_1\|u_h\|_1\|v_h\|_0. \tag{26}$$

Combining (26) with a trivial inequality

$$a_1(w_h, u_h, v_h) \le c\,\|w_h\|_1\|u_h\|_1\|v_h\|_1 \le c\,\tau^{-1/2}\,\|w_h\|_1\|u_h\|_1(\sqrt{\tau}\,\|v_h\|_1),$$

we conclude (25). □

We are going to prove the main results of this paper.

Proof of Theorem 1. Let (w_h^n, r_h^n) be the Stokes projection of $(u^n,\, p^n)$ with $\overline{\nu} \equiv \nu^n(\theta_h^{n-1})$, and set $\omega_h^n \equiv \mathit{\Pi}_{X_h}\theta^n$. Let $(e_{1h}^n, e_{2h}^n, e_{3h}^n)$ be the errors defined by

$$(e_{1h}^n, e_{2h}^n, e_{3h}^n) \equiv (u_h^n - w_h^n, p_h^n - r_h^n, \theta_h^n - \omega_h^n).$$

From (1), (10), and (21) the errors $(e_{1h}^n, e_{2h}^n, e_{3h}^n)$ satisfy the equations

$$\begin{cases} \big(\overline{D}_\tau e_{1h}^n, v_h\big) + a_0\big(\nu^n(\theta_h^{n-1}); e_{1h}^n, v_h\big) + b(v_h, e_{2h}^n) \\ \qquad\qquad = \langle R_{L1}{}_h^n, v_h\rangle + \langle R_{N1}{}_h^n, v_h\rangle & \text{for all } v_h \in V_h, \quad (27a) \\ b(e_{1h}^n, q_h) = 0 & \text{for all } q_h \in Q_h, \quad (27b) \\ \big(\overline{D}_\tau e_{3h}^n, \psi_h\big) + c_0\big(\kappa^n(\theta_h^{n-1}); e_{3h}^n, \psi_h\big) = \langle R_{L2}{}_h^n, \psi_h\rangle + \langle R_{N2}{}_h^n, \psi_h\rangle & \text{for all } \psi_h \in \Psi_h \quad (27c) \end{cases}$$

at each time step n, where $R_{L1}{}_h^n$, $R_{L2}{}_h^n$, $R_{N1}{}_h^n$, and $R_{N2}{}_h^n$ are the linear and the nonlinear remainder terms defined by

$$\begin{aligned} R_{L1}{}_h^n &\equiv \partial_t u^n - \overline{D}_\tau w_h^n, \\ R_{N1}{}_h^n &\equiv \big\{a_0\big(\nu^n(\theta^n); u^n, .\,\big) - a_0\big(\nu^n(\theta_h^{n-1}); u^n, .\,\big)\big\} \\ &\qquad + \big\{a_1(u^n, u^n, .\,) - a_1(u_h^{n-1}, u_h^n, .\,)\big\} \\ &\qquad + \big\{\big(\beta^n(\theta_h^{n-1})\,\theta_h^{n-1}, .\,\big) - \big(\beta^n(\theta^n)\,\theta^n, .\,\big)\big\} \\ &\equiv R_{D1}{}_h^n + R_{C1}{}_h^n + R_{B}{}_h^n, \\ R_{L2}{}_h^n &\equiv \partial_t\theta^n - \overline{D}_\tau\omega_h^n, \\ R_{N2}{}_h^n &\equiv \big\{c_0\big(\kappa^n(\theta^n); \theta^n, .\,\big) - c_0\big(\kappa^n(\theta_h^{n-1}); \omega_h^n, .\,\big)\big\} + \big\{c_1(u^n, \theta^n, .\,) - c_1(u_h^{n-1}, \theta_h^n, .\,)\big\} \\ &\equiv R_{D2}{}_h^n + R_{C2}{}_h^n. \end{aligned}$$

Substituting $(e_{1h}^n, e_{2h}^n, e_{3h}^n) \in V_h \times Q_h \times \Psi_h$ into (v_h, q_h, ψ_h) in (27), we find

$$\begin{cases} \big(\overline{D}_\tau e_{1h}^n, e_{1h}^n\big) + a_0\big(\nu^n(\theta_h^{n-1}); e_{1h}^n, e_{1h}^n\big) + b(e_{1h}^n, e_{2h}^n) = \langle {R_{L1}}_h^n, e_{1h}^n\rangle + \langle {R_{N1}}_h^n, e_{1h}^n\rangle, & \text{(28a)} \\ b(e_{1h}^n, e_{2h}^n) = 0, & \text{(28b)} \\ \big(\overline{D}_\tau e_{3h}^n, e_{3h}^n\big) + c_0\big(\kappa^n(\theta_h^{n-1}); e_{3h}^n, e_{3h}^n\big) = \langle {R_{L2}}_h^n, e_{3h}^n\rangle + \langle {R_{N2}}_h^n, e_{3h}^n\rangle. & \text{(28c)} \end{cases}$$

