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A numerical method to verify the invertibility of linear elliptic operators with applications to nonlinear problems

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Abstract

In this paper, we propose a numerical method to verify the invertibility of the second order linear elliptic operators. The invertibility condition is formulated as a uniqueness property of the solution for a certain linear fixed point equation and is reduced to a numerical condition based upon the existing verification method originally developed by one of the authors. As a useful application of the result, we present a new verification method of solutions for nonlinear elliptic problems, which enables to simplify the verification process . Verification techniques are followed by numerical examples which confirm us the actual effectiveness of the method.

AMS Subject Classifications: 35J25, 35J60, 65N25.

Key words: Numerical verification, Unique solvability of linear elliptic problem, finite element method.

1 Introduction

We consider the solvability of the linear elliptic boundary value problem of the form

$$\begin{aligned} \mathcal{L}u \equiv -\Delta u + b \cdot \nabla u + cu &= g \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

which is equivalent to the invertibility of the operator $\mathcal{L}$ on a certain function space. Here, for $n = 1, 2, 3$, we assume that $b \in (W^1_\infty(\Omega))^n$, $c \in L^\infty(\Omega)$, where $\Omega \subset R^n$ is a bounded convex domain with piecewise smooth boundary. By using this result, we present a procedure to compute the operator norm corresponding to the inverse $\mathcal{L}^{-1}$, and then, we formulate a numerical verification method of solutions for the following nonlinear elliptic problems:

$$\begin{array}{rcll} -\Delta u &=& f(x,u,\nabla u) & x \in \Omega, \\ u &=& 0 & x \in \partial\Omega. \end{array} \tag{1.2}$$

On the numerical verification methods of solutions for (1.2), e.g., in [3], [6] etc., several works have been presented based upon the principle originally found by one of the authors. They use the method which consists of two procedures, one is a finite dimensional Newton-like iterative process and another an error estimate in each iteration. In general, the method for finite dimensional part utilizes a kind of interval Newton method. And, it is recently observed that, in case of having the term with first order derivative ∇u, this iterative process sometimes fails due to the explosive enlargement of the width of intervals. In order to overcome this difficulty, we considered a refinement, in [7], which adopt a technique to avoid the direct interval computations for the finite dimensional part.

In the present paper, we propose a new approach which enables us the direct estimation of the norm of linearized inverse operators for (1.2) and yields further simplification of the verification procedures. This approach is considered as a refinement of the existing verification principle, and particularly, is an extension of the method presented in [7]. According to the direct verification of the invertibility of the linearized operator in the infinite dimensional sense, we can avoid the complicated interval computations which sometimes cause unexpected overestimates. Also note that the computational procedure in our new method would be much simpler compared with the existing methods such as, e.g., [2], [8] etc. which use the similar kind of evaluation of the linearized inverse operator.

In the below, we denote the L^2 inner product on Ω by $(\cdot,\cdot)$ and the norm by $\|\cdot\|_{L^2}$. And denote the usual L^2 Sobolev spaces on Ω by $H^k(\Omega)$ for any positive integer k. For the first order Sobolev space $H_0^1(\Omega)$ with homogeneous boundary condition, we define the norm by $\|v\|_{H_0^1} := \|\nabla v\|_{L^2}$, and also define the H^2 semi-norm on Ω by, e.g., when $n = 2$,

$$|u|_{H^2} = \left(\|u_{xx}\|_{L^2}^2 + 2\,\|u_{xy}\|_{L^2}^2 + \|u_{yy}\|_{L^2}^2\right)^{\frac{1}{2}}.$$

For $n = 1$ or $n = 3$, analogously defined.

2 Invertibility condition of linear elliptic operators

In the present section, we consider the verification condition, in the numerical sense, of invertibility of the operator $\mathcal{L}$ defined by (1.1), as well as we present a method to estimate the norm of the inverse operator corresponding to $\mathcal{L}^{-1}$.

