55.

ON THE PREROGATIVE OF A TERNARY DENOMINATIONAL SYSTEM OF COINAGE.

[*Johns Hopkins University Circulars*, I. (1881), p. 132.]

PROFESSOR SYLVESTER drew attention to the fact that a system of coinage in which each coin is three times the value of the one below it would possess a superiority *above every other* in so far as it would admit of all payments up to any assigned limit, being effected with the smallest possible number of pieces, this advantage increasing with the size of the limit. Thus suppose the limit of 10 dollars to be selected, two persons each possessed of 7 coins of the respective value of 1, 3, 9, 27, 81, 243, 729 cents could pay each other by interchange of their coins any sum from 1 cent up to this limit. The full amount so capable of being paid being of course, $\frac{3^7-1}{2}$ cents, that is, \$10.93. Whereas with 7 coins doubling at each step, the extreme limit would be \$1.27.

Again if each coin were quadruple the value of its antecedent, the extreme limit attainable with 8 coins would be only $2(1+4+16+64)$, or \$1.70. Or, if each coin were five times the value of its antecedent, the sum of the geometrical progression $1+5+25+125+625$ being 781, 10 coins at least would be required to be possessed by each of two persons to enable one of them to pay the other any amount from 1 cent up to \$7.81, whereas as previously shown, 7 would be more than sufficient to allow of this being done on the ternary scale.

Thus the absolutely perfect system of coinage, so far as this depends on the smallness of the number of coins necessary to be used, is that which proceeds in a geometrical progression according to the ternary scale.

The following problem in arithmetic is suggested by the preceding considerations.

What is the condition that the sums and differences of the integers a_1, a_2, a_3, ... a_n, *not subject to any defined law of progression, may comprise between them all the numbers from* 1 *up to* $a_1+a_2+a_3 \ldots + a_n$?

56.

ON THE MULTISECTION OF THE ROOTS OF UNITY.

[*Johns Hopkins University Circulars*, I. (1881), pp. 150, 151.]

IF p be a prime number, e a divisor of $p-1$, and the e periods into which the primitive pth roots of unity may be distributed are the roots of $\eta^e + B\eta^{e-1} + \ldots$, I call this last written function (say E), the e-period function to p. Every divisor of such function, it is well known, if not p itself or an eth power residue of p, must be a divisor of the discriminant of E.

Every divisor q of the discriminant is necessarily a divisor of E but may or may not be, according to circumstances, an eth power residue to p; if it is not, then q may be called an exceptional divisor of the period-function.

When $e=2$ the discriminant is p itself so that (as is well known) there are no exceptional factors to the two-period function. When $e=3$, it may be shown that every factor of the discriminant is necessarily a cubic residue of p.

This may be proved by the Law of Reciprocity for cubic residues, although obtained in quite a different manner. It follows that the three-period function has no exceptional divisor.

When $e=4$ it is better to distinguish between the two cases of $p=8i+1$ and $p=8i+5$.

In the former case 2 is not necessarily a biquadratic but may be only a quadratic residue of p, although a divisor of the 4-period function, and consequently 2 may be an exceptional divisor. When $p=8i+5$, if $p=f^2+4\gamma^2$, every divisor of γ is necessarily a divisor of the function inasmuch as γ is contained in the discriminant, but whilst divisors of γ of the form $4i+1$ are biquadratic, those of the form $4i-1$ will be only quadratic and not biquadratic residues of p. The results for $e=4$ so far as yet stated may be proved by the law of reciprocity for biquadratic residues.

But when $p = 8i + 5$, or in other words, when $p = f^2 + 4\gamma^2$ where γ is odd, it may be shown that $\frac{3p^2 + f^2}{16}$ is also that factor of the discriminant which is represented by $(\eta_0 - \eta_1)(\eta_1 - \eta_2)(\eta_2 - \eta_3)(\eta_3 - \eta_0)$, ($\eta_0, \eta_1, \eta_2, \eta_3$ being the four periods taken in natural order), and it is capable of proof that every divisor of this chain of products cannot but be a biquadratic residue to p*, or in other words, every divisor of $\frac{f^2 + 4\gamma^2}{4}$ is a biquadratic residue of $f^2 + 4\gamma^2$ when this last quantity is a prime number. This theorem, deduced from the method applied to the divisors of period-functions, does not appear to be referable to any known theorem concerning biquadratic residues. Professor Sylvester finally stated that he had under consideration the question of the existence or otherwise of exceptional factors to the e-period function in the general case of e being a prime number.

* For suppose q, a prime-number divisor of the "chain-product," to be not a biquadratic residue of p; then if q is a quadratic residue of p, it may be shown that q must be also a divisor of $(\eta_0 - \eta_2)^2 (\eta_1 - \eta_4)^2$ and therefore of $p\gamma^2$, which is impossible because γ is prime to f and p, and if q is a non-quadratic residue of p, it may be shown that all four roots of the congruence, which expresses that the 4-period function contains q, must be equal to one another, which admits of easy disproof. Hence q cannot but be a biquadratic residue of p.

57.

SUR LES DIVISEURS DES FONCTIONS DES PÉRIODES DES RACINES PRIMITIVES DE L'UNITÉ.

[*Comptes Rendus*, XCII. (1881), pp. 1084—1086.]

SOIT p un nombre premier égal à $ef + 1$; la fonction du $e^{\text{ième}}$ degré, dont les racines sont les e périodes entre lesquelles on peut distribuer les ef $p^{\text{ièmes}}$ racines primitives de l'unité, est ce que je désigne comme la fonction à e périodes par rapport à p.

On connaît bien que p et un $e^{\text{ième}}$ résidu quelconque par rapport à p sont toujours diviseurs de cette fonction. Tout autre diviseur se nomme *diviseur exceptionnel* de la fonction. On sait que tout diviseur exceptionnel d'une fonction de périodes doit être contenu comme facteur dans le discriminant de cette fonction et, de plus, que pour les cas où $f = 1$, ou $f = 2$, ou $e = 2$, il n'y a pas de facteurs exceptionnels. Si l'on en connaît davantage au sujet de ces facteurs exceptionnels, je n'en suis pas instruit. On ne trouve rien de plus dans le Livre classique de Bachmann (*Kreistheilung*, 1872)*.

Or je trouve facilement, pour le cas de $e = 3$, qu'il n'y a pas de facteurs exceptionnels, de sorte que tout diviseur premier de la fonction bien connue

$$\eta^3 + \eta^2 - \frac{p-1}{3}\eta + \ldots$$

est nécessairement ou p ou un résidu cubique de p. Pour $e = 4$, la même chose n'a pas lieu.

* Dans cet excellent ouvrage, M. Bachmann démontre que, si Ω est la fonction à e périodes par rapport à p et q nombre premier qui est une $e^{\text{ième}}$ puissance résidu de p, la congruence $\Omega \equiv 0$ (mod. q) aura e racines réelles, mais, chose extraordinaire, omet de démontrer ou même de dire que la même chose a lieu pour la congruence $\Omega \equiv 0$ (mod. q^i), i étant un nombre entier positif quelconque. En effet, cette propriété de q (que *toutes* ses puissances sont diviseurs) est le *caractère distinctif* de la classe *principale* de diviseurs, non pas seulement pour les fonctions des périodes de racines d'unité par rapport à un nombre premier, mais aussi pour les fonctions cyclotomiques en général. Dans le cas que nous considérons, ni p ni aucun *diviseur exceptionnel* ne possède cette propriété.

Quand $p = f^2 + g^2$, où f est impair, si g est divisible par 4, mais non pas par 8, le nombre 2 divisera la fonction des quatre périodes, mais ne sera pas (comme on sait bien) un résidu biquadratique, mais seulement un résidu quadratique de p; de plus, si g n'est pas divisible par 4, tout nombre premier contenu dans $\frac{g}{2}$ sera un diviseur de la fonction des périodes, et, si ce nombre premier est de la forme $4i+3$, il sera seulement un résidu quadratique et non biquadratique de p. Pour $e=4$, il n'y a pas d'autres diviseurs exceptionnels au delà de ceux que j'ai donnés ci-dessus. En établissant ce fait, j'ai été amené à cette proposition curieuse, qu'il serait difficile (il me semble) d'établir par un autre genre de considérations, mais qui est indubitablement vraie, c'est-à-dire :

Si $p = f^2 + (2g)^2$ (p étant un nombre premier et g impair), *tout nombre contenu dans le nombre impair* $\frac{f^2+3g^2}{4}$ *est un résidu biquadratique de* p.

Mais je passe à un théorème général, qui me paraît très intéressant et que voici :

1°. *Si* e (le nombre des périodes) *est un nombre premier de la forme* $2^{2^x}+1$, *le nombre* 2 *ne peut pas être un diviseur exceptionnel de la fonction des* e *périodes.*

2°. *Si* e *est un nombre premier, un facteur exceptionnel* K (si un tel cas peut exister) *doit entrer à la seconde puissance au moins comme facteur dans* $e-1$, *de sorte qu'on sait que, pour* $e = 2, 3, 5, 7, 11, 17$, *il n'existe pas de diviseur de la fonction des* e *périodes en dehors de* p *et des résidus* $e^{\text{ièmes}}$ *de* p.

Quand $e=19$, puisque $19-1$ contient 3^2, le théorème n'exclut pas la possibilité que 3 soit un diviseur de la fonction à dix-neuf périodes sans être une dix-neuvième puissance résidu de p. De même, quand $e=13$, le théorème ne dit rien sur le caractère du diviseur 2, dont le carré 4 est contenu dans 13. Cependant, je n'ai pas la moindre raison pour conclure que les diviseurs exceptés sont vraiment des facteurs exceptionnels.

On doit regarder le cas où, e étant un nombre premier, $e-1$ contient K^2, non pas comme un cas exceptionnel, mais comme un cas réservé pour un examen ultérieur.

58.

SUR LES COVARIANTS IRRÉDUCTIBLES DU QUANTIC BINAIRE DU HUITIÈME ORDRE.

[*Comptes Rendus*, XCIII. (1881), pp. 192—196, 365—369.]

M. VON GALL a récemment calculé les dérivées invariantives irréductibles qui appartiennent à la forme binaire du huitième ordre (voir *Mathem. Annalen*, 1880, 1881), et s'est mis en parfait accord avec les résultats que j'avais déjà obtenus, sinon qu'il a trouvé un covariant du *degré-ordre* 10.4 qu'il affirme ne pas avoir réussi à décomposer. Je vais donc démontrer que nul covariant irréductible de ce degré-ordre ne peut exister.

Selon M. von Gall et moi-même, on a un seul invariant irréductible de chacun des degrés 2, 3, 4, 5, 6, 7, 8; il y a aussi un seul invariant des degrés 9, 10 respectivement, dont je n'aurai pas besoin de parler. On a aussi un seul covariant irréductible du degré-ordre 2.4 et du degré-ordre 3.4 et deux des degrés-ordres 4.4, 5.4, 6.4, 7.4, 8.4 respectivement. Il y a aussi un covariant du degré-ordre 5.2, mais nul covariant quadratique d'un degré inférieur à 5 et, comme on le sait bien *à priori*, nul covariant de l'ordre impair 1 ou 3.

En combinant ensemble ces covariants et invariants, on peut obtenir* trente-deux covariants composés, chacun du degré-ordre 10.4, car

2 peut être formé avec......		2,
3	„	3,
4	„	4 et 2, 2,
5	„	5 et 3, 2,
6	„	6 ; 4, 2 ; 3, 3 ; 2, 2, 2,
7	„	7 ; 5, 2 ; 4, 3 ; 3, 2, 2,
8	„	8 ; 6, 2 ; 5, 3 ; 4, 4 ; 4, 2, 2 ; 3, 3, 2 ; 2, 2, 2, 2.

[* See p. 485, below. Also p. 509.]

Donc les covariants irréductibles des ordres 8, 7, 6, 5, 4 donnent naissance à $2(1+1+2+2+4)$, c'est-à-dire vingt covariants du degré-ordre 10 . 4, et les covariants irréductibles des ordres 2 et 3 à $4+7$, c'est-à-dire dix covariants de ce même degré-ordre, et il y en a aussi un de plus qui s'obtient en prenant le carré du covariant irréductible du degré-ordre 5 . 2. Le nombre total est donc $20+11+1=32$.

Je me propose d'établir catégoriquement que ces formes sont linéairement indépendantes entre elles. En suivant la notation de M. von Gall, on aura

	Degré-ordre
f	1 . 8
$i \;= (f, f)_4$	2 . 8
$k \;= (f, f)_6$	2 . 4
$\Delta = (k, k)_2$	4 . 4
$A = (f, f)_8$	2 . 0
$B = (f, i)_8$	3 . 0
$C = (k, k)_4$	4 . 0
$f_4 = (f, k)_4$	3 . 4
$i_4 = (i, k)_4$	4 . 4
$f_{k,2} = (f_4, k)_2$	5 . 4
$f_{k,3} = (f_4, k)_3$	5 . 2
$f_{k,k} = (f, k^2)_8$	5 . 0
$f_\Delta = (f, \Delta)_4$	5 . 4
$i_{k,2} = (i_4, k)_2$	6 . 4
$i_{k,k} = (i_4, k)_4$	6 . 0
$i_\Delta = (i, \Delta)_4$	6 . 4
$f_{k,\Delta} = (f_4, \Delta)_4$	7 . 0
$i_{k,\Delta} = (i_4, \Delta)_4$	8 . 0

Dans cette table, f représente la forme primitive $*(x, y)^8$ et, en général, $(\phi, \psi)_\mu$ signifie l'invariant linéo-linéaire par rapport à u, v des deux formes

$$\left(u\frac{d}{dx}+v\frac{d}{dy}\right)^\mu \phi, \quad \left(u\frac{d}{dx}+v\frac{d}{dy}\right)^\mu \psi.$$

Je ne donne pas la genèse du covariant du degré-ordre 7 . 4 ni de celui du degré-ordre 8 . 4, car, par la méthode dont je vais me servir, on n'aura pas occasion d'introduire explicitement ces deux formes dans le calcul.

Je commence en attribuant à f la forme spéciale

$$(0, r, s, 0, 0, 0, 0, 0, 1\between x, y)^8,$$

c'est-à-dire

$$8rx^7y + 28sx^6y^2 + y^8.$$

Cette supposition réduira à zéro, comme on va le voir, les trois invariants des degrés 2, 3, 4 respectivement.

En suivant les indications de la table donnée et en négligeant des multiplicateurs numériques, on trouvera facilement

$$\begin{aligned}
A &= (f, f)_8 = 0,\\
i &= (f, f)_4 = 3r^2x^8 + 4qx^3y^5 + br^2x^2y^6,\\
B &= (i, f)_8 = 0,\\
k &= (f, f)_6 = 2qxy^3 + ry^4,\\
i_4 &= (i, k)_4 = 84r^3x^4 - q^2y^4,\\
C &= (k, k)_4 = 0,\\
\Delta &= (k, k)_2 = q^2y^4,\\
i_{k,k} &= (i_4, k)_4 = r^4,\\
f_{k,k} &= (f, k^2)_8 = q^2r,\\
i_{k,\Delta} &= (i_4, \Delta)_4 = q^2r^3,\\
f_4 &= (f, k)_4 = q^2x^4 - 4qrx^3y,\\
f_{k,2} &= (f_4, k)_2 = 2q^3x^3y - q^2rx^2y^2 - 4qr^2xy^3,\\
f_{k,\Delta} &= (f_4, \Delta)_4 = q^4,\\
f_\Delta &= (f, \Delta)_4 = 2q^3x^3y + 3q^2rx^2y^2,\\
f_{k,3} &= (f_4, k)_3 = q^4xy + q^3ry^2.
\end{aligned}$$

Or, soit Ω la fonction (si une telle existe) du degré-ordre 10 . 4, composée avec les produits des invariants et covariants irréductibles de $*(x, y)^8$, qui est identiquement égale à zéro.

Les seuls termes dans Ω qui continueront à subsister pour la forme spéciale attribuée à f seront (en addition au carré du covariant du degré-ordre 5 . 2) des multiples numériques des produits des invariants irréductibles des degrés 5, 6, 7, 8 multipliés respectivement par chacun des deux covariants du degré 5, par chacun des deux covariants du degré 4, par le covariant du degré 3 et par le covariant du degré 2.

On obtiendra ainsi les sept expressions suivantes :

$$q^8x^2y^2 + 2q^7rxy^3 + q^6r^2y^4, \qquad (1)$$

$$2q^5rx^3y - q^4r^2x^2y^2 - 4q^3r^3xy^3, \qquad (2)$$

$$2q^5rx^3y + 3q^4r^2x^2y^2, \qquad (3)$$

$$84r^7x^4 - q^2r^4y^4, \qquad (4)$$

$$q^2r^4y^4, \qquad (5)$$

$$q^6x^4 - 4q^5rx^3y, \qquad (6)$$

$$2q^3r^3xy^3 + q^2r^4y^4. \qquad (7)$$

Supposons que les multiplicateurs numériques, dans Ω, de ces fonctions soient $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7$ respectivement. Alors, en se souvenant que Ω est identiquement zéro, on voit immédiatement que $\mu_1 = 0$, et à cause du terme seul r^7x^4 dans (4), que $\mu_4 = 0$, et, à cause du terme seul q^6x^4 dans (6), que $\mu_6 = 0$, et conséquemment, à cause des termes q^5rx^3y et $q^4r^2x^2y^2$ dans (2) et (3), que $\mu_2 = 0$, $\mu_3 = 0$. Il ne reste donc que μ_5 et μ_7 à considérer, lesquels évidemment, à cause du terme $q^3r^3xy^3$ dans (7), seront tous les deux zéro.

Ainsi on voit que l'expression Ω ne peut contenir que des multiples des invariants irréductibles A, B et C.

Pour démontrer que Ω ne contient pas de termes qui ne contiennent ni A ni B, considérons la forme spéciale nouvelle

$$f = x^8 + 8\lambda x^7 y + 8\mu xy^7 + y^8,$$

où je suppose que $\lambda\mu$ est égal à $\frac{1}{8}$.

Alors

$$\begin{aligned}
A &= (f, f)_8 = 1 - 8\lambda\mu = 0,\\
i &= (f, f)_4 = 4\mu x^5y^3 + x^4y^4 + 4\lambda x^3y^5,\\
B &= (i, f)_8 = 0,\\
K &= (f, f)_6 = 16\mu x^3y + 6x^2y^2 + 16\lambda xy^3,\\
C &= (k, k)_4 = -64\mu\lambda + 3 = -5,\\
\Delta &= (k, k)_2 = 96\mu^2x^4 + 48\mu x^3y - 6x^2y^2 + 48\lambda xy^3 + 96\lambda^2y^4,\\
i_4 &= (i, k)_4 = 20\mu^2x^4 - 4\mu x^3y + 24x^2y^2 - 4\lambda xy^3 + 20\lambda^2y^4,\\
i_\Delta &= (i, \Delta)_4 = (48\mu^2 - 30\mu)x^4 + \ldots + (48\lambda^2 - 30\lambda)y^4,\\
i_{k,2} &= (i_4, k)^2 = 28\mu^2x^4 + \ldots + 28\lambda^2y^4,
\end{aligned}$$

et les seuls termes qui peuvent subsister dans Ω (vu qu'on a démontré que $\mu_1, \mu_2, \ldots, \mu_7$ sont égaux chacun à zéro) seront des multiples numériques de C^2k, $Ci_{k,2}$, Ci_Δ.

Mais le terme μx^4 paraît seulement dans i_Δ, et μ^2x^4 ne paraît pas dans k; conséquemment les trois termes doivent disparaître spontanément.

Il s'ensuit que Ω peut être mis sous la forme $AV + BU$, où U et V sont des covariants du degré 7 et 8 respectivement, et, puisque $AV + BU$ est identiquement zéro, il faut que $\frac{V}{B} + \frac{U}{A} = 0$, où $\frac{V}{B}$ et $\frac{U}{A}$ sont tous les deux covariants entiers, c'est-à-dire qu'on aura une équation de l'ordre 5 entre les covariants irréductibles, ce qui impliquerait un rapport numériquement linéaire entre les valeurs générales de Af_4 et Bk, ce qui est évidemment absurde. Donc l'expression supposée Ω ne peut pas exister, et les trente-deux covariants composés du degré-ordre 10.4, dont j'ai parlé, seront linéairement indépendants entre eux, pourvu, du moins, que nul rapport linéaire ne lie ensemble les invariants dont nul n'est d'un degré aussi élevé que 8; évidemment, le seul rapport possible de cette nature serait de la forme $C^2 =$ une fonction de A et B, mais on a vu que A et B peuvent disparaître sans que C disparaisse. Donc un tel rapport ne peut pas avoir

lieu, et les trente-deux covariants dont il est question sont linéairement indépendants.

Or, par le théorème célèbre de M. Cayley (dont l'exactitude a été établie catégoriquement* dans le *Journal de Borchardt* et dans le *Philosophical Magazine*), on sait que le nombre total des covariants du degré-ordre 10.4 linéairement indépendants les uns des autres, appartenant au quantic de l'ordre 8, est représenté par $\left(\frac{ij-\epsilon}{2}; i, j\right) - \left(\frac{ij-\epsilon}{2} - 1; i, j\right)$, où $i = 8$, $j = 10$, $\epsilon = 4$, et où, en général, $(w : i, j)$ signifie le nombre de représentation de w comme somme de j ou moins de j, chiffres dont nul n'excède 8 en grandeur, c'est-à-dire, selon un théorème bien connu d'Euler, sera le coefficient de $t^{\frac{8.10-4}{2}}$, c'est-à-dire de t^{38}, dans le développement en série de puissances ascendantes de t de la fraction

$$\frac{(1-t^{11})(1-t^{12})(1-t^{13})(1-t^{14})(1-t^{15})(1-t^{16})(1-t^{17})(1-t^{18})}{(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6)(1-t^7)(1-t^8)},$$

ce qu'on trouvera égal à 32. Conséquemment, un covariant quelconque du degré-ordre 10.4 sera une fonction linéaire des trente-deux composés dont j'ai parlé, et ne peut pas être irréductible, ce qu'il fallait démontrer.

Il existe, dans la Note [p. 481, above] sur ce sujet insérée dans les *Comptes rendus* du 25 juillet dernier, des erreurs de calcul qui rendent la conclusion que je voulais établir tout à fait illusoire; cependant j'ai réussi, par le travail plus pénible qui suit, à parvenir au même résultat.

Je prends $(0, b, 2c, d, 0, 0, 0, 0, 1 \between x, y)^8$ avec la condition $bd = 3c^2$ pour la forme spéciale de f. Alors les invariants du deuxième et du troisième degré, comme on va voir, deviendront nuls; les invariants des degrés 3, 4, 5, 6, 7, 8 ne seront pas nuls, et, en les combinant avec les deux covariants des *degrés-ordres* 7.4, 6.4, 5.4 et avec les seuls covariants des degrés-ordres 4.4, 3.4 dans toutes les manières possibles pour former 1 covariant du degré-ordre 10.4, on aura 9 covariants de ce type, de sorte que, en y ajoutant le carré du covariant du degré-ordre 5.2, il y aura en tout 10 covariants composés du degré-ordre 10.4, dans lesquels les invariants des degrés 2 et 3 ne figurent pas.

Je vais démontrer qu'aucun de ces 10 covariants ne peut paraître dans la fonction Ω (voir *Comptes rendus*, p. 194) [p. 483, above], et conséquemment cette fonction, si elle existe, contiendra dans chaque terme ou l'invariant du deuxième degré ou l'invariant du troisième, et conduira à une équation syzygétique du degré-ordre 5.4, comme je l'ai déjà remarqué, à moins qu'elle ne soit un multiple exact de l'un ou l'autre de ces 2 invariants, dans lequel cas il conduira à une telle équation du degré-ordre 8.4 ou 7.4.

[* pp. 232, 117 above.]

Mais, en tout cas, il y aura un rapport syzygétique d'un degré-ordre inférieur à 10.4 entre les covariants composés, ce qui, selon la loi de Cayley dont j'ai parlé, aurait pour effet d'augmenter le nombre de covariants irréductibles trouvés également par M. Von Gall et moi-même, et dont l'exactitude n'a pas été discutée. Donc tout se ramène à prouver que les 10 covariants composés du degré-ordre 10.4, qui correspondent à la forme spéciale que j'ai adoptée, sont linéairement indépendants l'un de l'autre, ce que je vais établir.

On trouvera facilement, pour la forme spéciale supposée,

$$I_2=0,\quad I_3=bd-3c^2=0,\quad I_4=cd^2,\quad I_5=d^4,$$
$$I_6=c^4,\quad I_7=b^4+200c^3d^2,\quad I_8=b^2c^3,$$

où I_j signifie un invariant du degré j, et en suivant la notation et les procédés de M. Von Gall, négligeant, en outre, des multiplicateurs numériques,

$$k\ =(-20d^2,\ 0,\ 0,\ b,\ 4c\between x,\ y)^4,$$
$$\Delta\ =(0,\ 90c^2d,\ 40cd^2,\ 0,\ 3b^2\between x,\ y)^4,$$
$$f_4\ =(b^2,\ bc,\ c^2,\ -c\delta,\ 5\delta^2\between x,\ y)^4,$$
$$f_{k,2}=(-120c^2d^2,\ 3b^3+60cd^3,\ 6b^2c-100d^4,\ 9bc^2,\ 60c^3\between x,\ y)^4,$$
$$f_{k,3}=(b^3-20cd^3,\ b^2c+50d^4,\ bc^2\between x,\ y)^2,$$
$$f_\Delta\ =(40c^2d^2,\ b^3+80cd^3,\ 2b^2c,\ 3bc^2,\ 0\between x,\ y)^4,$$
$$i_\Delta\ =(240cd^4,\ 21bc^3,\ -6x^4,\ -120c^3d,\ -120c^2d^2\between x,\ y)^4,$$
$$i_4\ =(33bc^3,\ 92c^2d,\ 72cd^2,\ 140d^3,\ 4b^2\between x,\ y)^4,$$
$$i_{k,2}=(-360cd^4,\ 42bc^3-350d^5,\ 49c^4,\ 19c^3d,\ -138c^2d^2\between x,\ y)^4.$$

Désignons $b^6x^4+4b^5cx^3y+6b^4c^2x^2y^2$, c^7x^4, c^2d^6, $4c^6dx^3y$, $4cd^7x^3y_1$, $6c^5d^2x^2y^2$; $6d^8x^2y^2$, $4d^8xy^3$, $4b^3c^3xy^3$; $b^2c^4y^4$, $c^3d^4y^4$ par les lettres α, β, γ, δ, ϵ, η, ζ, θ, κ, λ, μ et, au lieu des valeurs actuelles des covariants composés du degré-ordre 10.4, prenons leurs valeurs par rapport au module 11; alors on trouvera que les valeurs peuvent être représentées par le Tableau suivant:

α.	η.	θ.	β.	λ.	γ.	δ.	ϵ.	ζ.	κ.	μ.
3	6	3	5	3	1	1	3	2	10	.
1	.	8	7	1	.	6	.	2	9	10
.	10	.	.	.	1	4	5	10	5	5
.	.	1	7	4	.	.	.	.	.	.
.	.	.	6	4	.	4	.	6	8	.
.	.	.	.	3	.	2	.	7	.	.
.	.	.	.	.	2	.	.	.	3	4
.	.	.	.	.	7	5	3	7	9	.
.	.	.	.	.	9	8	.	8*	1	9
.	.	.	.	.	3	5	2	5	8	5

[* See below, p. 517.]

où la première ligne des chiffres représente la fonction

$$3\alpha+6\eta+3\theta+\ldots+10\kappa,$$

la seconde,

$$\alpha+8\theta+\ldots+9\kappa+10\mu,$$

et ainsi pour toutes les autres.

Il ne reste donc qu'à examiner si les déterminants mineurs du *matrix* écrit au-dessus de l'ordre 10 s'évanouissent tous par rapport au module 11; sinon les 10 fonctions seront nécessairement indépendantes par rapport au module 11 et à plus forte raison absolument aussi: or ce petit problème numérique se ramène facilement à la question de déterminer si les déterminants mineurs de l'ordre 5 du *matrix*

$$\begin{vmatrix} 5 & 2 & \cdot & \cdot & 5 & \cdot \\ 2 & \cdot & \cdot & \cdot & 3 & 4 \\ 7 & 5 & 3 & 7 & 9 & \cdot \\ 9 & 8 & \cdot & 8 & 1 & 9 \\ 3 & 5 & 2 & 5 & 8 & 5 \end{vmatrix}$$

s'évanouissent tous par rapport au module 4, ce qui ne peut avoir lieu si le déterminant

$$\begin{vmatrix} 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 4 \\ 5 & 3 & 7 & 2 & \cdot \\ 8 & \cdot & 8 & 3 & 9 \\ 5 & 2 & 5 & 5 & 5 \end{vmatrix},$$

ou bien si le déterminant

$$\begin{vmatrix} \cdot & \cdot & 1 & 4 \\ 3 & 7 & 2 & \cdot \\ \cdot & 8 & 3 & 9 \\ 2 & 5 & 5 & 5 \end{vmatrix},$$

ou finalement si le déterminant

$$\begin{vmatrix} 3 & 7 & 8 \\ \cdot & 8 & 3 \\ 2 & 5 & 4 \end{vmatrix},$$

c'est-à-dire si le nombre $51-86$ ou bien -35 ne contient pas 11. Donc les 10 fonctions dont je parle sont linéairement indépendantes entre elles. Mais il serait très périlleux d'admettre cette preuve sans confirmation de l'exactitude des chiffres qui résultent de l'immense calcul dont je n'ai qu'indiqué la marche. En effet, j'ai consacré de longues heures à la confirmation de chaque pas de ce calcul, et j'ai appelé à mon aide un calculateur habile; mais ce qui est le plus important, j'ai pu le vérifier de la manière suivante.

J'ai calculé pour ma forme spéciale la valeur du covariant i_4'', donné par M. Von Gall [p. 518, below] et jusqu'à présent trouvé par lui irréductible; cette valeur est

$$\begin{pmatrix} -128520c^7 - 25600c^2d^6, & 37590c^6d^2, & -10920c^5d^2, \\ 63b^3c^3 - 25000c^4d^3, & 1638b^2c^4 + 9600c^3d^4 & \end{pmatrix}(x, y)^4.$$

En combinant cette fonction avec les dix autres du même type, on obtiendra un déterminant de l'ordre 11 qui doit s'évanouir si mes chiffres sont exacts.

J'ai calculé très consciencieusement la valeur de ce déterminant par rapport aux modules 11, 13, 17, et comme, dans les trois cas, j'ai trouvé que la valeur de ce déterminant se divise par le module, je crois que l'exactitude de mes chiffres est parfaitement démontrée, et qu'on peut rester tout à fait convaincu que l'existence d'un covariant irréductible du degré-ordre 10.4 appartenant au quantic octavique est impossible. Les détails du calcul vont être fournis au prochain fascicule de l'*American Mathematical Journal* [p. 509, below].

59.

TABLES OF THE GENERATING FUNCTIONS AND GROUNDFORMS OF THE BINARY DUODECIMIC, WITH SOME GENERAL REMARKS, AND TABLES OF THE IRREDUCIBLE SYZYGIES OF CERTAIN QUANTICS*.

[*American Journal of Mathematics*, IV. (1881), pp. 41—61.]

Generating Function for differentiants,

Denominator:

$$(1-a)(1-a^2)^2(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^8)(1-a^9)$$
$$(1-a^{10})(1-a^{11}).$$

Numerator:

$$1 + 4a^2 + 17a^3 + 49a^4 + 125a^5 + 285a^6 + 594a^7 + 1143a^8 + 2063a^9$$
$$+ 3517a^{10} + 5693a^{11} + 8817a^{12} + 13104a^{13} + 18769a^{14} + 25979a^{15}$$
$$+ 34830a^{16} + 45317a^{17} + 57327a^{18} + 70595a^{19} + 84730a^{20} + 99214a^{21}$$
$$+ 113430a^{22} + 126698a^{23} + 138345a^{24} + 147722a^{25} + 154297a^{26}$$
$$+ 157689a^{27} + 157689a^{28} + 154297a^{29} + 147722a^{30} + 138345a^{31}$$
$$+ 126698a^{32} + 113430a^{33} + 99214a^{34} + 84730a^{35} + 70595a^{36} + 57327a^{37}$$
$$+ 45317a^{38} + 34830a^{39} + 25979a^{40} + 18769a^{41} + 13104a^{42} + 8817a^{43}$$
$$+ 5693a^{44} + 3517a^{45} + 2063a^{46} + 1143a^{47} + 594a^{48} + 285a^{49} + 125a^{50}$$
$$+ 49a^{51} + 17a^{52} + 4a^{53} + a^{55}.$$

Generating Function for covariants, reduced form,

Denominator:

$$(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^8)(1-a^9)(1-a^{10})$$
$$(1-a^{11})(1-ax^2)(1-ax^4)(1-ax^6)(1-ax^8)(1-ax^{10})(1-ax^{12}).$$

* The tables of the duodecimic have been calculated by Mr F. Franklin in accordance with Professor Sylvester's second method (see this *Journal*†, Vol. III. p. 146), in pursuance of a grant made by the British Association for the Advancement of Science. The corresponding tables for the binary quantics of the first ten orders are given in this *Journal*, Vol. II. p. 223 [p. 283, above]; those for systems of quantics of the first four orders, taken two and two together, are given at page 293 of the same volume [p. 392, above].

[† *On the Calculation of the Generating Functions and Tables of Groundforms for Binary Quantics*, By F. Franklin, *American Journal of Mathematics*, III. (1880), pp. 128—153.]

Numerator:*

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}	x^{36}	x^{38}	x^{40}
a^0	1																				
a^1		-1	-1	-1	-1	-1															
a^2			1	1	2	2	3	2	2	1	1										
a^3			1		1	-1	-1	-2	-3	-3	-3	-3	-2	-1	-1						
a^4	1		3	1	3		1			1	1	2	2	3	2	2	1	1			
a^5	1	1	4	2	2	-1	-2	-2	-3	-2	-2		-1		-1	-1	-1	-1	-1	-1	
a^6	3	3	7	4	4		-1	-2	-3			3	1	2	1	1					1
a^7	4	7	10	7	4	-1	-6	-7	-8	-4	-3	1	-1	1		-1	-1	-1			
a^8	7	12	17	10	6	-2	-11	-13	-12	-5	-2	5	3	6	2	1			1		
a^9	9	23	25	18	7	-6	-21	-24	-22	-10	-3	7	7	8	2	-1	-1	-2		-1	
a^{10}	17	36	39	25	5	-15	-39	-45	-37	-16	-1	16	16	15	5	1	-2	-3		-1	1
a^{11}	21	56	53	32	-1	-32	-67	-72	-54	-19	9	31	34	24	9	-1	-6	-7	-2	-1	1
a^{12}	36	81	76	44	-7	-52	-100	-108	-72	-17	24	56	56	35	11	-5	-14	-13	-4		3
a^{13}	45	112	97	51	-27	-93	-158	-162	-101	-21	45	85	87	47	11	-14	-24	-22	-6	3	4
a^{14}	65	151	133	63	-45	-134	-216	-218	-120	-3	90	143	136	69	15	-22	-40	-32	-8	7	7
a^{15}	81	199	163	66	-83	-206	-309	-303	-157	5	139	199	187	83	4	-48	-66	-51	-14	13	9
a^{16}	110	251	206	69	-124	-282	-404	-386	-173	52	226	301	267	114		-69	-97	-68	-16	24	17
a^{17}	131	309	241	59	-188	-389	-532	-489	-196	97	323	403	345	127	-29	-120	-146	-100	-23	37	21
a^{18}	168	370	288	51	-253	-495	-653	-580	-193	188	460	550	446	150	-58	-169	-202	-127	-23	57	36
a^{19}	193	433	318	22	-347	-636	-808	-692	-196	274	604	691	539	149	-116	-251	-274	-168	-26	83	45
a^{20}	232	493	359	-6	-436	-759	-939	-770	-153	421	797	883	655	160	-176	-329	-352	-204	-19	115	65
a^{21}	256	551	377	-54	-546	-912	-1093	-861	-119	554	980	1045	742	128	-277	-448	-450	-254	-18	154	81
a^{22}	293	598	402	-97	-648	-1035	-1209	-909	-35	746	1201	1252	844	110	-372	-553	-543	-292	1	201	110
a^{23}	307	638	402	-161	-762	-1174	-1335	-956	43	914	1402	1410	907	45	-510	-697	-654	-343	14	251	131
a^{24}	336	667	407	-210	-852	-1266	-1404	-948	168	1131	1619	1594	972	-14	-637	-821	-752	-375	44	308	168
a^{25}	339	679	381	-280	-953	-1371	-1480	-944	274	1296	1791	1706	983	-119	-800	-972	-855	-417	69	369	193
a^{26}	351	678	367	-326	-1012	-1405	-1477	-867	430	1508	1972	1836	997	-209	-934	-1086	-937	-434	114	430	232
a^{27}	339	664	323	-382	-1070	-1446	-1479	-810	542	1632	2069	1860	937	-351	-1100	-1225	-1017	-457	148	490	256
a^{28}	336	635	291	-410	-1086	-1418	-1410	-692	688	1782	2164	1900	893	-453	-1211	-1301	-1056	-446	203	547	293
a^{29}	307	595	239	-445	-1093	-1393	-1342	-595	784	1837	2172	1837	784	-595	-1342	-1393	-1093	-445	239	595	307

* In the tabulated numerators, the *minus* sign is placed *over* the number which it affects.

Numerator—(*Continued.*)

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}	x^{36}	x^{38}	x^{40}
a^{30}	293	547	203	-446	-1056	-1301	-1211	-453	893	1900	2164	1782	688	-692	-1410	-1418	-1086	-410	291	635	336
a^{31}	256	490	148	-457	-1017	-1225	-1100	-351	937	1860	2069	1632	542	-810	-1479	-1446	-1070	-382	323	664	339
a^{32}	232	430	114	-434	-937	-1086	-934	-209	997	1836	1972	1508	430	-867	-1477	-1405	-1012	-326	367	679	351
a^{33}	193	369	69	-417	-855	-972	-800	-119	983	1706	1791	1296	274	-944	-1480	-1371	-953	-280	381	679	339
a^{34}	168	308	44	-375	-752	-821	-637	-14	972	1594	1619	1131	168	-948	-1404	-1266	-852	-210	407	667	336
a^{35}	131	251	14	-343	-654	-697	-510	45	907	1410	1402	914	43	-956	-1335	-1174	-762	-161	402	638	307
a^{36}	110	201	1	-292	-543	-553	-372	110	844	1252	1201	746	-35	-909	-1209	-1035	-648	-97	402	598	293
a^{37}	81	154	-18	-254	-450	-448	-277	128	742	1045	980	554	-119	-861	-1093	-912	-546	-54	377	551	256
a^{38}	65	115	-19	-204	-352	-329	-176	160	655	883	797	421	-153	-770	-939	-759	-436	-6	359	493	232
a^{39}	45	83	-26	-168	-274	-251	-116	149	539	691	604	274	-196	-692	-808	-636	-347	22	318	433	193
a^{40}	36	57	-23	-127	-202	-169	-58	150	446	550	460	188	-193	-580	-653	-495	-253	51	288	370	168
a^{41}	21	37	-23	-100	-146	-120	-29	127	345	403	323	97	-196	-489	-532	-389	-188	59	241	309	131
a^{42}	17	24	-16	-68	-97	-69		114	267	301	226	52	-173	-386	-404	-282	-124	69	206	251	110
a^{43}	9	13	-14	-51	-66	-48	4	83	187	199	139	5	-157	-303	-309	-206	-83	66	163	199	81
a^{44}	7	7	-8	-32	-40	-22	15	69	136	143	90	-3	-120	-218	-216	-134	-45	63	133	151	65
a^{45}	4	3	-6	-22	-24	-14	11	47	87	85	45	-21	-101	-162	-158	-93	-27	51	97	112	45
a^{46}	3		-4	-13	-14	-5	11	35	56	56	24	-17	-72	-108	-100	-52	-7	44	76	81	36
a^{47}	1	-1	-2	-7	-6	-1	9	24	34	31	9	-19	-54	-72	-67	-32	-1	32	53	56	21
a^{48}	1	-1		-3	-2	1	5	15	16	16	-1	-16	-37	-45	-39	-15	5	25	39	36	17
a^{49}		-1		-2	-1	-1	2	8	7	7	-3	-10	-22	-24	-21	-6	7	18	25	23	9
a^{50}			1			1	2	6	3	5	-2	-5	-12	-13	-11	-2	6	10	17	12	7
a^{51}				-1	-1	-1		1	-1	1	-3	-4	-8	-7	-6	-1	4	7	10	7	4
a^{52}	1					1	1	2	1	3			-3	-2	-1		4	4	7	3	3
a^{53}		-1	-1	-1	-1	-1	-1		-1		-2	-2	-3	-2	-2	-1	2	2	4	1	1
a^{54}				1	1	2	2	3	2	2	1	1			1		3	1	3		1
a^{55}							-1	-1	-2	-3	-3	-3	-3	-2	-1	-1	1		1		
a^{56}											1	1	2	2	3	2	2	1	1		
a^{57}																-1	-1	-1	-1	-1	
a^{58}																					1

Generating Function for covariants, representative form,

Denominator:

$$(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^8)(1-a^9)(1-a^{10})$$
$$(1-a^{11})(1-a^2x^4)(1-a^2x^8)(1-a^2x^{12})(1-a^2x^{16})(1-a^2x^{20})(1-ax^{12}).$$

Numerator:

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}
a^0	1																	
a^3			1	1	2	1	2	2	1	2	1	1	1	1		1		
a^4	1		3	2	4	3	4	4	3	4	2	3	1	2	1	1		1
a^5	1	2	5	6	7	8	9	9	5	6	3	4	1	1		1		2
a^6	3	4	9	11	13	15	13	15	9	11	5	6	2	1		1		3
a^7	4	10	16	21	23	27	23	24	18	15	10	6	3	2	1	5		4
a^8	7	16	28	34	40	46	40	37	27	22	12	6		5	5	10	5	7
a^9	9	30	44	58	64	71	64	55	39	27	13	3	8	18	16	22	14	13
a^{10}	17	45	71	89	99	110	97	77	51	29	5	15	30	40	39	42	30	23
a^{11}	21	73	106	133	148	156	137	101	62	20	10	50	65	83	76	74	54	38
a^{12}	36	102	153	191	208	218	187	123	61	1	53	102	129	146	136	120	89	59
a^{13}	45	148	214	265	287	288	240	143	55	44	113	190	220	239	218	181	131	75
a^{14}	65	196	290	353	377	373	299	152	21	114	218	309	352	363	327	258	178	89
a^{15}	81	264	379	460	486	460	357	147	33	226	357	483	524	528	467	344	227	90
a^{16}	110	332	486	577	601	558	408	118	129	376	558	694	750	725	627	439	266	68
a^{17}	131	419	602	707	728	647	442	61	261	587	805	972	1018	960	810	529	286	12
a^{18}	168	501	728	842	848	734	457	30	452	840	1122	1287	1330	1216	996	604	266	91
a^{19}	193	601	856	979	970	800	443	160	677	1160	1476	1654	1664	1489	1171	644	194	265
a^{20}	232	686	985	1106	1068	854	397	331	964	1516	1887	2038	2017	1751	1318	636	48	507
a^{21}	256	783	1102	1223	1158	867	318	541	1279	1920	2310	2451	2356	1995	1422	556	171	842
a^{22}	293	854	1209	1319	1207	865	203	785	1635	2332	2758	2831	2673	2183	1451	403	491	1259
a^{23}	307	931	1293	1388	1241	814	54	1050	1993	2763	3171	3200	2930	2310	1408	151	893	1768
a^{24}	336	974	1352	1430	1225	744	118	1330	2364	3155	3567	3490	3115	2352	1263	182	1393	2335
a^{25}	339	1015	1384	1437	1192	631	310	1607	2691	3525	3876	3719	3204	2299	1032	614	1953	2967
a^{26}	351	1017	1385	1409	1110	507	511	1867	2998	3812	4123	3834	3192	2144	698	1105	2575	3605
a^{27}	339	1015	1352	1353	1018	350	704	2095	3224	4033	4249	3860	3067	1893	301	1661	3199	4244
a^{28}	336	974	1294	1267	887	203	883	2274	3389	4139	4281	3752	2839	1550	175	2223	3824	4823
a^{29}	307	931	1210	1156	762	38	1032	2399	3452	4162	4183	3558	2518	1143	658	2788	4374	5329
a^{30}	293	854	1105	1031	611	100	1146	2456	3446	4059	3998	3251	2127	692	1158	3290	4850	5697
a^{31}	256	783	988	893	482	236	1219	2447	3333	3876	3701	2881	1690	226	1612	3730	5188	5942

Numerator—(*Continued.*)

x^{36}	x^{38}	x^{40}	x^{42}	x^{44}	x^{46}	x^{48}	x^{50}	x^{52}	x^{54}	x^{56}	x^{58}	x^{60}	x^{62}	x^{64}	x^{66}	x^{68}	x^{70}	
																		a^0
																		a^3
																		a^4
	2		1		1													a^5
	2		1		1													a^6
	1		1		2		2		1		1							a^7
2	2	1			2		1		1									a^8
7	4	3	2		3		1				1		1		1			a^9
13	8	2	2	3	5	2	3	1			1							a^{10}
22	6		10	7	11	6	7	2	1		1		1				1	a^{11}
28	3	11	20	20	22	15	11	4	1		1	1	1					a^{12}
32	15	31	45	40	43	29	18	7		1	5	2	3					a^{13}
18	46	75	85	78	72	51	27	9	1	5	9	7	4	1				a^{14}
8	106	139	149	130	115	79	37	9	10	14	19	12	8	2			1	a^{15}
73	199	247	237	209	169	116	46	2	23	31	34	24	13	4	1	2	1	a^{16}
177	344	392	367	311	243	161	51	11	50	58	55	37	21	5	3	4	3	a^{17}
348	540	602	526	445	326	206	48	43	90	100	88	63	28	7	8	10	4	a^{18}
582	815	861	740	601	427	250	29	94	155	163	132	89	42	7	14	16	7	a^{19}
906	1152	1192	980	780	525	279	12	179	244	250	193	132	51	5	27	30	9	a^{20}
1298	1574	1564	1264	962	621	287	91	298	375	365	271	174	69	3	42	45	17	a^{21}
1788	2048	1998	1553	1149	692	260	207	468	537	510	366	232	75	15	70	73	21	a^{22}
2334	2588	2443	1862	1312	734	191	378	680	748	684	476	283	88	40	105	102	36	a^{23}
2947	3140	2910	2135	1441	723	57	600	954	993	890	597	348	82	74	154	148	45	a^{24}
3568	3714	3338	2388	1510	656	134	886	1269	1285	1116	730	394	78	123	216	196	65	a^{25}
4201	4240	3733	2561	1511	513	399	1221	1639	1596	1360	860	444	48	189	293	264	81	a^{26}
4772	4721	4029	2667	1422	298	723	1617	2029	1942	1605	987	466	14	272	383	332	110	a^{27}
5285	5088	4239	2658	1250	2	1115	2036	2454	2276	1844	1098	480	53	370	490	419	131	a^{28}
5665	5349	4301	2561	981	359	1538	2489	2858	2612	2062	1183	454	124	488	606	501	168	a^{29}
5920	5442	4250	2331	634	780	2003	2925	3257	2905	2242	1237	417	233	615	731	601	193	a^{30}
6003	5403	4043	2023	225	1231	2446	3347	3592	3161	2374	1251	330	340	753	860	686	233	a^{31}

Numerator—(*Continued.*)

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}
a^{32}	232	686	860	753	340	$\overline{330}$	$\overline{1251}$	$\overline{2374}$	$\overline{3161}$	$\overline{3592}$	$\overline{3347}$	$\overline{2446}$	$\overline{1231}$	225	2023	4043	5403	6003
a^{33}	193	601	731	615	233	$\overline{417}$	$\overline{1237}$	$\overline{2242}$	$\overline{2905}$	$\overline{3257}$	$\overline{2925}$	$\overline{2003}$	$\overline{780}$	634	2331	4250	5442	5920
a^{34}	168	501	606	488	124	$\overline{454}$	$\overline{1183}$	$\overline{2062}$	$\overline{2612}$	$\overline{2858}$	$\overline{2489}$	$\overline{1538}$	$\overline{359}$	981	2561	4301	5349	5665
a^{35}	131	419	490	370	53	$\overline{480}$	$\overline{1098}$	$\overline{1844}$	$\overline{2276}$	$\overline{2454}$	$\overline{2036}$	$\overline{1115}$	2	1250	2658	4239	5088	5285
a^{36}	110	332	383	272	$\overline{14}$	$\overline{466}$	$\overline{987}$	$\overline{1605}$	$\overline{1942}$	$\overline{2029}$	$\overline{1617}$	$\overline{723}$	298	1422	2667	4029	4721	4772
a^{37}	81	264	293	189	$\overline{48}$	$\overline{444}$	$\overline{860}$	$\overline{1360}$	$\overline{1596}$	$\overline{1639}$	$\overline{1221}$	$\overline{399}$	513	1511	2561	3733	4240	4201
a^{38}	65	196	216	123	$\overline{78}$	$\overline{394}$	$\overline{730}$	$\overline{1116}$	$\overline{1285}$	$\overline{1269}$	$\overline{886}$	$\overline{134}$	656	1510	2388	3338	3714	3568
a^{39}	45	148	154	74	$\overline{82}$	$\overline{348}$	$\overline{597}$	$\overline{890}$	$\overline{993}$	$\overline{954}$	$\overline{600}$	57	723	1441	2135	2910	3140	2947
a^{40}	36	102	105	40	$\overline{88}$	$\overline{283}$	$\overline{476}$	$\overline{684}$	$\overline{748}$	$\overline{680}$	$\overline{378}$	191	734	1312	1862	2443	2588	2334
a^{41}	21	73	70	15	$\overline{75}$	$\overline{232}$	$\overline{366}$	$\overline{510}$	$\overline{537}$	$\overline{468}$	$\overline{207}$	260	692	1149	1553	1998	2048	1788
a^{42}	17	45	42	3	$\overline{69}$	$\overline{174}$	$\overline{271}$	$\overline{365}$	$\overline{375}$	$\overline{298}$	$\overline{91}$	287	621	962	1264	1564	1574	1298
a^{43}	9	30	27	$\overline{5}$	$\overline{51}$	$\overline{132}$	$\overline{193}$	$\overline{250}$	$\overline{244}$	$\overline{179}$	$\overline{12}$	279	525	780	980	1192	1152	906
a^{44}	7	16	14	$\overline{7}$	$\overline{42}$	$\overline{89}$	$\overline{132}$	$\overline{163}$	$\overline{155}$	$\overline{94}$	29	250	427	601	740	861	815	582
a^{45}	4	10	8	$\overline{7}$	$\overline{28}$	$\overline{63}$	$\overline{88}$	$\overline{100}$	$\overline{90}$	$\overline{43}$	48	206	326	445	526	602	540	348
a^{46}	3	4	3	$\overline{5}$	$\overline{21}$	$\overline{37}$	$\overline{55}$	$\overline{58}$	$\overline{50}$	$\overline{11}$	51	161	243	311	367	392	344	177
a^{47}	1	2	1	$\overline{4}$	$\overline{13}$	$\overline{24}$	$\overline{34}$	$\overline{31}$	$\overline{23}$	2	46	116	169	209	237	247	199	73
a^{48}	1			$\overline{2}$	$\overline{8}$	$\overline{12}$	$\overline{19}$	$\overline{14}$	$\overline{10}$	9	37	79	115	130	149	139	106	8
a^{49}				$\overline{1}$	$\overline{4}$	$\overline{7}$	$\overline{9}$	$\overline{5}$	$\overline{1}$	9	27	51	72	78	85	75	46	$\overline{18}$
a^{50}					$\overline{3}$	$\overline{2}$	$\overline{5}$	$\overline{1}$		7	18	29	43	40	45	31	15	$\overline{32}$
a^{51}					$\overline{1}$	$\overline{1}$	$\overline{1}$		1	4	11	15	22	20	20	11	$\overline{3}$	$\overline{28}$
a^{52}	1				$\overline{1}$		$\overline{1}$		1	2	7	6	11	7	10		$\overline{6}$	$\overline{22}$
a^{53}							$\overline{1}$			1	3	2	5	3	2	$\overline{2}$	$\overline{8}$	$\overline{13}$
a^{54}			$\overline{1}$		$\overline{1}$		$\overline{1}$				1		3		2	$\overline{3}$	$\overline{4}$	$\overline{7}$
a^{55}									1		1		2			$\overline{1}$	$\overline{2}$	$\overline{3}$
a^{56}							1		1		2		2		1		$\overline{1}$	
a^{57}													$\overline{1}$		$\overline{1}$		$\overline{2}$	
a^{58}													$\overline{1}$		$\overline{1}$		$\overline{2}$	
a^{59}																		
a^{60}																		
a^{63}																		

Numerator—(*Continued.*)

x^{36}	x^{38}	x^{40}	x^{42}	x^{44}	x^{46}	x^{48}	x^{50}	x^{52}	x^{54}	x^{56}	x^{58}	x^{60}	x^{62}	x^{64}	x^{66}	x^{68}	x^{70}	
5942	5188	3730	1612	$\overline{226}$	$\overline{1690}$	$\overline{2881}$	$\overline{3701}$	$\overline{3876}$	$\overline{3333}$	$\overline{2447}$	$\overline{1219}$	$\overline{236}$	482	893	938	783	256	a^{32}
5697	4850	3290	1158	$\overline{692}$	$\overline{2127}$	$\overline{3251}$	$\overline{3998}$	$\overline{4059}$	$\overline{3446}$	$\overline{2456}$	$\overline{1146}$	$\overline{100}$	611	1031	1105	854	293	a^{33}
5329	4374	2788	658	$\overline{1143}$	$\overline{2518}$	$\overline{3558}$	$\overline{4183}$	$\overline{4162}$	$\overline{3452}$	$\overline{2399}$	$\overline{1032}$	38	762	1156	1210	931	307	a^{34}
4823	3824	2223	175	$\overline{1550}$	$\overline{2839}$	$\overline{3752}$	$\overline{4281}$	$\overline{4139}$	$\overline{3389}$	$\overline{2274}$	$\overline{883}$	203	887	1267	1294	974	336	a^{35}
4244	3199	1661	$\overline{301}$	$\overline{1893}$	$\overline{3067}$	$\overline{3860}$	$\overline{4249}$	$\overline{4033}$	$\overline{3224}$	$\overline{2095}$	$\overline{704}$	350	1018	1353	1352	1015	339	a^{36}
3605	2575	1105	$\overline{698}$	$\overline{2144}$	$\overline{3192}$	$\overline{3834}$	$\overline{4123}$	$\overline{3812}$	$\overline{2998}$	$\overline{1867}$	$\overline{511}$	507	1110	1409	1385	1017	351	a^{37}
2967	1953	614	$\overline{1032}$	$\overline{2299}$	$\overline{3204}$	$\overline{3719}$	$\overline{3876}$	$\overline{3525}$	$\overline{2691}$	$\overline{1607}$	$\overline{310}$	631	1192	1437	1384	1015	339	a^{38}
2335	1393	182	$\overline{1263}$	$\overline{2352}$	$\overline{3115}$	$\overline{3490}$	$\overline{3567}$	$\overline{3155}$	$\overline{2364}$	$\overline{1330}$	$\overline{118}$	744	1225	1430	1352	974	336	a^{39}
1768	893	$\overline{151}$	$\overline{1408}$	$\overline{2310}$	$\overline{2930}$	$\overline{3200}$	$\overline{3171}$	$\overline{2763}$	$\overline{1993}$	$\overline{1050}$	54	814	1241	1388	1293	931	307	a^{40}
1259	491	$\overline{403}$	$\overline{1451}$	$\overline{2183}$	$\overline{2673}$	$\overline{2831}$	$\overline{2758}$	$\overline{2332}$	$\overline{1635}$	$\overline{785}$	203	865	1207	1319	1209	854	293	a^{41}
842	171	$\overline{556}$	$\overline{1422}$	$\overline{1995}$	$\overline{2356}$	$\overline{2451}$	$\overline{2310}$	$\overline{1920}$	$\overline{1279}$	$\overline{541}$	318	867	1158	1223	1102	783	256	a^{42}
507	$\overline{48}$	$\overline{636}$	$\overline{1318}$	$\overline{1751}$	$\overline{2017}$	$\overline{2038}$	$\overline{1887}$	$\overline{1516}$	$\overline{964}$	$\overline{331}$	397	854	1068	1106	985	686	232	a^{43}
265	$\overline{194}$	$\overline{644}$	$\overline{1171}$	$\overline{1489}$	$\overline{1664}$	$\overline{1654}$	$\overline{1476}$	$\overline{1160}$	$\overline{677}$	$\overline{160}$	443	800	970	979	856	601	193	a^{44}
91	$\overline{266}$	$\overline{604}$	$\overline{996}$	$\overline{1216}$	$\overline{1330}$	$\overline{1287}$	$\overline{1122}$	$\overline{840}$	$\overline{452}$	$\overline{30}$	457	734	848	842	728	501	168	a^{45}
$\overline{12}$	$\overline{286}$	$\overline{529}$	$\overline{810}$	$\overline{960}$	$\overline{1018}$	$\overline{972}$	$\overline{805}$	$\overline{587}$	$\overline{261}$	61	442	647	728	707	602	419	131	a^{46}
$\overline{68}$	$\overline{266}$	$\overline{439}$	$\overline{627}$	$\overline{725}$	$\overline{750}$	$\overline{694}$	$\overline{558}$	$\overline{376}$	$\overline{129}$	118	408	558	601	577	486	332	110	a^{47}
$\overline{90}$	$\overline{227}$	$\overline{344}$	$\overline{467}$	$\overline{528}$	$\overline{524}$	$\overline{483}$	$\overline{357}$	$\overline{226}$	$\overline{33}$	147	357	460	486	460	379	264	81	a^{48}
$\overline{89}$	$\overline{178}$	$\overline{258}$	$\overline{327}$	$\overline{363}$	$\overline{352}$	$\overline{309}$	$\overline{218}$	$\overline{114}$	21	152	299	373	377	353	290	196	65	a^{49}
$\overline{75}$	$\overline{131}$	$\overline{181}$	$\overline{218}$	$\overline{239}$	$\overline{220}$	$\overline{190}$	$\overline{113}$	$\overline{44}$	55	143	240	288	287	265	214	148	45	a^{50}
$\overline{59}$	$\overline{89}$	$\overline{120}$	$\overline{136}$	$\overline{146}$	$\overline{129}$	$\overline{102}$	$\overline{53}$	1	61	123	187	218	208	191	153	102	36	a^{51}
$\overline{38}$	$\overline{54}$	$\overline{74}$	$\overline{76}$	$\overline{83}$	$\overline{65}$	$\overline{50}$	$\overline{10}$	20	62	101	137	156	148	133	106	73	21	a^{52}
$\overline{23}$	$\overline{30}$	$\overline{42}$	$\overline{39}$	$\overline{40}$	$\overline{30}$	$\overline{15}$	5	29	51	77	97	110	99	89	71	45	17	a^{53}
$\overline{13}$	$\overline{14}$	$\overline{22}$	$\overline{16}$	$\overline{18}$	$\overline{8}$	$\overline{3}$	13	27	39	55	64	71	64	58	44	30	9	a^{54}
$\overline{7}$	$\overline{5}$	$\overline{10}$	$\overline{5}$	$\overline{5}$		6	12	22	27	37	40	46	40	34	28	16	7	a^{55}
$\overline{4}$		$\overline{5}$	1	$\overline{2}$	3	6	10	15	18	24	23	27	23	21	16	10	4	a^{56}
$\overline{3}$		$\overline{1}$		1	2	6	5	11	9	15	13	15	13	11	9	4	3	a^{57}
$\overline{2}$		$\overline{1}$		1	1	4	3	6	5	9	6	8	7	6	5	2	1	a^{58}
1		1	1	2	1	3	2	4	3	4	4	3	4	2	3		1	a^{59}
		1		1	1	1	1	2	1	3	2	1	2	1	1			a^{60}
																	1	a^{63}

Table of Groundforms.

Degree in the Coefficients.	Order in the Variables. 0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	34
1							1										
2	1		1		1		1		1		1						
3	1		1	1	2	1	2	2	1	2	1	1	1	1		1	
4	2		3	2	4	3	4	4	3	4	2	3	1	2	1	1	1
5	2	2	5	6	7	8	6	9	5	6	3	4	1	1			
6	4	4	9	11	12	14	10	12	3	5							
7	5	10	15	20	18	21	9	8									
8	7	16	24	29	21	21											
9	9	28	33	37	15												
10	14	39	41	30													
11	15	53	40														
12	19	56	7														
13	18	44															
14	12																

The total number of groundforms (counting in the absolute constant and the quantic itself) is 949.

The manuscript sheets containing the original calculations from which the preceding tables have been constructed (as is the case also with the calculations connected with all the similar tables which have appeared in this journal) are deposited in the iron safe of the Johns Hopkins University, Baltimore, where they can be seen and examined, or copied, by any one interested in the subject. From the manifold independent systematic tests* to which the work has been subjected, Mr Franklin estimates that the chance is far more than a million to one that the generating functions for the twelfthic as calculated do not contain a single numerical error. The highest order of any ground-covariant to the twelfthic it will be seen is 34, which is the superior limit of order given by M. Camille Jordan's formula for the ground-covariants to a system of an indefinite number of simultaneous binary forms of each of which the order is 12 or less: M. Jordan's "superior limit" in fact in this as in all the other calculated cases, being actually attained by one (and only one) ground-covariant to a single form*. It will also be noticed that for all orders of the primitive which have been calculated, namely, from 3 to 12 (with 11 omitted), the degree of the covariant of highest order is either 3 or 4. Looking at single quantics of the even orders 6, 8, 10, 12, it will be observed that the maximum order of their ground-covariants for any degree (from and after the 4th degree) diminishes, or, to speak more strictly, never increases as the degree increases. As regards quantics of the odd orders 5, 7, 9, the same rule applies for the maximum order of their groundforms of even degrees; and in respect to their groundforms of odd degrees, the maximum order from and after the 3rd degree diminishes or remains stationary as the degree increases. Also (alike for quantics of odd or even order) when (beginning with the 3rd degree) in passing from an odd to the next even or from an even to the next odd degree of the groundforms, an increase in the maximum order takes place, it is only to the extent of a single unit. These facts, which constitute a sort of *law of shrinkage*, assume practical importance when the successive tables of groundforms are compared together, with a view to track the ground-differentiants (or, in Mr Cayley's language, the ground-seminvariants or *sources* of covariants), as the order of the primitive quantic is increased. Some of these ground-sources retain their irreducible character permanently, others only up to a particular limit of order in the primitive. The former may be regarded as the irreducible differentiants to a quantic of an infinite order: such for instance are all the differentiants of the second and third degree. But when we consider differentiants of the 4th degree this is no longer true. Thus we have the well-known example of the discriminant to $(a, b, c, d \between x, y)^3$, namely, $a^2d^2 + 4ac^3 + 4df^3 - 3b^2c^2 - 6abcd$, which is irreducible for this quantic, but for the quantic $(a, b, c, d, e \between x, y)^4$ remains, it is obvious, a differentiant, but no longer a ground-differentiant, being expressible under the form of the difference of two products of lower differentiants, namely, as

$$(ac - b^2)(ae - 4bd + 3c^2) - a \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$$

* One of these tests depends upon the following property of the generating function, which has been disclosed by observation, and of which the significance is not yet known. On putting $a = 1$ in the numerator of the generating function, the coefficients of the various powers of x are integer multiples of the coefficient of x^0. Thus in the case of the duodecimic, the numerator of the reduced form becomes, on putting $a = 1$,

$$5663(1 + 2x^2 + x^4 - x^6 - 3x^8 - 4x^{10} - 4x^{12} - 2x^{14} + 2x^{16} + 5x^{18} + 6x^{20} + 5x^{22} + 2x^{24} - 2x^{26} - 4x^{28} - 4x^{30} - 3x^{32} - x^{34} + x^{36} + 2x^{38} + x^{40}).$$

Thus the numerical divisibility of the result of putting $a = 1$ furnishes a test for the sums of the *columns*, while the algebraic divisibility of the result of putting $x = 1$ (see this *Journal*†, Vol. III. p. 151) tests the sums of the *rows*; and the satisfaction of both tests makes the correctness of the result practically certain.

