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ON EXTREMAL QUASIMODULAR FORMS

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Abstract. We define and study 'extremal' quasimodular forms. Some explicit descriptions of such forms are given. Connections with certain differential equations and Atkin's orthogonal polynomials, and the positivity of the Fourier coefficients, are also discussed.

1. Introduction

The purpose of this paper is to introduce the notion of 'extremal' quasimodular forms and to study some of their properties. In particular, we discuss connections with certain differential equations and the Atkin orthogonal polynomials, which are closely related to supersingular elliptic curves.

Let $E_2(\tau)$, $E_4(\tau)$ and $E_6(\tau)$ be the standard Eisenstein series on the full modular group $\mathrm{PSL}_2(\mathbb{Z})$ of weights 2, 4 and 6, respectively:

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\bigg(\sum_{d|n} d\bigg)q^n,$$

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\bigg(\sum_{d|n} d^3\bigg)q^n,$$

$$E_6(\tau) = 1 - 504\sum_{n=1}^{\infty}\bigg(\sum_{d|n} d^5\bigg)q^n,$$

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where τ is the variable in the upper half-plane and $q = e^{2\pi i\tau}$ as usual. Of these, $E_4(\tau)$ and $E_6(\tau)$ are modular but $E_2(\tau)$ is not (it is quasimodular). A quasimodular form of weight k on $\mathrm{PSL}_2(\mathbb{Z})$ is an element of weight k in the graded ring $\mathbb{C}[E_2(\tau), E_4(\tau), E_6(\tau)]$. (See [**3**] for the definition of quasimodular forms.) In this paper we only consider forms on $\mathrm{PSL}_2(\mathbb{Z})$ and we often drop the reference to the group.

Any quasimodular form of weight k is uniquely written as

$$f = f_0 + f_1E_2 + f_2E_2^2 + \cdots + f_rE_2^r \tag{1}$$

with r an integer ≥ 0, f_i $(0 \leq i \leq r)$ a (true) modular form of weight $k - 2i$ and $f_r \neq 0$.

Definition 1.1. The integer r in (1) is referred to as the depth of f. The $\mathbb{C}$-vector space of all quasimodular forms of weight k and depth not greater than r is denoted by $QM_k^{(r)}$.

The space $QM_k^{(0)}$ is none other than the space of usual modular forms of weight k and we denote this space also by M_k.

Clearly, the depth r does not exceed $k/2$, and there exists no quasimodular form of weight k and depth $k/2-1$, due to the non-existence of modular forms of weight 2. Thus, the chain of inclusion of the spaces is as follows:

$$\begin{aligned} M_k = QM_k^{(0)} &\subseteq QM_k^{(1)} \subseteq \cdots \subseteq QM_k^{(k/2-2)} = QM_k^{(k/2-1)} \subseteq QM_k^{(k/2)} \\ &= QM_k^{(k/2+1)} = \cdots . \end{aligned}$$

We now define our main object of study.

Definition 1.2. Let $f = \sum_{n=0}^{\infty} a_nq^n$ be an element in $QM_k^{(r)} \setminus QM_k^{(r-1)}$, i.e. a quasimodular form of weight k and depth r. Denote the dimension of the space $QM_k^{(r)}$ by m: $m = \dim_{\mathbb{C}} QM_k^{(r)}$ $(= \sum_{i=0}^{r} \dim_{\mathbb{C}} M_{k-2i})$. We call f extremal if its first m Fourier coefficients satisfy

$$a_0 = a_1 = \cdots = a_{m-2} = 0,\ a_{m-1} \neq 0.$$

If, moreover, $a_{m-1} = 1$, f is said to be normalized.

Remark 1.3. There exists the notion of extremal modular forms (a modular form $\in M_k$ is extremal if it is $1 + O(q^m)$, where $m = \dim M_k$). The readers should not mix up the two extremalities.

Here are some examples in low weights.