We now introduce the finite dimensional subspace S_h of $H_0^1(\Omega)$ depending on the parameter h with a certain base functions $\{\phi_i\}_{1\le i\le N}$. And, for each $v \in H_0^1(\Omega)$, define the H_0^1-projection $P_h v \in S_h$ by

$$(\nabla(v - P_h v), \nabla\phi_h) = 0, \qquad \forall \phi_h \in S_h.$$

Further, we assume that there exists a positive constant C_0 which can be numerically estimated satisfying, for any $u \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\|u - P_h u\|_{H_0^1} \le C_0 h |u|_{H^2}. \tag{2.1}$$

Notice that the invertibility of the elliptic operator $\mathcal{L}$ defined in (1.1) is equivalent to the unique solvability of the fixed point equation:

$$u = Au. \tag{2.2}$$

Here, the compact operator $A : H_0^1 \longrightarrow H_0^1$ is defined by $Au := \Delta^{-1}(b \cdot \nabla u + cu)$, where Δ^{-1} stands for the solution operator of the Poisson equation with homogeneous boundary condition.

Now, according to the usual verification principle, e.g.,[6], we formulate, in numerical, a sufficient condition so that the equation (1.2) has a unique solution. As the preliminary, we define the matrices $\mathbf{G} = (\mathbf{G}_{i,j})$ and $\mathbf{D} = (\mathbf{D}_{i,j})$ by :

$$\begin{aligned} \mathbf{G}_{i,j} &= (\nabla\phi_j, \nabla\phi_i) + (b \cdot \nabla\phi_j, \phi_i) + (c\phi_j, \phi_i), \\ \mathbf{D}_{i,j} &= (\nabla\phi_j, \nabla\phi_i), \qquad \text{for} \quad 1 \le i, j \le N. \end{aligned}$$

Let $\mathbf{L}$ be a lower triangular matrix satisfying the Cholesky decomposition: $\mathbf{D} = \mathbf{L}\mathbf{L}^T$. And we denote the matrix norm induced from the Euclidean norm in R^N by $\|\cdot\|_E$.
Also, we define the following constants:

$$\begin{aligned} C_b &= \|b\|_{L^\infty}, \quad C_b' = \|\nabla b\|_{L^\infty}, \quad C_c = \|c\|_{L^\infty}, \\ C_1 &= C_0(\sqrt{n}C_b + nC_b'C_p), \qquad C_3 = C_b + C_cC_p, \\ C_2 &= C_0C_cC_p, \qquad C_4 = C_b + C_0C_ch, \end{aligned}$$

where $\|\cdot\|_{L^\infty}$ means L^∞ norm on Ω and C_p is a Poincaré constant such that $\|\phi\|_{L^2} \le C_p\|\phi\|_{H_0^1}$ for arbitrary $\phi \in H_0^1(\Omega)$. Then we have the following main result of this paper.

Theorem 2.1 *For the constants defined above, if*

$$C_0h(C_3M(C_1 + C_2)h + C_4) < 1,$$

then the operator $\mathcal{L}$ defined in (1.1) is invertible. Here, $M \equiv \|\mathbf{L}^T\mathbf{G}^{-1}\mathbf{L}\|_E$ and C_0 is the same constant in (2.1).

Proof: First, as usual, we decompose the equation $u = Au$ as

$$\begin{aligned} P_hu &= P_hAu, \\ (I - P_h)u &= (I - P_h)Au, \end{aligned}$$

where I implies the identity map on $H_0^1(\Omega)$.
Next, according to the same formulation to that in [4] [6]etc., we define two operators by

$$N_hu \equiv P_hu - [I - A]_h^{-1}P_h(I - A)u$$

and

$$Tu \equiv N_hu + (I - P_h)Au,$$

respectively, where $[I-A]_h^{-1}$ means the inverse of $P_h(I-A)|_{S_h} : S_h \longrightarrow S_h$.
We now, for positive real numbers α and γ, define *the candidate set* U which possibly enclose the solution of (2.2) by

$$\begin{aligned} U_h &:= \left\{ u_h \in S_h : \|u_h\|_{H_0^1} \le \gamma \right\}, \\ U_\perp &:= \left\{ u_\perp \in S_h^\perp : \|u_\perp\|_{H_0^1} \le \alpha, \quad \|u_\perp\|_{L^2} \le C_0 h\alpha \right\}, \end{aligned}$$

where $S_h^\perp$ stands for the orthogonal complement of S_h in $H_0^1(\Omega)$. And set $U \equiv U_h + U_\perp$. Then by the fact that $u = Au$ is equivalent to $u = Tu$, in order to prove the unique existence of a solution to (2.2) in the set U, it suffices, by Schauder's fixed point theorem and due to the linearity of the equation, to show the inclusion $TU \overset{\circ}{\subset} U$ which implies $\overline{TU} \subset \overset{\circ}{U}$, i.e., the closure of TU is included by the interior of U.
Further notice that a sufficient condition of this inclusion can be written as