[† See footnote, above, p. 489.]

* It is also particularly noticeable that the number of the successively positive and negative blocks in the table follows the law observed in the inferior cases, namely, for Quantics of orders 3 and 4 there is a single block, for Quantics of orders 5 and 6 two blocks, for order 8 three blocks, and for orders 9 and 10 four blocks, there being five distinct blocks alternately positive and negative in the instance before us of the Quantic of order 12.

Suppose a differentiant to be the source of a covariant of the deg-order $j.\epsilon$ considered as belonging to the quantic $(a_0, a_1, \ldots a_i \between x, y)^i$; then it is easily seen that it will be the source of a covariant of the deg-order $j.j+\epsilon$ in respect to the quantic $(a_0, a_1, \ldots a_{i+1} \between x, y)^{i+1}$. We can, therefore, in many cases by a mere inspection of successive tables of groundforms eliminate some at least of the transient ground-differentiants: that is, wherever there are K groundforms of deg-order $j.\epsilon$ to a quantic of the order i, but only $K-\Delta$ of the deg-order $j.\epsilon+\lambda j$ to the quantic of the order $i+\lambda$, we know that at least Δ of the sources to the K groundforms, that is, Δ ground-differentiants of degree j and weight $\frac{1}{2}(ij-\epsilon)$ are only transiently irreducible. Thus, for example, the table of groundforms for the quintic exhibits a groundform of deg-order 4.4, that is, of deg-weight 4.8; but the table of groundforms for the sextic contains no groundform of the same deg-weight, that is, of deg-order 4.8. Hence the differentiant of deg-weight 4.8, although irreducible when regarded as a function of 6 letters (the number of letters which actually appear in it), is reducible when regarded as a function potentially of 7 or more.

So, again, for a like reason, the ground-differentiants of 5 letters, of deg-orders (in respect to the quintic) 5.1 and 5.7, that is, of deg-weights 5.12, 5.9, are only transiently irreducible; and, what is very interesting, it will be seen at a glance (and here the law of shrinkage makes its importance felt) that the sources of all the groundforms to a quintic of a higher order than the 5th are only transitory (or provisional, so to say) ground-differentiants. So in like manner it will be recognized by comparing the tables of groundforms for the seventhic and eighthic, that of the 9 ground-sources of the degree 6 to the former, only two *can be* permanent, namely, one of the weight $\frac{1}{2}(6.7-2)$ and one of the weight $\frac{1}{2}(6.7-4)$, that is, of the deg-weights 6.20 and 6.19 respectively: all the others becoming resoluble when an additional letter is introduced into the quantic. Moreover, as the table for the eighthic contains no groundforms of deg-order 7.8, we see from the law of shrinkage that there can be no ground-source to the seventhic of a higher than the 6th degree which is permanently irreducible*.

A systematic weeding out of the transitory ground-sources from the published tables, which cannot in all cases for groundforms of earlier degrees be effected completely without an examination of a more searching kind than that illustrated by the above examples, must be reserved for a future occasion—after I shall have completed, as I hope soon to do, the study of a subject of higher interest and more pressing importance, which has for its object to determine not only the groundforms so called, but also the ground-syzygants, the ground-counter-syzygants, &c., of quantics from their

* For the 6th degree it will at once be seen that there can be no permanent differentiant to the seventhic except one of the 2nd and one of the 4th order.

generating functions by a purely arithmetical process, which I believe to be already substantially in my possession.

As the first fruits of this method, I may state that the invariantive ground-syzygants (or, if the expression is preferred, fundamental syzygies) to the octavian quantic $(x, y)^8$ are 5 in number, and of the degrees 16, 17, 18, 19, 20 respectively in the coefficients. As regards the ground-syzygants (invariantive and covariantive) of the quintic, my method furnishes the same list as that given in Professor Cayley's *Tenth Memoir on Quantics*. Their deg-orders may be found as follows.

By the supernumerary ground-types understand the deg-orders of the ground-covariants exclusive of those represented by the factors which appear in the denominator of the representative generating function*, which are therefore $23-6$, that is, 17 in number. Let these types be added each to itself and every other, thus giving rise to $\frac{17.18}{2}$ types: out of these sums strike out the types

8.4 9.5 10.2 10.4 11.3 12.2 14.4 16.2

and replace them by

13.5 14.6 15.3 15.5 16.4 17.3 19.5 21.3

The 153 types thus formed, together with the types, 26 in number, furnished by the negative terms in the numerator to the generating function (see this *Journal*, vol. II. p. 224 [p. 284, above]), 179 in all, will be the deg-orders of the fundamental syzygants. Mr Cayley founds this rule on his theory of the so-called Real Generating Function, which essentially consists in what may be termed the Dialytic Presentation of the Representative G. F. for the Quintic—namely as a sum of 26 pairs, each pair containing one positive and one negative term of the numerator divided by the denominator, so selected for conjunction that the developed expression of each pair shall be seen to be omni-positive by an obvious dialytic process.

The method followed by the eminent author in singling out the fundamental syzygants does not appear (as far as I can make out) to be explicitly stated in his memoir. The dialytic form (supposing, as is probably the case, it always exists for *finite* representative generating functions) is not easy to arrive at: a serious additional obstacle to the use of the dialytic method would arise in the case where (as for the seventhic) the numerator of the representative form becomes an infinite series. The method I employ does not require the use of the dialytic method, nor even of the *representative* form of the G. F., although the practical process is much simplified by the use of the representative form when it has a finite numerator. The result

* In such denominator the number of factors for a Quantic of any odd order $2i-1$ is $3i-3$, and for any even order $2i$ is $3i-2$ (i in each case being supposed greater than unity).

I obtain for the fundamental syzygants of the sextic is as follows: Take the 19 supernumerary ground-types (see* vol. II. p. 225), and add them each to each and to every other, as in the preceding case. Then strike out of the sums so formed the types of the deg-orders 6.4, 9.6, 8.4, 11.6, 10.4, 7.8, 8.6, 11.4, as well as one of the two sums 13.4 obtained from the addition of 5.2 and 8.2 or of 3.2 and 10.2 and replace the nine types so omitted by the eight types 12.8, 14.8, 13.6, 15.6, 10.10, 11.8, 14.6, 16.6. There will thus arise $19 \cdot \frac{20}{2} - 9 + 8$, or 189 types: to these adjoin the 29 types given by the negative terms in the numerator of the Rep. G. F.: the total number of types $189 + 29$ or 218 so obtained will be the deg-orders of the complete system of fundamental syzygants to the sextic. The two types of the deg-order 6.6 which appear among the supernumerary types, it will of course be understood, are to be treated as distinct types in forming the binary sums. It is just barely possible (but I think very unlikely) that I may have committed some oversight in the table of replacement in the above calculation, and that the true number of ground-syzygies may be $19 \cdot \frac{18}{2} + 29$ or 219 instead of 218†.

I subjoin a brief *aperçu* of the general theory.

A generating function (whatever its subject-matter) developed in a series consists of facients and coefficients, where any facient is a product of a finite set of letters each raised to a certain power. The totality of the exponents expressing these powers may be termed the type of the facient. In the generating functions to be referred to hereinunder, the letters employed are just as many in number as there are quantics in the system to be considered: namely, one letter corresponds to each quantic.

A generating function proper (with reference to the present theory) is defined to be one that is or can be developed into a series of facients whose coefficients and whose types are omni-positive integers, and where each such numerical coefficient is the number of linearly independent invariants whose degrees in the coefficients of the several quantics of the system are identical with the indices of the corresponding letters in the facient to which that numerical coefficient is attached‡. The type of the facient may be also styled the type of the connoted invariants. A binomial expression consisting

[* p. 285, above.]

† Nine binary sums of types are omitted, and are replaced by only eight other combinations. This is analogous to the loss of a unit in counting the irreducible syzygies to the invariants of an eighthic. The *supernumerary* invariants in this case are 3 in number; of degrees 8, 9, 10 respectively. Their binary combinations would give 6, but the true number of irreducible syzygies is only 5.

‡ I speak designedly (for greater facility of expression) of invariants only, which can be done for binary quantics without any loss of generality, inasmuch as covariants may be regarded as invariants of a given system of quantics with a linear quantic superadded.

of unity followed by a facient and separated from it by the negative sign may be termed a *generator**.

A proper generating function to a system of quantics may always by known methods (see this *Journal*, vol. III. p. 133)† be expressed by a fraction whose numerator is a finite series of facients with numerical coefficients and its denominator a finite product of generators.

It may also be expressed (according to a definite process), and in one way only, by a fraction whose numerator and denominator alike consist of a finite or infinite (except in a few trivial cases, an infinite) product of generators‡.

A finite product of generators (or powers of generators) may be termed a generator-group.

For greater uniformity of statement in regard to what follows, let us agree to understand by a syzygant of the grade zero, an irreducible invariant. Then the two infinite products above referred to (whose ratio is algebraically equal to the generating function) may each be resolved into a product (usually infinite) of collect-groups, such that the totality of the types of the 1st, 2nd, ... ith groups of the denominator shall respectively represent the totality of the types of irreducible syzygants of the grades $0, 2, \ldots (2i-2)$ and the totality of the types of the 1st, 2nd, ... ith groups of the numerator the totality of the types of irreducible syzygants of the grades $1, 3, 5, \ldots (2i-1)$, so that each group may be said to be related to or to represent a complete system of irreducible syzygants of a certain grade (invariants being regarded as zero-graded syzygants)—that is to say, as many times as any generator is repeated in a group so many (and no more) irreducible syzygants of that type will there be of the corresponding grade.

Let G be a proper generating function to a system of quantics, $\Gamma_0, \Gamma_1, \Gamma_2 \ldots$ generator-groups such that

$$G = \frac{1 \cdot \Gamma_1 \cdot \Gamma_3 \cdot \Gamma_5 \ldots}{\Gamma_0 \cdot \Gamma_2 \cdot \Gamma_4 \cdot \Gamma_6 \ldots};$$

then, as suggested to me by Mr Franklin, in order that the Γ series may be

* If $a, b, c, \ldots$ are facients, $1 - a^\alpha b^\beta c^\gamma \ldots$ is a *generator*, and $\alpha, \beta, \gamma \ldots$ (taken in a definite order) is its *type*.

[† See above, p. 489, footnote.]

‡ For instance let G be the generating function proper to the invariants of an eighthic.

Then $$G = \frac{1 + a^8 + a^9 + a^{10} + a^{18}}{(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)}$$
$$= [(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^8)(1-a^9)(1-a^{10})]^{-1}$$
$$\cdot (1-a^{16})(1-a^{17})(1-a^{18})(1-a^{19})(1-a^{20})$$
$$\cdot [(1-a^{25})(1-a^{26})(1-a^{27})(1-a^{28})(1-a^{29})]^{-1}$$
$$\cdot (1-a^{33})(1-a^{34})(1-a^{35})^2(1-a^{36})^2(1-a^{37})^2(1-a^{38})(1-a^{39})$$
$$\cdot [(1-a^{41})(1-a^{42})^2(1-a^{43})^3(1-a^{44})^4(1-a^{45})^4(1-a^{46})^4(1-a^{47})^3(1-a^{48})^2]^{-1}$$
$$\cdot \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

representative of complete systems of irreducible syzygants of the successive grades, it is *necessary* that $\frac{1}{\Gamma_0} - G$; $\frac{\Gamma_1}{\Gamma_0} - G$; $\frac{\Gamma_1\Gamma_3}{\Gamma_0} - G$; $\frac{\Gamma_1\Gamma_3}{\Gamma_0\Gamma_2} - G$; ... shall, when developed in series of facients with omni-positive indices, be alternately omni-positive and omni-negative. But the existence of these inequalities, although a *necessary*, is not a *sufficient* condition in order that the Γ's shall be so representative; for example, $\Gamma_0 . \Gamma_2$ and $\Gamma_1 . \Gamma_3$ might evidently be regarded as single groups and the inequalities would still be satisfied; but suppose we further limit the Γ's in succession by the following rule, namely, that on withdrawing any one of the generator-factors from Γ_0 and calling Γ_0' the group so reduced $\frac{1}{\Gamma_0'} - G$ is no longer omni-positive, this will serve to define Γ_0 absolutely; Γ_0 being so determined, Γ_1 may in like manner be limited by the condition that its quotient by any one of its generators being called Γ_1, $\frac{\Gamma_1'}{\Gamma_0} - G$ shall be no longer omni-negative; then Γ_1 is accurately determined, and, proceeding in like manner with each group in succession, the whole system of groups becomes exactly defined, and thus we obtain the necessary and sufficient condition of group-representation.

Calling $\frac{1}{\Gamma_0}, \frac{\Gamma_1}{\Gamma_0}, \frac{\Gamma_1\Gamma_3}{\Gamma_0}, \frac{\Gamma_1\Gamma_3}{\Gamma_0\Gamma_2}, \ldots \nu_0, \nu_1, \nu_2, \nu_3 \ldots$

respectively, the ν series of quantities stand to G in somewhat the same relation as the complete quotients of a continued fraction to its complete value. Observe that $\nu_0 - 1$, $\nu_1 - 1$, $\nu_2 - 1$, ... each vanish when the variables in G are each zero, and become infinite when the variables in G are each unity.

When each such variable has any value intermediate between 0 and 1, I think it almost certain that no two of the ν's can become equal, so that for all values of the variables inside those limits the parabolic lines or surfaces or hyper-surfaces, &c., represented (after introducing a new variable ω) by the equations $\omega - \nu_0 = 0$, $\omega - \nu_1 = 0$, $\omega - \nu_2 = 0$, ... (which coincide for the limiting values of the original variables at the origin and at a point at infinity) will never intersect, so that within the prescribed limits $\nu_0 - \nu_2$, $\nu_2 - \nu_4$, $\nu_4 - \nu_6$, ... will be always positive and $\nu_1 - \nu_3$, $\nu_3 - \nu_5$, ... will be always negative, the limited boundaries represented by

$$\omega - G, \quad \omega - \nu_0, \quad \omega - \nu_2, \quad \omega - \nu_4, \ldots$$

being each external to the one that precedes it on one side of $\omega - G$, and

$$\omega - G, \quad \omega - \nu_1, \quad \omega - \nu_3, \quad \omega - \nu_5, \ldots$$

following the same law on the other side. It is possible, moreover, that a more stringent condition than the above may be verified, namely, that

$$\nu_0 - G, \quad \nu_2 - \nu_0, \quad \nu_4 - \nu_2 \ldots$$
$$G - \nu_1, \quad \nu_1 - \nu_3, \quad \nu_3 - \nu_5 \ldots$$

may each be developable into omni-negative functions, and again (to complete the analogy with the parallel theory of continued fractions or converging continued products) that

$$\nu_0 - G, \quad G - \nu_1, \quad \nu_2 - G, \quad G - \nu_3, \quad \nu_4 - G, \ldots$$

shall form a single series of continually decreasing quantities, or even in their developed state, of functions in which the corresponding coefficients to each facient form a continually decreasing (or, at least, never-increasing) series of numbers. Then in the case of a single quantic, within the limits defined by the facient a being 0 and 1 the curves $\omega - \nu_1$, $\omega - \nu_3$, ... $\omega - G$, ... $\omega - \nu_2$, $\omega - \nu_0$, will form an infinite series of loops having one common asymptote and one common point of intersection, and except at that one point keeping clear of each other.

I annex tables (pp. [506, 507, below]) of the fundamental syzygants* or (if one pleases so to say) irreducible syzygies for the quintic and sextic, rendered more complete by inserting entries corresponding to the fundamental in- and- covariants. The positive integers correspond to these latter, the negative integers (the negative sign being set over the figure) to the irreducible syzygants. Thus, for example, in the table to the sextic the positive integer 2 found in the 6th line and 6th column, indicates that there are 2 ground-covariants of deg-order 6 . 6. The negative integer $\bar{7}$ found in the 12th line and 12th column indicates that there are 7 irreducible syzygies of deg-order 12 . 12†. The negative sign is appropriate, inasmuch as every independent syzygy of any deg-order lowers by a unit the number of linearly independent in- or- covariants of that deg-order that can be produced out of the inferior groundforms, so that syzygants may be regarded as negative existences in regard to groundforms: carrying on the same idea, counter-syzygants might be numbered by integers carrying two negative signs contradicting each other, and so on indefinitely.

* N.B.—A syzygant to a Quantic is a rational integer function of its in- or- covariants which, expressed as a function of the coefficients, vanishes identically, but we may still understand its "degree in the coefficients" to mean the degree of any one of the terms of which it is the sum.

† If j or e exceed the highest degree or order respectively found in any table, or, if without that being the case there is a blank space in the jth line and eth column of the table, the meaning is that there is no irreducible groundform or syzygy of the deg-order $j . e$. In the tables exhibited it will be seen that the deg-order $j' . e'$ of each syzygant is superior to the deg-order $j . e$ of every groundform: that is, the differences $j' - j$, $e' - e$ are neither of them less and one of them is greater than zero. The same is true for all quantics which have a finite Rep. G. F., but not necessarily and probably never actually so in other cases; thus, for example, to the seventhic belongs an irreducible invariant of degree 22 and an irreducible syzygy of degree 20, so that here the $j' . e'$ (20 . 0) is inferior to the $j . e$ (22 . 0). The fact of every $j' . e'$ being superior to the $j . e$ can be expressed by saying that the invariantive syzygetic portions of a Rep. G. F. table are not intermingled but lie totally apart and may be divided from each other by a single continuous cut.

The method of partitions or generating functions, which leads to these surprising constructions, looks at invariants and their connexions solely with regard to their deg-order or type without taking any account of their content; in other words it deals only with the *idea* or *notion* of these beings and their relations, and may therefore, I think, suitably be termed the Idealistic method*. I cannot see the faintest possibility of the symbolic method serving to determine a complete system of syzygies in any but the trivial cases of quantics of the 3rd or 4th order—the only cases where the infinite procession of beings (syzygants, counter-syzygants, anti-counter-syzygants, &c.), rising out of each other, comes to a stop—there being for those cases no procession after the 1st step, as is also true of invariants (as distinguished from covariants) for quantics of the 6th order. This is how it came to pass in the infancy of the theory that the number of ground-covariants was supposed to become infinite for quantics beyond the fourth and their ground-invariants for quantics beyond the 6th order.

I think it may be interesting to some of the readers of the *Journal* to be put in possession of the complete system of irreducible syzygies to a system of two or more quantics, and I select as an easy example the case of a combined quadratic and cubic, reserving the other combinations of which the groundform tables have been published for a subsequent number of the *Journal*. The supernumerary groundforms for the quadri-cubic system (see

* My proof in the *Phil. Trans.*, founded on the canonical form of the Quintic, of its 4th, 8th, 12th and 18th-degreed invariants forming a complete system, the late Mr Boole's discovery of the cubinvariant to the Quartic, the various disproofs in the *Comptes Rendus* and in this *Journal* of the existence of supposed groundforms, are all exemplifications of the Realistic point of view. The Symbolic lies between this and the Idealistic aspect of the subject, in so far as the operations by which invariants are engendered constitute a new and so to say finer subject-matter, capable of being itself operated upon in all respects like ordinary algebraical substance. In Professor Cayley's *Tenth Memoir on Quantics* there is a sort of half return from the Idealistic to the Realistic view—a kind of substantiality being attributed to the groundforms themselves as primary elements in the study of their syzygetic interconnections. It may be well to notice, for the benefit of the readers of that memoir (*Phil. Trans.* 1878), that in the Representative Form given at p. 657 two terms are omitted by an oversight, namely, $-a^{17}x^4$ and a^3x^{12}. I need hardly add (since the publication of my tables in this *Journal*), with reference to a doubt expressed by Prof. Cayley (*loc. cit.*), that I *had* obtained the form referred to in the paragraph following the R. G. F. in question, though *not* by dividing out the common factors from the numerator and denominator of the R. G. F.; on the contrary, the N. G. F. is first obtained from the generating function in its crude form (which if left in that form would lead to a bivergent series), and then the R. G. F. is obtained from this, through multiplying its numerator and denominator by the factors needed to render the denominator a product of representative groundforms.

The Symbolic and the Idealistic (which I formerly called the fatalistic or peprotic) method alike, as far as is known, owe their conception to the same (unnecessary to be named) acute and capacious intellect. Whether very much that is essential remains to be added to the great discoveries of Gordan and Jordan in the direction of the former may reasonably be doubted, but no such misgiving can be entertained with respect to the latter, which already has given rise to many more questions than it has settled (of a kind, too, of which a solution sooner or later may reasonably be anticipated).

this *Journal**, vol. II. pp. 295, 296), are of the deg-deg-orders 3.4.0, 1.1.1, 2.1.1, 1.3.1, 2.3.1, 1.2.2, 1.1.3, 0.3.3, where the first and second numbers express the degrees in the coefficients of the quadric and cubic respectively, and the last number expresses the order in the variables. Adding each of these triads to itself and every other, rejecting the combinations 2.2.2, 3.2.2, 2.4.2, which appear in the numerator of the G. F. (and arise from the additions 1.1.1+1.1.1, 1.1.1+2.1.1, 1.1.1+1.3.1), replacing them by the higher combinations 1.1.1+1.1.1+1.1.1, 1.1.1+1.1.1+2.1.1, 1.1.1+1.1.1+1.3.1, that is, 3.3.3, 4.3.3, 3.5.3, and adding in the 12 types furnished by the negative terms in the numerator of the G. F., the totality of the irreducible syzygies (48 in number) to the binary quadri-cubic system is arrived at and exhibited in the annexed table, in which the exponents attached to any type signify the number of irreducible syzygies of the corresponding deg-deg-order.

Table of Irreducible Syzygies to the Quadri-cubic System.

6.8.0,	4.5.1,	4.7.1,	5.5.1,	5.7.1,	2.6.2,	$(3.4.2)^2$,
$(3.6.2)^2$,	4.2.2,	$(4.4.2)^2$,	$(4.6.2)^2$,	1.5.3,	2.3.3,	$(2.6.3)^2$,
$(3.3.3)^2$,	$(3.5.3)^2$,	3.6.3,	3.7.3,	4.3.3,	4.5.3,	1.4.4,
1.6.4,	2.2.4,	$(2.4.4)^4$,	2.6.4,	3.2.4,	3.4.4,	3.6.4,
$(1.5.5)^2$,	2.3.5,	2.5.5,	3.5.5,	0.6.6,	1.3.6,	1.4.6,
2.2.6,	4.7.6,					

there being thus one irreducible invariantive syzygy and 4, 10, 12, 11, 5, 5 covariantive syzygies of orders 1, 2, 3, 4, 5, 6 respectively.

It may be worth while just to notice that the types to the complete system of irreducible syzygies to a simultaneous linear and quartic form will consist simply of the sums of the 13 supernumerary types, (*A. M. J.* vol. II. p. 295†), 6.3.0, 3.1.1, 3.2.1, 5.3.1, 2.1.2, 2.2.2, 4.3.2, 1.1.3, 1.2.3, 3.3.3, 2.3.4, 1.3.5, 0.3.6, added each to itself and every other, together with the 14 types taken from the negative terms in the numerator of the G. F., namely, 7.3.1, 6.3.2, 5.3.3, 4.3.4, 6.4.4, 6.5.4, 3.3.5, 5.4.5, 5.5.5, 2.3.6, 4.4.6, 4.5.6, 1.3.7, 7.6.7, making $\frac{13.14}{2}+14$, that is, 105 in all. In this instance there is no rejection or substitution of sums called for.

A word or two seems necessary to leave unambiguous the meaning of the term syzygants of any specified grade in what precedes.

In- or- covariants may be termed syzygants of grade zero (as already stated). Syzygants of the first grade are defined to be rational integer

[* p. 394, above.] [† p. 393, above.]

ORDER IN THE VARIABLES.

Degree in the Coefficients	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1						1													
2			1				1												
3				1		1				1									
4	1				1		1												
5		1		1				1				$\bar{1}$							
6			1		1		$\bar{1}$		$\bar{1}$		$\bar{1}$		$\bar{1}$		$\bar{2}$				$\bar{1}$
7		1				1		$\bar{1}$		$\bar{3}$		$\bar{1}$		$\bar{2}$		$\bar{2}$			
8	1		1				$\bar{2}$		$\bar{2}$		$\bar{3}$		$\bar{3}$				$\bar{1}$		
9				1		$\bar{1}$		$\bar{5}$		$\bar{2}$		$\bar{2}$		$\bar{3}$					
10					$\bar{1}$		$\bar{3}$		$\bar{4}$		$\bar{5}$				$\bar{2}$				
11		1				$\bar{4}$		$\bar{3}$		$\bar{3}$		$\bar{3}$							
12	1				$\bar{2}$		$\bar{4}$		$\bar{5}$		$\bar{1}$		$\bar{2}$						
13		1		$\bar{2}$		$\bar{3}$		$\bar{3}$		$\bar{4}$									
14			$\bar{1}$		$\bar{2}$		$\bar{6}$		$\bar{1}$		$\bar{3}$								
15				$\bar{2}$		$\bar{3}$		$\bar{4}$		$\bar{1}$									
16					$\bar{5}$		$\bar{2}$		$\bar{2}$		$\bar{2}$								
17				$\bar{2}$		$\bar{3}$		$\bar{2}$		$\bar{1}$									
18	1		$\bar{2}$		$\bar{2}$		$\bar{2}$		$\bar{2}$		$\bar{1}$								
19				$\bar{2}$		$\bar{3}$		$\bar{1}$											
20			$\bar{2}$		$\bar{1}$		$\bar{2}$		$\bar{1}$										
21				$\bar{3}$		$\bar{1}$				$\bar{1}$									
22			$\bar{1}$		$\bar{2}$		$\bar{1}$												
23		$\bar{1}$		$\bar{1}$				$\bar{1}$				$\bar{1}$							
24			$\bar{2}$		$\bar{1}$														
25		$\bar{1}$				$\bar{1}$													
26			$\bar{2}$																
27				$\bar{1}$															
28																			
29		$\bar{1}$																	
30																			
31		$\bar{1}$																	
32																			
33																			
34																			
35																			
36	$\bar{1}$																		

ORDER IN THE VARIABLES.

ORDER IN THE VARIABLES.

Degree in the Coefficients	0	2	4	6	8	10	12	14	16	18	20	22	24
1				1									
2	1		1		1								
3		1		1	1		1						
4	1		1	1		1							
5		1	1		1				$\bar{1}$				
6	1			2	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{1}$	$\bar{1}$		$\bar{1}$
7		1	1	$\bar{1}$		$\bar{1}$	$\bar{4}$	$\bar{1}$	$\bar{2}$	$\bar{2}$		$\bar{1}$	
8		1			$\bar{2}$	$\bar{4}$	$\bar{2}$	$\bar{4}$	$\bar{3}$		$\bar{2}$		
9			1		$\bar{4}$	$\bar{2}$	$\bar{5}$	$\bar{4}$		$\bar{3}$			
10	1	1	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{6}$	$\bar{5}$	$\bar{2}$	$\bar{4}$				
11					$\bar{6}$	$\bar{5}$	$\bar{2}$	$\bar{4}$					
12		1	$\bar{1}$	$\bar{4}$	$\bar{3}$	$\bar{3}$	$\bar{7}$	$\bar{1}$	$\bar{1}$				
13			$\bar{1}$	$\bar{2}$	$\bar{5}$	$\bar{5}$	$\bar{1}$	$\bar{3}$					
14			$\bar{1}$	$\bar{4}$	$\bar{6}$	$\bar{2}$	$\bar{3}$						
15	1		$\bar{3}$	$\bar{3}$	$\bar{2}$	$\bar{4}$	$\bar{1}$	$\bar{2}$					
16			$\bar{1}$	$\bar{4}$	$\bar{4}$	$\bar{1}$	$\bar{2}$						
17			$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{2}$		$\bar{1}$					
18		$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{4}$		$\bar{1}$						
19			$\bar{2}$	$\bar{3}$		$\bar{1}$							
20		$\bar{1}$	$\bar{3}$		$\bar{1}$				$\bar{1}$				
21				$\bar{3}$									
22		$\bar{1}$	$\bar{2}$										
23		$\bar{1}$											
24			$\bar{2}$										
25		$\bar{1}$											
26													
27		$\bar{1}$											
28													
29													
30	$\bar{1}$												

ORDER IN THE VARIABLES.

functions of those of grade zero which vanish when the latter are expressed in terms of the original coefficients. It is not *necessary* to define *these* syzygants as functions of *irreducible* ones of grade zero (which vanish under the condition aforesaid), because every in- or- covariant is a rational integer function of the irreducible in- or- covariants. But when we come to syzygants of the second grade (since those of the first grade are not necessarily functions of the irreducible ones of that grade, but may be so of the in- or- covariants as well), it becomes necessary to define syzygants of the second grade (*aliter* counter-syzygants) as rational integer functions of *irreducible* ones of the first grade which vanish when they are expressed in terms of the quantities (here the in- or- covariants) which immediately precede them in the scale of generation. And so, in general, following out the defining process step by step, by a syzygant of the $(i+1)$th grade for the purpose of this theory, is to be understood a rational integer function of the *irreducible* ones of the ith grade which vanishes when these latter are expressed in terms of those of the grade $i-1$. Such at least is my present impression; but, supposing that I am labouring under a misconception on this point, it will in nowise affect the validity of the theory in what regards the computation of the irreducible in- or- covariants and the syzygants of the first grade.

60.

A DEMONSTRATION OF THE IMPOSSIBILITY OF THE BINARY OCTAVIC POSSESSING ANY GROUNDFORM OF DEG-ORDER 10.4.

[*American Journal of Mathematics*, IV. (1881), pp. 62—84.]

Dr von Gall has rendered an inestimable service to algebraical science by working out, according to Gordan's method, the complete system of groundforms to the octavian binary quantic $[(x, y)^8]$. His results, published in the *Mathematische Annalen*, were at first widely discordant from those which have appeared in this *Journal*, but eventually have been brought by their author into perfect agreement with them, with the sole exception that his table includes a covariant of *deg-order* 10.4, not included in my list, which he states that he has not been able to decompose: it is the object of the present communication to bring the two tables into exact accord by demonstrating that no irreducible covariant to $(x, y)^8$ of that deg-order can exist. The total number of covariants of deg-order 10.4 obtained by multiplying together the irreducible covariants of an inferior deg-order (which appear equally in von Gall's table and in my own, and whose existence therefore may be taken for granted*) will be seen to be 32, which is the number of linearly independent covariants of that deg-order given by Cayley's law, [see p. 525, below]; hence, by the fundamental postulate, the 32 compounds in question must not be supposed subject to any linear relation, so that, according to that postulate, there exists no groundform of the deg-order in question; but my object is to use this instance as another exemplification of the validity of that same very reasonable postulate—as I have done on the three former occasions where the tables of Clebsch, Gordan and Gundelfinger comprised groundforms extraneous to the tables obtained by me on the assumption of its truth; the proof, however, on the present occasion, is much lengthier than any that has ever hitherto been employed, and involves arithmetical computations of considerable prolixity,

* As I have elsewhere remarked, since no groundforms can exist exterior to the tables furnished by Gordan's method, and no reducible forms can be contained in the tables furnished by the English method, it follows, even without assuming the truth of the fundamental postulate, that wherever the tables furnished by the two methods accord, they must, of logical necessity, be correct, *mere errors of calculation excepted.*

all necessity for which I had, in previous cases, been able to evade. It is, I may add, only after repeated trials and discomfitures, that I have succeeded at length in devising a special method adequate to prove the important point at issue.

The irreducible invariants and covariants of deg-order *inferior* to 10.4, (that is, whose degree in the coefficients and whose order in the variables are not each as great as 10 and 4 respectively), and which also can enter as factors of a covariant of deg-order 10.4 (this excludes the necessity of considering invariants of degrees 9 or 10) are as follows: the invariants are of degrees 2, 3, 4, 5, 6, 7, 8, one of each degree; the covariants are one of deg-orders 5.2, 2.4, 3.4 respectively, and two of deg-orders 4.4, 5.4, 6.4, 7.4, 8.4 respectively. We may denote the invariants by 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, and the covariants by 5.2, 2.4, 3.4, 4.4, 4.4*, 5.4, 5.4*, 6.4, 6.4*, 7.4, 7.4*, 8.4, 8.4*, and it is an easy arithmetical calculation to show (see† *Comptes Rendus*, July 25, 1881) that there are (as already stated) 32 different ways in which these duads, by their combination, can give rise to the duad 10.4. Out of these 32 it is important, with a view to what follows, to isolate those in which neither 2.0 nor 3.0 appears; their number will easily be seen to be 10, as shown in the scheme below—

4.0 + 6.4	4.0 + 6.4*	4.0 + 4.0 + 2.4	5.0 + 5.4	5.0 + 5.4*
6.0 + 4.4	6.0 + 4.4*	7.0 + 3.4	8.0 + 2.4	5.2 + 5.2.

What I have to prove is, that no equation $\Omega = 0$ exists, where Ω is a linear function of the 32 products in question, connected by numerical coefficients. Suppose it can be shown that Ω does not contain any of the 10 functions above indicated. Then Ω is either of the form $(2.0)U$ or $(3.0)V$, or is a linear function of $(2.0)U$ and $(3.0)V$. In the former two cases we should obtain $U = 0$ or $V = 0$ respectively; and in the third case the equation $\lambda(2.0)U + \mu(3.0)V = 0$, since 2.0 and 3.0 have no common factor, implies the existence of an integral equation $\lambda\dfrac{U}{(3.0)} + \mu\dfrac{V}{(2.0)} = 0$. Hence, in the three cases supposed, there would exist a syzygy of the deg-order 8.4, 7.4, 5.4 respectively between composite covariants of the inferior deg-orders; but if this were so, the number of irreducible covariants of one or another of these deg-orders would not be what it is at present, but, in order to satisfy Cayley's law, would have to be increased by a unit: or, in other words, results obtained by my method and coincident with those resulting, or capable of resulting, from the German method, would be erroneous, which never can be the case‡. Hence, the non-existence of $\Omega = 0$ will be demonstrated if it can be shown that, for some particular form of the general primitive $(x, y)^8$

[† p. 481, above.]

‡ Towards the end of this paper I establish the same conclusion by a more direct method, in which nothing extraneous to Dr von Gall's own table is assumed, except the one fact of the linearly independent covariants of deg-order 10.4 being 32 in number.

which causes the invariants of the second and third degrees each to vanish, the particular values then assumed by the 10 compounds which remain in Ω are not subject to any linear relation. Of course the converse would not be true; the fact of the existence of a syzygy between these 10, or even between the whole 32 compounds for a special form of the primitive, would not establish the existence of a syzygy between them in the general case.

The great practical gain of making the first two invariants vanish is that it leads to a computation in which only 10 instead of 32 linear functions have to be handled—but it is not possible *à priori* before the calculations have been gone through, to feel at all assured that the particular form assumed may not be such as to lead only to nugatory results. Such happily, however, turns out not to be the case with the form I am about to employ which leads to the expression of the 10 compounds as homogeneous linear functions of 11 arguments*, giving rise to a rectangular matrix 11 places wide and 10 places deep of which it can be shown that the complete minors (determinants of the 10th order) do not all vanish, so that the 10 functions cannot be subject to any syzygy; and consequently, if $\Omega = 0$ were a really existing syzygy, Ω must consist exclusively of 22 terms, every one of which contains one or both of the two first invariants; but this has been shown to be impossible, so that the non-evanescence of the minors referred to at once establishes the non-existence of a syzygy of deg-order 10.4, and, therefore, the non-existence of a *groundform of that deg-order*.

I take for the primitive the special form $(0, b, 2c, d, 0, 0, 0, 0, 1 \between x, y)^8$, that is to say, $8bx^7y + 56cx^6y^2 + 56dx^5y^3 + y^8$, with the relation $bd = 3c^2$, and proceed to form the required derivatives in conformity with von Gall's scheme of derivation. I use, as the best practical method of obtaining the "alliance" of the ith order between any two forms ϕ, ψ (of the orders μ, ν) denoted by $(\phi, \psi)_i$, the lineo-linear quadrinvariant (with respect to the variables of emanation) of the ith emanant of ϕ combined into a system with the ith emanant of ψ, taking care to reduce the result to the *parenthetical* form $(\ldots\ldots \between x, y)^{\mu+\nu-2i}$, containing only integer coefficients free from any common numerical factor. For the sake of brevity, too, I omit in general the symbolical factor containing (x, y): so that $(a_0, a_1, a_2, \ldots, a_i)$ will indicate the same thing as $(a_0, a_1, a_2, \ldots, a_i \between x, y)^i$. I shall adhere, in what follows, to the notation employed by Dr von Gall.

We have then, according to this notation,

$$f = (0, b, 2c, d, 0, 0, 0, 0, 1) \qquad (1)$$

$$\begin{aligned} i = (f, f)_4 &= (4bx^3y + 12cx^2y^2 + 4dxy^3)\,y^4 - 4dx^4(bx^4 + 8cx^3y + 6dx^2y^2) \\ &\qquad + 3(2cx^4 + 4dx^3y)^2 \\ &= [12c^2 - 4bd, 16cd, 24d^2, 0, 0, 4b, 12c, 4d, 0][x, y]^8, \end{aligned}$$

* One of these arguments is itself a linear function of 3 combinations of the coefficients and variables, the total number of such combinations which appear in the 10 compounds being 13.

where the square bracket is employed to signify the same thing as would be indicated by the use of the round clamp, with the exception that the binomial coefficients are suppressed. We have, therefore, introducing the multipliers

$$\frac{14}{1},\ \frac{14}{8},\ \frac{14}{28},\ \frac{14}{56},\ \frac{14}{70},\ \frac{14}{56},\ \frac{14}{28},\ \frac{14}{8},\ \frac{14}{1},$$

$$i = (0,\ 28cd,\ 12d^2,\ 0,\ 0,\ b,\ 6c,\ 7d,\ 0) \qquad (2)$$

$$\begin{aligned} k = (f, f)_6 &= (2bxy + 2cy^2)\,y^2 - 10d^2x^4 \\ &\equiv (\overline{20}d^2,\ 0,\ 0,\ b,\ 4c)^* \end{aligned} \qquad (3)$$

$$\begin{aligned} \Delta = (k, k)_2 &= \overline{20}d^2x^2(2bxy + 4cy^2) - b^2y^4 \\ &\equiv (0,\ 90c^2d,\ 40cd^2,\ 0,\ 3b^2) \end{aligned} \qquad (4)$$

$$C = (k, k)_4 = 20d^2 . 4c \equiv cd^2 \qquad (5)$$

$$\begin{aligned} f_4 = (f, k)_4 &= 4c\,(4bx^3y + 12cx^2y^2 + 4dxy^3) - 4b\,(bx^4 + 8cx^3y + 6d)\,x^2y^2 \\ &\qquad + \overline{20}d^2y^4 \\ &= -4b^2x^4 - 16bcx^3y + (48c^2 - 24bd)\,x^2y^2 + 16cdxy^3 - 20d^2y^4 \\ &\equiv (b^2,\ bc,\ c^2,\ \overline{cd},\ 5d^2) \end{aligned} \qquad (6)$$

$$\begin{aligned} f_{k,2} = (f_4, k)_2 &= (b^2x^2 + 2bcxy + c^2y^2)(2bxy + 4cy^2) - 2by^2(bcx^2 + 2c^2xy - cdy^2) \\ &\qquad + \overline{20}d^2x^2(c^2x^2 - 2cdxy + 5d^2y^2) \\ &= [\overline{20}c^2d^2,\ 2b^3 + 40cd^3,\ 6b^2c + \overline{100}d^4,\ 6bc^2,\ 4c^3 + 2bcd]\,[x, y]^4 \\ &\equiv (\overline{120}c^2d^2,\ 3b^3 + 60cd^3,\ 6b^2c + 100d^4,\ 9bc^2,\ 60c^3) \end{aligned} \qquad (7)$$

$$\begin{aligned} f_{k,3} = (f_4, k)_3 &= (b^2x + bcy)(bx + 4cy) - 3by\,(bcx + c^2y) - \overline{20}d^2x\,(\overline{cd}x + 5d^2y) \\ &= (b^3 + \overline{20}cd^3)\,x^2 + (2b^2c + 100d^4)\,xy + bc^2y^2 \end{aligned}$$

$$\begin{aligned} (f_{k,3})^2 &= (b^6 + \overline{40}b^3cd^3 + 400c^2d^6)\,x^4 + (4b^5c\,[\overline{80}b^2c^2d^3 + 200b^3d^4] - 4000cd^7)\,x^3y \\ &\qquad + (6b^4c^2 + 400b^2cd^4 - 40bc^3d^3 + 10000d^8)\,x^2y^2 \\ &\qquad + (4b^3c^3 + 200bc^2d^4)\,xy^3 + b^2c^4y^4 \\ &= [b^6 + \overline{1080}c^7 + \overline{400}c^2d^6,\ 4b^5c + 4680c^6d + \overline{4000}cd^7, \\ &\qquad 6b^4c^2 + 3480c^5d^2 + 10000d^8,\ 4b^3c^3 + 600c^4d^3,\ b^2c^4]\,[x, y]^4 \\ &\equiv (3b^6 + \overline{3240}c^7 + 1200c^2d^6,\ 3b^5c + 3510c^6d + \overline{3000}cd^7, \\ &\qquad 3b^4c^2 + 1740c^5d^2 + 5000d^8,\ 3b^3c^3 + 450c^4d^3,\ 3b^2c^4) \end{aligned} \qquad (8)$$

$$\begin{aligned} (f_\Delta) = (f, \Delta)_4 &= 3b^2(4bx^3y + 12cx^2y^2 + 4dxy^3) + 6\,(40cd^2)(2cx^4 + 4dx^3y) \\ &\qquad - 120bd^2\,(dx^4) \\ &= [480c^2d^2 - 120bd^3,\ 12b^3 + 960cd^3,\ 36b^2c,\ 12b^2d,\ 0]\,[x, y]^4 \\ &\equiv (40c^2d^2,\ b^3 + 80cd^3,\ 2b^2c,\ 3bc^2,\ 0) \end{aligned} \qquad (9)$$

$$\begin{aligned} i_\Delta = (i, \Delta)_4 &= -120bd^2(6bx^2y^2 + 24cxy^3 + 7dy^4) \\ &\qquad + 240cd^2(12d^2x^4 + 4bxy^3 + 6cy^4) + 3b^2(112cdx^3y + 72d^2x^2y^2) \\ &= [2880cd^4,\ 336b^2cd,\ \overline{504}b^2d^3,\ \overline{1920}bcd^2,\ 1440c^2d^2 \\ &\qquad + \overline{840}bd^3]\,[x, y]^4 \\ &\equiv (240cd^4,\ 21bc^3,\ \overline{63}c^4,\ \overline{120}c^3d,\ \overline{90}c^2d^2) \end{aligned} \qquad (10)$$

* The sign of equivalence ($\equiv$) is used in the above and in what follows in the sense of "may be superseded by."

$$\begin{aligned} i_4 = (i, k)_4 &= \overline{20}d^2(4bx^3y + 36cx^2y^2 + 28dxy^3) - 4 . b\,(28cdx^4 + 48d^2x^3y + by^4) \\ &\qquad + (4c)(112cdx^3y + 72d^2x^2y^2) \\ &= [\overline{112}bcd,\ 448c^2d + \overline{272}bd^2,\ \overline{432}cd^2,\ \overline{560}d^3,\ \overline{4}b^2]\,[x, y]^4 \\ &\equiv (336c^3,\ 92c^2d,\ 72cd^2,\ 140d^3,\ 4b^2) \end{aligned} \qquad (11)$$

$$\begin{aligned} i_{k,2} = (i_4, k)_2 &= \overline{20}d^2x^2(72cd^2x^2 + 280d^3xy + 4b^2y^2) \\ &\qquad - 2by^2(92c^2dx^2 + 144cd^2xy + 140d^3y^2) \\ &\qquad + (2bxy + 4cy^2)(336c^3x^2 + 184c^2dxy + 72cd^2y^2) \\ &= [\overline{1440}cd^4,\ 672bc^3 + \overline{5600}d^5,\ 184bc^2d - 80b^2d^2 + 1344c^4, \\ &\qquad 736c^3d + \overline{144}bcd^2,\ 288c^2d^2 + \overline{280}bd^3]\,[x, y]^4 \\ &\equiv (\overline{360}cd^4,\ 42bc^3 + \overline{350}d^5,\ 49c^4,\ 19c^3d,\ 138c^2d^2) \end{aligned} \qquad (12)$$

$$\begin{aligned} f_{k,\Delta} = (f_4, \Delta)_4 &= -4\,(30bd^2)(\overline{cd}) + 6\,(40cd^2)\,c^2 + 3b^2 . b^2 \\ &= 3b^4 + 120bcd^3 + 240c^3d^2 \\ &\equiv b^4 + 200c^3d^2 \end{aligned} \qquad (13)$$

$$\begin{aligned} i_{k,\Delta} = (i_4, \Delta)_4 &= -4\,(30bd^2)(140d^3) + 6\,(40cd^2)(72cd^2) + (3b^2)(336c^3) \\ &= 1008b^2c^3 + \overline{33120}c^2d^4 \equiv 7b^2c^3 - 230c^2d^4. \end{aligned}$$

The term involving c^2d^4 being a multiple of the square of C (the invariant of the 4th degree) may be neglected, and, instead of $i_{k,\Delta}$, we may write the irreducible invariant of the 8th degree (say)

$$I_8 = b^2c^3. \qquad (14)$$

That of the 7th degree we have just found $= b^4 + 200c^3d^2$; and obviously the quadrinvariant of f is identically zero, or say

$$I_2 = 0. \qquad (15)$$

Also the cubinvariant $I_3 = (f, i)_8$, where

$$f = (0,\ b,\ 2c,\ d,\ 0,\ 0,\ 0,\ 0,\ 1)$$

and $$i = (0,\ 28cd,\ 12d^2,\ 0,\ 0,\ b,\ 6c,\ 7d,\ 0).$$

Hence $$I_3 = -56bd + 336c^2 - 168bd = 504c^2 - 168bd = 0, \qquad (16)$$

and we have found $I_4 \equiv cd^2$.

Also, $I_5 = (f, k^2)_8$ where

$$\begin{aligned} k^2 &= (10d^2x^4 - 2bxy^3 - 2cy^4)^2 \\ &= 100d^4x^8 - 40bd^2x^5y^3 - 40cd^2x^4y^4 + 4b^2x^2y^6 + 8bcxy^7 + 4c^2y^8. \end{aligned}$$

Hence $$I_5 = 100d^4 + 2c . 4b^2 - b . 8bc \equiv d^4\,{}^*.$$

* It will of course be recognized that the lineo-linear quadrinvariant to the system

$$(a_0,\ a_1,\ a_2, \ldots a_i)(x, y)^i,\ [b_0,\ b_1,\ b_2, \ldots b_i]\,[x, y]^i$$

is simply $$a_0b_i - a_1b_{i-1} + a_2b_{i-2} \ldots \pm a_ib_0:$$

the disappearance of the argument b^2c from companionship with d^4 in I_5 is rather remarkable, and could not have been predicted. This circumstance considerably simplifies the subsequent calculations.

The only remaining invariant required for present purposes is I_6, represented by $(i_4, k)_4$ where

$$k = [\overline{10}d^2, 0, 0, 2b, 2c][x, y]^4,$$

and $$i_4 = (336c^3, 92c^2d, 72cd^2, 140d^3, 4b^2 \between x, y)^4.$$

Hence $$I_6 = \overline{40}b^2d^2 - (2b)\,92c^2d + 2c\,(336c^3)$$
$$= (-360 - 552 + 672)\,c^4 \equiv c^4.$$

On proceeding to form the 10 compound covariants of deg-order 10.4 obtained by suitable combinations of the invariants and covariants of inferior deg-order, it will be found that the following 13 arguments will make their appearance, in which, for greater brevity, x and y are each taken equal to unity, which in nowise affects (favourably or unfavourably) the course of the reasoning: these arguments are

$$b^6, c^7, c^2d^6;\ b^5c, c^6d, cd^7;\ b^4c^2, c^5d^2, d^8;\ b^3c^3, c^4d^3;\ b^2c^4, c^3d^4,$$

where the 5 groups of arguments, separated from one another by semicolons, are elements of the coefficients of x^4, x^3y, x^2y^2, xy^3, y^4, and when supplemented by such powers of k (of weight 8) as will bring their degrees up to the number 10, are of the respective weights 38, 39, 40, 41, 42, which is right, since the weight of the differentiant of deg-order 10.4 to $(x, y)^8$ is $\frac{10.8-4}{2}$, that is, 38; for greater brevity (in what precedes) k, the coefficient of y^8 in f, has been made unity, and it is worthy of notice that all the arguments that can appear consistently with the law of weight are represented by these 13, upon the understanding that any power of bd in an argument is replaceable by the like power of c^2.

But it is further noticeable that the 10 compounds in question, although apparently linear functions of 13 arguments, are virtually such of only 11; for it will be seen that $b^6 + 4b^5c + 6b^4c^2$ may be regarded as a single argument, none of the three simpler arguments which appear in it occurring except in two of the 10 compounds, and their coefficients in each of those two being in the ratio 1 : 4 : 6.

Had the contraction in the number of really independent arguments extended two steps further, so that the 10 compounds had been linear functions of only 9 quantities (as might, for anything that could be known *à priori*, have been the case), they would necessarily have been linearly connected, and no inference could have been drawn from the particular value assigned to f: moreover, had the 10 compounds been linear functions of only 10 quantities, although the particular form might have been sufficient for drawing a positive inference as to the non-existence of the general syzygy $\Omega = 0$, still there would have been no room for applying the all-important *test* of the correctness of the arithmetical computations upon which that inference would have reposed; and it would have been very

unsatisfactory and unphilosophical to have made so important a conclusion rest upon the negative fact of a determinant of the 10th order *not vanishing*, when the undisproved existence of a single error committed in the many hundreds (or even—it might be said—thousands) of arithmetical steps involved in the calculations of the elements of that determinant would have been sufficient to account for its value differing from zero.

Fortunately, as will be seen, the correctness of the calculations may be *verified* (thanks to the existence of elements one more than barely sufficient —namely, 11 instead of 10) by the *positive* fact of a certain determinant of the 11th order being found equal to zero. It has often seemed to me that a special providence or pre-established harmony in the intellectual world brings it about that honest labour, persevering in the pursuit of an important truth in the face of doubts and difficulties and repeated disappointments, shall not in the end lose its due reward†.

Let us now denote the quantities $b^6 + 4b^5c + 6b^4c^2$, c^7, c^2d^6; $4c^6d$, $4cd^7$; $6c^5d^2$, $6d^8$; $4b^3c^3$, $4c^4d^3$; b^2c^4, c^3d^4 by A, β, γ, δ, ϵ, ζ, η, θ, κ, λ, μ, respectively, and denote the covariants of the order 4 that have been calculated in what precedes according to their deg-order—namely, let us call

$$(f_{k,3})^2;\quad i_{k,2};\quad i_\Delta;\quad f_{k,2};\quad f_\Delta;\quad \Delta;\quad i_4;\quad f_4;\quad k$$

10.4; 6.4; 6.4*; 5.4; 5.4*; 4.4; 4.4*; 3.4; 2.4 respectively,

then the values of 10.4, $I_4 \times 6.4$, $I_4 \times 6.4*$, $I_5 \times 5.4$, $I_5 \times 5.4*$, $I_6 \times 4.4$, $I_6 \times 4.4*$, $I_7 \times 3.4$, $I_4^2 \times 2.4$, $I_8 \times 2.4$, will be as shown in the table annexed

A	β	γ	δ	ϵ	ζ	η	θ	κ	λ	μ	
3	$\overline{3240}$	1200	3510	$\overline{3000}$	1740	5000	3	450	3	.	.........(1)
.	.	$\overline{360}$	126	$\overline{350}$	49	.	.	19	.	$\overline{138}$	.........(2)
.	.	240	63	.	$\overline{63}$	.	.	$\overline{120}$	.	$\overline{90}$	.........(3)
.	.	$\overline{120}$	81	60	54	$\overline{100}$	.	27	.	60	.........(4)
.	.	40	27	80	18	.	.	9	.	.	.........(5)
.	.	.	90	.	40	.	.	.	3	.	.........(6)
.	336	.	92	.	72	.	.	140	4	.	.........(7)
1	1800	.	600	.	200	.	$\overline{3}$	$\overline{200}$	45	1000	.........(8)
.	.	$\overline{20}$	.	.	.	.	.	3	.	4	.........(9)
.	$\overline{180}$	.	.	.	.	.	1	.	4	.	.......(10)

Line (1) of course signifies $3A - 3240\beta + \ldots + 3\lambda$,

(2) .... $-360\gamma + 126\delta \ldots - 138\mu$,

† I began with taking as a special form $ax^8 + by^8 + cz^8$, with the relation $x + y + z = 0$ (which, like the form f, contains two arbitrary ratios), and went through the very considerable labour of calculating all its inferior derivatives capable of entering into the composition of a covariant of deg-order 10.4, but the result turned out altogether nugatory.

and so for all the other lines, each being a linear function of the 11 quantities $A, \beta, \ldots, \lambda, \mu$.

If these 10 linear functions are linearly connected, all the *complete* minors of the rectangular matrix (11 by 10) must vanish.

It is not so difficult as it might at first sight appear, to calculate the actual value of any one of these minors, convenient combinations of the lines and columns having been previously effected; this arises from the number of zeros which appear in the matrix. Mr Morgan Jenkins, of the London Mathematical Society, and myself actually calculated two of them in the course of an hour or two; but the same object may be reached more expeditiously and quite as satisfactorily by proving that the minors do not vanish in respect to some judiciously or fortunately chosen modulus. I find that the number 11, taken as modulus, will accomplish the end in view. It will be found convenient to change the order of sequence of the lines and columns; to take the lines in the order 1, 8, 4, 10, 7, 6, 9, 5, 3, 2, and the columns in the order $A, \eta, \theta, \beta, \lambda, \gamma, \delta, \epsilon, \zeta, \kappa, \mu$. These transpositions having been effected, and the least positive residue of each element in respect to 11 being substituted in place of the element, the rectangular matrix above given will be replaced by the following:

$$\begin{array}{ccccccccccc}
3 & 6 & 3 & 5 & 3 & 1 & 1 & 3 & 2 & 10 & . \\
1 & . & 8 & 7 & 1 & . & 6 & . & 2 & 9 & 10 \\
. & 10 & . & . & . & 1 & 4 & 5 & 10 & 5 & 5 \\
. & . & 1 & 7 & 4 & . & . & . & . & . & . \\
. & . & . & 6 & 4 & . & 4 & . & 6 & 8 & . \\
. & . & . & . & 3 & . & 2 & . & 7 & . & . \\
. & . & . & . & . & 2 & . & . & . & 3 & 4 \\
. & . & . & . & . & 7 & 5 & 3 & 7 & 9 & . \\
. & . & . & . & . & 9 & 8 & . & 8 & 1 & 9 \\
. & . & . & . & . & 3 & 5 & 2 & 5 & 8 & 5
\end{array}$$

It is easy to see that by proceeding as if to eliminate A between the two first lines, then β between the new line so formed and the third line, then γ between the new line again so formed and the fourth line, and so on, (always substituting the remainders to modulus 11 in lieu of the numbers themselves that arise in the process,) the first six lines may be replaced successively by the six following:

$$\begin{array}{ccccccccccc}
3 & 6 & 3 & 5 & 3 & 1 & 1 & 3 & 2 & 10 & . \\
 & 5 & 10 & 5 & . & 10 & 6 & 8 & 4 & 6 & 8 \\
 & & 10 & 5 & . & 4 & 4 & . & 10 & 9 & . \\
 & & & 10 & 7 & 7 & 7 & . & 1 & 2 & . \\
 & & & & 9 & 2 & 9 & . & 10 & 2 & . \\
 & & & & & 5 & 2 & . & . & 5 & .
\end{array}$$

Consequently, it only remains to ascertain whether the complete minors all disappear in the matrix of the dimensions (6×5) given below, namely:

$$\begin{array}{cccccc}
5 & 2 & . & . & 5 & . \\
2 & . & . & . & 3 & 4 \\
7 & 5 & 3 & 7 & 9 & . \\
9 & 8 & . & 3 & 1 & 9 \\
3 & 5 & 2 & 5 & 8 & 5
\end{array}$$

If all the complete minors of this matrix contain 11, the same must be true of the determinant formed by subtracting the first column in the above from the fifth and substituting the difference in place of the fifth column, that is,

$$\left|\begin{array}{ccccc}
2 & . & . & . & . \\
. & . & . & 1 & 4 \\
5 & 3 & 7 & 2 & . \\
8 & . & 3 & 3 & 9 \\
5 & 2 & 5 & 5 & 5
\end{array}\right| \text{ and therefore } \left|\begin{array}{cccc}
. & . & 1 & 4 \\
3 & 7 & 2 & . \\
. & 3 & 3 & 9 \\
2 & 5 & 5 & 5
\end{array}\right|$$

should contain 11, and (as we may see by substituting the excess of 4 times the 3rd column over the fourth in place of the 3rd) the same must be true of

$$\text{the determinant } \left|\begin{array}{ccc}
3 & 7 & 8 \\
. & 3 & 3 \\
2 & 5 & 4
\end{array}\right| \text{ of which the value is } 3(12-15)+2(21-24),$$

that is, -15, and as this does not contain 11, it follows that the complete minors of the matrix which expresses the 10 compounds as linear functions of the 11 arguments $A, \beta, \gamma \ldots \lambda, \mu$ are not all zero, and they are consequently not linearly connected*. But, obviously, the calculations on which this proof depends imperatively call for a verification, as nothing would be more easy than to bring out some or all of the minors different from zero by a single error of calculation or slip of the pen. To this end I calculate the

* In the *Comptes Rendus* for 22nd August of this year, I have given a brief *résumé* of the contents of this paper. At page 367 of that fascicule, (third line from foot†), in the last line but one of the matrix, I have written 9 8 . 8 1 9 in error for 9 8 . 3 1 9 (having mistaken a 3, *covered with a blot*, for 8); consequently, the calculations which follow page 368 of the *C. R.* are erroneous. Fortunately, I did not repeat the mistake in calculating the value of the determinant subsequently given of the 11th order, in proving that it contains the divisor 11. Moreover, this determinant, or rather its remainder to modulus 11, has been calculated by an entirely different process by Mr Morgan Jenkins (whose work is before my eyes), and with the same result of its being divisible by 11. This instance shows how unsafe it would have been to have trusted to the fact of the minors not vanishing, unsupported by the positive evidence which the determinant of the 11th order affords of the preceding calculations, as regards the values of the groundforms, being unaffected with *one single error* in spite of the vast number of processes of addition, subtraction, multiplication, division, transposition, transcription and change of sign employed in working them out.

[† p. 486, above.]

value of von Gall's undecomposed covariant for the assumed special form f, and shall show that the 10 compounds and this 11th function do become linearly connected, that is, subject to a syzygy, on the assumption that the arithmetical values of the coefficients have been correctly calculated.

The function in question, Dr von Gall's i_4'', is obtained as follows:

$i'' = (i, \Delta)_2$ of deg-order 6 . 8 is equal to

$$(168cdx^5y + 180d^2x^4y^2 + 6bxy^5 + 6cy^6)(40cd^2x^2 + 3b^2y^2)$$
$$-(180c^2dx^2 + 160cd^2xy)(28cdx^6 + 72d^2x^5y + 15bx^2y^4 + 36cxy^5 + 7dy^6)$$
$$+(12d^2x^6 + 20bx^3y^3 + 90cx^2y^4 + 42dxy^5)(180c^2dxy + 40cd^2y^2)$$
$$= [\overline{5040}c^3d^2,\ \overline{8560}c^2d^3,\ \overline{3840}cd^4,\ 504b^2cd,\ 7560c^4,\ 5640c^3d,\ 4380c^2d^2,$$
$$18b^3 + 560cd^3,\ 18b^2c] . [x, y]^8$$

which, multiplied by 28, will be seen to be equivalent to

$$(\overline{141120}c^3d^2,\ \overline{29960}c^2d^3,\ \overline{3840}cd^4,\ 756bc^3,\ 3024c^4,\ 2820c^3d,\ 4380c^2d^2,$$
$$63b^3 + 1960cd^3,\ 504b^2c).$$

Finally,

$$i_4'' = (i'', \Delta)_4 = 3b^2(\overline{141120}c^3d^2x^4 + \overline{119840}c^2d^3x^3y + \overline{23040}cd^4x^2y^2$$
$$+ 3024bc^3xy^3 + 3024c^4y^4)$$
$$+ 6 . 40cd^2(\overline{3840}cd^4x^4 + 3024bc^3x^3y + 18144c^4x^2y^2 + 11280c^3dxy^3 + 4380c^2d^2y^4)$$
$$- 4 . 90c^2d[756bc^3x^4 + 12096c^4x^3y + 16920c^3dx^2y^2 + 17520c^2d^2xy^3$$
$$+ (63b^3 + 1960cd^3)]y^4$$

which, dividing out by 144,

$$\equiv (\overline{32130}c^7 + \overline{6400}c^2d^6)x^4 + \overline{37590}c^6dx^3y + \overline{16380}c^5d^2x^2y^2$$
$$+ (63b^3c^3 + \overline{25000}c^4d^3)xy^3 + \left(\frac{\overline{819}}{2}b^2c^4 + 2400c^3d^4\right)y^4$$
$$\equiv (\overline{128520}c^7 + \overline{25600}c^2d^6,\ \overline{37590}c^6d,\ \overline{10920}c^5d^2,\ 63b^3c^3 + \overline{25000}c^4d^3,$$
$$\overline{1638}b^2c^4 + 9600c^3d^4).$$

Here it will be noticed that the arguments collected in what I have designated by A, namely, b^6, b^5c, b^4c^2, do not appear at all in i_4''. Had they made their appearance with other than coefficients bearing to each other the ratios of $1:4:6$, i_4'' could not have been a linear function of the 10 compounds which are linear functions of A and of 10 other arguments. This is in itself, to some extent, a verification of a portion at least of the preceding calculations: i_4'', as it turns out, is a linear function of only 8 out of the 11 arguments which appear in the other 10 compound covariants, namely, of $\beta, \gamma, \delta, \zeta, \theta, \kappa, \lambda, \mu$, neither A, ϵ nor η appearing in i_4''.

If the figuring throughout is correct, the determinant represented by the matrix constituted of the coefficients of the 11 compounds, ought to vanish identically; but it will be sufficient for all reasonable purposes (that is, to

satisfy any reasonable doubts on the subject) if I show that this is the case for the value of that determinant in respect to three consecutive prime numbers 11, 13, 17 taken almost at hazard.