Example 1.4. Weight 2, depth 1:

$$E_2 = 1 - 24q - 72q^2 - 96q^3 - 168q^4 - 144q^5 - 288q^6 - \cdots .$$

Weight 4, depth 2:

$$\frac{E_4 - E_2^2}{288}\left(= -\frac{E_2'}{24}\right) = q + 6q^2 + 12q^3 + 28q^4 + 30q^5 + 72q^6 + \cdots .$$

Weight 6, depth 1:

$$\frac{E_4E_2 - E_6}{720}\left(= \frac{E_4'}{240}\right) = q + 18q^2 + 84q^3 + 292q^4 + 630q^5 + 1512q^6 + \cdots .$$

Weight 6, depth 3:

$$\frac{5E_2^3 - 3E_4E_2 - 2E_6}{51\,840} = q^2 + 8q^3 + 30q^4 + 80q^5 + 180q^6 + 336q^7 + \cdots .$$

Weight 8, depth 1:

$$\frac{E_4^2 - E_6E_2}{1008}\left(= -\frac{E_6'}{504}\right) = q + 66q^2 + 732q^3 + 4228q^4 + 15\,630q^5 + 48\,312q^6 + \cdots .$$

Weight 8, depth 2:

$$\frac{5E_4^2 + 2E_6E_2 - 7E_4E_2^2}{362\,880} = q^2 + 16q^3 + 102q^4 + 416q^5 + 1308q^6 + 3360q^7 + \cdots .$$

Weight 8, depth 4:

$$\frac{5E_4^2 + 16E_6E_2 + 14E_4E_2^2 - 35E_2^4}{11\,612\,160} = q^3 + \frac{21}{2}q^4 + 54q^5 + 192q^6 + 546q^7 + 1323q^8 + \cdots .$$

In the case of modular forms (on $\mathrm{PSL}_2(\mathbb{Z})$), an element in M_k is uniquely determined by its first $\dim M_k$ Fourier coefficients and one can prescribe these coefficients arbitrarily. However, whether the corresponding statements for quasimodular forms hold true is not at all obvious and we do not know the answer in general. Our first question is then

> Does there always exist an extremal quasimodular form of given weight k and depth r, provided k and r satisfy the necessary constraint $0 \le r \le k/2,\ r \ne k/2 - 1$? And is it unique when normalized?

We shall show the existence in the case of depth 1 and 2 in Sections 2 and 3, and in Section 4 we give some conjectures on the general case. It is hoped that, at least in some cases, extremal quasimodular forms have arithmetic significance. There are some signs supporting such a prediction, but the study has just begun and for the most part it is still to be explored.

2. Depth 1 case

Define two sequences of polynomials $P_n(x)$, $P_n^*(x)$ $(n = 0, 1, 2, \dots)$ by

$$\begin{aligned} &P_0(x) = 1, \quad P_1(x) = x, \quad P_{n+1}(x) = x P_n(x) + \mu_n P_{n-1}(x) \quad (n = 1, 2, \dots), \\ &P_0^*(x) = 1, \quad P_1^*(x) = x, \quad P_{n+1}^*(x) = x P_n^*(x) + \mu_n^* P_{n-1}^*(x) \quad (n = 1, 2, \dots), \end{aligned}$$

where μ_n and μ_n^* are given by

$$\mu_n = 12\left(6 + \frac{1}{n}\right)\left(6 - \frac{1}{n+1}\right), \quad \mu_n^* = 12\left(6 - \frac{1}{n}\right)\left(6 + \frac{1}{n+1}\right).$$

The first few examples are

$$\begin{aligned} &P_2(x) = x^2 + 462, \quad P_3(x) = x^3 + 904x, \quad P_4(x) = x^4 + 1341x^2 + 201\,894, \dots, \\ &P_2^*(x) = x^2 + 390, \quad P_3^*(x) = x^3 + 808x, \quad P_4^*(x) = x^4 + 1233x^2 + 165\,750, \dots. \end{aligned}$$

$P_n(x)$ and $P_n^*(x)$ are even (respectively odd) polynomials when n is even (respectively odd). We also define $Q_n(x)$, $Q_n^*(x)$ by the same recursion with different initial values

$$\begin{aligned} &Q_0(x) = 0, \quad Q_1(x) = 1, \quad Q_{n+1}(x) = x Q_n(x) + \mu_n Q_{n-1}(x) \quad (n = 1, 2, \dots), \\ &Q_0^*(x) = 0, \quad Q_1^*(x) = 1, \quad Q_{n+1}^*(x) = x Q_n^*(x) + \mu_n^* Q_{n-1}^*(x) \quad (n = 1, 2, \dots). \end{aligned}$$

The first examples are

$$\begin{aligned} &Q_2(x) = x, \quad Q_3(x) = x^2 + 442, \quad Q_4(x) = x^3 + 879x, \\ &Q_5(x) = x^4 + 1314x^2 + 192\,270, \dots, \\ &Q_2^*(x) = x, \quad Q_3^*(x) = x^2 + 418, \quad Q_4^*(x) = x^3 + 843x, \\ &Q_5^*(x) = x^4 + \frac{6354}{5}x^2 + \frac{894\,102}{5}, \dots, \end{aligned}$$

and the parity is opposite to that of $P_n(x)$ and $P_n^*(x)$.