$$\|N_h U\|_{H_0^1} \equiv \sup_{u\in U} \|N_h u\|_{H_0^1} < \gamma, \tag{2.3}$$

$$\begin{aligned} \|(I-P_h)AU\|_{H_0^1} &\equiv \sup_{u\in U} \|(I-P_h)Au\|_{H_0^1} \\ &\le C_0 h \sup_{u\in U} |Au|_{H^2} \\ &\le C_0 h \sup_{u\in U} \|b\cdot\nabla u + cu\|_{L^2} < \alpha, \end{aligned} \tag{2.4}$$

where we have used the estimate (2.1) and well known inequality $|\phi|_{H^2} \le \|\Delta\phi\|_{L^2}$ for the convex domain.

Therefore, in the below, by using various constants defined above, we try to estimate norms $\|N_h u\|_{H_0^1}$ and $\|b\cdot\nabla u + cu\|_{L^2}$ in (2.3) and (2.4), respectively.
First, for arbitrary $u = u_h + u_\perp \in U_h + U_\perp$, setting $\psi_h := N_h(u_h + u_\perp)$, we have

$$\begin{aligned} \psi_h &= u_h - [I-A]_h^{-1} P_h (I-A)(u_h + u_\perp) \\ &= [I-A]_h^{-1} P_h A u_\perp. \end{aligned} \tag{2.5}$$

Now, setting $v_h := P_h A u_\perp \in S_h$, ψ_h is given as the solution of the following weak form:

$$(\nabla\psi_h, \nabla\phi_h) + (b\cdot\nabla\psi_h, \phi_h) + (c\psi_h, \phi_h) = (\nabla v_h, \nabla\phi_h), \qquad \forall \phi_h \in S_h. \tag{2.6}$$

Denoting

$$\psi_h := \sum_{i=1}^N w_i \phi_i, \qquad v_h := \sum_{i=1}^N v_i \phi_i, \tag{2.7}$$

from (2.6) we have a matrix equation of the form

$$\mathbf{G}\vec{w} = \mathbf{D}\vec{v}. \tag{2.8}$$

Here, $\vec{w} = (w_1, w_2, \cdots, w_N)^T$, $\vec{v} = (v_1, v_2, \cdots, v_N)^T$ are coefficient vectors of ψ_h, v_h. Therefore, from (2.7) and (2.8), it follows that

$$\begin{aligned}
\|\psi_h\|_{H_0^1}^2 &= \vec{w}^T \mathbf{D} \vec{w} \\
&= \vec{w}^T \mathbf{D} \mathbf{G}^{-1} \mathbf{D} \vec{v} \\
&= (\mathbf{L}^T \vec{w})^T (\mathbf{L}^T \mathbf{G}^{-1} \mathbf{L})(\mathbf{L}^T \vec{v}) \\
&\leq \|\mathbf{L}^T \vec{w}\|_E \|\mathbf{L}^T \mathbf{G}^{-1} \mathbf{L}\|_E \|\mathbf{L}^T \vec{v}\|_E \\
&= \|\psi_h\|_{H_0^1} \|\mathbf{L}^T \mathbf{G}^{-1} \mathbf{L}\|_E \|v_h\|_{H_0^1}.
\end{aligned}$$

Thus, defining $M \equiv \|\mathbf{L}^T \mathbf{G}^{-1} \mathbf{L}\|_E$, we obtain the estimate

$$\begin{aligned}
\|\psi_h\|_{H_0^1} &\leq M \|P_h A u_\perp\|_{H_0^1} \\
&= M \|P_h \Delta^{-1}(b \cdot \nabla u_\perp + c u_\perp)\|_{H_0^1} \\
&\leq M \|\Delta^{-1}(b \cdot \nabla u_\perp + c u_\perp)\|_{H_0^1}.
\end{aligned} \tag{2.9}$$