It must be understood that the vanishing of the determinant in question adds *no additional strength whatever* to the proof—which, by Cayley's law, is perfect without it—provided that the figures in the coefficients of the 10 compounds (excluding i_4'') have been correctly calculated. It is to authenticate these figures, and not to verify the legitimacy of the argument, that the 11th compound is calculated, and the determinant formed by all the eleven shown to contain any number taken at will. It must be remembered that the calculations have been most carefully conducted and verified at each step: consequently, if any person, after the evidence that will be given, entertains any doubt of the correctness of the result, the duty is incumbent on him to put his finger upon some one of the coefficients of the 10 first compounds and prove it to be incorrectly stated.

First, for the modulus 11. In respect to this modulus, the coefficients in i_4'' of

$$A,\ \eta,\ \theta,\ \beta,\ \lambda,\ \gamma,\ \delta,\ \epsilon,\ \zeta,\ \kappa,\ \mu$$

are congruous to

$$0,\ 0,\ 8,\ 4,\ 1,\ 8,\ 8,\ 0,\ 3,\ 3,\ 8.$$

Hence, (making use of the transformations already calculated of the upper half of the rectangular matrix), it has to be shown that 11 is a divisor of the determinant of the 9th order

$$\begin{array}{ccccccccc}
8 & 4 & 1 & 8 & 8 & . & 3 & 3 & 8 \\
10 & 5 & . & 4 & 4 & . & 10 & 9 & . \\
 & 10 & 7 & 7 & 7 & . & 1 & 2 & . \\
 & & 9 & 2 & 9 & . & 10 & 2 & . \\
 & & & 5 & 2 & . & . & 5 & . \\
 & & & 2 & . & . & . & 3 & 4 \\
 & & & 7 & 5 & 3 & 7 & 9 & . \\
 & & & 9 & 8 & . & 3 & 1 & 9 \\
 & & & 3 & 5 & 2 & 5 & 8 & 5
\end{array}$$

The first and second lines of this matrix combined give rise to the line $1\ 7\ 7\ .\ 6\ 9\ 8$, and this, combined with the 4th, to the line $5\ 1\ .\ .\ 9\ 5$ under which last, writing the 5 remaining lines

$$\begin{array}{cccccc}
5 & 1 & . & . & 9 & 5 \\
5 & 2 & . & . & 5 & . \\
2 & . & . & . & 3 & 4 \\
7 & 5 & 3 & 7 & 9 & . \\
9 & 8 & . & 3 & 1 & 9 \\
3 & 5 & 2 & 5 & 8 & 5
\end{array}$$

it has to be shown that the determinant to the above matrix of the 6th order contains 11.

Let the fourth line be replaced by 3 times itself + the last line, which, to the modulus 11, reduces the third column to the form of five zeros followed by 2. This shows that we may use, instead of the above, the determinant

$$\begin{array}{ccccc} 5 & 1 & . & 9 & 5 \\ 5 & 2 & . & 5 & . \\ 2 & . & . & 3 & 4 \\ 2 & 9 & 4 & 2 & 5 \\ 9 & 8 & 3 & 1 & 9; \end{array}$$

and again, replacing the fourth line of the new matrix by its double + the last line, we fall upon the matrix

$$\begin{array}{cccc} 5 & 1 & 9 & 5 \\ 5 & 2 & 5 & . \\ 2 & . & 3 & 4 \\ 2 & 4 & 5 & 8, \end{array}$$

for which we may substitute

$$\begin{array}{cccc} 5 & 1 & 4 & 5 \\ 5 & 2 & . & . \\ 2 & . & 1 & 4 \\ 2 & 4 & 3 & 8, \end{array}$$

or (as may be seen by replacing the second column by 3 times itself + the first column) $\begin{vmatrix} 8 & 4 & 5 \\ 2 & 1 & 4 \\ 3 & 3 & 8 \end{vmatrix}$, in which (to modulus 11) the first line is 4 times the second. Hence, the test is satisfied as regards the modulus 11.

I will next take the modulus 13.

The residues to modulus 13 of the coefficients in i_4'' of

$$\begin{array}{ccccccccc} \theta & \beta & \lambda & \gamma & \delta & \epsilon & \zeta & \kappa & \mu \end{array}$$

will be seen to be

$$\begin{array}{ccccccccc} 11 & 11 & . & 10 & 6 & . & . & 12 & 6 \end{array}$$

and the matrix corresponding to the one of the same dimensions (11 × 10), previously calculated for modulus 11, will, in respect to modulus 13, become

$$\begin{array}{ccccccccccc} 3 & 8 & 3 & 10 & 3 & 4 & . & 3 & 11 & 8 & . \\ 1 & . & 10 & 6 & 6 & . & 2 & . & 5 & 8 & 12 \\ & 4 & . & . & . & 10 & 3 & 8 & 2 & 1 & . \\ & & 1 & 2 & 4 & . & . & . & . & . & . \\ & & & 11 & 4 & . & 1 & . & 7 & 10 & . \\ & & & & 3 & . & 12 & . & 1 & 1 & . \\ & & & & & 6 & . & . & . & 3 & 4 \\ & & & & & 1 & 1 & 2 & 5 & 9 & . \\ & & & & & 6 & 11 & . & 2 & 10 & 1 \\ & & & & & 4 & 9 & 1 & 10 & 6 & 5. \end{array}$$

In place of the first six of the above lines, applying the same process as before, we may substitute

$$\begin{array}{ccccccccccc} 3 & 8 & 3 & 10 & 3 & 4 & . & 3 & 11 & 8 & . \\ & 5 & 1 & 8 & 2 & 9 & 5 & 10 & 4 & 3 & 10 \\ & & 9 & 7 & 5 & 1 & 8 & . & 7 & 6 & 12 \\ & & & 11 & 5 & 12 & 5 & 12 & 6 & 7 & 1 \\ & & & & 2 & 11 & 8 & 11 & 11 & 7 & 2 \\ & & & & & 6 & . & 7 & 8 & 7 & 7. \end{array}$$

Combining the i_4'' line (that is, the coefficients of $\theta\ \beta\ \lambda \ldots \mu$ in i_4'' above given) with the third of these, we obtain the line

$$\begin{array}{cccccccc} 4 & 3 & 12 & 8 & . & 12 & 10 & . \end{array}$$

which, again combined with the fourth of the same, gives rise to the line

$$\begin{array}{cccccc} 7 & 10 & 9 & 9 & 9 & 4. \end{array}$$

Adding on the sixth line, namely $6\ \ .\ \ 7\ \ 8\ \ 7\ \ 7$ and the four last lines of the first matrix, namely, the lines marked with an asterisk,

$$\begin{array}{cccccc} 7 & 10 & 9 & 9 & 9 & 4 \\ 6 & . & 7 & 8 & 7 & 7 \\ {}^*6 & . & . & . & 3 & 4 \\ {}^*1 & 1 & 2 & 5 & 9 & . \\ {}^*6 & 11 & . & 2 & 10 & 1 \\ {}^*4 & 9 & 1 & 10 & 6 & 5, \end{array}$$

the arithmetical problem to be solved reduces itself to showing that the above determinant vanishes to modulus 13.

Substituting for the 1st column twice the 1st less three times the 6th, and for the 5th column twice the 5th less the 1st, and neglecting the factor 3, we fall upon the determinant

$$\begin{vmatrix} 2 & 10 & 9 & 9 & 11 \\ 4 & . & 7 & 8 & 8 \\ 2 & 1 & 2 & 5 & 4 \\ 9 & 11 & . & 2 & 1 \\ 6 & 9 & 1 & 10 & 8 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 2 & 10 & 9 & 9 & 2 \\ 4 & . & 7 & 8 & . \\ 2 & 1 & 2 & 5 & 12 \\ 9 & 11 & . & 2 & 12 \\ 6 & 9 & 1 & 10 & 11 \end{vmatrix}.$$

Then in this last, substituting for the 4th column the 4th less twice the 1st, say M, and for the 3rd column 5 times the 1st less the 3rd, say N, we descend in like manner upon the determinant

$$\begin{array}{cccc} 5 & 2 & 2 & 1 \\ 1 & 2 & 12 & 8 \\ 10 & 9 & 12 & 6 \\ 11 & 6 & 11 & 3 \end{array}$$

where the 1st column is the M with the zero in it left out, and the 4th column the N with the zero in it left out.

This, by elimination (so to say) of the first variable to the left between the successive pairs of lines, gives rise to the determinant

$$\begin{array}{rrr} 8 & 6 & . \\ 2 & 9 & 4 \\ . & 4 & 3 \end{array}$$

which (to modulus 13) $\equiv 8.1 - 8.3 - 6.6 \equiv 8 - 11 - 10 \equiv 0$.

It remains only to apply the 3rd proposed test, using 17 as the modulus.

The i_4'' line here becomes

$$\begin{array}{rrrrrrrrr} 12 & 0 & 11 & 2 & 14 & . & 11 & 7 & 12 \end{array}$$

and the grand rectangular matrix becomes

$$\begin{array}{rrrrrrrrrrr} 3 & 2 & 3 & 7 & 3 & 10 & 8 & 9 & 6 & 8 & . \\ 1 & . & 14 & 15 & 11 & . & 5 & . & 13 & 4 & 14 \\ & 2 & . & . & . & 16 & 13 & 9 & 3 & 10 & 9 \\ & & 1 & 7 & 4 & . & . & . & . & . & . \\ & & & 13 & 4 & . & 7 & . & 4 & 4 & . \\ & & & & 3 & . & 5 & . & 6 & . & . \end{array}$$

with 4 more lines, which will be presently supplied in their proper place. For those above written may be substituted

$$\begin{array}{rrrrrrrrrrr} 3 & 2 & 3 & 7 & 3 & 10 & 8 & 9 & 6 & 8 & . \\ & 15 & 5 & 4 & 13 & 7 & 7 & 8 & 16 & 4 & 8 \\ & & 7 & 9 & 8 & 5 & 11 & . & 13 & 6 & . \\ & & & 6 & 3 & 12 & 6 & . & 4 & 11 & . \\ & & & & 2 & 14 & 15 & . & 6 & . & . \\ & & & & & 9 & 16 & . & 11 & . & . \end{array}$$

Rejecting the first two lines, and writing over the remaining ones the i_4'' line, there results

$$\begin{array}{rrrrrrrrr} 12 & 0 & 11 & 2 & 14 & . & 11 & 7 & 12 \\ 7 & 9 & 8 & 5 & 11 & . & 13 & 6 & . \\ & 6 & 3 & 12 & 6 & . & 4 & 11 & . \\ & & 2 & 14 & 15 & . & 6 & . & . \\ & & & 9 & 16 & . & 11 & . & . \end{array}$$

which may be replaced by

$$\begin{array}{rrrrrrrrr} 12 & 0 & 11 & 2 & 14 & . & 11 & 7 & 12 \\ & 6 & 2 & 12 & . & . & 11 & 6 & 1 \\ & & 6 & . & 2 & . & 9 & 13 & 11 \\ & & & 16 & 1 & . & 1 & 8 & 12 \\ & & & 9 & 16 & . & 11 & . & . \\ & & & {}^*14 & . & . & . & 3 & 4 \\ & & & {}^*6 & 10 & 12 & 1 & 9 & . \\ & & & {}^*2 & 12 & . & 5 & 16 & 12 \\ & & & {}^*14 & 7 & 7 & 15 & 2 & 15 \end{array}$$

to which I add in the 4 pretermitted lines distinguished by asterisks,

and the determinant, represented by the square matrix (6×6) exhibited by the 6 lines last appearing above, ought to contain the modulus 17 as a divisor. Instead of the 3rd line from the bottom we may substitute its double less the last line, and thus, neglecting the factor 7, fall upon the matrix

$$\begin{array}{rrrrr} 16 & 1 & 1 & 8 & 12 \\ 9 & 16 & 11 & . & . \\ 14 & . & . & 3 & 4 \\ 15 & 13 & 4 & 16 & 2 \\ 2 & 12 & 5 & 16 & 12. \end{array}$$

Substituting for the 4th column the sum of itself and the 1st, and for the 5th column 5 times itself + the 1st, and neglecting the factor 14, we obtain the determinant

$$\begin{array}{rrrr} 1 & 1 & 7 & 8 \\ 16 & 11 & 9 & 9 \\ 13 & 4 & 14 & 8 \\ 12 & 5 & 1 & 11. \end{array}$$

Subtracting the 2nd column from the 1st and the 4th from the 2nd + the 3rd, we obtain the matrix

$$\begin{array}{rrrr} 0 & 0 & 1 & 7 \\ 5 & 11 & 11 & 9 \\ 9 & 10 & 4 & 14 \\ 7 & 12 & 5 & 1, \end{array}$$

and replacing the 3rd column by 7 times the 3rd less the 4th, we descend upon the determinant

$$\begin{array}{rrr} 5 & 11 & . \\ 9 & 10 & 14 \\ 7 & 12 & . \end{array}$$

where the 1st line to modulus 17 equals 8 times the 3rd.

Hence the determinant in question contains 17, as was to be shown.

It seems needless to multiply these tests—the object being, as before stated, not a confirmation of the argument, which is wholly unnecessary, but a verification of the accuracy of the arithmetic: for this reason it has seemed to me essential that the calculations, authenticating the figures previously obtained, should be set out in considerable detail.

Instead of founding anything upon the concordance (as far as it extends) between Dr von Gall's table and my own, the proof of the non-existence of the 10.4 irreducible covariant may be inferred exclusively from the former and completed as follows.

I have proved that the syzygetic function Ω of the deg-order 10.4, if it exists, must be a consequence of the existence of a like function of the deg-order 8 . 4, 7 . 4, or 5 . 4. The last hypothesis may at once be rejected as implying an equation of the form $\frac{2\,.\,4}{3\,.\,4}$ = a numerical multiple of $\frac{2\,.\,0}{3\,.\,0}$.

Next, for the deg-order 7 . 4, again using for the primitive the same special form f, which causes 2 . 0 and 3 . 0 to vanish, the only non-vanishing arguments in the supposed syzygetic function Ω' for the particular form f will be 4 . 0 × 3 . 4 and 5 . 0 × 2 . 4, that is, $cd^2(b^2, bc, c^2, -cd, 5d^2)$, and $d^4(20d^3, 0, 0, b, 4c)$, between which obviously no syzygy is possible, so that neither of them can appear in the general form of Ω'. Hence the terms in the general form of Ω' must be divisible all by 2 . 0 or all by 3 . 0, or some by 2 . 0 and some by 3 . 0, and consequently there must exist a syzygy of the deg-order 5 . 4, 4 . 4, or 2 . 4. The first of these hypotheses has already been shown to be impossible, and the remaining two need not even have been mentioned, as there is only a single compound of the deg-order 4 . 4, namely, 2 . 0 × 2 . 4, and none of the deg-order 2 . 4. Lastly, for the deg-order 8 . 4, still using the same special form of f, the arguments in the supposed syzygetic functions which do not vanish are 4 . 0 × 4 . 4, 4 . 0 × 4 . 4*, 5 . 0 × 3 . 4, and 6 . 0 × 2 . 4, that is,

$$cd^3(0,\ 90c^2d,\ 40cd^2,\ 0,\ 3b^2)$$
$$cd^3(336c^3,\ 92c^2d,\ 72cd^2,\ 140d^3,\ 4b^2)$$
$$d^4(b^2,\ bc,\ c^2,\ -cd,\ 5d^2)$$

and
$$c^4(-20d^3,\ 0,\ 0,\ b,\ 4c).$$

The argument d^4 in the 3rd of these quantities has no equivalent in any of the other 3. Hence the 3rd quantity does not appear in the syzygy: moreover, the 4th compound contains one argument, namely, bc^4, which does not rationally contain d^2c $\left(\text{for } \frac{bc^3}{d^2} = \frac{b^2c}{3d}\right)$. Hence this compound also disappears, and obviously no syzygy connects together the first two. Hence in the supposed general syzygy there exist no compounds containing neither 2 . 0 nor 3 . 0, and by the same reasoning as before, this supposed syzygetic function must imply the existence of one of the deg-order 6 . 4 or 5 . 4 or 4 . 4. The two last of the three suppositions have already been seen to be impossible, and the first would imply a linear relation between 2 . 0 × 4 . 4, 2 . 0 × 4 . 4*, 3 . 0 × 3 . 4, 4 . 0 × 2 . 4, the last of which we see, by taking f for the primitive, cannot appear in the general syzygy, and the remaining 3 arguments would imply that the general covariant 3 . 4 would contain the invariant 2 . 0, which is absurd. Hence it follows from Dr von Gall's own results that the existence of a groundform of deg-order 10 . 4 is impossible. The only principle extraneous to his results made use of is Cayley's all-important rule, of which an irrefragable demonstration has been given by the author of this paper, but which still, as far as he is aware, remains unutilized, and is almost passed over in silence by invariantists of the German school.

It may be as well to make this article self-contained by showing that the number of compound irreducible groundforms of deg-order 10 . 4 is, as stated, 32, namely the same as the number of linearly-independent covariants of that deg-order requisitioned by Cayley's rule.

Using then, for brevity's sake, i to represent the invariant i . 0, it is easy to see that the following is an exhaustive enumeration of all the compounded irreducibles of deg-order 10 . 4:

$(5\,.\,2)^2$; 8 × 2 . 4; 7 × 3 . 4; 6 × 4 . 4; 6 × 4 . 4*; 5 × 5 . 4; 5 × 5 . 4*; 4 × 6 . 4; 4 × 6 . 4*; 4 × 4 × 2 . 4; 3 × 7 . 4; 3 × 7 . 4*; 3^2 × 4 . 4; 3^2 × 4 . 4*; 3 × 4 × 3 . 4; 3 × 5 × 2 . 4; 2 × 8 . 4; 2 × 8 . 4*; 2 × 3 × 5 . 4; 2 × 3 × 5 . 4*; 2 × 4 × 4 . 4; 2 × 4 × 4 . 4*; 2 × 5 × 3 . 4; 2 × 6 × 2 . 4; 2 × 3^2 × 2 . 4; 2^2 × 6 . 4; 2^2 × 6 . 4*; 2^2 × 3 × 3 . 4; 2^2 × 4 × 2 . 4; 2^3 × 4 . 4; 2^3 × 4 . 4*; 2^4 × 2 . 4.

The same number 32, it is all-important to bear in mind, is also the number of linearly independent covariants of deg-order 10 . 4 given by Cayley's law. For this number is represented by $(w:8,\ 10) - (w':8,\ 10)$ where $w = \frac{10\,.\,8 - 4}{2} = 38$, $w' = w - 1 = 37$; that is, (by Euler's Theorem) is the coefficient of t^{38} in the development of

$$\frac{(1-t^{11})(1-t^{12})(1-t^{13})(1-t^{14})(1-t^{15})(1-t^{16})(1-t^{17})(1-t^{18})}{(1-t^{2})(1-t^{3})(1-t^{4})(1-t^{5})(1-t^{6})(1-t^{7})(1-t^{8})},$$

which may be calculated as follows: The numerator is

$$1 - t^{11} - t^{12} - t^{13} - t^{14} - t^{15} - t^{16} - t^{17} - t^{18} + t^{23} + t^{24} + 2t^{25} + 2t^{26} + 3t^{27} + 3t^{28}$$
$$+ 4t^{29} + 3t^{30} + 3t^{31} + 2t^{32} + 2t^{33} + t^{34} + t^{35} - t^{36} - t^{37} - 2t^{38} \ldots.$$

Dividing this by $1 - t^8$, the quotient by $1 - t^7$, and so on for $1 - t^6, \ldots 1 - t^2$, we have for the numerator and the successive quotients so obtained the following values respectively:

t^0	t^1	t^2	t^3	t^4	t^5	t^6	t^7	t^8	t^9	t^{10}	t^{11}	t^{12}	t^{13}	t^{14}	t^{15}	t^{16}	t^{17}	t^{18}	t^{19}
1	0	0	0	0	0	0	0	0	0	0	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	0
1	0	0	0	0	0	0	0	1	0	0	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	$\bar{1}$	$\bar{1}$	$\bar{1}$
1	0	0	0	0	0	0	1	1	0	0	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	0	0	$\bar{1}$	$\bar{2}$	$\bar{2}$
1	0	0	0	0	0	1	1	1	0	0	$\bar{1}$	0	0	1	0	0	$\bar{2}$	$\bar{2}$	$\bar{2}$
1	0	0	0	0	1	1	1	1	0	1	0	1	1	1	1	0	$\bar{1}$	$\bar{1}$	$\bar{1}$
1	0	0	0	1	1	1	1	2	1	2	1	3	2	3	2	3	1	2	1
1	0	0	1	1	1	2	2	3	3	4	4	6	6	7	8	9	8	10	10
1	0	1	1	2	2	4	4	7	7	11	11	17	17	24	25	33	33	43	43

t^{20}	t^{21}	t^{22}	t^{23}	t^{24}	t^{25}	t^{26}	t^{27}	t^{28}	t^{29}	t^{30}	t^{31}	t^{32}	t^{33}	t^{34}	t^{35}	t^{36}	t^{37}	t^{38}
0	0	0	1	1	2	2	3	3	4	3	3	2	2	1	1	$\bar{1}$	$\bar{1}$	$\bar{2}$
$\bar{1}$	$\bar{1}$	$\bar{1}$	0	1	1	1	2	2	3	2	3	3	3	2	3	1	2	0
$\bar{2}$	$\bar{1}$	$\bar{1}$	0	0	$\bar{1}$	$\bar{1}$	0	1	2	2	3	2	2	2	4	3	4	3
$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	0	0	0	0	0	1	2	4	3	4	3
0	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	1	0	1	1
3	0	0	$\bar{2}$	0	$\bar{3}$	$\bar{3}$	$\bar{5}$	$\bar{3}$	$\bar{6}$	$\bar{6}$	$\bar{8}$	$\bar{6}$	$\bar{8}$	$\bar{7}$	$\bar{7}$	$\bar{6}$	$\bar{7}$	$\bar{6}$
11	10	10	9	10	7	6	5	4	0	$\bar{1}$	$\bar{4}$	$\bar{6}$	$\bar{9}$	$\bar{11}$	$\bar{13}$	$\bar{15}$	$\bar{18}$	$\bar{19}$
54	53	64	62	74	69	80	74	84	74	83	70	77	61	66	48	51	30	32

Hence the required coefficient is 32.

It is obvious that the particular method adopted in treating the grand determinant made up of 11^2 places employed in the foregoing investigation furnishes or indicates a good practical process for determining 10 out of the 32 numerical coefficients which enter into the expression of Dr von Gall's covariant i_4'' as a linear function of the 32 linearly independent covariants of its own deg-order; but, as this calculation possesses no point either of intrinsic theoretical interest or practical importance, I leave it to those who may feel any curiosity on the subject, to go through the calculations necessary to attain that end.

It may be supposed that the long calculations rendered necessary by the quadrinomial form f, attributed to the primitive in the preceding investigation, might have been evaded by using a trinomial form (of which several exist) possessing the same property of causing the two first invariants to vanish, and not less general, inasmuch as containing three independent coefficients in place of four connected by a homogeneous equation; for example, we might assume for the primitive $(0, b, 0, 0, 0, f, 0, 0, i\between x, y)^8$, where the weights of b, f, i are respectively 1, 5, 8.

The quadrinvariant vanishes because no binary combination of 1, 5, 8, with or without repetitions, will make up the required weight 8, and the cubinvariant because no ternary combination of the same will make up the weight 12. It may, however, easily be shown that such form will lead only to a nugatory conclusion, as not supplying the necessary number of arguments (10 at least are wanted) to support the independence of the 10 surviving compound covariants of deg-order 10 . 4. This may be seen as follows.

The weights of the coefficients of $x^4, x^3y, x^2y^2, xy^3, y^4$ in a 10 . 4 covariant are respectively 38, 39, 40, 41, 42. Let us ascertain in how many ways 10 numbers, consisting exclusively of the numbers 1, 5, 8, can be put together to make up these totals. I use the notation $a^\alpha . b^\beta . c^\gamma$ to indicate a sum of α numbers a, β numbers b, and γ numbers c.

Then the sole admissible representations of 38 are $8^4 . 1^6,\ 5^7 . 1^3$,

„ 39 „ $8^3 . 5^2 . 1^5$,

„ 40 „ $8^2 . 5^4 . 1^4$,

„ 41 „ $8 . 5^6 . 1^3$,

„ 42 „ $8^4 . 5 . 1^5,\ 5^8 . 1^2$,

that is, there are only at utmost 7 arguments contained in the expressions for the 10 compounds.

So, in like manner, if we assumed for the primitive

$$(0, b, 0, 0, 0, 0, g, 0, i\between x, y)^8$$

to find the number of independent arguments possible in a 10 . 4 covariant, we must ascertain the sum of the numbers of similar representations to the foregoing of the same integers 38, 39, 40, 41, 42, with 10 integers confined to be 1, 6 or 8, and we shall find that the sole representations of that kind are $8^4 . 1^6$; $8^2 . 6^3 . 1^5$; $6^6 . 1^4$; $8^4 . 6 . 1^3$; $8^3 . 6^2 . 1^5$; $8 . 6^5 . 1^4$, that is, 6 representations in all. In like manner it will be found that all the other trinomial forms of the primitive so taken that the first two invariants are null, will be incapable of yielding as many as 10 arguments to any covariant of deg-order 10 . 4†, so that the 10 compounds appurtenant to such special form will be bound to be linearly related, and no inference can be drawn from any such assumption. I have reason for believing that the quadrinomial form employed in the foregoing investigation is the most convenient and economical, as leading to the simplest calculations of any that could have been employed for the same purpose.

† On an exhaustive examination, it will be found that the only trinomial forms of the primitive which will cause the first two invariants to disappear, are those in which the surviving coefficients are

$b, f, i;\ b, g, i$

or the complementary ones

$a, b, c;\ a, b, d;\ a, c, d;\ b, c, d,$

$g, d, a;\ h, c, a$

$i, h, g;\ i, h, f;\ i, g, f;\ h, g, f,$

which, of course, are substantially equivalent to the former.

Confining our attention, then, to the upper group, it will readily be seen that the four last will cause not only the quadrinvariant and the cubinvariant, but all the other invariants as well, to vanish. Since, then, it has been shown that the b, f, i; b, g, i forms are insufficient to support the independence of the 10 compound covariants with which the reasoning is concerned, it follows that *no trinomial form* will be adequate to do so.

It may be asked what would have been the effect of using the form in which b, c, d, i are the surviving coefficients, but b, c, d are supposed mutually independent, instead of being subject to the condition employed in the refutation above: on this supposition the quadrinvariant, but not the cubinvariant, will vanish; and an easy calculation will show that of the 32 representations of the covariant of deg-order 10 . 4 as a product of inferior groundforms there will be only 16 in which the quadrinvariant does not appear as a factor. And, again, it will be found that the number of ways of representing 38, 39, 40, 41, 42, as the sum of 4 numbers, each of which is either 1, 2, 3 or 8 is 29. Hence there would arise a matrix of 16 lines and 29 columns, and to disprove the existence of the 10 . 4 groundform it would be sufficient to prove that some one of the *complete* minor determinants of this matrix differs from zero. The work involved in dealing with this and the subsequent verificatory matrix of 17 lines and 29 columns would evidently be vastly greater and more liable to error than when (as in the text) we assign the relation between b, c, d so as to make the cubinvariant vanish.

In the absence of the information as to the number of linearly independent 10 . 4's given by Cayley's rule, the direct mode of refutation would have required the calculation of the 32 compound 10 . 4's and the problematical one of von Gall for the general form of the Octavic, subject only to the simplification of taking two of the coefficients zero. There would then have remained to show that the leading terms of these 33 forms were linearly connected, which would necessarily imply that the same was true of the 33 entire forms themselves; a colossal task, probably transcending the sphere of human ability to execute.

It may be well (by way of confirmation) to determine *à priori* the number of possible arguments that can belong to the 10 . 4 covariants of the quadrinomial form of $(x, y)^8$ employed in the antecedent investigation. Since c^2 may be replaced by a numerical multiple of bd, it follows that each argument may be brought to a form in which c does not enter at all, or in which it enters only in the first degree. The total possible number (which turns out to be the actual number) of arguments is, consequently, the number of ways in which 38, 39, 40, 41, 42 can be composed with 10 parts each of them 1, 3 or 8 + the number of ways in which 36, 37, 38, 39, 40 can be composed with 9 parts, each of them also 1, 3 or 8. All the possible different compositions of these kinds are exhibited in the annexed table.

$38 = 4.8 + 6.1 = 2.8 + 7.3 + 1.1$	$36 = 3.8 + 3.3 + 3.1$
$39 = 3.8 + 4.3 + 3.1$	$37 = 4.8 + 5.1 = 2.8 + 7.3$
$40 = 4.8 + 5.1 + 1.3 = 2.8 + 8.3$	$38 = 3.8 + 4.3 + 2.1$
$41 = 3.8 + 5.3 + 2.1$	$39 = 4.8 + 1.3 + 4.1$
$42 = 4.8 + 4.1 + 2.3$	$40 = 3.8 + 5.3 + 1.1.$

There are thus 7 + 6, that is, 13 distinct arguments, that is, the number which actually appear distributed among the 10 surviving covariants of deg-order 10 . 4 as previously shown—it being at the same time remembered that three of the 13 enter as elements of a fixed linear combination into the 10 functions, which are thus virtually functions of only 11 independent arguments.

The method employed in what precedes suggests a mode of calculating in part at least the discriminant of the eighthic in terms of the subordinate groundforms. Thus, suppose we take for our special form,

$$(0,\ b,\ c,\ d,\ 0,\ 0,\ 0,\ 0,\ i \between x,\ y)^8$$

with b, c, d independent.

Then the quadrinvariant will vanish, and there will be no very great effort of calculation required to express the 8 remaining invariants as functions of b, c, d, i.

The discriminant is of the 14th degree and 14 may be made up in 10 (and no more than 10) ways as a sum of numbers each limited to be 3, 4, 5, 6, 7, 8, 9, or 10; as exhibited in the exhaustive table

$$14 = 10 + 4 = 9 + 5 = 8 + 6 = 8 + 3 + 3 = 7 + 7 = 7 + 4 + 3 = 6 + 4 + 4$$
$$= 6 + 5 + 3 = 5 + 5 + 4 = 4 + 4 + 3 + 3.$$

Again the weight of the discriminant is 56, and the number of ways of compounding 56 with 14 numbers each limited to be 1, 2, 3 or 8 is 11, as shown in the exhaustive table

$$56 = 6.8 + 8.1 = 5.8 + 7.2 + 2.1 = 5.8 + 3.3 + 1.2 + 5.1 = 5.8 + 2.3$$
$$+ 3.2 + 4.1 = 5.8 + 1.3 + 5.2 + 3.1 = 5.8 + 7.2 + 2.1 = 4.8 + 7.3$$
$$+ 3.1 = 4.8 + 6.3 + 2.2 + 2.1 = 4.8 + 5.3 + 4.2 + 1.1 = 4.8 + 4.3$$
$$+ 6.2 = 3.8 + 10.3 + 1.2.$$

Now there will be no difficulty at all in finding by substitution and multiplication the discriminant of the assumed quantic, say Q, which is in fact the same as the resultant of $\frac{dQ}{dy}$ and $bx^6y + 3cx^5y^2 + 5dx^4y^3$. Hence there will be 11 equations for determining the coefficients of the 10 invariants of the 14th degree which are products of the inferior invariants (the quadrinvariant excepted); consequently there will be sufficient or more than sufficient equations for the purpose, unless it should (unfortunately and contrary to probability) turn out to be the case that the 10 products, although linear functions of 11 arguments, are expressible as linear functions of only 9 linear functions of those arguments.

61.

ON TCHEBYCHEFF'S THEORY OF THE TOTALITY OF THE PRIME NUMBERS COMPRISED WITHIN GIVEN LIMITS.

[*American Journal of Mathematics*, IV. (1881), pp. 230—247.]

If it be admitted that Legendre's approximate formula for the number of prime numbers inferior to a given number, which has been confirmed by direct enumeration of the number of prime numbers contained in the first few millions, can be extended to those remote regions of number which transcend the limits and even the possibilities of human experience, it will follow as a consequence that the average density of the distribution of prime numbers in the neighbourhood of a large quantity x approximates to $\frac{1}{\log x}$, and consequently that the number of primes included between x and $(1+\epsilon)x$, or if we like to say so, between $x+A$ and $(1+\epsilon)x+B$, will be approximately equal to $\frac{\epsilon x}{\log x}$, and therefore will become indefinitely great, however small ϵ may be taken. Although there can hardly be a doubt that such is the fact, no step had been taken previous to Tchebycheff's researches towards establishing this proposition demonstratively. Tchebycheff has succeeded in proving it, not, it is true, in an absolute sense, but for all values of ϵ exceeding the fraction $\frac{1}{5}$. He has done more, inasmuch as he has given formulae for actually ascertaining a number x for all values superior to which there will be at least any specified number K of primes included between $x+A$ and $(1+\epsilon)x+B$ when ϵ has any positive value superior to $\frac{1}{5}$, and A and B are any quantities positive or negative. He may not perhaps have actually stated this proposition in so many words, but it is an immediate inference from the limits (expressed in terms of x, $x^{\frac{1}{2}}$ and $\log x$) which he has obtained to the number of prime numbers not exceeding x. The object of what follows is to make a little further advance in the same

direction, and to show upon Tchebycheff's own principles that the proposition remains true when ϵ is conditioned no longer to be inferior to the fraction $\frac{1}{5}$, but to the fraction $\frac{1}{6}+\frac{1}{4642\frac{10}{11}}$, so that the excess above unity (the region so to say of darkness) is scarcely more than five-sixths of what it is for the first named fraction. This conclusion is arrived at by aid exclusively of Tchebycheff's own formulae.

Tchebycheff's method may be regarded as the *first* approximation to the inferior and superior limits of a quantity ψx subject to the conditions

$$Vx > Ax + F\log x,$$
$$Vx < Ax + F_1 \log x,$$

where
$$Vx = \psi x - \psi\frac{x}{6} + \psi\frac{x}{7} - \psi\frac{x}{10} \text{ etc.,}$$

(see Serret's *Cours d'Algèbre supérieure*, 4th Ed., Vol. II., pp. 230—233), and to the further conditions that ψx is not less than $\psi x'$ if $x > x'$, and that $\psi x = 0$ when $x < 1$.

The limits obtained for ψx depend exclusively on these definitions, and would be applicable to any function ψx whatever that satisfied them.

The advance made in this article consists in pursuing the approximation through an indefinite number of steps, so as to bring the superior and inferior limits to ψx continually nearer and nearer to each other as regards the *principal* term (a multiple of x) which enters into each of them: the remaining terms over and above this multiple of x in the expressions for the limits always continue to be positive integer powers of $\log x$, and consequently the ratio of the limits becomes as nearly as we please identical with the ratio of the principal terms (that is of their coefficients) when x is taken sufficiently great: this ratio as given in the first approximation is $\frac{6}{5}$, but as the approximation is continued continually converges to but never reaches the fraction

$$\frac{7}{6}+\frac{1}{4642\frac{10}{11}}.$$

Such, and such only, is the small but not unimportant contribution here supplied to Tchebycheff's remarkable theory. As no allusion is made to the possibility of this contraction of the limits in a work published so recently as 1879, by an author so competent as M. Serret, I presume that it has hitherto remained unnoticed; but of this I cannot speak with certainty, inasmuch as it was enough for M. Serret's purpose to obtain for the ratio of the principal terms a number less than 2; that being sufficient for the object he had in view, which was to prove M. Bertrand's celebrated postulate that at least one prime number must be included (for all values of x greater than $\frac{7}{2}$) between x and $2x-2$.

Although I might confine myself exclusively to the determination of the limits to ψx which flow from the conditions above given, it is, I think, desirable to supply a brief summary of M. Tchebycheff's method, so as to point out the connexion between the determination of these limits and the limits to "the totality of the prime numbers comprised within a given range." In so doing I shall adopt for the convenience of reference the notation which I find in M. Serret's able exposition of the subject (*Alg. sup.*, Vol. II. pp. 225—239).

θx stands for the sum of the logarithms of all the *prime* numbers not exceeding x.

$$\psi x = \theta x + \theta x^{\frac{1}{2}} + \theta x^{\frac{1}{3}} + \theta x^{\frac{1}{4}} + \theta x^{\frac{1}{5}} + \ldots,$$

$$Tx = \psi x + \psi \frac{x}{2} + \psi \frac{x}{3} + \psi \frac{x}{4} + \psi \frac{x}{5} + \ldots,$$

and, as a consequence founded on purely arithmetical considerations, Tx is the sum of the logarithms of *all* the numbers not exceeding x, and therefore, as an easy deduction from Stirling's theorem, it follows that for all values of x superior to unity,

$$Tx < x \log x - x + \frac{1}{2} \log x + \left\{\log \sqrt{(2\pi)} + \frac{1}{12}\right\}$$

$$Tx > x \log x - x - \frac{1}{2} \log x + \log \sqrt{(2\pi)}.$$

If then Vx (a notation not in Serret) be used to denote

$$Tx - T\frac{x}{2} - T\frac{x}{3} - T\frac{x}{5} + T\frac{x}{30}$$

(where it should be noticed that $1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{30} = 0$), limits for Vx can be found in which $x \log x$ will not appear, and expressed solely in terms of x and $\log x$: it may in fact be shown that for all values of x superior to unity,

$$Vx > A\,(x-1) - \frac{5}{2} \log x$$

$$Vx < A\,(x-1) + \frac{5}{2} \log x,$$

where $\quad A = \frac{1}{2} \log 2 + \frac{1}{3} \log 3 + \frac{1}{5} \log 5 - \frac{1}{30} \log 30 = \cdot 92129202\ldots.$

The limits actually employed, however, are the slightly wider ones,

$$Vx > Ax - \frac{5}{2} \log x - 1$$

$$Vx < Ax + \frac{5}{2} \log x.$$

If now we take an infinite succession of numbers separable into batches of sixteen, such that every $(i+1)$th batch may be got by adding $30i$ to each of the numbers in the first batch, those numbers being

1, 6, 7, 10, 11, 12, 13, 15, 17, 18, 19, 20, 23, 24, 29, 30

(where it is perhaps worth noticing that leaving out the last number 30, the remaining 15 consist of a middle term 15 and pairs of numbers whose sum is always 30, disposed symmetrically about that middle term), it will readily be seen to follow from the expression for V in terms of the T's and of T in terms of the ψ's, that

$$\begin{aligned} Vx = \psi x &- \psi\frac{x}{6} + \psi\frac{x}{7} - \psi\frac{x}{10} + \psi\frac{x}{11} - \psi\frac{x}{12} + \psi\frac{x}{13} - \psi\frac{x}{15} \\ &+ \psi\frac{x}{17} - \psi\frac{x}{18} + \psi\frac{x}{19} - \psi\frac{x}{20} + \psi\frac{x}{23} - \psi\frac{x}{24} + \psi\frac{x}{29} - \psi\frac{x}{30} \\ &+ \psi\frac{x}{31} - \psi\frac{x}{36} \cdots\cdots\cdots\cdots - \psi\frac{x}{45} \\ &+ \psi\frac{x}{47} - \psi\frac{x}{48} \cdots\cdots\cdots\cdots - \psi\frac{x}{60} \\ &+ \psi\frac{x}{61} \cdots\cdots\cdots\cdots\cdots\cdots \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

just in the same way as if supposing $\omega x = \psi x - 2\psi\frac{x}{2}$ we should find

$$\omega x = \psi x - \psi\frac{x}{2} + \psi\frac{x}{3} - \psi\frac{x}{4} + \psi\frac{x}{5} \cdots;$$

or as if supposing $\Omega x = \psi x - \psi\frac{x}{2} - \psi\frac{x}{3} - \psi\frac{x}{6}$ we should find

$$\Omega x = \psi x + \psi\frac{x}{5} - 2\psi\frac{x}{6} + \psi\frac{x}{7} + \psi\frac{x}{11} - 2\psi\frac{x}{12} + \ldots.$$

From the limits to which Vx is subject (Vx being now regarded as representing the series of ψ's above written) limits can be found to ψx of the form $mx + R_1(\log x)$, $nx + R_2(\log x)$, where the R's signify rational integer forms of function. In the first approximation, for the inferior and superior limits respectively, $m = A$, $n = 6\frac{A}{5}$; R_1 is a linear and R_2 a quadratic form of function. In the approximation of the ith order m and n will become functions of i, and R_1, R_2 will be of the ith and $(i+1)$th orders respectively in $\log x$.

The limits of ψx being supposed to be given (say $\psi' x$ the superior and $\psi_1 x$ the inferior limit), $\psi' x$ will serve as a superior and $\psi_1 x - 2\psi' x^{\frac{1}{2}}$ as an inferior limit to θx. But instead of $\psi' x$ we may use (although not at all necessary for the object in view) the slightly closer limit $\psi' x - \psi_1 x^{\frac{1}{2}}$, which is what M. Serret employs, and equally instead of $\psi_1 x - 2\psi' x^{\frac{1}{2}}$ we might use the slightly closer limit

$$\psi_1 x - \psi' x^{\frac{1}{2}} - \psi' x^{\frac{1}{3}} - \psi' x^{\frac{1}{5}} + \psi_1 x^{\frac{1}{30}},$$

which, probably as leading to calculations needlessly complicated (as regards the object in view), M. Serret does not employ. In any case, following the same notation as before to distinguish the two limits, we shall obtain

$$\theta' x = nx + F(x^{\frac{1}{2}}, \log x),$$
$$\theta_1 x = mx + F'(\ldots, \log x),$$

where F, F' are rational integer forms of function, and the dots in the F' may be filled in either with $x^{\frac{1}{2}}$ or with $x^{\frac{1}{2}}, x^{\frac{1}{3}}, x^{\frac{1}{5}}, x^{\frac{1}{30}}$; and we shall have

$$\theta' x = nx(1 + \epsilon_x), \quad \theta_1 x = mx(1 + \eta_x),$$

where ϵ_x and η_x vanish when $x = \infty$.

To come to our ultimate object, it is obvious that the number of primes between x and $(1+\rho)x$ will be greater than $[\theta_1(1+\rho)x - \theta' x] \div \log x$. It will therefore be greater than $\frac{[m(1+\rho)-n]x + \delta_x}{\log x}$, where $\delta_x = 0$ when $x = \infty$. Hence we may find a value of x so great that the number of primes shall be at least K by finding a number x sufficiently large to make

$$\theta_1(1+\rho)x - \theta' x - (K-1)\log x > 0,$$

which it must always be possible to do provided that $m(1+\rho) > n$, that is, that $\rho > \left(\frac{n}{m} - 1\right)$. Hence the importance of diminishing what I call the asymptotic ratio $\frac{n}{m}$, that is, the ratio of the coefficients in the principal terms of the superior and inferior limits to ψx. That is what I shall now proceed to accomplish, but first it is necessary to establish a certain easy lemma.

Suppose the equation $fx - f\frac{x}{c} = Ax^m$ is to be satisfied; this can be done by writing $fx = A\frac{c^m}{c^m - 1}x$, and in particular if $m = 1$, the only case that the present theory demands, $fx = \frac{c}{c-1}Ax$. Again if the equation

$$fx - f\frac{x}{c} = P(\log x)^\mu$$

is to be satisfied, this may be done by making

$$fx = P_0(\log x)^{\mu+1} + P_1(\log x)^\mu + P_2(\log x)^{\mu-1} + \ldots + P_\mu \log x,$$

for since $\log\frac{x}{c} = (\log x - \log c)$, $fx - f\frac{x}{c}$ will then obviously become a function

of $\log x$ of the μth order, which may be identified with $P(\log x)^\mu$ by properly assigning the values of the $(\mu+1)$ disposable constants $P_0, P_1, P_2, \ldots P_\mu$. In fact the equation might easily (if it were worth while to do so) be turned into an equation of differences, and the general values of the P's be expressed once for all in terms of Bernoulli's numbers for any value of μ. Hence it follows that the equation

$$fx - f\frac{x}{c} = Nx + R_\mu{}^* \log x,$$

where R_μ is a rational integer form of function of the μth order, may be satisfied by making

$$fx = \frac{c}{c-1}Nx + R_{\mu+1}\log x,$$

where the second term on the right hand side of the equation is a *known* function of $\log x$ of the $(\mu+1)$th order.

Suppose now that the inequality $\psi x - \psi\frac{x}{c} < Nx + R_\mu \log x$, where $c > 1$, is given, and it is desired to extract from this inequality an inferior limit to ψx. It is only necessary to get a solution of the equation

$$fx - f\frac{x}{c} = Nx + R_\mu \log x.$$

We shall then have

$$\psi x - \psi\frac{x}{c} < fx - f\frac{x}{c},$$
$$\psi\frac{x}{c} - \psi\frac{x}{c^2} < f\frac{x}{c} - f\frac{x}{c^2},$$
$$\psi\frac{x}{c^2} - \psi\frac{x}{c^3} < f\frac{x}{c^2} - f\frac{x}{c^3},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

and consequently

$$fx - f\frac{x}{c^q} > \psi x - \psi\frac{x}{c^q}.$$

If then q be supposed to be taken such that $\frac{x}{c^q}$, say z, lies between 0 and 1, we shall have

$$fx - \psi x > fz,$$

and *à fortiori* $> R_{\mu+1}\log z$ (if N be positive, as is the case throughout the present investigation), where the right hand side of the inequality is a known rational integer function of $\log z$. If then M be a number less than the least value that $R_{\mu+1}\xi$ can assume between the limits $\xi = 0$, $\xi = -\log c$, we shall have $\psi x < fx - M$, and an inferior limit will have been obtained to ψx.

* The reader's attention is called to the fact that R_μ is used throughout to denote a *form of function*, and not, like P_μ, a *coefficient*.

In the first approximation (Serret, p. 234), where $\mu = 1$ and $c = 6$,

$$R_2\xi = \frac{5}{4\log 6}\xi^2 + \frac{5}{4}\xi,$$

the minimum value of which is got by taking $2\xi = -\log 6$ or $\xi = -\log\sqrt{6}$ (which happens to lie between the limits of $\log 1$ and $-\log 6$) and gives $M = \frac{-5\log 6}{16}$, so that $\psi x < fx + \frac{5\log 6}{16}$. The actual value employed for the superior limit, as sufficiently near and more convenient for use, is $fx + 1$.

So in the general case we shall have $fx - \psi x > M$ where M is any number less than the least value of $R_{i+1}\xi$ for values of ξ lying between 0 and $-\log c$. It may or may not be the absolute minimum of $R_{i+1}\xi$ that has to be taken according as the value of ξ which gives this absolute minimum does or does not lie between 0 and $-\log c$. In the latter case it may be either some other minimum, or one of the values of $R_{i+1}\xi$ corresponding to the extreme values $\xi = 0$ and $\xi = -\log c$, which might be found by trial. But a method practically better and sufficient for the demands made by the present investigation, would be to substitute zero in place of any term in the function of ξ of the form $+K\xi^{2m}$ or $-K\xi^{2m+1}$, and for any term of the form $-K\xi^{2m}$ or $+K\xi^{2m+1}$ to substitute $-K(\log c)^{2m}$ and $-K(\log c)^{2m+1}$ respectively.

For instance, in the case just considered we might have written

$$M = -\frac{5}{4}\log 6,$$

and the superior limit instead of being $fx + 1$ would have been $fx + \frac{5}{4}\log 6$, which would practically have been just as good. With a view to a remark which will subsequently be made it is well to notice that the inequality

$$\psi x - \psi\frac{x}{c} > Nx + R_\mu \log x$$

may also be solved precisely in the same manner, and will give for an inferior limit to ψx (using fx to signify the very same function as before) $fx - M_1$, where (N being supposed positive) $M_1 = -N\log c +$ any quantity not less than the greatest value of a known rational integer function of a variable conditioned to lie between 0 and $-\log c$, which may either be found by an exact algebraical process or by substituting 0 in those two cases where previously $-\log c$, and $-\log c$ in those other two cases where previously 0 was to be substituted for the variable.

The lemma needful for our purposes may now accordingly be stated in the following terms: *If* $\psi x - \psi\frac{x}{c}$ *is less or greater than* $Nx +$ *a given rational integer function of* $\log x$ *of any given order,* ψx is less or greater than $\frac{c}{c-1}Nx +$ a known (and easily determinable) rational integer function of $\log x$ of the order next superior.

If the coefficients of x in the superior and inferior limits to ψx at any stage of the investigation be called u and v, I shall show that these values will serve to give (step by step) other superior and inferior limits where u and v are replaced by quantities u', v', such that $u' < u$, $v' > v$; u', v' being known linear functions of u, v. We shall thus be led to a system of two simultaneous linear equations of differences in order to obtain the effect of those changes repeated any number, finite or infinite, of times: but for greater clearness I shall begin with supposing that one of the two expressions u, v, namely, v (which undergoes far less modification than the other) is kept constant. There will then result a single scheme of successive substitutions leading to the construction of a single linear equation in differences.

The first step will then be as follows:

$$\psi x - \psi\frac{x}{6} < Ax + \frac{5}{2}\log x - \psi\frac{x}{7} + \psi\frac{x}{10}$$

$$< Ax + \frac{5}{2}\log x - \left(A\frac{x}{7} - \frac{5}{2}\log\frac{x}{7} - 1\right) + \frac{6}{50}Ax + \frac{5}{4\log 6}\left(\log\frac{x}{10}\right)^2 + \frac{5}{4}\log\frac{x}{10}$$

or writing $\lambda = \log 6,\quad \mu = \log 7,\quad \nu = \log 10,$

$$\psi x - \psi\frac{x}{6} < \frac{171}{175}Ax + \frac{5}{4\lambda}(\log x)^2 + \left\{\left(\frac{25}{4} - \frac{5\nu}{2\lambda}\right)\log x + \frac{5\nu^2}{4\lambda} - \frac{5}{2}\mu - \frac{5}{4}\nu + 1\right\}.$$

Hence $\psi x < \frac{1026}{875}Ax + P(\log x)^3 + Q(\log x)^2 + Rx - M,$

where first to find P, Q, R, we have the three equations

$$3P\lambda = \frac{5}{4\lambda}$$

$$-3P\lambda^2 + 2Q\lambda = \frac{25}{4} - \frac{5\nu}{2\lambda}$$

$$P\lambda^3 - Q\lambda^2 + R\lambda = \frac{5\nu^2}{4\lambda} - \frac{5}{2}\mu - \frac{5}{4}\nu + 1,$$

that is, $P = \frac{5}{12\lambda^2}$; $Q = \frac{15}{4\lambda} - \frac{5\nu}{4\lambda^2}$; $R = -\frac{5}{12} + \frac{15}{4} - \frac{5\nu}{2\lambda} + \frac{5\nu^2}{4\lambda^2} - \frac{5\mu}{2\lambda} + \frac{1}{\lambda}$.

Here P is positive; Q, whose sign depends on that of $3 - \frac{\log 10}{\log 6}$, is also positive; and

$$R = \frac{10}{3} + 5\left(\frac{\nu}{2\lambda} - \frac{1}{2}\right)^2 - \frac{5\mu - 2}{2\lambda} - \frac{5}{4}$$
$$= 3{\cdot}33333\ldots + {\cdot}10160\ldots - 2{\cdot}1570\ldots - 1{\cdot}25$$
$$= 3{\cdot}43493\ldots - 3{\cdot}4070\ldots,$$ which is also positive.

Hence we may make

$$M = -P\lambda^3 - R\lambda,$$

or

$$-M = 1 + \frac{15}{4}\lambda - \frac{5(\mu+\nu)}{2} + \frac{5\nu^2}{4\lambda} = 1{\cdot}2947.$$

It is quite possible, and even most likely, that the minimum of

$$P\lambda^3 - Q\lambda^2 + R\lambda$$

(within the prescribed limits) would be found to exceed -1 were it worth while to go through the arithmetical calculations necessary to obtain it, but it is quite sufficiently near for all practical purposes to use the value above determined, or even to take $-M$ as great as 2 and to adopt for our new superior limit

$$\frac{171}{175} Ax + P(\log x)^3 + Q(\log x)^2 + R \log x + 2.$$

In like manner this new limit will enable us to find another, and it is obvious that the general form of the limit obtained after i of these steps have been gone through will be $u_i Ax + R_{i+2} \log x$, where

$$u_i = \frac{6}{5}\left(1 - \frac{1}{7} + \frac{u_{i-1}}{10}\right), \quad \text{that is,} \quad u_i - \frac{3u_{i-1}}{25} = \frac{36}{35}.$$

Putting $$u_i = \omega_i + h$$

and making $$\frac{22}{25} h = \frac{36}{35}, \quad \text{that is,} \quad h = \frac{90}{77},$$

we have $$\omega_i - \frac{3}{25}\omega_{i-1} = 0.$$

Hence $$u_i = C\left(\frac{3}{25}\right)^i + \frac{90}{77}.$$

The ultimate value of u_i is therefore $\frac{90}{77}$, and accordingly, by repeating the process indicated a sufficient number of times, we shall have for a superior limit $\left(\frac{90}{77} - \epsilon_i\right) Ax + R_{i+2} \log x$, where ϵ_i may be made as small as we please by taking i sufficiently great, and thus the ultimate asymptotic ratio of the two limits is $\frac{90}{77}$ instead of $\frac{6}{5}$.

Another mode of approximation may be used, as shown in what follows.

Since $$\psi x - \psi\frac{x}{10} < Ax - \psi\frac{x}{6} + \psi\frac{x}{7};$$

if we have found $$\psi x < u_i' Ax + R_{i+2} \log x$$

we shall have $$\psi x - \psi\frac{x}{10} < Ax + u_i' A\frac{x}{6} - A\frac{x}{7} + R'_{i+2} \log x,$$

and therefore $$\psi x < u'_{i+1} Ax + R_{i+3} \log x,$$

where $$u'_{i+1} = \frac{10}{9}\left\{1 - \frac{1}{7} + \frac{1}{6}u_i'\right\}$$

that is, $$u'_{i+1} - \frac{5}{27}u_i' = \frac{20}{21};$$

or $$u_i' = K\left(\frac{5}{27}\right)^i + h'$$

where $$h' = \frac{27}{22} \cdot \frac{20}{21} = \frac{90}{77}.$$

Thus $h' = h$ and consequently also, if we suppose each of the two sorts of approximation to start from the same point, $K = C$.

Hence the ultimate value of u_i and u_i' is the same, but the former method of approximation is to be preferred, as the same number of steps, that is, the same value of i, makes $C\left(\frac{5}{27}\right)^i + h$ always $> C\left(\frac{3}{25}\right)^i + h$. The corresponding values of u_i, u_i' have the same initial and final values, but for every intermediate value of i, $u_i < u_i'$. In fact u_i, u_i' are ordinates to the same abscissa of two non-intersecting curves, having a common starting point and a common asymptote.

The maximum value of $u_i' - u_i$ is found by making $\left(\frac{5}{27}\right)^i - \left(\frac{3}{25}\right)^i$ a maximum, which takes place when i is the integer next above or next below the value

$$\frac{\log\log\frac{25}{3} - \log\log\frac{27}{5}}{\log\frac{25}{3} - \log\frac{27}{5}}, \text{ which is obviously less than unity.}$$

Hence after the *first* approximation u_i and u_i' are always drawing closer together.

We may now proceed to the more (but only very slightly more) advantageous method of approximation, namely, that in which the principal terms in both limits are simultaneously varied, decreasing as before in the superior, and now at the same time increasing in the inferior limit.

Suppose then that we have found

$$\psi x < u_i Ax + R_{i+2} \log x$$
$$\psi x > v_i Ax + R_{i+1} \log x;$$

observing that $\frac{v_i}{24} - \frac{u_i}{29}$ is always positive, we shall succeed in increasing the principal term of the inferior limit by writing

$$\psi x > Ax + v_i A\frac{x}{24} - u_i A\frac{x}{29} + R_{i+2} \log x,$$

and slightly more than previously diminishing the principal term in the superior limit by writing

$$\psi x - \psi\frac{x}{6} < Ax - v_i A\frac{x}{7} + u_i A\frac{x}{10} + R'_{i+2} \log x.$$

We shall thus easily derive

$$\psi x > v_{i+1} A x + R_{i+2} \log x$$

$$\psi x < u_{i+1} A x + R_{i+3} \log x$$

where

$$v_{i+1} = 1 + \frac{v_i}{24} - \frac{u_i}{29}$$

$$u_{i+1} = \frac{6}{5}\left(1 - \frac{v_i}{7} + \frac{u_i}{10}\right) = \frac{6}{5} - \frac{6}{35} v_i + \frac{3}{25} u_i$$

or, making

$$v_i = v_i' + f, \quad u_i = u_i' + e,$$

$$v'_{i+1} - \frac{1}{24} v_i' + \frac{1}{29} u_i' = 0$$

$$u'_{i+1} - \frac{3}{25} u_i' + \frac{6}{35} v_i' = 0,$$

if

$$\frac{23}{24} f + \frac{1}{29} e = 1, \quad \frac{6}{35} f + \frac{22}{25} e = \frac{6}{5}.$$

So that, calling ρ_1, ρ_2 the roots of

$$\begin{vmatrix} \rho - \frac{1}{24}; & \frac{1}{29} \\ \frac{6}{35}; & \rho - \frac{3}{25} \end{vmatrix} = 0,$$

$$u_i = C_1 \rho_1^i + C_2 \rho_2^i + e$$

$$v_i = C_1' \rho_1^i + C_2' \rho_2^i + f.$$

The equation for finding ρ_1, ρ_2 is

$$\rho^2 - \frac{97}{600} \rho - \frac{37}{40600} = 0,$$

whence

$$\rho_1 = \cdot167253\ldots, \quad \rho_2 = \cdot005637\ldots.$$

Also the equations in e, f give e, f (the values of u_∞, v_∞) as follows:

$$e = \frac{59595}{50999}, \quad f = \frac{51072}{50999}.$$

If there were any use in obtaining the values of the disposable constants they could of course be obtained from the equations

$$C_1 + C_2 + e = u_0 = \frac{6}{5}, \quad C_1 \rho_1 + C_2 \rho_2 + e = u_1 = \frac{1026}{875},$$

$$C_1' + C_2' + f = v_0 = 1, \quad C_1' \rho_1 + C_2' \rho_2 + f = v_1 = \frac{3481}{3480}.$$

The asymptotic ratio of the two limits is

$$\frac{e}{f} = \frac{59595}{51072} = \frac{6}{7} + \frac{11}{51072}.$$

Various other modes of approximation may be adopted, but it will be found that no smaller value can be obtained for the asymptotic ratio than that above given: the value of u_∞ cannot be made less than $\frac{59595}{50999}$, nor the value of v_∞ greater than $\frac{51072}{50999}$.

Thus for example, making use of the inequality

$$\psi x - \psi \frac{x}{6} > A x - \psi \frac{x}{7} + \psi \frac{x}{24} - \psi \frac{x}{29} + R (\log x),$$

we might by the lemma obtain

$$\psi x > \frac{6}{5} A \left(1 - \frac{u_i}{7} - \frac{u_i}{29} + \frac{v_i}{24}\right) + R_{i+3} \log x,$$

and consequently

$$v_{i+1} = \frac{6}{5}\left(1 - \frac{u_i}{7} - \frac{u_i}{29} + \frac{v_i}{24}\right);$$

combining which with the previous equation for u_{i+1} we should have for finding u_∞, v_∞, say e', f', the two equations,

$$\frac{19}{24} f' + \frac{36}{203} e' = 1,$$

$$\frac{6}{35} f' + \frac{22}{25} e' = \frac{6}{5},$$

and consequently

$$e' = \frac{331905}{284029}, \quad f' = \frac{284424}{284029}.$$

Reduced to decimals

$$e = 1\cdot16855\ldots, \quad f = 1\cdot00143\ldots,$$

$$e' = 1\cdot16856\ldots, \quad f' = 1\cdot00125\ldots.$$

It may be noticed that $eA = 1\cdot006774\ldots$, $fA = \cdot992619\ldots$ of which the sum is nearly $1\cdot999394$, and their mean nearly $\cdot999697$, whereas the mean of A and $\frac{6A}{5}$ (the original coefficients of x in the limits) is nearly $1\cdot01342$. Thus the new mean is more than 44 times nearer than the latter to the true asymptotic value deducible from the empirical formula.

Were it desired merely to find superior and inferior limits to ψx in the form obtained in Tchebycheff's method, it would (as already indicated) have been sufficient to have taken for Vx, $Tx - 2T\frac{x}{2}$, which would have led to the inequalities

$$\psi x > (\log 2) x + R_1 \log x,$$

$$\psi x < 2 (\log 2) x + R_2 \log x,$$

but the asymptotic ratio being here 2, these limits could not have conducted to a proof of M. Bertrand's postulate. If, however, we were to take $Vx = Tx - T\frac{x}{2} - T\frac{x}{3} - T\frac{x}{6}$ we should obtain

$$Vx > Bx + R_1 \log x,$$
$$Vx < Bx + R_1' \log x,$$

where $$B = \frac{1}{2}\log 2 + \frac{1}{3}\log 3 + \frac{1}{6}\log 6 = 1{\cdot}0114043,$$

and $$Vx = \psi x + \psi\frac{x}{5} - 2\psi\frac{x}{6} + \psi\frac{x}{7} + \psi\frac{x}{11} - 2\psi\frac{x}{12} + \ldots,$$

when we should obtain

$$\psi x - \psi\frac{x}{6} < Bx + R_1' \log x,$$
$$\psi x < \frac{6}{5}Bx + R_2 \log x,$$

and again $$\psi x + \psi\frac{x}{5} > Bx + R_1 \log x,$$
$$\psi x > B\left(1 - \frac{6}{25}\right)x + R_2' \log x.$$

Here the asymptotic ratio of the two limits is $\frac{30}{19}$, which being less than 2, the formulae above indicated would suffice to prove M. Bertrand's postulate, and would lead to an equation somewhat simpler in form than that led to by M. Tchebycheff's process, but whose greatest root would be considerably larger than that found by the established method; so that there would be a larger number of verifications of the postulate to be made for the lower numbers: this, however, is really a matter of very trifling importance, as the needful verifications could be made even up to 100,000 if necessary, by throwing a rapid glance over a few leaves of Burckhardt's tables.

It is noticeable that the limits above found by giving Vx the form $Tx - 2T\frac{x}{2}$ are the *only* limits that can be got in such case; no process of successive approximation being here possible, on account of the too close contiguity of the successive denominators in $\psi x - \psi\frac{x}{2} + \psi\frac{x}{3}\ldots$.

Such, however, would not be the case were we to use Vx to signify $Tx - T\frac{x}{2} - T\frac{x}{3} - T\frac{x}{6}$, and consequently

$$Vx = \psi x + \psi\frac{x}{5} - 2\psi\frac{x}{6} + \psi\frac{x}{7} + \psi\frac{x}{11} - 2\psi\frac{x}{12} + \psi\frac{x}{13}\ldots$$

The limits expressed by the inequalities

$$\psi x < u_i Bx + \ldots,$$
$$\psi x > v_i Bx + \ldots,$$

would lead to the narrower limits

$$\psi x < u_{i+1} Bx + \ldots,$$
$$\psi x > v_{i+1} Bx + \ldots,$$

where $$u_{i+1} = \frac{6}{5}\left(1 - \frac{v_i}{7} + \frac{u_i}{12}\right),$$
$$v_{i+1} = 1 - \frac{u_i}{5} + \frac{v_i}{6} - \frac{u_i}{11},$$

that is to say $$u_{i+1} - \frac{u_i}{10} + \frac{6v_i}{35} - \frac{6}{5} = 0,$$
$$\frac{u_i}{5} + \frac{u_i}{11} + v_{i+1} - \frac{v_i}{6} - 1 = 0.$$

Hence, using as before e, f to indicate the ultimate values of u_i, v_i, we should have

$$21e + 4f - 28 = 0,$$
$$96e + 275f - 330 = 0,$$

and consequently $$e = \frac{6380}{5391}, \quad f = \frac{4242}{5391},$$

and $$\frac{e}{f} = \frac{6380}{4242} = \frac{3}{2} + \frac{1}{249\frac{9}{17}},$$

which is the ultimate value of the asymptotic ratio, of which the initial value was $\frac{30}{19}$, that could be found by this method.