From (8), (9), (11), (28b), and the elementary identity

$$2b(b-a) = b^2 - a^2 + (b-a)^2 \qquad \text{for all } \ a, b \in \mathbb{R}$$

we obtain

$$\begin{aligned} &\big(\overline{D}_\tau e_{1h}^n, e_{1h}^n\big) + a_0\big(\nu^n(\theta_h^{n-1}); e_{1h}^n, e_{1h}^n\big) + b(e_{1h}^n, e_{2h}^n) \\ &\qquad \geq \frac{1}{2}\big(\overline{D}_\tau \|e_{1h}^n\|_0^2 + \tau \|\overline{D}_\tau e_{1h}^n\|_0^2\big) + \nu_0 \alpha^* \|e_{1h}^n\|_1^2, \qquad \text{(29a)} \\ &\big(\overline{D}_\tau e_{3h}^n, e_{3h}^n\big) + c_0\big(\kappa^n(\theta_h^{n-1}); e_{3h}^n, e_{3h}^n\big) \\ &\qquad \geq \frac{1}{2}\big(\overline{D}_\tau \|e_{3h}^n\|_0^2 + \tau \|\overline{D}_\tau e_{3h}^n\|_0^2\big) + \kappa_0 \gamma^* \|e_{3h}^n\|_1^2. \qquad \text{(29b)} \end{aligned}$$

Moreover, the conventional estimate of the time-difference quotient leads to

$$\begin{aligned} &\langle {R_{L1}}_h^n, e_{1h}^n\rangle \\ &\qquad \leq c\left\{\sqrt{\tau}\, \|\partial_t^2 u\|_{L^2(t_{n-1},\, t_n;\, L^2)} + \frac{h^k}{\sqrt{\tau}} \|(\partial_t u, \partial_t p)\|_{L^2(t_{n-1},\, t_n;\, H^{k+1}\times H^k)}\right\} \|e_{1h}^n\|_0, \qquad \text{(30a)} \\ &\langle {R_{L2}}_h^n, e_{3h}^n\rangle \leq c\left\{\sqrt{\tau}\, \|\partial_t^2 \theta\|_{L^2(t_{n-1},\, t_n;\, L^2)} + \frac{h^k}{\sqrt{\tau}} \|\partial_t \theta\|_{L^2(t_{n-1},\, t_n;\, H^k)}\right\} \|e_{3h}^n\|_0. \qquad \text{(30b)} \end{aligned}$$

From Lemma 2 with

$$\begin{aligned} &(\phi_1, \overline{\phi}, \phi_2) = \big(D(u^n), D(u^n), D(u^n)\big), \\ &(\theta_1, \theta_2) = (\theta^n,\, \theta_h^{n-1}), \quad \text{and} \quad (\lambda, \psi) = \big(\nu^n,\, D(e_{1h}^n)\big) \end{aligned}$$

we have

$$\begin{aligned} \langle {R_{D1}}_h^n, e_{1h}^n\rangle &\leq \max\big\{\|\nu\|_{\mathscr{C}(\overline{\Omega}\times\mathbb{R};\,\mathbb{R}^+)},\, \|D(u^n)\|_{0,\,\infty} |\nu|_{\mathscr{C}^{0,1}(\overline{\Omega}\times\mathbb{R};\,\mathbb{R})}\big\} \|\theta^n - \theta_h^{n-1}\|_0 \|D(e_{1h}^n)\|_0 \\ &\leq c^*\big(\sqrt{\tau}\, \|\partial_t \theta\|_{L^2(t_{n-1},t_n;L^2)} + h^k \|\theta\|_{\mathscr{C}([t_{n-1},t_n];H^k)} + \|e_{3h}^{n-1}\|_0\big) \|e_{1h}^n\|_1. \qquad \text{(31a)} \end{aligned}$$

Similarly, from Lemma 2 with

$$\begin{aligned} &(\phi_1, \overline{\phi}, \phi_2) = (\nabla\theta^n, \nabla\theta^n, \nabla\omega_h^n), \\ &(\theta_1, \theta_2) = (\theta^n,\, \theta_h^{n-1}), \quad \text{and} \quad (\lambda, \psi) = \big(\kappa^n,\, \nabla e_{3h}^n\big) \end{aligned}$$

we have

$$\begin{aligned} \langle {R_{D2}}_h^n, e_{3h}^n\rangle &\leq \max\big\{\|\kappa\|_{\mathscr{C}(\overline{\Omega}\times\mathbb{R};\,\mathbb{R}^+)},\, \|\nabla\theta^n\|_{0,\,\infty} |\kappa|_{\mathscr{C}^{0,1}(\overline{\Omega}\times\mathbb{R};\,\mathbb{R})}\big\} \\ &\qquad \times \big(\|\theta^n - \theta_h^{n-1}\|_0 + \|\nabla(\theta^n - \omega_h^n)\|_0\big) \|\nabla e_{3h}^n\|_0 \\ &\leq c^*\big(\sqrt{\tau}\, \|\partial_t \theta\|_{L^2(t_{n-1},t_n;L^2)} + h^k \|\theta\|_{\mathscr{C}([t_{n-1},t_n];H^{k+1})} + \|e_{3h}^{n-1}\|_0\big) \|e_{3h}^n\|_1. \qquad \text{(31b)} \end{aligned}$$