We now estimate $\|\Delta^{-1}(b \cdot \nabla u_\perp)\|_{H_0^1}$ and $\|\Delta^{-1} c u_\perp\|_{H_0^1}$, which yields the desired estimation of $\|\psi_h\|_{H_0^1}$ by (2.9). For the first term, letting $\psi_1 := \Delta^{-1}(b \cdot \nabla u_\perp)$, we have, by some simple calculations

$$\begin{aligned}
\|\psi_1\|_{H_0^1}^2 = (\nabla \psi_1, \nabla \psi_1) &= (-\Delta \psi_1, \psi_1) \\
&= (-b \cdot \nabla u_\perp, \psi_1) \\
&\leq \|u_\perp\|_{L^2} \|\mathrm{div}(b \psi_1)\|_{L^2} \\
&\leq C_0 h \alpha (\sqrt{n} C_b + n C_b' C_p) \|\psi_1\|_{H_0^1},
\end{aligned} \tag{2.10}$$

where we have used the fact that $\|u_\perp\|_{L^2} \leq C_0 h \alpha$. For the second term, setting $\psi_2 := \Delta^{-1}(c u_\perp)$ and by applying the similar argument as above, we have

$$\|\psi_2\|_{H_0^1}^2 = C_c C_p C_0 h \alpha \|\psi_2\|_{H_0^1} \tag{2.11}$$

Thus, by (2.9) – (2.11), we obtain the following estimate for the finite dimensional part

$$\|N_h U\|_{H_0^1} \leq M (C_1 + C_2) h \alpha, \tag{2.12}$$

where $C_1 \equiv C_0(\sqrt{n} C_b + n C_b' C_p)$, $C_2 \equiv C_c C_p C_0$. Next, observe that

$$\begin{aligned}
\|b \cdot \nabla u_h + c u_h\|_{L^2} &\leq C_b \|u_h\|_{H_0^1} + C_c C_p \|u_h\|_{H_0^1} \\
&\leq (C_b + C_c C_p) \gamma, \\
\|b \cdot \nabla u_\perp + c u_\perp\|_{L^2} &\leq C_b \|u_\perp\|_{H_0^1} + C_c \|u_\perp\|_{L^2} \\
&\leq (C_b + C_0 C_c h) \alpha.
\end{aligned}$$

Therefore, by using (2.4) and the triangle inequality, we have

$$\|(I - P_h) A U\|_{H_0^1} \leq C_0 h (C_3 \gamma + C_4 \alpha), \tag{2.13}$$

where $C_3 \equiv C_b + C_c C_p$, $C_4 \equiv C_b + C_0 C_c h$.

Now from (2.12) and (2.13), we can write the invertibility condition (2.3) and (2.4) as

$$
\begin{aligned}
M(C_1 + C_2)h\alpha &< \gamma, && (2.14)\\
C_0 h(C_3\gamma + C_4\alpha) &< \alpha. && (2.15)
\end{aligned}
$$

For arbitrary small $\varepsilon > 0$, if we set $\gamma := M(C_1 + C_2)h\alpha + \varepsilon$, then the condition (2.14) clearly holds. Therefore, by substituting it to (2.15) we have

$$
C_0 h(C_3(M(C_1 + C_2)h\alpha + \varepsilon) + C_4\alpha) < \alpha,
$$

which is equivalent to

$$
1 - C_0 h(C_3 M(C_1 + C_2)h + C_4) > 0.
$$

Thus the desired conclusion is obtained. ∎

When the coefficient function b of the first order term is not differentiable, we have the following alternative condition.

Corollary 1 *For the operator $\mathcal{L}$ defined in (1.1), let $b \in (L^\infty(\Omega))^n$. If*

$$
C_0 h(C_3 M(\hat{C}_1 + C_2 h) + C_4) < 1,
$$

then the operator $\mathcal{L}$ defined in (1.1) is invertible. Here, $\hat{C}_1 = \sqrt{n} C_b C_p$.

Proof: The difference from the arguments in the proof of Theorem 2.1 is only the part concerning the estimates (2.10). Corresponding estimates are now

$$
\begin{aligned}
\|\psi_1\|_{H_0^1}^2 = (-\Delta\psi_1, \psi_1) &= (-b \cdot \nabla u_\perp, \psi_1)\\
&\le \sqrt{n}\|b\|_\infty \|u_\perp\|_{H_0^1} \|\psi_1\|_{L^2}\\
&\le \sqrt{n} C_b C_p \alpha \|\psi_1\|_{H_0^1},
\end{aligned}
$$

which yields the corollary. ∎

Now our next purpose is, by using the result obtained in Theorem 2.1, to estimate the operator norm $\|(I - A)^{-1}\|_{H_0^1}$, corresponding to the H_0^1-norm for $\mathcal{L}^{-1}$.

