Dynamic Asset Allocation with Event Risks under Value-at-Risk Regulation

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1 Introduction

In the problem to estimate the probability corresponding to rare events we must focus on the tail of the distribution functions, and sometime we are puzzled by so-called fat-tail and long-tail distributions [1]-[14]. Fat-tail and long-tail lead us to overestimation or underestimation of rare events, then there occurs serious accidents such as large losses and damages in financial assets [1]-[10]. Similar problems are found in the control of network traffic [11]-[14].

It is assumed in a common hypothesis about the behavior of asset prices in perfect market that the returns show the random walk or the geometric Brownian motions in the continuous time form [21]. The model implies that asset prices are stationary and log-normally distributed. However, a number of investigators in the field of stock and commodity prices have questioned the accuracy of the hypothesis [15]-[20]. The assumption about the independent increments and the stationarity have been criticized , and also from nonacademic aspects stock price patterns and related trading rules called technical analysis have been investigating on the conditions with presupposed departure from random price changes.

Moreover, one of the inherent hazards of investing in financial market is the risk incurred by the sudden large shock in security prices and volatilities. With the event-related jumps, the investors must also consider the effects of large security prices and volatility changes in selecting dynamic portfolio strategy. We must keep the portfolio to be optimal enough for large returns as well as for small returns in event risks.

In addition to dynamic portfolio selections, the impact of market risk regulation on optimal portfolio must be taken into account to maintain and improve the safety of financial institutions [15]-[20]. In 1996, VaR-based risk management had already emerged as common market practice. In the 1996 Amendment on market risk regulation, the Bank of International Settlement (BIS) chose VaR as the regulatory reporting tool for the market risk of the banks' trading book.

This paper deals with the implications of event-related jumps in security prices and the dynamic portfolio strategies [15]-[20]. At the same time, the impact of VaR-based regulation on the dynamic portfolio is also discussed to examine the deviation from the equilibrium. Based on the incomplete market model different form normally distributed returns, we can analyze the role of regulation under worse market conditions. In the investment horizon, the problem of intertemporal optimization problem under VaR constraints is resolved. Then, the method proposes individuals risk management within the framework of equilibrium analysis with heterogeneous banks.

In the model, the security price follows jump-diffusion processes which are triggered by a Poisson event. Because of the tractability provided by the affine structure of the model, we can reduce the Hamilton-Jacobi-Bellman partial differential equations which are allowing us to obtain an analytical solution. In the model, it is assumed that VaR is bounded at time t by an exogenous limit proportional to the current wealth directly for a given time horizon, then the problem becomes to be tractable enough. By using the first-order approximation of the wealth process, we find the optimal dynamic portfolio in which we switch the weight for the risky asset depending on the boundaries of weight. As a result, the equilibrium incentive of VaR regulation can lead banks to increase their risk exposure in high-volatility states.

In the followings, in Section 2, we show the basics of the asset price dynamics and budget equation. Section 3 gives modeling of impact of VaR regulation. In Section 4, we describe examples of conventional works for the dynamic portfolio selection and VaR regulations. Section 5 shows the application for the dynamic asset allocation with event risks under VaR regulation.

2 Asset Price Dynamics and Budget equation

2.1 Processes of asset prices (without Jump processes)

At first, we describe the asset price dynamics and budget equation following the Merton's result [18]-[20]. However, for the first step, we only explain the formalization for the cases where the price processes include no jump processes.

To apply the dynamic programming technique in a continuous-time model, the state variable dynamics must be expressible as Markov stochastic processes defined over time intervals of small length h. The two types of the processes are functions of Gauss-Wiener Brownian motions which are continuous in the space variables, and the Poisson processes (the jump-diffusion processes) which are discrete in the space variables. A particular class of continuous-time Markov processes called Ito process are defined as the solution of the stochastic differential equation.

$$dP = f(P,t)dt + g(P,t)dz \tag{1}$$

where P, f, g are n vectors and $z(t)$ is an n vector of standard normal random variables. Then, $dz(t)$ is called a multi-dimensional Wiener process (Brownian motion).

Throughout the paper, it is assumed that all assets are of the limited liability type, that there exist continuously trading perfect markets with no transaction costs for all assets, and that the prices per share $P_i(t)$ are generated by Ito processes.

$$\frac{dP_i}{P_i} = \alpha_i(P,t)dt + \sigma_i(P,t)dz_i \tag{2}$$

where α_i is the instantaneous conditional expected percentage change in price per unit time, and σ_i^2 is the instantaneous conditional variance per unit time. In the particular case where the geometric Brownian motion hypothesis is assumed to hold for asset prices, α and σ will be constants.

To derive the correct budget equation, it is necessary to examine the discrete-time formulation of the model and then to take limits carefully to obtain the continuous time form. Consider a period model with periods of length h, where all income is generated by the capital gains, and wealth, $W(t)$ and $P_i(t)$ are known at the beginning of period t. Let the decision variables be indexed such that the indices coincide with the period in which the decision are implemented. Namely, let

$N_i(t)$:number of shares of asset i purchased during period between t and $t+h$(called period t)

$C(t)$:amount of consumption per unit time during period between t and $t+h$

The model assumes that the individual "comes into" period t wealth invested in assets so that

$$W(t) = \sum_{1}^{n} N_i(t-h)P_i(t) \tag{3}$$

Notice that it is $N_i(t-h)$ because $N_i(t-h)$ is the number of shares purchased for the portfolio in the period $(t-h)$, and it is $P_i(t)$ because $P_i(t)$ is the current value of share of the i-th asset. The amount of consumption C for the period t, namely $C(t)h$ and the new portfolio, $N_i(t)$ are simultaneously chosen, and if it it is assumed that all trades are made at known current prices, then we have that

$$-C(t)h = \sum_1^n [N_i(t) - N_i(t-h)]P_i(t) \tag{4}$$

Incrementing equation (3) and equation (4) by h to eliminate backward differences, we have following two equations.

$$-C(t+h)h = \sum_1^n [N_i(t+h) - N_i(t)]P_i(t+h) \tag{5}$$

$$W(t+h) = \sum_1^n N_i(t)P_i(t+h) \tag{6}$$

The first equation is equal to

$$-C(t+h)h = \sum_1^n [N_i(t+h) - N_i(t)][P_i(t+h) - P_i(t)] + \sum_1^n [N_i(t+h) - N_i(t)]P_i(t) \tag{7}$$

Taking the limits as $h \to 0$, we arrive at the continuous version of equations (6) and (7), respectively.

$$-C(t)dt = \sum_i^n dN_i(t)dP_i(t) + \sum_1^n dN_i(t)P_i(t) \tag{8}$$

$$W(t) = \sum_i^n N_i(t)P_i(t) \tag{9}$$

Using Ito lemma, we differentiate equation (9) to get

$$dW = \sum_1^n N_i dP_i + \sum_1^n dN_i P_i + \sum_1^n dN_i dP_i \tag{10}$$

The last terms $\sum_1^n dN_i P_i + \sum_1^n dN_i dP_i$ are the net value of additions to wealth from sources other than capital gains. Then,we have

$$-C(t)dt = \sum_1^n dN_i P_i + \sum_1^n dN_i dP_i \tag{11}$$

From equations (2) and (10), the budget or accumulation equation is written as

$$dW = \sum_1^n N_i(t)dP_i - C(t)dt \tag{12}$$

For convenience, we define a new variable $w_i(t) = N_i(t)P_i(t)/W(t)$, the percentage of wealth invested in the i-th asset at time t. Substituting dP_i/P_i from equation (2) , we can write equation (12) as

$$dW = \sum_1^n w_i W \alpha_i dt - Cdt + \sum_1^n w_i W \sigma_i dz_i \tag{13}$$

If one of the n-asset is risk-free, by convenience, the n-th asset ,then $\sigma_n = 0$, the instantaneous rate of return α will be called r and is rewritten as

$$dW = \sum_{1}^{m} w_i(\alpha_i - r)Wdt + (rW - C)dt + \sum_{1}^{m} w_i\sigma_i W dz_i \tag{14}$$

where $m = n-1$, and $w_n = 1 - \sum_1^m w_i$ will ensure that the identity constraints in equation (14) is satisfied.