In every such kind of series as I have denoted by Vx, it is obvious that the sum of the multiples of x under the sign of ψ in Vx is equal to the coefficient of x in either limit to Vx. Thus, for example, in Tchebycheff's series, if we take n a multiple of 30, and make $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$, the sum of n terms of $1 - \frac{1}{6} + \frac{1}{7} - \frac{1}{10} + \frac{1}{11}\ldots$

$$= \left(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{30}\right)S_n + \frac{1}{2}\left(\frac{1}{\frac{1}{2}n+1} + \frac{1}{\frac{1}{2}n+2} + \ldots + \frac{1}{n}\right)$$
$$+ \frac{1}{3}\left(\frac{1}{\frac{1}{3}n+1} + \frac{1}{\frac{1}{3}n+2} + \ldots + \frac{1}{n}\right) + \frac{1}{5}\left(\frac{1}{\frac{1}{5}n+1} + \frac{1}{\frac{1}{5}n+2} + \ldots + \frac{1}{n}\right)$$
$$- \frac{1}{30}\left(\frac{1}{\frac{1}{30}n+1} + \frac{1}{\frac{1}{30}n+2} + \ldots + \frac{1}{n}\right);$$

and the multiplier of S_n being always 0, it follows that the sum of an infinite number of the consecutive terms

$$=\frac{1}{2}\log 2+\frac{1}{3}\log 3+\frac{1}{5}\log 5-\frac{1}{30}\log 30=A.$$

It may not unreasonably be conjectured that whilst nothing more can be done with the Tchebycheffian Vx, it may be possible to find such other form of function in lieu of it, or such infinite succession of different forms of function, as may either directly or by successive approximation bring the coefficients of x in the two limits as near as we please to one another, at the expense, of course, of proportionally lengthening out the residues, or tails as they might be termed, of the two limits. Could this be done, it is easy to demonstrate that the limit thus continually approached from opposite sides must be unity, as indicated in advance by Legendre's empirical formula. For this purpose it will be sufficient to use the simplest form of Vx, namely, $Tx-2T\frac{x}{2}$, whence we obtain

$$\psi x-\psi\frac{x}{2}+\psi\frac{x}{3}\ldots>\log 2\,.\,x(1+\epsilon_x),$$

$$\psi x-\psi\frac{x}{2}+\psi\frac{x}{3}\ldots<\log 2\,.\,x(1+\eta_x),$$

ϵ_x, η_x being known logarithmic quantities which vanish when $x=\infty$.

For suppose it possible to prove that, with a value of h capable of being made less than any assignable quantity,

$$\psi x>Q(1-h)\,x+Gx,$$

$$\psi x<Q(1+h)\,x+Fx,$$

where $\frac{Fx}{x}$, $\frac{Gx}{x}$ may be made as small as we please by taking x sufficiently large, (I mean by taking x greater than some certain value ξ). Then

$$(1+\epsilon_x)\log 2\,.\,x<\psi x-\psi\frac{x}{2}+\psi\frac{x}{3}\ldots-\psi\frac{x}{2m}$$

$$<Q(1+h)\,x\left(1-\frac{1}{2}+\frac{1}{3}\ldots-\frac{1}{2m}\right)$$

$$+Fx-F\frac{x}{2}+F\frac{x}{3}\ldots-F\frac{x}{2m}.$$

Let ξ be taken so great that for all values of x greater than $\frac{\xi}{2m}$, $\frac{Fx}{x}$ shall be less in absolute numerical value than $\frac{k}{2m}$, where k is an *arbitrary* positive quantity: then, if we take $x>\xi$, the sum of the absolute values of Fx, $F\frac{x}{2}$, $F\frac{x}{3}$, ... $F\frac{x}{2m}$, is less than kx; and *à fortiori*

$$Fx-F\frac{x}{2}+F\frac{x}{3}\ldots-F\frac{x}{2m}<kx.$$

Therefore $$Q(1+h)\log 2\,.\,x>(1+\epsilon_x)\log 2\,.\,x-kx.$$

Hence, Q being greater than $\frac{1+\epsilon_x}{1+h}-\frac{k}{(1+h)\log 2}$, and ϵ_x, h, k being all three capable of becoming indefinitely small, $1-Q$ cannot be a finite positive quantity; which amounts to saying that $1-Q$ cannot be positive.

In precisely the same manner, dealing with the other limit to Vx and stopping in its development at the term $\psi\frac{x}{2m+1}$ $\left(\text{instead of stopping at the term } -\psi\frac{x}{2m}\right)$ it may be proved that $1-Q$ cannot be negative. Hence $1-Q$ must be zero, that is, $Q=1$. Q. E. D.

We have thus determined what is the common limit to which the principal terms in the superior and in the inferior limits of ψx are bound to approximate, on the supposition of the possibility of formulae being discoverable admitting of the interval between these principal terms being capable of being made as small as we please. But to pronounce with certainty upon the existence of such possibility, we shall probably have to wait until some one is born into the world as far surpassing Tchebycheff in insight and penetration as Tchebycheff has proved himself superior in these qualities to the ordinary run of mankind.

62.

ON THE SOLUTION OF A CERTAIN CLASS OF DIFFERENCE OR DIFFERENTIAL EQUATIONS.

[*American Journal of Mathematics*, IV. (1881), pp. 260—265.]

CASTING my eye over Mr Moulton's valuable edition of Boole's *Treatise on Finite Differences* (see pp. 229—231), I was gratified to find that he had embalmed in it a solution that I had given* many years ago, of an equation in differences, of the simple but very general form expressed by equating to zero or to Pm^x the persymmetrical determinant

$$\begin{vmatrix} u_x & u_{x+1} & \cdots u_{x+i} \\ u_{x+1} & u_{x+2} & \cdots u_{x+i+1} \\ u_{x+2} & u_{x+3} & \cdots u_{x+i+2} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ u_{x+i} & u_{x+i+1} & \cdots u_{x+2i} \end{vmatrix}$$

which is of the ith degree and $2i$th order.

To fix the ideas, let us consider the simple case

$$u_x u_{x+2} - u^2_{x+1} = Pm^x,$$

of which, when $P=0$, the solution is $u_x = A\alpha^x$, A and α being both arbitrary, but for P not zero is expressed by $u_x = \pm(A\alpha^x + B\beta^x)$ with the conditions

$$\alpha\beta = m, \quad AB(\alpha-\beta)^2 = P$$

which solution as an *obiter dictum* I may remark may easily be converted into the simpler and more explicit form

$$(\sin\beta)^2 u_x^2 + P\{\sin(\alpha+\beta x)\}^2 m^{x-1} = 0$$

where α, β are arbitrary constants.

If we proceed now to verify the solution in its original form, we shall immediately be led to perceive a certain generalization which the given equation may be made to undergo without ceasing to be soluble—the solution however becoming narrowed from a general to a special one: whether particular or singular I shall not discuss.

[* This Reprint, Vol. II., pp. 308, 313.]

If we write $u_x = A\alpha^x + B\beta^x$, the determinant becomes

$$\begin{vmatrix} A\alpha^x + B\beta^x, & A\alpha^{x+1} + B\beta^{x+1} \\ A\alpha^{x+1} + B\beta^{x+1}, & A\alpha^{x+2} + B\beta^{x+2} \end{vmatrix}$$

which is equal to $AB(\alpha-\beta)^2(\alpha\beta)^x$; this is the verification spoken of: but, as a consequence, it is apparent that we must have

$$\begin{vmatrix} A\alpha^x + B\beta^x + C\gamma^x, & A\alpha^{x+1} + B\beta^{x+1} + C\gamma^{x+1} \\ A\alpha^{x+1} + B\beta^{x+1} + C\gamma^{x+1}, & A\alpha^{x+2} + B\beta^{x+2} + C\gamma^{x+2} \end{vmatrix}$$
$$= AB(\alpha-\beta)^2(\alpha\beta)^x + BC(\beta-\gamma)^2(\beta\gamma)^x + CA(\gamma-\alpha)^2(\gamma\alpha)^x.$$

Hence we can solve the equation

$$u_x u_{x+2} - u^2_{x+1} = Pl^x + Qm^x + Rn^x,$$

namely, we may write $\quad u_x + A\alpha^x + B\beta^x + C\gamma^x = 0,$

where $$\beta\gamma = l, \quad \gamma\alpha = m, \quad \alpha\beta = n,$$

$$AB(\alpha-\beta)^2 = R, \quad BC(\beta-\gamma)^2 = P, \quad CA(\gamma-\alpha)^2 = Q,$$

that is to say $\quad \alpha = \sqrt{\left(\frac{mn}{l}\right)}, \quad \beta = \sqrt{\left(\frac{nl}{m}\right)}, \quad \gamma = \sqrt{\left(\frac{lm}{n}\right)},$

$$A = \sqrt{\left(\frac{QR}{P}\right)} \frac{(\beta-\gamma)}{(\alpha-\beta)(\alpha-\gamma)}, \quad B = \sqrt{\left(\frac{RP}{Q}\right)} \frac{(\gamma-\alpha)}{(\beta-\alpha)(\beta-\gamma)},$$

$$C = \sqrt{\left(\frac{PQ}{R}\right)} \frac{(\alpha-\beta)}{(\gamma-\alpha)(\gamma-\beta)};$$

or calling $$\sqrt{(lmn)} = g, \quad \sqrt{(PQR)} = G,$$

$$\alpha = \frac{g}{l}, \quad \beta = \frac{g}{m}, \quad \gamma = \frac{g}{n},$$

$$A = \frac{G}{g} \cdot \frac{(n-m)l^2}{(l-m)(l-n)P}, \quad B = \frac{G}{g} \cdot \frac{(l-n)m^2}{(m-n)(m-l)Q},$$

$$C = \frac{G}{g} \cdot \frac{(m-l)n^2}{(n-l)(n-m)R}.$$

The result therefore in its rational unambiguous form is

$$PQR(lm-mn)^2(mn-nl)^2(nl-lm)^2(lmn)^{2x-1}u_x^2 = [\Sigma(lm-ln)QR(mn)^x]^2.$$

When any of the quantities P, Q, R vanish, or any of the quantities l, m, n vanish or become equal to one another, the solution fails.

We shall, however, easily obtain a compensatory form of equation supplying the place of two of the exponentials, and another supplying the place of all three becoming identical, and the solution of these substituted forms may be deduced from that of the original form of the equation.

Thus, first, let

$$\left.\begin{array}{ll} m=(1+\epsilon)\mu, & n=(1-\epsilon)\mu \\ Q=\frac{1}{2}\left(S+\frac{T}{\epsilon}\right), & R=\frac{1}{2}\left(S-\frac{T}{\epsilon}\right) \end{array}\right\} \text{ where } \epsilon \text{ is an infinitesimal.}$$

Then the equation becomes

$$u_x u_{x+2} - u^2_{x+1} = Pl^x + S\mu^x + Tx\mu^x$$

and the solution in its unreduced form is

$$u_x = A\alpha^x + B\beta^x + C\gamma^x,$$

where $\alpha = \sqrt{\frac{\mu^2}{l}}, \quad \beta = (1-\epsilon)\sqrt{l}, \quad \gamma = (1+\epsilon)\sqrt{l},$

and

$$A = \sqrt{\left(\frac{PR}{Q}\right)} \frac{\beta-\gamma}{(\alpha-\beta)(\alpha-\gamma)} = \frac{T}{\sqrt{(-P)}} \frac{\sqrt{l}}{\left\{\sqrt{\left(\frac{\mu^2}{l}\right)}-\sqrt{l}\right\}^2} = \frac{T}{\sqrt{(-P)}} \cdot \frac{l^{\frac{3}{2}}}{(\mu-l)^2}$$

$$B = \sqrt{\left(\frac{PR}{Q}\right)} \frac{\gamma-\alpha}{(\beta-\alpha)(\beta-\gamma)}$$

$$= \sqrt{(-P)}\left(1-2\epsilon\frac{S}{T}\right) \frac{\sqrt{l}-\sqrt{\left(\frac{\mu^2}{l}\right)}+\sqrt{(l)}\epsilon}{\sqrt{l}-\sqrt{\left(\frac{\mu^2}{l}\right)}-\sqrt{(l)}\epsilon} \div (-2\epsilon\sqrt{l})$$

$$= \sqrt{(-P)}\left\{\frac{-1}{2\sqrt{(l)}\epsilon} + \frac{S}{T\sqrt{l}} - \frac{\sqrt{l}}{l-\mu}\right\}$$

$$C = \sqrt{(-P)}\left\{\frac{1}{2\sqrt{(l)}\epsilon} + \frac{S}{T\sqrt{l}} - \frac{\sqrt{l}}{l-\mu}\right\}.$$

$$\text{Hence}\quad B\beta^x + C\gamma^x = \sqrt{(-P)}\, l^{\frac{x-1}{2}} \left\{\begin{array}{l} (1+x\epsilon)\left(\frac{1}{2\epsilon}+\frac{S}{T}-\frac{l}{l-\mu}\right) \\ +(1-x\epsilon)\left(-\frac{1}{2\epsilon}+\frac{S}{T}-\frac{l}{l-\mu}\right) \end{array}\right\}$$

$$= \sqrt{(-P)}\left\{2\left(\frac{S}{T}-\frac{l}{l-\mu}\right)+x\right\} l^{\frac{x-1}{2}};$$

so that $\sqrt{(-lP)}\, u_x = T\left(\frac{l}{\mu-l}\right)^2 \left(\frac{\mu^2}{l}\right)^{\frac{x}{2}} - P\left(\frac{2S}{T}-\frac{2l}{l-\mu}+x\right) l^{\frac{x}{2}}$

or $Pl^{x+1}u_x^2 + \left\{T\left(\frac{l}{\mu-l}\right)^2 \mu^x - P\left(\frac{2S}{T}+\frac{2l}{\mu-l}+x\right) l^x\right\}^2 = 0$

will satisfy the given equation

$$u_x u_{x+2} - u^2_{x+1} = Pl^x + S\mu^x + Tx\mu^x.$$

When $T=0$ the solution fails, as we know *à priori* it ought to do.

When $S=0$ it takes the form

$$Pl^{x+1}u_x^2 + \left\{T\left(\frac{l}{\mu-l}\right)^2 \mu^x - P\left(\frac{2l}{\mu-l}+x\right) l^x\right\}^2 = 0.$$

We might, by an analogous process, writing $(1+\epsilon)$, $(1+\rho\epsilon)$, $(1+\rho^2\epsilon)$ in lieu of l, m, n, and giving P, Q, R appropriate values involving ϵ^2 as well as ϵ, render ΣPl^x a finite function of the form $(S+Tx+Ux^2)\lambda^x$, and deduce the solution of $u_x u_{x+2} - u^2_{x+1} = (S+Tx+Ux^2)\lambda^x$ as a particular case of the solution of the general equation. But as we can easily see that the unreduced form of the solution must be $u_x = \lambda^{\frac{x}{2}}(A+Bx+Cx^2)$, it will be easier to find A, B, C immediately from the equation

$$\begin{vmatrix} A+Bx+Cx^2 & A+B(x+1)+C(x+1)^2 \\ A+B(x+1)+C(x+1)^2 & A+B(x+2)+C(x+2)^2 \end{vmatrix}$$

or

$$\begin{vmatrix} A+Bx+Cx^2 & A+B(x+1)+C(x+1)^2 \\ B+C+2Cx & B+3C+2Cx \end{vmatrix}$$

or

$$\begin{vmatrix} A+Bx+Cx^2 & B+C+2Cx \\ B+C+2Cx & 2C \end{vmatrix} = S+Tx+Ux^2.$$

Hence $-2C^2 = U, \quad -2BC-4C^2 = T, \quad 2AC-(B+C)^2 = S.$

Hence

$$C = \sqrt{\left(\frac{-U}{2}\right)}, \quad B = -2C - \frac{T}{2C} = -\sqrt{(-2U)} - \frac{T}{\sqrt{(-2U)}} = \frac{2U+T}{\sqrt{(-2U)}},$$

$$A = \frac{S+(B+C)^2}{2C} = \frac{S+\left\{\sqrt{\left(\frac{-U}{2}\right)}+\frac{T}{\sqrt{(-2U)}}\right\}^2}{\sqrt{(-2U)}} = \frac{-2SU+(T+U)^2}{-2U\sqrt{(-2U)}},$$

or $8U^3u_x^2 + \{2U^2x^2 + (4U^2-2UT)x + 2SU - (T+U)^2\}\lambda^x = 0$

is the required primitive of the given equation.

The method may obviously be extended to any equation of the given form: that is to say when the persymmetrical determinant which it contains is of the degree i and is equated to $(i+1)$ multiples of exponentials each of the form Pl^x an integral of it can be found, and if these i exponentials be subdivided into partial groups of $\epsilon, \epsilon', \epsilon''\ldots$ terms in a group, then instead of the ϵ multiples of exponentials belonging to any group may be substituted

$$(P_1 + P_2x + P_3x^2 + \ldots + P_\epsilon x^{\epsilon-1})\, l^x,$$

and the solution of the equation so modified may be deduced from the solution first mentioned as a particular case thereof.

It will be sufficient for all reasonable purposes of illustration briefly to consider the case of

$$\begin{vmatrix} u_x & u_{x+1} & u_{x+2} \\ u_{x+1} & u_{x+2} & u_{x+3} \\ u_{x+2} & u_{x+3} & u_{x+4} \end{vmatrix} = Pl^x + Qm^x + Rn^x + Sp^x.$$

An integral of this may be found by writing

$$u_x = A\alpha^x + B\beta^x + C\gamma^x + D\delta^x,$$

where $$\beta\gamma\delta = l, \quad \alpha\gamma\delta = m, \quad \alpha\beta\delta = n, \quad \alpha\beta\gamma = p,$$

$$BCD\zeta(\beta, \gamma, \delta) = P, \quad ACD\zeta(\alpha, \gamma, \delta) = Q, \quad ABD\zeta(\alpha, \beta, \delta) = R,$$

$$ABC\zeta(\alpha, \beta, \gamma) = S,$$

ζ meaning the product of the squared differences of the letters which it governs. We have thus

$$\alpha = \frac{g}{l}, \quad \beta = \frac{g}{m}, \quad \gamma = \frac{g}{n}, \quad \delta = \frac{g}{p},$$

where $$g = \sqrt[3]{(lmnp)}$$

and $$A^3B^3C^3D^3[\zeta(\alpha, \beta, \gamma, \delta)]^2 = PQRS,$$

so that writing

$$G = \left\{\frac{PQRS}{[\zeta(\alpha, \beta, \gamma, \delta)]^2}\right\}^{\frac{1}{3}} = \frac{1}{g^4}\frac{(PQRS)^{\frac{1}{3}}}{\left\{\zeta\left(\frac{1}{l}, \frac{1}{m}, \frac{1}{n}, \frac{1}{p}\right)\right\}^{\frac{2}{3}}},$$

$$A = \zeta(\beta, \gamma, \delta)\frac{G}{P}; \quad B = \zeta(\alpha, \gamma, \delta)\frac{G}{Q}; \quad C = \zeta(\alpha, \beta, \delta)\frac{G}{R}; \quad D = \zeta(\alpha, \beta, \gamma)\frac{G}{S};$$

and thus

$$(PQRS)^2(lmnp)^8\left\{\zeta\left(\frac{1}{l}, \frac{1}{m}, \frac{1}{n}, \frac{1}{p}\right)\right\}^2 u_x^3 = (lmnp)^x\{\Sigma QRS\zeta(\beta, \gamma, \delta)\,l^{-x}]^3$$

or $$\left\{PQRS\zeta\left(\frac{1}{l}, \frac{1}{m}, \frac{1}{n}, \frac{1}{p}\right)\right\}^2 u_x^3 = (lmnp)^{x-2}\left\{\Sigma QRS\zeta\left(\frac{1}{m}, \frac{1}{n}, \frac{1}{p}\right)l^{-x}\right\}^3.$$

It is scarcely necessary to add that all the above conclusions continue to hold, when, on the left hand side of the equation for u_{x+h} we write $\left(\frac{d}{dx}\right)^h y$ and at the same time for any exponential l^x on the right hand side substitute e^{lx}.

Thus for instance we may in general find an integral of

$$yy'' - y'^2 = Ae^{hx} + Be^{kx}\cos(\alpha x + \beta)$$

or again of

$$(yy'' - y'^2)\,y'''' - y\,(y''')^2 + 2y'y''y''' - y''^3 = Ae^{hx}\cos(\alpha x + \beta) + Be^{kx}\cos(\gamma x + \delta).$$

63.

NOTE ON THE THEORY OF SIMULTANEOUS LINEAR DIFFERENTIAL OR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS.

[*American Journal of Mathematics*, IV. (1881), pp. 321—326.]

THIS theory is virtually the same for differential as for finite-difference equations. The mere verbal part of the exposition being somewhat easier for the former of the two, I shall prefer in the first instance to deal with them, although the applications are more interesting when made to bear on the latter. Simple to the last degree as are the method of solution and the nature of the result, I do not find the one or the other set out, or even indicated, except in the most perfunctory manner, in the ordinary text-books. This brief notice, designed for the junior readers of the *Journal*, is intended to supply the lacuna.

Let $u_{j,k}$ denote a linear function, with constant coefficients, of ω_k and of its first ϵ_j derivatives in respect to t.

Let
$$u_{1,1} + u_{1,2} + \ldots + u_{1,i} = 0,$$
$$u_{2,1} + u_{2,2} + \ldots + u_{2,i} = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$u_{i,1} + u_{i,2} + \ldots + u_{i,i} = 0,$$

be the system of differential equations proposed for integration.

Call $$\epsilon_1 + \epsilon_2 + \ldots + \epsilon_i = \sigma.$$

The process of arriving at the reducing equation for any one of the variables is after the manner of the dialytic method of elimination, namely:

Along with the first equation take each of its $(\sigma - \epsilon_1)$th derivatives, with the second equation each of its $(\sigma - \epsilon_2)$th derivatives, ... and with the ith equation each of its $(\sigma - \epsilon_i)$th derivatives.

There will thus come into existence $(\sigma + 1) i - (\epsilon_1 + \epsilon_2 + \ldots + \epsilon_i)$, that is, $i(\sigma + 1) - \sigma$ equations between the $i(\sigma + 1)$ quantities

$$\begin{array}{l} \omega_1, \quad \delta_t \omega_1, \ldots \delta_t^{\sigma} \omega_1, \\ \omega_2, \quad \delta_t \omega_2, \ldots \delta_t^{\sigma} \omega_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \omega_i, \quad \delta_t \omega_i, \ldots \delta_t^{\sigma} \omega_i. \end{array}$$

If we omit those which appear in any one of the lines above written, there will remain $(\sigma + 1)(i - 1)$ or $i(\sigma + 1) - \sigma - 1$ which might be eliminated between the $i(\sigma + 1) - \sigma$ equations, and there would thus result an equation between the quantities contained in the omitted line. This elimination, it will presently be seen, there is no occasion to perform; the noticeable algebraical fact about it is, that supposing it were performed, the form of the equation resulting between $\omega_k, \delta_t \omega_k, \ldots \delta_t^{\sigma} \omega_k$ is invariable, whichever of the numbers $1, 2, 3, \ldots i$ be the value assigned to k.

Let the order of the highest derivative of each ω be reduced by one unit below the highest order previously taken, then there will be $i\sigma - \sigma$ or $(i - 1)\sigma$ equations connecting the $i\sigma$ quantities

$$\begin{array}{l} \omega_1, \quad \delta_t \omega_1, \ldots \delta_t^{\sigma-1} \omega_1, \\ \omega_2, \quad \delta_t \omega_2, \ldots \delta_t^{\sigma-1} \omega_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \omega_i, \quad \delta_t \omega_i, \ldots \delta_t^{\sigma-1} \omega_i, \end{array}$$

and accordingly, if we omit the σ quantities which appear in any one (say the first) of the above lines, the remaining $(i - 1)\sigma$ quantities may each of them be expressed as linear functions of ω_1 and its $(\sigma - 1)$ derivatives: but the elimination previously indicated would lead to a homogeneous linear equation between ω_1 and its σ derivatives, and if in that, each argument $\delta_t^{\lambda} \omega_1$ be replaced by h^{λ} and $\lambda_1, \lambda_2, \ldots \lambda_{\sigma}$ be the σ roots of the algebraical equation so formed, it follows from the ordinary theory for a single equation that ω_1 (provided the given equations, and consequently the resulting ones, be left in their general form) will be of the form

$${}^1C_1 e^{h_1 t} + {}^1C_2 e^{h_2 t} + \ldots + {}^1C_{\sigma} e^{h_{\sigma} t},$$

and consequently by virtue of the previous remark $\omega_2, \omega_3, \ldots \omega_k$ will be of the same form as ω_1 (but, of course, with different coefficients), that is to say, the σ roots $h_1, h_2, \ldots h_{\sigma}$ are the same for the equation in σ_k as for the equation in σ_1, so that the coefficients in the equation between ω_k and its σ derivatives are, as premised, independent of the value of k.

Finally, to determine the equation whose roots are $h_1, h_2, \ldots h_{\sigma}$, let ${}^1Ce^{ht}$, one of the terms in the general value, be taken as a particular value of ω_1, which with corresponding values of the other ω's will serve to satisfy the given equations; $\omega_2, \omega_3, \ldots \omega_i$ being each of them linear functions of ω and derivatives of ω, must be of the forms ${}^2Ce^{ht}, {}^3Ce^{ht}, \ldots {}^iCe^{ht}$, so that $\omega_1, \omega_2, \ldots \omega_i$ and the derivatives of each of them will contain the common factor e^{ht}, and by substitution in the original equations we shall obtain a system of simultaneous algebraical equations leading to the equation

$$\begin{vmatrix} R_{1,1}, & R_{1,2} \ldots R_{1,i} \\ R_{2,1}, & R_{2,2} \ldots R_{2,i} \\ \ldots\ldots & \ldots\ldots \\ R_{i,1}, & R_{i,2} \ldots R_{i,i} \end{vmatrix} = 0,$$

where in general $R_{p,q}$ is what $u_{p,q}$ becomes on writing h^{μ} in place of $\delta_t^{\mu} \omega_q$.

The above determinant of the ith order will be of degree $\epsilon_1 + \epsilon_2 + \ldots + \epsilon_i$, that is, of the degree σ (for the general case) in h, and the roots of the equation will give the σ values $h_1, h_2, \ldots h_{\sigma}$.

It follows, therefore, that the result of the hypothetical elimination in the first instance referred to will be a linear function of $\delta_t^{\sigma} \omega_k, \delta_t^{\sigma-1} \omega_k, \ldots \delta_t \omega_k, \omega_k$ of which the coefficients will be identical with the coefficients of $h^{\sigma}, h^{\sigma-1}, \ldots h, 1$ in the above determinant. Hence no matter now what special values may be attributed to the coefficients of the given equations, the result last obtained remains of *universal* validity—without excepting those cases in which the result of the hypothetical elimination would be such that the corresponding algebraical equation possess equal roots, although in those cases the form assumed in the course of the argument for the value of ω_1 (namely, a linear function of exponentials) ceases to hold good. Neither for the same reason need any exception be made for those cases where the number of terms in the equation to ω_k falls below σ on account of one or more of the leading coefficients in the result of the hypothetical elimination becoming zero: the degree to which h rises in the determinant will be in all cases the right degree, whether it reaches the extreme possible limit σ or falls below it.

The result obtained may be briefly summarized as follows.

If

$$\begin{array}{l} (\phi_1 \delta_t) x + (\phi_2 \delta_t) y + \ldots + (\phi_i \delta_t) z = 0, \\ (\psi_1 \delta_t) x + (\psi_2 \delta_t) y + \ldots + (\psi_i \delta_t) z = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (\omega_1 \delta_t) x + (\omega_2 \delta_t) y + \ldots + (\omega_i \delta_t) z = 0, \end{array}$$

(each $\phi, \psi, \ldots, \omega$ standing for a rational-integral functional form) then will

$$(R\delta_t) x = 0, \quad (R\delta_t) y = 0, \ldots (R\delta_t) z = 0,$$

where $R(\delta_t)$ is the resultant in respect to $x, y, \ldots z$ of what the above equations become when δ_t is treated as an ordinary algebraical quantity; under which form the proposition (by virtue of Euler's method of multipliers) becomes so nearly intuitive as to abrogate all necessity for any other demonstration*.

To pass to the parallel and more important theory in finite differences, it is only necessary to interpret $u_{j,k}$ to signify a linear function, with constant coefficients, of $(\omega_k)_t, (\omega_k)_{t+1}, \ldots (\omega_k)_{t+\epsilon_j}$, where t is the integer independent variable, (*say* $(\omega_k)_t$ and its ϵ_j difference-augmentatives), and instead of taking the differential derivatives of any one of the given equations, to take the corresponding difference-augmentatives. Then by precisely the same reasoning as before we shall have

$$\omega_{t+\sigma} + B\omega_{t+\sigma-1} + \ldots + L\omega_t = 0,$$

$B, C, \ldots L$ being so taken as that $h^\sigma + Bh^{\sigma-1} + \ldots + L$ shall be the determinant represented by the same form of matrix expressed by R's as before, but where $R_{p,q}$ is obtained from $u_{p,q}$ by writing h^θ in lieu of any argument $\omega_t + \theta$ which occurs in it.

The simplest example that can be given is where $i = 2$, $\epsilon_1 = \epsilon_2 = 1$,

$$u_{1,1} = -\eta_{t+1} + a\eta_t, \qquad u_{1,2} = b\theta_t,$$
$$u_{2,1} = c\eta_t \qquad\qquad u_{2,2} = -\theta_{t+1} + d\theta_t;$$

this was the case which occurred in the article on the extension of Tchebycheff's theorem, in the last number of the *Journal* [p. 530, above], leading to the equation

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0,$$

and to expressions for η_t, θ_t as linear functions of λ_1^t, λ_2^t.

It may also be remarked that this same case gives an instantaneous solution of the problem proposed and successfully treated by Babbage in his *Calculus of Functions*, more than half a century ago, and since revived in connection with the theory of substitutions (Serret, *Alg. Sup.* 4 ed., tom. 2, pp. 256—262). The problem is to find $\phi x = \dfrac{ax+\alpha}{\beta x + b}$ so that $\phi^i x$, say $\dfrac{a_i x + \alpha_i}{\beta_i x + b_i}$, shall equal x for a given value of i.

* I regret that this simple reflection did not present itself to my mind before the preceding investigation, the necessity for which it does away with, had been set up in print. It of course applies equally well to the analogous proposition for finite-difference equations ($u_t, v_t, \ldots$ being substituted for $x, y \ldots$, and $1+\Delta$ for δ_t). This last named proposition, limited to the case of equations of the first order, is the foundation-stone of my new theory of Matrices regarded as Quantities, that is, as subject to every kind of functional operation which ordinary arithmetical or algebraical quantities are or can be subject to: but though so important and so easily established, I know not where it can be found explicitly stated.

To find in general $\phi^i x$ it is only necessary to solve the difference equations

$$u_i = a u_{i-1} + \alpha v_{i-1},$$
$$v_i = \beta u_{i-1} + b v_{i-1},$$

and then u_i, v_i will, if $u_0 = 1$, $v_0 = 0$, coincide with a_i, β_i, and if $u_0 = 0$, $v_0 = 1$ with α_i, b_i.

Thus calling ρ_1, ρ_2 the two roots of

$$\begin{vmatrix} -\rho + a & \alpha \\ \beta & -\rho + b \end{vmatrix} = 0,$$

α_i will be of the form $C(\rho_1^i - \rho_2^i)$ and β_i of the same form except as to C, say $\Gamma(\rho_1^i - \rho_2^i)$. Also a_i, b_i will be of the forms $C_1\rho_1^i + C_2\rho_2^i$, $\Gamma_1\rho_1^i + \Gamma_2\rho_2^i$, where $C_1 + C_2 = 1$, $\Gamma_1 + \Gamma_2 = 1$, and the required condition will be fulfilled, provided only that $\rho_1^i = \rho_2^i$, or say

$$\rho_1 = K\left(\cos\frac{\lambda\pi}{i} + \sqrt{(-1)}\sin\frac{\lambda\pi}{i}\right)$$
$$\rho_2 = K\left(\cos\frac{\lambda\pi}{i} - \sqrt{(-1)}\sin\frac{\lambda\pi}{i}\right)$$

that is, if $(a+b)^2 - 4(ab - \alpha\beta)\left(\cos\dfrac{\lambda\pi}{i}\right)^2 = 1$, λ having any integer value (which without loss of generality may be taken inferior to i) except zero*.

If $\lambda = 0$, the two roots of the equation in ρ become equal and the form of the solution changes into

$$u_i = (C_1 + C_2 i)\rho^i, \qquad v_i = (C_1' + C_2' i)\rho^i.$$

When $u_0 = 1$ and $v_0 = 0$ then $u_1 = a$, $v_0 = \beta$,

$$C_1 = 1,\ C_1' = 0, \qquad C_2 = \frac{a}{\rho} - 1,\ C_2' = \frac{\beta}{\rho},$$

and when $u_0 = 0$, $v_0 = 1$, $u_1 = \alpha$, $v_0 = b$,

$$C_1 = \frac{\alpha}{\rho},\ C_1' = \frac{b}{\rho} - 1, \qquad C_2 = 0,\ C_2' = 1,$$

and $\phi^i x = \dfrac{[\rho + i(a-\rho)]x + i\alpha}{i\beta x + \rho + (b-\rho)i}$, which cannot be periodic for any value of i, and when $i = \infty$ becomes

$$\frac{(a-\rho)x + \alpha}{\beta x + b - \rho} = \frac{a-\rho}{\beta} = \frac{\alpha}{b-\rho}, \text{ that is, } = \frac{a-b}{2\beta} \text{ or } \frac{2\alpha}{a-b},$$

so that $\phi^i x$ in this case continually converges to a constant limit.

I may add that $\phi^i x$ converges to a constant limit not merely when the roots ρ_1, ρ_2 of

$$\begin{vmatrix} a-\rho & \alpha \\ \beta & b-\rho \end{vmatrix}$$

* There will thus be $(i-1)$ values of λ which will each give a distinct admissible solution of the problem of periodicity, but of course only those values of λ which are relatively prime to i will give primitive solutions. If $i = i'\delta$ the effect of making $\lambda = \lambda'\delta$ will be to make $\phi^i x = x$ by virtue o its making $\phi^{i'} x = 0$.

are equal, but whenever they are real. For the general form of $\phi^i x$, it may easily be found, is

$$\frac{[(\rho_2 - a)\rho_1^i + (\rho_1 - a)\rho_2^i]\,x + \alpha(\rho_1^i - \rho_2^i)}{\beta(\rho_1^i - \rho_2^i)\,x + [(\rho_2 - b)\rho_1^i + (\rho_1 - b)\rho_2^i]},$$

which if $\rho_2 > \rho_1$ when $i = \infty$ becomes $\frac{(\rho_1 - a)x - \alpha}{-\beta x + \rho_1 - b} = \frac{a - \rho_1}{\beta}$ or $\frac{\alpha}{b - \rho_1}$ where ρ_1 signifies the smaller of the two roots ρ_1, ρ_2; or in other words when $a - b > 2\sqrt{(\alpha\beta)}$, the limiting value to $\phi^i x$, when ϕx represents $\frac{ax + \alpha}{\beta x + b}$, is $\frac{(a-b) + \sqrt{\{(a-b)^2 - 4\alpha\beta\}}}{2\beta}$, with the understanding that the quantity under the radical sign is to be taken positive.

So, if

$$x_{i+1} : y_{i+1} : z_{i+1} = ax_i + by_i + cz_i : a'x_i + b'y_i + c'z_i : a''x_i + b''y_i + c''z_i,$$

when all the roots of the determinant

$$\begin{vmatrix} a - \lambda & b & c \\ a' & b' - \lambda & c' \\ a'' & b'' & c'' - \lambda \end{vmatrix}$$

are real, the point x_i, y_i, z_i, as i increases, will be found to approach indefinitely near to a fixed straight line; and if all the roots are equal, to a fixed point.

The condition of the system of ratios $x_i : y_i : z_i$ being periodic and having a period m is tantamount to the condition that the mth power of the matrix

$$\begin{matrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{matrix}$$

shall be the matrix

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1. \end{matrix}$$

The complete solution of this problem, and of the more general one of extracting the mth root of any unit-matrix (that is, a matrix in which each element in the principal diagonal is unity, and the rest zero), which constitutes the ultimate generalization of Babbage's problem and is soluble by the same method, will probably appear in a memoir on matrices, in the forthcoming number of the *Journal*.

In general, for a matrix of the order ω, the number of mth roots is m^ω and each of them is perfectly determinate. But when the matrix is a unit-matrix or a zero-matrix (the latter meaning one in which every element is zero) there are distinct genera and species of such roots, and every species contains its own appropriate number of arbitrary constants.

64.

NOTE ON MECHANICAL INVOLUTION.

[*American Journal of Mathematics*, IV. (1881), pp. 336—340.]

MECHANICAL involution is the name invented by me to signify the relation between six lines in space, so situated that forces may be made to act along them whose statical sum is zero. The definition may be extended to comprise an indefinite number of lines, any six of which have this property.

I shall use $[p, q]$ for the present to denote the moment of a unit of force acting along the directed line p about the directed line q, taken positive or negative according as to a spectator looking in the given direction (or sense) of q, a force in the given direction (or sense) of p tends to produce a right-handed or a left-handed rotation, which tendency, by a property of our mental constitution, we know is not affected in kind by the lines p and q becoming interchanged—a fact which might also be anticipated with a high degree of probability from the circumstance that the unit-moment is measured by the product of the perpendicular distance from each other, of the two lines, multiplied by the sine of the angle between them, so that *each* factor of this product changes its sign when the relation or aspect of the two lines to each other is reversed. Hence it follows that $[p, q] = [q, p]$.

Three lines in a plane, it may be noticed, are in involution when they intersect in the same point, or, as a particular case, are parallel to each other.

Let a, b, c, d, e, f be any six lines in space, $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ six forces capable of balancing when acting along the lines 1, 2, 3, 4, 5, 6 supposed to be in involution.

Then by the equation of moments in regard to each of the first series of lines taken successively as axes of rotation, we must have

$$\lambda_1[1, a]+\lambda_2[2, a]+\lambda_3[3, a]+\lambda_4[4, a]+\lambda_5[5, a]+\lambda_6[6, a]=0$$
$$\lambda_1[1, b]+\ldots\ldots\ldots\ldots\ldots\ldots+\lambda_6[6, b]=0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\lambda_1[1, f]+\lambda_2[2, f]+\lambda_3[3, f]+\lambda_4[4, f]+\lambda_5[5, f]+\lambda_6[6, f]=0$$

and consequently the determinant

$$\begin{vmatrix} [1, a] \ldots\ldots\ldots\ldots\ldots\ldots [6, a] \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ [1, f] \ldots\ldots\ldots\ldots\ldots\ldots [6, f] \end{vmatrix} = 0.$$

Consequently we may find quantities $\mu_a, \mu_b, \mu_c, \mu_d, \mu_e, \mu_f$ such that

$$\mu_a[a, 1]+\mu_b[b, 1]+\mu_c[c, 1]+\mu_d[d, 1]+\mu_e[e, 1]+\mu_f[f, 1]=0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\mu_a[a, 6]+\mu_b[b, 6]+\mu_c[c, 6]+\mu_d[d, 6]+\mu_e[e, 6]+\mu_f[f, 6]=0.$$

Thus it becomes evident by regarding $\mu_a, \mu_b, \mu_c, \mu_d, \mu_e, \mu_f$ as the magnitudes of forces acting along the lines a, b, c, d, e, f, that the equations of moments of a given set of forces about six lines which are in general independent, become linearly related when the six axes are in involution—a conclusion which springs also immediately from the consideration that the law of statical composition of directed lengths is the same whether they be regarded as representing forces or as representing the axes of couples. So much by way of introduction.

I now pass to the formation of the intrinsic equation of condition to be satisfied in the case of involution.

To obtain this, let the lines a, b, c, d, e, f be made identical with 1, 2, 3, 4, 5, 6.

In each of these latter lines (say in i) let two points be taken at the distance $\frac{1}{l_i}$ apart, whose quadriplanar coordinates are respectively $i_x, i_y, i_z, i_t, i_x', i_y', i_z', i_t'$, and let (i, j)—where j is another of the lines in involution—denote the determinant

$$\begin{vmatrix} i_x & i_y & i_z & i_t \\ i_x' & i_y' & i_z' & i_t' \\ j_x & j_y & j_z & j_t \\ j_x' & j_y' & j_z' & j_t' \end{vmatrix}$$

This determinant will represent (enlarged six-fold) a tetrahedron, two of whose opposite edges are the lengths intercepted between the pairs of points on i, j respectively, and consequently $l_i l_j (i, j)$ will serve to represent (on the same scale) the quantities previously represented by $[i, j]$.

Hence the determinant of the sixth order above written becomes

$$\begin{vmatrix} 0\ , & l_1 l_2(1, 2), & l_1 l_3(1, 3), & l_1 l_4(1, 4), & l_1 l_5(1, 5), & l_1 l_6(1, 6) \\ l_2 l_1(2, 1), & 0\ , & l_2 l_3(2, 3), & l_2 l_4(2, 4), & l_2 l_5(2, 5), & l_2 l_6(2, 6) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ l_6 l_1(6, 1), & l_6 l_2(6, 2), & l_6 l_3(6, 3), & l_6 l_4(6, 4), & l_6 l_5(6, 5), & 0 \end{vmatrix}$$

and this equated to zero gives the intrinsic condition of involution.

Imagining this equation to be formed, the terms in each line and also the terms in each column will have some common factor, removing which, by a two-fold scheme of division, all the quantities l will disappear, so that now regarding each of the pairs of points on the lines 1, 2, 3, 4, 5, 6 respectively as *any two non-coincident points whatever*, the intrinsic condition is represented by the evanescence of the following symmetrical invertebrate (that is, zero-axial) compound determinant

$$\begin{vmatrix} 0 & (1, 2) & (1, 3) & (1, 4) & (1, 5) & (1, 6) \\ (2, 1) & 0 & (2, 3) & (2, 4) & (2, 5) & (2, 6) \\ (3, 1) & (3, 2) & 0 & (3, 4) & (3, 5) & (3, 6) \\ (4, 1) & (4, 2) & (4, 3) & 0 & (4, 5) & (4, 6) \\ (5, 1) & (5, 2) & (5, 3) & (5, 4) & 0 & (5, 6) \\ (6, 1) & (6, 2) & (6, 3) & (6, 4) & (6, 5) & 0 \end{vmatrix}$$

where each pair of numbers within a parenthesis represents a determinant of the fourth order*.

Just as the equations of moments of a system of forces about six lines in space are in general independent, but cease to be so if (and only if) these lines are in involution, so the equations of moments of a system of forces in a plane about three points are in general independent, and only cease to be so when the three points lie in a right line. Thus under the two-fold aspect of a system of force-directions and a system of axes of moments, six lines in involution in space are on the one hand the analogues of three force-directions in a plane in involution, that is, meeting in a point, and on the other hand of three points (centres of moments) lying in a right line; and as *concurrence* is the polar correlative to *collineation* we ought to expect to

* This determinant (which is sufficiently obvious, I have found since going to press) has been given by Professor Cayley in his memoir on line-coordinates, *Camb. Phil. Trans.*, 1861, which is avowedly based upon my constructions connected with the problem of Involution.

find involution in space to be its own polar correlative; that is, that the polar reciprocal of a system of lines in involution in respect to a general quadric should be another such system: and such is the fact: for, as I have shown in the *Comptes Rendus**, the necessary and sufficient condition of six lines being in involution is that they shall respectively intersect pairs of corresponding rays in two homographic pencils lying in two planes whose intersection contains the centres and two corresponding (coincident) rays of the two pencils—a condition which will not be affected by any polar transformation.

This leads to the remark that we may change the signification of the symbol (i, j) in the equation last indicated without destroying its validity as the condition of involution: namely, we may suppose two planes to be drawn through each line instead of two points being fixed upon it: and then if we understand by the determinant of two lines in space the determinant formed by the coefficients of the two pairs of equations which denote the lines, we may interpret (i, j) to mean the determinant of i, j and sum up the result obtained in the following proposition:

The determinants formed by six lines in involution, taken two and two together, are related in precisely the same manner as the squared distances from one another of six points in four-dimensional space.

The legitimacy of the second reading of (i, j) may be proved directly, as follows. For greater clearness let (i, j) when read with reference to pairs of planes through i and j, be called (I, J). Then

$$\begin{matrix} i_x & i_y & i_z & i_t \\ i_x' & i_y' & i_z' & i_t' \\ I_x & I_y & I_z & I_t \\ I_x' & I_y' & I_z' & I_t' \end{matrix}$$

will constitute an example of what in the *Johns Hopkins University Circular* for May, 1882†, I have called a *split matrix*, inasmuch as each of the first two lines multiplied term for term by each of the latter two gives

[* Vol. II. of this Reprint, p. 237.]

† Baltimore: John Murphy & Co.—It is interesting to notice (as there indicated) that the same theory of the split matrix here applied to mechanical involution has an important, although quite a different kind of bearing on the theory of algebraical involution. The two theories of involution have a considerable affinity to each other—groundforms and their coefficients in the equation of linear connection in the one theory, being regarded as the analogues of space-directions and the force-magnitudes acting along them in the other. (See *J. H. U. Circular*, June, 1882.) It was the sense of this connection which caused me to throw a retrospective glance on the theory of mechanical involution, abandoned by me since the remote date of the appearance of my papers on the subject in the *Comptes Rendus*. I ought to mention that I owe the idea of applying the split-matrix theory to the proof of the polar property of an involution-system, to a suggestion of Professor Cayley.

products whose sum is zero. Hence by virtue of the property of such a matrix, each complete minor of the upper pair will bear to the opposite complete minor in the lower pair the ratio of (i) to (I), where

$$(i)^2 = \begin{vmatrix} \Sigma i_x^2 & \Sigma i_x i_x' \\ \Sigma i_x i_x' & \Sigma i_x'^2 \end{vmatrix} \text{ and } (I)^2 = \begin{vmatrix} \Sigma I_x^2 & \Sigma I_x I_x' \\ \Sigma I_x I_x' & \Sigma I_x'^2 \end{vmatrix},$$

and of course the same conclusions apply *mutatis mutandis* when j, J take the place of i, I; from which it immediately follows that

$$(i, j) : (I, J) = (i)(j) : (I)(J).$$

Let now in the (i, j) determinant, which is equated to zero, each element in any θth column be multiplied by $\frac{I_\theta}{i_\theta}$, and then again each element in any θth row by the same; these multiplications will not affect the equality to zero of the determinant so modified, but the effect of the combined multiplications will be to change the element in the ith row and jth column, namely, (i, j), into $\frac{(I)(J)}{(i)(j)}(i, j)$, that is into (I, J). Thus it is proved that we may pass from the first reading of the (i, j) determinant to the second; and this in its turn serves to prove that if six lines are in involution their polars in respect to any quadric must also be in involution.

The theory of involution may of course be extended to a system of $\frac{n(n+1)}{2}$ lines in n-dimensional space.

65.

SUR LES PUISSANCES ET LES RACINES DE SUBSTITUTIONS LINÉAIRES.

[*Comptes Rendus*, XCIV. (1882), pp. 55—59.]

On sait ce que veut dire un déterminant de substitution. Ces déterminants ne diffèrent nullement, dans leur forme extérieure, des déterminants ordinaires, que l'on peut nommer *déterminants absolus*, mais les lois de combinaison ne sont pas les mêmes dans les deux cas. Ainsi, par exemple, l'inverse du déterminant absolu $\begin{vmatrix} a & \alpha \\ \beta & b \end{vmatrix}$ est

$$\begin{vmatrix} \frac{b}{\Delta} & -\frac{\beta}{\Delta} \\ -\frac{\alpha}{\Delta} & \frac{a}{\Delta} \end{vmatrix}$$

où

$$\Delta = ab - \alpha\beta,$$

tandis que pour ce même déterminant, envisagé comme déterminant de substitution, l'inverse est

$$\begin{vmatrix} \frac{b}{\Delta} & -\frac{\alpha}{\Delta} \\ -\frac{\beta}{\Delta} & \frac{a}{\Delta} \end{vmatrix},$$

et ainsi, en général, l'inverse d'un déterminant de substitution est ce que l'on peut nommer le *transversal* de l'inverse d'un déterminant absolu, c'est-à-dire ce que ce déterminant devient quand, en prenant la diagonale qui joint le premier au dernier terme comme axe, on fait décrire à l'inverse ordinaire une demi-révolution autour de cet axe. De même pour la multiplication de deux déterminants de substitutions A et B, chacun de l'ordre n; pour obtenir le produit de A par B, il faut multiplier ensemble le transversal de A par B, selon la règle ordinaire, ce qui donnera un déterminant C'; C, le transversal de C', sera le produit de la substitution A par la substitution B.

Ainsi, tandis que le carré d'un déterminant absolu quelconque est un déterminant symétrique, le carré d'un déterminant non symétrique de substitution reste asymétrique.

Soit un déterminant quelconque donné, et ajoutons le terme $-\lambda$ à chaque terme diagonal; on obtient ainsi une fonction de λ; je nomme les racines de cette fonction racines *lambdaïques* du déterminant donné, et j'obtiens facilement les deux théorèmes suivants:

(1) *Les racines* lambdaïques *de l'inverse d'un déterminant sont les réciproques des racines* lambdaïques *du déterminant lui-même.*

(2) *i étant un nombre entier et positif quelconque, les $i^{èmes}$ puissances des racines* lambdaïques *d'un déterminant* de substitution *sont identiques avec les racines* lambdaïques *de la puissance $i^{ème}$ du déterminant.*

En réunissant ces deux énoncés, on parvient à ce théorème plus général:

i étant une quantité commensurable quelconque, les $i^{èmes}$ puissances des racines lambdaïques *d'un déterminant de substitution sont identiques avec les racines* lambdaïques *de $i^{ème}$ puissance du déterminant.*

Si le déterminant est symétrique, on n'a pas besoin de le définir comme représentant une substitution, car, pour les déterminants symétriques (qu'ils soient envisagés comme absolus ou comme substitutifs), les lois d'opération deviennent identiques.

Avec l'aide du théorème sur les racines lambdaïques, je parviens facilement à la résolution de ce beau problème:

Extraire la racine $\mu^{ième}$, ou plus généralement trouver la puissance $i^{ième}$ d'une substitution donnée, i étant un nombre commensurable quelconque.

Voici la solution. Soit n l'ordre du déterminant de substitution donné.

Soient K un terme quelconque dans ce déterminant, K_θ le terme qui occupe, dans la puissance $\theta^{ième}$ du déterminant, la même position que K dans le déterminant lui-même. De plus, soient $K_0 = 1$ quand K est un terme dans la diagonale, et $K_0 = 0$ dans tout autre cas. Alors je dis que, pour une valeur commensurable quelconque de i, positive ou négative, en nommant la somme des quantités $\lambda_2, \lambda_3, \ldots, \lambda_n$, S_1, leur produit, S_{n-1}, et en général la somme de leurs combinaisons binaires, ternaires, etc., S_2, S_3, ... on aura

$$K_i = \Sigma \frac{K_{n-1} - S_1 . K_{n-2} + S_2 . K_{n-3} - \ldots \pm S_{n-2} . K_1 \mp S_{n-1} . K_0}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\ldots(\lambda_1 - \lambda_n)} \lambda_1^{\,i},$$

où $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ sont les racines lambdaïques du déterminant donné.

Si l'on fait $i = \frac{1}{\mu}$, où μ est un nombre entier, on voit que le nombre des $\mu^{ièmes}$ racines est μ^n et consistera en μ^{n-1} groupes de μ matrices pour chaque groupe, ou pour le même groupe on passe d'une matrice à une autre, en multipliant chacun des n^2 éléments qu'il contient par la même racine $\mu^{ième}$ de l'unité.

Il peut arriver que les racines *lambdaïques* du déterminant ne soient pas toutes inégales; alors la formule générale pour K_i subira une modification qu'on déduit facilement du théorème général, au moyen de l'introduction de différences infinitésimales entre les racines.

Il y a cependant un cas très particulier qu'on ne doit pas manquer de signaler: c'est le cas où le nombre de solutions devient infini pour une valeur finie de i, où, en effet, le problème à résoudre devient un véritable porisme; dans ce cas, des n^2 quantités qu'on cherche, $n^2 - n$, c'est-à-dire tous les termes qui ne sont pas en diagonale, restent absolument arbitraires. C'est le cas où le déterminant donné est de la forme la plus simple possible, c'est-à-dire où tous les termes qui ne se trouvent pas dans la diagonale du déterminant donné sont des zéros, et tous les termes qui sont dans la diagonale égaux entre eux. Pour plus de clarté, supposons que tous les termes qui ne disparaissent pas sont des *unités*.

(1) Pour que le problème soit résoluble, il faut que μ ne soit pas moindre que n.

(2) μ n'étant pas inférieur à n, la seule condition nécessaire et suffisante pour que la $\mu^{\text{ième}}$ puissance du déterminant Δ soit de la forme proposée est que les *racines lambdaïques* de Δ soient égales respectivement à μ *racines distinctes* (choisies à volonté) de l'unité.

Par exemple, si $n = 2$, pour que la $\mu^{\text{ième}}$ puissance de la substitution $\begin{vmatrix} a & \alpha \\ \beta & b \end{vmatrix}$ soit de la forme $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, on n'a qu'à faire les racines de

$$\lambda^2 - (a+b)\lambda + ab - \alpha\beta = 0,$$

égale à $$\cos\frac{2r\pi}{\mu} + i\sin\frac{2r\pi}{\mu}, \quad \cos\frac{2s\pi}{\mu} + i\sin\frac{2s\pi}{\mu},$$

respectivement.

Si l'on veut seulement que la $\mu^{\text{ième}}$ puissance de $\begin{vmatrix} a & \alpha \\ \beta & b \end{vmatrix}$ soit de la forme $\begin{vmatrix} A & 0 \\ 0 & A \end{vmatrix}$, A étant arbitraire, il suffira que les deux racines de λ soient dans le rapport de 1 à une $\mu^{\text{ième}}$ racine imaginaire quelconque de l'unité, de sorte qu'on peut mettre

$$\lambda_1 = k\left(\cos\frac{r\pi}{\mu} + i\sin\frac{r\pi}{\mu}\right),$$

$$\lambda_2 = k\left(\cos\frac{r\pi}{\mu} - i\sin\frac{r\pi}{\mu}\right),$$

ce qui donne pour la seule condition nécessaire et suffisante

$$(a+b)^2 = 4\left(\cos\frac{r\pi}{\mu}\right)^2 (ab - \alpha\beta).$$

C'est la solution bien connue du problème soulevé et résolu par le célèbre M. Babbage, dans son traité *Sur le Calcul des Fonctions: Trouver*

$$\phi(x) = \frac{ax + \alpha}{bx + \beta},$$

tel que $\phi^\mu(x) = x$. La même question a été bien plus récemment considérée de nouveau par M. Serret (voir son *Cours d'Algèbre supérieure*, t. II. pp. 356—362).

66.

SUR LES RACINES DES MATRICES UNITAIRES.

[*Comptes Rendus*, XCIV. (1882), pp. 396—399.]

Une matrice dont les termes sont tous des zéros, sauf toutefois les termes de la diagonale principale, qui ont des unités, constitue ce que je nomme une *matrice unitaire*.

Je suppose une telle matrice (assujettie à la loi de multiplication donnée par la combinaison des substitutions linéaires) de l'ordre n. On peut demander quelle est la forme d'une autre matrice M du même ordre n, telle que la $i^{\text{ème}}$ puissance de M soit une matrice unitaire.

J'ai donnée une solution de cette question dans ma précédente Note*.

Cette solution n'exige que n conditions, qui doivent être remplies par n^2 éléments de M; mais, chose remarquable, ce n'est pas la solution la plus générale. Je vais à présent donner toutes les solutions dont la question est susceptible. Soient $\nu_1, \nu_2, \nu_3, \ldots, \nu_k$ des nombres entiers et positifs quelconques dont la somme est n, et $\rho_1, \rho_2, \ldots, \rho_k$, k quelconques des $i^{\text{èmes}}$ racines de l'unité. Soit M_λ la matrice M affectée de l'indice λ, c'est-à-dire modifiée par l'addition de $-\lambda$ à chacun des n termes de la diagonale.

Considérons les systèmes de matrices mineurs de M, de l'ordre

$$n - \nu_1 + 1,\ n - \nu_2 + 1, \ldots,\ n - \nu_k + 1$$

respectivement; et prenons M tel que

$$\lambda - \rho_1,\ \lambda - \rho_2, \ldots,\ \lambda - \rho_k$$

soient facteurs de chaque mineur du premier, du second, ..., du $k^{\text{ième}}$ de ces systèmes respectivement; alors M sera une racine $i^{\text{ème}}$ de la matrice unitaire de l'ordre n.

Ainsi, pourvu que i soit égal ou supérieur à n, il y aura autant de *genres* de racines $i^{\text{èmes}}$ de cette matrice qu'il y a de partitions indéfinies de n.

[* p. 562, above.]

Le genre principal (*summum genus*) sera celui qui correspond à la partition de n en n unités, et le nombre de conditions exigées sera n.

Le genre le plus bas (*infimum genus*) sera celui où n est laissé sans décomposition, et le nombre de conditions pour ce cas sera n^2.

En général, à $n = \nu_1 + \nu_2 + \ldots + \nu_k$ on aura une solution pour laquelle le nombre de conditions exigées sera $\nu_1^2 + \nu_2^2 + \ldots + \nu_k^2$, de sorte qu'il restera $2\Sigma \nu_1 \nu_2$ constantes arbitraires dans M.

Si i est moindre que n, quelques-uns des genres manqueront, mais il y en aura toujours quelques-uns qui resteront. Ainsi, par exemple, si $n=3$ et $i=2$, le *summum genus*, qui suppose l'existence de trois racines distinctes des racines quadratiques de l'unité, cesse nécessairement d'exister; mais on aura une valeur de M pour laquelle tous les mineurs premiers de M_λ contiendront le facteur $\lambda - 1$, et une autre valeur de M (du même genre) pour laquelle tous ces déterminants contiendront le facteur $\lambda + 1$.

En effet, la matrice trouvée par M. Cayley, dans son Mémoire sur les matrices (*Philosophical Transactions*, 1858),

$$\begin{vmatrix} \dfrac{\alpha}{\alpha+\beta+\gamma}, & \dfrac{-(\beta+\gamma)}{\alpha+\beta+\gamma}\dfrac{\nu}{\mu}, & \dfrac{-(\beta+\gamma)}{\alpha+\beta+\gamma}\dfrac{\nu}{\mu} \\ -\dfrac{(\gamma+\alpha)}{\alpha+\beta+\gamma}\dfrac{\mu}{\nu}, & \dfrac{\beta}{\alpha+\beta+\gamma}, & \dfrac{-(\gamma+\alpha)}{\alpha+\beta+\gamma}\dfrac{\lambda}{\mu} \\ -\dfrac{\alpha+\beta}{\alpha+\beta+\gamma}\dfrac{\mu}{\nu}, & \dfrac{-(\alpha+\beta)}{\alpha+\beta+\gamma}\dfrac{\nu}{\lambda}, & \dfrac{\gamma}{\alpha+\beta+\gamma} \end{vmatrix},$$

sera la matrice M, telle que chaque mineur de M_ρ contiendra $(\rho-1)$; de même chaque mineur de $(-M)_\rho$ contiendra $\rho+1$; on remarquera que 1 et -1 sont les racines carrées de l'unité, et l'on vérifiera aisément que M^2 ou, ce qui revient au même, $\Phi(-M)^2$ ont tous les deux la forme

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

Le genre *infime* de solution sera

$$M = \begin{vmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{vmatrix}$$

où $$p^2=1, \quad q^2=1, \quad r^2=1.$$

Il y a une théorie analogue pour l'extraction des racines de la matrice *zéroïdale*, c'est-à-dire où tous les termes de la matrice sont des zéros, ce qui constitue encore un nouveau cas de porisme dans la théorie de l'extraction des racines des matrices.

Je n'entrerai pas dans les détails de cette question: il suffit de l'indiquer par le cas le plus frappant; je dis que, si M est une matrice de l'ordre n telle que le déterminant de M_ρ soit de la forme ρ^n (ce qui n'exige que la satisfaction de n conditions entre les n^2 termes de M), M^n sera une matrice zéroïdale. Ainsi, par exemple,

$$\begin{vmatrix} a & a\dfrac{\lambda}{\mu} \\ -a\dfrac{\mu}{\lambda} & -a \end{vmatrix}^2 = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}.$$

De même, comme solution particulière du cas de $n=3$, on trouve que si 1, ρ, ρ^2 sont les trois racines de l'unité,

$$\begin{vmatrix} (\rho-\rho^2)(c-b) & \dfrac{\mu}{\lambda}(a+\rho b+\rho^2 c) & -\dfrac{\nu}{\mu}(a+\rho^2 b+\rho c) \\ -\dfrac{\lambda}{\mu}(a+\rho^2 b+\rho c) & (\rho-\rho^2)(a-c) & \dfrac{\nu}{\mu}(a+\rho b+\rho^2 c) \\ \dfrac{\lambda}{\mu}(a+\rho b+\rho^2 c) & -\dfrac{\mu}{\lambda}(a+\rho^2 b+\rho c) & (\rho-\rho^2)(b-a) \end{vmatrix}^2 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

Je terminerai en ajoutant que j'ai déjà établi une théorie fonctionnelle générale des matrices, et que je ne regarde plus celles-ci comme des schemata d'éléments, mais comme des communautés, ou, si l'on veut, comme des quantités complexes.

Cette théorie n'est pas même bornée au cas de matrices simples. On peut faire subir à des lois générales d'Analyse les quantités complexes où chaque terme d'un complexe de l'ordre m est lui-même un complexe de l'ordre m', et chaque élément de ces nouveaux complexes encore un complexe de l'ordre m'', etc., de sorte qu'on a des complexes de rangs successifs qu'on peut prolonger indéfiniment.

67.

ON SUBINVARIANTS, THAT IS, SEMI-INVARIANTS TO BINARY QUANTICS OF AN UNLIMITED ORDER.

[*American Journal of Mathematics*, v. (1882), pp. 79—136.]

Er macht kein System, sondern es wird, es concrescirt in ihm, wie das Kind im Mutterleibe. (Schopenhauer) *Deutsche Rundschau*, July, 1882, p. 69.

§ 1. Proem.

Any rational integer function ϕ of the letters $a, b, c, \ldots$ indefinitely continued, which satisfies the partial differential equation

$$(a\delta_b + 2b\delta_c + 3c\delta_d \ldots)\,\phi = 0$$

may be termed a subinvariant in respect to the elements $a, b, c, \ldots$ or simply a subinvariant to or *quâ* those elements. It follows from this definition that any rational integer function of one or more subinvariants is itself one.

The same function of the letters $a, b, c, \ldots$ which, when regarded as the coefficient of the highest power of the first variable x in a covariant to the quantic $(a, b, c, \ldots \between x, y)^i$ or the polynomial $(a, b, c, \ldots \between x, 1)^i$ is termed a differenciant of the quantic or polynomial, when regarded as an individual of the infinite scale to which ϕ belongs, assumes the name of a subinvariant in respect to the letters $a, b, c, \ldots$.