The depth 1 extremal quasimodular forms of weight k are described according to the congruence classes of k modulo 6 as follows.

THEOREM 2.1.

(1) *Suppose that* $k = 6n$ $(n = 1, 2, 3, \dots)$. *Then the form*

$$\sqrt{\Delta(\tau)}^{n-1} P_{n-1}\left(\frac{E_6(\tau)}{\sqrt{\Delta(\tau)}}\right)\frac{E_4'(\tau)}{240} - \sqrt{\Delta(\tau)}^{n} Q_{n-1}\left(\frac{E_6(\tau)}{\sqrt{\Delta(\tau)}}\right) \quad \left(\Delta = \frac{E_4^3 - E_6^2}{1728}\right)$$

is an extremal quasimodular form of weight k *and depth* 1 *on* $\mathrm{PSL}_2(\mathbb{Z})$, *and is a solution of the differential equation*

$$f''(\tau) - \frac{k}{6}E_2(\tau)f'(\tau) + \frac{k(k-1)}{12}E_2'(\tau)f(\tau) = 0 \quad \left({}' = \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}\right).$$

(2) *Suppose that* $k = 6n + 2$ $(n = 1, 2, 3, \dots)$. *Then the form*

$$\sqrt{\Delta(\tau)}^{n-1} P_{n-1}^{*}\left(\frac{E_6(\tau)}{\sqrt{\Delta(\tau)}}\right)\left(-\frac{E_6'(\tau)}{504}\right) - \sqrt{\Delta(\tau)}^{n} Q_{n-1}^{*}\left(\frac{E_6(\tau)}{\sqrt{\Delta(\tau)}}\right)E_2(\tau)$$

is an extremal quasimodular form of weight k *and depth* 1 *on* $\mathrm{PSL}_2(\mathbb{Z})$, *and is a solution of*

$$f'' - \left(\frac{k}{6}E_2 - \frac{1}{3}\frac{E_6}{E_4}\right)f' + \left(\frac{k(k-1)}{12}E_2' - \frac{k-1}{18}\frac{E_6'}{E_4}\right)f = 0.$$

(3) *An extremal quasimodular form of weight* $k \equiv 4 \bmod 6$ *and depth* 1 *is obtained from the form in (1) with* k *replaced by* $k - 4$, *by multiplying* $E_4(\tau)$. *The differential equation it satisfies is*

$$f'' - \left(\frac{k}{6}E_2 - \frac{2}{3}\frac{E_6}{E_4}\right)f' + \left(\frac{k(k-1)}{12}E_2' - \frac{k-1}{9}\frac{E_6'}{E_4} - \frac{2}{9}\left(E_4 - \frac{E_6^2}{E_4^2}\right)\right)f = 0.$$

Remark 2.2. (i) The expressions in (1) and (2) contain $\sqrt{\Delta(\tau)}$, but the square root is superficial because of the parities of $P_n(x)$ and $Q_n(x)$. Hence, they define elements in $\mathbf{Q}[E_2, E_4, E_6]$ (note also Ramanujan's formulas $E_4' = (E_2E_4 - E_6)/3$, $E_6' = (E_2E_6 - E_4^2)/2$).

(ii) The differential equation in (1) (with k replaced by $k + 1$) is that which was studied extensively in [**4**] and [**2**].

Proof. We note that by the standard dimension formula for M_k we have $\dim QM_k^{(1)} = [k/6] + 1$. For (1), that the given form satisfies the differential equation as indicated has been proved in [**2**, Theorem 2]. By looking at the exponent of that solution, we conclude it is extremal (obviously of depth 1). The proof of (2) goes in a similar manner.

For (3), it is enough to note that, when $k \equiv 4 \bmod 6$, we have $\dim QM_k^{(1)} = \dim QM_{k-4}^{(1)}$ and, hence, $QM_k^{(1)} = E_4 \cdot QM_{k-4}^{(1)}$. The differential equation can be deduced from (1) with the aid of Ramanujan's formulas mentioned above. □

If we compute the forms in Theorem 2.1 separately according to the parity of n (in order to avoid the expression $\sqrt{\Delta(\tau)}$), we encounter the Atkin orthogonal polynomials. We shall describe the resulting expressions when k is congruent to 0 or 2 modulo 6.