Theorem 2.2 *Under the same assumptions in Theorem 2.1, provided that*

$$
\kappa \equiv C_0 h(C_3 M(C_1 + C_2)h + C_4) < 1,
$$

then the following estimation follows

$$
\|(I - A)^{-1}\|_{H_0^1} \le (R^2 + S^2)^{\frac{1}{2}} =: \mathcal{M}. \tag{2.16}
$$

Here, R and S are defined by

$$
R = (C_0 C_3 h M + 1)/(1 - \kappa), \qquad S = \{(C_1 + C_2)hR + 1\}M.
$$

Proof: Let ψ be an arbitrary element in $H_0^1(\Omega)$. Then, by the invertibility of $(I-A)$, there exists a unique element $u \in H_0^1(\Omega)$ satisfying $(I-A)u = \psi$. When we set

$$\begin{aligned} N_h u &:= P_h u - [I-A]_h^{-1} P_h((I-A)u - \psi), \\ Tu &:= N_h u + (I - P_h)(Au + \psi). \end{aligned}$$

Then notice that $(I-A)u = \psi$ is equivalent to $Tu = u$. Using the unique decompositions $u = u_h + u_\perp$ and $\psi = \psi_h + \psi_\perp$ in $H_0^1(\Omega) = S_h \oplus S_h^\perp$, by some simple calculations, we have

$$\begin{aligned} u_h &= [I-A]_h^{-1}(P_h A u_\perp - \psi_h), \\ u_\perp &= (I-P_h)A(u_h + u_\perp) + \psi_\perp. \end{aligned} \tag{2.17}$$

Hence, taking notice of $M = \|[I-A]_h^{-1}\|_{H_0^1}$ and the estimates in the proof of Theorem2.1, we have by (2.17)

$$\begin{aligned} \|u_h\|_{H_0^1} &\le M\|P_h A u_\perp - \psi_h\|_{H_0^1} \\ &\le M(C_1 + C_2)h\|u_\perp\|_{H_0^1} + M\|\psi_h\|_{H_0^1}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} \|u_\perp\|_{H_0^1} &\le \|(I-P_h)A(u_h + u_\perp)\|_{H_0^1} + \|\psi_\perp\|_{H_0^1} \\ &\le C_0 h(C_3\|u_h\|_{H_0^1} + C_4\|u_\perp\|_{H_0^1}) + \|\psi_\perp\|_{H_0^1}. \end{aligned} \tag{2.19}$$

Substituting the estimate of $\|u_h\|_{H_0^1}$ in (2.18) into the last right-hand side of (2.19) and solving it with respect to $\|u_\perp\|_{H_0^1}$, we get

$$\begin{aligned} \|u_\perp\|_{H_0^1} &\le \{(C_0 C_3 h M + 1)/(1-\kappa)\}\|\psi\|_{H_0^1} \\ &= R\|\psi\|_{H_0^1} \end{aligned} \tag{2.20}$$

Thus we also have by (2.18)

$$\begin{aligned} \|u_h\|_{H_0^1} &\le M(C_1 + C_2)hR\|\psi\|_{H_0^1} + M\|\psi_h\|_{H_0^1} \\ &= S\|\psi\|_{H_0^1}. \end{aligned} \tag{2.21}$$

Therefore, we immediately obtain the desired conclusion from (2.20)and (2.21). ∎

We also have the following estimates corresponding to the corollary 1.

Corollary 2 *Under the same assumption as in the corollary 1, if*

$$\hat{\kappa} \equiv C_0 h(C_3 M(\hat{C}_1 + C_2 h) + C_4) < 1,$$

then

$$\|(I-A)^{-1}\|_{H_0^1} \le (\hat{R}^2 + \hat{S}^2)^{\frac{1}{2}} =: \hat{\mathcal{M}}. \tag{2.22}$$

Here, $\hat{R}$ and $\hat{S}$ are defined as

$$\hat{R} := (C_0 C_3 h M + 1)/(1-\hat{\kappa}) \quad \textit{and} \quad \hat{S} := \{(\hat{C}_1 + C_2 h)R + 1\}M.$$