2.2 Optimal portfolio and consumption rules

The problem of choosing optimal portfolio and consumption rules for an individual who lives T years is formulated as follows.

$$\max E_0[\int_0^T U(C(t),t)dt + B(W(T),T)] \tag{15}$$

subject to $W(0) = W_0$. We must note that the budget constraint equation in the case of a risk-free asset becomes equation (14), and the utility function is assumed to be strictly concave in C. It is also noted that the bequest function B is assumed also to be concave in W.

To derive the optimal rules, the technique of stochastic dynamic programming is used. For simplicity, we assume that we have one risky asset and one risk-free asset with interest rate r whose ratio in the wealth W is defined as w. Also, we assume that tentatively the parameters α and σ in the price process are constants (independent from P).

We define

$$J(W,t) = \max_{C,w} E_t[\int_t^T U[C(s)]ds + B(W(T),T)] \tag{16}$$

where E_t is the conditional expectation operator, conditional on $W(t) = W$ and $P_i(t) = P_i$. Therefore,

$$J(W(T),T) = B[W(T),T] \tag{17}$$

To derive the optimality equations, we restate equation (16) in a dynamic programming form so that the Bellman's principle of optimality can be applied.

In the following, it is assumed that the measure $J(W,t)$ is only the function of wealth $W(t)$ and time t, at first. In general, from definition (1),

$$J(W(t_0),t_0) = \max E_0[\int_{t_0}^t U[C(s)]ds + J(W(t),t)] \tag{18}$$

and in particular, (18) can be rewritten as

$$J(W_0,0) = \max E[\int_0^t U[C(s)]ds + J(W(t),t] \tag{19}$$

if $t = t_0 + h$ and the third partial derivatives of $J[W(t_0),t_0)]$ are bounded, then by Taylor's theorem and the mean value theorem for integrals, equation (19) can be rewritten as

$$J(W(t_0,t_0) = \max_{C,w} E_0[U(C(t) + J[W(t_0),t_0)] + \frac{\partial J[W(t_0),t_0)]}{\partial t} + \frac{\partial J[W(t_0),t_0)]}{\partial W}[W(t) - W(t_0)]$$

$$+\frac{1}{2}\frac{\partial J^2[W(t_0),t_0)]}{\partial W^2}[(W(t) - W(t_0)^2]] \tag{20}$$

by neglecting small term $O(h^2)$. In equation (20), take the E_0 operator onto each term and, noting that $J([W(t_0),t_0)] = E_0[J[W(t_0),t_0)]$, subtract $J[W(t_0),t_0)]$ from both sides. Substitute relations $E_0[W(t)-W(t_0)] = [w(t_0)(\alpha-r)+r]W(t_0)$ and $E_0[(W(t)-W(t_0))^2] = w(t_0)^2W(t_0)^2\sigma^2h$ for $E_0[W(t)-W(t_0)]$ and $E_0[(W(t)-W(t_0))^2]$, and then derive the equation by h. Take the limit of the resultant equations as $h \to 0$ and (20) becomes a continuous-time version of the Bellman-Drefus fundamental equation of optimality as follows.

$$0 = \max_{C(t),w(t)}[U(C(t) + \frac{\partial J_t}{\partial t} + \frac{\partial J_t}{\partial W}[(w(t)(\alpha-r)+r)W(t) - C(t)] + \frac{1}{2}\frac{\partial J_t^2}{\partial W^2}\sigma^2 w(t)^2 W(t)^2 \quad (21)$$

where J_t is short for $J[W(t),t)]$ and the subscript on t_0 has been dropped to reflect that (21) hold for any $t \in [0,T]$.

If we define $\phi(w,C,W,t)$ using the Dynkin operator $L[J]$

$$\phi(w,C,W,t) = U(C) + L[J] \quad (22)$$

then, equation (21) can be rewritten in the more compact form as

$$\max_{C,w} \phi(w,C,W,t) = 0 \quad (23)$$

Now, we extend the model to general cases where the measure $J(.)$ includes also the price variable $P(t)$. Namely, the parameters α and σ in the price process are dependent form P. In these cases, the Dynkin operator is defined as

$$L = \frac{\partial}{\partial t} + [\sum_1^n w_i\alpha_i W - C]\frac{\partial}{\partial W} + \sum_1^n \alpha_i P_i \frac{\partial}{\partial P_i} + \frac{1}{2}\sum_1^n\sum_1^n \sigma_{ij}w_iw_jW^2\frac{\partial^2}{\partial W^2}$$
$$+\frac{1}{2}\sum_1^n\sum_1^n P_iP_j\sigma_{ij}\frac{\partial^2}{\partial P_iP_j} + \sum_1^n\sum_1^n P_iWw_j\sigma_{ij}\frac{\partial^2}{\partial P_i\partial W} \quad (24)$$

In the definition, we assume that the measure $J(.)$ is not only the function of the wealth $W(t)$ and time t, but also the function of asset prices $P_i(t)$. The weight of these asset in the wealth $W(.)$ is defined again as $w_i, i = 1,2,..,n$. It is easily shown the extension of the definition is relevant.

From the theory of stochastic dynamic programming, we have a set of optimal rules (controls) w^* and C^* satisfying $\sum_1^n w_i^* = 1$ and $J(W,P,t) = B(W,T)$, where it is assumed that the $P_i(t)$ are generated by a strong diffusion process, $U(C)$ is strictly concave in C, and $B(W)$ is concave in W.

Following the usual fashion of maximization under constraint, we define the Lagrangian, $K = \phi + \lambda[1 - \sum_1^m w_i]$ where λ is the multiplier and find the extreme points from the first-order conditions.

$$0 = K_C(C^*,w^*) = U_C(C^*,t) - J_W \quad (25)$$

$$0 = K_{w_k}(C^*,w^*) = -\lambda + J_W\alpha_kW + J_{WW}\sum_1^n \sigma_{kj}w_j^*W^2 + \sum_1^n J_{jW}\sigma_{kj}P_jW \quad (26)$$

$$0 = K_\lambda(C^*,w^*) = 1 - \sum_1^n w_i^* \quad (27)$$

We have notations for partial derivatives as $J_W = \partial J/\partial W, J_t = \partial J/\partial t J_i = \partial J/\partial P_i, J_{ij} = \partial J^2/\partial P_iP_j, J_{iW} = \partial J^2/\partial P_i\partial W, U_C = \partial U/\partial C$.

Because $K_{CC} = \phi_{CC} = U_{CC.} < 0, K_{Cw_k} = \phi_{Cw_k} = 0, K_{w_k w_k} = \sigma_k^2 W_k^2 J_{WW}, K_{w_k w_j} = 0$, for $k \neq j$, a sufficient condition for a unique interior maximum is that $J_{WW} < 0$.

To solve explicitly for C^* and w^*, we solve $n+2$ nondynamic implicit equations (25)～(27) for C^*, w^* and λ as functions of $J_W, J_{WW}, J_{jW}, W, P$, and t. Then, C^* and w^* are substituted in (15) which is now becomes a second-order partial differential equation for J, subject to the boundary condition $J(W,P,T) = B(W,T)$.