Again, from Lemma 2 with

$$\begin{aligned} &(\phi_1, \overline{\phi}, \phi_2) = (\theta^n,\, \theta^{n-1},\, \theta_h^{n-1}), \\ &(\theta_1, \theta_2) = (\theta^n,\, \theta_h^{n-1}), \quad \text{and} \quad (\lambda, \psi) = \big(\beta^n,\ e_{1h}^n\big) \end{aligned}$$

we have

$$\begin{aligned}\langle R_{B}{}_h^n, e_{1h}^n\rangle &\le \max\{\|\beta\|_{\mathscr{C}(\overline{\Omega}\times\mathbb{R};\,\mathbb{R}^d)},\ \|\theta^{n-1}\|_{0,\infty}|\beta|_{\mathscr{C}^{0,1}(\overline{\Omega}\times\mathbb{R};\,\mathbb{R}^d)}\}\\ &\qquad\times\big(\|\theta^n-\theta_h^{n-1}\|_0+\|\theta^n-\theta^{n-1}\|_0+\|\theta^{n-1}-\theta_h^{n-1}\|_0\big)\|e_{1h}^n\|_0\\ &\le c^*\big(\sqrt{\tau}\,\|\partial_t\theta\|_{L^2(t_{n-1},t_n;L^2)}+h^k\|\theta\|_{\mathscr{C}([t_{n-1},t_n];H^k)}+\|e_{3h}^{n-1}\|_0\big)\|e_{1h}^n\|_0. \qquad (32)\end{aligned}$$

For the estimates of the convection terms we apply Lemma 4.5 in [19] to obtain

$$\begin{aligned}\langle R_{C1}{}_h^n, e_{1h}^n\rangle &\le c^*\big(\sqrt{\tau}\,\|\partial_t u\|_{L^2(t_{n-1},\,t_n;\,L^2)}\\ &\qquad+h^k\|(u,p)\|_{\mathscr{C}([t_{n-1},\,t_n];\,H^{k+1}\times H^k)}+\|e_{1h}^{n-1}\|_0\big)\|e_{1h}^n\|_1, \qquad (33a)\\ \langle R_{C2}{}_h^n, e_{3h}^n\rangle &\le c^*\big(\sqrt{\tau}\,\|\partial_t u\|_{L^2(t_{n-1},\,t_n;\,L^2)}+h^k\|(u,p)\|_{\mathscr{C}([t_{n-1},\,t_n];\,H^{k+1}\times H^k)}\\ &\qquad+h^k\|\theta\|_{\mathscr{C}([t_{n-1},\,t_n];\,H^k)}+\|e_{1h}^{n-1}\|_0\big)\|e_{3h}^n\|_1. \qquad (33b)\end{aligned}$$

Combining (31), (32), and (33), we find that the nonlinear remainder terms $R_{N1}{}_h^n$ and $R_{N2}{}_h^n$ are bounded as follows:

$$\begin{aligned}\langle R_{N1}{}_h^n, e_{1h}^n\rangle &\le c^*\big\{\sqrt{\tau}\,\|\partial_t u\|_{L^2(t_{n-1},\,t_n;\,L^2)}+\sqrt{\tau}\,\|\partial_t\theta\|_{L^2(t_{n-1},\,t_n;\,L^2)}\\ &\qquad+h^k\|(u,p)\|_{\mathscr{C}([t_{n-1},\,t_n];\,H^{k+1}\times H^k)}+h^k\|\theta\|_{\mathscr{C}([t_{n-1},\,t_n];\,H^k)}\\ &\qquad+\|e_{1h}^{n-1}\|_0+\|e_{3h}^{n-1}\|_0\big\}\|e_{1h}^n\|_1, \qquad (34a)\\ \langle R_{N2}{}_h^n, e_{3h}^n\rangle &\le c^*\big\{\sqrt{\tau}\,\|\partial_t u\|_{L^2(t_{n-1},\,t_n;\,L^2)}+\sqrt{\tau}\,\|\partial_t\theta\|_{L^2(t_{n-1},\,t_n;\,L^2)}\\ &\qquad+h^k\|(u,p)\|_{\mathscr{C}([t_{n-1},\,t_n];\,H^{k+1}\times H^k)}+h^k\|\theta\|_{\mathscr{C}([t_{n-1},\,t_n];\,H^{k+1})}\\ &\qquad+\|e_{1h}^{n-1}\|_0+\|e_{3h}^{n-1}\|_0\big\}\|e_{3h}^n\|_1, \qquad (34b)\end{aligned}$$

Collecting (12), (28)–(30), (34), and the elementary inequality

$$ab\le\frac{1}{4\varepsilon}a^2+\varepsilon b^2\qquad\text{for all } a,b\in\mathbb{R}$$

with $\varepsilon=\min\{\nu_0\alpha^*,\,\kappa_0\gamma^*\}/4$, we get

$$\begin{aligned}&\overline{D}_\tau\|e_{1h}^n\|_0^2+\overline{D}_\tau\|e_{3h}^n\|_0^2+\tau\|\overline{D}_\tau e_{1h}^n\|_0^2+\tau\|\overline{D}_\tau e_{3h}^n\|_0^2\\ &\qquad+\nu_0\alpha^*\|e_{1h}^n\|_1^2+\kappa_0\gamma^*\|e_{3h}^n\|_1^2\le c_\varepsilon^*\big(\|e_{1h}^{n-1}\|_0^2+\|e_{3h}^{n-1}\|_0^2+\delta_n\big), \qquad (35)\end{aligned}$$

where

$$\begin{aligned}\delta_n\equiv{}&\tau\,\|\partial_t^2 u\|_{L^2(t_{n-1},\,t_n;\,L^2)}^2+\tau\,\|\partial_t^2\theta\|_{L^2(t_{n-1},\,t_n;\,L^2)}^2\\ &+\frac{h^{2k}}{\tau}\|(\partial_t u,\partial_t p)\|_{L^2(t_{n-1},\,t_n;\,H^{k+1}\times H^k)}^2+\frac{h^{2k}}{\tau}\|\partial_t\theta\|_{L^2(t_{n-1},\,t_n;\,H^k)}^2\\ &+\tau\|\partial_t u\|_{L^2(t_{n-1},\,t_n;\,L^2)}^2+\tau\|\partial_t\theta\|_{L^2(t_{n-1},\,t_n;\,L^2)}^2\\ &+h^{2k}\|(u,p)\|_{\mathscr{C}([t_{n-1},\,t_n];\,H^{k+1}\times H^k)}^2+h^{2k}\|\theta\|_{\mathscr{C}([t_{n-1},\,t_n];\,H^{k+1})}^2.\end{aligned}$$