Put

$$\lambda_n = \begin{cases} 1, & n = 0, \\ 720, & n = 1, \\ 12\left(6 + \dfrac{(-1)^n}{n-1}\right)\left(6 + \dfrac{(-1)^n}{n}\right), & n \geq 2, \end{cases}$$

and

$$\lambda_n^* = \begin{cases} 1, & n = 0, \\ 1008, & n = 1, \\ 12\left(6 - \dfrac{(-1)^n}{n-1}\right)\left(6 - \dfrac{(-1)^n}{n}\right), & n \geq 2. \end{cases}$$

Define the sequences of polynomials $A_n(X)$, $B_n(X)$, $\widetilde{A}_n(X)$, $\widetilde{B}_n(X)$, $A_n^*(X)$, $B_n^*(X)$, $\widetilde{A}_n^*(X)$, $\widetilde{B}_n^*(X)$ by the recursion relations

$$A_{n+1}(X) = (X - (\lambda_{2n+1} + \lambda_{2n}))A_n(X) - \lambda_{2n}\lambda_{2n-1}A_{n-1}(X) \quad (n \geq 1),$$
$$B_{n+1}(X) = (X - (\lambda_{2n+1} + \lambda_{2n}))B_n(X) - \lambda_{2n}\lambda_{2n-1}B_{n-1}(X) \quad (n \geq 1),$$

$$\widetilde{A}_{n+1}(X) = (X - (\lambda_{2n} + \lambda_{2n-1}))\widetilde{A}_n(X) - \lambda_{2n-1}\lambda_{2n-2}\widetilde{A}_{n-1}(X) \quad (n \geq 1),$$
$$\widetilde{B}_{n+1}(X) = (X - (\lambda_{2n} + \lambda_{2n-1}))\widetilde{B}_n(X) - \lambda_{2n-1}\lambda_{2n-2}\widetilde{B}_{n-1}(X) \quad (n \geq 1),$$

$$A_{n+1}^*(X) = (X - (\lambda_{2n+1}^* + \lambda_{2n}^*))A_n^*(X) - \lambda_{2n}^*\lambda_{2n-1}^*A_{n-1}^*(X) \quad (n \geq 1),$$
$$B_{n+1}^*(X) = (X - (\lambda_{2n+1}^* + \lambda_{2n}^*))B_n^*(X) - \lambda_{2n}^*\lambda_{2n-1}^*B_{n-1}^*(X) \quad (n \geq 1),$$

$$\widetilde{A}_{n+1}^*(X) = (X - (\lambda_{2n}^* + \lambda_{2n-1}^*))\widetilde{A}_n^*(X) - \lambda_{2n-1}^*\lambda_{2n-2}^*\widetilde{A}_{n-1}^*(X) \quad (n \geq 1),$$
$$\widetilde{B}_{n+1}^*(X) = (X - (\lambda_{2n}^* + \lambda_{2n-1}^*))\widetilde{B}_n^*(X) - \lambda_{2n-1}^*\lambda_{2n-2}^*\widetilde{B}_{n-1}^*(X) \quad (n \geq 1)$$

with initial conditions

$$\begin{aligned}
&A_0(X) = 1, \quad A_1(X) = X - 720, \quad B_0(X) = 0, \quad B_1(X) = 1,\\
&\widetilde{A}_0(X) = 0, \quad \widetilde{A}_1(X) = 1, \quad \widetilde{B}_0(X) = -1, \quad \widetilde{B}_1(X) = 1,\\
&A_0^*(X) = 1, \quad A_1^*(X) = X - 1008, \quad B_0^*(X) = 0, \quad B_1^*(X) = 1,\\
&\widetilde{A}_0^*(X) = 0, \quad \widetilde{A}_1^*(X) = 1, \quad \widetilde{B}_0^*(X) = -1, \quad \widetilde{B}_1^*(X) = 1.
\end{aligned}$$

Let $j(\tau) = E_4(\tau)^3/\Delta(\tau)$ be the elliptic modular invariant.

THEOREM 2.3.

(1) *The form*

$$\frac{1}{N_n^*}(\Delta(\tau)^n A_n^*(j(\tau)) - E_2(\tau)E_4(\tau)E_6(\tau)\Delta(\tau)^{n-1}B_n^*(j(\tau)))$$
$$(N_n^* = \lambda_{2n}^*\lambda_{2n-1}^* \cdots \lambda_2^*\lambda_1^*)$$

is a normalized extremal quasimodular form of weight $12n$ *and depth* 1.