For the case where one of the asset is risk-free, the equations are simplified because the problem can be solved directly as an unconstrained maximum by eliminating w_n. Then, the optimal proportions in the risky assets are

$$w_k^* = -\frac{J_W}{J_{WW}W}\sum_1^m v_{kj}(\alpha_j - r) - \frac{J_{kW}P_k}{J_{WW}W}, k = 1, 2, ..., m \tag{28}$$

where the matrix v_{ij} is defined as the inverse of covariance matrix σ_{ij} as

$$[v_{ij}] = \Omega, \Omega = [\sigma_{ij}] \tag{29}$$

Then, by substituting equations (28) and (32) into the equation (15), we have the corresponding partial differential equation for J as

$$0 = U[G,T] + J_t + J_W[rW - G] + \sum_1^m J_i\alpha_i P_i + \frac{1}{2}\sum_1^m\sum_1^m J_{ij}\sigma_{ij}P_iP_j - \frac{J_W}{J_{WW}}\sum_1^m J_{jW}P_j(\alpha_j - r)$$

$$+\frac{1}{2}\frac{J_W^2}{J_{WW}}\sum_1^m\sum_1^m v_{ij}(\alpha_i - r)(\alpha_j - r) - \frac{1}{2J_{WW}}\sum_1^m\sum_1^m J_{iW}J_{jW}\sigma_{ij}P_iP_j \tag{30}$$

subject to the boundary condition $J(W,P,T) = B(W,T)$, where the function $G(.)$ is defined as the inverse function of $U(C)$.

$$G = [U(C)]^{-1} \tag{31}$$

Then, we have

$$C^* = G(J_W, t) \tag{32}$$

However, the original equations including general proportion of assets lead themselves to hard problems to be solved. The equation have very deep complexity based on the nonlinearity of the equations and the large number of state variables.

2.3 Optimal portfolio with prices having Poisson (Jump) processes

Returning to the consumption-portfolio problem, assume that one asset is a common stock whose price is log-normally distributed, and the other asset is a risky bond which pays a instantaneous rate of interest when not in default, but in the event of default, the price of the bond becomes zero.

The process which generates the bond's price P can be written as

$$dP = rPdt + \sigma dz - Pdq \tag{33}$$

where dq is defined as a Poisson process. The simplest independent Poisson process defines the probability of an event occurring in the time interval $(t, t+h)$ (where h is as small as you like) as follows.

$$Prob(t) = \begin{cases} 1 - \lambda h + O(h), & \text{event does not occur;} \\ \lambda h + O(h), & \text{event occurs once} \end{cases}. \tag{34}$$

Then, the budget equation becomes as

$$dW = [wW(\alpha - r) + rW - C]dt + w\sigma W dz - (1 - w)W dq \tag{35}$$

We have three optimality equations as

$$0 = U(C^*, t) + J_t(W, t) + \lambda[J(w^*W, t) - J(W, t)] + J_W(W, t)[(w^*(\alpha - r) + r)W - C^*] + \frac{1}{2}J_{WW}(W, t)\sigma^2 w^{*2}W^2 \tag{36}$$

$$0 = U_C(C^*, t) - J_W(W, t) \tag{37}$$

$$0 = \lambda J_W(w^*W, t) + J_W(W, t)(\alpha - r) + J_{WW}(W, t)\sigma^2 w^{*2}W \tag{38}$$

3 Modeling of impact of VaR regulation

3.1 Simplified VaR limit

Usually, VaR is defined as the probability level

$$Prob[W(t) - W(t + \tau)] \geq L_{loss} \tag{39}$$

for a given loss probability L_{loss} and the time horizon τ. For reporting purposes the time horizon τ is typically one day or 10 days. But, we must note that we work with fixed relative portfolio weights, although the regulator requires to assume fixed absolute weights for a given holding period. Most particular implementations of VaR calculations, such as RiskMetrics calculate VaR over a one-day holding period and scale it accordingly to obtain the 10-days VaR required for regulatory reporting. For the one-day holding period, it is typically assumed that the drift of portfolio changes is equal to zero. Under this assumption, the fixed absolute weights and the fixed relative weight assumptions are indeed equivalent. We utilize these characteristics for the portfolio evaluation under VaR regulations.

There exist several models for evaluating VaR limit such as the Variance-Covariance method (called Delta method), but these rigid definitions are not relevant to estimate the impact of VaR regulations. Following the research by Leippold et al., we use the definition of VaR as follows [16].

$$VaR = \beta W(t) \tag{40}$$

We work with a VaR limit proportional to current wealth $W(t)$. Even though the VaR limit in equation (40) is a legitimate but certainly not unique choice. In practice, different risk limit specifications are used. However, the definition in equation (8) has some nice tractability properties when we perform the optimization. Then, we restrict our analysis to a proportional VaR limit to mimic the regulation framework.

We must note that the wealth dynamics depends on the stochastic state variable $P(t)$, then we cannot expect to obtain closed form solutions for the bank's intertemporal decision problem in the presence of VaR regulations. To retain analytical tractability, we approximate the VaR constraint shown in equation (40).

So as to approximate the VaR constraints implied by equation (40), we apply the Ito Tayler expansion formula to define the first-order approximation.

$$\log W(t + \tau) \simeq \log W(t + \tau)^{(1)} = \log W(t) + [r(P(t)) + w(t)\lambda(P) - \frac{1}{2}w^2\sigma(P(t))^2] \tag{41}$$

As Leippold et al. discussed, the approximation error using the first-order approximation is relatively good. They define the approximation error as the probability of the first-order approximation $W(t+\tau)^{(1)}$ for the value $W(t)$ of a fixed weight portfolio with initial weight $w(t)$ which is bounded by

$$Prob[\log W(t+\tau)^{(1)} - \log W(t+\tau)] \geq M] \tag{42}$$

As they suggested the conditional probability that the logarithmic difference between the approximated wealth and the true wealth exceeds the amount M at time $t+\tau$ can be bounded by a measure R. If we assume the mean-reverting geometric Brownian motion for the volatility process, the experimental results show the approximation error M is usually bounded below 1%.

The quality of the approximation ensures us to use the VaR approximation to investigate the constrained dynamic portfolio. Moreover, market practice usually confines itself to regulatory VaR figures reported based on a conditional normal distribution. The approximation implies us the possibility of the direct portfolio bounds on the optimal policy of VaR-constrained bank.

3.2 Upper and lower bounds of weight

It is shown that under the approximation in equation (41), the constraint VaR is equivalent to the following upper and lower bounds on the fraction $w(t)$ of wealth invested in the risky asset.

$$w^-(P) \leq w(t) \leq w^+(P) \tag{43}$$

It is seen from equation (8), for the VaR limit a bound on the optimal portfolio fraction that is wealth independent. In general, the VaR limit lead of wealth-dependent VaR boundaries under the above approximation procedure. It is also seen that $w^+(P) \geq 0$ and $w^-(P) \leq 0$. These inequalities hold for all functional forms $\lambda(P(t))$ and $\sigma(P(t))$. It is also found that the portfolio bound are functions of the interest rate r, and equity expected returns $\alpha(P(t))$ and volatilities $\sigma(P(t))$.

3.3 Partial equilibrium (without Jump processes)

At the beginning for the discussion of optimal portfolio selection, we at first treat the case where the price process includes no jump diffusion, and follows ordinary Brownian motion, however time dependent characteristics of parameters, according to the result obtained by Leippold et al. [18]. To reduce the bank optimal behavior under the VaR constraints to adjust the weight $w(t)$, we start with assuming next two assumptions. Also we assume that there is no consumption C for a while.

Assumption 1

The utility function from final wealth $W(T)$ is defined by a CRRA-utility function

$$U(W) = \frac{W^\gamma - 1}{\gamma}, \gamma < 1 \tag{44}$$

Assumption 2

The stochastic process $P(t)$ follows a mean reverting process given by

$$dP(t)/P(t) = \alpha(P)dt + rdt + \sigma(P)dZ(t) \tag{45}$$

We assume that the model parameters are chosen to ensure that the process $P(t)$ is a strictly positive process. The wealth dynamics is written as

$$\frac{dW(t)}{W(t)} = (w(t)\zeta(P) + r)dt + w(t)\sigma(P)dZ(t) \tag{46}$$

where $\zeta(P) = \alpha(P) - r$.