Of course a differenciant derives its name from reference to the fact that when multiplied by a suitable power of a it may be regarded as a function of the differences of the roots of any one of the infinite series of polynomials, of some covariant of each of which it is the principal coefficient.

It follows also from the definition that if any composite function is a subinvariant, each of its factors must be so too. For if the function be $P^\alpha . Q^\beta . R^\gamma \ldots$ writing $a\delta_b + 2b\delta_c + \ldots = E$, we must have

$$\alpha\frac{EP}{P} + \beta\frac{EQ}{Q} + \gamma\frac{ER}{R} + \ldots = 0,$$

which for denominators $P, Q, R, \ldots$ relatively prime to each other is obviously impossible unless $EP = 0$, $EQ = 0$, $ER = 0 \ldots$, that is, $P, Q, R \ldots$ are subinvariants.

Again, suppose U, V, Ω to be three subinvariants so related that the equation $XU + VU = \Omega$ is capable of being satisfied at all. I say that it must be capable of being satisfied by subinvariantive values of X, Y*.

For from the equation it follows that $EX . U + EY . V = 0$, of which the most general solution is

$$EX = K\left(\frac{V}{\Delta}\right), \quad EY = -K\left(\frac{U}{\Delta}\right).$$

Hence $$X = \left(\frac{V}{\Delta}\right)E^{-1}K + U_1, \quad Y = -\left(\frac{U}{\Delta}\right)E^{-1}K + V_1,$$

where U_1, V_1 are subinvariants. Substituting these values of X, Y in the original equation, there results $U_1U + V_1V = \Omega$, as was to be shown possible. The same or a similar manner of proof will serve to show that if for three functions U, V, W, $XU + YV + ZW = 0$, X, Y, Z, are, or may be replaced by subinvariants. I do not know for certain, but think that the proposition may be extended to any number of given functions $U, V, W, \ldots$.

It is scarcely necessary to add the fundamental theorem that if for the elements $a, b, c, \ldots$ be substituted the elements a, $a\lambda + b$, $a\lambda^2 + 2\lambda b + c, \ldots$ where λ is arbitrary, any subinvariant remains unchanged; the proof being that if such a change be made in the elements of any function F, ΔF (the change in F) is expressible by $(e^E - 1)F$, which, when F is a subinvariant, so that $EF = 0$, vanishes identically. Hence it is that subinvariants become differenciants†.

It may be worth while here to notice that if in place of the operator on ϕ in the above equation *any* numerical linear function of $a\delta_b$, $b\delta_c$, $c\delta_d \ldots$ be substituted‡, the value of ϕ which satisfies the transformed equation will be a subinvariant *quâ* the elements $a, b, c, \ldots$ divided respectively by appropriate

* For instance, in the above equation, U, V may be supposed to be two subinvariants of equal extent, exceeding by a unit that of Ω, their resultant in respect to their final letter. We know, by a principle demonstrated further on in the text, that Ω must be a subinvariant. The present theorem shows that X and Y also are (or may be replaced by) subinvariants.

† Or more simply for any number of letters $\alpha_1, \alpha_2, \ldots \alpha_i$, not fewer than the number of ratios between $a, b, c, \ldots$, if

$$a\Sigma\alpha_1 = ib, \quad a\Sigma\alpha_1\alpha_2 = \frac{i(i-1)}{2}c, \quad a\Sigma\alpha_1\alpha_2\alpha_3 = \frac{i(i-1)(i-2)}{2 . 3}d \ldots \text{ then } a\delta_b + 2b\delta_c + 3c\delta_d \ldots = \Sigma\frac{d}{d\alpha},$$

because $$\Sigma\frac{db}{d\alpha} = a, \quad \Sigma\frac{dc}{d\alpha} = 2b, \quad \Sigma\frac{dd}{d\alpha} = 3c \ldots.$$

Hence any subinvariant to the letters $a, b, c, \ldots$ is a function of the differences of $\alpha_1, \alpha_2, \ldots \alpha_i$.

‡ So, for example, $(a\delta_b + b\delta_c + c\delta_d \ldots)^{-1}\,0$ is a subinvariant *quâ* the elements

$$a, \; b, \; \frac{c}{1 . 2}, \; \frac{d}{1 . 2 . 3} \ldots.$$

numbers; namely, if the linear function be $pa\delta_b + qb\delta_c + rc\delta_d$, these numbers will be 1, p, $\frac{p.q}{1.2}$, $\frac{p.q.r}{1.2.3}$, as will be evident by making

$$\alpha = a,\quad p\beta = b,\quad \frac{p.q}{1.2}\gamma = c,\quad \frac{p.q.r}{1.2.3}\delta = d, \ldots$$

which being done the operator last above written may be changed into

$$\alpha\delta_\beta + 2\beta\delta_\gamma + 3\gamma\delta_\delta \ldots.$$

As a consequence of this it will readily be seen that if $\phi(a, b, c, d, \ldots)$ be a subinvariant to the elements $a, b, c, d \ldots$

$$\phi(0, b, c, d, \ldots),\quad \phi(0, 0, c, d, \ldots),\quad \phi(0, 0, 0, d, \ldots)$$

will respectively be subinvariants *quâ* the elements

$$b, \frac{c}{2}, \frac{d}{3}, \frac{e}{4}, \ldots,$$

$$c, \frac{d}{3}, \frac{e}{6}, \ldots,$$

$$d, \frac{e}{4}, \ldots,$$

and so on, the denominators following the law of figurate numbers.

This theorem, although foreign to the original and primary object of the present paper, as given in § 4, is of some considerable importance to the method of deduction. I mean the method (theoretically perfect but practically very difficult of application for quantics beyond the 4th order) according to which all the groundforms of a quantic, or which is the same thing, their ground-differenciants*, may be deduced by an exhaustive algebraical process in successive strata or categories from one another beginning with the known forms a, $ac - b^2$, $a^2d - 3abc + 2b^3, \ldots$ as the first category. See § 3.

It follows from the definition above given that a subinvariant may contain any given number of letters, and the number which it actually contains, less one (that is, the weight of the most advanced letter which appears in it), may be called its *extent*. Any subinvariant will then be a differenciant to a quantic whose order is not less than such extent.

Of course the definition of subinvariant may be extended to sets of letters $a, b, c \ldots;\ a', b', c' \ldots;\ a'', b'', c'' \ldots$. Any function ϕ of these sets of letters may be called a subinvariant, or when necessary, by way of distinction, a pluri-subinvariant, which satisfies the equality

$$(a\delta_b + 2b\delta_c + \ldots + a'\delta_{b'} + 2b'\delta_{c'} + \ldots + a''\delta_{b''} + 2b''\delta_{c''} \ldots)\phi = 0.$$

* I shall frequently use the term groundform to signify the leading coefficient of what is ordinarily so termed.

But for greater simplicity, except when a necessity arises for enlarging the horizon, I shall, in what follows, confine myself to the case of a single set of letters, that is, of uni-subinvariants*.

By an irreducible subinvariant is of course to be understood one which cannot be expressed as a rational integer function of any others. A differenciant to an irreducible quantic is of necessity a subinvariant, but not necessarily or even generally an irreducible subinvariant in the absolute sense in which the word is employed above; it will, however, be inexpressible as a rational integer function of any other subinvariants whose extent does not exceed the order of the quantic concerned, and may thus be said to be *relatively* irreducible. Thus, for example, the subinvariant

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd$$

is irreducible, relatively to the extent 3 or *quâ* the letters a, b, c, d, that is to say, cannot be expressed as a rational integer function of subinvariants whose elements are limited to a, b, c, d, but it is not an irreducible subinvariant in the absolute sense of the term, because it can be represented by a combination of the subinvariants

$$a,\ ac - b^2,\ ae - 4bd + 3c^2,\ (ac - b^2)e + 2bcd - ad^2 - c^3,$$

the letter e being eliminated by the process of taking the difference between the product of the 2nd and 3rd and that of the 1st and 4th of the preceding groundforms†.

Here I may take occasion to state a theorem of wide generality suggested by the above decomposition. It is well known that if ϕ be a subinvariant extending to the letter l as the highest letter which it contains, all the successive derivatives of ϕ in respect to l will also be subinvariants, as is evident from the fact that if $(a\delta_b + 2b\delta_c + \ldots + ik\delta_l)\phi$ is zero, the same must be true of $(\delta_l)(a\delta_b + 2b\delta_c + \ldots + ik\delta_l)\phi$, or what is the same thing, of

$$(a\delta_b + 2b\delta_c + \ldots + ik\delta_l)\delta_l\phi.$$

Suppose then that $\phi, \psi, \omega, \ldots$ are any number of subinvariants limited to l as their highest letter, and regarded, each of them, as a homogeneous function of l and 1, then I say that any differenciant in respect to l of this system of quantics will be a subinvariant *quâ* the elements $a, b, c, \ldots k$. For we know that any differenciant of the system $\phi(x), \psi(x), \ldots$ say

$$(\alpha, \beta, \gamma \ldots \lambda \between x, 1)^i;\ (\alpha', \beta', \gamma' \ldots \lambda' \between x, 1)^{i'}, \ldots$$

* Eventually I am inclined to substitute the word binariant for subinvariant, and to speak of simple, double, treble or multiple binariants. The functions similarly related to ternary forms will then be styled simple or multiple ternariants, and so in general.

† So it may be shown that the subinvariants of deg-orders 5 . 7, 5 . 1, 5 . 5 to the Quintic (which are perfectly determinate), may be regarded as the resultants in respect to g of the sextic groundforms 2 . 0 and 4 . 6, 2 . 0 and 4 . 0, 2 . 0 and 4 . 4 respectively, all four of which are linear in g. See Sextic Germ Table, § 2. [p. 578, below.]

remains unaltered when α, $\alpha+\beta x$, $\alpha+2\beta x+\gamma x^2 \ldots \phi$, respectively, are substituted for $\alpha, \beta, \gamma \ldots \lambda$, and at the same time

$$\alpha', \ \alpha'+\beta' x, \ldots \psi, \text{ for } \alpha', \ \beta', \ldots \lambda',$$

respectively, and so on; that is to say, any subinvariant of the equation above written may be regarded as a function of

$$\phi x, \ \phi' x, \ \phi'' x, \ldots; \ \psi x, \ \psi' x, \ \psi'' x, \ldots; \ldots.$$

Hence in regard of the system of subinvariants any of its differenciants is a function of the members of the system, and the successive derivatives in respect to l of each member, all of which are subinvariants. Hence the differenciant in question may be regarded as a function exclusively of subinvariants, and is therefore a subinvariant of the letters $a, b, c, \ldots k$. As a particular application of the theorem we see that the resultant in regard to their last letter of two subinvariants of like extent and the discriminant of any subinvariant in regard to its last letter are subinvariants. Thus, for example, if the discriminant of a cubic be exhibited as a quadratic function of d, namely, under the form $a^2d^2+(4b^3-6abc)\,d+(4ac^3-3b^2c^2)$, its discriminant, namely,

$$(2b^3-3abc)^2-a^2(4ac^3-3b^2c^2), \text{ that is, } 4\,(b^3-3ab^3c+3a^2b^2c^2-a^3x^3)$$

is as it ought to be a subinvariant, namely, it is $4\,(b^2-ac)^3$. So more generally, if we regard any number of pluri-subinvariants (all of the same extent in each set of letters) as a system of multi-partite polynomials in the extreme letter of each set, any differenciant of such system will be a subinvariant (of course with diminished extent in each set) in regard to the original letters. The simple instance already given will serve as a diagram to make the reason self-evident. The invariant in respect to d of the discriminant of the cubic is the same as in respect to x of

$$a^2(x+d)^2+(4b^3-6abc)(x+d)+(4ac^3-3b^2c^2),$$

that is, of $a^2x^2+2\,(a^2d-3abc+2b^3)\,x+(a^2d^2+4b^3d-6abcd+4ac^3-3b^2c^2)$,

hence being a function of the three coefficients, which are all of them subinvariants, it is itself a subinvariant*.

It has been shown above that the same form which regarded as a differenciant is irreducible, that is, is incapable of being decomposed into products of other differenciants of no higher extent than its own, when regarded as a subinvariant may be, and as a matter of fact, far oftener than not will be decomposable into products of subinvariants of higher extent. Thus the irreducible differenciants to any quantic naturally resolve themselves into two classes, those which are absolutely irreducible and those which are only relatively so; and it would seem that in any natural method of proof of Gordan's theorem these would, it is likely, have to be considered separately.

* The method of proof here employed, it will be seen, is the same in kind as that employed in the ordinary proof of Taylor's theorem.

There is comparatively little difficulty in proving that the first class are finite in number; the proof of the second class being likewise finite, must depend upon the fact that they are the resultants of a finite number of functions.

I use the word resultant in the above paragraph in an enlarged sense. If $U, V, W, \ldots$ are any given polynomials in $x, y, \ldots z, t, \ldots u$, I call any quantity not containing $x, y, \ldots z$ capable of being exhibited under the form of the syzygetic function $U_1U+V_1V+W_1W \ldots$ a resultant of the given polynomials in respect to $x, y, \ldots z$. For resultants thus defined, the following important proposition admits of easy proof, namely: Every such resultant is capable of being represented as a sum of products U_1U, $V_1V, \ldots$ of which the orders in $x, y, \ldots z$ are *limited* in extent, and consequently the most general representation of such resultant can contain only a finite number of arbitrary parameters. When the number of the eliminables $x, y, \ldots$ is one less than the number of the given functions which contain them, we fall back upon the ordinary kind of resultant, having only one arbitrary parameter. When there is but one eliminable x, and any number of polynomials $U, V, W, \ldots$ of orders $\alpha, \beta, \gamma, \ldots$ in x, the order in x of each syzygetic product U_1U, $V_1V, \ldots$ in a syzygetic function of $U, V, W, \ldots$ which is competent to represent any resultant of the system, is (if I mistake not) at most one unit less than the sum of the two highest (or of the two as high as any) of the numbers $\alpha, \beta, \gamma \ldots$.

The orders of the syzygetic multipliers being once determined, the number of indeterminate constants is known, and these will be subject to satisfy a known number of *linear* equations, namely, a number greater by unity than the order of the $U_1U+V_1V \ldots$ polynomial, and thus the problem of finding the complete system of resultants of the original system of polynomials in one variable is brought to depend upon the problem of finding the complete system of resultants of a system of homogeneous *linear* functions of several variables, a problem of which the solution and the number of arbitrary parameters which at most can appear in it are perfectly well known and need not be here set forth.

The syzygetic products U_1U, $V_1V, \ldots$ whose sum is competent to express every resultant of $U, V \ldots$, I have said, need none of them be taken of an order so high as the sum of the two greatest of the quantities $\alpha, \beta, \gamma \ldots$. Thus for instance in the case of $U, V, W, \ldots$ being linear functions, the syzygetic multipliers, as is well known, need only to be taken as constants; or again when $\alpha, \beta, \gamma, \ldots$ form a descending series, the syzygetic products need only to be all of them made of the same order as the highest of the given functions. Take, to fix the ideas, three functions, U, V, W, all of them quadratics in x. The syzygetic multipliers may be taken all linear functions in x: there will thus arise six disposable constants subject to three conditions, inasmuch as the coefficients of x^3, x^2, x, must vanish in the sum of the products: if two of the multipliers, say of U, V, were made quadratic functions, there would be eight disposable constants subject to four conditions, since an additional coefficient, namely, of x^4, would have to vanish in the sum of the products: there would therefore be one additional arbitrary parameter, namely, $8-4$ instead of $6-3$, but the form of the *resultant* would be not more general than on the preceding supposition, because if to U_1, V_1 (the most general values of the linear multipliers of U, V), λV, $-\lambda U$ respectively be added, there will then be four arbitrary parameters, and consequently the solution must be the same as on the second supposition, but the value of the resultant remains unaltered by the change made in U_1, V_1.

Or again if U, V, W were the two first quadratics and the second a linear function in x, their syzygetic multipliers might be taken two constants and a linear function respectively: by raising the orders of any two of these multipliers by a unit, an additional arbitrary constant would be gained, but the sum of the products resulting therefrom would not thereby gain in generality, as may be shown by the same method as in the preceding example.

It might probably not be difficult to give a universal rule for determining the lowest orders of the syzygetic multipliers required for expressing the resultant in its most general form, of functions of one or even of several variables, but this is an inquiry which it is necessary to postpone, as it might lead to too long a deviation from the immediate purpose in view, and there are some difficulties attending the subject more than present themselves at first sight.

It is enough to know, and that only for the case of a single eliminable, the existence of a limit to the orders of the multipliers, which it is quite easy to demonstrate. That being premised, it will follow as an easy consequence, that any combination *inter se* of subinvariants of any given extent and each containing the highest letter corresponding thereto can only give rise to a limited number of subinvariants of lower extent, and from that it is easy by repeated applications of the same *principle of the limit* to infer that only a finite number of *relatively* irreducible subinvariants of any given extent (that is, irreducible into combinations of subinvariants of the same or lower extent) can arise from the combinations of a finite number of subinvariants of any given higher extent; but it will appear in the sequel that the degree and consequently that the number of irreducible subinvariants of any given extent is subject to a limit; consequently if the number of relatively irreducible subinvariants of any given extent (or which is the same thing, if the covariants of a quantic of any given order) were unlimited in number, this could only be in consequence of there being no extent so large but

that subinvariants of that extent and containing the most advanced letter corresponding thereto, would be needed in order to exhibit the composition of the relatively irreducible, but in an absolute sense, reducible subinvariants referred to.

In § 4 I propose to show how to obtain the types (that is, deg-weights) of the absolutely irreducible subinvariants of the first few degrees. Besides the intrinsic interest of the inquiry, the result obtained without going beyond subinvariants of the 7th degree will serve to show conclusively that *it is not true* "that syzygants and groundforms of the same degree and order cannot appertain to the same binary quantic," but that when the order of the quantic is sufficiently elevated there *must* appertain to it, syzygants (*compound* ones) and groundforms of the same degree and order.

Let it be observed that the proposition here about to be disproved is not coextensive with the law of parsimony, but goes considerably beyond it—that is, implies much more than that law gives warrant for.

Let us for the moment call the number of linearly independent forms of the deg-order (j, ω) to a given quantic given by Cayley's rule, the denumerator to the type (j, ω), and the number of forms of such type that can be obtained by compounding together groundforms of lower types, the aggregator to the same type. Let us further suppose that the duad (j, ω) may be compounded of (j', ω'), (j'', ω'')*.

Suppose further that the aggregator to the type (j', ω') exceeds its denumerator, and also that there exists one or more, say Δ' linearly independent invariantive forms of the deg-order (ω'', j''), but that (*if possible*) the aggregator to the type (j, ω) is equal to or less than its denumerator, the difference being Δ. Obviously if such a case can occur, the law of parsimony (that is, the Newtonian rule of not assuming more causes to exist than are necessary to the explanation of a phenomenon or set of phenomena) will, on such a supposition, lead to the conclusion, not that there are Δ groundforms and *no* syzygies, but $\Delta+\Delta'$ groundforms and Δ' syzygies. Such a case does not present itself for quantics of the lower orders; it seems natural and logical therefore to seek for it in the case of a quantic of an infinite order, that is, in the case of subinvariants unlimited in *extent*. If it can be shown (as in § 4 it will be shown) that with an unlimited number of letters, an irreducible subinvariant and a compound syzygy of subinvariants coexist for a given degree and for the weight ω, it will follow from the nature of the process employed in what follows, that the same conclusion must hold when the *extent* of the subinvariants is limited, provided (at the very worst) that the limit is not less than ω, for it will be seen that no letter of higher weight than ω enters into the process which leads to the result under con-

* I mean that $j=j'+j''$, $\omega=\omega'+\omega''$.

sideration. It is in all human probability true that the proposition holds good in the form in which it was originally presented, namely, that *irreducible* syzygants and irreducible invariantive derivatives of the same type, to the same quantic cannot coexist; but whether the proposition so limited is sufficient to support the substitution of the process of tamisage performed upon the numerator of the representative generating fraction, in lieu of tamisage performed upon the development of that fraction in an infinite series, or how the method of substitutive tamisage, if at present inexact, may be modified *pari passu* with the needful modification in brute tamisage so as to recover its validity, is a matter which must be reserved for future consideration.

§ 2. Germs.

Before proceeding to the more immediate object of this paper I think it will be profitable to insert the following table of the multipliers of the highest letter or power of the highest letter f in the relatively irreducible subinvariants of the extent 5 (that is, the leading coefficients in the groundforms of the quintic), and a similar table for the groundforms of the sextic arranged according to the powers of g*. For many purposes these tables will be found as serviceable as the entire function of the letters or even as the entire covariant written out at length. Those relating to the quintic may be verified by comparison with the tables (as far as they extend) contained in the *Formes Binaires* of M. Faà de Bruno, but the order of arrangement of the terms in those tables is not what my method of representation points out as the most natural, and proceeds upon some principle not easy to divine. It is also necessary to state that there are very many errors and misprints in those tables. With regard to the particular choice of the groundforms of any deg-order I believe that in all cases but one the tables of M. de Bruno are in accordance with those employed by myself, and which are on the face of them the *simplest* that can be employed, with one exception, namely, in the expression for the covariant of deg-order 9 . 3 the multiplier of the power of f, or *germ* as it may well be styled, is $(ac-b^2)^3$, whereas in the extended tables of M. de Bruno the germ will be found to be some numerical linear function (its exact value I have forgotten) of

$$(ac-b^2)^3,\ a^2(ac-e^2)(ae-4bd+3c^2),\ \text{and}\ a^3\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$$

or which comes to the same thing, of the two former and

$$a^2d^2+4ac^3+4db^3-3b^2c^2-6abcd;$$

the covariant thus given of deg-order 9 . 3 is accordingly more complicated than it need have been.

* Any such multiplier I call the *germ* of the form to which it appertains.

It may be well to notice that whenever two consecutive terms in either table occur with the same germ but different powers of the last letter, the complete subinvariant of the antecedent is (to a numerical factor *près*) the differential derivative of the consequent in respect to that letter; thus, for example, the leading coefficient in the covariant to the quintic of the deg-order 7 . 5 will be found by simply differentiating the invariant of the degree 8 and dividing the result by the number 3.

In the table immediately following (c), (d), (e), (e'), Δ stand for

$$a,\ ac-b^2,\ a^2d-3abc+2b^3,\ ae-4bd+3c^2,\ ace-ad^2+2bcd-c^3-ad^2$$

and

$$a^2d^2+4ac^3+4db^3-3b^2c^2-6abcd$$

respectively. The quantities which appear in the outside vertical column are the germs; the double figures which fill the occupied spaces are the deg-orders. Thus, for example, 7 . 5 being opposite to the germ $(c)(d)$ and in the column headed by f^2, indicates that the covariant to the quintic of degree 7 and order 5 has for its differenciant a quantity of the form

$$(ac-b^2)(a^2d-3abc+2b^3)^2+\text{a linear function of } f,$$

and so in general.

Germ Table to the Quintic.

	1	f	f^2	f^3	f^4	f^7
a	1 . 5					
(c)	2 . 6					
(d)	3 . 9	4 . 4				
(e)	2 . 2					
(e')	3 . 3					
a^2		3 . 5	4 . 0			
$a(c)$		4 . 6	5 . 1			
$a(d)$			6 . 4			
$(c)^2$		5 . 7	6 . 2			
$(c)(d)$			7 . 5	8 . 0		
$3a(e')-2(c)(e)$		5 . 3				
$a^2(c)$				7 . 1		
$a(c)^2$				8 . 2		
$(c)^3$				9 . 3		
$(c)^2(d)$					11 . 1	
$(c)^2$					12 . 0	
$(c)^3(e')$					13 . 1	
$a(c)^5$						18 . 0

In the annexed table $(c)(d)(e)(e')(f)(\Delta)$ retain their previous significations. The additional symbols (cf), (c^2f), (df), (cef) represent respectively the differenciants to the quintic of the deg-orders 4 . 6, 5 . 7, 4 . 4, 5 . 3, all of which are linear functions of f (see preceding table).

GERM TABLE TO THE SEXTIC.

	1	g	g^2	g^3	g^5	g^6
a	1.6	2.0				
(c)	2.8	3.2				
(d)	3.12	4.6				
(Δ)			6.0			
(e)	2.4					
(e')	3.6	4.0				
(f)	3.8					
$a(c)$			5.2			
$a(d)$			6.6			
$a(e)$		4.4				
$(c)(d)$				8.2		
$(d)(e)$			7.4			
$(c)(f)$	4.10	5.4				
$(d)^3$						15.0
$a(d)(e)$				9.4		
$(c)^2(f)$		6.6*				
$a(d)(f)$			7.2			
$(c)(e)^2(f)$				10.2		
$(c)(e)(f)$	5.8					
$a^2(c)(d)$					12.2	
a^5					10.0	
$(c)(c^2f)$				10.2		

§ 3. GROUNDFORMS.

Quantitative Deduction of their Categories.

I will now proceed to explain what I mean by the exhaustive or quantitative method of deducing the ground differenciants to a given quantic, referred to in the course of the preceding observations.

The well-known functions of alternately the second and third degrees $ac-b^2$, $a^2d-3abc+2b^3$, $ae-4bd+3c^2$, ... limited in extent to the order of the quantic under consideration, may be called the protomorphs or primaries.

Suppose then the groundforms to the cubic are to be deduced. The primaries or protomorphs, omitting a, are $ac-b^2$, $a^2d-3abc+2b^3$, and the residues (meaning thereby the remainders when these quantities are divided by a) are $-b^2$, $2b^3$. Hence $(a^2d-3abc+2b^3)^2+4(ac-b^2)^3$ will divide out by a (as it happens by a^2) and give the new groundform

$$a^2d^2+4ac^3+6abcd+4b^3d-3b^2c^2.$$

Between its residue $4b^3d-3b^2c^2$, and the two former, it is obvious that no new relation can arise. Hence the four forms

$$a,\ ac-b^2,\ a^2d-3abc+2b^3,\ a^2d^2+4ac^3-6abcd+4b^3d+3b^2c^2$$

constitute the complete system of ground differenciants, and the corresponding co- and- invariants comprehend the complete system of such for the cubic.

Proceeding to the quartic, a new protomorph or base-form comes into view, namely, $ae-4bd+3c^2$, whose residue is $-4bd+3c^2$ in addition to the antecedent ones $4b^3d-3b^2c^2$, $2b^3$, $-b^2$, and since the second of these is the product of the first and last it follows that

$$-(a^2d^2+\ldots)+(ac-b^2)(ae-4bd+3c^2)$$

must contain the factor a, and on performing the division there emerges the new groundform

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$$

so that $(a^2d^2+\ldots)$ being equal to this multiplied by a less the product of two other groundforms, ceases itself to be one, and the groundforms now subsisting are the one last named in addition to the base-forms

$$a,\ ac-b^2,\ a^2d-3abc+2b^3,\ ae-4bd+3c^2,$$

which, since the new one is the only one of the *five* containing the letter e, can enter into no combination with them of which the residue is zero, and consequently the *deduction* is at an end and the five named constitute the complete system of groundforms.

Beyond this point the method of deduction has not hitherto been pushed, nor could it have been, without the use of the theorem concerning the subinvariantive character of the residues, in consequence of their enormous complexity when regarded as simple functions of the letters. In what follows the deduction is extended to the case of the quintic*.

Algebraical Deduction of the Groundforms of the Quintic†.

The complete system of groundforms to be deduced may be denoted by the deg-order or the deg-weight: viewed as subinvariants, the latter is the more natural mode of designation: if j and ω are the degree and weight, the order ϵ will be $5j - 2\omega$. For greater facility of reference to the known list of groundforms, it will be convenient to set out the order as well as the degree; the complete system of the designating $j; \epsilon . \omega$, of the twenty-three groundforms, that is, of the twenty-three relatively irreducible subinvariants of *extent* not exceeding five, will then be as follows: 1;5.0, 2;2.4, 2;6.2,

* In Salmon's *Modern Algebra*, 3rd Ed., pp. 170—1, 195—6, the base-forms employed in the deduction of the quartic groundforms are not identical with those employed above, the third one being of the fourth instead of the second degree in the letters, and consequently not a groundform, whereby the deduction is rendered somewhat longer than that given in the text. The most eligible base-forms to employ in any case are alternately of the second and third degrees, whereas those given by Prof. Cayley, the author of this important method, are of degrees continually increasing by a unit.

† By algebraical, I mean in this connection, that which deals only with the ordinary algebraical processes of addition, multiplication and division, as contradistinguished from transcendental processes involving differential operation, or which is substantially the same thing, symbolical resolution.

The preceding deduction for the Cubic and the Quartic is by far the simplest mode of obtaining the complete systems of groundforms for these quantics, and proving their completeness, which, at an earlier period of the theory, was regarded as a problem of some little difficulty. See Faà de Bruno's *Formes Binaires*, Chapter 7, pp. 260—263, where the same results are obtained through the medium of "*Formes Associées.*" I cannot but think that sooner or later this method, first discovered by the eagle-gaze of Cayley, will lead to the object which I presume he had in view when he originated it, namely, a proof of Gordan's theorem by ordinary algebra.

I think I see looming in the not far distance such a proof, depending ultimately upon the fact of a certain succession of increasing integer multiplets, subject to stated laws of limitation, not being capable of being indefinitely produced. To render sensible the sort of arithmetical theorem which I have in view, I subjoin a theorem *ejusdem generis* concerning singlets (simple integers), which, as far as I know, is new, and admits of easy proof.

A succession of integers of which no one is a multiple of one nor the sum of the multiples of two others cannot be continued ad infinitum.

To prove this we may begin with the case where one of the integers written down is a prime number, for which case the proof is immediate. Then it is easy from this to show that if the theorem is true for the case where one of the integers is a product of only i-primes, it must be true for the case where one of the integers is a product of only $(i+1)$ primes; for this case, by virtue of the supposition made, may easily be reduced to the case where one of the numbers is a relative prime to all the others, for which case the theorem is true, for the same reason as if the number in question were an absolute prime. Consequently the theorem is true universally.

By the quotient of a duad (in what follows) is to be understood the quotient of the second element by the first; by the sum of two duads, the duad whose elements are the sums of the

3;5.5, 3;9.3, 3;3.6, 4;0.10, 4;4.8, 4;6.7, 5;1.12, 5;3.11, 5;7.9, 6;2.14, 6;4.13, 7;1.17, 7;5.15, 8;0.20, 8;2.19, 9;3.21, 11;1.27, 12;0.30, 13;1.32, 18;0.45. The protomorphs or base-forms are the five first of these, namely,

1;5.0 is a, 2;6.2 is $ac - b^2$, 3;9.3 is $a^2d - 3abc + 2b^3$,

2;2.4 is $ae - 4bd + 3c^2$, 3;5.5 is $a^2f - 5abe + 2acd + 8b^2d - 6bc^2$.

Again, 3;3.6 is the determinant

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$$

corresponding elements of the two, and by a multiple of a duad the duad whose elements are the elements of that duad multiplied each by the same *integer*. The foregoing theorem may then be extended as follows:

A succession of duads, the quotients of all which but two are intermediate to the quotients of those two, and such that no duad is a multiple of any one or the sum of the multiples of any two or three of the others, cannot be indefinitely continued.

Again, one couple of quantities may be said to be *intermediate* to three others when the point representing the first is situated within the triangle whose apices represent the other three; a point being said to represent the two quantities which are equal to its two coordinates in respect to any two given axes. So a triplet of quantities, by aid of an analogous representation in space, may be said to be intermediate to four others when its representative point lies inside the pyramid whose apices represent those four.

It will readily be understood that these definitions may be translated into conditions of inequality between determinants, and thus translated may be extended so as to yield a definition of one pollad of $n-1$ elements being *intermediate* to n, or indeed to any number of other such pollads. Also the quotient-system of an n-ad will be understood to mean the system of $(n-1)$ quotients got by dividing the first element of the n-ad into the $n-1$ others. The following general theorem may then be enunciated:

A succession of n*-ads such that the quotient-systems of all but* n *of them are intermediate to the quotient-systems of those* n cannot be indefinitely continued, if every n-ad which is either a multiple of some one or a sum of multiples of 2, 3, ..., n or $n+1$ of the others, is excluded from the succession.

More generally, and with a less stringent negative condition, *a succession of* n*-ads such that the quotient-systems of all but* ν *given ones (*ν *being any number) are intermediate to the quotient-systems of those* ν*, cannot be indefinitely continued, if every* n*-ad which is a multiple or a sum of multiples of any or all of the* n*-ads of a group of* $\nu+1$ *others (whereof* ν *are the given ones) is excluded from the succession.*

The hypothetical ground of connection between this theorem and Gordan's algebraical one is as follows: It may be shown to be implied in the method of deduction, that if the number of groundforms to the quintic were infinite, then there must exist a certain infinite succession of products, *[some of the form $b^x Q^y R^z S^t$, the others of the form $b^\xi Q^\eta R^\zeta S^\tau T$, such that neither any product $b^x Q^y R^z S^t$ nor any product $b^\xi Q^\eta R^\zeta S^\tau$ could be (a power of one or) a product of powers of any number of the products not involving T. If then it could be shown that there exists a set of quadruplets of the kind x, y, z, t such that every other one of that kind and also every one of the kind ξ, η, ζ, τ is *intermediate* to that] set, the existence of such a succession would be impossible by virtue of the arithmetical theorem, and the possibility of the existence of an infinite number of groundforms would consequently be disproved. A similar kind of proof could conceivably, but with more difficulty, be extended to quantics of any order.

[* For the words placed in square brackets, see the correction, p. 621, below.]

This, not involving the letter f, has been previously deduced, and it has been shown that its integrating factor (that is, the power of a by which it must be multiplied to give a rational integer function of the base-forms) is a^3; it has, in fact, been shown (dropping the second integer and dealing only with deg-weights) that

$$(1.0)^3(3.6)=(1.0)^2(2.2)(2.4)-4(2.2)^3+(3.3)^2.$$

I shall denote the residue of any form ϕ by the symbol $\mathfrak{R}\phi$; each such residue is a function of the *five* letters b, c, d, e, f, being in fact a subinvariant in regard to the letters $b, \frac{c}{2}, \frac{d}{3}, \frac{e}{4}, \frac{f}{5}$, and therefore of the four groundforms proper to the diminished extent 4, that is, of the five following functions

$$b,\ \frac{bd}{3}-\frac{c^2}{4},\ \frac{b^2e}{2}-3\frac{bcd}{6}+2\frac{c^3}{8},\ \frac{bf}{5}-4\frac{ce}{8}+3\frac{d^2}{9},$$

$$\begin{vmatrix} b, & \frac{c}{2}, & \frac{d}{3} \\ \frac{c}{2}, & \frac{d}{3}, & \frac{e}{4} \\ \frac{d}{3}, & \frac{e}{4}, & \frac{f}{5} \end{vmatrix}$$

or (getting rid of the denominators) of

$$b,\ 4bd-3c^2,\ 2b^2e-bcd+2c^3,\ 6bf-15ce+10d^2,$$

$$\begin{vmatrix} 3b, & 3c, & 2d \\ 3c, & 4d, & 3e \\ 10d, & 15e, & 12f \end{vmatrix}$$

of which the deg-weights are 1.1, 2.4, 3.6, 2.6, 3.9 respectively; the first of these is b, the others I shall call Q, T, R, S respectively. In all that follows I shall denote a numerical linear function of two or more quantities by enclosing them in brackets with commas interposed*; thus, for example, (ϕ, ψ, θ) will mean $\lambda\phi+\mu\psi+\nu\theta$, where λ, μ, ν are certain determinate (but unexpressed) numbers.

We know from the theory of the groundforms of extent 4 (that is, differenciants of a quartic) that the above five quantities are not algebraically independent, but are connected by an equation of the form

$$T^2=(Q^3,\ b^3S,\ b^2QR).$$

We have also the following expressions for the residues of the groundforms denoted by their deg-orders, and their first deduct, namely,

$$\mathfrak{R}(2;6)=b^2,\ \mathfrak{R}(3;9)=b^3,\ \mathfrak{R}(2;2)=Q,\ \mathfrak{R}(3;5)=bQ,\ \mathfrak{R}(3;3)=T,$$

or, using deg-weights instead of deg-orders,

$$\mathfrak{R}(2.2)=b^2,\ \mathfrak{R}(3.3)=b^3,\ \mathfrak{R}(2.4)=Q,\ \mathfrak{R}(3.5)=bQ,\ \mathfrak{R}(3.6)=T.$$

* The brackets will sometimes for convenience be omitted.

Since $b^3.Q=b^2.bQ$, that is, $\mathfrak{R}(3, 9)\,\mathfrak{R}(2, 2)-\mathfrak{R}(2.6)\,\mathfrak{R}(3, 5)=0$, it follows that $((3;9)(2;2),\ (2;6)(3;5))$ must contain a.

Also it is obvious that the effect of throwing out a from a differenciant to the quintic which contains it, is to diminish the degree by one unit, leaving the weight unaltered, and therefore diminishes the order by five units.

Hence $$\frac{1}{a}((3;9)(2;2),\ (2;6)(3;5))=4;6.$$

It will be more convenient here and hereafter to use exclusively deg-weights instead of deg-orders to denote the forms; the above equation thus expressed becomes

$$\frac{1}{a}((3.3)(2.4),\ (2.2)(3.5))=4.7.$$

Turning now to the deg-weights of the residues, it will be seen that 4.7 can only be composed of 1.1 and 3.6.

Hence $\mathfrak{R}(4.7)=bT$, which is not a product of residues; so 4.7 must be a new groundform. Again, (adhering to the use of deg-weights) we have

$$(\mathfrak{R}(3.5))^2=b^2Q^2=\mathfrak{R}(2.2)\,(\mathfrak{R}(2.4))^2.$$

Hence $$\frac{1}{a}((3.5)^2,\ (2.2).(2.4)^2)=5.10.$$

The only mode of resolving 5.10 into sums of the duads 1.1, 2.4, 3.6, 2.6, 3.9, is by the addition of 2.4 and 3.6.

Hence $\mathfrak{R}(5.10)$ is a numerical multiple of QT, that is, of $\mathfrak{R}(2.4)$ and $\mathfrak{R}(3.6)$. Hence $((5.10),\ (2.4)(3.6))$ contains a; consequently 5.10 is not a groundform, but we shall have $\frac{1}{a}((5.10),\ (2.4)(3.6))=4.10$, and 4.10 can be resolved into $1.1+3.9$ and $2.4+3.6$. Hence $\mathfrak{R}(4.10)=(bS,\ QR)$ and 4.10* will be a new groundform.

So again $(3.3)(3.5)=b^3.bQ$, and $(2.2)^2(2.4)=(b^2)^2Q$. Hence

$$\frac{1}{a}\{(3.3)(3.5),\ (2.2)^2(2.4)\}=5.8,$$

which can be resolved in only one way into a sum of the duads 1.1, 2.4, 3.6, 2.6, 3.9, namely, into $1.1+1.1+3.6$. Hence

$$\mathfrak{R}(5.8)=\mathfrak{R}(2.2)\,\mathfrak{R}(3.6),$$

and consequently 5.8 is not a groundform, but

$$\frac{1}{a}\{5.8,\ (2.2)(3.6)\}=[4.8],$$

which, in respect to the duads above mentioned, is resoluble (and only resoluble) into $2.4+2.4$ and $1.1+1.1+2.6$.

* 4.10 which is the same (using deg-orders) as 4.0 obviously cannot undergo further depression, and is consequently a groundform.

Hence $\mathfrak{R}(4.8) = (Q^2, b^2R)$, and since $Q = \mathfrak{R}(2.4)$ we have

$$\mathfrak{R}((4.8), (2.4)^2) = b^2R;$$

and since $([4.8], (2.4)^2)$ is of the deg-weight 4.8, we see that there is a form 4.8 such that $\mathfrak{R}(4.8) = b^2R$, which is consequently a groundform, since b^2R is not a rational integer function of any of the previous residues. Thus, then, from the base-forms 2.2, 3.3, 2.4, 3.5, besides the groundform not containing f, namely, 3.6, we have derived the three additional groundforms 4.7, 4.10, 4.8. Of these 4.7 and 4.8 belong to the same category as 3.6, being like it derived immediately from the base-forms. Whereas, in obtaining 4.10 it has been necessary to employ 3.6, so that it belongs to a more distant category. If we call the base-forms primaries, 3.6, 4.7, 4.8 will be secondaries, and 4.10 a tertiary. So again we shall find

$$\mathfrak{R}(3.3)\,\mathfrak{R}(3.6) = b^2.T, \text{ and } \mathfrak{R}(2.2)\,\mathfrak{R}(4.7) = b^2.bT.$$

Hence $\frac{1}{a}\{(3.3)(3.6), (2.2)(4.7)\} = 5.9$, and $\mathfrak{R}(5.9) = b^3.R$,

which cannot be compounded out of the preceding residues, so that (5.9) is another tertiary.

Again $\mathfrak{R}(4.7)\,\mathfrak{R}(2.4) = Q.bT$, and $\mathfrak{R}(3.5)\,\mathfrak{R}(3.6) = bQ.T$.

Hence $\frac{1}{a}((4.7)(2.4), (3.5)(3.6)) = 5.11$, and $\mathfrak{R}(5.11) = (b^3S, bQR)$,

for 5.11, in regard to the oft-quoted duads, is resoluble only into

$$1.1 + 1.1 + 3.6 \text{ and } 1.1 + 2.4 + 2.6.$$

Hence 5.11 is also a tertiary groundform.

Again

$$\mathfrak{R}(2.2)\,\mathfrak{R}(2.4)\,\mathfrak{R}(3.6) = b^2.Q.T, \text{ and } \mathfrak{R}\,4.7\,\mathfrak{R}\,3.5 = bT.bQ.$$

Hence $$\frac{1}{a}((2.2)(2.4)(3.6), (4.7), (3.5)) = [6.12],$$

and the duad 6.12 is resoluble into

$$3.9 + (1.1)^{2*}(2.6) + (2.4) + (1.1)^2(3.6)^2 \text{ and } (2.4)^3,$$

corresponding to b^3S, b^3QR, Q^3, T^2. Now Q, T, b^2R are all residues, as already shown, and since b^2 and (bS, QR) are residues $(b^3S, b^2R.Q)$, and therefore b^3S is a residue.

Hence a form denotable by 6.12 which shall be a linear function of [6.12] and of the combinations of inferior groundforms, will have a residue zero, and consequently [6.12] will not be a groundform, but the 6.12 last spoken of will be divisible by a, and the quotient will give a groundform 5.12, whose residue corresponding to the composition 3.6 + 2.6 is RT. We shall thus have obtained for our tertiary or third batch of groundforms (descendants,

* It will be often found convenient to use $(p.q)^i$ to mean the sum of i duads $p.q$.

that is, in the second degree from the base-forms) the subinvariants denoted by 4.10, 5.9, 5.11, 5.12.

Again $\mathfrak{R}(3.3)\,\mathfrak{R}(4.10) = b^3(bS, QR)$;

$$\mathfrak{R}(2.2)\,\mathfrak{R}(5.11) = b^2(b^3S, bQR);\ \mathfrak{R}(2.4)\,\mathfrak{R}(5.9) = Q(b^3R).$$

Hence between these three equations the two arguments b^4S, b^3QR may be eliminated, and there results

$$\frac{1}{a}\{(3.3)(4.10), (2.2)(5.11), (2.4)(5.9)\} = 6.13,$$

and 6.13 will be resoluble only into

$$3.6 + 2.6 + 1.1, \text{ so that } \mathfrak{R}\,6.13 = bRT.$$

Again $\mathfrak{R}\,3.5\,\mathfrak{R}\,5.12 = bQ.RT$; $\mathfrak{R}\,3.6\,\mathfrak{R}\,5.11 = T(b^3S, bQR)$;

$$(\mathfrak{R}\,4.7).(\mathfrak{R}\,4.10) = bT(bS.QR),$$

on the right-hand side of which three equations $bQRT$, b^3ST are the only two arguments appearing, so that

$$(\mathfrak{R}\,3.5, \mathfrak{R}\,5.12, \mathfrak{R}\,3.6, \mathfrak{R}\,5.11, \mathfrak{R}\,4.7, \mathfrak{R}\,4.10)$$

may be made equal to zero. Hence we have a new deduct 7.17, and $\mathfrak{R}\,7.17$ will be found $= (Q^2S, b^2RS, bQR)$, and 7.17 will be a groundform, as is apparent at once from the fact that it is the same (using a deg-order instead of deg-weight) as 7 ; 1 which is obviously indecomposable into any inferior forms.

But it may be objected that conceivably there might exist a syzygy between (3.5)(5.12), (3.6)(5.11), (4.7)(4.10), so that the form 7.17 obtained by dividing a linear combination of the three products by a may really be a null quantity. But not to mention the unlikelihood that a syzygy should occur between so low a number as only three products of groundforms of elevated degrees, the existence of such a syzygy may be directly disproved as follows: (3.6)(6.11) will contain only the first power of f, and writing

$$5.12 = Lf^2 + 2Mf + N, \quad 4.10 = Pf^2 + 2Qf + R,$$

we shall have $4.7 = Lf + M, \qquad 3.5 = Pf + Q,$

so that if the supposed syzygy exists we must have $LQ - MP = 0$, but

$$L = -a^2,\ M = 5abc - 2acd + 8b^2d + 6bc^2,\ P = (a^2c - ab^2),\ Q = \ldots.$$

Hence since M does not contain a as a factor, MP cannot equal LQ, so that the conceivable syzygy does not exist, and the groundform 7.17 is correctly deduced*.

* I shall eventually supersede this proof of the non-existence of the syzygy under discussion by a method involving no algebraical computation. It is a remarkable feature in this deduction that although it is in its nature quantitative, no algebraical computations whatever need to nor will be employed in working it out and establishing its validity at each stage, thanks to the use made of the factors of integration, as will presently appear.

Again $\mathfrak{R}\,5.9\,\mathfrak{R}\,3.5 = b^3R.bQ$, $\mathfrak{R}\,3.3\,\mathfrak{R}\,5.11 = b^3(b^2S,\ bQR)$,

$$(\mathfrak{R}\,2.2)^2.(\mathfrak{R}\,4.10) = b^4(bS,\ QR),$$

between which equations b^5S, b^4QR can be eliminated; thus there will be a form [7.14] deduced from

$$\frac{1}{a}\left((5.9)(3.5),\ (3.3)(5.11),\ (2.2)^2(4.10)\right).$$

Also the sole components of $\mathfrak{R}\,7.14$ will be easily seen to be

$$(3.6\times(2.4)^2,\ 3.6\times 2.6\times(1.1)^2).$$

Hence $\mathfrak{R}\,[(7.14)] = (Q^2T,\ b^2RT)$, in which each of the two arguments is a residue*. Hence we may find a 7.14 which will be divisible by a and thus obtain a form 6.14, which (since 5.14 is necessarily non-existent) cannot be further depressed.

That this is not a *null* form will presently be demonstrated. It results that 6.14 is a new groundform, and we have now completed a new (quatertiary) group, that is, the third in order of descent from the primaries, namely, the group 5.12, 6.13, 7.17, 6.14.

Here, having reached the middle of this long deduction, it will be expedient to pause for a while and take stock of the relations so far established between the base-forms and their deducts.

I enclose, in what follows, the deg-weight numbers within square brackets, in order to indicate that the forms which they represent are not necessarily identical with the simplified forms represented by the same numbers, but are the immediate quotients which present themselves after dividing out by a or a power of a in the course of the deduction. We have thus

$$a^3[3.6] - a^2[2.2].[2.4] = (2.2)^3,\ (3.3)^2 \qquad (3)$$
$$a\,[4.7] = [2.2][3.5],\ [2.4][3.3] \qquad (1)$$
$$a^2[4.8] + a\,(?) = [3.3][3.5],\ [2.2]^2[2.4] \qquad (2)$$
$$a^2[4.10] + a\,(?) = [3.5]^2,\ [2.4]^2[2.2] \qquad (2)$$
$$a\,[5.9] = [4.7][2.2],\ [3.3][3.6] \qquad (4)$$
$$a\,[5.11] = [4.7][2.4],\ [3.5][3.6] \qquad (4)$$
$$a^2[5.12] + a\,(?) = [3.6][2.4][2.2],\ [4.7][3.5] \qquad (5)$$
$$a\,[7.17] = [5.12][3.5],\ [3.6][5.11],\ [4.10][4.7] \qquad (6)$$
$$a\,[6.13] = [4.10][3.3],\ [2.2][5.11],\ [2.4][5.9] \qquad (5)$$
$$a^2[6.14] = [5.9][3.5],\ [5.11][3.3],\ [2.2]^2[4.10] \qquad (6)$$

In the above table the quantities connected by one or more commas represent a linear function of themselves, and the sign of interrogation means

* For Q^2, b^2R, T are each of them residues.

"some known rational integral function of the base-forms." The numerals to the right (beginning with (3) and ending with (6)) indicate the power of (a) by which each corresponding deduct has to be multiplied in order to become an integral function of the base-forms, and which may be called its integrating factor. Thus for example the integrating factor of [5.9] is a^4, because the integrating factors of the two arguments in the linear function expressing $a\,[5.9]$ are a, a^3 respectively; so again a^5 is the integrating factor of [5.12], because the integrating factors of the arguments of the linear function which expresses $a^2[5.12] + a\,(?)$ are a^3, a respectively. So again the arguments corresponding to $a\,[7.17]$ having the integrating factors a^5, a^4, a^3 respectively, the integrating factor of [7.17] will be $1 + 5$ (the dominant of the numbers 3, 4, 5), that is, 6. This will be sufficient to show how the integrating factors are to be successively obtained, it being of course borne in mind that the integrating factor of a product of deducts is the product of the integrating factors of the deducts taken separately. With the aid of this table we may see *à priori* that the linear forms representing [7.17], [6.13], [6.14] cannot be identically *nulls*. In the preceding cases no proof is required because we know subinvariants can only be decomposed in one way into factors.

Thus, firstly, for [7.17], the integrating factors of the three arguments being a^5, a^4, a^3; for if a syzygy existed between them we should have $B_1 + aB_2 + a^2B_3 = 0$, where each B is a rational integer function of the base-forms not containing a as a factor.

Secondly, for [6.13], the separate integrating factors being a^2, a^4, a^4 respectively, did a syzygy exist, we should have $a^2B + B_1 + B_2 = 0$, and consequently [2.2][5.11] would be in syzygy with [2.4][5.9], which is impossible.

Thirdly, for [6.14], the separate integrating factors being a^4, a^4, a^2, the syzygy is impossible, for the same reason as in the preceding case.

I pass on now to the fifth group, that is, to the deducts four degrees of succession removed from the base-forms.

$\mathfrak{R}\,2.2\,\mathfrak{R}\,6.13 = b^2.bRT$, $\mathfrak{R}\,3.6\,\mathfrak{R}\,5.9 = T.b^3R$. Hence there is a deduct [7.15]. Its integrating factor will be a into the dominant of the integrating factors of 6.13, 5.9, which are a^4, a^5, that is, it will be a^6. Also in regard to the duads 1.1, 2.4, 2.6, 3.9, 3.6, the compositions of 7.15 are

$$(1.1)^3 + (2.6)^2,\ (1.1)^2 + (2.4) + (3.9),\ (1.1) + (2.4)^2 + (2.6),$$

or b^3R^2, b^2QS, bQ^2R, and the two latter being residues we may write $\mathfrak{R}\,7.15 = b^3R^2$. Its integrating factor is a into the dominant of the integrating factors of 6.13, 5.9 (which are a^5, a^4), and is therefore a^6; 7.15 is necessarily a groundform, for b^3R^2 is obviously indecomposable into simpler residues.

Again $\mathfrak{R}\,3.6\,\mathfrak{R}\,6.13 = T.bRT$, and $\mathfrak{R}\,5.12\,\mathfrak{R}\,4.7 = RT.bT$. Hence 8.19 is a deduct, and its decompositions in respect to the customary duads being

$3.6 \times 3.9 \times 2.4$, $3.6 \times (2.6)^2 \times 1.1$, we have $\mathfrak{R}\, 8.19 = (QST, bR^2T)$. Also 8.19 is a groundform, for the existence of such a form as 7.19 is impossible, inasmuch as 5 times 7 is less than the double of 19. Its integrating index will be the dominant of those of $(3.6)(6.13)$ and $(4.7)(5.12)$ [which are $3+5$ and $1+5$ respectively] increased by unity, that is, is 9. I use here and shall in future use the phrase "index of integration" to signify the index of the power of a which is the integrating factor.

Again, $\mathfrak{R}\, 4.7\, \mathfrak{R}\, 6.13 = bT.b.RT$, $\mathfrak{R}\, 5.12\, \mathfrak{R}\, 3.6\, \mathfrak{R}\, 2.2 = RT.T.b^2$. Hence there is a deduct [9.20].

The resolutions of the duad 9.20 in respect to 3.6, 3.9, 2.6, 2.4, 1.1 are

$$3.6 + 3.9 + 2.4 + 1.1,\quad 3.6 + (2.6)^2 + (1.1)^2,\quad 3.6 + (2.4)^2 + 2.6,$$

corresponding to $bQST$, b^2R^2T, Q^2RT. Now Q^2, b^2R, RT are already known to be residues, and $\mathfrak{R}\, 2.2\, \mathfrak{R}\, 3.3\, \mathfrak{R}\, (4.0) = (bQST, Q^2RT)$. Hence b^2R^2T, Q^2RT, $bQST$ are all residues. Hence there exists a deduct 9.20 such that $\mathfrak{R}\, 9.20 = 0$, and consequently there is a deduct 8.20 which must be a groundform *, since 7.20 is *à priori* known to be impossible. Its resolutions (regarded as a duad) in respect to the customary duads are

$$(1.1)^2 + (3.9)^2,\ (1.1) + (3.9) + (2.4) + (2.6),\ (2.4)^2 + (2.6)^2,\ (1.1)^2 + (2.6)^2,$$

so that $\mathfrak{R}\, 8.20 = b^2S^2$, $bQRS$, Q^2R^2, b^2R^2. The index of integration to $(4.7)(6.13)$ is $1 + 5 = 6$, and of $(5.12)(3.6)(2.2)$ is $5 + 3 = 8$. Hence the index of integration to 8.20 is $2 + 8$ or 10.

We have now obtained a new group of ground-deducts, fourth in descent from the primaries, namely, 7.15, 8.19, 8.20, whose integrating factors are a^6, a^9, a^{10} respectively.

Again, we have, firstly, the following group

$$\mathfrak{R}\, 2.2\, \mathfrak{R}\, 5.12\, \mathfrak{R}\, 6.13 = b^2.RT.bRT,\quad (\mathfrak{R}\, 3.6)^2\, \mathfrak{R}\, 7.15 = T^2.b^2R^2.$$

Hence there is a deduct [12.27].

In writing out the decomposition table (*quâ* 1.2, 2.4, 2.6, 3.9, 3.6 of 12.27), no account need be taken of $(3.6)^2$, inasmuch as T^2 which it represents is a rational integral function of b, Q, R, S, consequently $(3.6)^2$ will not appear therein.

The table will thus be

$$3.6 + (3.9)^2 + (1.1)^2,\ 3.6 + 3.9 + 2.6 + 2.4 + (1.1)^2,$$
$$3.6 + 3.9 + (2.4)^2,\ 3.6 + (2.6)^2 + (1.1)^2.$$

* I have accidentally omitted here (and may possibly have done so in some other cases) the usual proof by means of the indices of integration, that the deduct is not a null.

Hence $\mathfrak{R}\, [12.27] = (b^3S^2T,\ b^2QRST,\ Q^3ST,\ b^3R^3T)$. But b^2R^2, b^2QS, RT have all been seen to be residues, hence b^3R^3T, b^2QRST are residues.

Also $(\mathfrak{R}\, 4.10)^2 = (b^2S^2, bQRS, Q^2R^2)$ is a residue, as is also bT. Hence $(b^3S^2T, b^2QS.RT, bQ^2R.RT)$ is a residue, and $bQ\,(bS, QR)$, $Q(b^2S, bQR)$ being each of them residues, b^2QS, bQ^2R are each of them separately residues. Hence b^3S^2T is a residue. Also $Q^2\,\mathfrak{R}\, 8.2 = (Q^3ST, bQ^2R^2T)$ is a residue, and bQ^2R^2T is a residue, because bQ^2R, RT are residues. Hence Q^3ST is a residue. Hence all the arguments in expression for $\mathfrak{R}\, [12.27]$, namely, b^3R^3T, b^2QRST, $b^3S^2T^2$, Q^3ST are residues; consequently a deduct 12.27 may be found such that $\mathfrak{R}\, 12.27 = 0$, and there will be a deduct 11.27 which cannot be still further reducible, because 10.27 is necessarily non-existent. Its index of integration will be two greater than the dominant of those of $(5.12)(6.13)$ and 7.15, which are $5 + 5$ and 6, that is, it is 12. Its residue $\mathfrak{R}\, 11.27$ will easily be seen to be

$$(b^3R^4,\ b^3RS^2,\ bQ^2R^3,\ bQ^2S^2,\ b^2QSR^2,\ Q^3RS).$$

Again, secondly, $\mathfrak{R}\, 5.9\ \mathfrak{R}\, 5.12 = RT.b^2R$,

$$\mathfrak{R}\, 3.6\ \mathfrak{R}\, 7.15 = T.b^2R^2.$$

Hence there is a deduct 9.21 which cannot be further depressed, because 8.21 is necessarily non-existent, and it will readily be found that

$$\mathfrak{R}\, 9.21 = (b^3S^2,\ b^3R^3,\ b^2QRS.Q^3S),$$

and that the index of integration is $1 + 4 + 5$, that is, is 10.

Again, thirdly,

$$\mathfrak{R}\, 6.13\ \mathfrak{R}\, 7.17 = bRT\,(Q^2S, b^2RS, bQR^2)$$
$$\mathfrak{R}\, 5.11\ \mathfrak{R}\, 8.19 = (b^2S, bQR)\,(QST, bR^2T)$$
$$\mathfrak{R}\, 3.6\ \mathfrak{R}^2\, 5.12 = T.(RT)^2 = R^2T(Q^3, b^2QR, b^3S)$$
$$\mathfrak{R}\, 5.12\ \mathfrak{R}\, 4.8\ \mathfrak{R}\, 4.10 = RT.b^2R(bS, QR)$$
$$\mathfrak{R}\, 2.4\ \mathfrak{R}\, 3.6\ \mathfrak{R}^2\, 4.10 = Q.T.(bS, QR)^2$$
$$\mathfrak{R}\, 2.4\ \mathfrak{R}\, 3.6\ \mathfrak{R}\, 8.20 = Q.T(b^2S^2, bQRS, Q^2RQ^2, b^2R^2).$$

Hence it will be seen that the arguments on the right-hand side of the equation are the five following, namely, bQ^3RST, b^3R^2ST, b^2QR^3T, b^2QS^2T, Q^3R^2T, and no others. Hence the six products on the left may be linearly combined so as to give a result zero, and there will consequently be a deduct 12.30.

To prove that this is not a null, take the integrating factors of

$$(6.13)(7.17),\ (5.11)(8.19),\ (3.6)(5.12)^2,\ (5.12)(4.8)(4.10),$$
$$(2.4)(3.6)(4.10)^2,\ (2.4)(3.6)(8.20).$$

These will be found to be

$$5 + 6,\ 4 + 9,\ 3 + 5 + 5,\ 5 + 2 + 2,\ 3 + 2 + 2,\ 3 + 10, \text{ or } 11, 13, 13, 9, 7, 13.$$

Hence if there were any syzygy between these products it must be between the 2nd, 3rd and 6th, which have a common integrating factor a^{13}, but the

3rd and 6th products have a common factor 3.6; hence the three cannot be syzygetically connected, and consequently 12.30 is a *bona-fide* existing deduct, and being incapable of further depression, is necessarily a ground-form.

The index of integration will be a unit greater than the dominant of the indices last found, that is, it is 14.

Its residue will be found to be of the form

$$(b^3S^3,\ b^3R^3S,\ b^2QR^4,\ b^2QRS^3,\ bQ^2R^2S,\ Q^3R^3,\ Q^3S^2).$$

Again, fourthly,

$$\mathfrak{R}\,6.13\ \mathfrak{R}\,8.19 = bRT.(QST,\ bR^2T)$$

$$\mathfrak{R}^2 5.12\ \mathfrak{R}\,4.\ 8 = R^2T^2.b^2R$$

$$\mathfrak{R}^2 5.12\ \mathfrak{R}^2 2.\ 4 = R^2T^2.Q^2$$

$$\mathfrak{R}^2 3.\ 6\ \mathfrak{R}^2 4.10 = T^2.(bS,\ QR)^2$$

$$\mathfrak{R}\,2.4\ \mathfrak{R}\,3.6\ \mathfrak{R}\,5.12\ \mathfrak{R}\,4.10 = QT.RT.(bS,\ QR).$$

In these five equations the arguments on the left-hand side are four in number, namely, $b^2R^3T^2$, $b^2S^2T^2$, $bQRST^2$, $Q^2R^2T^2$. Accordingly, a linear combination of the five quantities on the right-hand side will be zero, and there is a deduct 13.32 which cannot be further depressed (since 12.32 is necessarily non-existent), and may be easily seen to be an actual quantity and not a null, inasmuch as the indices of integration of the products of which the quantities to the left are the residues (the anti-residues as they may be termed), are

$$5+9,\ 5+5+2,\ 5+5,\ 3+3+2,\ 3+5+2, \text{ that is, } 14,\ 12,\ 10,\ 8,\ 10,$$

of which only *a pair are equal*. Its index of integration is one unit more than the dominant of these numbers, that is, is 15.

Finally $\mathfrak{R}\,13.32 = (b^2RT,\ b^2RS^2T,\ bQR^2ST,\ Q^2R^3T,\ Q^2S^2T)$. The four last deducts 11.27, 9.21, 12.30, 13.32 form the batch fifth in descent from the primaries, and their indices of integration have been shown to be 12, 10, 14, 15.

We are now within sight of the goal of our wearisome pilgrimage. We may form eight equations leading to 18.45, the skew-invariant, as follows:

$$\mathfrak{R}\,4.10\ \mathfrak{R}\,7.17\ \mathfrak{R}\,3.6\ \mathfrak{R}\,5.12 = (bS,\ QR)(Q^2S,\ b^2RS,\ bQR^2).T.R.T \quad (1)$$

$$\mathfrak{R}^2 4.10\ \mathfrak{R}\,3.6\ \mathfrak{R}\,8.19 = (bS,\ QR)^2.T.(QST,\ bR^2T) \quad (2)$$

$$\mathfrak{R}^2 4.10\ \mathfrak{R}\,6.13\ \mathfrak{R}\,5.12 = (bS,\ QR)^2.bRT.RT \quad (3)$$

$$\mathfrak{R}\,8.20\ \mathfrak{R}\,3.6\ \mathfrak{R}\,8.19 = (b^2S^2,\ b^2R^3,\ bQRS,\ Q^2R^2)\,T(QST.bR^2T) \quad (4)$$

$$\mathfrak{R}\,8.20\ \mathfrak{R}\,6.13\ \mathfrak{R}\,5.12 = (b^2S^2,\ b^2R^3,\ bQRS,\ Q^2R^2)bRT.RT \quad (5)$$

$$\mathfrak{R}\,11.27\ \mathfrak{R}\,3.6\ \mathfrak{R}\,5.12 = (b^2R^4,\ bQ^2R^3,\ bQ^2S^3,\ b^2QSR^2)\,T.RT \quad (6)$$

$$\mathfrak{R}\,6.13\ \mathfrak{R}\,13.32 = bRT(b^2R^4T,\ b^2RS^2T,\ bQR^2ST,\ Q^2R^3T,\ Q^2S^2T) \quad (7)$$

$$\mathfrak{R}\,9.21\ \mathfrak{R}^2 5.12 = (b^3S^2,\ b^3R^3,\ b^2QRS,\ Q^3S)R^2T^2. \quad (8)$$

The arguments on the right-hand side of these equations will be seen to be the seven following: $T^2b^3R^5$, $T^2b^3R^2S^2$, $T^2b^2QR^3S$, $T^2b^2QS^3$, $T^2bQ^2R^4$, $T^2bQ^2RS^3$, $T^2Q^3R^2S$. Hence a linear function of the anti-residues to the eight products to the left can be made zero, and the sums of each set of duads being 19.45, there emerges the deduct 18.45 corresponding to the skew-invariant 18;0.