(2) *The form*

$$\frac{1}{N_n}(E_2(\tau)\Delta(\tau)^n A_n(j(\tau)) - E_4(\tau)^2E_6(\tau)\Delta(\tau)^{n-1}B_n(j(\tau)))$$
$$(N_n = \lambda_{2n}\lambda_{2n-1} \cdots \lambda_2\lambda_1)$$

is a normalized extremal quasimodular form of weight $12n+2$ *and depth* 1.

(3) *The form*

$$\frac{1}{\widetilde{N}_n}(E_2(\tau)E_4(\tau)\Delta(\tau)^n \widetilde{A}_{n+1}(j(\tau)) - E_6(\tau)\Delta(\tau)^n\widetilde{B}_{n+1}(j(\tau)))$$
$$(\widetilde{N}_n = \lambda_{2n+1}\lambda_{2n} \cdots \lambda_2\lambda_1)$$

is a normalized extremal quasimodular form of weight $12n+6$ *and depth* 1.

(4) *The form*

$$\frac{1}{\widetilde{N}_n^*}(E_4(\tau)^2\Delta(\tau)^n \widetilde{A}_{n+1}^*(j(\tau)) - E_2(\tau)E_6(\tau)\Delta(\tau)^n\widetilde{B}_{n+1}^*(j(\tau)))$$
$$(\widetilde{N}_n^* = \lambda_{2n+1}^*\lambda_{2n}^* \cdots \lambda_2^*\lambda_1^*)$$

is a normalized extremal quasimodular form of weight $12n+8$ *and depth* 1.

Proof. The relations between μ_n, μ_n^* and λ_n, λ_n^* are given by (we put $\mu_0 = 720$ and $\mu_0^* = 1008$)

$$\begin{aligned}
&\lambda_{2n+1} + \lambda_{2n} = 1728 - (\mu_{2n}^* + \mu_{2n-1}^*), \quad \lambda_{2n}\lambda_{2n-1} = \mu_{2n-1}^*\mu_{2n-2}^* \quad (n \geq 1),\\
&\lambda_{2n} + \lambda_{2n-1} = 1728 - (\mu_{2n-1} + \mu_{2n-2}), \quad \lambda_{2n-1}\lambda_{2n-2} = \mu_{2n-2}\mu_{2n-3} \quad (n \geq 2),\\
&\lambda_{2n+1}^* + \lambda_{2n}^* = 1728 - (\mu_{2n} + \mu_{2n-1}), \quad \lambda_{2n}^*\lambda_{2n-1}^* = \mu_{2n-1}\mu_{2n-2} \quad (n \geq 1),\\
&\lambda_{2n}^* + \lambda_{2n-1}^* = 1728 - (\mu_{2n-1}^* + \mu_{2n-2}^*), \quad \lambda_{2n-1}^*\lambda_{2n-2}^* = \mu_{2n-2}^*\mu_{2n-3}^* \quad (n \geq 2).
\end{aligned}$$

By using these, that the forms given in the theorem coincide up to normalizing factors to those in Theorem 2.1 is easily seen. For instance, in the case of weight $12n+2$, we can show that the identities

$$\begin{aligned}
A_n(x^2 + 1728) &= xP_{2n-1}^*(x) + 1008Q_{2n-1}^*(x)\\
B_n(x^2 + 1728) &= P_{2n-1}^*(x)/x
\end{aligned}$$

hold by checking that both sides of each identity satisfy the same recursion and initial conditions. Then by noting $j(\tau) = (E_6(\tau)/\sqrt{\Delta(\tau)})^2 + 1728$ we obtain the assertion in this case. The remaining cases can be shown similarly.

To determine the normalizing factors, we use the argument in [**4**, Section 4]. We only look at the case (2), the other cases being similar. Put

$$\Phi = \frac{E_2E_4}{E_6 j} = \frac{1}{j} + \frac{720}{j^2} + \cdots .$$

In [**4**], it is shown that

$$\frac{B_n(j)}{A_n(j)} = \Phi - \frac{N_n}{j^{2n+1}} + O\left(\frac{1}{j^{2n+2}}\right),$$

where N_n is the number given in the theorem and is equal to the inner product (A_n, A_n) associated to Φ (in this case it is equal to the Atkin scalar product). From this we have

$$E_2\Delta^n A_n(j) - E_4^2E_6\Delta^{n-1}B_n(j) = N_nq^{2n} + O(q^{2n+1}),$$

which shows the left-hand side divided by N_n is a normalized extremal quasimodular form ($\dim QM_{12n+2}^{(1)} = 2n+1$). □

Remark 2.4. The polynomials $\{A_n(X)\}$ is so-called Atkin's orthogonal polynomials. For any prime p, denoting by n_p the number of isomorphism classes of supersingular elliptic curves in characteristic p, the roots of $A_{n_p}(X)$ mod p exactly give the j-invariants of supersingular elliptic curves (see [**4**]).