According to Assumption 1, the bank derives utility from only terminal wealth. Because, here we are interested in the partial equilibrium impact of VaR constraints in the presence of a stochastic opportunity set.

Based on the regulatory VaR constraints on the approximated VaR constraint denoted as $BR(W, P)$ the budget constraints, the optimal portfolio selection problem becomes

$$J(W, P, t) = \max_{w \in BR(W,P)} E[U(W)] \tag{47}$$

Intuitively, we see the solution of the problem must provide optimal investment strategies characterized by tha region where the VaR constraint binds and a region where it does not bind.

Consider the optimal control problem under the assumptions 1 and 2, the optimal portfolio is given by

$$w(P,t) = \begin{cases} w^+(P(t)), & \text{if } w_f(P(t)) \geq w^+; \\ w^-(P(t)), & \text{if } w_f(P(t)) \leq w^-; \\ w_f(P(t)), & \text{otherwise} \end{cases} \tag{48}$$

where

$$w_f(P(t), t) = -\frac{\zeta(P) J_W}{W J_{WW} W} - P(t)\frac{J_{PW}}{W J_{WW}} \tag{49}$$

is the solution for the constraint free optimal strategy. The two limits $w^-(P), w^+(P)$ are given by the solution of the following second-order equation.

We note that the VaR is equivalent to the form by using the first-order approximation of $\log W(t+\tau)$ at the confidence level v over the time horizon

$$VaR = W(t)[1 - \exp Q(P, w)] \tag{50}$$

$$Q(P, w) = \log(1-\beta) - [r + w\zeta(P) - \frac{1}{2}w^2\sigma(P)^2]\tau + vw\sigma(P)\sqrt{\tau} \tag{51}$$

with $\beta = L/W(t)$. Then, the restriction for the VaR is equivalent to

$$Q(P, w) \leq 0 \tag{52}$$

$$Q(P, w) = \log(1-\beta) - [r + w\zeta(P) - \frac{1}{2}w^2\sigma(P)^2]\tau - w\sigma(P)\sqrt{\tau}N^{-1}(\nu) \tag{53}$$

where $N^{-1}(\nu)$ is the inverse function of the integral of normal distribution function $N(\nu)$.

The outline of the proof is shown along the result given by Leippold et al. Under the VaR constraint, the HJB equations for the control problem is given as

$$0 = \max_w [J_t + W(r + w\zeta(P))J_W + \alpha(P)PJ_P + \frac{1}{2}\sigma^2 w^2 W^2 J_{WW} +$$

$$\frac{1}{2}\sigma^2 P^2 J_{PP} + wW\sigma^2 PJ_{WP} - \phi Q(w(t))] \tag{54}$$

or in the simplified form as

$$0 = \max_w [-\phi Q(w) + L[J] \tag{55}$$

with the terminal condition $J(W, P, t) = U(W, P)$, where $L[J]$ is the corresponding Dynkin operator.

The first-order conditions for the problem are $0 = (\partial/\partial w)(-\phi Q(w) + L[J]), 0 = \phi Q(w)$, and $Q(w) \leq 0$. From the conditions, we have

$$w(W^2 J_{WW} - \tau\phi) = -\frac{1}{\sigma^2}(\zeta\tau\phi + \sigma\sqrt{\tau}\phi N^{-1}(\nu) + W\lambda J_W + W\sigma^2 P J_{PW}) \tag{56}$$

Since the terms in the brackets are functions of W, X, and t there exist a function ϕ satisfying the first-order condition. The inequality $Q(w) \leq 0$ is equivalent to $w^- \leq w_f(t) \leq w^+$. If the VaR constraint does not bind, slackness implies

$$w_f(t) = \frac{W\eta J_W + \sigma^2 P J_{WP}}{W\sigma^2 J_{WW}} \tag{57}$$

If $w_f \geq w^+$, then J solves

$$0 = -J_t + \alpha J_P + \frac{1}{2}\sigma^2 P^2 J_{PP} + (r + \zeta w^+) W J_W + w^+ \sigma^2 P W J_{WP} + \frac{1}{2}\sigma^2 (w^+)^2 W^2 J_{WW} \tag{58}$$

. The same PDE holds if $w_f \leq w^-$ with w^+ replaced by w^-.

3.4 Optimal portfolio and consumption

As the next step, we extend the method of VaR regulation analysis for the cases where we also include the optimal consumption in the model. So as to obtain closed form solution, we restrict ourselves to the cases with HARA family form of the utility function.

As is shown in previous sections, the price process of the risky asset follows $dP/P = \alpha dt + \sigma dz$. The wealth process is given as

$$dW(t) = [W(t)(r + \zeta w(t) - C]dt + w(t)\sigma W(t) dz(t) \tag{59}$$

In the application of the Ito Taylor formula to define the first-order approximation of $\log W(t+\tau)$ we must note that the variable $C(t)$ is not multiplied by $W(t)$.

Therefore, if the functional form $V(C)$ in the utility function is not simple, then the closed form solution is not available. Then, in this case we assume that

$$V(C) = \frac{1-\gamma}{\gamma}(\frac{\beta C}{1-\gamma} + \eta)^{\gamma}, \eta = 0 \tag{60}$$

in the definition of $V(C)$.

Then, we have

$$\log W(t+\tau) \simeq \log W(t+\tau)^{(1)} = \log W(t) + [r + \zeta w(t) - H - \frac{1}{2}w^2\sigma^2]\tau \tag{61}$$

$$H = \frac{[\rho - \gamma\nu]}{\delta(1 - exp[\frac{(\rho-\gamma\nu)}{\delta}(t-T)])} \tag{62}$$

where $\delta = 1-\gamma, \nu = r + (\alpha - r)^2/2\delta\sigma^2$. The reduction of the definition is shown later.

The accompanied function $Q(w)$ becomes as

$$Q(w) = \log(1-\beta) - [r + w\zeta - H - \frac{1}{2}w^2\sigma^2]\tau - w\sigma\sqrt{\tau}N^{-1}(\nu) \tag{63}$$

The solutions of $Q(w) = 0$ denoted as w^-, w^+ give the parameters of optimal portfolio selection.

The same control scheme using w^-, w^+ is applied to the optimal portfolio selection, however the values of w^-, w^+ are given by the solution of equation (63).

$$w(t) = \begin{cases} w^+, & \text{if } w_f \geq w^+; \\ w^-, & \text{if } w_f \leq w^-; \\ w_f, & \text{otherwise} \end{cases} \tag{64}$$

where the symbol w_f is equal to the optimal value w^* without VaR regulation in previous discussion.

3.5 Partial equilibrium (with jump diffusion processes)

Now, we extend the partial equilibrium without jump diffusion in the price process to the cases with jump diffusion in the price processes.

We also see that the solution of the problem must provide optimal investment strategies characterized by the region where the VaR constraint binds and a region where it does not bind. Then, the calculation of $w^-(P), w^+(P)$ may be affected by the diffusion processes. we assume again that the bond price process can be written as

$$dP = rPdt + \sigma(P)dz - dq \tag{65}$$

where dq is defined as a Poisson process which is previously defined. The wealth process is written as

$$dW(t)/W(t) = (r + \zeta w(t))dt + w(t)\sigma dz(t) - (1 - w(t))dq \tag{66}$$

We apply the Ito Taylor formula to define the first-order approximation. Then, we have

$$\log W(t+\tau) \simeq \log W^{(1)}_{t+\tau} = \log W_t + [r + w(t)\zeta(P) - \lambda(1-w) - \frac{1}{2}w^2\sigma(P)^2]\tau \tag{67}$$

where we assume that the expectation of dq in the time interval dt is equal to λdt.