That this is not a null may be shown in the usual manner as follows: The indices of integration of the several anti-residues are

$$2+6+3+5,\ 2+2+3+9,\ 2+2+5+5,\ 10+3+9,\ 10+5+5,\ 12+3+5,$$
$$5+15,\ 10+5, \text{ that is, } 16,\ 16,\ 14,\ 22,\ 20,\ 20,\ 20,\ 15.$$

The 5th, 6th and 7th indices constitute the only triad of equal indices, but the 5th, 6th and 7th anti-residues cannot be in syzygy, inasmuch as the two first of them have the factor 5.12 in common. Hence the value of 18.45 found as above will not be null.

Its index of integration will be one unit more than the dominant of the above numbers, that is, it is 23, and its residue will be of the form

$$(b^3R^6T,\ b^3R^3S^2T,\ b^3S^4T,\ b^2QR^4ST,\ b^2QRS^3T,\ bQ^2R^5T,\ bQ^2R^2S^2T,$$
$$Q^3R^3ST,\ Q^3S^3T).$$

We ought now to be able to show that there exists no other deduct of which the residue is not a rational integral function of the 22 residues which have been determined in order to prove that the system of groundforms obtained is complete. But this inquiry is one of considerable difficulty, and must be reserved for future consideration.

I will now bring together the several steps of the deduction (several of which, especially in the earlier stages, would admit of abridgement), separating the successive strata from one another and substituting the more familiar designation of deg-orders for the equivalent deg-weights. The single numbers on the left-hand side are the indices of integration to the corresponding deducts.

Table of Deduction for the Quintic.

(3) $a^3(3;3) + a^2(?) = (2;6)^3,\ (3;9)^2$

(1) $a(4;6) = (2;6)(3;5),\ (2;2)(3;9)$

(2) $a^2(4;4) + a(?) = (3;9)(3;5);\ (2.6)^2(2.2)$

(2) $a^2(4;0) + a(?) = (3;5)^2,\ (2;2)^2(2;6)$

(4) $a(5;3) = (4;6)(2.2),\ (3;5)(3;3)$

(5) $a^2(5;1) + a(?) = (3;3)(2;2)(2;6),\ (4.6)(3.5)$

(4) $a(5;7) = (4;6)(2;6),\ (3;9)(3;3)$

(5) $(a)(6;4)=(4;0)(3;9),\ (2;6)(5;3),\ (2;2)(5;7)$

(8) $a(7;1)=(3;5)(5;1),\ (3;3)(5;3),\ (4;6)(4;0)$

(6) $a^2(6;2)+a(?)=(5;7)(3;5),\ (3;9)(5;3),\ (2.6)^2(4.0)$

(8) $a(7;5)=(2;6)(6;4),\ (3;3)(5;7)$

(9) $a(8;2)=(3;3)(6;4),\ (5;1)(4;6)$

(10) $a^2(8;0)+a(?)=(4;6)(6;4),\ (5;1)(3;3)(2;6)$

(12) $a^3(11;1)+a(?)=(2;6)(5;1)(6;4),\ (3;3)^2(7;5)$

(10) $a(9;3)=(3;3)(7;5),\ (5;7)(5;1)$

(14) $a(12;0)=(6;4)(7;1),\ (5;3)(8;2),\ (3;3)(5;1)^2(5;1)(4.4)(4.0),$
$(2;2)(3;3)(8;0)$

(15) $a(13;1)=(6;4)(8;2),\ (5;1)^2(6;4),\ (5;1)^2(2;2)^2,\ (3;3)^2(4;0)^2,$
$(2;2)(3;3)(5;1)(4;0)$

(23) $18;0=(4;0)(7;1)(3;3)(5;1),\ (4;0)(3;3)(8;2),$
$(4;0)^2(6;4)(5;1),\ (8;0)(6;4)(5;1),\ (8;0)(3;3)(8;2),$
$(6;4)(13;1),\ (9;3),\ (5;1)^2$

In addition to the deducts which appear in the above table, the groundform 1.5 and the four protomorphs $2;2\ \ 2;6\ \ 3;5\ \ 3;9$ have to be taken into account. Thus the twenty-three groundforms to the quintic will be seen to be distributed among seven batches or categories containing respectively 1, 4, 3, 4, 3, 3, 4, 1 individuals.

It was my intention to have simplified some of the steps of the deduction, and to have supplied the omissions, to show in one or two cases that the deducts as obtained are actual and not null forms*, but unfortunately the proof-sheets have been kept back, owing to the necessities of the printing-office, for some weeks, and in the meanwhile my attention has been drawn off to other parts of the subject, and I am unable to give sufficient time to call back to mind the intended ameliorations or rectifications of the text.

§ 4. Perpetuants.

On Absolutely Irreducible Binary Subinvariants.

Any rational integral value of $(\lambda a\delta_b+\mu b\delta_c+\nu c\delta_d\ldots)^{-1}0$ is a binary subinvariant. If none of the numerical coefficients $\lambda,\ \mu,\ \nu\ldots$ are zero, the subinvariant is simple. If in the series of coefficients $\lambda,\ \mu,\ \nu,\ \pi,\ \rho\ldots$, any number i of breaks occur in consequence of i non-contiguous terms $\nu,\ \rho\ldots$ vanishing, it becomes a multiple subinvariant corresponding to a semi-invariant

* When the deduct is a zero instead of a possible new groundform, it indicates a syzygy between anterior groundforms.

of i distinct binary quantics. If, however, the subinvariant is to appertain to a system of quantics, all of unlimited order, it would be necessary for the breaks in the series to be each of them at an infinite distance from the initial term and from one another.

In what follows I shall confine my attention to simple binary subinvariants, and investigate the types, that is, the deg-weights (order ceases to be predicable) of those of them which are absolutely indecomposable, that is, incapable of being expressed as rational integral functions of others of lower types of any extent whatever.

It may be convenient to give a name to absolutely indecomposable subinvariants, and I propose, until an apter word presents itself, to call them perpetuants*. The present section then will be occupied with the successive determination of the types of all possible simple binary perpetuants up to a certain limit of degree.

We know, by Cayley's rule, that the number of linearly independent binariants of degree j and weight w is the difference between the number of partitions of w into j parts, and the number of partitions of $w-1$ into such parts, and therefore by Euler's law of reciprocity is the difference between the number of partitions of w into parts none exceeding j, and the number of partitions of $w-1$ into such parts; it is therefore the coefficient of x^w in

$$\left\{\frac{1}{(1-x)(1-x^2)\ldots(1-x^j)}-\frac{x}{(1-x)(1-x^2)\ldots(1-x^j)}\right\}$$

or the coefficient of x^w in $\dfrac{1}{(1-x^2)(1-x^3)\ldots(1-x^j)}$, which I shall call the generating function for the degree j of the linearly independent subinvariants.

Thus for the degree 1 the generating function is simply 1, and there will be one subinvariant (a) of the degree 1 and weight zero.

For the degree 2 the generating function is $\dfrac{1}{1-x^2}$, which expanded gives the series $1+x^2+x^4+\ldots$; there is consequently one semi-invariant of the degree 2 for every even weight 0, 2, 4, 6 ...; but the first of these will be merely the square of the one of degree 0 and weight 1; hence the generating function for the perpetuants of degree 2 is $\dfrac{1}{1-x^2}-1$ or $\dfrac{x^2}{1-x^2}$ giving rise to the deg-weights $2.2\ \ 2.4\ \ 2.6\ldots$ corresponding to the well-known series of quadrinvariants or quadri-semi-invariants $ac-b^2,\ ac-4bd+3c^2,\ldots$. Again,

* Perhaps *Revenants* would be more expressive to signify the forms (or ghosts of forms, if one pleases to say so) which never die out, but *continually return* as the leading coefficients of irreducible covariants. Such I need not say is not the case with conditionally irreducible integrals of the above partial differential equation (as for instance the discriminants to the cubic), which sooner or later die out and are seen no more as sources of irreducible covariants to quantics of a superior order.

for $j=3$ the generating function to the linearly independent binariants, or for brevity sake say the *total* generating function is $\frac{1}{(1-x^2)(1-x^3)}$.

To find the irreducible forms, or say the *limited* generating function, we must take away the cube of the one of degree 1 and weight zero, and the product of this one and each indecomposable one of the degree 2, and consequently the limited generating function will be

$$\frac{1}{(1-x^2)(1-x^3)}-\left(\frac{x^2}{1-x^2}+1\right) \text{ that is } \frac{x^3}{(1-x^2)(1-x^3)};$$

thus we obtain perpetuants of the deg-weights $3.i$, where the least value of i is 3 and the number of such for $i=3, 4, 5, 6, 7, 8$; $9, 10, 11, 12, 13, 14$; $15, 16, 17, \ldots$ will be $1, 0, 1, 1, 1, 1$; $2\ 1\ 2\ 2\ 2\ 2$; $3, 2, 3, \ldots$.

Again, for $j=4$, the total generating function is $\frac{1}{(1-x^2)(1-x^3)(1-x^4)}$.

To determine the subtrahend consider the total partitions of 4 (the number itself not counting as a partition). These are 1^4, $1^2.2$, 1.3, 2^2. The three former will give rise to the partial subtrahends $1, \frac{x^2}{1-x^2}, \frac{x^3}{(1-x^2)(1-x^3)}$, but for 2^2, that is, 2.2 the case is different.

Taking the development of $\frac{x^2}{1-x^2}$, that is, $x^2+x^4+x^6+x^8+\ldots$ the function corresponding to 2.2 to be subtracted is not $\left(\frac{x^2}{1-x^2}\right)^2$, but the sum of the *homogeneous products* of the second order of the infinite succession $x^2, x^4, x^6, x^8, \ldots$, or calling s_1 the sum of the terms and s_2 the sum of their squares, is $\frac{s_1^2+s_2}{2}$, that is, is

$$\frac{1}{2}\left\{\left(\frac{x^2}{1-x^2}\right)^2+\frac{x^4}{1-x^4}\right\} \text{ or } \frac{x^4(1+x^2)+x^4(1-x^2)}{(1-x^2)(1-x^4)},$$

that is,

$$\frac{x^4}{(1-x^2)(1-x^4)}.$$

Hence the *limited* generating function for the degree 4 is

$$\frac{1}{(1-x^2)(1-x^3)(1-x^4)}-\left(\frac{x^3}{(1-x^2)(1-x^3)}+\frac{x^2}{1-x^2}+1\right)-\frac{x^4}{(1-x^2)(1-x^4)},$$

which is

$$\frac{1}{(1-x^2)(1-x^3)(1-x^4)}\{1-(1-x^4)-x^4(1-x^3)\},$$

that is

$$\frac{x^7}{(1-x^2)(1-x^3)(1-x^4)}.$$

Let us pause a moment in the deduction to draw an inference from this result. The lowest power of x in the development of the limited generating function for the degree 4 being x^7, we see that an absolutely indecomposable binariant of the 4th degree cannot be of lower weight than 7. Consider any semi-invariant of degree 4 to a quantic of order i. Its weight must be less

than $2i$. Hence if it is indecomposable, 7 must be less than $2i$, or i must be at least 4. Thus we see that there can be no absolutely indecomposable binariant of the 4th degree appertaining to a cubic. This shows *à priori* that the discriminant to the cubic, regarded as a subinvariant, is decomposable, as we know is the case*.

So in general if we know that no perpetuant of the degree j is of lower weight than k, we may be assured that no invariant or semi-invariant to a quantic of the degree j can be absolutely indecomposable if the order of the quantic is less than $\frac{2k}{j}$.

Agreeing to call the weight of any subinvariant divided by its degree its relative weight, we may put this result into words, by saying no quantic can possess an absolutely indecomposable invariant or semi-invariant of a given degree unless its order is at least twice as great as the minimum relative weight of a perpetuant of that degree. We may see further that the quartic can have no indecomposable invariant or semi-invariant of the degree 4, for its weight would be 8, but x^8 does not appear in the development of

$$\frac{x^7}{(1-x^2)(1-x^3)(1-x^4)}.$$

Pass we on now to the case of the 5th degree.

The indefinite partitions of 5 (leaving 5 itself out of the number) are 4.1, 3.2, $3.1.1$, $2.2.1$, 2.1^3, 1^5 which obviously give rise to the subtrahends

$$\frac{x^7}{(1-x^2)(1-x^3)(1-x^4)}, \frac{x^3}{(1-x^2)(1-x^3)}, \frac{x^2}{1-x^2}, \frac{x^3}{(1-x^2)(1-x^3)},$$
$$\frac{x^4}{(1-x^2)(1-x^4)}, \frac{x^2}{1-x^2}, 1.$$

But from the mode in which the deduction has been carried on, it will be obvious on reflection that the sum of all these except the second which corresponds to a partition not ending with a unit will be equal to the total generating function for the case of the degree 4. So that the total subtrahend is

$$\frac{1}{(1-x^2)(1-x^3)(1-x^4)}+\frac{x^5}{(1-x^2)(1-x^2)(1-x^3)}.$$

Hence the limited generating function for the degree 5 is

$$\frac{x^5}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)}-\frac{x^5}{(1-x^2)(1-x^2)(1-x^3)},$$

that is, is $\frac{x^5\{1-(1+x^2)(1-x^5)\}}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)}$, which is $\frac{-x^7+x^{10}+x^{12}}{(2)(3)(4)(5)}$, where for brevity I use in general (q) to denote $1-x^q$.

* It may easily be collected from the course of the ensuing investigation that *every* binary discriminant is decomposable into subinvariants of lower degrees than its own.

Here, for the first time, a new feature presents itself, namely, the presence of a negative coefficient in the numerator, and consequently of a series of such in the development in an infinite series of the generating function.

Each negative term $-kx^t$ in the development will obviously indicate the existence of k general syzygies of the degree 5 and weight t, or as we might call them, *privative* groundforms. The number of such terms will be finite, and they will be most readily obtained by writing the *l. g. f.* (limited generating function) under the form

$$\frac{-x^7(1-x^3)(1-x^5)+x^{15}}{(2)(3)(4)(5)}, \text{ that is } \frac{-x^7}{(2)(4)}+\frac{x^{15}}{(2)(3)(4)(5)}.$$

To find them it will be observed that the number of ways of composing 0, 2, 4, 6, 8, 10, 12, 14, 16 with the elements 2 and 4 are respectively 1, 1, 2, 2, 3, 3, 4, 4, 5, and that 1, 1, 2, 3, 5 are the number of ways of composing 0, 2, 4, 6, 8, with the elements 2, 3, 4, 5. Hence there will exist the negative terms

$$-x^7,\ -x^9,\ -2x^{11},\ -2x^{13},\ -2x^{15},\ -2x^{17},\ -2x^{19},\ -x^{21}\,*,$$

the sum of which is

$$-\frac{x^7+x^{11}-x^{21}-x^{23}}{1-x^2}.$$

Adding this with its sign changed to $\dfrac{-x^7+x^{10}+x^{12}}{(2)(3)(4)(5)}$ there results

$$\frac{x^{18}+x^{20}-x^{21}-x^{23}+x^{24}+x^{25}+2x^{26}-x^{29}-2x^{30}-x^{31}-x^{32}+x^{33}+x^{35}}{(2)(3)(4)(5)},$$

which may be thrown under the form

$$x^{18}\left\{\frac{(3)+x^2(2)(3)+x^4+x^6(8)+x^8(3)(4)+x^8(4)(5)}{(2)(3)(4)(5)}\right\}.$$

It is therefore omni-positive in its development, which shows that no negative terms have been omitted, but that the 13 syzygies of odd weights ranging from 7 to 21 typically represented by $-\dfrac{x^7+x^{11}-x^{21}-x^{23}}{1-x^2}$ (say $-R_5$) constitute their entire aggregate. We see also that the minimum weight of a perpetuant of the 5th degree is 18, so that the double of the minimum relative weight is $\dfrac{36}{5}$, and accordingly there can exist no absolutely indecomposable binary subinvariants of the 5th degree, until we come to Quantics of the 8th order or upwards.

Proceeding to the degree 6, the total subtrahend from the *t. g. f.* (total generating function) for that degree would be *ut suprà* the *t. g. f.* for the

* The numbers 1 1 2 2 2 . 2 1 are got by subtracting from the figures 1 1 2 2 3 3 4 4 5 the figures 1 1 2 3 5

degree one below (here 5), less expressions depending on the partitions of 6 not concluding with a unit, were it not for the presence of the negative terms represented by $-R_5$; the quantity to be subtracted corresponding to the partition 5 . 1, being now not the *l. g. f.* for degree 5, $\dfrac{-x^7+x^{10}+x^{12}}{(2)(3)(4)(5)}$, but this quantity rendered omni-positive in its development by the addition of R_5.

Hence the total subtrahend will be $\dfrac{1}{(2)(3)(4)(5)}+R_5+$ the quantities depending on the partitions 2 . 4 2 . 2 . 2 3 . 3.

To 2 . 4 will correspond the subtrahend $\dfrac{x^2}{(2)}\cdot\dfrac{x^7}{(2)(3)(4)}$.

To 3 . 3 will correspond $\dfrac{\phi x^2+(\phi x)^2}{2}$ where $\phi x=\dfrac{x^3}{(1-x^2)(1-x^3)}$, and to 2 . 2 . 2 by Crocchi's theorem*, will correspond the representative of the homogeneous products of the 3rd order of the terms in

$$\psi x=\frac{x^2}{1-x^2}, \text{ that is, } \frac{(\psi x)^3+3\psi x\psi x^2+2\phi x^3}{2\,.\,3}.$$

There might for a moment be felt a hesitation in applying the formula for homogeneous products to ϕx, in consequence of the coefficients in its development being no longer exclusively unities; but the force of this objection vanishes as soon as it is borne in mind that we may replace any term kx^t in the development of ϕx by k separate terms x^t, each of which corresponds to a distinct subinvariant.

Thus then to 3 . 3 will correspond the partial subtrahend

$$\frac{x^6}{2}\left\{\frac{1}{(1-x^2)^2(1-x^3)^2}+\frac{1}{(1-x^4)(1-x^6)}\right\} \text{ or } x^6\,\frac{(1+x^2)(1+x^3)+(1-x^2)(1-x^3)}{2(2)(3)(4)(6)},$$

that is,

$$\frac{x^6+x^{11}}{(2)(3)(4)(6)},$$

and to 2 . 2 . 2 will correspond

$$x^6\,\frac{(1+x^2)(1+x^2+x^4)+3(1-x^6)+2(1-x^2)(1-x^4)}{6(2)(4)(6)}, \text{ or } \frac{x^6}{(2)(4)(6)}.$$

It may be remarked, in passing, that for any degree $2i$ the subtrahend corresponding to the partition consisting of i parts (each of the value 2), is $\dfrac{x^{2i}}{(2)(4)\ldots(2i)}$, as may be shown, *à priori*, thus: using y in place of x^2 we have to find the sum of all the quantities ky^t where k is the number of ways of generating y^t as a product of i of the powers $1, y, y^2, y^3 \ldots$, that is, k is the number of ways of composing t with i or less than i of the indefinite

* See for an instantaneous proof of this theorem, the *Johns Hopkins University Circular* for November 1882 [below, p. 653].

series of natural numbers, which by Euler's theorem, already cited, is the same as that of compounding t out of any number of parts none exceeding i. Hence the denominator of the subtrahend required will be

$$\frac{1}{(1-y)(1-y^2)\dots(1-y^i)}, \text{ that is, } \frac{1}{(2)(4)\dots(2i)}.$$

The numerator is obviously x^{2i}, and the complete value $\frac{x^{2i}}{(2)(4)\dots(2i)}$ as was to be found.

I may add, that this theorem (which is one concerning homogeneous product-sums expressed as functions of power-sums of the same elements), by an easy deduction from Crocchi's theorem, serves to show if the ith power-sum of a set of elements is $\frac{1}{1-c^i}$ (I substitute c for y) then the ith elementary symmetric function of the elements is

$$\frac{c^{\frac{i^2-i}{2}}}{(1-c)(1-c^2)\dots(1-c^i)}$$

and reversing the terms of this proposition we may say, that if

$$z^q - \frac{1}{1-c}z^{q-1} + \frac{c}{(1-c)(1-c^2)}z^{q-2}\dots \pm \frac{c^{\frac{n^2-n}{2}}}{(1-c)(1-c^2)\dots(1-c^n)}z^{q-n} + \dots = 0,$$

then the sum of the ith powers of z (q being not less than i) is $\frac{1}{1-c^i}$, to which may be added that the sum of the ith homogeneous products of z is

$$\frac{1}{(1-c)(1-c^2)\dots(1-c^i)},$$

as, for example, if $i=2$ the first of these sums, namely,

$$\frac{1}{(1-c)^2} - 2\frac{c}{(1-c)(1-c^2)} = \frac{1}{1-c^2}$$

and the other, namely,

$$\frac{1}{(1-c)^2} - \frac{c}{(1-c)(1-c^2)} = \frac{1}{(1-c)(1-c^2)}.$$

But this is a mere digression, a wild flower gathered on the wayside. Returning to the determination of the *l. g. f.** for the degree 6, we see that it will be

$$\frac{1}{(2)(3)(4)(5)(6)} - \frac{1}{(2)(3)(4)(5)} - \frac{x^5}{(2)(2)(3)(4)} - \frac{x^6+x^{11}}{(2)(3)(4)(6)} - \frac{x^6}{(2)(4)(6)} - R_5,$$

or $\frac{N}{(2)(3)(4)(5)(6)} - R_5$, where

* I repeat that *t. g. f.* stands for total generating function, and *l. g. f.* for limited generating function.

$$\begin{aligned} N &= x^6 - (1+x^2+x^4)(1-x^5)x^9 - (x^6 - x^{16}) - x^6(1-x^3)(1-x^5) \\ &= x^6 + x^{14} + x^{16} + x^{18} + x^{16} + x^9 + x^{11} \\ &\quad - x^9 - x^{11} - x^{13} - x^6 - x^6 - x^{14} \\ &= -x^6 - x^{13} + 2x^{16} + x^{18}. \end{aligned}$$

Thus the *l. g. f.* for the degree 6 is

$$-R_5 + \frac{-x^6 - x^{13} + 2x^{16} + x^{18}}{(2)(3)(4)(5)(6)}.$$

$-R_5$ represents the fourteen compound syzygants of the degree 6; the fraction to which $-R_5$ is annexed, when developed, will give rise to only a *limited* number of terms with negative coefficients corresponding to the ground-syzygies; the remainder of the terms, infinite in number, will represent the infinite succession of groundforms. It may be well here to notice, as a universal fact, that in the development of the fraction $\frac{R(x)}{(2)(3)\dots(n)}$ (where $R(x)$ is rational integral function of x) the number of negative terms or the number of positive terms will be finite according as $R(1)$ is positive or negative, and, as in the above fraction, $R(1)=1$, it follows that there are only a finite number of negative terms, and consequently only a limited number of ground-syzygies, an important conclusion which will easily be seen to apply not only to the use of the degree 5 (in which syzygies first make their appearance) and 6, as here shown, but for all higher degrees, it being a universal law that the irreducible syzygies for subinvariants of any given degree, and therefore of any degree not exceeding a given limit, are finite in number.

The law that the development of $\frac{R(x)}{(1-x^2)(1-x^3)\dots(1-x^n)}$, commencing from a certain point is omni-positive or omni-negative, according as $\phi 1$ is positive or negative when n exceeds 2, admits of easy proof. Of course the law could not be true when $n=2$, as, for example, for $\frac{1-2x}{1-x^2}$ which remains *neutral*, that is, neither omni-positive nor omni-negative (which latter, if the law did apply, it ought eventually to become) throughout its entire extent.

Beginning with $\frac{Rx}{(1-x^2)(1-x^3)}$ the coefficient of x^i [where $i = 6t+\tau$ $(\tau < 6)$] will be not less than t, and not greater than $t+1$ in the development of

$$\frac{1}{(1-x^2)(1-x^3)}.$$

Hence in the development $\frac{-K+(K+\epsilon)x^\delta}{(1-x^2)(1-x^3)}$ the coefficient of x^i will be not less than $-K(t+1)+(K+\epsilon)\left(t-\frac{\delta}{6}-1\right)$, and consequently for a sufficiently large value of i must be positive. *A fortiori* the same will be true for

$\dfrac{R(x)}{(1-x^2)(1-x^3)}$ when $K+\epsilon$ is the sum of the positive coefficients in Rx of powers of x none of whose indices are higher than δ, and K the sum of the negative coefficients of any powers of x; this proves the law for $\dfrac{R(x)}{(1-x^2)(1-x^3)}$ when $R(1)$ is supposed to be positive, and moreover the series will be omni-positive after a certain point in the strict sense of the following coefficients being neither negative nor zero.

Hence the law will be true for $\dfrac{Rx}{(1-x^2)(1-x^3)(1-x^4)}$, for we may divide $\dfrac{Rx}{(1-x^2)(1-x^3)}$, when expanded, into four series, whose indices $\equiv 0, 1, 2, 3$ respectively to modulus 4, and the negative terms in each of these being finite in number, it is clear that the effect of dividing any one of them by $1-x^4$ will be to give rise to a series omni-positive after a certain point, because each coefficient in the quotient of any one of the series divided by $\dfrac{1}{1-x^4}$ will at worst contain only the sum of a finite number of given negative coefficients, and a number of terms all greater than zero, whose sum, when that number is taken great enough, must exceed the arithmetical value of the former sum. Hence $\dfrac{R(x)}{(1-x^2)(1-x^3)(1-x^4)}$ will be the sum of four series, each omni-positive from a certain point, and will therefore be omni-positive from the most advanced of those points. In like manner

$$\frac{Rx}{(1-x^2)(1-x^3)(1-x^4)(1-x^5)}$$

may be shown to be the sum of five series, each with an infinite omni-positive branch, and consequently will be itself of the same character, and so in general. Of course the same reasoning would show the truth of the law when $R(x^i)$ is negative, and that it may be extended to any denominator of the form $(1-x^i)(1-x^j)(1-x^k)\ldots$ provided any two of the indices $i, j, k \ldots$ are prime to one another. And of course a similar conclusion obtains (*mutatis mutandis*) when $R(x)$ is negative. The law might be proved more scientifically and more briefly as a consequence of the general algebraical representation of the denumerant of any equation in integers

$$[l_1x_1 + l_2x_2 + \ldots + l_ix_i = n]$$

as a sum of a non-periodical and of periodical parts, whereof the former is always of a higher dimension in n than any of the latter, except when all the l quantities have a common factor. See the annexed Excursus [p. 605, below].

I now proceed to find the lowest power of x in the fraction

$$\frac{-x^6 - x^{13} + 2x^{16} + x^{18}}{(2)(3)(4)(5)(6)},$$

say F, in which the coefficient is positive, in order to ascertain the minimum weight of an absolutely irreducible subinvariant of the 6th degree.

I think the easiest practical mode of proceeding to effect this is to use the table in my possession (having been previously calculated for me by Mr Franklin for another purpose) which gives the coefficients of the powers of x in $\dfrac{1}{(2)(3)(4)(5)(6)}$; those coefficients used as they stand, then advanced seven steps, then five steps further, then taken back two steps, and at the same time doubled, will give four series of numbers, the sum of the 1st and 2nd of which subtracted from the sum of the 3rd and 4th will give the successive coefficients from x^6 upwards in the development of F.

The four series are as underwritten:

(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)
1,	0,	1,	1,	2,	2,	4,	3,	6,	6;	9,	9,	14,	13,	19,	20,	26,	27,	36,	36;	47,
							1,	0,	1,	1,	2,	2,	4,	3,	6,	6;	9,	9,	14,	13,
												1,	0,	1,	1,	2,	2,	4,	3,	6,
										2,	0,	2,	2,	4,	4,	8,	6,	12,	12,	18,

(21)	(22)	(23)	(24)	(25)	(26)	(27)	(28)	(29)	(30)	(31)	(32)	(33)	(34)	(35)	(36)
49,	60,	63,	78,	80,	97,	102,	120,	126;	149,	154,	180,	189,	216,	227,	260,
19,	20,	26,	27,	36,	36;	47,	49,	60,	63,	78,	80,	97,	102,	120,	126;
6;	9,	9,	14,	13,	19,	20,	26,	27,	36,	36;	47,	49,	60,	63,	78,
18,	28,	26,	38,	40,	52,	54,	72,	72;	94,	98,	120,	126,	156,	160,	194,

(37)	(38)	(39)	(40)	(41)	(42)	(43)	(44)	(45)	(46)	(47)	(48)	(49)	(50)	(51)
270,	307,	322;	361,	378,	424,	441,	492,	515,	568,	594,	656,	682;	750,	783,
149,	154,	180,	189,	216,	227,	260,	270,	307,	322;	361,	378,	414,	441,	492,
80,	97,	102,	120,	126;	149,	154,	180,	189,	216,	227,	260,	270,	307,	322;
204,	240,	252;	298,	308,	360,	378,	432,	454,	520,	540,	614,	644;	722,	756,

(52)	(53)	(54)	(55)	(56)	(57)	(58)	(59)	(60)	(61)	(62)	(63)
854,	891,	972,	1010,	1098,	1144,	1236,	1287;	1391,	1443,	1555,	1617,
515,	568,	594,	656,	682;	750,	783,	854,	891,	972,	1010,	1098,
361,	378,	424,	441,	492,	515,	568,	594,	656,	682;	750,	783,
848,	882,	984,	1030,	1136,	1188,	1312,	1364;	1500,	1566;	1708,	1782,

(64)	(65)	(66)	(67)	(68)	(69)	(70)	(71)	(72)	(73)	(74)
1734,	1802,	1932,	2002,	2142,	2223;	2369,	2457,	2618,	2709,	2881,
1144,	1236,	1287;	1391,	1443,	1555,	1617,	1734,	1802,	1932,	2002,
854,	891,	972,	1010,	1098,	1144,	1236,	1287;	1391,	1443,	1555,
1944,	2020,	2196,	2288,	2472,	2574;	2782,	2886,	3100,	3234,	3468,

(75)	(76)	(77)	(78)	(79)
2985,	3164,	3276,	3472,	3588;
2142,	2223;	2369,	2457,	2618,
1617,	1734,	1802,	1932,	2002,
3604,	3864,	4004,	4284,	4446;

The first group of four numbers in which the 3rd and 4th terms combined exceed the 1st and 2nd will easily be seen to be,

$$\left.\begin{matrix}2369\\1617\\1236\\2782\end{matrix}\right\}$$

which is 70 places from the first term, and for which the difference is 4018 less 3986 or 32. Starting from this point the series for F will be seen to be

$$32x^{76} - 18x^{77} + 81x^{78} + 36x^{79} + 188x^{80} + 94x^{81} + 211x^{82} + 161x^{83} + 287x^{84} + 242x^{85} + \ldots;$$

so that there can be no practical doubt of the series being omni-positive from and after the 78th power of x*.

The relative weight of any one of the irreducible subinvariants corresponding to $32x^{76}$ is $\frac{76}{6}$, the double of which is $25\frac{1}{3}$. Hence there can be no irreducible semi-invariant of the 6th degree to a quantic below the 26th order, and, on account of the coefficient of x^{77} being negative, we see that a quantic of the 26th order can have no groundforms of the 6th degree in the coefficients except such as are invariants or quart-invariants.

As regards the syzygies irrespective of the compound ones represented by $-R_5$, we see that there will be primitive ones of all weights from 6 to 77 inclusive, with the exception of the weights 7 and 76, but that there will be no syzygies, whether reducible or irreducible, of the same weights as the irreducible subinvariants. Let us now pass on to the case of the 7th degree†.

The partitions of seven itself and those ending in unity excluded are 5.2 4.3 2.2.3.

Hence calling R_6 the sum of the negative terms in $\frac{-x^6 - x^{13} + 2x^{16} + x^{18}}{(2)(3)(4)(5)(6)}$, the *l. g. f.* for 7 will be

$$\frac{x^7}{(2)(3)(4)(5)(6)(7)} - \frac{-x^7 + x^{10} + x^{12}}{(2)(3)(4)(5)}\,\frac{x^2}{(2)} - \frac{x^7}{(2)(3)(4)}\,\frac{x^3}{(2)(3)} - \frac{x^4}{(2)(4)}\,\frac{x^3}{(2)(3)} - R_5\frac{x^2}{1-x^2} - R_6.$$

If we call this
$$\frac{x^7 + N}{(2)(3)(4)(5)(6)(7)} - R_5\,\frac{x^2}{1-x^2} - R_6,$$

$$N = x^7(1-x^7)\{(1+x^2+x^4)(-x^9+x^{12}+x^{14}) + x^{10}(1+x+x^2+x^3+x^4)(1-x+x^2) + x^7(1-x^5)(1+x^2+x^4)\} = -(1-x^7)P,$$

where $P = \Sigma x^t - \Sigma x^\tau$, t having the values 12 14 16, 14 16 18; 10 11 12 13 14, 12 13 14 15 16; 7 9 11, and τ having the values 9 11 13; 11 12 13 14 15; 12 14 16.

* This conclusion will be strictly proved in the sequel with the aid of my general partition formulæ, in Section V.

[† For the 7th degree, cf. J. Hammond, *American Journal*, Vol. v. (1882), p. 225, under the heading: Disproof of Prof. Sylvester's Fundamental Postulate.]

Hence
$$P = x^7 + x^{10} + x^{12} + 2x^{14} + 2x^{16} + x^{18},$$
and
$$x^7 + N = -x^{10} - x^{12} - x^{14} - 2x^{16} - x^{18} + x^{17} + x^{19} + 2x^{21} + 2x^{23} + x^{25}.$$

The first term in the development of $\frac{x^7+N}{(2)(3)\ldots(7)}$ is $-x^{12}$, indicating that the first irreducible syzygy is of the weight 12; it is not until a very high power of x is reached that a positive coefficient corresponding to a perpetuant makes its appearance.

The tables set out in a subsequent section exhibit *inter alia* the coefficients in the developments of $\frac{1}{(2)(3)\ldots(7)}$ and $\frac{1}{(2)(3)\ldots(6)}$, say F_7 and F_6 as far as the 174th power of x. Using instead of $\frac{x^7+N}{(2)\ldots(7)}$ the equivalent value $x^7F_7 - PF_6$, if the coefficient of x^{q+7} in this is positive, the coefficient of x^q in F_7 must be greater than that of x^q in $(1 + x^3 + x^5 + 2x^7 + 2x^9 + x^{11})F_6$, and *à fortiori* greater than that of x^q in $8x^{11}F_6$, that is, greater than 8 times that of x^{q-11} in F_6. But a glance at the tables* for the developments of F_7, F_6 will show that this is never the case within the limits of q, furnished by the tables, that is, for any value of q not exceeding 174. It is certain, therefore, that the value of the lowest index of x^r, for which in $\frac{N}{(2)\ldots(7)}$ the coefficient is positive, must considerably exceed 181, as indeed one might have anticipated from the series of similar exponents 2, 3, 7, 18, 76 corresponding to the cases previously considered, the ratio of increase in these numbers going on continually increasing†. To ascertain the value of the exponent in question there is left no resource but to endeavour to elicit it (as I shall presently proceed to do) from the general algebraical value of the coefficient. But before doing so it will be well to notice a very important inference that may be drawn from the form of the generating function, namely,

$$\frac{N}{(2)(3)(4)(5)(6)(7)} - \frac{R_5}{(2)} - R_6.$$

$\frac{R_5}{(2)}$ or $(1 + x^2 + x^4 + \ldots)(x^7 + x^9 + 2x^{11} + 2x^{13} + 2x^{15} + 2x^{17} + 2x^{19} + x^{21})$ will represent the deg-weights of the compound syzygies corresponding to the multiplication of the syzygies of the deg-weights 5.7 5.9 5.11 5.13 5.15 5.17 5.19 5.21 5.23 by the groundforms of every even weight.

There will thus be seen to exist compound syzygies of every odd weight (no less than 13 in fact of weight 21 or any higher odd number). If then ω' be the lowest power of x in $\frac{N}{(2)(3)(4)(5)(6)(7)}$ with a positive coefficient and

* *Vide* the numerical tables at end of Section V of this Memoir.

† Subsequent calculations, however, have revealed to me that this ratio does not go on continually increasing.

with an odd exponent, there will coexist groundforms and syzygies of the same degree and weight appertaining to the quantic of an infinite order for every weight denoted by an odd number not less than ω'. From this it is easy to infer that there must exist syzygies and groundforms of the same deg-weight (and therefore of the same deg-order) for one or more quantics of an order not exceeding ω'; [and it may be added that ω' being a high number (not a number less than 23) there will be 13 syzygies of every odd weight equal to or greater than ω'].

For suppose that Q is a quantic of order i. In determining its ground-semi-invariants of the successive degrees the same process may be applied as in calculating the perpetuants, that is, the ground semi-invariants to a quantic of an unlimited order, except that in lieu of the complete development of the generating function $\frac{1}{(1-x^2)(1-x^3)\ldots(1-x^j)}$ only such powers of x must be retained as are not higher than x^i. For the number of linearly independent subinvariants of the weight w and degree j will now be the difference between the number of ways of making up w with j parts none greater than i, less the number of ways of so making up $(w-1)$ which will be the difference between the number of ways of making up w and of making up $(w-1)$ with i parts none greater than j, which, if w does not exceed i, will be the same as if i were infinite. So far then as weights not superior in value to i are concerned, the total generating function for a quantic of the order i will be the same as for a quantic of an unlimited order, and consequently up to the weight i (inclusive) the generating functions for the ground subinvariants (to be obtained, be it remembered, by combining the total generating functions in the same manner, whatever the value of i may be) will be the same for a quantic of the ith as for the quantic of an unlimited order. Hence there must of necessity appertain irreducible covariants and compound syzygants of the same degree and order (namely, of the deg-order $7 . 5\omega'$) to a quantic of the order ω', and of course there is nothing to prevent such coexistence holding good for a quantic of an order very much lower than ω', the least value of which number say i, as far as I am able at present to see, can only be determined by putting each quantic of an order inferior to i successively upon its trial, a work of exceedingly great labour to undertake.

I use ω' to signify the lowest *odd* power of x in the development of the *g.f.* to perpetuants of the 7th degree affected with a positive coefficient, reserving ω to signify the lowest power (whether odd or even) so affected. Until further investigation we cannot say whether ω is equal to or less than ω', but we know that no absolutely irreducible subinvariant of the 7th degree can appertain to a quantic of an order lower than $\frac{2\omega}{7}$, a number whose exact

value we shall eventually succeed in ascertaining with the aid of a partition formula obtained by the method which will form the subject of the annexed "excursus."

Inasmuch as the theory is precisely the same for fractions in general as for those which correspond to denumerants (the name I give to the number of solutions in integers of one or more linear equations), I shall show how to find the general term in the development of any rational fraction, limiting myself however, for the present, to the theory of rational functions of a single variable, which covers the case with which alone we are here concerned, of denumerants of a single linear equation, or which is the same thing, the problem of exhibiting the number of modes of composing a general number n with given smaller numbers as an algebraico-exponential function of n.

When analysis is sufficiently advanced to admit of a perfectly methodical distribution of its subject-matter, the theorem for the expansion of rational functions, about to be given, will, it seems to me, take its place immediately after Newton's binomial theorem, as the second leading theorem of Algebra; my method of partitions (as stated and applied in Tortolini's *Ann.* Vol. VIII. 1856, and in the *Quarterly Mathematical Journal*, 1855, Vol. I. p. 141, to neither of which I have at present means of access*, but the latter of which is referred to by Prof. Cayley in the *Phil. Trans.* for 1880, footnote p. 47) virtually amounted to an enunciation of the theorem for the case of the reciprocal of a rational integral function all of whose roots are roots of unity, under such a form as almost of necessity to lead to the supposition of its remaining true (*mutatis mutandis*) in the general case; the actual averment of the generalization was, I believe, first made by Prof. Cayley†.

Excursus.

On Rational Fractions and Partitions.

The method of finding the general term in the development of a rational fraction of a single variable in a series of ascending powers of the same may be regarded as a corollary to the following lemma, the proof of which is an instantaneous consequence of the fact that the coefficient of $\frac{1}{x}$, or to use Cauchy's word, the residue of $\frac{1}{(1-e^x)^i}$ developed in ascending powers of x

[* See Vol. II. of this Reprint, p. 90.]

† On second thoughts, and after more deliberate reflection, it occurs to me that I may have overstated in the text above the importance of the general theorem viewed as a theorem *an sich*; and that it is only from its special application to rational fractions whose infinity-roots are all of them roots of unity, that it derives its claim to be regarded as a cardinal theorem in Algebra.

when i is any positive integer is always -1: that this is so will be seen at once from the fact that the effect of changing i into $i+1$ in the above fraction is to increase it by $\frac{e^x}{(1-e^x)^{i+1}}$, that is, by the differential derivative of $\frac{1}{i(1-e^x)^i}$, whose residue is obviously zero, so that the residue of $\frac{1}{(1-e^x)^i}$ will be unaffected by continually decreasing i by a unit until it becomes unity; and obviously therefore the residue in question is always -1.

The lemma may be stated as follows:

The constant term in any proper algebraical fraction developed in ascending powers of its variable is the same as the residue with its sign changed of the sum of the fractions obtained by substituting in the given fraction in lieu of the variable its exponential multiplied in succession by each of its values (zero excepted, if there be such) which makes the given fraction infinite.

Any value of a variable which makes a function infinite may conveniently be called an infinity root, and if it is not zero, a finite-infinity root. So too, a factor whose vanishing makes a function vanish may be termed an infinity factor.

Suppose Fx is a *proper* Algebraical fraction, then we may write

$$Fx = \Sigma\Sigma \frac{c_{\lambda,\mu}}{(a_\mu - x)^\lambda} + \Sigma \frac{\gamma_\lambda}{x^\lambda},$$

where $\lambda = 1, 2, \ldots$; $\mu = 1, 2, \ldots j$ and of course any of the coefficients in either sum may be made zero, and then (using in general here and hereafter co_n to signify the coefficient of x^n in an ascending expansion of the function with which it is in regimen) we have

$$\text{co}_{-1} \Sigma F(a_\nu e^x) \text{ [where } \nu = 1, 2, \ldots j]$$

$$= \text{co}_{-1} \Sigma\Sigma\Sigma \frac{c_{\lambda,\mu}}{(a_\mu - a_\nu e^x)^\lambda} + \text{co}_{-1} \Sigma\Sigma \frac{\gamma_\lambda}{a_\nu^\lambda e^{\lambda x}}$$

$$= \text{co}_{-1} \Sigma\Sigma \frac{c_{\lambda,\mu}}{(a_\mu - a_\mu e^x)^\lambda} = -\frac{c_{\lambda,\mu}}{a_\mu^\lambda} = -\text{co}_0 Fx$$

which proves the lemma.

Hence the coefficient of x^n in a rational function fx, which is the same as $\text{co}_0 \frac{fx}{x^n}$ will be $-\text{co}_{-1} \Sigma (r^{-n} e^{-nx} fre^x)$ or $\text{co}_{-1} \Sigma \{r^{-n} e^{nx} f(re^{-x})\}$, [$r$ meaning each finite-infinity root of fx taken in turn], provided only that $\frac{fx}{x^n}$ is a proper algebraical function, that is, provided that n is greater than the degree of $f(x)$.

As for instance, if the degree of the fraction is zero, the theorem will not give the constant, but will give every coefficient of positive powers in the

ascending expansion of fx, and if it is negative, the theorem will give all but the coefficients of negative powers.

This theorem, as observed by Prof. Cayley, *Phil. Trans.*, 1856, p. 139, may be obtained "from the known theorem," that if fx be resolved into simple partial fractions, the sum of those which have any power of $a - x$ in their denominator will be the residue of

$$\frac{f(a+\zeta)}{x - a - \zeta}\text{*}.$$

Prof. Cayley quotes as "a theorem of Cauchy's and Jacobi's, that the coefficient of $\frac{1}{z}$ in $Fz =$ coefficient of $\frac{1}{t}$ in $\psi' t F \psi t$."

This is obviously not true in general, for we might take $Fz = \frac{1}{z}$ and $\psi t = a + t$ or t^2 and the alleged equality would not exist. It is, however, true whenever ψt is of the form $at + bt^2 +$ etc., as may be proved instantaneously by supposing Fz resolved into partial fractions, and making $z = \psi t$, so that $\int dz\, Fz = \int dt\, \psi' t F \psi t$, and observing that if the expansion of $\psi' t F \psi t$ contains $\frac{k}{t}$, that of $\int dz Fz$ must contain $\frac{k}{z}$, since otherwise when this integral is expressed as a function of t, it would not contain (as it is bound to do) the term $k \log t$. The theorem so limited is sufficient for the purpose in view, since on writing, in place of ζ, $-a(1 - e^{-t})$ we see that the residue of $\frac{f(a+\zeta)}{x - a - \zeta}$ is the same as the residue of $\frac{f(ae^{-t})}{(1 - ae^{-t}x)}$, and consequently the coefficient of x^n in so far as it depends on the infinity root a, will be the residue of $(a^{-n} e^{nt}) f(ae^{-t})$ as has been shown above to be the case. It may, possibly, be thought somewhat surprising that those familiar with the known theorem referred to and the general principle of transformation of residues should not have recognized, previous to the divulgation of my theorem, that the two things put together were competent to give a complete solution of the much ventilated problem of simple denumeration. But, perhaps, even supposing the mental conjunction of the two facts to have taken place, there would still have been needed an act of imagination (such as Kant justly remarks is at the bottom of every advance in geometry, where in reality the proof lies in the construction †) to have led to the choice of the particular transformation

* In his *Cours d'Algèbre*, Edition 1877, Vol. I. pp. 497—499, M. Serret obtains the same result under the form of the value (for ζ = zero) of

$$\frac{1}{\pi(m-1)} \left[\left(\frac{d}{d\zeta}\right)^{m-1} \frac{\zeta^m f(a+\zeta)}{x - a - \zeta} \right],$$

where m is the degree to which $(x - a)$ rises in the denominator of fx.

† Take as an example the theorem that the sum of the three angles of a triangle is equal to two right-angles: as soon as by a stroke of the imagination a line is conceived as drawn from one angle parallel to the opposite side, the truth of the proposition becomes virtually self-evident.

employed in this case, and to have entailed the consequences that are implied in it*.

In applying this theorem to finding the value of the denumerant to the equation $ax + by + \ldots + lt = n$, which I denote by $\dfrac{n}{a, b, \ldots l,}$ and is the same thing as the coefficient of x^n in the expansion of the rational fraction

$$\frac{1}{(1-x^a)(1-x^b)\ldots(1-x^l)}$$

or more generally to finding the value of the denumerant

$$\frac{n}{a_1, a_2, \ldots a_\alpha, b_1, b_2, \ldots b_\beta, \ldots l_1, l_2, \ldots l_\lambda,}$$

(where each letter has a fixed value independent of its subindex), that is, the coefficient of x^n in the development of $\dfrac{1}{(1-x^a)^\alpha (1-x^b)^\beta \ldots (1-x^l)^\lambda}$, say Fx, the first thing to be done is to determine and arrange in convenient groups the infinity roots of these functions. To effect this we have only to write down all the divisors of the set of numbers $a, b, \ldots l$, that is, all the integers which divide one or more of those numbers, say $\delta_1, \delta_2, \ldots \delta_\mu$. These divisors necessarily include the indices $a, b, \ldots l$ and *unity*, which latter we may suppose to be δ_1.

Giving then i every value from 1 to μ, the primitive δ_ith roots of unity will obviously be the infinity roots required, and we may separate the required function of n into μ distinct portions or waves, as I term them, where supposing $\nu_1, \nu_2, \ldots \nu_{\phi(\delta_i)}$ [$\phi(\delta_i)$ being the *totient* of δ_i, that is, the number of integers less than δ_i and prime to it] to be the primitive δ_ith roots of unity, the ith period or wave, say W_i, will be equal to the residue of

$$\Sigma r_q^{-n} e^{nt} F(r_q e^{-t}) \; [q = 1, 2, \ldots, (\phi\delta_i)].$$

Since every primitive root r_q is either equal to or is mated with its reciprocal, the above expression may be replaced by the somewhat more convenient one $\Sigma (r_q^n e^{nt}) F(r_q^{-1} e^t)$.

This again admits of a very important transformation, namely, we may write $\nu = n + \frac{1}{2}(\alpha a + \beta b + \ldots + \lambda l)$ and then

$$W_i = \text{co}_{-1} \Sigma \frac{r_q^\nu e^{\nu t}}{P\left(r_q^{\frac{a}{2}} e^{\frac{at}{2}} - r_q^{-\frac{a}{2}} e^{-\frac{at}{2}}\right)^\alpha}$$

* Thus, for example, the supposed investigator might have chosen to write $\sin t$ or $\log(1+t)$ in lieu of $1-e^t$ and the theorem thereby obtained would have been perfectly valid, but of little if any use, and the great bulk of transformations would certainly be of no use whatever; indeed, it is safe to say that the substitution practised, namely, that of $1-e^{\lambda t}$ [λ being taken at will] is the only one that would lead to a practical solution of the question.

(where P is used to signify that the product is to be taken of terms of like form to the one which is in regimen with it).

From this it follows that every wave W_i expressed as a function of ν, when ν is changed into $-\nu$, becomes $(-)^{\alpha+\beta+\ldots+\lambda-1} W_i$, that is, retains its value absolutely or else merely changes its algebraic sign. To prove this it may be observed that whatever the index of the wave the above sum may be replaced by

$$\frac{1}{2}\text{co}_{-1}\Sigma\left\{\frac{r_q^\nu e^{\nu t}}{P\left(r_q^{\frac{a}{2}} e^{\frac{at}{2}} - r_q^{-\frac{a}{2}} e^{-\frac{at}{2}}\right)^\alpha} + \frac{r_q^{-\nu} e^{\nu t}}{P\left(r_q^{-\frac{a}{2}} e^{\frac{at}{2}} - r_q^{\frac{a}{2}} e^{-\frac{at}{2}}\right)^\alpha}\right\}.$$

This is a consequence of r being either identical with $\frac{1}{r}$ as is the case for W_1 and W_2, or else being mated with it as belonging to the same group of primitive roots of unity.

Hence r_q may be changed into r_q^{-1}, and the expression to be residuated will undergo no change.

Again, if t is changed into $-t$, the residue changes its sign, and finally if r_q, t, and ν are simultaneously changed into $r_q^{-1}, -t, -\nu$ the expression to be residuated remains unaltered, except that it takes up a factor $(-)^{\Sigma\alpha}$. Consequently the effect of changing ν into $-\nu$, leaving everything else unaltered, will be to introduce the factor $(-)^{\Sigma\alpha-1}$; and this being true of every portion of the value of $\dfrac{n}{a\ldots, b\ldots, l\ldots,}$ it follows that when that denumerant is expressed under the form $F\nu$, where $\nu = n + \frac{1}{2}\Sigma\alpha a$, $F(-\nu) = (-)^{-1+\Sigma\alpha} F(\nu)$.

There is consequently an enormous advantage gained, as well in the abbreviation of the calculations as in the conciseness of the result, by putting such a denumerant under the form of a function of the *augmented* argument ν instead of the original argument n; when so expressed I speak of the denumerant being in its canonical form.

In future, for greater simplicity, I shall disuse the indices $\alpha, \beta \ldots$ it being understood (unless the contrary is stated) that any of the indices $a, b, c \ldots$ in the denominator of the denumerant $\dfrac{n}{a, b, c, \ldots l,}$ or in its generating function $\dfrac{1}{(1-x^a)(1-x^b)\ldots(1-x^l)}$ may be made equal to one another.

It is perhaps not unworthy of notice that the denumerant $\dfrac{n}{a, b, \ldots, l,}$ may be expressed as the residue of a double sum without knowing the divisors of the indices. For it is obvious that we may express it as the sum of an infinite number of waves whose indices take in all values from unity up to infinity (since all those whose indices are non-divisors will be equal to zero)*,

* By a process, so to say, of *natural selection*.

and consequently as the residue of a sum of quantities obtained by substituting for r in the expression

$$\frac{r^{\nu} e^{\nu x}}{P\left(r^{\frac{a}{2}} e^{\frac{x}{2}} - r^{-\frac{a}{2}} e^{-\frac{x}{2}}\right)},$$

every primitive root of unity of every order up to the ωth inclusive, where ω is any number not less than the greatest of the quantities a, and therefore, if we please, equal to Σa, which saves the necessity of distinguishing the relative magnitudes of the several quantities a (ω it should be noticed must not be taken infinity, because that would render the sum to be residuated infinite). Thus then we see that the denumerant $\dfrac{n}{a, b, \ldots, l,}$ is the residue of

$$\Sigma \frac{e^{(t+2\pi ik)\nu}}{P\left\{e^{a\left(\frac{t}{2}+\pi ik\right)} - e^{-a\left(\frac{t}{2}+\pi ik\right)}\right\}},$$

where k represents every distinct quantity expressible by a proper fraction whose denominator is equal to or less than Σa*.

The result previously found concerning the relation of $F\nu$ to $F-\nu$ is in accordance with the observation due, I believe, to Jacobi, that if ϕn, ψn be the coefficients of x^n [n positive or negative] in the ascending and descending expansions of a proper rational fraction, then $\psi n = -\phi n$. For, in the particular fraction we are considering, it is obvious that calling the number of the factors (our former $\alpha + \beta + \ldots + \lambda$) i and $a + b + \ldots + l = s$, we shall have

$$\psi(-n-s) = (-)^i \phi n.$$

Therefore $\phi n = (-)^{i-1} \phi(-n-s)$ by Jacobi's observation.

If then $\nu = n + \dfrac{s}{2}$ and $\phi n = F\nu$ so that $\phi(-n-s) = F\left(-n - \dfrac{s}{2}\right) = F(-\nu)$ we shall have $F\nu = (-)^{i-1} F(-\nu)$, as already shown.

It is also a part of the same observation and shown in the same way that ϕn, used in the same sense as above, is zero for all values of negative n between *zero* and the *degree* of the fraction (*exclusive*); hence $F(\pm\nu)$ is zero for all values of ν from 0 to $\dfrac{s}{2} - 1$ inclusive if s be even, and from $\dfrac{1}{2}$ to $\dfrac{s}{2} - 1$ inclusive if s be odd†.

This fact alone is sufficient to give exactly the number of homogeneous equations required to determine (to a numerical factor près) the algebraico-

* The number of terms in this sum will be the sum of the totients of all the numbers up to the limit, an empirical expression for which (if my memory is not in fault) has been recently investigated by Mr Merrifield.

† In order not to break up the text, the footnote (which ought to come here) regarding the two statements above, as to the coefficient-functions of any proper fraction, is transferred to the last page of this Excursus [p. 621 below].

exponential form $F(\nu)$, that is, the effective* *trivial* zero values of $F(\nu)$ are exactly equal in number to the number of terms which that form contains, as I will proceed to show.

The number of the indices $a, b, c, \ldots$ in which any divisor is contained may be termed its frequency in respect to those numbers, and it is a very simple arithmetical fact that if the totient of every divisor of a set of given numbers be multiplied by its frequency in respect to the set, the sum of the products so obtained will be equal to the sum of the given numbers. When the set reduces to a single term this theorem becomes the familiar one, that any number is equal to the sum of the totients of all its several divisors, and from this to the general case there is but a step, for we may suppose the set of numbers written out in a line, and under every one of them which contains a divisor j the totient of j to be written, and every value from 1 upwards as far as the highest number of the set to be given to j. The rectangle (partly filled with totients and partly vacant) so formed, read off in columns, will, by the preceding case, give the sum of the set of numbers, and read off in lines, the sum of the products of each divisor by its frequency.

Let us now inquire into the number of the terms contained in the several waves. W_1, which always exists, will be the coefficient of $\dfrac{1}{t}$ in $\dfrac{e^{\nu t}}{P\left(e^{\frac{at}{2}} - e^{-\frac{at}{2}}\right)}$, and therefore (always supposing the number of indices a to be i) will be the coefficient of t^{i-1} in the product of $\left(1 + \nu t + \nu^2 \dfrac{t^2}{1.2} + \ldots\right)$ into the ascending development of $\dfrac{1}{P\left(\dfrac{e^{\frac{at}{2}} - e^{-\frac{at}{2}}}{t}\right)}$, and will therefore be a function of ν consisting of multiples of $\nu^{i-1}, \nu^{i-3}, \ldots$ until a multiple of ν or a constant is reached, and therefore containing $E\dfrac{i+1}{2}$ terms, the first of which it may be well to notice (using $a_1, a_2 \ldots a_i$ in lieu of $a, b, \ldots l$ as the indices) will obviously always be $\dfrac{1}{\Pi(i-1) a_1 . a_2 . \ldots a_i}$†.

In like manner it will be obvious that for W_2 the degree of ν will be the frequency of 2 diminished by a unit, and the form of W_2 will be $(-)^n$ into a polynomial function of ν of that degree.

* I say *effective* because it will presently be seen that in a certain case one of the trivial zero values will be ineffective, that is, will only lead to an identity and not to an equation between the coefficients in question.

† The highest power of ν in any other wave (which is its frequency diminished by unity) will *in general be* less than $i - 1$, and consequently the sign of the terms in the development of any rational fraction beyond a certain point must be unvarying, and the development from that

Again, any other wave W_i of frequency f_i will consist of a set of products of polynomial functions of ν of the degree $f_i - 1$ each multiplied by a sum of exponential quantities consisting of pairs of the form $c\Sigma(\rho^{\nu+\delta} + \rho^{\nu-\delta})$ or $c\Sigma(\rho^{\nu+\delta} - \rho^{\nu-\delta})$ according as $i - f_i$ is even or odd, where δ will be half the number of primitive ith roots of unity, say $\frac{\tau(i)}{2}$, where the numerator is the totient of i.

Hence the total number of constants to be determined in the algebraico-exponential function representing $\frac{n}{a_1, a_2, \ldots a_i}$ will be

$$E\frac{f_1+1}{2} + E\frac{f_2+1}{2} + \Sigma\frac{\phi\lambda f_\lambda}{2} \quad [\lambda = 3, 4, \ldots \infty].$$

(1) Suppose that f_1 and f_2 are not both even.

Then remembering that $\frac{f_1}{2} + \frac{f_2}{2} + \frac{f_3 . \tau_3}{2} + \frac{f_4 . \tau_4}{2} + \ldots = \frac{s}{2}$, the antecedent expression $= E\left(\frac{s}{2} + 1\right)$, for when f_1, f_2 are both odd, the two first terms on the left-hand side of this equation exceed the corresponding ones in the equation above it by $\frac{1}{2}, \frac{1}{2}$ respectively, and $E\left(\frac{s}{2} + 1\right)$ will exceed $\frac{s}{2}$ by unity (because $f_1 - f_2$ the number of the odd elements in the sum of all of them being even, s is even). And if f_1, f_2 are one odd and the other even, the right as well as the left-hand side of each equation will be increased $\frac{1}{2}$, for s will be now odd.

(2) Suppose that f_1, f_2 are both even, then

$$E\frac{f_1+1}{2} + E\frac{f_2+1}{2} + \frac{f_3\tau(3)}{2} + \ldots = \frac{f_1}{2} + \frac{f_2}{2} + \frac{f_3\tau(3)}{2} + \ldots = \frac{s}{2}.$$

Hence the number of constants to be determined is $1 + E\frac{s}{2}$, except when f_1, f_2 are both even, in which case it is $\frac{s}{2}$.

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point omni-positive or omni-negative, according as the numerator, on substituting unity for the variable, is positive or negative. The case of exception is when all the indices have a common numerant, say δ, for then the frequency of δ will be the same as of unity, and W_δ be of the same degree as W_1 in ν, so that the reason for uniformity of sign (at a sufficient distance from the origin) no longer subsists. This is the proof referred to at p. [600], in what precedes.

It is worth while imprinting on the memory the rule that the asymptotic value of

$$\frac{n}{a_1, a_2, \ldots a_i} \div n^{i-1} \text{ is } \frac{1}{\{1 . 2 . 3 \ldots (i-1)\} a_1 . a_2 \ldots a_i},$$

which ought, I imagine, to be susceptible of some simple proof or illustration by the method of nodes or cross-gratings, such as employed by Eisenstein to prove the law of reciprocity for quadratic residues, and by myself (*Johns Hopkins Circulars*, Nos. 13 and 14, pp. 179, 180, 209)* to demonstrate the impossibility of the existence of trebly periodic functions.

[* Below, pp. 635, 644.]

On the first supposition the trivial values of ν which make $F(\nu)$ zero are $0, 1, 2, \ldots \frac{s}{2} - 1$ when s is even, and $\frac{1}{2}, \frac{3}{2}, \ldots \left(\frac{s}{2} - 1\right)$ when s is odd, the number of such being $E\left(\frac{s}{2}\right)$ in either case, and there will be $E\left(\frac{s}{2}\right)$ homogeneous equations for finding the ratios of $E\left(\frac{s}{2}\right) + 1$ coefficients, which is exactly the right number.

On the second supposition, that is, when f_1, f_2 are both even, the number of the trivial values in question will be $\frac{s}{2}$, the same as the number of the coefficients, so that at first sight there would appear to be one superfluous equation—such, however, is not really the case—because the value 0 attributed to ν will lead not to a homogeneous equation between the coefficients but to the identity $0 = 0$. For evidently W_1, W_2 becoming odd functions of ν, will vanish when $\nu = 0$, and every other wave will also vanish; for when $\nu = 0$ it will consist exclusively of pairs of terms of the form $c(\rho^\delta - \rho^{-\delta})$ (because by hypothesis f_i the number of the elements is even), and since ρ and $\frac{1}{\rho}$ may be interchanged, it follows that the sum of such pairs must be zero. Hence whatever the relation of the number of odd and the number of even elements to the modulus 2, there will be just as many homogeneous equations as are required for determining the ratios of the coefficients in the form which expresses the denumerant. The absolute values of the coefficients may be found by writing $F\left(\frac{s}{2}\right)$ = coefficient of x^0 in the generating function $= 1$, or by virtue of the observation made above, that the leading coefficient in W_1 for the elements $a_1, a_2, \ldots a_i$ is $\frac{1}{\pi(i-1) a_1, a_2, \ldots a_i}$.

When the denumerant is regarded as a function of n and not of ν, it is obvious *à priori* that being a particular integral of an equation in finite differences of the order s, its coefficients must be determinable in relative magnitude by the knowledge of $(s-1)$ values of the variable for which it vanishes, and this is almost but not quite sufficient in itself to establish the preceding result regarding the canonical form.

I will illustrate this method presently by one or two easy examples, but previously it will, I think, be desirable to give greater precision and uniformity to the nomenclature of simple denumerants.

If any such be denoted by $\frac{n}{a, b, \ldots l}$, (I have sometimes here or elsewhere referred to n as the numerator or denumerator or partible number, and to $a, b, \ldots l$, variously as the denominators or as the indices or as the elements of the denumerant), in future I shall call n the componend, and $a, b, \ldots l$ the components of the denumerant.

A denumerant with a single component as $\frac{n}{a,}$ which I call an elementary denumerant, deserves special attention, for it will presently be seen that every given simple denumerant is expressible as a sum of powers of its componend multiplied respectively by linear functions of elementary denumerants whose several components are the divisors of the components of the given one.