3. Depth 2 case

We shall describe extremal quasimodular forms of depth 2 when the weights are divisible by 4.

Define polynomials $P_n(x)$, $Q_n(x)$, $R_n(x)$ by initial values

$$\begin{aligned} &P_0(x) = 0, \quad P_1(x) = -120, \quad P_2(x) = 420x, \\ &Q_0(x) = 0, \quad Q_1(x) = 0, \quad Q_2(x) = -420, \\ &R_0(x) = 1, \quad R_1(x) = 0, \quad R_2(x) = 0 \end{aligned}$$

and the same recursions

$$\begin{aligned} P_{n+1}(x) &= a_n x P_n(x) + b_n x^2 P_{n-1}(x) + c_n P_{n-2}(x) \quad (n \geq 2), \\ Q_{n+1}(x) &= a_n x Q_n(x) + b_n x^2 Q_{n-1}(x) + c_n Q_{n-2}(x) \quad (n \geq 2), \\ R_{n+1}(x) &= a_n x R_n(x) + b_n x^2 R_{n-1}(x) + c_n R_{n-2}(x) \quad (n \geq 2), \end{aligned}$$

where

$$\begin{aligned} a_n &= 16n^3 - 20n^2 + 2n - 1, \\ b_n &= -(4n-9)(4n-1)(2n-1)^2(n-1)^2, \\ c_n &= 8(4n-13)(4n-9)(4n-7)(4n-5)(4n-1)(2n-3)^2(2n-1)^2. \end{aligned}$$

THEOREM 3.1. *Suppose that* $k = 4n$ $(n = 0, 1, 2, \ldots)$. *Then the form*

$$\begin{aligned} &\sqrt[3]{\Delta(\tau)}^{\,n-1} P_n\left(\frac{E_4(\tau)}{\sqrt[3]{\Delta(\tau)}}\right)\left(-\frac{E_2'(\tau)}{24}\right) \\ &\quad + \sqrt[3]{\Delta(\tau)}^{\,n-2} Q_n\left(\frac{E_4(\tau)}{\sqrt[3]{\Delta(\tau)}}\right)\left(-\frac{E_6'(\tau)}{504}\right) + \sqrt[3]{\Delta(\tau)}^{\,n} R_n\left(\frac{E_4(\tau)}{\sqrt[3]{\Delta(\tau)}}\right) \end{aligned}$$

is an extremal quasimodular form of weight k *and depth* 2, *and satisfies the differential equation*

$$f''' - \frac{k}{4}E_2 f'' + \frac{k(k-1)}{4}E_2' f' - \frac{k(k-1)(k-2)}{24}E_2'' f = 0.$$

Proof. We can mimic the argument given in [**2**, Section 3], although computations naturally become more tedious. First, it is convenient to re-write the differential equation in the theorem as

$$\partial_{k+2}\partial_k\partial_{k-2}(f) - \frac{3k^2-4}{144}E_4\partial_{k-2}(f) - \frac{(k-2)^2(k+1)}{864}E_6 f = 0, \qquad (\sharp)_k$$

where ∂_k is the operator given by

$$\partial_k(f) = q\frac{df}{dq} - \frac{k}{12}E_2 f.$$

We show that we can construct a quasimodular solution of $(\sharp)_k$ inductively as follows. Put $f_0 = 1$, $f_4 = 5E_2'$, $f_8 = 420(E_6'/504 - E_4E_2'/24) = 5(5E_4^2 + 10E_6E_2 - 7E_4E_2^2)/24$, and define

$$f_{k+4} = a_k E_4 f_k + b_k E_4^2 f_{k-4} + c_k \Delta f_{k-8} \quad (k \geq 8)$$

with

$$\begin{aligned} a_k &= \tfrac{1}{4}(k^3 - 5k^2 + 2k - 4), \\ b_k &= -\tfrac{1}{64}(k-9)(k-4)^2(k-2)^2(k-1), \\ c_k &= \tfrac{1}{2}(k-13)(k-9)(k-7)(k-6)^2(k-5)(k-2)^2(k-1). \end{aligned}$$

Then f_k is a solution of $(\sharp)_k$. To show this, we need the following lemma, which corresponds to lemmas and propositions in [**2**, Section 3]. Since the argument is completely parallel, we omit the proof.