From (12) it holds that

$$\tau\sum_{i=1}^n\delta_i\le c^*\,(\tau^2+h^{2k}). \qquad (36)$$

Applying the discrete Gronwall inequality to (35) with (13) and (36), we have

$$\begin{aligned}&\|e_{1h}^n\|_0^2+\|e_{3h}^n\|_0^2\\ &\qquad+\tau\sum_{i=1}^n\big(\|e_{1h}^i\|_1^2+\|e_{3h}^i\|_1^2+\tau\|\overline{D}_\tau e_{1h}^i\|_0^2+\tau\|\overline{D}_\tau e_{3h}^i\|_0^2\big)\le c^*(\tau^2+h^{2k}). \qquad (37)\end{aligned}$$

Using $u-u_h = u-w_h-e_{1h}$, $\theta-\theta_h=\theta-\omega_h-e_{3h}$, and (22), we conclude (14). □

Next we prove the error estimate (15) of the pressure.

Proof of Theorem 2. From Hypothesis 1, (11), and (27a) it holds that

$$\begin{aligned}\|e_{2h}^n\|_0 &\le \frac{1}{\beta^*}\sup_{v_h\in V_h}\frac{b(v_h,e_{2h}^n)}{\|v_h\|_1}\\ &\le \frac{1}{\beta^*}\sup_{v_h\in V_h}\frac{1}{\|v_h\|_1}\big\{\langle R_{L1}{}_h^n,v_h\rangle+\langle R_{N1}{}_h^n,v_h\rangle-\big(\overline{D}_\tau e_{1h}^n,v_h\big)-a_0\big(\nu^n(\theta_h^{n-1});e_{1h}^n,v_h\big)\big\}\\ &\le c^*\Big\{\|R_{L1}{}_h^n\|_{V_h'}+\|R_{N1}{}_h^n\|_{V_h'}+\frac{1}{\sqrt{\tau}}\big(\sqrt{\tau}\,\|\overline{D}_\tau e_{1h}^n\|_0\big)+\|e_{1h}^n\|_1\Big\}. \end{aligned}\tag{38}$$

We evaluate $\|R_{C1}{}_h^n\|_{V_h'}$ to estimate the second term in the right-hand side of (38). Applying Lemma 4.5 in [19], we have for $v_h\in V_h$

$$\begin{aligned}\langle R_{C1}{}_h^n,v_h\rangle \le c^*\big(&\sqrt{\tau}\,\|\partial_t u\|_{L^2(t_{n-1},t_n;L^2)}+h^k\|(u,p)\|_{\mathscr{C}([t_{n-1},t_n];H^{k+1}\times H^k)}\\ &+\|e_{1h}^{n-1}\|_0\big)\|v_h\|_1+|a_1(e_{1h}^{n-1},e_{1h}^n,v_h)|.\end{aligned}$$

From Hypothesis 3 and (37) we obtain

$$\|e_{1h}^n\|_1\le c\,\min\big(h^{-1}\|e_{1h}^n\|_0,\|e_{1h}^n\|_1\big)\le c^*\min\big(h^{-1}(\tau+h^k),\tau^{-1}(\tau+h^k)\big)\le c^*,$$

which implies

$$|a_1(e_{1h}^{n-1},e_{1h}^n,v_h)|\le c\|e_{1h}^{n-1}\|_1\|e_{1h}^n\|_1\|v_h\|_1\le c^*\|e_{1h}^{n-1}\|_1\|v_h\|_1.$$

Hence it holds that

$$\|R_{C1}{}_h^n\|_{V_h'}\le c^*\big(\sqrt{\tau}\,\|\partial_t u\|_{L^2(t_{n-1},t_n;L^2)}+h^k\|(u,p)\|_{\mathscr{C}([t_{n-1},t_n];H^{k+1}\times H^k)}+\|e_{1h}^{n-1}\|_1\big).$$

Estimating the other terms of (38) in similar ways to (30a), (31a), and (32), we obtain from (37)

$$\|e_{2h}\|_{\ell^2(L^2)}\le c^*\tau^{-1/2}(\tau+h^k).$$

Using the identity $p-p_h=p-r_h-e_{2h}$ and (22), we conclude (15). □

At the end of this section we give the proofs of Corollaries 1 and 2. Before beginning the proof we prepare a Poisson projection. Let $\overline{\kappa}$ be a function in $\mathscr{C}^{0,1}(\overline{\Omega})$ such that the inequalities

$$\overline{\kappa}_0\le\overline{\kappa}(x)\le\overline{\kappa}_1\qquad\text{for all } x\in\overline{\Omega}$$

hold with positive constants $\overline{\kappa}_0$ and $\overline{\kappa}_1$. For each $\theta\in X$ we define a Poisson projection $\omega_h\in\Psi_h(\theta|_\Gamma)$ to be the finite element solution of the problem

$$c_0(\overline{\kappa};\omega_h,\psi_h)=c_0(\overline{\kappa};\theta,\psi_h)\qquad\text{for all } \psi_h\in\Psi_h.\tag{39}$$

Corresponding to (22) and (23) we have for $\theta\in H^{k+1}(\Omega)$

$$\|\theta-\omega_h\|_1\le c\,h^k\|\theta\|_{k+1},\tag{40a}$$

$$\|\omega_h\|_{0,\infty}+\|\omega_h\|_{1,3}\le c\,\|\theta\|_2.\tag{40b}$$

Proof of Corollary 1. Here we take ω_h^n as the Poisson projection of θ^n with $\overline{\kappa}\equiv\kappa(x)$, which makes the proof simpler. Since the coefficients ν and κ are independent of t and θ, the remainder terms $R_{D1}{}_h^n$ and $R_{D2}{}_h^n$ in (27) vanish.