The elementary denumerant $\frac{n}{a,}$ being the number of solutions in positive integers of the equation $ax = n$, is obviously 1 or 0 according as n does or does not contain a. But we may also regard $\frac{n}{a,}$ as an analytical function and define it as the mean of the a values of ρ^n where ρ is any root of the equation $\rho^a - 1 = 0$, and so construed it will preserve a meaning even when n is taken a negative integer, and will mean 1 or 0, provided that n be an integer of either kind, according as it does or does not contain a without a remainder. It is in this extended sense that $\frac{n}{a,}$ or $\frac{\nu}{a,}$ will be employed in what follows.

Supposing r to be a primitive ith root of unity, W_i will consist of a sum of powers of ν each multiplied by the sum of quantities of the form $cr^{n+\delta}$ (where for the moment for greater clearness of elucidation I purposely retain n instead of using its augmentative ν). On giving n all values from $-\delta$ to $-\delta + i - 1$ inclusive, this sum will take i successive values to be determined from the equation containing the primitive roots, say $\epsilon_0, \epsilon_1, \dots \epsilon_{i-1}$, so that its general value will be expressible under the form

$$\epsilon_0 \frac{n+\delta}{i,} + \epsilon_1 \frac{n+\delta-i}{i,} + \dots + \epsilon_{i-1} \frac{n+\delta-i+1}{i,}.$$

We may then replace n by $\nu - \frac{s}{2}$, and on so doing and further replacing (where requisite) any numerator by its residue in respect to i, shall obtain a sum of the form

$$\eta_0 \frac{\nu}{i,} + \eta_1 \frac{\nu - 1}{i,} + \dots + \eta_{i-1} \frac{\nu - i + 1}{i,} \text{ when } s \text{ is even,}$$

and of the form

$$\eta_0 \frac{\nu - \frac{1}{2}}{i,} + \eta_1 \frac{\nu - \frac{3}{2}}{i,} + \dots + \eta_{i-1} \frac{\nu - i + \frac{1}{2}}{i,} \text{ if } s \text{ is odd.}$$

On this being done, remembering the extension given to the sense of an elementary denumerant and the theorem that the analytical value $F\nu$ of a denumerant is equal to $\pm F(-\nu)$, we see that in either case the above sums will be reducible to a sum of pairs of terms of the form $\eta\left(\frac{\nu+k}{i,} \pm \frac{\nu'-k}{i,}\right)$

[the same + or − sign subsisting throughout the whole series for any specified power of ν] but subject to the exception that when i is even, two of the pairs will be replaced by single terms, multiples of $\frac{\nu \pm \frac{i}{2}}{i,}$ and of $\frac{\nu}{i,}$ respectively, which become zero when the negative sign is the one to be employed*.

Thus taking $i = 2$, W_2 takes the form $(-)^n R\nu$, that is, $\frac{n}{2,} - \frac{n-1}{2,}$. W_1 it is scarcely necessary to repeat will contain no elementary denumerants, being purely an algebraical function of the resolvent. W_2 is such a function multiplied by $(-1)^n$. This multiplier is expressible under the form $\left(\frac{n}{2,} - \frac{n \pm 1}{2,}\right)$ which is always a function of n that remains unchanged when n is changed into $-n$. But when the two denumerants are expressed as functions of ν the case is different; if s (the sum of the components) is an even number, the above pair of terms becomes $(-)^{\frac{s}{2}}\left(\frac{\nu}{2,} - \frac{\nu \pm 1}{2,}\right)$ which is unaltered by the change of ν into $-\nu$, but when s is odd it becomes $(-)^{\frac{s-1}{2}}\left(\frac{\nu - \frac{1}{2}}{2,} - \frac{\nu + \frac{1}{2}}{2,}\right)$ which changes its sign when ν is changed into $-\nu$.

Before quitting the subject of nomenclature I may just observe that it will be convenient to call denumerants, when their resolvents are the natural numbers commencing with unity, *natural denumerants*, and when the natural numbers commencing with 2, curtate natural, or for greater brevity simply *curtate* denumerants, the highest number reached in either case being termed the order; D_i and Δ_i may then be used to denote natural and curtate denumerants of the order i†.

I now return to the application of the method of indeterminate coefficients to finding the value of denumerants whose components are given. This method is not practically applicable when the sum of the components is considerable, because that sum measures the number of linear equations to be solved. In the following section I shall work out in full, by the regular process, the case where the components are 2, 3, 4, 5, 6, 7, of which the result

* The sign is positive or negative according as the number of the components less the power of ν in question is odd or even, and it is easy also to see that the sum of all the coefficients of the elementary denumerants in the multiplier of each power of ν will be always zero.

† It is curtate denumerants which are almost exclusively required in the applications to the theory of invariants. If necessary to bring into evidence the componend we may use the more explicit notation $\overset{n}{D}_i$, $\overset{n}{\Delta}_i$ to signify natural and curtate denumerants of the order i with the componend n. Thus we may write $\overset{n}{D}_i - \overset{n-1}{D}_i = \overset{n}{\Delta}_i$ and $\overset{n}{D}_i - \overset{n}{D}_{i-1} = \overset{n-1}{\Delta}_i$.

It may be as well to notice that for curtate, as well as for natural denumerants, the divisors of the components are the natural numbers from unity to the order of the denumerant inclusive, so that the number of the waves for either of these sort of denumerants is equal to the order.

is more especially required for the purposes of the preceding section, and which has not previously been calculated. The other algebraical formulae for denumerants in their canonical form I shall give without exhibiting the work; the accuracy of most of them can be ascertained by comparison with Prof. Cayley's values of the same, exhibited as functions of the unaugmented componend in the *Phil. Trans.* for 1856 and 1858.

Let us suppose 1, 2, 3 to be the components,

we may write $\frac{n}{1,\,2,\,3,} = A\nu^2 + B + (-)^\nu C + \Sigma(\rho^{\nu+1} + \rho^{\nu-1})D,$

where $\rho^2 + \rho + 1 = 0$, or more simply, $A\nu^2 + B + (-)^\nu C - D\Sigma\rho^\nu = 0$.

Hence making $\nu = 0, 1, 2$ we have

$$B + C - 2D = 0$$
$$A + B - C + D = 0$$
$$4A + B + C + D = 0,$$

so that $2C + 3A = 0 \quad 3D + 4A = 0 \quad B + \left(\frac{8}{3} - \frac{3}{2}\right) A = 0,$

or $A = 6\sigma \quad B = -7\sigma \quad C = -9\sigma \quad D = -8\sigma;$

and to find σ, making $\nu = 3$, we obtain

$$(54 - 7 + 9 + 16)\sigma = 1 \quad \text{or } \sigma = \frac{1}{72}.$$

Hence $$\frac{n}{1,\,2,\,3,} = \frac{\nu^2}{12} - \frac{7}{72} - \frac{1}{8}\left(\frac{\nu}{2,} - \frac{\nu-1}{2,}\right) + \frac{1}{9}\left(2\frac{\nu}{3,} - \frac{\nu+1}{3,} - \frac{\nu-1}{3,}\right)$$

monomial denumerants being used to replace the exponential quantities $(-1)^\nu$; $\Sigma\rho^\nu$.

The leading coefficient $\frac{1}{12}$ it will be observed $= \frac{1}{(1\,.\,2)(1\,.\,2\,.\,3)}$, as it ought to be by the general rule.

The maximum negative value of $\frac{n}{1,\,2,\,3,} - \frac{\nu^2}{12}$ is $\frac{7}{72} + \frac{1}{8} - \frac{1}{9}$ or $\frac{1}{9}$, and its maximum positive value $\frac{2}{9} + \frac{1}{8} - \frac{7}{72}$ or $\frac{1}{4}$. Hence the value of $\frac{n}{1,\,2,\,3,}$ is always the nearest integer to $\frac{(n+3)^2}{12}$.

But by Euler's theorem of reciprocity $\frac{n}{1,\,2,\,3,}$ is the number of ways of resolving n into three or less than three parts, and consequently $\frac{n-3}{1,\,2,\,3,}$ is the number of ways of resolving n into exactly three parts, this therefore is always the nearest integer to $\frac{n^2}{12}$, as first observed I believe by the late lamented Prof. De Morgan.

Take as another case the components 1, 2, 3, 4 which give $\nu = n + 5$. We may write

$$\frac{n}{1,\,2,\,3,\,4,} = A\nu^3 + B\nu + (-)^\nu C\nu + D\Sigma(\rho^{\nu+1} - \rho^{\nu-1}) + E\Sigma(i^{\nu+1} - i^{\nu-1})$$

where $\rho^2 + \rho + 1 = 0$, $i^2 + 1 = 0$. Hence giving ν the successive values 1, 2, 3, 4, (omitting $\nu = 0$, which would lead to $0 = 0$) we obtain

$$A + B - C - 3D - 4E = 0$$
$$8A + 2B + 2C + 3D = 0$$
$$27A + 3B - 3C + 4E = 0$$
$$64A + 4B + 4C - 3D = 0.$$

Hence $72A + 6B + 6C = 0$, and $36A + 6B - 2C = 0$,

consequently $2C + 9A = 0 \quad 2B + 15A = 0 \quad -3D + 16A = 0$

or $A = 6\sigma \quad B = -45\sigma \quad C = -27\sigma \quad D = 32\sigma \quad E = -27\sigma.$

Finally making $\nu = 5$ $\quad \sigma(750 - 225 + 135 + 96 + 108) = 1$, or $\sigma = \frac{1}{864}$,

and $$\frac{n}{1,\,2,\,3,\,4,} = \frac{1}{144}\nu^3 - \frac{5}{96}\nu - \frac{1}{32}\left(\frac{\nu}{2,} - \frac{\nu-1}{2,}\right)$$
$$+ \frac{1}{9}\left(\frac{\nu-1}{3,} - \frac{\nu+1}{3,}\right) + \frac{1}{8}\left(\frac{\nu-1}{4,} - \frac{\nu-3}{4,}\right).$$

The principal coefficient is $\frac{1}{144}$ or $\frac{1}{\Pi 3\,.\,1\,.\,2\,.\,3\,.\,4}$, as it ought to be, according to the general rule, and this serves as a verification of the correctness of the whole work.

It will be found convenient to append here, instead of reserving for the following section, the analytical expression for the first wave of a general denumerant, which stands out markedly from the rest, inasmuch as it can be expressed once for all as an algebraical function of the componend and components without any regard being had to the arithmetical form of the latter.

Let $C(\tau_1\tau_2 \ldots \tau_j)$, $H(\tau_1\tau_2 \ldots \tau_j)$ or more briefly $C_j\tau$ $H_j\tau$ be understood to denote the perfectly well-known functions of $\tau_1, \tau_2, \ldots \tau_j$ which represent the elementary symmetric function and the sum of the homogeneous products of the jth order of those quantities of which τ_q represents the sum of the qth powers, so that, for example, $C_2\tau$, $H_2\tau$ will serve to denote $\frac{\tau_1^2 - \tau_2}{2}$, $\frac{\tau_1^2 + \tau_2}{2}$ respectively, upon which supposition we may write

$$e^{\tau_1 t + \tau_2 \frac{t^2}{2} + \tau_3 \frac{t^3}{3} + \ldots} = 1 + \tau_1 t + \frac{\tau_1^2 + \tau_2}{2} t^2 + \ldots + H_q\tau t^q + \ldots$$

Also let it be observed preliminarily that as a direct inference from Maclaurin's theorem, if ϕ represent any function of x but does not contain ν,

$$\text{co}_j e^{\nu x + \phi} = \text{co}_j e^\phi + \text{co}_{j-1} e^\phi \nu + \text{co}_{j-2} e^\phi \frac{\nu^2}{1\,.\,2} + \ldots$$

Furthermore for greater brevity let us agree to express the W_1 for j components $a_1, a_2, \ldots, a_j$ under the form $W_{1,j}$, and write it equal to $\frac{V_j}{\pi_j}$ where π_j indicates the product of the j components.

We may then write

$$V_j = \pi_j \, \text{co}_{-1} \frac{e^{\nu x}}{P(e^{a\frac{x}{2}} - e^{-a\frac{x}{2}})}.$$

Now from the known expression for $\log \sin \theta$, we may write

$$\log (e^{\frac{\theta}{2}} - e^{-\frac{\theta}{2}}) = \log \theta + \beta_1 \theta^2 - \beta_2 \theta^4 + \ldots \pm \beta_q \theta^{2q} + \ldots$$

where
$$\beta_q = \frac{1}{\Pi 2q} \cdot \frac{B_{2q-1}}{2q}.$$

Hence
$$V_j = \text{co}_{j-1} \, e^{\nu x - 2\tau_1 \frac{x^2}{2} + 2\tau_2 \frac{x^4}{4} - 2\tau_3 \frac{x^6}{6} \cdots}$$

where $2\tau_q = \frac{B_{2q-1}}{\Pi 2q} \sigma_{2q}$ and the latter factor indicates the sum of the $2q$th powers of the components.

Hence writing $x^2 = t$ we have $V_j = \text{co}_{j-1} \, e^{\nu x - \tau_1 t + \tau_2 \frac{t^2}{2} - \tau_3 \frac{t^3}{3} \cdots}$

and consequently making $T = -\tau_1 t + \tau_2 \frac{t^2}{2} - \tau_3 \frac{t^3}{3} \cdots$

$$V_j = \text{co}_{j-1} T + \text{co}_{j-2} T \,.\, \nu + \text{co}_{j-3} T \cdot \frac{\nu^2}{1\,.\,2} + \text{co}_{j-4} T \cdot \frac{\nu^3}{1\,.\,2\,.\,3} \cdots$$

$$= \frac{\nu^{j-1}}{\Pi (j-1)} - H_1 \tau \frac{\nu^{j-3}}{\Pi (j-3)} + H_2 \tau \frac{\nu^{j-3}}{\Pi (j-3)} \cdots$$

the series ending with ν or with a constant according as j is even or odd.

Thus
$$V_2 = \nu,$$
$$V_3 = \frac{\nu^2}{2} - H_1 (\tau),$$
$$V_4 = \frac{\nu^3}{6} - H_1 (\tau) \nu,$$
$$V_5 = \frac{\nu^4}{24} - H_1 (\tau) \frac{\nu^2}{2} + H_2 (\tau),$$
$$V_6 = \frac{\nu^5}{120} - H_1 (\tau) \frac{\nu^3}{6} + H_2 (\tau) \nu, \text{ and so on,}$$

each V being an integral with respect to ν of the one which precedes it.

Substituting for each τ its value in terms of the Bernouillian numbers B and the σ's, and giving the former their arithmetical values we shall obtain

$$V_2 = \nu,$$
$$V_3 = \frac{\nu^2}{2} - \frac{\sigma_2}{24},$$
$$V_4 = \frac{\nu^3}{6} - \frac{\sigma_2}{24} \nu,$$
$$V_5 = \frac{\nu^4}{24} - \frac{\sigma_2}{48} \nu^2 + \left(\frac{\sigma_2^2}{1152} + \frac{\sigma_4}{2880}\right),$$
$$V_6 = \frac{\nu^5}{120} - \frac{\sigma_2}{144} \nu^3 + \left(\frac{\sigma_2^2}{1152} + \frac{\sigma_4}{2880}\right) \nu,$$
$$V_7 = \frac{\nu^6}{720} - \frac{\sigma_2}{576} \nu^4 + \left(\frac{\sigma_2^2}{2304} + \frac{\sigma_4}{5160}\right) \nu^2 - \left(\frac{\sigma_2^3}{82944} + \frac{\sigma_2 \sigma_4}{103680} + \frac{\sigma_6}{181440}\right),$$
$$V_8 = \int_\gamma^0 d\nu \, V, \text{ and so on.}$$

Such are the expressions for V best adapted for actual use, since it is desirable to express $W_{1,j}$, that is, $\frac{V_j}{a_1 \,.\, a_2 \ldots a_j}$ explicitly in terms of powers of ν; but there is another somewhat noteworthy form which can be given to the V with an even subindex as follows:

It is obvious that

$$V_{2k} = \text{co}_{-1} \frac{\frac{1}{2}(e^{\nu x} - e^{-\nu x}) + \frac{1}{2}(e^{\nu x} + e^{-\nu x})}{P(e^{\frac{a}{2}x} - e^{-\frac{a}{2}x})} = \text{co}_{-1} \frac{\frac{1}{2}(e^{\nu x} - e^{-\nu x})}{P(e^{\frac{a}{2}x} - e^{-\frac{a}{2}x})}$$

for the neglected part of the numerator will contribute nothing to the residue*.

We may now calculate the logarithm of the entire quantity to be residuated instead of merely the denominator, and take the residue of its exponential ν; on so doing it will be obvious on reflection that we shall obtain the product of ν into a quantity of the very same form as the constant term in V_{2k-1}, when instead of σ_{2q} in the value of τ_q we substitute $-(2n)^{2q} + \sigma_{2q}$. If then we write $2U_q = \frac{B_{2q-1}}{\Pi 2q} \{(2n)^{2q} - \sigma_{2q}\}$ it is easy to see that we shall have $V_{2k} = \nu C_{k-1}(U)$.

* For V_{2k} the *effective* numerator of the residuand is a *sine* form, and may be subjected to the same treatment as its fellows in the denominator. The case is different with V_{2k-1}, for which the effective numerator of the residuand is a *cosine* form. But we may write

$$V_{2k-1} = \frac{d}{d\nu} V_{2k} = C_{k-1} U + \nu \delta C_{k-1} U,$$

and if we turn to account the fact that in $C_{k-1} U$ along with $(2\nu)^2, (2\nu)^4, \ldots (2\nu)^q \ldots$ are associated $-s_2, -s_4, \ldots -s_{2q} \ldots$ and choose to write $-\nu^{2q} \frac{d}{ds_{2q}} = \Delta^q$, it will be found that the above expression may be transformed so as to give the symbolical equation (more curious perhaps than useful) $V_{2k-1} = \left(\frac{1+\Delta}{1-\Delta}\right)^2 C_{k-1} U$, whereas as previously found $V_{2k} = \nu C_{k-1} U$.

Thus, for example, suppose $2k=6$, we may write V_6 under the form

$$\nu\left\{\frac{(4\nu^2-s_2)^2}{1152}-\frac{(16\nu^4-s_4)}{2880}\right\}$$

to verify which it will be observed that

$$\frac{16}{1152}-\frac{16}{2880}=\frac{1}{72}-\frac{1}{180}=\frac{1}{120} \text{ and } \frac{8}{1152}=\frac{1}{144},$$

so that

$$V_6=\frac{\nu^5}{120}-\frac{s_2}{144}\nu^3+\left(\frac{s_2^2}{1152}+\frac{s_4}{2880}\right)\nu, \text{ as previously found.}$$

Before having done with this outline it may be well to call attention to the circumstance that the distribution of the infinity-roots into groups determined by the divisors of the components is not in all cases the best mode of grouping to adopt.

Thus suppose that the components $(a_1, a_2, \ldots a_i)$ are all prime relatively to each other, it will in such case be most expeditious, after taking out the algebraical part W_1, to separate what remains into i portions, referring respectively to *all* the non-unity a_1th, a_2th, ... a_ith roots of unity*.

This view enables us to give a concise answer to a question of some interest, namely, as to what is the number of solutions of the inequality

$$a_1x_1+a_2x_2+\ldots+a_ix_i<\mu(a_1a_2\ldots a_i),$$

say $\mu\pi_i$, where μ is any positive integer and the coefficients are relatively prime each to each.

Certainly this number is no other than the denumerant $\dfrac{\mu\pi_i}{1, a_1, a_2, \ldots a_i}$ which might be calculated by the general formula, but would give a result neither concise nor elegant; we may on the other hand regard it as a sum of denumerants, say $\Sigma\dfrac{\mu\pi_i-\delta}{a_1, a_2, \ldots a_i}$, where δ takes all values from 0 to $\mu\pi_i-1$. Now each such denumerant will consist of a purely algebraical and a purely periodical part, and it is very easy to see according to the view just indicated that the sum of all the latter will be zero. Hence the number required will be

$$\Sigma^0_{\mu\pi_i-1}\frac{V_1}{\pi_i}.$$

I may illustrate this by the very simplest imaginable case, where there are but two components p, q and the number required is that of the solutions in integers of the inequality $px+qy<pq$ where p and q are relative primes.

Calling $pq=n$, the rule laid down will give for the number sought

$$\Sigma_0^{n-1}\frac{\nu}{n}, \text{ that is, } \Sigma_0^{n-1}\frac{n+\frac{p+q}{2}}{n}=\frac{pq-p-q-1}{2}.$$

* This is tantamount to blending into one all the waves corresponding to the non-unity divisors of each component.

This result admits of a somewhat *piquant* verification. The number of integers less than pq and containing neither p nor q is $(p-1)(q-1)$, and if every two of these which are supplementary to one another (I mean whose sum is pq) be made into a pair, it is an easily demonstrable, but by no means an unimportant fact, that one of the pair will be a compound and the other a non-compound of p and q. Hence the total number of non-compounds is $\frac{1}{2}(p-1)(q-1)$, and therefore the total number of solutions of $px+qy<pq$ will be the remainder when the above is subtracted from pq, that is, $\frac{1}{2}(pq+p+q-1)$ as previously determined.

I will embrace this opportunity of noticing a correction that should be made to the long footnote in Section 3 given in the preceding number of the *Journal*. In lieu of the words* in the last paragraph of that note following the word *products*, line 3 and preceding the word *set*, line 8, read as follows:

Of the form $b^xQ^yR^zS^tT^u$ such that no one of them could be (a power of one or) a product of powers of any of the others. If then it could be shown that there exists in the succession a set of quintuplets x, y, z, t, u, such that the quotient system of every other quintuplet in the succession is intermediate to the quotient system of that

It may also be as well to notice here that the method of expressing in terms of ordinary space the intermediateness of a quadruplet, a triplet or a couplet, to four, three or two other such respective multiplets, may be profitably simplified by the use of quadriplanar, trilinear and bi-punctual coordinates, in flat spaces of three, two and one dimension respectively; for we may then without having recourse to quotient-systems regard each element of the multiplet as a coordinate of its representative point, inasmuch as the affection concerned being one relative exclusively to the inwardness or outwardness of a point in regard to a closed environment, obviously remains unchanged by projection.

What follows is the footnote referred to at foot of page [610] *where it was meant to be inserted.*

Each of the two statements regarding the coefficient-functions becomes next to self-evident when the coefficient of x^n in the reciprocal of $(1-ax)(1-\beta x)\ldots(1-\lambda x)$ is put under the form of a sum of terms similar to $a^n\div\left(1-\frac{\beta}{a}\right)\left(1-\frac{\gamma}{a}\right)\ldots\left(1-\frac{\lambda}{a}\right)$ interpreted (when necessary) as meaning the function of $(n;\, a, \beta\ldots\lambda)$ indefinitely near to the value of what such sum becomes when any equal elements are made to undergo arbitrary infinitesimal variations. Jacobi's proof of the theorem, I rather think, is got by proving it directly for each of the simple partial fractions into which any given proper fraction may be supposed to have been resolved.

A third method is to form the equation between $u_n, u_{n-1}, \ldots u_{n+j-1}$, and between

$$v_{-n}, v_{-n-1}, \ldots v_{-n-j-1}$$

[* p. 581 above.]

[u_n being the general coefficient in the ascending and v_{-n} in the descending development of $1 \div R(x)$]; the two equations become identical on changing u and n into v and $-n$, and $j-1$ homogeneous equations which help to determine the constants will be the same in both, namely, those got by making $n=-1, -2, \ldots -(j-1)$, consequently the two particular integrals u_n, v_n can differ only by a factor independent of n; if we write then $u_n : v_n :: P : Q$ and call the first and last coefficients in the denominator A and L, and pay attention to the fact that u_n, v_n can only become infinite when A, L vanish, and also to the indifference of the relation of R regarded as a quantic in x and 1 to the two sorts of development, it is plain to see that $P : Q :: A^{\mu} : \pm L^{\mu}$, but the x-weight of u_n is n and of v_{-n} is $-n$; hence $\mu=0$ and $u_n : v_n$ is independent, not only of n but of the coefficients in R, and to determine its value we may make $R=x^{j-1}-x^j$, which gives at once $u_n=-v_n$. This being true for all values of n, it is obvious that the relation will continue to subsist, when instead of unity any polynomial function of x of lower degree than that of the denominator (see below) is taken for the numerator.

Moreover, if the degree of the numerator be $j-\delta$, u_q and v_q will be seen (from what goes before) to vanish for every value of q common to the series

$$-1, -2, \ldots -(j-1) : 0, -1, \ldots -(j-2) : \ldots : (j-\delta-1), (j-\delta-2), \ldots -(\delta-1),$$

namely, for the values $-1, -2, \ldots -(\delta-1)$ or in other words either coefficient-function of the index of any power of the variable which appears neither in the ascending nor the descending development of a rational fraction is equal to zero.

Unless the fraction is a proper one u_n and v_n (the coefficient-functions) will not be continuous functions of n throughout; hence arises the necessity of this limitation in dealing with the generalized equation $u_n=-v_n$. Thus, for example, for the improper fraction $\frac{1+2x^2}{(1-x)^2}$, u_0 and v_0 are 1 and 2, but for any positive or negative value of n other than 0, u_n and v_n will be $3n-1$ and $-(3n-1)$ respectively. It may be added that the theorem will continue to subsist even for an improper fraction, provided that on freeing its numerator from a power of the variable, it becomes a proper one, for then the coefficient-functions remain continuous throughout.

This last proof, although more laboured than the preceding ones, seems to me the best because it goes straight to the heart of the question and does not depend on any apparently accidental results of calculation, but (so to say) compares the two twin functions in their nascent state, in the very act of birth.

The relation of the two coefficient-functions to one another and to the two general terms in the actual expansions becomes more clear if we use ϕn, ψn to denote the two former, reserving u_n, v_n for the two latter. Then besides the equation $\phi n + \psi n = 0$ which is absolute, we have the equations $u_n = \phi n$, $v_n = \psi n$, limited as follows. Call Δ the deficiency of the numerator of the generating proper fraction, that is, the number of units that it stops short of its maximum possible value: then the first of these two equations holds good for all values of n not less than $-\Delta$, the latter for all values of n not greater than -1; if Δ is not zero, that is, if the degree of the numerator is not the integer next below that of the denominator, these two ranges will overlap for the values $-1, -2, \ldots, -\Delta$ of n, and for those values $\phi n = u_n = 0$, $\psi n = v_n = 0$. In the use made of these theorems in the text, the numerator is a mere constant, so that Δ has its maximum value, namely it is one unit less than the sum of the components (that sum being the degree of the generating function to a denumerant).

The general theorem may be brought into more distinct relief as follows: A finite fraction may be conceived as containing any number of powers of x positive or negative in numerator and denominator, and its two developments may be supposed to touch or be separate or to intersect one another. In the *last* case two coefficient-functions ϕn, $-\phi n$ exist applicable to all terms outside but inapplicable to any term inside the overlap. In the second case such functions exist which (besides being applicable, as in the case of contact, to all terms belonging to either of the two developments) vanish for all values of n in the chasm which separates them.

68.

TABLES OF GENERATING FUNCTIONS, REDUCED AND REPRESENTATIVE, FOR CERTAIN TERNARY SYSTEMS OF BINARY FORMS.

[*American Journal of Mathematics*, v. (1882), pp. 241—250.]

THE annexed tables have been calculated under my directions by Messrs Durfee and Ely, out of the fund placed at my disposition by the British Association for the Advancement of Science in the year 1881. Subsequent investigation will be necessary in order to ascertain whether there exist or not extra tabular groundforms which escape the operation of tamisage.

G. F. it will be understood stands for the words Generating Function.

SYSTEM OF TWO QUADRATICS AND ONE QUARTIC.

G. F. for invariants, reduced form.

Denominator: $(1-b^2)(1-\beta^2)(1-d^2)(1-d^3)(1-b\beta)(1-bd)$
$(1-\beta d)(1-b^2d)(1-\beta^2 d)$.

Numerator:

		d^0	d^1	d^2	d^3	d^4
	β^0	1				
b^0	β^1		$\bar{1}$			
	β^2			1		
	β^0		$\bar{1}$			
b^1	β^1		1	2		
	β^2		1		$\bar{1}$	
	β^3			$\bar{1}$		

		d^0	d^1	d^2	d^3	d^4
	β^1			$\bar{1}$		
b^3	β^2				1	
	β^3					$\bar{1}$
	β^0			1		
b^2	β^1		1		$\bar{1}$	
	β^2			$\bar{2}$	$\bar{1}$	
	β^3				1	

G. F. for invariants, representative form.

Denominator: $(1-b^2)(1-\beta^2)(1-d^2)(1-d^3)(1-b\beta)(1-b^2d^2)$
$(1-\beta^2d^2)(1-b^2d)(1-\beta^2d)$.

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5
	β^0	1					
b^0	β^1						
	β^2						
	β^3				1		
	β^4						
	β^0						
	β^1		1	1			
b^1	β^2		1	1	1		
	β^3						
	β^4				$\bar{1}$		
	β^0						
b^2	β^1		1	1	1		
	β^2						
	β^3				$\bar{1}$	$\bar{1}$	$\bar{1}$
	β^4						

		d^0	d^1	d^2	d^3	d^4	d^5	d^6
	β^0							
b^4	β^1				$\bar{1}$			
	β^2							
	β^3							
	β^4							$\bar{1}$
	β^0				1			
	β^1							
b^3	β^2				$\bar{1}$	$\bar{1}$	$\bar{1}$	
	β^3					$\bar{1}$	$\bar{1}$	
	β^4							

TABLE OF GROUNDFORMS.

Deg. in coeff's of quadratic	Deg. in coeff's of quadratic	Deg. in coeff's of quartic. 0	1	2	3
0	0			1	1
	1				
	2	1	1	1	
	3				1
1	0				
	1	1	1	1	
	2		1	1	1
2	0	1	1	1	
	1		1	1	1
3	0				1

SYSTEM OF QUADRATIC, CUBIC, AND QUARTIC.

G. F. for invariants, reduced form.

Denominator: $(1-b^2)(1-c^4)(1-d^2)(1-d^3)(1-bc^2)(1-b^2c^2)$
$(1-bd)(1-b^2d)(1-c^2d)(1-c^2d^3)(1-c^4d)$
$(1-c^4d^3)$.

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}
	c^0	1										
	c^2		$\bar{1}$									
b^0	c^4			2	2	2	1					
	c^6			1	1		$\bar{1}$	$\bar{1}$				
	c^8				$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$				
	c^{10}								1			
	c^{12}									$\bar{1}$		
	c^0		$\bar{1}$									
	c^2		2	4	2	1						
	c^4		2	2		$\bar{1}$	$\bar{2}$	$\bar{1}$				
b^1	c^6			$\bar{2}$	$\bar{3}$	$\bar{3}$	$\bar{1}$	1	1			
	c^8			$\bar{1}$	$\bar{1}$		1	1	1			
	c^{10}				1	1		$\bar{1}$	$\bar{2}$	$\bar{2}$		
	c^{12}								$\bar{1}$		1	
	c^{14}									1		
	c^0			1								
	c^2		2	1	$\bar{1}$	$\bar{1}$	$\bar{1}$					
	c^4		1	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$					
	c^6			$\bar{1}$			1	1				
b^2	c^8			$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^{10}					$\bar{1}$	$\bar{1}$			1	1	
	c^{12}					1	2	2	1	2	1	
	c^{14}										$\bar{1}$	
	c^2		1	$\bar{1}$	$\bar{1}$	$\bar{1}$						
	c^4	1	1		1			$\bar{1}$				
	c^6		$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{3}$	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{1}$		
b^3	c^8		$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{1}$		1	2	1	1	
	c^{10}			1	1	2	1	2	2	3	2	
	c^{12}					1			$\bar{1}$		$\bar{1}$	$\bar{1}$
	c^{14}							1	1	1	$\bar{1}$	

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}
	c^4			1								
	c^6				$\bar{1}$							
b^6	c^8					2	2	2	1			
	c^{10}					1	1		$\bar{1}$	$\bar{1}$		
	c^{12}						$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$		
	c^{14}										1	
	c^{16}											$\bar{1}$
	c^2			$\bar{1}$								
	c^4		$\bar{1}$		1							
	c^6			2	2	1		$\bar{1}$	$\bar{1}$			
b^5	c^8				$\bar{1}$	$\bar{1}$	$\bar{1}$		1	1		
	c^{10}				$\bar{1}$	$\bar{1}$	1	3	3	2		
	c^{12}					1	2	1		$\bar{2}$	$\bar{2}$	
	c^{14}							$\bar{1}$	$\bar{2}$	$\bar{4}$	$\bar{2}$	
	c^{16}										1	
	c^2		1									
	c^4		$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^6		$\bar{1}$	$\bar{1}$			1	1				
	c^8					1	2	2	1	1		
b^4	c^{10}					$\bar{1}$	$\bar{1}$			1		
	c^{12}						1	1	1	1	$\bar{1}$	
	c^{14}						1	1	1	$\bar{1}$	$\bar{2}$	
	c^{16}									$\bar{1}$		

G. F. for invariants, representative form.

Denominator: $(1-b^2)(1-c^4)(1-d^2)(1-d^3)(1-bc^2)(1-b^3c^2)(1-b^2d^2)$
$(1-b^2d)(1-c^4d^2)(1-c^2d^3)(1-c^4d)(1-c^4d^3)$.

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}
	c^0	1											
	c^2												
	c^4			1	2	2	1						
b^0	c^6			1	3	2	1						
	c^8					$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$				
	c^{10}					$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^{12}												
	c^{14}										$\bar{1}$		
	c^2		2	3	2	1							
	c^4		2	4	5	3	1						
	c^6												
b^1	c^8			$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{2}$	$\bar{1}$				
	c^{10}						$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$			
	c^{12}					1	1		$\bar{2}$	$\bar{2}$	$\bar{1}$		
	c^{14}												
	c^{16}										1		
	c^2		2	3	3	1							
	c^4		1	3	4	2	$\bar{1}$	$\bar{2}$	$\bar{1}$				
	c^6				$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{3}$	$\bar{1}$				
b^2	c^8			$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{5}$	$\bar{2}$	1	2	1		
	c^{10}				$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$				
	c^{12}					1	2	2	1				
	c^{14}						1	2	2	1	1	1	1
	c^0				1								
	c^2		1	1		$\bar{1}$	$\bar{1}$	$\bar{1}$					
	c^4	1	1	2	1	$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{1}$				
	c^6		$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{1}$			
b^3	c^8		$\bar{1}$	$\bar{3}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{2}$		1			
	c^{10}					$\bar{1}$	$\bar{2}$	$\bar{1}$	1	4	4	2	
	c^{12}				1	2	3	2	2	3	4	3	1
	c^{14}						1	2	3	2			
	c^{16}								1	1	1	$\bar{1}$	$\bar{1}$

		d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}	d^{12}
	c^4			1									
	c^6												
	c^8					1	2	2	1				
	c^{10}					1	3	2	1				
b^7	c^{12}							$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$		
	c^{14}							$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$		
	c^{16}												
	c^{18}												$\bar{1}$
	c^2			$\bar{1}$									
	c^4												
	c^6			1	2	2		$\bar{1}$	$\bar{1}$				
b^6	c^8				2	3	2	1					
	c^{10}					1	2	3	3	3	1		
	c^{12}												
	c^{14}							$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{4}$	$\bar{2}$	
	c^{16}								$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	
	c^4	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$					
	c^6					$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^8					1	2	3	2	1			
b^5	c^{10}			$\bar{1}$	$\bar{2}$	$\bar{1}$	2	5	5	3	1		
	c^{12}					1	2	3	2	1			
	c^{14}					1	2	1	$\bar{2}$	$\bar{4}$	$\bar{3}$	$\bar{1}$	
	c^{16}								$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{2}$	
	c^2	1	1	$\bar{1}$	$\bar{1}$	$\bar{1}$							
	c^4				$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$					
	c^6	$\bar{1}$	$\bar{3}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$			
b^4	c^8		$\bar{2}$	$\bar{4}$	$\bar{4}$	$\bar{1}$	1	2	1				
	c^{10}				$\bar{1}$		2	4	5	6	3	1	
	c^{12}				1	2	2	2	2	2	2	1	
	c^{14}					1	3	3	1	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{1}$
	c^{16}						1	1	1		$\bar{1}$	$\bar{1}$	
	c^{18}									$\bar{1}$			

TABLE OF GROUNDFORMS.

Deg. in coeff's of Quadratic	Deg. in coeff's of Cubic.	Degree in coeff's of Quartic. 0	1	2	3	4	5
0	0			1	1		
	2				1		
	4	1	1	2	3	2	1
	6			1	3	2	1
1	2	1	2	3	2	1	
	4		2	4	5	3	1
2	0	1	1	1			
	2		2	3	3	1	
	4		1				
3	0				1		
	2	1	1	1			
	4	1	1				
4	2		1	1			

SYSTEM OF ONE QUADRATIC AND TWO QUARTICS.

G. F. for invariants, reduced form.

Denominator: $(1-b^2)(1-\delta^2)(1-\delta^3)(1-d^2)(1-d^3)(1-b\delta)(1-b^2\delta)$
$(1-bd)(1-b^2d)(1-\delta d)(1-\delta^2 d)(1-\delta d^2)$.

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5	d^6			d^0	d^1	d^2	d^3	d^4	d^5	d^6
	δ^0	1								δ^2			1				
	δ^1									δ^3							
b^0	δ^2			1					b^5	δ^4					1		
	δ^3									δ^5							
	δ^4					1				δ^6							1
	δ^0		$\bar{1}$							δ^1			$\bar{1}$				
	δ^1	$\bar{1}$	1	1	1					δ^2		$\bar{1}$					
b^1	δ^2		1	1					b^4	δ^3				1		1	
	δ^3		1		1					δ^4					1	1	
	δ^4						$\bar{1}$			δ^5				1	1	1	$\bar{1}$
	δ^5					$\bar{1}$				δ^6						$\bar{1}$	
	δ^0			1						δ^1		1	$\bar{1}$				
	δ^1		2		$\bar{1}$	$\bar{1}$				δ^2		$\bar{1}$				$\bar{1}$	
b^2	δ^2	1		$\bar{1}$	$\bar{2}$				b^3	δ^3					$\bar{2}$	$\bar{1}$	
	δ^3		$\bar{1}$	$\bar{2}$						δ^4				$\bar{2}$	$\bar{1}$		1
	δ^4		$\bar{1}$				$\bar{1}$			δ^5			$\bar{1}$	$\bar{1}$		2	
	δ^5					$\bar{1}$	1			δ^6					1		

G. F. for invariants, representative form.

Denominator: $(1-b^2)(1-\delta^2)(1-\delta^3)(1-d^2)(1-d^3)(1-b^2\delta^2)(1-b^2\delta)$
$(1-b^2d^2)(1-b^2d)(1-\delta d)(1-\delta^2 d)(1-d^2\delta)$.

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5	d^6			d^1	d^2	d^3	d^4	d^5	d^6	d^7
	δ^0	1								δ^3			1				
	δ^1									δ^4							
b^0	δ^2			1					b^7	δ^5					1		
	δ^3									δ^6							
	δ^4					1				δ^7							1
	δ^1		1	1	1					δ^4				1	1	1	
b^1	δ^2		1	1	1				b^6	δ^5				1	1	1	
	δ^3		1	1	1					δ^6				1	1	1	
	δ^1		1	1						δ^1			$\bar{1}$				
	δ^2		1	1						δ^2			$\bar{1}$				
b^2	δ^3				1	1			b^5	δ^3	$\bar{1}$	$\bar{1}$		1			
	δ^4				1		$\bar{1}$	$\bar{1}$		δ^4			1	1			
	δ^5					$\bar{1}$				δ^5					1	1	
	δ^6					$\bar{1}$				δ^6					1	1	
	δ^0				1					δ^2				$\bar{1}$	$\bar{1}$	$\bar{1}$	
	δ^1		1	1		$\bar{1}$	$\bar{1}$			δ^3				$\bar{2}$	$\bar{2}$	$\bar{1}$	
b^3	δ^2		1	1	$\bar{1}$	$\bar{2}$	$\bar{1}$		b^4	δ^4		$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$		1
	δ^3	1		$\bar{1}$	$\bar{3}$	$\bar{2}$	$\bar{1}$			δ^5		$\bar{1}$	$\bar{2}$	$\bar{1}$	1	1	
	δ^4		$\bar{1}$	$\bar{2}$	$\bar{2}$					δ^6		$\bar{1}$	$\bar{1}$		1	1	
	δ^5		$\bar{1}$	$\bar{1}$	$\bar{1}$					δ^7				1			

TABLE OF GROUNDFORMS.

Deg. in coeff's of quadratic.	Deg. in coeff's of quartic.	Deg. in coeff's of quartic. 0	1	2	3
0	0			1	1
	1		1	1	
	2	1	1	1	
	3	1			
1	0				
	1		1	1	1
	2		1	1	1
	3		1	1	
2	0	1	1	1	
	1	1	1	1	
	2	1	1		
3	0				1
	1		1	1	
	2		1		
	3	1			

SYSTEM OF THREE QUARTICS.

G. F. for invariants, reduced form.

Denominator: $(1-\partial^2)(1-\partial^3)(1-\delta^2)(1-\delta^3)(1-d^2)(1-d^3)$
$(1-\partial\delta)(1-\partial d)(1-\delta d)(1-\partial^2 d)(1-\partial d^2)$
$(1-\partial^2\delta)(1-\partial\delta^2)(1-\delta^2 d)(1-\delta d^2)$.

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8			d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8
	δ^0	1										δ^4				$\bar{1}$				
	δ^1											δ^5								
	δ^2			1								δ^6						$\bar{1}$		
∂^0	δ^3										∂^8	δ^7								
	δ^4					1						δ^8								$\bar{1}$
	δ^1		1	1								δ^2				1				
	δ^2		1	1								δ^3				1				
	δ^3				1	1						δ^4		1	1		$\bar{1}$			
∂^1	δ^4				1		$\bar{1}$	$\bar{1}$			∂^7	δ^5				$\bar{1}$	$\bar{1}$			
	δ^5					$\bar{1}$						δ^6						$\bar{1}$	$\bar{1}$	
	δ^6					$\bar{1}$						δ^7						$\bar{1}$	$\bar{1}$	
	δ^0			1								δ^1				$\bar{1}$				
	δ^1		1	1								δ^2		$\bar{1}$	$\bar{1}$		1	1		
	δ^2	1	1	1			$\bar{1}$	$\bar{1}$				δ^3		$\bar{1}$	$\bar{1}$	2	2	1		
∂^2	δ^3				1	$\bar{1}$	$\bar{2}$	$\bar{1}$			∂^6	δ^4	$\bar{1}$		2	3	1			
	δ^4				$\bar{1}$	$\bar{3}$	$\bar{2}$		1			δ^5		1	2	1	$\bar{1}$			
	δ^5			$\bar{1}$	$\bar{2}$	$\bar{2}$	1	1				δ^6		1	1			$\bar{1}$	$\bar{1}$	$\bar{1}$
	δ^6			$\bar{1}$	$\bar{1}$		1	1				δ^7						$\bar{1}$	$\bar{1}$	
	δ^7					1						δ^8						$\bar{1}$		
	δ^1				1	1						δ^1				$\bar{1}$				
	δ^2				1	$\bar{1}$	$\bar{2}$	$\bar{1}$				δ^2		$\bar{1}$	$\bar{2}$	$\bar{2}$	1	1		
∂^3	δ^3		1	1		$\bar{4}$	$\bar{3}$	$\bar{1}$			∂^5	δ^3		$\bar{2}$	$\bar{3}$		3	2		
	δ^4		1	$\bar{1}$	$\bar{4}$	$\bar{5}$		2	1			δ^4	$\bar{1}$	$\bar{2}$		5	4	1	$\bar{1}$	
	δ^5			$\bar{2}$	$\bar{3}$		3	2				δ^5		1	3	$\bar{4}$		$\bar{1}$	$\bar{1}$	
	δ^6			$\bar{1}$	$\bar{1}$	2	2	1				δ^6		1	2	1	$\bar{1}$			
	δ^7					1						δ^7				$\bar{1}$	$\bar{1}$			

Numerator—*Continued:*

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8
	δ^0					1				
	δ^1				1		$\bar{1}$	$\bar{1}$		
	δ^2				$\bar{1}$	$\bar{3}$	$\bar{2}$		1	
∂^4	δ^3		1	$\bar{1}$	$\bar{4}$	$\bar{5}$		2	1	
	δ^4	1		$\bar{3}$	$\bar{5}$		5	3		$\bar{1}$
	δ^5		$\bar{1}$	$\bar{2}$		5	4	1	$\bar{1}$	
	δ^6		$\bar{1}$		2	3	1			
	δ^7			1	1		$\bar{1}$			
	δ^8					$\bar{1}$				

Representative form same as reduced form.

TABLE OF GROUNDFORMS.

Deg. in coeff's of quartic	Deg. in coeff's of quartic	Deg. in coeff's of quartic. 0	1	2	3
0	0			1	1
	1		1	1	
	2	1	1	1	
	3	1			
1	0		1	1	
	1	1	1	1	
	2	1	1	1	
2	0	1	1	1	
	1	1	1	1	
	2	1	1		
3	0	1			

69.

ON A CERTAIN INTEGRABLE CLASS OF DIFFERENTIAL AND FINITE DIFFERENCE EQUATIONS.

[*Johns Hopkins University Circulars*, I. (1882), p. 178.]

IN Mr Moulton's edition of Boole's *Finite Differences* will be found quoted from the author of this notice a certain class of equations of which the *general* integral can be found as for example

$$\begin{vmatrix} u_x & u_{x+1} \\ u_{x+1} & u_{x+2} \end{vmatrix} = A\alpha^x,$$

that is,

$$u_x u_{x+2} - u^2_{x+1} = A\alpha^x,$$

or again

$$\begin{vmatrix} u_x & u_{x+1} & u_{x+2} \\ u_{x+1} & u_{x+2} & u_{x+3} \\ u_{x+2} & u_{x+3} & u_{x+4} \end{vmatrix} = A\alpha^x,$$

and so on for a persymmetrical determinant of any order (n) constructed on the same principle as the two foregoing ones; an equation of the nth degree and $2n$th order will thus arise.

In this communication to the Seminarium the writer pointed out that an integral (but without any arbitrary constants) may be found for an equation of the same form as that above indicated on the left hand side but with $(n+1)$ different exponentials instead of a single one on the right hand side as for example

$$u_x u_{x+2} - u^2_{x+1} = A\alpha^x + B\beta^x + C\gamma^x$$

can be integrated provided that there are really three and not merely two distinct terms as would happen if A or B or C were one of them to vanish. But any number of the exponentials may be made indefinitely near to each other and the integral still hold good; in this way other integrable forms of equations can be obtained. As for instance

$$u_x u_{x+2} - u^2_{x+1} = A\alpha^x + (B + Cx)\beta^x,$$

$$u_x u_{x+2} - u^2_{x+1} = (A + Bx + Cx^2)\alpha^x$$

are integrable.

The same conclusions in all respects apply both as regards the general and the special integral case when any term u_x is replaced by y and u_{x+i} by $\left(\frac{d}{dx}\right)^i y$. The form of the special integral whether for differential or difference equations is rather too long to produce in this abstract but will be given in full in a future number of the *American Journal of Mathematics* [above, p. 546].

70.

ON A QUESTION IN PARTITIONS.

[*Johns Hopkins University Circulars*, I. (1882), p. 179.]

CLOSELY connected with the theory of the contacts or special intersections of quadric figures in space of any number of dimensions, and also with the more general but allied theory of the different genera and species of the roots of unitary matrices, is the question of the number of series that can be formed commencing with zero and ending with a given number i subject to the condition that each intermediate term of any such series shall be not greater than the mean between its antecedent and consequent. By arranging each of the indefinite partitions of i according to an ascending order of magnitude, it was shown that there was a one to one correspondence between each such arrangement and each such series, and, consequently, that the number of the series is equal to the number of indefinite partitions of the given final term i.

71.

ON A GEOMETRICAL PROOF OF A THEOREM IN NUMBERS.

[*Johns Hopkins University Circulars*, I. (1882), pp. 179, 180.]

THE theorem in question is the well-known one that if a, b are incommensurable and x, y integers $ax + by + c$ may be made positively and negatively indefinitely small. This is tantamount to showing that on the plane of a reticulation*, nodes may be found indefinitely near to and on each side of an irrational straight line, that is, a line not parallel to any line of nodes. The proof is based on the Lemma that no infinite parallelogram, each side of which is an irrational line containing a node, can be vacuous of nodes in its interior. If this were not true a succession of shifts of the figure in the direction of the line joining the two nodes would lead to the absurd conclusion that the whole reticulation consists of a single line of nodes.

(1) Suppose the irrational line L contains a node and that there is no other node at less than a finite distance from it on one side of it, say to the right. Let it be moved to the right parallel to itself until it passes through another node N', then there will be a vacuous parallelogram of the kind declared impossible by the Lemma. [To this it may be objected that when L has moved from the left to M through a distance δ, M might be supposed to be an asymptote to an infinite series of nodes to its right. But if this were the case a node P might be found at a less distance than δ from M, and a node, Q, nearer to M than P is; if this line of nodes PQ be followed up until we reach the first node T on the other side of M, the most elementary geometry seems to show that T in *any* case is nearer to M than P is and consequently there would be a node between L and M contrary to hypothesis.] Hence there must be a node indefinitely near to L on each side of it.

(2) Suppose the irrational line L not to contain a node. If the theorem to be proved is not true, L may as before be moved parallel to itself (through

* By a reticulation is to be understood a pair of systems of an infinite number of indefinite equidistant parallel lines in a plane whose intersections form the nodes.

a finite distance) until it pass through a single node and there would be a vacuous parallelogram of which one side contains nodes, which has already been shown to be impossible.

Dr Story and Dr Franklin took part in the discussion and the valuable critical observations of the latter, led to the consideration of the objection stated and disposed of in the passage within brackets above. Professor Cayley made a remark to the effect that the diamond point in a graver's tool however fine, drawn in a straight direction across the face of a double grating must either pass through none of the intersections of the two systems of parallel lines or through an infinite number of them. The principle established in the bracketed passage admits of being stated in the following terms: "It is impossible for a straight line in the plane of a reticulation to be asymptotic in regard to nodes on one side of it and not so in regard to the nodes on the other side"; this proposition and the Lemma being conceded, the existence of any indefinite *vacuous* strip bounded by irrational parallel lines is disproved by imagining it distended on both sides, still retaining its form (in case neither bounding line contains a node), or in the contrary case on one side only (that is, in the direction away from the nodal line) until the distended figure passes through two nodes. The asymptotic rule shows that this construction would be possible—the Lemma that it leads to an impossible result. From this it follows that every irrational line is asymptotic in respect to the nodes lying on *each* side of it which is the thing to be proved.

Let a line be termed mono-asymptotic when it is asymptotic in regard to any scheme of points lying on one side of it,—amphi-asymptotic when it is so for schemes of points lying on each side of it. The foregoing argument may then be summed up as follows. Any irrational right line in the plane of a reticulation, must be amphi-asymptotic as regards the nodes. For if not, a line parallel to it must (under pain of contradicting the Lemma) be conceded to exist, which shall be mono-asymptotic in respect to them, but the existence of such a line has been proved to be impossible*. Similarly, it may be shown for a solid network, that no indefinite open prism whose parallel edges are doubly irrational (that is, neither parallel to a nodal line nor to a plane of nodes) can be vacuous of nodes, and also that no plane can be mono-asymptotic—from which, by very similar reasoning to that previously used, may be deduced the law, that no prism of finite dimensions, *vacuous* of nodes, can be constructed about an irrational line as its axis and that consequently any such line may be regarded as a sort of asymptotic axis to a

* The form of proof is a somewhat unusual combination of an *Ex-absurdo* with a *Dilemma*. A *denial* of the amphi-asymptoticism of an irrational straight line either dashes itself against the impossibility of the existence of a vacuous parallelogram or against the equal impossibility of the existence of a mono-asymptotic line.

helical spiral of nodes. Hence it follows that if a, b, c (taken two and two) are incommensurable with each other, the quadratic function

$$\{b(z-\gamma)-c(y-\beta)\}^2+\{c(x-\alpha)-a(z-\gamma)\}^2+\{a(y-\beta)-b(x-\alpha)\}^2,$$

and as a particular case

$$(bz-cy)^2+(cx-az)^2+(ay-bx)^2$$

may be made indefinitely small with integer values of x, y, z.

Nor is this all, for not only can a node be found indefinitely near to the doubly irrational line $x:y:z::a:b:c$, but such node may be successfully sought for within any infinitesimal sector of space contained within two planes drawn through that line, or in other words a node can be found indefinitely near to the irrational line and to any plane drawn through it, that is, to any plane

$$bc(m-n)x+ca(n-l)y+ab(l-m)z,$$

where l, m, n are any quantities whatever*, so that x, y, z integer numbers can be found which shall simultaneously cause

$$(bz-cy)^2+(cx-az)^2+(ay-bx)^2$$

to be less than any positive quantity k^2 and

$$(b-c)x+(c-a)y+(a-b)z$$

to lie between 0 and h, or 0 and $-h$ where h is also any assigned quantity. And of course the proposition can be stated in more general terms by considering an irrational line which does not pass through a node.

The same geometrical property admits of being defined under another form, namely, through the assertion that if any two given planes be drawn through a completely irrational line in an infinite Nodal Block†, a node may be formed indefinitely near to each of them—and this statement translates itself into the arithmetical proposition following:

If no linear equation

$$\lambda(b\gamma-c\alpha)+\mu(c\alpha-a\gamma)+\nu(a\gamma-b\alpha)=0$$

exists for integer values of λ, μ, ν, the two expressions

$$ax+by+cz+d,$$

and

$$\alpha x+\beta y+\gamma z+\delta$$

may simultaneously be made less than any given quantity k, by integer values of x, y, z.

* A particular form of this is $(b-c)x+(c-a)y+(a-b)z$. In order to give the theorem its greatest generality it is only necessary to substitute for $x, y, z, x-\alpha, y-\beta, z-\gamma$ where α, β, γ, are any real quantities whatever.

† By a Nodal Block is to be understood three systems in space of an indefinite number of equidistant parallel planes whose intersections are the nodes.

Although it would be difficult to follow the theory of nodal schemes into regions transcending the sensible dimensions of space, there need be no hesitation in accepting the truth of the generalized arithmetical theorem corresponding to this bolder than Icarian flight, namely, that

Any number of linear functions of one more than that number of integer Variables such that the determinant to the Matrix formed by the coefficients of the Variables supplemented by a line of arbitrary integers is incapable of being made zero, can by a right assignment of the Variables be brought to lie each of them between any assigned (indefinitely narrow) limits.

This proposition admits of a partial removal of the condition imposed in the above statement.

For an irrational line even if singly irrational, that is, parallel to a *nodal* plane although not so to a line of nodes, will be asymptotic to a series of nodes *if it lies in a nodal plane*, the only difference in this case being that the nodal sheath will be plain instead of being helical. Hence the two functions

$$ax + by + cz + d, \quad a'x + b'y + c'z + d',$$

can be made simultaneously indefinitely small, even though integer numbers $A, B, C,$ *can* be found such that the determinant $\begin{vmatrix} a & b & c \\ a' & b' & c' \\ A & B & C \end{vmatrix}$ is zero, provided that a rational number D can also be found, which will cause *all* the complete minors of the Matrix $\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ A & B & C & D \end{vmatrix}$ to vanish.

A particular case of this arises when d and d' are each zero. Consequently the two twin functions

$$ax + by + cz \text{ and } a'x + b'y + c'z,$$

may *in all cases* be made each of them simultaneously to vanish, or else to become indefinitely small for integer values of x, y, z. Thus then, we have an immediate and intuitive proof of Jacobi's celebrated proposition for proving the impossibility of the existence of trebly periodic functions*.

Those gifted with the powers of a Stringham, a Newcomb, or a Charles S. Peirce to feel their way about in supersensible space, may, in like manner, obtain if not an *intuitive*, at least an immediate, or non-mediated proof of the theorem that: *Any number of* homogeneous *functions of one more than that number of integer variables may be made either* to vanish simultaneously or else *to become simultaneously less than any assignable quantity.*

* See M. Hermite's admirable *Note sur le calcul différentiel et le calcul intégral*, Paris, 1862, pp. 5—8, where the proposition in question is established by means of the theory of ternary and binary quadratic forms.

Whilst in the course of writing out the above matter the following note, addressed to him, from Dr F. Franklin, was received by Professor Sylvester:

"Your proof may be put into the following form:

Theorem.—In any stripe bounded by irrational parallels there must be a node.

For if not, let N and N' be *any* two nodes. Repeat the stripe a finite number of times, namely, until the aggregate of the stripes shall have included N and N'. No stripe can contain two nodes ν, ν', for if it did, by producing $\nu\nu'$ we see that each of the stripes must contain at least one node, which is contrary to the hypothesis. Hence we have an open parallelogram containing two nodes N, N', and only a finite number of others, which is absurd; for since the parallelogram intercepts a distance greater than NN', it must intercept on *every* nodal line parallel to NN' at least one node. Thus the theorem is proved.

It may be noted that while a stripe of finite width bounded by *rational* lines contains either no nodes or a *singly* infinite number of them, a stripe bounded by *irrational* lines always contains a *doubly* infinite number of nodes; which, although easily explicable, might at first sight strike one as paradoxical, inasmuch as the probable number in a *given finite portion* is the same for one sort of stripe as for the other."

One word in conclusion. The modes above given of presenting the theory with reference to planes passing through a singly or doubly irrational line ought not to be allowed to draw away attention from the image afforded by a doubly irrational line surrounded by an asymptotic spiral sheath (a point-helix winding round a fish-bellied-torpedo-like bobbin or core) tapering off to an indefinitely fine point in both directions, nor from the extension of the theory of continued fractions to which that image points.

Taking for greater simplicity the case of such a line passing through a node at the origin, the question invites solution to *devise an Algorithm* for finding the integer values of x, y, z which shall give the successive minima (corresponding to nodes of nearest approach) of the function

$$(bz - cy)^2 + (cx - az)^2 + (ay - bx)^2,$$

that being the problem next in the order of natural succession to the solved one of finding the successive absolute minima of $ay - bx$.

In the latter case, a and b are supposed to be incommensurable—in the former, no linear equation with rational coefficients is supposed to exist between a, b, c*.

* Which is tantamount to saying that the line $x : y : z :: a : b : c$ must be doubly irrational.

72.

ON THE GEOMETRICAL FORMS CALLED TREES.

[*Johns Hopkins University Circulars*, I. (1882), pp. 202, 203.]

[In connexion with the reference to his name in the above] Professor Sylvester stated that to M. Camille Jordan was due the credit of being the first to discover the existence of the centre or centre-pair of each kind described in the above note. In entire ignorance of M. Jordan's work he rediscovered for himself the centre or centre-pair of the first kind, and was the first to make use of the method immediately flowing therefrom to solve the problem of finding the forms and the number of tree-graphs* corresponding to a hydro-carbon or hypothetical hydro-boron series with a given number of carbon atoms. His results, which he communicated from time to time to Professor Tait, of Edinburgh, were however as regards the ascertainment of the number of such graphs, purely arithmetical, but giving all the different forms of the so-called trees or (more properly speaking) ramifications for different values of the number of atoms up to a certain arithmetical limit. The problem was subsequently taken up from this point by Professor Cayley, who obtained general generating-function formulæ for effecting the denumeration of the graphs. Mr Sylvester then proceeded to explain his method of arriving at the first kind of centre or centre-pair of any given tree or ramification.

To this end he supposes all the terminal branches of the tree removed. A tree with a less number of nodes is thus brought into evidence which is subjected (if possible) to like treatment and so a third tree with still fewer nodes is arrived at. As this process cannot be indefinitely continued (for if so a finite number could be continually diminished) we must at length come

* In accordance with the nomenclature employed above, the writer uses here occasionally the word *tree*, but considers his original word *ramification* more correct. A tree is a ramification with one point fixed as a root or origin, and no such fixed origin is supposed to exist in the graphs in question.

to a tree or ramification whose terminal branches cannot be removed without leaving nothing in the form of a tree remaining. So long as not less than three nodes remain, since they must not form a triangle, for that would be inconsistent with their appertaining to a ramification, the process of lopping off terminals cannot be brought to a close. Eventually, therefore, this process must lead to a system of branches all radiating out from a single point, or which being removed, only an isolated point remains, or else to a sort of double-headed mop or broom consisting of two such radiating systems stuck into the two ends of an axis. This is the case of bicentric or axial, the former of a monocentric ramification. Thus every ramification may be said to belong either to a central or an axial class. He concluded with suggesting that some general chemical or physical property or set of such properties might reasonably be supposed to exist serving to distinguish between these two classes or genera in the case of the well developed series of the hydro-carbons.

73.

ON THE 8-SQUARE IMAGINARIES.

[*Johns Hopkins University Circulars*, I. (1882), p. 203.]

[WITH reference to the above communication] Professor Sylvester referred to the general question of representing the product of sums of two, four or eight squares under the form of a like sum, and mentioned that Professor Cayley had been the first to demonstrate, by an exhaustive investigation, the impossibility of extending the law applicable to 2, 4 and 8 to the case of 16 squares. The new kind of so-called imaginaries referred to by Professor Cayley are, as far as Mr Sylvester is aware, the first example of the introduction into Analysis of locative symbols not subject to the strict law of association, and he considers the law regulating the connexion of the two products represented by a succession of three such symbols, most interesting, inasmuch as such products are either identical, or if not identical, of the same absolute value, but with contrary signs: most persons, before this example had been brought forward, would have felt inclined to doubt the possibility of locative symbols (*vulgo* imaginary quantities*) whose multiplication table should give results inconsistent with the common associative

* Using θ, h, t, u to denote thousands, hundreds, tens, units, the year of grace in which we live may be represented by $\theta+8h+8t+2u$, θ, h, t, u, being locative symbols which it would be absurd to style *imaginary quantities*; but they are as much entitled to that name as the i, j, k, or any like set of symbols—the only essential difference being that the one set of symbols is limited, the other unlimited in number—and accordingly the law of combination of the one set is given by a finite and of the other set by an infinite *multiplication table*. We might mark off the specific difference between the two cases, by defining the latter set as *unlimited*, the former as *recurrent* or *periodic* locatives or locators; the *locatives* indicate out of what *basket*, so to say, the *quantities* appearing in an analytical expression are to be selected—the multiplication table determines the basket into which their product is to be thrown. Under a purely analytical point of view this is all that is wanted—but in the application of quaternions to problems in nature, it becomes necessary to give special significance to the baskets or rubrics (which would do as well) to which the quantities belong and understand them to signify that certain geometrical processes of *setting* are to be performed.

The true analytical theory of quaternions has nothing to do with this setting part of the

law, being capable of forming the groundwork of any real accession to algebraical science—the results of Professor Cayley referred to above, seem to show that such doubts are open to question. Mr Sylvester mentioned as bearing upon the subject of so-called imaginary quantities, that in his recent researches in Multiple Algebra he had come upon a system of Nonions, the exact analogues of the Hamiltonian Quaternions and like them capable of being represented by square matrices. Mr Charles S. Peirce, it should be stated, had to the certain knowledge of Mr Sylvester arrived at the same result many years ago in connexion with his theory of the *logic of relatives*; but whether this result had been published by Mr Peirce, he was unable to say*.

business, and regards quaternions as matrices of the second order of a certain determinate form, and accordingly the whole analytical side of the theory of quaternions merges into a particular case of the general theory of *Multiple Algebra*.