To state the lemma, we need to introduce the Rankin–Cohen bracket $[\cdot,\cdot]_n^{(k,l)}$, which is defined for integers $k, l, n \geq 0$ and functions f, g on the upper half-plane by

$$[f,g]_n^{(k,l)} := \sum_{i=0}^{n} (-1)^i \binom{n+k-1}{n-i}\binom{n+l-1}{i} f^{(i)} g^{(n-i)},$$

where $f^{(i)}$ (respectively $g^{(n-i)}$) is the ith (respectively $(n-i)$th) derivative of f (respectively g) with respect to $2\pi i\tau$. We have $[f,g]_0^{(k,l)} = fg$, $[f,g]_1^{(k,l)} = kfg' - lf'g$, etc. When f and g are modular of respective weights k and l on some group $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$, the bracket $[f,g]_n^{(k,l)}$ is modular of weight $k+l+2n$ on Γ (see [**6**] for properties of the Rankin–Cohen bracket).

LEMMA 3.2.
(1) *If f is a solution of $(\sharp)_k$, then*

$$\begin{aligned} \partial_{k+6}([f,E_4]_2^{(k-2,4)}) &= \tfrac{5}{63}(k-3)[f,E_6]_2^{(k-2,6)}, \\ \partial_{k+8}([f,E_6]_2^{(k-2,6)}) &= \tfrac{7}{20}(k-4)E_4[f,E_4]_2^{(k-2,4)} - 42(k-2)^2(k-1)\Delta f. \end{aligned}$$

(2) *If f is a solution of $(\sharp)_k$, then $[f,E_4]_2^{(k-2,4)}/\Delta$ is a solution of $(\sharp)_{k-4}$.*

(3) *Suppose that f_k, f_{k-4} and f_{k-8} are solutions of $(\sharp)_k$, $(\sharp)_{k-4}$ and $(\sharp)_{k-8}$, respectively. Then, the form*

$$f_{k+4} = E_4 f_k + E_4^2 f_{k-4} + \Delta f_{k-8}$$

is a solution of $(\sharp)_{k+4}$ if and only if

$$\begin{aligned}[f_k, E_6]_2^{(k-2,6)} &+ 2E_4[f_{k-4}, E_6]_2^{(k-6,6)}\\ &= -\tfrac{7}{8}\Delta(27\,648\partial_{k-6}(f_{k-4}) + 12(k-2)E_4\partial_{k-10}(f_{k-8})\\ &\quad + (k^2-6k+20)E_6 f_{k-8}).\end{aligned}$$

The way to obtain the description of f_k as in Theorem 3.1 is also similar to that in [**2**]. This time, the polynomials $P_n,\ Q_n,\ R_n$ have the form $x^i\,\widetilde{P}_n(x^3)$, etc., according to the value n mod 3. From this, the appearance of $\sqrt[3]{\Delta(\tau)}$ turns out to be superficial and the expression in the theorem defines an element in $\mathbb{C}[E_2, E_4, E_6]$. □

The differential equations in Theorems 2.1 and 3.1 can be neatly written in terms of the operator $\theta_k^{(r)}$ defined by

$$\theta_k^{(r)}(f) := f^{(r+1)} - \frac{k+r}{12}[E_2, f]_r^{(2,k)}.$$

When $r = 0$, the operator

$$\theta_k^{(0)}(f) = f' - \frac{k}{12}E_2 f$$

(identical to the ∂_k above) is sometimes referred to as Serre's operator and sends modular forms of weight k to those of weight $k+2$. When $r = 1$ and 2, we have

$$\theta_{k-1}^{(1)}(f) = f'' - \frac{k}{6}E_2 f' + \frac{k(k-1)}{12}E_2' f$$

and

$$\theta_{k-2}^{(2)}(f) = f''' - \frac{k}{4}E_2 f'' + \frac{k(k-1)}{4}E_2' f' - \frac{k(k-1)(k-2)}{24}E_2'' f.$$

These are nothing but the differential operators that appeared in Theorems 2.1(1) and 3.1. In the next section we give a speculation for the cases $r = 3$ and 4.

We record here one general property of the operator $\theta_k^{(r)}$, which might be of considerable interest.