In (27) we substitute $(\overline{D}_\tau e^n_{1h}, \overline{D}_\tau e^n_{2h}, \overline{D}_\tau e^n_{3h}) \in V_h \times Q_h \times \Psi_h$ into (v_h, q_h, ψ_h). The diffusion terms of the left-hand side in (27) have the following bounds:

$$a_0(\nu; e^n_{1h}, \overline{D}_\tau e^n_{1h}) \geq \frac{1}{2}\Big\{\overline{D}_\tau a_0(\nu; e^n_{1h}, e^n_{1h}) + \tau\nu_0\alpha^*\|\overline{D}_\tau e^n_{1h}\|_1^2\Big\}, \tag{41a}$$

$$c_0(\kappa; e^n_{3h}, \overline{D}_\tau e^n_{3h}) \geq \frac{1}{2}\Big\{\overline{D}_\tau c_0(\kappa; e^n_{3h}, e^n_{3h}) + \tau\kappa_0\gamma^*\|\overline{D}_\tau e^n_{3h}\|_1^2\Big\}. \tag{41b}$$

For ${R_{L1}}^n_h$ and ${R_B}^n_h$ we use estimate corresponding to (30a) and (32). Again, the conventional estimate of the time-difference quotient leads to

$$\langle {R_{L2}}^n_h, \overline{D}_\tau e^n_{3h}\rangle \leq c\left\{\sqrt{\tau}\,\|\partial_t^2\theta\|_{L^2(t_{n-1},t_n;L^2)} + \frac{h^k}{\sqrt{\tau}}\|\partial_t\theta\|_{L^2(t_{n-1},t_n;H^{k+1})}\right\}\|\overline{D}_\tau e^n_{3h}\|_0. \tag{42}$$

For the estimates of the convection terms we again apply Lemma 4.5 in [19] to obtain

$$\begin{aligned}\langle {R_{C1}}^n_h, \overline{D}_\tau e^n_{1h}\rangle \leq\ & c^*\big(\sqrt{\tau}\,\|\partial_t u\|_{L^2(t_{n-1},t_n;H^1)} \\ & + h^k\|(u,p)\|_{\mathscr{C}([t_{n-1},t_n];H^{k+1}\times H^k)} + \|e^{n-1}_{1h}\|_1\big)\|\overline{D}_\tau e^n_{1h}\|_0 \\ & + |a_1(e^{n-1}_{1h}, e^n_{1h}, \overline{D}_\tau e^n_{1h})|,\end{aligned} \tag{43a}$$

$$\begin{aligned}\langle {R_{C2}}^n_h, \overline{D}_\tau e^n_{3h}\rangle \leq\ & c^*\big(\sqrt{\tau}\,\|\partial_t u\|_{L^2(t_{n-1},t_n;H^1)} + h^k\|(u,p)\|_{\mathscr{C}([t_{n-1},t_n];H^{k+1}\times H^k)} \\ & + h^k\|\theta\|_{\mathscr{C}([t_{n-1},t_n];H^{k+1})} + \|e^{n-1}_{1h}\|_1\big)\|\overline{D}_\tau e^n_{3h}\|_0 \\ & + |c_1(e^{n-1}_{1h}, e^n_{3h}, \overline{D}_\tau e^n_{3h})|.\end{aligned} \tag{43b}$$

From Lemma 3 the last terms of the right-hand sides in (43) can be estimated as

$$|a_1(e^{n-1}_{1h}, e^n_{1h}, \overline{D}_\tau e^n_{1h})| \leq c\,\alpha_n\|e^{n-1}_{1h}\|_1(\|\overline{D}_\tau e^n_{1h}\|_0 + \sqrt{\tau}\|\overline{D}_\tau e^n_{1h}\|_1), \tag{44a}$$

$$|c_1(e^{n-1}_{1h}, e^n_{3h}, \overline{D}_\tau e^n_{3h})| \leq c\,\gamma_n\|e^{n-1}_{1h}\|_1(\|\overline{D}_\tau e^n_{3h}\|_0 + \sqrt{\tau}\|\overline{D}_\tau e^n_{3h}\|_1), \tag{44b}$$

where

$$\alpha_n \equiv \min(h^{-d/6}, \tau^{-1/2})\|e^n_{1h}\|_1, \qquad \gamma_n \equiv \min(h^{-d/6}, \tau^{-1/2})\|e^n_{3h}\|_1.$$