Note that, by using $\|(I-A)^{-1}\|_{H_0^1}$, we have the following a priori estimate of the solution to (1.1)

Theorem 2.3

$$\|u\|_{H_0^1} \le C_p \|(I-A)^{-1}\|_{H_0^1} \|g\|_{L^2}.$$

Indeed, defining $\psi := -\Delta^{-1} g$, then taking account that $(I-A)u = \psi$ and that

$$\|\psi\|_{H_0^1}^2 = (\nabla\psi, \nabla\psi) = (-\Delta\psi, \psi) = (g, \psi) \le \|g\|_{L^2} \|\psi\|_{L^2},$$

the conclusion follows from the Poincaré inequality.

3 Applications

In this section, we describe the actual applications of the results obtained in the previous section to the verification of solutions for nonlinear elliptic problem (1.2). We suppose that, in (1.2), the nonlinear map $f(u) \equiv f(\cdot, u, \nabla u)$ is a continuous and bounded map from $H_0^1(\Omega)$ into $L^2(\Omega)$.

3.1 Preliminary

First, we transform the original boundary value problem (1.2) into the so-called residual equation by using an approximate solution $\hat{u}_h \in S_h \subset H_0^1(\Omega)$ satisfying

$$(\nabla \hat{u}_h, \nabla \phi_h) = (f(\hat{u}_h), \nabla \phi_h), \qquad \forall \phi_h \in S_h. \tag{3.1}$$

For the effective computation of the solution for (3.1) with guaranteed accuracy, refer, for example, [1], [10] etc.
Next, we define the $\bar{u} \in H_0^1(\Omega) \cap H^2(\Omega)$ by the solution of Poisson's equation

$$\begin{aligned} -\Delta \bar{u} &= f(\hat{u}_h) \quad \text{in} \quad \Omega, \\ \bar{u} &= 0 \qquad\quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.2}$$

Further, let define residues by

$$u - \hat{u}_h = (u - \bar{u}) + (\bar{u} - \hat{u}_h), \qquad w := u - \bar{u}, \quad v_0 := \bar{u} - \hat{u}_h. \tag{3.3}$$

Note that v_0 is an unknown function but the estimation of its norm can be computed by an a priori and a posteriori techniques (e.g., see [5], [11]). Thus, using the residues in (3.3), concerned problem is reduced to the verification for the solution w of the residual form

$$\begin{aligned} -\Delta w &= f(w + v_0 + \hat{u}_h) - f(\hat{u}_h) \quad \text{in} \quad \Omega, \\ w &= 0 \qquad\qquad\qquad\qquad\qquad\quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.4}$$

Hence, denoting the Fréchet derivative at $\hat{u}_h$ by $f'(\hat{u}_h)$, the Newton-type residual equation for (3.4) is written as:

$$\begin{aligned} -\Delta w - f'(\hat{u}_h) w &= g_r(w) \quad \text{in} \quad \Omega, \\ w &= 0 \qquad\quad \text{on} \quad \partial\Omega, \end{aligned} \tag{3.5}$$

where $g_r(w) \equiv f(w + v_0 + \hat{u}_h) - f(\hat{u}_h) - f'(\hat{u}_h)w$.

In the above, we assumed that the approximate solution $\hat{u}_h$ is defined as an element in $H_0^1(\Omega)$, i.e., C^0-element. When we use the function satisfying $\hat{u}_h \in H^2(\Omega)$, i.e., C^1-element, we can get more simpler residual Newton-type equation without v_0 of the form

$$\begin{array}{rcll} -\Delta w - f'(\hat{u}_h)w &=& g_d(w) & \text{in} \quad \Omega, \\ w &=& 0 & \text{on} \quad \partial\Omega, \end{array} \tag{3.6}$$

where $w := u - \hat{u}_h$ and $g_d(w) := f(w + \hat{u}_h) + \Delta\hat{u}_h - f'(\hat{u}_h)w$.