PROPOSITION 3.3. *For all $n \geq 0$, if $f \in QM_{k+n}^{(n)}$, then $\theta_k^{(r)}(f) \in QM_{k+n+2(r+1)}^{(n)}$. In particular, if f is modular of weight k, then $\theta_k^{(r)}(f)$ is modular of weight $k+2(r+1)$.*

Proof. To prove the proposition, we introduce three operators D, ∂ and H on $\mathbb{C}[E_2, E_4, E_6]$ as follows. We let D and ∂ be the differential operators

$$D = q\frac{d}{dq} \quad \text{and} \quad \partial = -12\frac{\partial}{\partial E_2},$$

and H the Euler operator

$$H(f) = kf \quad (k = \text{ the weight of } f).$$

Alternatively, if we write $x = E_4$, $y = E_6$ and $z = E_2$, these operators on the ring $\mathbb{C}[x, y, z]$ are given by (use Ramanujan's formulas for D)

$$\begin{aligned} D &= \frac{xz-y}{3}\frac{\partial}{\partial x} + \frac{yz-x^2}{2}\frac{\partial}{\partial y} + \frac{z^2-x}{12}\frac{\partial}{\partial z}, \\ \partial &= -12\frac{\partial}{\partial z}, \\ H &= 4x\frac{\partial}{\partial x} + 6y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}. \end{aligned}$$

It is easy to check that these operators form an '$\mathfrak{sl}_2$-triple'†, i.e. they satisfy the commutation relations $[D, \partial] = H, [H, D] = 2D$, and $[H, \partial] = -2\partial$. By using these relations, we can prove by induction the relation

$$\partial D^n = D^n\partial - nD^{n-1}H - n(n-1)D^{n-1} \quad (n \geq 0).$$

With this we can easily deduce

$$\partial\theta_k^{(r)}(f) = \theta_k^{(r)}(\partial f) + (r+1)(k - wt(f))\theta_{k+1}^{(r-1)}(f) \quad (wt(f) = \text{ weight of } f)$$

and then inductively

$$\partial^{n+1}\theta_k^{(r)}(f) = \sum_{i=0}^{n+1}\binom{n+1}{i}(r+1)_i(k+n-wt(f))_i\,\theta_{k+i}^{(r-i)}(\partial^{n+1-i}f)$$

where $(a)_i = a(a-1)\cdots(a-i+1)$.

Now, note that for a quasimodular form $f \in \mathbb{C}[E_2, E_4, E_6]$, f has depth at most r if and only if $\partial^{r+1}f = 0$. Assume $f \in QM_{k+n}^{(n)}$. Then we have $(k+n-wt(f))_i = 0$ for all $i > 0$ and $\partial^{n+1}f = 0$, hence $\partial^{n+1}\theta_k^{(r)}(f) = 0$. This shows $\theta_k^{(r)}(f) \in QM_{k+n+2(r+1)}^{(n)}$. □

Note that the operator $\theta_k^{(r)}$ does not increase the depth despite of differentiations.

†We learned this fact from Don Zagier. Also, an equivalent statement in a more representation theoretic setting is given in [**5**].

4. Observations and conjectures

As far as our numerical computations show, extremal quasimodular forms always exist (if weight and depth satisfy necessary conditions) and unique if normalized. For depth less than 5, we have the following.

CONJECTURE 1. *If $r = 3$ and $k \equiv 0 \bmod 6$, or $r = 4$ and $k \equiv 0 \bmod 12$, the extremal quasimodular form of weight k and depth r is a solution of*

$$\theta_{k-r}^{(r)}(f) = 0.$$

We have shown in Theorems 2.1 and 3.1 that this is so for $r = 1$, $k \equiv 0 \bmod 6$ and $r = 2$, $k \equiv 0 \bmod 4$.

By inspecting exponents, we see that any extremal quasimodular forms of weight k and depth $r \geq 5$ cannot satisfy the equation $\theta_{k-r}^{(r)}(f) = 0$. The case $r \leq 4$ is somehow special, so it seems, as the following conjecture supported by numerical evidence also shows.

CONJECTURE 2. *If the depth is at most 4, the Fourier coefficients of any normalized extremal quasimodular forms of weight greater than 2 are always positive. Moreover, no denominator of such coefficients has prime factors greater than the weight.*

One may expect some meaning of the positive coefficients, say as counting numbers of some objects. Indeed, for the normalized extremal quasimodular form of weight 6 and depth 3,

> the coefficient of q^d is equal to the number of simply ramified coverings of genus 2 and degree d of an elliptic curve over $\mathbb{C}$.

(See [**1**] and [**3**] for a more precise definition and statement.) We do not know any other example of extremal quasimodular forms whose Fourier coefficients allow such interpretation as counting numbers or something of that sort. It would be very interesting to have such an interpretation.

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