Then, the HJB equation with VaR restriction and second-order function $Q(w)$ corresponding to equation (54) and equation (63) become to be

$$0 = \max_w [J_t + \zeta[J(wW,t) - J(W,t)] + W(r + w\zeta(P))J_W + \alpha P J_P + \frac{1}{2}\sigma^2 w^2 W^2 J_{WW} +$$

$$\frac{1}{2}\sigma^2 P^2 J_{PP} + wW\sigma^2 P J_{WP} - \phi Q(P, w(t)) + L[J]] \tag{68}$$

$$Q(P,w) = \log(1-\beta) - [r + w\zeta(P) - \lambda(1-w) - \frac{1}{2}w^2\sigma(P)^2]\tau - w\sigma(P)\sqrt{\tau}N^{-1}(\nu) \tag{69}$$

The solutions of $Q(P,w) = 0$ denoted as w^-, w^+ give the parameters of optimal portfolio selection.

The same control scheme using w^-, w^+ is applied to the optimal portfolio selection, however the values of w^-, w^+ are given by the solution of equation (69).

3.6 Optimal portfolio and consumption with jump diffusion price processes

Then, we finally obtain the optimal control scheme for the cases where the price processes include the jump diffusion processes and also the optimal consumption is allowed in the dynamic behavior. As is shown by Merton, in this case the closed form solution is obtained only for HARA family of the function $V(C)$. We also assume that the parameters defining price processes are constant values.

In these cases, the VaR regulation free solution is given by the optimal combination of C and w (denoted as C^*, w^*). Moreover the wealth process is written as

$$dW(t) = [W(t)(r + \zeta w(t) - C]dt + w(t)W(t)\sigma W(t)dz(t) - (1 - w(t))W(t)dq \tag{70}$$

In the application of the Ito Taylor formula to define the first-order approximation of $\log W(t+\tau)$ we must note that the variable $C(t)$ is not multiplied by $W(t)$. Therefore, if the functional form $V(C)$ in the utility function is not simple, then the closed form solution is not available. Then, in this case we assume that $eta = 0$ in the definition of $V(C)$, and more simplified form is given as $U(C) = C^\gamma/\gamma$.

Then, we have

$$\log W(t+\tau) \simeq \log W^{(1)}_{t+\tau} = \log W_t + [r + \zeta w(t) - M - (1 - w) - \frac{1}{2}w^2\sigma^2]\tau \tag{71}$$

where

$$M = A/(1-\gamma)(1 - exp[A(t-T)/(1-\gamma)]), \tag{72}$$

$$A = -\gamma[\frac{(\alpha - r)^2}{2\sigma^2(1-\gamma))} + r] + \lambda[1 - \frac{(2-\gamma)}{\gamma}(w^*)^\gamma + \frac{\gamma(\alpha - r)}{2\sigma^2(1-\gamma)}(w^*)^{\gamma-1}] \tag{73}$$

$$w^* = \frac{(\alpha - r)}{\sigma^2(1-\gamma)} + \frac{\lambda}{\sigma^2(1-\gamma)}(w^*)^{\gamma-1} \tag{74}$$

The reduction of these values is given later. The accompanied function $Q(w)$ becomes as

$$Q(w) = \log(1-\beta) - [r + w\zeta - M - (1-w) - \frac{1}{2}w^2\sigma^2]\tau - w\sigma\sqrt{\tau}N^{-1}(\nu) \tag{75}$$

The solutions of $Q(w) = 0$ denoted as w^-, w^+ give the parameters of optimal portfolio selection.

The same control scheme using w'w^+ is applied to the optimal portfolio selection, however the values of w^-, w^+ are given by the solution of equation (72).

$$w(t) = \begin{cases} w^+, & \text{if } w_f \geq w^+; \\ w^-, & \text{if } w_f \leq w^-; \\ w_f, & \text{otherwise} \end{cases} \tag{76}$$

where the symbol w_f is equal to the optimal value w^* without VaR regulation in previous discussion.

4 Examples of conventional works

4.1 Explicit solutions for a particular class utility functions

We firstly show examples given by Merton showing explicit solutions for a particular class utility functions. Assume that the utility function for the individual, $U(C,t)$ can be written as $U(C,t) = e^{-\rho t}V(C)$, where V is a member of the family of utility functions whose measure of absolute risk aversion is positive and hyperbolic in consumption, namely,

$$A(C) = -V''/V' = 1/[\frac{C}{1-\gamma} + \eta/\beta] > 0 \tag{77}$$

subject to the restrictions

$$\gamma \neq 1, \beta > 0, (\frac{\beta C}{1-\gamma} + \eta) > 0, \eta = 1 if \gamma = -\infty \tag{78}$$

The family of functions is defined as the HARA family which is the concave functions having following forms. All member of the these HARA (hyperbolic absolute risk-aversion) family can be expressed as

$$V(C) = \frac{1-\gamma}{\gamma}(\frac{\beta C}{1-\gamma} + \eta)^{\gamma} \tag{79}$$

The function can realize a utility function with absolute or relative risk aversion increasing, decreasing or constant.

Assuming that there are one risky asset with return r and one risky asset whose price is log-normally distributed, and the parameters α, σ included in the price process are constant. From the optimality equation for J, we have

$$0 = \frac{(1-\gamma)^2}{\gamma} e^{-\rho t}[\frac{e^{\rho t}J_W}{\beta}]^{\gamma/(\gamma-1)} + J_t + [(1-\gamma)\eta/\beta + rW]J_W - \frac{J_W^2}{J_{WW}}\frac{(\alpha-r)^2}{\alpha\sigma^2} \tag{80}$$

subject to $J(W,T) = 0$. Then, the equations for the optimal consumption and portfolio rules are reduced to

$$C^*(t) = \frac{(1-\gamma)}{\beta}[\frac{e^{\rho t}J_W}{\beta}]^{1/(\gamma-1)} - \frac{(1-\gamma)\eta}{\beta} \tag{81}$$

$$w^*(t) = -\frac{J_W^2}{J_{WW}}\frac{(\alpha-1)}{\sigma^2} \tag{82}$$

where $w^*(t)$ is the optimal proportion of wealth invested in the risky asset at time t. Then, we have the explicit solution for the differential equations.

$$J(W,t) = \delta\beta^{-\gamma}e^{-\rho t}[\frac{\delta(1-e^{-(\rho-r\nu)(T-t)/\delta}}{\rho-\gamma\nu}]^{\delta}[\frac{W}{\delta} + \frac{\eta}{\beta r}(1-e^{-r(T-t)})]^{\gamma} \tag{83}$$

where $\delta = 1-\gamma, \nu = r + (\alpha-r)^2/2\delta\sigma^2$. From these equations, the optimal consumption and portfolio rules can be written in explicit form as

$$C^*(t) = \frac{[\rho-\gamma\nu][W(t) + \frac{\delta\eta}{\beta r}(1-e^{r(t-T)})]}{\delta(1-exp[\frac{(\rho-\gamma\nu)}{\delta}(t-T)])} - \frac{\delta\eta}{\beta} \tag{84}$$

$$w^*(t)W(t) = \frac{(\alpha-r)}{\delta\sigma^2}W(t) + \frac{\eta(\alpha-r)}{\beta r\sigma^2}(1-e^{r(t-T)}) \tag{85}$$

The important characteristics of the solution is that the demand functions are linear in wealth. The fact implies us that the a family of the functions is the only class leading to the linear solutions.