3.2 Verification conditions

When, according to the above Newton-type formulation with residual form, we write down again the nonlinear boundary value problem as follows:

$$\begin{array}{rcll} \mathcal{L}w \equiv -\Delta w - f'(\hat{u}_h)w &=& g(w) & \text{in} \quad \Omega, \\ w &=& 0 & \text{on} \quad \partial\Omega, \end{array} \tag{3.7}$$

where $g(w) \equiv g_r(w)$ or $g(w) \equiv g_d(w)$, if $\mathcal{L}$ is invertible, then (3.7) is rewritten as the fixed point form

$$w = F(w) \left(\equiv \mathcal{L}^{-1}g(w)\right). \tag{3.8}$$

Notice that the Newton-like operator F in (3.8) is compact on $H_0^1(\Omega)$ from the assumptions on f, and that it is expected to be a contraction map on some neighborhood of zero.
Therefore, we now consider the candidate set of the form $W \equiv \{w \in H_0^1(\Omega) : \|w\|_{H_0^1} \le \alpha\}$.
First, in order to verify the existence of solutions, we need to choose the set W, which is equivalent to determine a positive number α, satisfying the following criterion based on the Schauder fixed point theorem:

$$F(W) \subset W. \tag{3.9}$$

And next, for the proof of uniqueness, the following contraction property is needed on the same set W in (3.9):

$$\|F(w_1) - F(w_2)\|_{H_0^1} \le k\|w_1 - w_2\|_{H_0^1}, \quad \forall w_1, w_2 \in W, \tag{3.10}$$

where $0 < k < 1$.
For (3.9), from the theorem 2.3, a sufficient condition can be written as

$$\|F(W)\|_{H_0^1} \equiv \sup_{w\in W} \|F(w)\|_{H_0^1} \;\le\; \mathcal{M}_1 \sup_{w\in W} \|g(w)\|_{L^2} \;<\; \alpha, \tag{3.11}$$

where $\mathcal{M}_1 \equiv C_p\mathcal{M}$, and $\mathcal{M}$ is the H_0^1-norm of the operator $\mathcal{L}^{-1}$ defined in the theorem 2.2.
On the other hand, for the uniqueness verification, in general, using the following deformation:

$$g(w_1) - g(w_2) = \Phi(w_1, w_2)(w_1 - w_2),$$

the inequality of the form

$$\mathcal{M}_1\|\Phi(w_1, w_2)(w_1 - w_2)\|_{L^2} \le k\|w_1 - w_2\|_{H_0^1} \tag{3.12}$$

is proved with $0 < k < 1$.

4 Numerical examples

Example 4.1 (Emden's equation)

$$\begin{aligned} -\Delta u &= u^2 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{4.1}$$

In this case, $\mathcal{L}$ and $g(w)$ in (3.7) are given as follows

$$\begin{aligned} \mathcal{L}w &\equiv -\Delta w - 2\hat{u}_h w, \\ g_r(w) &\equiv w^2 + 2v_0 w + v_0^2 + 2\hat{u}_h v_0. \end{aligned} \tag{4.2}$$

Therefore, for the candidate set $W_\alpha = \{w \in H_0^1(\Omega) : \|w\|_{H_0^1} \le \alpha\}$, the condition (3.11) is given by

$$\mathcal{M}_1 \sup_{w \in W_\alpha} \|w^2 + 2v_0 w + v_0^2 + 2\hat{u}_h v_0\|_{L^2} \ < \ \alpha. \tag{4.3}$$

By (4.3) and some calculations using the several kinds of norms, e.g., [4], [11] etc., we obtain the existential condition (3.11) of the form

$$\mathcal{M}_1(K_2\alpha^2 + K_1\alpha + K_0) < \alpha, \tag{4.4}$$

where $K_i,\ 0 \le i \le 2$, are constants dependent on the norms of $\hat{u}_h$ and v_0. It implies that, for any positive number α satisfying the quadratic inequality (4.4), there exists at least a solution in the set of the form $\hat{u}_h + v_0 + W_\alpha$. Also, notice that a sufficient condition corresponding to the relation (3.12) can be similarly and readily treated, and it leads to a simple linear inequality in α such as $\kappa_1\alpha + \kappa_0 < 1$. Thus, we can determine two nonnegative intervals, *E-range* and *U-range*, of α for which we assure the existence and the uniqueness of solutions, respectively. Table 1. shows the computational results for the domain $\Omega = (0,1) \times (0,1)$ using piecewise quadratic C^0 finite element as S_h with mesh size $h = 0.1$ and $h = 0.05$.