By virtue of (18b) we have for $n \geq 1$

$$b(\overline{D}_\tau e^n_{1h}, \overline{D}_\tau e^n_{2h}) = 0.$$

Accordingly, the estimates (41)–(44) yield the inequality

$$\begin{aligned}&\overline{D}_\tau a_0(\nu; e^n_{1h}, e^n_{1h}) + \overline{D}_\tau c_0(\kappa; e^n_{3h}, e^n_{3h}) \\ &\quad + \tau\nu_0\alpha^*\|\overline{D}_\tau e^n_{1h}\|_1^2 + \tau\kappa_0\gamma^*\|\overline{D}_\tau e^n_{3h}\|_1^2 + \|\overline{D}_\tau e^n_{1h}\|_0^2 + \|\overline{D}_\tau e^n_{3h}\|_0^2 \\ &\quad \leq c_\varepsilon^*\left\{(\alpha_n^2 + \gamma_n^2)\|e^{n-1}_{1h}\|_1^2 + \delta_n\right\},\end{aligned} \tag{45}$$

where

$$\begin{aligned}\delta_n \equiv\ & \tau\,\|\partial_t^2 u\|^2_{L^2(t_{n-1},t_n;L^2)} + \tau\,\|\partial_t^2\theta\|^2_{L^2(t_{n-1},t_n;L^2)} \\ & + \frac{h^{2k}}{\tau}\|(\partial_t u, \partial_t p)\|^2_{L^2(t_{n-1},t_n;H^{k+1}\times H^k)} + \frac{h^{2k}}{\tau}\|\partial_t\theta\|^2_{L^2(t_{n-1},t_n;H^{k+1})} \\ & + \tau\,\|\partial_t u\|^2_{L^2(t_{n-1},t_n;H^1)} + \tau\,\|\partial_t\theta\|^2_{L^2(t_{n-1},t_n;L^2)} \\ & + h^{2k}\|(u,p)\|^2_{\mathscr{C}([t_{n-1},t_n];H^{k+1}\times H^k)} + h^{2k}\|\theta\|^2_{\mathscr{C}([t_{n-1},t_n];H^{k+1})} \\ & + \|e^{n-1}_{1h}\|_1^2 + \|e^{n-1}_{3h}\|_0^2.\end{aligned}$$

From (37) and (12) it holds that

$$\begin{aligned}\tau\sum_{i=1}^{n}(\alpha_i^2+\gamma_i^2) &\le \min\{h^{-d/3},\,\tau^{-1}\}\,\tau\sum_{i=1}^{n}\big(\|e_{1h}^i\|_1^2+\|e_{3h}^i\|_1^2\big)\\ &\le c^*\min\{h^{-d/3},\,\tau^{-1}\}\,(h^{2k}+\tau^2)\\ &\le c^*(h^{2k-d/3}+\tau)\\ &\le c^*, \qquad (46a)\end{aligned}$$

$$\tau\sum_{i=1}^{n}\delta_i \le c^*\,(\tau^2+h^{2k}). \qquad (46b)$$

Applying the discrete Gronwall inequality to (45) with (18a) and (46), we conclude (19). □

Proof of Corollary 2. From Corollary 1 the time-difference quotient can be estimated as

$$\|\overline{D}_\tau e_{1h}\|_{\ell^2(L^2)} \le c^*(\tau+h^k). \qquad (47)$$

Introducing (47) to estimate the term $\|\overline{D}_\tau e_{1h}\|_0$ in (38), we obtain the desired result. □

5 Practical schemes

When $k=1,\,2$, we present schemes modified by approximate coefficients, which are used conveniently in practical computations.

Hypothesis 4 *Let $k=1,2$. There exist a finite dimensional subspace Z_h of $L^2(\Omega)$, and an operator $\Pi_h^{k-1}\in\mathscr{L}\big(H^k(\Omega);\,Z_h\big)$ such that for every $\psi\in H^k(\Omega)$*

$$\|\,\psi-\Pi_h^{k-1}\psi\,\|_0 \le c\,h^k\,\|\psi\|_k\,, \qquad (48a)$$

$$\|\Pi_h^{k-1}\psi\,\|_0 \le c\,\|\psi\|_0, \qquad (48b)$$

$$\Pi_h^{k-1}\psi\ge 0 \quad \text{if } \psi\ge 0, \qquad \Pi_h^{k-1}1=1. \qquad (48c)$$

Remark 7 We give an example of Z_h and Π_h^{k-1} satisfying Hypothesis 4 with $k=1,\,2$. Suppose the case mentioned in Remark 2. We define Z_h by the P_{k-1}-finite element space and Π_h^{k-1} by the interpolation operator introduced by Clément [5]. Such a choice enables us to prove (48).

Let us denote by $\nu_h^{n,m}\in Z_h$ the approximate viscosity $\Pi_h^{k-1}\big[\nu^n(\theta_h^m)\big]$, and $\kappa_h^{n,m}$ and $\beta_h^{n,m}$ are defined similarly. We consider a modified approximate problem instead of (10): Setting $u_h^0\in V_h(u_D)$ and $\theta_h^0\in\Psi_h(\theta_D)$ as approximations to u^0 and θ^0, respectively, we find $\big\{(u_h^n,p_h^n,\theta_h^n)\in V_h(u_D)\times Q_h\times\Psi_h(\theta_D);\ n=1,\dots,N_T\big\}$ such that for $n=1,\dots,N_T$

$$\begin{cases}(\overline{D}_\tau u_h^n,v_h)+a_0(\nu_h^{n,n-1};u_h^n,v_h)+a_1(u_h^{n-1},u_h^n,v_h)\\ \qquad\qquad +b(v_h,p_h^n)-(\beta_h^{n,n-1}\,\theta_h^{n-1},\,v_h)=(f^n,\,v_h) & \text{for all } v_h\in V_h, \quad (49a)\\ b(u_h^n,q_h)=0 & \text{for all } q_h\in Q_h, \quad (49b)\\ (\overline{D}_\tau\theta_h^n,\psi_h)+c_0(\kappa_h^{n,n-1};\theta_h^n,\,\psi_h)+c_1(u_h^{n-1},\,\theta_h^n,\,\psi_h)=(g^n,\,\psi_h) & \text{for all } \psi_h\in\Psi_h. \quad (49c)\end{cases}$$