4.2 Optimal portfolio and consumption including Poisson processes

Merton also gives a closed-form solution for the case where the price includes the Poisson processes as the event risk. We also assume that there are one risky asset with return r and one risky asset whose price is log-normally distributed, and the parameters α, σ included in the price process are constant. To see the effect of default on the portfolio and consumption decisions, consider the particular case when $U(C,t) = C^\gamma/\gamma$ for $\gamma < 1$. The solutions are obtained as follows.

$$C^*(t) = AW(t)/(1-\gamma)(1-exp[A(t-T)/(1-\gamma)]) \tag{86}$$

where

$$A = -\gamma[\frac{(\alpha-r)^2}{2\sigma^2(1-\gamma))} + r] + \lambda[1 - \frac{(2-\gamma)}{\gamma}(w^*)^\gamma + \frac{\gamma(\alpha-r)}{2\sigma^2(1-\gamma)}(w^*)^{\gamma-1}] \tag{87}$$

$$w^* = \frac{(\alpha-r)}{\sigma^2(1-\gamma)} + \frac{\lambda}{\sigma^2(1-\gamma)}(w^*)^{\gamma-1} \tag{88}$$

It is seen from the solutions that the demand of the common stock is an increasing function of λ, and $\lambda > 0, w^*$ holds for all values of α, r, σ^2

4.3 Constant volatility and deterministic jump size

Liu et al. show an application of dynamic asset allocation with event risk basically following the Merton's reduction. They assume one risk-free asset and one risky asset whose price process is subject to event-related jumps. They originally use the model of price changes including time-variate variances (volatility), however, we show only cases with steady (constant) volatility. The dynamics of the price changes is written as

$$dP(t) = (r + \eta V_0 - \mu\lambda V_0)P(t)dt + \sqrt{V_0}P(t)dZ + \mu P(t)dN(t) \tag{89}$$

where all parameters are assumed to be constant. The term $dN(t)$ denotes the jump (Poisson) process corresponding to the event. The amplitude (size) of the price jumps is assumed to be deterministic (constant). We assume indirect utility function as

$$J(W,t) = \frac{1}{1-\gamma}W^{1-\gamma}\exp[A(t) + B(t)V] \tag{90}$$

Then, the optimal weight w^* is obtained from the first order condition

$$0 = (\eta - \mu\lambda)V + \rho\sigma B - \gamma w^* V + \lambda V E[(1 + w^* X)^{-\gamma} X e^B] \tag{91}$$

In this case, the optimal solution of w^* is given as the solution of following equation

$$w^* = \frac{\eta - \mu\lambda}{\gamma} + \frac{\mu\lambda}{\gamma}(1 + \mu w^*)^{-\gamma} \tag{92}$$

They show the simulation studies for the optimal portfolio by depicting the plots of weight as a function of the value of the jump size μ. From the result, it is explained that the optimal portfolio is highly sensitive to the size of the jump μ. If the jump is in the downward direction, the investors takes a smaller position on the risky asset than he would if jumps does not occur. Surprisingly, however, the investor also takes a smaller position when the is in the upward direction. The rationale for this is related to the effects of jumps on the variance and skewness of the distribution o terminal wealth.

4.4 VaR regulation without jump diffusion

Leippold et al. shows the simulation results for the optimal portfolio selection under the VaR regulation. However, in this case, the process of the risky asset in restricted to a kind of Brownian motion, and includes no jump (Poisson) process.

$$dP(t) = (\theta - \kappa P(t))dt + \sigma_P P(t)dZ \tag{93}$$

Again, we show the CRRA utility function as $U(W) = [W^\gamma - 1]/\gamma, \gamma < 1$. Then, the optimal solution under the VaR regulation is given by $w_f(P(t), t)$ characterized by the upper bound w^- and lower bound w^+.

They discussed the simulation result for the optimal portfolio under the VaR regulation by comparing the result where the VaR constraints are not imposed. They give the plots of optimal portfolio strategies of a VaR-constrained and a VaR-unconstrained investor for $\gamma = 0.5$ as function of volatility σ. From the result, it is seen that for a one year investment horizon, the difference between the two portfolio strategies is small. However, if for longer years investment horizon is assumed, the risk-exposure of a VaR-constrained bank has already been substantially reduced before the VaR constraint becomes binding. Therefore, it implies us that the VaR constraint might become binding in the future leads a reduction of the bank's exposure in the risky asset.

5 Applications

5.1 Two cases of utility function

In the application of the VaR regulation analysis, we assume types of utility functions to use the closed form solutions. Additionally, we assume that the parameters used for modeling price processes are to be constant (time -invariant) so that we see the basic performance of the dynamic portfolio selection under the VaR regulations. We consider following two cases of utility functions.
Case I : CRRA utility function.

The utility is given by the final wealth $W(T)$ as $U(W) = (W^\gamma - 1)/\gamma, \gamma < 1$. However, in this case we dot include the optimal consumption in the portfolio selection. Initial conditions for the parameters of the model are given as follows:
$\alpha = 0.20, r = 0.10, \beta = 0.05, N^{-1}(\nu) = -1.64, \gamma = 0.5, \tau = 0.01$
σ changes from 0.1 to 2.00.
Case II : HARA utility function.

In this case, the utility function is written by the HARA family as $U(C, t) = e^{\rho t}V(C)$ including the optimal consumption where $V(C)$ is characterized as $V(C) = \frac{1-\gamma}{\gamma}(\frac{\beta C}{1-\gamma})^\gamma$ introduced by Merton. Initial conditions for t the parameters of the model is given as follows.
$\alpha = 0.20, r = 0.10, \beta = 0.05, N^{-1}(\nu) = -1.64, \gamma = 0.5, \tau = 0.01$
σ changes from 0.1 to 2.00.

5.2 Partial equilibrium without jump diffusion price process

At first, we show the partial equilibrium under the VaR regulation for Case I where the price processes including no jump diffusion. We are mainly interested in the effect of variance σ included in the price process. Therefore, we examine the behavior of w^* along the value of σ.
Case I

Then, we have the value w_f which is the solution for the constraint free optimal strategy and we can define the upper and lower bound w^-, w^+ as the solution of equation $Q(w) = 0$.

The solution w_f is equal to the optimal value $w^* = w_f$ within the ragne $w^- \leq w_f \leq w^+$. It is imagined if the variance σ^2 increases, the risk exposure of the asset becomes large, then the investor decrease the weight for the risky asset.

Fig.1 shows the change of w^* along the variance σ by comparing it with the value w_f. It is seen from Fig.1, the value w_f is larger than w^+ in the region with smaller σ, and the value w_f is lower than w^+ in the region with larger σ. The fact is not the same as our first conjecture that the value w_f is suppressed in the region with larger σ , because we may experience large risk exposure in the region with large σ. However, the fact reflects the control scheme of the stochastic dynamic programming, namely, if the variances of price dynamics is large, then the optimization process adjusts the weight of risky asset to smaller value. On the contrary, in the region with smaller σ, the optimal solution goes outside of the VaR regulation. In nay way, from a certain point on the axis, the investment to the risky asset (the weight w^*) becomes to be very small.

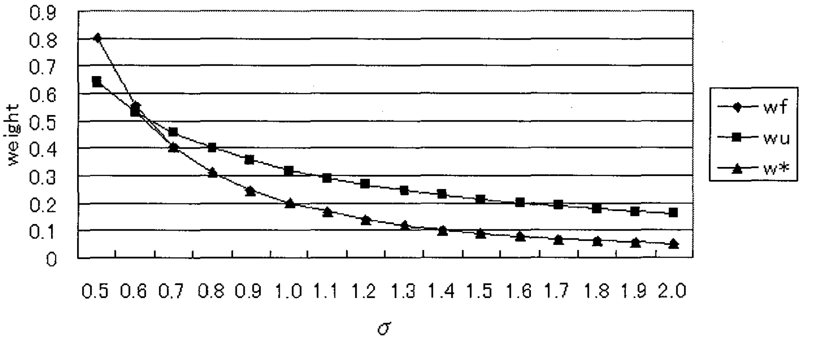

Figure 1: Change of w^* along σ

5.3 Optimal portfolio and consumption without jump diffusion price process

We also treat the optimal portfolio and consumption where the price processes including no jump diffusion.