Table 1: Verification results for example 4.1

$1/h$	$\mathcal{M}_1$	$\|v_0\|_{H_0^1}$	E-range	U-range
10	0.9099	0.5481	[0.5377, 9.6402]	[0, 5.0889]
20	0.7663	0.1389	[0.0524, 12.5603]	[0, 6.3063]

Example 4.2 (Burgers equation)

$$\begin{aligned} \Delta u &= \lambda(u \cdot \nabla)u + \varphi(x,y) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{4.5}$$

where λ *is a parameter and* $\varphi(x,y) \equiv x$.

In this case, we need to consider a candidate set, a little bit of different from the previous case, of the form

$$W_\alpha \equiv \left\{ w \in H_0^1(\Omega) \cap L^\infty(\Omega) : \max\{\|\nabla w\|_{L^2}, \|w\|_\infty\} \le \alpha \right\}. \tag{4.6}$$

Namely, we enclose the solution of (4.5) in the Banach space $W^1 \equiv H_0^1(\Omega) \cap L^\infty(\Omega)$ with norm $\|w\|_{W^1} \equiv \max\{\|\nabla w\|_{L^2}, \|w\|_\infty\}$. Further we need the inverse norm estimates in the following L^∞ sense:

$$\|v\|_{L^\infty} \le \mathcal{M}_\infty \|\mathcal{L}v\|_{L^2}, \quad \forall v \in H^2(\Omega),$$

where $\mathcal{M}_\infty$ can be computed by using $\mathcal{M}_1$ in the section 2 and the constructive approach to the imbedding theory described in [8], [9].
Thus the existential condition is written as

$$\max(\mathcal{M}_1, \mathcal{M}_\infty) \sup_{w \in W_\alpha} \|g(w)\|_{L^2} \le \alpha. \tag{4.7}$$

Then, the linearized operator $\mathcal{L}$ and the right-hand side $g(w)$ of (3.7) are as follows:

$$\begin{aligned} \mathcal{L}w &\equiv -\Delta w + (\hat{u} \cdot \nabla) w + (w \cdot \nabla)\hat{u}_h, \\ g_r(w) &\equiv -\lambda \left[((w + v_0) \cdot \nabla)(w + v_0) + (\hat{u}_h \cdot \nabla) v_0 + (v_0 \cdot \nabla)\hat{u}_h \right]. \end{aligned} \tag{4.8}$$

The verification conditions using α is similarly represented as in the previous example. That is, it also leads to the inequality in α of the quadratic form such that $c_2\alpha^2 + c_1\alpha + c_0 < \alpha$, where $c_i,\ 0 \le i \le 2$, are computed by the estimates of the norms for $\hat{u}_h$ and v_0. Particularly, in order to get efficient computation, we used the L^∞ residual method for v_0([5]). And the uniqueness condition is also similarly given as before. The verification results for the parameter $\lambda = 10$ are shown in Table 2.

Table 2: Verification results for example 4.2 for $\lambda = 10$

$1/h$	$\mathcal{M}_Y$	$\|v_0\|_{H_0^1}$	$\|v_0\|_{L^\infty}$	E-range$\times 10^2$	U-range$\times 10^2$
5	0.8012	4.7171e-3	1.1216e-2	[1.2544, 5.9782]	[0, 3.6163]
10	0.7446	1.2531e-3	2.9668e-3	[0.1969, 8.8787]	[0, 4.5377]
20	0.7201	3.0232e-4	7.1951e-4	[0.0411, 9.6771]	[0, 4.8591]

Remark 1. Note that the computational efficiency of the above results, in the example 4.1, is almost similar to that the existing methods up to now, e.g., comparing with [11]. But, the determination of the range for existence and/or uniqueness as shown in the tables might be impossible for those methods up to now. Particularly, we can find rather wide range which contains no solutions. For example, from Table 1. and Table 2., we can conclude that there are no solutions at all for α in $[0.0524, 6.3063]$ and in $[0.000411, 0.048591]$, respectively. This property should be useful and powerful for the purpose to prove the nonexistence theorem in various kinds of problems.

Remark 2. For the present cases, we separately verified the existence and uniqueness by the criteria (3.9) and (3.10), respectively. We can also use another method to prove them simultaneously. Namely, the condition

$$F(0) + F'(W)W \overset{\circ}{\subset} W$$

is satisfied for the candidate set W in H_0^1, then it implies that a locally unique solution is enclosed in W([12]).

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