Remark 8 From (11) and (48c) the inequalities

$$\nu_0\le\nu_h^{n,n-1}\le\nu_1,\quad \kappa_0\le\kappa_h^{n,n-1}\le\kappa_1,\quad\text{and}\quad \big|\beta_h^{n,n-1}\big|\le\beta_1 \qquad \text{for all } x\in\overline{\Omega}$$

hold. Therefore the bilinear forms $a_0(\nu_h^{n,n-1};\,.\,,\,.\,)$ and $c_0(\kappa_h^{n,n-1};\,.\,,\,.\,)$ are coercive on V_h and on Ψ_h, respectively. Thus the solution $\{(u_h^n,p_h^n,\theta_h^n);\ n=1,\dots,N_T\}$ is obtained step by step from u_h^0 and θ_h^0.

Theorem 3 *Suppose that the assumptions in Theorem* 1 *hold with a positive number* h_0 *and* $k = 1, 2$. *Furthermore, assume Hypothesis* 4 *with the same* k *and suppose that the functions* ν, κ, *and* β *satisfy*

$$\nu,\ \kappa \in H^k(\Omega \times (0,T) \times \mathbb{R};\, \mathbb{R}^+), \tag{50a}$$

$$\beta \in H^k(\Omega \times (0,T) \times \mathbb{R};\, \mathbb{R}^d). \tag{50b}$$

Then, for any $h \in (0, h_0]$ *the velocity* u_h *and the temperature* θ_h *of* (49) *satisfy*

$$\|u - u_h\|_{\ell^\infty(L^2)\cap\ell^2(H^1)},\ \|\theta - \theta_h\|_{\ell^\infty(L^2)\cap\ell^2(H^1)} \le c^*(\tau + h^k). \tag{51}$$

Theorem 4 *Under the assumptions in Theorem* 3 *the pressure* p_h *of* (49) *satisfies*

$$\|p - p_h\|_{\ell^2(L^2)} \le c^*\tau^{-1/2}(\tau + h^k). \tag{52}$$

Taking $\tau = ch^k$, *we have*

$$\|p - p_h\|_{\ell^2(L^2)} \le c^* h^{k/2}. \tag{53}$$

Corollary 3 *Replacing* (50a) *by*

$$\nu,\ \kappa \in H^k(\Omega;\, \mathbb{R}^+) \tag{54}$$

and (13) *by* (18), *we keep the other assumptions in Theorem* 3. *Then, for any* $h \in (0, h_0]$ *the velocity* u_h *and the temperature* θ_h *of* (49) *satisfy*

$$\|\partial_t u - \overline{D}_\tau u_h\|_{\ell^2(L^2)},\ \|u - u_h\|_{\ell^\infty(H^1)},$$
$$\|\partial_t \theta - \overline{D}_\tau \theta_h\|_{\ell^2(L^2)},\ \|\theta - \theta_h\|_{\ell^\infty(H^1)} \le c^*(\tau + h^k). \tag{55}$$

Corollary 4 *Under the assumptions in Corollary* 3 *the pressure* p_h *of* (49) *satisfies*

$$\|p - p_h\|_{\ell^2(L^2)} \le c^*(\tau + h^k). \tag{56}$$

Proof of Theorem 3. Reset the Stokes projection (w_h^n, r_h^n) of $(u^n,\, p^n)$ with $\overline{\nu} = \nu_h^{n,\,n-1}$. From (49) and the modification of the Stokes projection the nonlinear remainder terms R_{D1h}, R_{Bh}, and R_{D2h} in (27) are replaced by

$$\begin{aligned} R_{D1_h}^n &= a_0\big(\nu^n(\theta^n); u^n, \,.\,\big) - a_0(\nu_h^{n,\,n-1}; u^n, \,.\,), \\ R_{D2_h}^n &= c_0\big(\kappa^n(\theta^n); \theta^n, \,.\,\big) - c_0(\kappa_h^{n,\,n-1}; \omega_h^n, \,.\,), \\ R_{B_h}^n &= (\beta_h^{n,\,n-1}\,\theta_h^{n-1},\ .\,) - \big(\beta^n(\theta^n)\,\theta^n,\ .\,\big). \end{aligned}$$

A simple calculation gives us

$$\begin{aligned} \langle R_{D2_h}^n, e_{3h}^n\rangle = &\Big\{c_0\big(\kappa^n(\theta^n);\, \theta^n,\, e_{3h}^n\big) - c_0\big(\Pi_h^{k-1}\kappa^n(\theta^n);\, \theta^n,\, e_{3h}^n\big)\Big\} \\ &+ \Big\{c_0\big(\Pi_h^{k-1}\kappa^n(\theta^n);\, \theta^n,\, e_{3h}^n\big) - c_0\big(\Pi_h^{k-1}\kappa^n(\theta^{n-1});\, \theta^n,\, e_{3h}^n\big)\Big\} \\ &+ \Big\{c_0\big(\Pi_h^{k-1}\kappa^n(\theta^{n-1});\, \theta^n,\, e_{3h}^n\big) - c_0\big(\kappa_h^{n,\,n-1};\, \theta^n,\, e_{3h}^n\big)\Big\} \\ &+ \Big\{c_0\big(\kappa_h^{n,\,n-1};\, \theta^n,\, e_{3h}^n\big) - c_0\big(\kappa_h^{n,\,n-1};\, \omega_h^n,\, e_{3h}^n\big)\Big\} \\ \equiv\ & I_1 + I_2 + I_3 + I_4. \end{aligned}$$