Case II

In this case, we can obtain a set of values of w_f and C_f which correspond to the optimal solution for the optimal portfolio and consumption without jump diffusion price process under no Var regulation. Then, it is possible to take several conditions for checking the effect of VaR regulation depending on situations whether we change the value w^* from w_f, or change C^* form C_f. It is also possible to change both of them simultaneously.

We at first examine the effect of VaR regulation by fixing the value of C^* to C_f, and by changing the value of w^* from w_f so that the solutions are limited in the VaR regulation. Fig.2 show the change of w^* along the variance σ where the value C^* is equal to C_f. In Fig.2 the optimal value for w^* is depicted only for the time $t = 3$ with $T = 5$, but we see similar performances for other t.

Similar to Fig.1, it is seen from Fig.2, the value w^f is larger than w^+ in the region with smaller σ, and the value w_f is lower than w^+ in the region with larger σ. The fact implies us that the variances of price dynamics is large, then the optimization process adjusts the weight of risky asset to smaller value. From a certain point on the axis, the investment to the risky asset (the weight w^*) becomes to be very small.

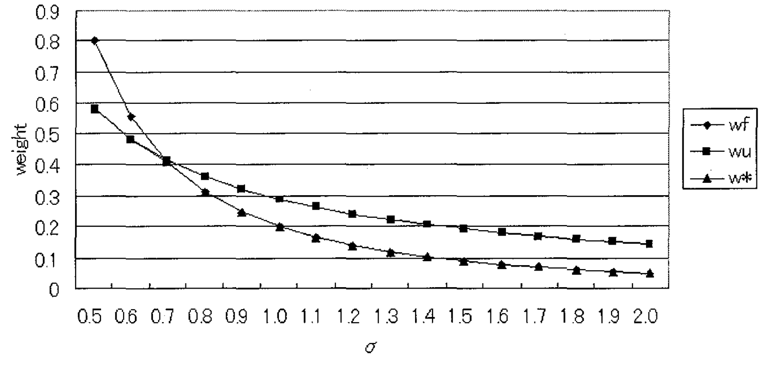

Figure 2: Change of w^* along σ (with fixed C^*)

In the next step, we examine the effect of VaR regulation by changing the value of C^* rather than the value w^*. We change the value of C^* from C_f along the time t by fixing the value of w^* to optimal solution w_f without VaR regulation. To attain the VaR regulation, the following inequality should be hold.

$$Q(w_f, C^*) = \log(1-\beta) - [r + \zeta w_f - H(t) - \frac{1}{2} w_f \sigma^2]\tau - w_f \sigma \sqrt{\tau} N^{-1}(\nu) \leq 0, H(t) = C^*(t)/W(t) \tag{94}$$

The inequality means if the VaR regulation is not attained for a given $w^* = w_f$, we must decrease the value of C^* so that $Q(w_f, C^*) \leq 0$ will be hold.

Fig.3 (the left figure) shows the optimal $C(t)^*$ along the time t with $T = 20$ by comparing the value with C_f, and the right figure shows the optiomized value of $Q(w)^*$ along time t which is suppresed under 0. In the simulation, we fixed the value of σ to 0.7, and w_f to $w_f = 0.408$. As is seen from Fig.3, from time $t = 17$ to $t = 18$, the value of $Q(w)$ becomes positive if $C(t)^*$ is not changed from C_f, then we slightly decrease the value of $C(t)^*$, then the VaR regulation is attained.

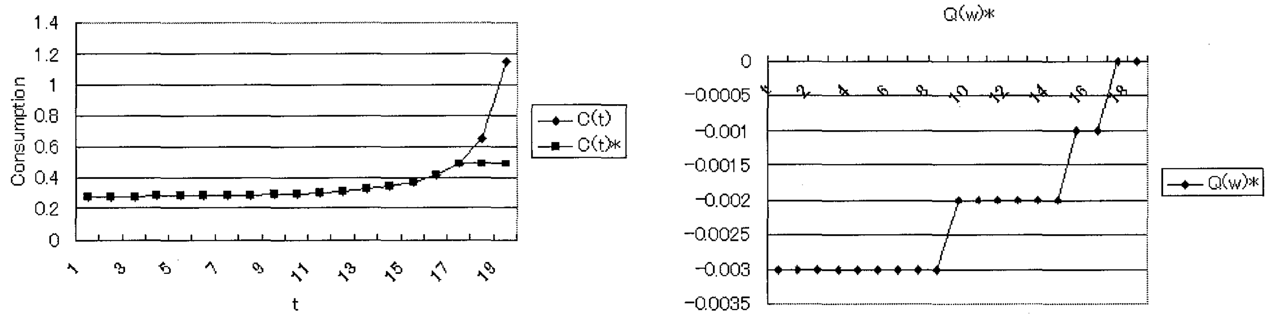

Figure 3: Left:Change of $C(t)^*$ along t, Right:Change of $Q(w)^*$ along t (with fixed $w^* = w_f$)

5.4 Partial equilibrium with jump diffusion price process

As the next application, we show the optimal portfolio under VaR regulation for the cases where the price processes have also the jump diffusion.

Case I

In this case, we can obtain values of w_f which correspond to the optimal solution for the optimal portfolio with jump diffusion price process for a given λ under no Var regulation. Then, it is possible to see the effect of VaR regulation by changing λ. For simplicity, we select at first λ as a fixed value and only change the value of w^* from w_f to attain VaR regulation.

Fig.4 shows the change of w^* along the value of σ where we fix the value of λ as $\lambda = 0.1$. The value w_f is always larger than w^+ in all region of σ. The fact implies us that if a certain jump diffision (dudden fall of price) is found in the bond price, then the optimization process adjusts the weight of risky asset to smaller value in any condition. Off course, if we choose $\lambda = 0$, then the result is the same as the case of no-jump diffusion. In other words, there is no VaR regulation scheme if a jump diffusion is found in bond prices.

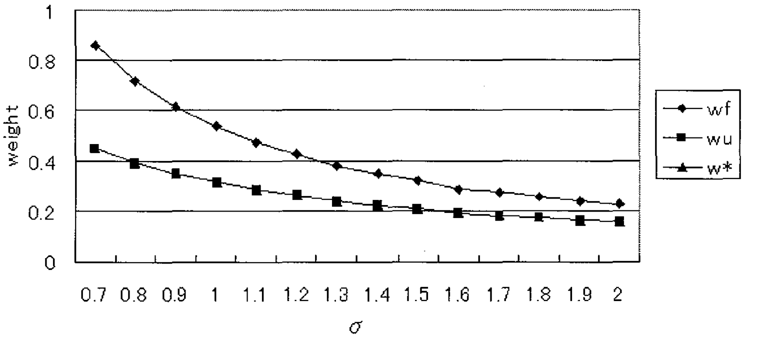

Figure 4: Change of w^* along σ (with fixed λ)

Then, we examine the effect of the probability of jump diffusion λ on the VaR regulation by changing the value of λ rather than the value σ. Fig.5 shows the optimal w^* along the value λ by comparing the value with w_f. In the simulation, we fixed the value of σ to $\sigma = 1.0$. As is seen from Fig.5, the value of w_f becomes larger if the λ becomes larger. The fact means of the bond price becomes to be unstable due to the increse of default ptobability, the the weight of risky asset should be increases. However, the value of w^+ is always lower than w_f, and then the investor must decrease and suppress the weight for the risky asset to attain the purpose of VaR regulations, and as a result we have $w^* = w^+$.

5.5 Optimal portfolio and consumption with jump diffusion price process

Then, we consider the cases of optimal portfolio and consumption with jump diffusion price processes.

Case II

In this case, we can obtain values of w_f and C_f which correspond to the optimal solution for the optimal portfolio with jump diffusion price process for a given $^{l}ambda$ under no Var regulation. Then, it is possible to take several conditions for checking the effect of VaR regulation depending on situations whether we change the value w^* from w_f, or change C^* form C_f. It is also possible to change both of them simultaneously. Moreover, the probability of jump diffusion λ is also a parameter to affect the VaR regulation.

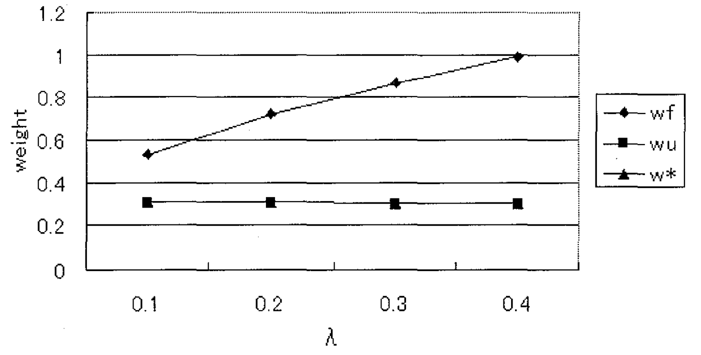

Figure 5: Change of w^* along λ (for fixed σ and C^*)

For simplicity, at first we select λ as a fixed value 0.1 and also set C^* to C_f and only change the value of w^* form w_f to attain VaR regulation. Fig.6 shows the change of w^* along the variance σ. The value w^f is always larger than w^+ in all region of σ. The fact implies us that with jump diffuiosn price processes, the optimization process adjusts the weight of risky asset to smaller value for Var rgulation.

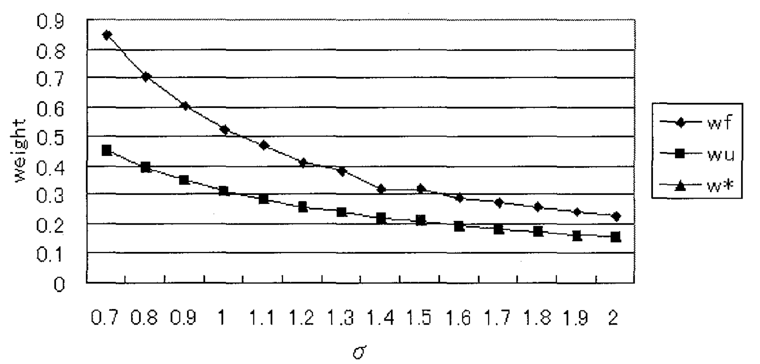

Figure 6: Change of w^* along σ (for fixed λ and C^*)

Then, we examine the effect of the probability of jump diffusion λ on the VaR regulation by changing the value of λ rather than the value σ. Fig.7 shows the optimal w^* along the value λ by comparing the value with w_f. In the simulation, we fixed the value of σ to $\sigma = 1.0$, and $C(t)^* = C_f$. As is seen from Fig.7, the value of w_f becomes larger if the λ becomes larger. The fact means of the bond price becomes to be unstable due to the increse of default ptobability, the the weight of risky asset should be increases. However, the value of w^+ is always lower than w_f, and then the investor must decrease and suppress the weight for the risky asset to attain the purpose of VaR regulations, and as a result we have $w^* = w^+$.

In the final example, we examine the effect of VaR regulation by changing the value of C^* rather than the value w^*. We change the value of C^* from C_f along the time t by fixing the value of w^* to optimal solution w_f without VaR regulation. To attain the VaR regulation, the following inequality should be hold.

$$Q(w_f, C^*) = \log(1-\beta) - [r + \zeta w_f - M(t) - \frac{1}{2} w_f \sigma^2]\tau - w_f \sigma \sqrt{\tau} N^{-1}(\nu) \leq 0, M(t) = C^*(t)/W(t) \tag{94}$$

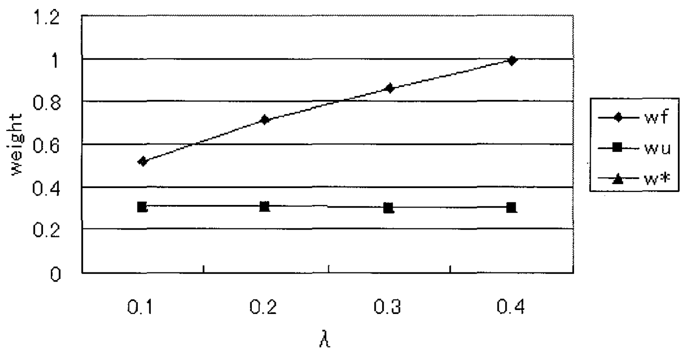

Figure 7: Change of w^* along λ (for fixed σ and C^*)

The inequality means if the VaR regulation is not attained for a given $w^* = w_f$, we must decrease the value of C^* so that $Q(w_f, C^*) \leq 0$ will be hold.

Fig.8 (the left figure) shows the optimal $C(t)^*$ along the time t with $T = 20$ by comparing the value with C_f, and the right figure shows the optiomized value of $Q(w)^*$ along time t which is suppresed under 0. In the simulation, we fixed the value of σ to 0.7, and w_f to $w_f = w^* = 0.302$, and the selection is different from the result in Fig.3. Because, in this case, the value of w_f is always larger than w^*, and then the VaR regulation is attained only if we choose $w^* = w^+$. As is seen from Fig.8, from time $t = 17$ to $t = 18$, the value of $Q(w)$ becomes positive if $C(t)^*$ is not changed from C_f, then we slightly decrease the value of $C(t)^*$, then the VaR regulation is attained.

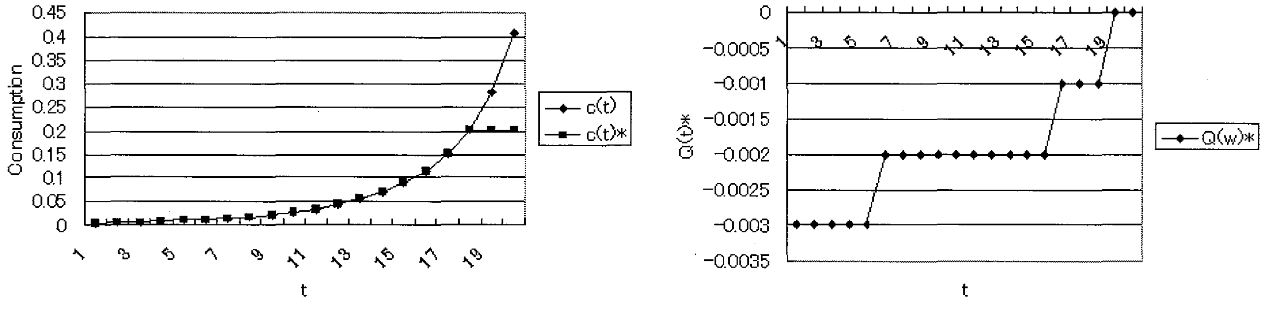

Figure 8: Left:Change of $C(t)^*$ along t, Right:Change of $Q(w)^*$ along t (with fixed $w^* = w_f$)

6 Conclusion

This paper showed the implications of event-related jumps in security prices and the dynamic portfolio strategies under VaR-based regulation. Based on the incomplete market model different form normally distributed returns, we formalized the the optimization problem under VaR constraints. With the jump-diffusion processes triggered by a Poisson event, we reduced the Hamilton-Jacobi-Bellman partial differential equations. By assuming that VaR is proportional to current wealth directly, first-order approximation of the wealth process was used. Then, we found the optimal dynamic portfolio by switch the weight for the risky asset depending on the boundaries of weight. We described examples application for the proposed methods.

For future works, it is necessary to extend the method to various fields of investment problems. Further researches will be done by the authors.

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