From Hölder inequality and Hypothesis 4 we obtain

$$
\begin{aligned}
I_1 &\le \|(I-\Pi_h^{k-1})\kappa^n(\theta^n)\|_0\|\nabla\theta^n\|_{0,\infty}\|\nabla e_{3h}^n\|_0\\
&\le c^* h^k \|\kappa(\theta)\|_{\mathscr{C}([t_{n-1},t_n];H^k)}\|e_{3h}^n\|_1,\\
I_2 &\le \|\Pi_h^{k-1}\big(\kappa^n(\theta^n)-\kappa^n(\theta^{n-1})\big)\|_0\|\nabla\theta^n\|_{0,\infty}\|\nabla e_{3h}^n\|_0\\
&\le c\,|\kappa^n|_{\mathscr{C}^{0,1}(\overline{\Omega}\times\mathbb{R};\,\mathbb{R})}\|\theta^n-\theta^{n-1}\|_0\|\nabla\theta^n\|_{0,\infty}\|e_{3h}^n\|_1\\
&\le c^*\sqrt{\tau}\,\|\partial_t\theta\|_{L^2(t_{n-1},t_n;L^2)}\|e_{3h}^n\|_1,\\
I_3 &\le \|\Pi_h^{k-1}\big(\kappa^n(\theta^{n-1})-\kappa^n(\theta_h^{n-1})\big)\|_0\|\nabla\theta^n\|_{0,\infty}\|\nabla e_{3h}^n\|_0\\
&\le c\,|\kappa^n|_{\mathscr{C}^{0,1}(\overline{\Omega}\times\mathbb{R};\,\mathbb{R})}\|\theta^{n-1}-\theta_h^{n-1}\|_0\|\nabla\theta^n\|_{0,\infty}\|e_{3h}^n\|_1\\
&\le c^*(h^k\|\theta\|_{\mathscr{C}([t_{n-1},t_n];H^k)}+\|e_{3h}^{n-1}\|_0)\|e_{3h}^n\|_1,\\
I_4 &\le \|\Pi_h^{k-1}\kappa^n(\theta_h^{n-1})\|_{0,\infty}\|\nabla(\theta^n-\omega_h^n)\|_0\|\nabla e_{3h}^n\|_0\\
&\le c\,\kappa_1\|\theta^n-\omega_h^n\|_1\|e_{3h}^n\|_1\\
&\le c^* h^k\|\theta\|_{\mathscr{C}([t_{n-1},t_n];H^{k+1})}\|e_{3h}^n\|_1.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\langle R_{D2}{}_h^n, e_{3h}^n\rangle \le c^*\Big\{&\sqrt{\tau}\|\partial_t\theta\|_{L^2(t_{n-1},t_n;L^2)}\\
&+h^k\big(\|\kappa(\theta)\|_{\mathscr{C}([t_{n-1},t_n];H^k)}+\|\theta\|_{\mathscr{C}([t_{n-1},t_n];H^{k+1})}\big)+\|e_{3h}^{n-1}\|_0\Big\}\|e_{3h}^n\|_1.
\end{aligned}
$$

Similar arguments give us

$$
\begin{aligned}
\langle R_{D1}{}_h^n, e_{1h}^n\rangle \le c^*\Big\{&\sqrt{\tau}\|\partial_t\theta\|_{L^2(t_{n-1},t_n;L^2)}\\
&+h^k\big(\|\nu(\theta)\|_{\mathscr{C}([t_{n-1},t_n];H^k)}+\|\theta\|_{\mathscr{C}([t_{n-1},t_n];H^{k+1})}\big)+\|e_{3h}^{n-1}\|_0\Big\}\|e_{1h}^n\|_1.
\end{aligned}
$$

$$
\begin{aligned}
\langle R_{B}{}_h^n, e_{1h}^n\rangle \le c^*\Big\{&\sqrt{\tau}\|\partial_t\theta\|_{L^2(t_{n-1},t_n;L^2)}\\
&+h^k\big(\|\beta(\theta)\|_{\mathscr{C}([t_{n-1},t_n];H^k)}+\|\theta\|_{\mathscr{C}([t_{n-1},t_n];H^k)}\big)+\|e_{3h}^{n-1}\|_0\Big\}\|e_{1h}^n\|_0.
\end{aligned}
$$

These bounds lead to the desired results. □

We omit the proofs of Theorem 4 and Corollaries 3 and 4. They are similar to those of Theorem 2 and Corollaries 1 and 2 .

6 Concluding remarks

In this paper we have continued the study of series on finite element analysis of flow problems, the Navier–Stokes equations and thermal convection problems; see [12], [15], [17]–[20]. As mentioned in Section 1 the variable coefficients play an important role in the formation of the convection patterns. Thus we have focused our attention on the variable coefficients case, whose mathematical model is the most general among these studies. The error estimates obtained in this paper are best possible except for the pressure in the time- or temperature-dependent coefficient cases.

For the practical use we have also presented finite element schemes with interpolated variable coefficients, which maintain the same convergence order when the approximation order k is equal to 1 or 2. It is enough in practical computations.

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