As far as the present writer is aware, Professor Cayley in his memoir on Matrices, (*Phil. Trans.* 1858), was the first to recognize the parallelism between quaternions and matrices, but the idea and method of effecting their complete identification is due to the late Prof. Benjamin Peirce or to his son Mr C. S. Peirce.

* Mr C. S. Peirce gave a form of this Algebra in a paper "On a Notation for the Logic of Relatives," published in 1870. The class of Associative Algebras to which this belongs were termed *quadrates* by the late Professor Clifford.

74.

ON A GEOMETRICAL TREATMENT OF A THEOREM IN NUMBERS.

[*Johns Hopkins University Circulars*, I. (1882), p. 209.]

THE author made some remarks additional to those made on the same subject at the preceding meeting of the seminarium. In a plane reticulation four cases present themselves, namely, a line may be drawn through a line of nodes, or through a solitary node, or parallel to a line of nodes, or so as neither to pass through any node nor to be parallel to a line of nodes. In the third case the distance of the nodes of nearest approach is constant: in the second and fourth cases it approximates continually to zero. So in a solid reticulation eight cases present themselves, namely, four in addition to those last detailed: for without lying in a nodal plane, the line of flight may (α) pass through a single node, or (β) it may be parallel to a line of nodes, or (γ) it may be parallel to a nodal plane but not to a nodal line, or (δ) it may not pass through any node. In case (β) the distance of the nodes of nearest approach is constant; in case (γ) it approximates to a constant finite limit: in cases (α) and (δ) it approximates to zero.

There are thus four cases in all for which the distance from the nodes of nearest approach is a continually decreasing infinitesimal, namely: two for which the line of flight does not pass through any node, and two for which it does pass through a node—these latter two being those which serve to establish the theorem relating to the non-existence of trebly periodic functions.

The author further drew attention to the singular metamorphosis undergone by the geometrical setting forth of this theorem. It may be put under the form of asserting that a trilateral whose three sides are conditioned to be exact multiples of, and parallel to, three given straight lines lying in a plane may either be made to form a closed triangle or else such that the line closing the trilateral shall be less than any assigned quantity. On the other hand, the very same fact lends itself to, and is absolutely equivalent in substance to the statement that an arrow let fly from a node of a solid reticulation whether it speed along a nodal plane or be shot miscellaneously at the stars must (the law of gravity being supposed to be suspended) pass *indefinitely near* an infinite number of nodes in the course of its flight. The corresponding theorem for space of five dimensions serves to show that Quaternion Functions cannot have a higher than a quadruple periodicity.

75.

ON THE PROPERTIES OF A SPLIT MATRIX.

[*Johns Hopkins University Circulars*, I. (1882), pp. 210, 211.]

SUPPOSE a square matrix split into two sets of lines which need not be contiguous and may be called ranges, say ABC, $DEFG$. Let the sum of the products of the corresponding elements of any two lines be called their product. It is well known (see Salmon's *Higher Alg.*, 3rd Ed., p. 82) that if the product of each line in the first range by every line in the other is zero, the opposite complete minors of the two ranges will be in a constant ratio, say in the ratio $l:\lambda$. Call the content of the matrix Δ: then it follows, if S, Σ denote the sums of the squares of the complete minors in the two ranges respectively, that

$$\frac{\lambda}{l}S = \frac{l}{\lambda}\Sigma = \Delta.$$

But by a theorem of Cauchy concerning rectangular matrices S is equal to the determinant $(A, B, C)^2$, that is, to the determinant

$$\begin{vmatrix} AA & AB & AC \\ BA & BB & BC \\ CA & CB & CC \end{vmatrix}$$

and similarly $$\Sigma = (D, E, F, G)^2$$

so that $$\lambda^2 : l^2 :: (D, E, F, G)^2 : (A, B, C)^2$$

and $$S\Sigma = \Delta^2.$$

Suppose now that the product of *every* two lines in the entire matrix is zero. Then into whatever two ranges the matrix be divided the ratio $\lambda^2 : l^2$ (since all but the diagonal terms in the matrices which express the ratio $l^2 : \lambda^2$ vanish) will be expressed by the ratio of one simple product to another: thus for example for the ranges $ABC : DEFG$

$$\lambda^2 : l^2 :: D^2 . E^2 . F^2 . G^2 : A^2 . B^2 . C^2; \text{ also } \Delta^2 = A^2 . B^2 . C^2 . D^2 . E^2 . F^2 . G^2.$$

If we now further suppose that the sum of the squares of the elements in each line is unity, that is, that

$$A^2 = B^2 = C^2 = D^2 = E^2 = F^2 = G^2 = 1,$$

it will follow that every minor whatever divided by its opposite will be equal to Δ $\left(\text{for on the hypothesis made, } \frac{\lambda}{l} = \frac{\Delta}{S} = \Delta\right)$.

Also Δ will be plus or minus unity since $\Delta^2 = 1$. Thus it is seen that we may pass by a natural transition from the theory of a split to that of an orthogonal or self-reciprocal matrix—to show which was the principal motive to the present communication. It is by aid of the theorem of the *split matrix* that I prove a remarkable theorem in Multiple Algebra, namely, that if the product of two matrices of the same order is a complete null, the sum of the nullities of the two factors must be at least equal to the order of the matrix—the nullity of a matrix of the order ω being regarded as unity, when its determinant simply is zero, as 2 when each first minor simply is zero, as 3 when each second minor is zero ... as $(\omega - 1)$ when each quadratic minor is zero and as ω (or absolute) when every element is zero. This theorem again is included in the more general and precise one following—*If any number of matrices of the same order be multiplied together, the nullity of their product is not less than the nullity of any single factor and not greater than the sum of the nullities of all the several factors.*

In Professor Cayley's memoir on Matrices (*Phil. Trans.*, 1858) the very important proposition is stated that if

$$\begin{matrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \\ a''' & b''' & c''' & d''' \end{matrix}$$

be any matrix of substitution, say m (here taken by way of illustration of the order 4) the determinant

$$\begin{vmatrix} a - m & b & c & d \\ a' & b' - m & c' & d' \\ a'' & b'' & c'' - m & d'' \\ a''' & b''' & c''' & d''' - m \end{vmatrix}$$

is identically zero; or in other words, its nullity is complete. By means of the above theorem it may be shown that the nullity of any i distinct algebraical factors of such matrix is equal to i, i having any value from unity up to the number which expresses the order of the matrix, inclusive.

76.

A WORD ON NONIONS.

[*Johns Hopkins University Circulars*, I. (1882), pp. 241, 242; II. (1883), p. 46.]

In my lectures on Multiple Algebra I showed that if u, v are two matrices of the second order, and if the determinant of the matrix $(z + yv + xu)$ be written as

$$z^2 + 2bxz + 2cyz + dx^2 + 2exy + fy^2$$

then the necessary and sufficient conditions for the equation $vu + uv = 0$ are the following, namely,

$$b = 0, \quad c = 0, \quad e = 0.$$

If to these conditions we superadd $d = 1$, $f = 1$, and write $uv = w$, then

$$u^2 = -1, \quad v^2 = -1, \quad w^2 = -1, \quad uv = -vu = w, \quad vw = -wv = u, \quad wu = -uw = v;$$

and 1, u, v, w form a quaternion system. The conditions above stated will be satisfied if

$$\text{Det.}\,(z + yv + xu) = z^2 + y^2 + x^2,$$

which will obviously be the case if

$$v = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}, \quad u = \begin{vmatrix} 0 & \theta \\ \theta & 0 \end{vmatrix},$$

where $\theta = \sqrt{(-1)}$. For then

$$z + yv + xu = \begin{vmatrix} z & y + x\theta \\ -y + x\theta & z \end{vmatrix}.$$

Hence the matrices

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \begin{vmatrix} 0 & \theta \\ \theta & 0 \end{vmatrix} \begin{vmatrix} -\theta & 0 \\ 0 & \theta \end{vmatrix}$$

construed as complex quantities are a linear transformation of the ordinary quaternion system 1, i, j, k; that is to say, if we form the multiplication table

	λ	μ	ν	τ
λ	λ	μ	0	0
μ	0	0	λ	μ
ν	ν	τ	0	0
τ	0	0	ν	τ

$$\lambda+\tau=1 \qquad -\mu+\nu=i$$
$$-\theta\lambda+\theta\tau=k \qquad \theta\mu+\theta\nu=j.$$

Since u, v contain between them 8 letters subject to the satisfaction of 5 conditions, the most general values of λ, μ, ν, τ ought to contain 3 arbitrary constants; but it is well-known that any particular (i, j, k) system may be superseded by a $\lambda(i', j', k')$ system, where i', j', k' are orthogonally related linear functions of i, j, k; and as this substitution introduces just 3 arbitrary constants, we may, by aid of it, pass from the system of matrices above given, to the most general form. The general expression for the matrices containing 3 arbitrary constants may also be found directly by the method given in my lectures, which will be reproduced in the memoir on Multiple Algebra in the *Mathematical Journal*. What goes before is by way of introduction to the *word* on Nonions which follows.

Just as the necessary and sufficient condition that u, v, two matrices of the second order, may satisfy the equations $vu=-uv$, $u^2=1$, $v^2=1$, is that the determinant to $z+yv+xu$ may be $z^2+y^2+x^2$, so I have proved that the necessary and sufficient condition, in order that we may have $vu=\rho uv$, $u^3=1$, $v^3=1$ (u, v being matrices of the third order, and ρ an imaginary cube root of unity) is that the determinant to $z+yu+xv$ may be $z^3+y^3+x^3$; but if we make

$$u=\begin{vmatrix} 0 & 0 & 1 \\ \rho & 0 & 0 \\ 0 & \rho^2 & 0 \end{vmatrix}, \qquad v=\begin{vmatrix} 0 & 0 & 1 \\ \rho^2 & 0 & 0 \\ 0 & \rho & 0 \end{vmatrix},$$

then
$$z+yu+xv=\begin{vmatrix} z & 0 & y+x \\ \rho y+\rho^2 x & z & 0 \\ 0 & \rho^2 y+\rho x & z \end{vmatrix}$$

of which the determinant is

$$z^3+(y+x)(\rho y+\rho^2 x)(\rho^2 y+\rho x)=z^3+y^3+x^3.$$

Hence there will be a system of Nonions (precisely analogous to the known

system of quaternions) represented by the 9 matrices

$$\begin{array}{ccccc} & & 1 & & \\ & u & & v & \\ u^2 & & uv & & v^2 \\ & u^2v & & uv^2 & \\ & & u^2v^2 & & \end{array}$$

and just as in the preceding case the 8 terms $\pm 1, \pm u, \pm v, \pm uv$ form a closed group, so here the 27 terms obtained by multiplying each of the above 9 by $1, \rho, \rho^2$ will form a closed group. The values of the 9 matrices will easily be found to be

$$\begin{array}{ccccc} & & \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} & & \\ & \begin{vmatrix} 0 & 0 & 1 \\ \rho & 0 & 0 \\ 0 & \rho^2 & 0 \end{vmatrix} & & \begin{vmatrix} 0 & 0 & 1 \\ \rho^2 & 0 & 0 \\ 0 & \rho & 0 \end{vmatrix} & \\ \begin{vmatrix} 0 & \rho^2 & 0 \\ 0 & 0 & \rho \\ 1 & 0 & 0 \end{vmatrix} & & \begin{vmatrix} 0 & \rho & 0 \\ 0 & 0 & \rho \\ \rho & 0 & 0 \end{vmatrix} & & \begin{vmatrix} 0 & \rho & 0 \\ 0 & 0 & \rho^2 \\ 1 & 0 & 0 \end{vmatrix} \\ & \begin{vmatrix} \rho & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} & & \begin{vmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho \end{vmatrix} & \\ & & \begin{vmatrix} 0 & 0 & \rho \\ \rho & 0 & 0 \\ 0 & \rho & 0 \end{vmatrix} & & \end{array}$$

These forms can be derived from an algebra given by Mr Charles S. Peirce (*Logic of Relatives*, 1870).

I will only stay to observe that as the condition of the Determinant to $z+uy+vx$ (which for general values of u, v is a general cubic with the coefficient of z^3 unity) assuming the form $z^3+y^3+x^3$, implies the satisfaction of 9 conditions, and as u, v between them contain 18 constants, the most general form of a system of Nonions must contain $18-9$, or 9 arbitrary constants; but how these can be obtained from the particular form of the system above given, remains open for further examination.

[*Note*. For the remark made above] "These forms can be derived from an algebra given by Mr Charles S. Peirce (*Logic of Relatives*, 1870)," read "Mr C. S. Peirce informs me that these forms can be derived from his *Logic of Relatives*, 1870." I know nothing whatever of the fact of my own personal knowledge*. I have not read the paper referred to, and am not

* I have also a great repugnance to being made to speak of Algebras in the plural; I would as lief acknowledge a plurality of Gods as of Algebras.

acquainted with its contents. The mistake originated in my having left instructions for Mr Peirce to be invited to supply in my final copy for the press, such reference as he might think called for. He will be doing a service to Algebra by showing in these columns how he derives my forms from his logic*. The application of Algebra to Logic is now an old tale—the application of Logic to Algebra marks a far more advanced stadium in the evolution of the human intellect; the same may be said as regards the application by Descartes of Analysis to Geometry, and the reverse application by Eisenstein, Dirichlet, Cauchy, Riemann, and others, of Geometry to Analysis—so that if Mr Peirce accomplishes the task proposed to him (his ability to do which I do not call into question), he will have raised himself as far above the level of the ordinary Algebraic logicians as Riemann's mathematical stand-point tops that of Descartes.

It is but justice to Boole's memory to recall the fact that, in one of his papers in the *Philosophical Transactions*, he has made a reverse use of logic to establish a certain theorem concerning inequalities, which is very far from obvious, and which I think he states it took him ten years to deduce from purely algebraical considerations, having previously seen it through logical spectacles—I mean, by the aids to vision afforded him by his logical calculus: this theorem I believe (or at least did so when it was present to my mind) must of necessity admit of a much more comprehensive form of statement.

* I had understood Mr Peirce to say that these forms were actually contained in his memoir.

77.

ON MECHANICAL INVOLUTION.

[*Johns Hopkins University Circulars*, I. (1882), pp. 242, 243.]

MANY years ago I gave in the *Comptes Rendus* of the Institute of France, one or more geometrical constructions of the problem of Mechanical Involution.

When forces can be introduced along six given lines in space whose statical sum is zero, a certain geometrical condition must be fulfilled by the 6 lines which are then said to be in involution. If two homographic pencils of rays in different planes have two corresponding rays coincident (but their centres apart), any six lines, each of which cuts two corresponding rays, will form an involution system. In the communication to the Society I showed that the analytical condition of involution might be expressed by means of equating to zero a certain compound determinant. I have found since that this determinant is given by Cayley in the *Cam. Phil. Soc. Tr.* 1861, part 2.

Let 1, 2, 3, 4, 5, 6 be the six lines and on each of them let two arbitrary points be taken; let the quadri-planar coordinates of the two arbitrary points on any of the lines, say j, be called j_x, j_y, j_z, j_t; j_x', j_y', j_z', j_t', respectively, the condition of involution referred to will be

$$\begin{vmatrix} & 1.2 & 1.3 & 1.4 & 1.5 & 1.6 \\ 2.1 & & 2.3 & 2.4 & 2.5 & 2.6 \\ 3.1 & 3.2 & & 3.4 & 3.5 & 3.6 \\ 4.1 & 4.2 & 4.3 & & 4.5 & 4.6 \\ 5.1 & 5.2 & 5.3 & 5.4 & & 5.6 \\ 6.1 & 6.2 & 6.3 & 6.4 & 6.5 & \end{vmatrix} = 0$$

where any binary combination $ij = ji$, and where either of them represents the determinant

$$\begin{vmatrix} i_x, & i_y, & i_z, & i_t \\ i_x', & i_y', & i_z', & i_t' \\ j_x, & j_y, & j_z, & j_t \\ j_x', & j_y', & j_z', & j_t' \end{vmatrix}$$

Six lines in involution represent indifferently lines along which forces or axes of couples can be introduced, whose statical sum is zero. Consequently such a system is the analogue in space at one and the same time to three force-lines converging to a point, or to three points in a line regarded as centres of moments, in a plane. But *in plano* the concurrence of right lines is the polar property to the collineation of points. Hence we ought to expect that the polar reciprocal in respect to any quadric of an involution system, should also be an involution system; and such is obviously the case by virtue of the fact that the correspondence of the rays in the two homographic pencils, referred to above, will not be affected when for each ray in either pencil is substituted its polar in respect to any quadric. (A direct proof will be found in the *American Mathematical Journal*, Vol. IV., part 4 *.) I concluded with pointing out the analogy between the problem of Mechanical Involution and what I call Algebraical Involution, which takes place when x, y being each of them matrices of the order ω, a linear equation connects the ω^2 ground-forms represented by the distinct terms of the product

$$(1, x, x^2, \ldots x^{\omega-1} \between 1, y, y^2, \ldots y^{\omega-1}).$$

Mechanical involution in a plane, in 3-dimensional, in 4-dimensional space, etc., is the analogue of algebraical involution between two matrices of the order 2, 3, 4, etc.; the $\frac{1}{2}(\omega^2+\omega)$ directions in ω-dimensional space being the analogues of the ω^2 ground-forms of matrices of the order ω. Each of the two problems consists of two parts: to obtain the condition of involution being the one part, to assign the relative magnitudes, in the one case, of the forces which cause their statical sum to vanish, and in the other case of the coefficients which enter into the linear function, the other part of the problem. The form of the solution of this second part of the algebraical problem (subject only to a certain ambiguity) has been given in my lectures, and will appear in the Memoir on Multiple Algebra in the *American Journal of Mathematics*; but the former part of the algebraical problem, that is, the determination of the *condition* of Algebraical Involution, except for the case of matrices of the second order, I have not yet succeeded in solving.

[* Cf. p. 560, above.]

78.

ON CROCCHI'S THEOREM.

[*Johns Hopkins University Circulars*, II. (1883), p. 2.]

In *Battaglini's Journal* for July, 1880, Signor Crocchi has given a theorem which may be stated in the following terms. If s_i, σ_i, h_i denote respectively the sum of the elementary combinations, of the powers, and of the homogeneous products each of the ith order of any number of elements, then h_i is the same function of $\sigma_1, \overline{\sigma}_2, \sigma_3, \overline{\sigma}_4, \ldots$ that s_i is of $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \ldots$.

Signor Crocchi's proof is very elegant but a little circuitous. An instantaneous proof may be derived from the relation of reciprocity which connects s and h, namely, that if

$$h_i = f(s_1, s_2, \ldots s_i) \text{ then } s_i = f(h_1, h_2, \ldots h_i),$$

which is an immediate deduction from the well-known fact that

$$(1 + s_1 y + s_2 y^2 + s_3 y^3 + \ldots)(1 - h_1 y + h_2 y^2 - h_3 y^3 + \ldots) = 1.$$

For from this relation spring the equations

$$s_1 - h_1 = 0, \quad s_2 - h_1 s_1 + h_2 = 0, \quad s_3 - h_1 s_2 + h_2 s_1 - h_3 = 0 \ldots$$

which equations continue unaltered when the letters s and h are interchanged; for when such interchange takes place, the functions equated to zero of an even rank remain unaltered and those of an odd rank merely change their sign.

Returning to the immediate object in view, if $a, b, c, \ldots$ are the elements subject to the s, h, σ symbols, we may write

$$\Sigma \log(1 + ay) = \log(1 + s_1 y + s_2 y^2 + s_3 y^3 + s_4 y^4 + \ldots)$$

or,

$$\Sigma \log(1 - ay) = -\log(1 + h_1 y + h_2 y^2 + h_3 y^3 + h_4 y^4 + \ldots).$$

The first equation by differentiation performed in each side gives

$$\sigma_1 - \sigma_2 y + \sigma_3 y^2 - \sigma_4 y^3 + \ldots = \frac{s_1 + 2s_2 y + 3s_3 y^2 + 4s_4 y^3 + \ldots}{1 + s_1 y + s_2 y^2 + s_3 y^3 + \ldots},$$

and similarly the second equation gives

$$\sigma_1 + \sigma_2 y + \sigma_3 y^2 + \sigma_4 y^3 + \ldots = \frac{h_1 + 2h_2 y + 3h_3 y^2 + 4h_4 y^3 + \ldots}{1 + h_1 y + h_2 y^2 + h_3 y^3 + \ldots},$$

that is, $(\sigma_1 - \sigma_2 y + \sigma_3 y^2 - \ldots)(1 + s_1 y + s_2 y^2 + \ldots) = s_1 + 2s_2 y + 3s_3 y^2 + \ldots$

and $\quad (\sigma_1 + \sigma_2 y + \sigma_3 y^2 + \ldots)(1 + h_1 y + h_2 y^2 + \ldots) = h_1 + 2h_2 y + 3h_3 y^2 + \ldots.$

By comparison of coefficients of the powers of y, the first of these two equations affords the means of finding any σ in terms of the s quantities, and the second of these any σ in terms of the h quantities. But if we change s into h and $\sigma_2, \sigma_4, \ldots$ into $-\sigma_2, -\sigma_4, \ldots$ the first equation becomes the second. Hence if

$$s = f(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \ldots)$$

$$h = f(\sigma_1, \bar{\sigma}_2, \sigma_3, \bar{\sigma}_4, \ldots). \quad \text{Q.E.D.}$$

It is not without interest to set out the reciprocity of the 6 relations which exist between s, σ, h. The synoptical scheme of such reciprocity may be exhibited symbolically as follows:

$$h/s = s/h, \quad h/\sigma = s/\pm\sigma, \quad \sigma/h = \pm\sigma/s.$$

As an illustration of the second of these symbolic equalities take

$$s_3 = \frac{\sigma_1^3 - 3\sigma_1\sigma_2 + 2\sigma_3}{6}, \quad s_4 = \frac{\sigma_1^4 - 6\sigma_1^2\sigma_2 + 8\sigma_1\sigma_3 + 3\sigma_2^2 - 6\sigma_4}{24};$$

the corresponding equations are

$$h_3 = \frac{\sigma_1^3 + 3\sigma_1\sigma_2 + 2\sigma_3}{6}, \quad h_4 = \frac{\sigma_1^4 + 6\sigma_1^2\sigma_2 + 8\sigma_1\sigma_3 + 3\sigma_2^2 + 6\sigma_4}{24},$$

and it is worthy of observation that the sum of the numerical coefficients is always (as in the above examples) zero for the function of the σ quantities which gives an s of any order, and unity for the function of the same which expresses any h*.

* This statement is proved instantaneously by taking one of the elements equal to unity and all the rest zero; and the latter part of it gives a new proof of Cauchy's theorem which he obtains by a consideration of all the possible cyclic representations of the substitutions of n elements. The theorem is that if n elements be divided in every possible way into λ set of l, μ set of m, ν set of n ... elements, then

$$\Sigma \frac{1}{\pi\lambda\pi\mu\pi\nu \ldots l^\lambda m^\mu n^\nu \ldots} = 1.$$

For we know by a theorem of Waring that

$$s_n = \Sigma \pm \frac{1}{\pi\lambda\pi\mu \ldots l^\lambda m^\mu \ldots} \sigma_l^\lambda \sigma_m^\mu \ldots.$$

Hence by Crocchi's theorem the sum of the coefficients in h_n expressed in σ's is equal to

$$\Sigma \frac{1}{\pi\lambda\pi\mu \ldots l^\lambda m \ \ldots}$$

but it is also equal to unity. Cauchy's theorem is therefore proved.

Frequent occasion presents itself (especially in the theory of numbers) for expressing any s in terms of σ's, but probably up to the time when Signor Crocchi wrote on the subject there had never been any occasion to express h in terms of the σ's: for had such occasion ever arisen it seems almost impossible that the relation between the two corresponding sets of formulæ could have escaped observation.

In some recent researches, however, of the writer of this note on the irreducible semi-invariants of a quantic of an unlimited order, it becomes indispensable to convert homogeneous products into sums of powers, and Crocchi's theorem comes into play. (See sec. 4 of Article on Subinvariants, *Am. Math. Journ.*, Vol. v., part 2 [p. 597, above].)

The relation $\sigma/h = \pm\sigma/s$ is interesting under the point of view that virtually it contains an example of a sort of *invariance* of form which may possibly contain within itself the germ of an important theory. It informs us that if, in the function of h's which expresses any σ, in lieu of each h the function of s quantities to which it is equal be substituted, the form of the σ function will remain unchanged, except that when the order of the σ is an even number, its algebraical sign is reversed. Thus, for example,

$$\sigma_3 = h_1^3 - 3h_1h_2 + 3h_3, \quad h_1 = s_1, \quad h_2 = s_1^2 - s_2, \quad h_3 = s_1^3 - 2s_1s_2 + s_3.$$

Consequently if we write $\phi = x^3 - 3xy + 3z$, and for y and z substitute $x^2 - y$, $x^3 - 2xy + z$, respectively, the value of ϕ remains unaltered. So in like manner if we write

$$\phi = x^4 - 4x^2y + 4xz + 2y^2 - 4t,$$

and substitute for y, z, t;

$$x^2 - y, \quad x^3 - 2xy + z, \quad x^4 - 3x^2y + 2xz + y^2 - t,$$

respectively, no change ensues in ϕ except that it undergoes a change of sign.

So in general the σ functions with even and those with odd subindices may be regarded as the analogues of symmetrical and skew-invariants, respectively.

Again in the formula for s the sign *plus* or *minus* depends on the oddness or evenness of $\lambda + \mu + \ldots$. Hence if in

$$\Sigma \frac{1}{\pi\lambda\pi\mu \ldots l^\lambda m^\mu \ldots}$$

only those values of $\lambda, \mu, \ldots$ are admitted which make $\lambda + \mu + \ldots$ always odd or always even, either sum so formed will be equal to $\frac{1}{2}$, because the difference of the two sums is zero and their sum unity.

This theorem can, of course, be deduced like the former one from the method of cycles applied now, not to the entire number of the substitutions, but to that half of them which correspond to the *alternate group* of each, of which the number of representative cycles (monomial ones included) is always odd or else always even, according as the number of elements is one or the other.

79.

ON CERTAIN SUCCESSIONS OF INTEGERS THAT CANNOT BE INDEFINITELY CONTINUED.

[*Johns Hopkins University Circulars*, II. (1883), pp. 2, 3.]

A SUCCESSION of decreasing integers we know cannot be indefinitely continued, but there are also successions of increasing integers subject to certain stated conditions, but otherwise arbitrary, which are similarly incapable of indefinite extension.

The following is a simple instance of the kind. Suppose integers to be written down one after the other, no one of which is a multiple of any other, nor the sum of a multiple of any other and of a multiple of a specified one. *Such a succession cannot be indefinitely continued.*

Let a be the specified integer.

(1) Suppose that all the other integers of the succession are prime to a.

Then if b be any other of the integers, the equation $ax + by = c$ is soluble in integers if c is greater than ab, as follows at once from the consideration that the numbers $c - b, c - 2b, c - 3b \ldots c - ab$ must be all distinct residues to the modulus a, inasmuch as the difference of any two of them being of the form $(i - j)b$ where $i - j$ is less than and b prime to a, cannot be divisible by a.

But if the succession could be indefinitely produced, it must contain a number greater than ab. Hence the theorem is proved for the case where a is prime to every other integer in the succession.

(2) Suppose the theorem to be true for the case where the quotient of a divided by i prime numbers (not necessarily all distinct) is prime to all the other terms of the series: it must be true when the number of such prime numbers is $i + 1$. For let p be one of them and $a = pa'$, consider all the terms of the succession divisible by p apart from the rest.

Let $pa', pb', pc' \ldots$ be those terms. By the law of the succession the equation $pa'x + pb'y = pc'$ cannot be satisfied for any values of b', c', and consequently $a'x + b'y = c'$ cannot be satisfied.

Hence by hypothesis since a' divided by i factors is prime to $b', c' \ldots$ the succession of terms divisible by p must be finite in number, and this will be true for every factor p. Hence the succession $b, c, \ldots$ will contain only a finite number of terms having any factor in common with a. Moreover the succession containing a and terms prime to a exclusively, must also be finite by the preceding case. Consequently the whole succession will be finite, and the theorem if assumed to be true for $i = 0$, or any positive integer, is true for $i + 1$.

But by the preceding case the proposition is true when $i = 0$. Hence it is true universally.

In the long footnote to the Article on Subinvariants in Vol. v., pp. 92, 93 of the *Am. Journal of Math.*, will be found given the mode of extending this theorem to the case of successions of complex integers or multiplets, when a proper restriction is laid upon the ratios to one another of the simple numbers which constitute the multiplets, and a possible connexion pointed out between the finiteness of such successions and that of the system of ground-forms to a binary quantic [p. 580, above].

80.

ON THE FUNDAMENTAL THEOREM IN THE NEW METHOD OF PARTITIONS.

[*Johns Hopkins University Circulars*, II. (1883), p. 22.]

THE new method of partitions which I gave to the world more than a quarter of a century ago is an application of a theorem which, I think it must be conceded, is, after Newton's Binomial Theorem, the most important organic theorem which exists in the whole range of the Old Algebra. What Newton's theorem effects for the development of *radical*—that theorem accomplishes for the development of *fractional* forms of algebraical functions.

One (but not the most perfect) form in which it can be presented is the following. If Fx be any proper algebraical fraction in x, whose infinity roots (that is, the values of x which make Fx infinite) are $a, b, \ldots l$, quantities all supposed to differ from zero, then the coefficient of x^n for any value of n will be the residue, that is, the coefficient of $\frac{1}{x}$ in

$$\Sigma(\lambda^{-n}e^{nx})\,F(\lambda e^{-x}) \quad [\lambda = a, b, \ldots, l].$$

By supposing Fx broken up into proper simple fractions of the form $\Sigma\frac{fx}{(a-x)^i}$ it is very easy to see that the theorem will be true in general if true for $\frac{fx}{(a-x)^i}$, and from this it is but a step to see that the theorem will be true in general if true for the simplest form of rational function, that is, $\frac{1}{(1-x)^i}$.

All then that remains to do is to show that the coefficient of x^n in this fraction is the same as the coefficient of $\frac{1}{x}$ in $\frac{e^{nx}}{(1-e^{-x})^i}$ which may be done as follows:

$$\frac{1}{(1-e^{-x})^i} = \left(1-\frac{\delta_x}{1}\right)\left(1-\frac{\delta_x}{2}\right)\left(1-\frac{\delta_x}{3}\right)\ldots\left(1-\frac{\delta_x}{(i-1)}\right)\left(\frac{1}{1-e^{-x}}\right)$$

$$= (1 - A\delta_x + B\delta_x^2 - C\delta_x^3 \ldots)\left(\frac{1}{x} + \ldots\right)$$

$$= \left(\frac{1}{x} + \frac{A}{x^2} + \frac{1.2B}{x^3} + \frac{1.2.3C}{x^4} + \ldots\right) + \text{positive powers of } x.$$

Therefore the coefficient of $\frac{1}{x}$ in

$$\frac{1 + nx + \frac{n^2}{1.2}x^2 + \frac{n^3}{1.2.3}x^3 + \ldots}{(1-e^{-x})^i}$$

$$= (1 + An + Bn^2 + Cn^3 + \ldots)$$

$$= \left(1+\frac{n}{1}\right)\left(1+\frac{n}{2}\right)\left(1+\frac{n}{3}\right)\ldots\left(1+\frac{n}{i-1}\right)$$

$$= \frac{(n+1)(n+2)\ldots(n+i-1)}{1, 2, \ldots (i-1)} = \text{coefficient of } x^n \text{ in } \frac{1}{(1-x)^i}. \quad \text{Q.E.D.}$$

This method of proof, however, is not the simplest or best; as soon as we mould the theorem into a form most easily admitting of being expressed in general terms that very form itself suggests a simpler (nay, so to say, an instantaneous) proof, and moreover relaxes an unnecessarily stringent condition in the previous statement of the theorem.

Of course by a finite infinity root of a function no one can fail to understand a value of the variable differing from zero which makes the function infinite. This then is the true statement of the theorem in general terms.

In any proper-fractional function developed in ascending powers of a variable, the constant term is equal to the Residue (with its sign changed) of a sum of functions obtained by substituting in the given function in place of the variable the product of each, in succession, of its finite infinity roots into the exponential of the variable.

That is to say, if we take the proper-fraction

$$Fx = \frac{\phi x}{x^i(x-a)^j(x-b)^k\ldots(x-l)^\omega},$$

the constant term (with its sign changed) in this fraction developed in ascending powers of x is the same as the Residue of $\Sigma F(\lambda e^x)\,[\lambda = a, b, \ldots l]$.

To prove this it is only necessary to suppose the fraction Fx separated into simple partial fractions with constant numerators and the theorem becomes self-evident*.

* It must, however, previously be shown that the residue of $\frac{1}{(1-e^x)^i}$, where i is a positive

It follows, therefore, writing n in place of i that the coefficient of x^n in ascending-power series for the fraction

$$Gx = \left(\frac{\phi x}{(x-a)^{\lambda} \dots (x-l)^{\mu}}\right)$$

will be the Residue with its sign changed of $\Sigma (a^{-n} e^{-nx}) G(ae^x)$, or which is the same thing is the Residue of $\Sigma a^{-n} e^{nx} G(ae^{-x})$, which theorem we now see is true not merely for the case where G is a proper-fraction, that is, is a function of x whose degree is a negative integer, but remains true when the degree of G is any number inferior to n, for when that condition is satisfied $\frac{G}{x^n}$ is a proper fraction, which is all that is required in order for the parent theorem to apply.

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integer, is the same as that of $\frac{1}{1-e^x}$, that is, is -1; this becomes obvious from the consideration that the change of i into $i+1$ alters the quantity to be residuated by $\frac{e^x}{(1-e^x)^{i+1}}$, that is, by the differential derivative of $\frac{1}{(1-e^x)^i}$ divided by i, of which the residue is necessarily zero—that being true for the differential derivative of any series of powers of a variable.

81.

NOTE ON THE PAPER OF MR DURFEE'S.

[*Johns Hopkins University Circulars*, II. (1883), pp. 23, 24; 42, 43.]

MR DURFEE'S very elegant and interesting theorem above given may, by help of Euler's law of reciprocity, be expressed in the following terms.

Let fx and ϕx represent respectively:

$$\frac{1}{1-x} + \frac{x^4}{(1-x)(1-x^2)(1-x^3)} + \frac{x^{12}}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)} + \dots + \frac{x^{2i^2+2i}}{(1-x)\dots(1-x^{2i+1})} + \dots$$

and

$$\frac{x^2}{(1-x)(1-x^2)} + \frac{x^8}{(1-x)(1-x^2)(1-x^3)(1-x^4)} + \dots + \frac{x^{2i^2}}{(1-x)\dots(1-x^{2i})} + \dots,$$

then the number of self-conjugate partitions of $2m+1$ and of $2m$ are the coefficients of x^m in the ascending expansions of fx, ϕx, respectively.

Thus, suppose $2m+1=13$, the coefficient of x^6 in fx developed, that is, $\frac{6}{1,} + \frac{2}{1,\ 2,}$ or 3 is the number of self-conjugate partitions of 13.

These will be found to be 7 1 1 1 1 1 1, 4 4 3 2, 5 3 3 1 1. To find the conjugate to any partition $a, b, c \dots, l$, the most expeditious method is to find n_i, the number of the elements in the partition not less than i: $n_1, n_2, \dots, n_l$ (l being supposed to be the largest value of any element) will then be its opposite.

Thus, for example, for the partition 5 3 3 1 1, $n_1 = 5$, $n_2 = 3$, $n_3 = 3$, $n_4 = 1$, $n_5 = 1$, and $n_1\ n_2\ n_3\ n_4\ n_5$ reproduces 5 3 3 1 1.

If $2m = 12$ we have to find the value of $\frac{4}{1,\ 2,}$ which is again 3, and the 3 self-conjugate or self-opposite partitions of 12 will be seen to be 4 4 2 2; 5 3 2 1 1; 6 2 1 1 1 1.

In M. Faà de Bruno's tables of symmetric functions, which are only complete for the case of equations of not higher than the 11th degree, the number of self-conjugate partitions which appear among the headings and sidings of the tables is either 1 or 2, and it was therefore reasonable to try the effect of making arrangement of the partitions such as to bring the self-conjugate or pair of self-conjugate partitions into the centre of the line or column; but as soon as that degree is passed such a kind of principle (the rule founded upon which M. de Bruno does not state) becomes *prima facie* inapplicable at all events without undergoing modifications of which at present we know nothing.

Thus M. de Bruno's tables end just where his proposed principle of arrangement becomes inapplicable, stopping short at the case of the 12th degree, which has since been tabulated by Mr Durfee in the *American Journal of Mathematics.*

The term "opposite" or "conjugate" is used by Mr Durfee in the sense in which I am in the habit of employing it to signify the relation between what M. Faà de Bruno calls *combinaisons associées.* I think it right to recall attention to the fact that the credit of calling into being this kind of conjugate relation, is due to Dr Ferrers (the present Master of Gonville and Caius College, Cambridge), who some 30 years ago or more was the first to apply it to obtain an intuitive proof of Euler's great law of reciprocity, the very same as that which I have here employed to transform Mr Durfee's theorem. Euler demonstrated his law by help of his favourite instrument of generating functions.

By instituting in the case of combinations of *unrepeated* elements quite another and more exquisite kind of conjugate relation applicable to all such with the exception of those which belong to the infinite succession 1, 2, 2 3, 3 4, 3 4 5, 4 5 6, 4 5 6 7, 5 6 7 8, Mr Franklin, of this University, succeeded in finding an instantaneous demonstration of another well-known but very much more recondite theorem in partitions, also due to Euler, expressible by the statement that the indefinite product

$$(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots$$

has for its development

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26}\ldots,$$

where the indices are the complete series of direct and retrograde pentagonal numbers.

By a singular oversight in my note in the last *Circular*, I omitted to state that Mr Durfee's rule is tantamount to affirming that the number of self-conjugate partitions or (which is the same thing) of symmetrical partition graphs for n, is the coefficient of x^n in the series

$$1 + \ldots + \frac{x^{i^2}}{(1-x^2)(1-x^4)\ldots(1-x^{2i})} + \ldots$$

and since this series is identical with the infinite product

$$(1+x)(1+x^3)(1+x^5)\ldots$$

the number of self-conjugate partitions is the number of ways of distributing n into unrepeated odd-integers, a result which can be obtained directly by regarding any symmetrical partition graph as made up of a set of successively diminishing equilateral elbows or say carpenters' rules, each of which necessarily contains an odd number of points: the number of such elbows for any given graph will be the same as the number of points in the side of Mr Durfee's square nucleus, and consequently we have an intuitive proof of the theorem that the infinite product

$$(1+ax)(1+ax^3)(1+ax^5)\ldots$$

is equal to the infinite series

$$1 + \ldots + \frac{x^{i^2}}{(1-x^2)(1-x^4)\ldots(1-x^{2i})}a^i + \ldots$$

because the coefficient of $a^i x^n$ is the same in both expressions. By a similar method I obtain an intuitive and almost instantaneous solution of the problem to expand in infinite series the infinite products which express a Theta Function and its *reciprocal*, and many other questions of a similar nature.

It was the anticipation of the parallelism between the expressions for the number of special partitions in the unrepeated-numbers and the repeated-numbers theories which led me to find *à priori* the partition-into-odd-integers expression for the number of self-conjugate partitions, and thus started me on the track of the graphical method of transforming infinite products into infinite series: the light of analogy may sometimes "lead astray" but it is more often "light from heaven."

82.

ON DR F. FRANKLIN'S PROOF OF EULER'S THEOREM CONCERNING THE FORM OF THE INFINITE PRODUCT $(1-x)(1-x^2)(1-x^3)\ldots$.

[*Johns Hopkins University Circulars*, II. (1883), p. 42.]

REVOLVING in my mind Mr Franklin's remarkable proof of Euler's theorem concerning the above infinite product inserted in the *Comptes Rendus* of the Institute of France for 1880, I have found it useful to employ a certain terminology to enable myself to seize some of the points which it contains with a firmer grasp and to clothe it in what seems to me a more purely discursive, as distinguished from what, by analogy to geometrical processes, I am wont to call a diagrammatic form of reasoning; thinking that others may find advantage in what has been useful to myself, I avail myself of the pages of the *Circular* to give it publicity.

Let us agree to understand by a *distribution* of n any combination of *unrepeated* integers in descending order, whose sum is equal to n. The number of such component integers may be termed the *order* of the distribution.

If the initial components of such distributions be $m-1, m-2, \ldots, (m-i)$ [where i may be equal to but cannot exceed the order] not followed by an element $m-i-1$, I call i (the number of terms in such initial sequence) the *consecutant* and the final (that is, the least) component, the *concluant* of the distribution.

Lemma. Any distribution of a given integer, which does not form a single sequence whereof the concluant is either equal to or greater by a unit than the consecutant, may be converted by one or the other (but not by either) of two reversible processes (say of loading or unloading) into another distribution in which the order is diminished or increased by a single unit.

By loading is to be understood the process of taking away the concluant (say ω) and increasing the ω first terms of the initial sequence each by a

unit; and by unloading, that of taking away a unit from each of the components in the initial sequence and adding on an element equal to the consecutant as the new concluant.

1st. Suppose that the distribution does not form a single sequence.

If the concluant is equal to or less than the consecutant it is obvious that loading will be possible but not unloading, because the latter would give rise to a new concluant equal to or greater than the original one.

On the other hand, if the concluant is greater than the consecutant, unloading will be possible but not loading, because there will be too few terms in the initial sequence to exhaust (by the addition of one unit to each) the number of units in the concluant.

2nd. Suppose that the entire distribution forms a single sequence.

If the concluant is *less* than the consecutant loading will still be possible, because the number of terms in the sequence after taking away the concluant will still be not greater than the concluant.

Again, if the concluant is more than a unit greater than the consecutant, unloading will still be possible because the new concluant will be less than the original one even after it has lost a unit by the process of unloading.

Hence the Lemma is proved.

And as a Post-lemma, it may be stated that when the distribution forms a single sequence such that the concluant is equal to or only one unit greater than the consecutant, neither loading nor unloading will be possible. The loading on the first supposition is defeated by the fact that the diminished sequence will be one too few in number to absorb the units which make up the concluant—and the unloading on the second supposition is defeated by the fact that the new concluant will be equal to (that is, will be a repetition of) the old one when by the act of unloading it is diminished by a unit.

From the lemma and post-lemma combined, it follows as an inference that all the distributions of any number n may be taken in pairs (consisting of one of an even and one of an odd order), unless it should be the case that one of such distributions is a term in the series

$$1,\quad 2,\quad 3.2,\quad 4.3,\quad 5.4.3,\quad 6.5.4,\ \ldots,\qquad (2i-1).(2i-2)\ldots i,\quad 2i.(2i-1)\ldots(i+1),\ \ldots$$

which represent distributions of the several integers

$$1,\ 2,\ 5,\ 7,\ 12,\ 15,\ \ldots\ \frac{3i^2-i}{2},\ \frac{3i^2+i}{2},\ \ldots$$

to which the process either of loading or unloading (contraction or expansion by a unit) is inapplicable.

Hence if we denote by n_o, n_e, the number of distributions of n, into an odd and even number of unrepeated parts, we must have $n_o - n_e = 0$, except when $n = \frac{3i^2 \mp i}{2}$, in which case $n_e - n_o = (-)^i 1$.

Consequently we have

$$(1-x)(1-x^2)(1-x^3)\ldots,$$

that is,

$$1 + \ldots + (n_e - n_o)x^n + \ldots = \sum_{i=+\infty}^{i=-\infty} (-)^i x^{\frac{3i^2+i}{2}},$$

which is Euler's theorem.

To make the demonstration absolutely objection-proof it ought to be shown that if X is convertible into Y by loading or unloading, Y will be convertible into X by the reverse process—but this is almost self-obvious; for if X has become Y by loading, the new consecutant cannot be greater than the old one and will therefore not be greater than the new concluant, but equal to or less than it, and therefore the process of *unloading* is the one applicable to Y, and if X has become Y by unloading, the new consecutant cannot be less than the old one and will therefore be greater than the new concluant, and therefore the process of *loading* is the one applicable to Y; this completes the proof, and leaves I think nothing further to be desired.

In Mr Durfee's question, treated of in the last number of the *Circulars*, the object of research is the number of self-conjugate partitions (with repeated or unrepeated components) of a given integer n; in Mr Franklin's, the object sought for is the number (1 or 0) of (so to say celibate or) unconjugate distributions of an integer: the Ferrers-law of conjugation is of universal application to all partitions—the Franklin-law only to partitions with unrepeated components.

There is, however, a singular parallelism between the two theories; let us agree to call the self-conjugate in the one, and the non-conjugate partitions in the other, in each case alike *special* partitions—and denote the number of distributions of n into an odd number and into an even number of *unrestricted* parts by $(n)_o$ and $(n)_e$ respectively. Then just as the difference between n_o and n_e is the number of special partitions in the one, so it may be shown that the difference between $(n)_o$ and $(n)_e$ (which is well-known to be the same as the total number of partitions of n into unrepeated odd parts) is the number of special partitions in the cognate theory.

83.

ON THE USE OF CROSS-GRATINGS TO OBTAIN CERTAIN DEVELOPMENTS CONNECTED WITH THE THEORY OF ELLIPTIC FUNCTIONS.

[*Johns Hopkins University Circulars*, II. (1883), pp. 43, 44.]

It will be convenient to regard the components of any partition as arranged in a natural, say a descending order of magnitude: a partition graph means a series of points, say the knots in a web or the intersections of a cross-grating, lying in lines parallel to two fixed lines: the number of points, or lines parallel to one of the boundaries chosen at will, will represent the successive components of the partition and the number of the lines themselves will be the number of parts in the partition.

The lines in question may for greater distinctness be termed *magnitude* lines and the crossing ones, *part* lines. The graph may be termed regular when the magnitude lines never increase as they recede from the rectilinear boundary to which they are parallel. This, we see intuitively, cannot happen without the same condition being true of lines parallel to the *part* boundary: so that we may say that a regular partition graph is one in which the lines and columns of points neither of them ever increase in length as they recede from their respective boundaries. If such a graph corresponds to a partition *without* repetitions, the lines drawn in the magnitude direction must continually contract (that is, contain fewer and fewer points) as they recede from their maximum boundary.

The correlation referred to in the preceding paragraph is tantamount to saying that if there be two systems of partitions in one of which a given number is set out in every possible way as a sum of i parts none exceeding j in magnitude, and another in which the same number is set out in every possible way as a sum of j parts none exceeding i in number, such partitions arranged in natural order will have a one-to-one correspondence, being representable by the same regular graphs with the names of the magnitude and part boundaries interchanged, so that there will be the same number of partitions in the two systems.

A partition is self-conjugate when its representative graph, after an interchange of the names of the part- and magnitude-lines, gives the same reading.

Such a graph, therefore, must be symmetrical.

Suppose the partible number to be n.

Then its graph may be resolved into i angles fitting into one another, consisting of continually decreasing odd numbers; and the number of such graphs will be the number of ways of composing n with unrepeated odd numbers: but it may also be analyzed into a square containing i^2 points and two similar and equal appendages each containing $\frac{n-i^2}{2}$ points; and consequently their number will be the number of ways in which $\frac{n-i^2}{2}$ may be made up with the numbers 1, 2, ... i, or what is the same thing $n-i^2$ with the numbers 2, 4, ... $2i$; it is consequently the coefficient of n in the development of

$$\frac{x^{i^2}}{(1-x^2)(1-x^4)\dots(1-x^{2i})};$$

but it is also by virtue of the preceding remark the coefficient of $x^n a^i$ in the continued product

$$(1+ax)(1+ax^3)(1+ax^5)\dots.$$

Hence this continued product

$$=1+\frac{x}{1-x^2}a+\frac{x^4}{(1-x^2)(1-x^4)}a^2+\frac{x^9}{(1-x^2)(1-x^4)(1-x^6)}a^3$$
$$+\dots+\frac{x^{i^2}}{(1-x^2)(1-x^4)\dots(1-x^{2i})}a^i+\dots.$$

The expansion of the reciprocal of

$$(1-ax)(1-ax^3)(1-ax^5)\dots$$

may be obtained in a similar manner; the coefficient of $x^n a^j$ in this product is the number of ways in which n can be composed with j free odd numbers. If we construct a graph with j angles or elbows fitting into one another, such that the number of nodes in each such angle from the outermost inward corresponds with any such partition in descending order, the graph so constructed will be still symmetrical but no longer regular; a line of nodes instead of being necessarily equal or less in number than an antecedent one may jut one degree beyond it, but if the j points in the diagonal be removed (as in the example following, the points

$$\begin{array}{cccccc}
1 & \cdot & \cdot & \cdot & \cdot & \\
\cdot & 2 & \cdot & \cdot & \cdot & \\
\cdot & \cdot & 3 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 4 & \cdot & \\
\cdot & \cdot & \cdot & \cdot & & \\
 & & \cdot & & &
\end{array}$$

whose places are supplied by the numbers 1, 2, 3, 4) then the figure that is left is decomposable into two regular graphs with one boundary line horizontal, or vertical, and the other oblique. Hence the fraction above given expanded in powers of a becomes

$$1+\frac{x}{1-x^2}a+\dots+\frac{x^i}{(1-x^2)(1-x^4)\dots(1-x^{2i})}a^i+\dots$$

the only difference from the preceding case being that i points now instead of i^2 are taken away from the graph.

I might give numerous other exemplifications of the power and grasp of this method, but the following two may suffice for the present. I propose first to prove the equation between the reciprocal of

$$(1-ax)(1-ax^2)(1-ax^3)\dots$$

and the infinite series

$$1+\frac{x}{1-x}\cdot\frac{a}{1-ax}+\frac{x^4}{(1-x)(1-x^2)}\cdot\frac{a^2}{(1-ax)(1-ax^2)}+\dots$$
$$+\frac{x^{i^2}}{(1-x)(1-x^2)\dots(1-x^i)}\cdot\frac{a^i}{(1-ax)(1-ax^2)\dots(1-ax^i)}+\dots.$$

The coefficient of $x^n a^j$ in the continued product is the number of regular graphs that can be formed with n nodes, containing j lines of them with no limitations to the number of the columns. We may suppose, therefore, the number of columns to be made successively 1, 2, 3, ... Consider the case where there are i columns forming a square; the graph being regular the lines and columns will intersect in i^2 nodes, and there will be left $n-i^2$ nodes to be made up of $j-i$ quantities none greater than i (namely, the horizontal filaments of nodes in the columns underlying the square), and of other quantities not greater than i but otherwise unlimited (namely, the vertical filaments of nodes in the hollowed out indefinite parallelogram abutting alongside of the square): that number we well know is the coefficient of $x^n a^{j-i}$ in

$$\frac{1}{(1-ax)(1-ax^2)\dots(1-ax^i)}\cdot\frac{1}{(1-x)(1-x^2)\dots(1-x^i)}x^{i^2}.$$

Hence for every value of j the coefficient of $x^n a^j$ in the infinite product is the coefficient of $x^n a^j$ in the infinite series, and consequently the two forms when developed must be identical.

Not to weary my readers I hurry on to the development in an infinite series of the product of the two infinite products

$$(1+ax)(1+ax^3)(1+ax^5)\dots \text{ and } (1+a^{-1}x)(1+a^{-1}x^3)(1+a^{-1}x^5)\dots.$$

Here it will be expedient to explain what I mean by a parallel bipartition of n; it is simply a couple of sets of numbers written on opposite sides of a line of demarcation, so that the number of the numbers on the left always exceeds that on the right by a given difference δ, which may be any number from zero upwards, and such that the sum of all the elements collectively is equal to n.

When this difference is zero, such a bipartition may be called equi-parallel, in other cases parallel with a difference δ.

It is then clear that the coefficient of $x^n a^j$ or $x^n a^{-j}$ in the above product is nothing else but the number of parallel bipartitions of n to the difference j limited to contain only odd numbers which must not appear in the same arrangement more than once on the same side of the line of demarcation.

In particular the coefficient of x^n in the term not containing a will be the number of equi-parallel bipartitions of n restricted to odd numbers not repeated on the same side of the separating line.

Form a graph as follows: Supposing one of the bipartitions to consist of θ parts on each side, say $a, b, c, \ldots l$; $\alpha, \beta, \gamma, \ldots \lambda$; the parts being on each side taken in descending order, construct angles or elbows in which the horizontal sides contain

$$\frac{a+1}{2},\ \frac{b+1}{2},\ \ldots\ \frac{l+1}{2},$$

and the vertical sides

$$\frac{\alpha+1}{2},\ \frac{\beta+1}{2},\ \ldots\ \frac{\lambda+1}{2}$$

points, then these will contain

$$\frac{a+\alpha}{2},\ \frac{b+\beta}{2},\ \ldots\ \frac{l+\lambda}{2}$$

points respectively; on fitting them into one another we shall obtain a regular graph with θ lines or columns made up of $\frac{n}{2}$ points, and conversely every regular graph of $\frac{n}{2}$ points may be resolved into angles with sides p, p'; q, q'; $r, r' \ldots$ corresponding to an equi-parallel unrepeated odd-number bipartition

$$2p-1,\ 2q-1,\ 2r-1,\ \ldots;\ 2p'-1,\ 2q'-1,\ 2r'-1,\ \ldots.$$

Hence the coefficient of x^n in the term not containing a in the development is the number of regular graphs that can be formed with $\frac{n}{2}$ points; and therefore the term not containing a is

$$\frac{1}{(1-x^2)(1-x^4)(1-x^6)\ldots}.$$

Now consider the term containing a^j to which corresponds a parallel bipartition with j more elements to the left than to the right of the separating

line: arrange the sets on each side of the line in descending order, strike off the j highest on the left-hand side and construct a graph G with the sets which remain, as in the last case; then subtract 1, 3, 5, $(2j-1)$ respectively from the j elements [struck off] to the left, and place, taken in ascending order, half the numbers of points remaining, over the top line of the graph G; there will result a regular graph G'; and by an obvious reverse process every such graph can be made to correspond with a bipartition of unrepeated odd numbers having j more numbers to the left than to the right. Hence the number of the parallel bipartitions to the difference j will be the number of indefinite partitions of

$$\tfrac{1}{2}\{n-(1+3+\ldots+2j-1)\} \text{ or } \frac{n-j^2}{2},$$

that is, the coefficient of x^n in

$$\frac{x^{j^2}}{(1-x^2)(1-x^4)(1-x^6)\ldots}.$$

Hence the given bi-product when developed must be identical with

$$\frac{1}{(1-x^2)(1-x^4)\ldots}\{1+x(a+a^{-1})+x^4(a^2+a^{-2})+x^9(a^3+a^{-3})+\ldots\}.$$

In the preceding volume of the *Circular* I showed how the self-same method of points (but very differently applied) serves to establish and leads to wide generalizations of the theorem of Jacobi, upon which depends the proof of the impossibility of the existence of 3-period functions.

In a future number of the *Circular*, or else in the *American Journal of Mathematics*, I propose to show how to obtain intuitively by a graphical construction the expression for the product of the two infinite products

$$\frac{1-a^k}{1-a}\cdot\frac{1-a^{3k}}{1-a^3}\cdot\frac{1-a^{5k}}{1-a^5}\ldots \text{ and } \frac{1-a^{-k}}{1-a^{-1}}\cdot\frac{1-a^{-3k}}{1-a^{-3}}\cdot\frac{1-a^{-5k}}{1-a^{-5}}\ldots.$$

The *true inwardness* of this powerful analytical method depends in the first place on the idea of *correspondence*, assisted in the second place (in some but not in all instances) by the idea of graphical representation of partition numbers restrained to assume a natural order of succession.

Mr Ferrers' method, which has lain so long dormant and sterile in mathematical soil, has after an interval of 30 years begun to germinate, and seems to be about to burst forth into luxuriant growth and efflorescence.

It is Mr Durfee's graphical determination of the number of self-conjugate partitions of n, inserted in a preceding *Circular*, that has let in the light and air and supplied the fertilizing influence needful to bring this about.

84.

ON THE NUMBER OF FRACTIONS IN THEIR LOWEST TERMS WHOSE NUMERATORS AND DENOMINATORS ARE LIMITED NOT TO EXCEED A CERTAIN NUMBER.

[*Johns Hopkins University Circulars*, II. (1883), pp. 44, 45.]

THE fractions for greater simplicity may be supposed to be proper fractions, except that it is expedient to count in $\frac{1}{1}$ as one of them. To any given limit or argument as it may be called, n, corresponds a finite system of fractions in their lowest terms, which may be arranged in order of magnitude; when so arranged the system will be found to possess some remarkable properties, first apparently noticed by Mr Farey, an English mathematician, in 1816, subsequently made the subject of a proof by Cauchy in the same year (reproduced in his *Exercices de Mathématiques*, t. I. 1826), and again demonstrated and extended by Mr J. W. L. Glaisher in an interesting paper in the *London and Edinburgh Philosophical Magazine* for 1879, the same journal in which the subject was first broached.

I am under the impression that I have seen somewhere the names of one, if not two, English mathematicians who have endeavoured to obtain an empirical law for the number of fractions corresponding to any given limit, but all my endeavours to come upon the traces of those investigations, if such exist, have hitherto proved fruitless. Had anything been done in this direction prior to 1879, there is little doubt that reference would have been made to it by Mr Glaisher, who goes carefully in his paper of that date into the bibliography of the subject*.

This number for the limit or argument j is obviously no other than the sum of the *totients* of all the numbers from unity up to j. I shall use Tx

* Mr Glaisher, however, takes no notice of M. Halphen's important extension of Farey's theory, published in the *Proceedings of the Mathematical Society of France*, and followed by another on the same subject by M. Lucas, nor of Herzer's table in 1864, nor Hrabak's, 1876.

to denote the sum of all the totients of all the integers not exceeding x, where x is any positive quantity whatever, and show how to make Tx the subject of a functional equation, from which limiting functions to its value may be deduced. To this end consider the two sets of terms $1, 2, 3, \ldots i$ and $i+1, i+2, \ldots j$, where $j=2i$ or $2i+1$ indifferently.

The number of times that an integer r is contained in any given set of quantities, or rather the number of quantities in the set which contain r, I call the *frequency* of r in respect to the set.

Looking to the two sets here in question it is easily seen that the frequency of any integer *quâ* the second set must either be equal to its frequency *quâ* the first set or exceed it by a single unit. The equi-frequent and unequi-frequent integers are determinable by the following theorem which I call the theorem of harmonic division.

Let j_μ in general denote the integer part of j/μ if it is a fraction, or the whole of it if it is an integer.

Write down the successive ranges

$$j,\ j-1,\ j-2,\ \ldots\ j_2+1;\ j_2,\ j_2-1,\ \ldots\ j_3+1;$$
$$j_3,\ j_3-1,\ \ldots\ j_4+1;\ j_4,\ j_4-1,\ \ldots\ j_5+1;\ \ldots$$

where it will be understood that if $j_k=j_{k+1}$, the range $j_k \ldots$ becomes abortive.

Any number which appears in the first, third, fifth ... range is equi-frequent and any number which appears in the second, fourth ... range is unequi-frequent in respect to the two given series

$$1,\ 2,\ \ldots\ i;\ i+1,\ i+2,\ \ldots\ j.$$

This theorem will be found to be demonstrable without the slightest difficulty.

The second theorem required is one of which a demonstration almost instantaneous and conclusive is given in the Excursus on Rational Fractions and Partitions (*Am. Jour. of Math.*, Vol. v., No. 2*), namely, that the sum of the products formed by multiplying the frequency of any integer in respect to a given set of quantities by its totient is equal to the sum of the quantities contained in the set.

This proposition shows that if fr, $f'r$ be the frequencies of r in respect to the two last-named sets and τr its totient

$$\sum_{r=1}^{r=\infty}(f'r-fr)\,\tau r=(1+2+\ldots+j)-2\,(1+2+\ldots+i),$$

and the theorem of harmonic division shows that the left-hand side of this equation is equal to

$$\sum_{\lambda=j_2+1}^{\lambda=j}\tau\lambda+\sum_{\lambda=j_4+1}^{\lambda=j_3}\tau\lambda+\sum_{\lambda=j_6+1}^{\lambda=j_5}\tau\lambda+\ldots$$

[* Above, p. 611.]

because $f'r - fr = 1$ for the odd-ordered, and $= 0$ for the even-ordered harmonic ranges.

The separate sums above written are obviously the same respectively as

$$Tj - T\frac{j}{2},\ T\frac{j}{3} - T\frac{j}{4},\ T\frac{j}{5} - T\frac{j}{6},\ \ldots.$$

Hence, if we write

$$\vartheta j = Tj - T\frac{j}{2} + T\frac{j}{3} - T\frac{j}{4} + T\frac{j}{5} \ldots \textit{ ad inf.}$$

when j is an even integer

$$\vartheta j = (1 + 2 + \ldots + 2i) - 2(1 + 2 + \ldots + i) = i^2 = \frac{j^2}{4},$$

and when j is an odd integer

$$\vartheta j = \{1 + 2 + \ldots + (2i+1)\} - 2(1 + 2 + \ldots + i) = (i+1)^2 = \frac{(j+1)^2}{4}.$$

If now we write for any positive quantity x,

$$\vartheta x = Tx - T\frac{x}{2} + T\frac{x}{3} - T\frac{x}{4} + \ldots,$$

it may be shown by aid of the above results that for all values of x not less than unity,

$$\vartheta x = \text{ or } > \frac{x^2}{4} - \frac{x}{2},\quad \vartheta x = \text{ or } < \frac{x^2}{4} + \frac{x}{2} + \frac{1}{4},$$

and from these two inequalities limiting values to $T(x)$ may be deduced by a process of successive approximation in principle the same as that employed by me in the *Am. Jour. of Math.*, Vol. v., No. 2, pp. 124, 125*, in connexion with Tchebycheff's theory, but differing from it considerably in the mode of application and in the character of the results to which it leads.

The subject has been for so very short a time studied by me that I feel it desirable to reserve this part of its development for a future communication, but I am in a position to state that it is possible to find superior and inferior limits to Tx, say Lx and Λx, such that Lx shall be of the form $Mx^2 + Nx +$ a rational integral function of $\log x$ and Λx of a similar form, $M'x^2 + N'x +$ another rational integral function of x, where M, M' differ by a quantity less than any quantity that may be assigned from one another and from a number λ found from the equation

$$\lambda\left(1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} \ldots\right) = \frac{1}{4},$$

that is, equal to $3/\pi^2$.

Accordingly the ultimate, or so to say asymptotic, value of $\frac{Tx}{x^2}$ is $3/\pi^2$ where Tx is the number of pairs of integers not exceeding x, which are relatively prime to each other; consequently since the total number of such

[* Above, p. 530.]

pairs is ultimately in a ratio of equality to $\frac{x^2 + x}{2}$, it will follow, if the above assertion is correct, that the chance of two arbitrary independent integers, being relatively prime to one another, is $\frac{6}{\pi^2}$; the odds in favour of two such numbers being relatively prime, are thus very nearly expressed by the ratio of 60792 to 39268; that is to say, are pretty nearly as 76 to 49 or a trifle better than 3 to 2.

In what precedes I have used the simplest means or formula sufficient for obtaining a functional equation to the sum-totient Tx, but the theorem of harmonic division admits of a very wide generalization, and accordingly the functional equation admits of an indefinite number of distinct presentations.

Thus instead of j belonging to the series $2i$, $2i + 1$ it may be considered as belonging to the series ki, $ki + 1$, ..., $ki + (k-1)$, and the theorem of harmonic division then is as follows: calling the range commencing with j_λ and ending with $j_{\lambda+1} + 1$, range number λ, where λ may be understood to be any integer from 0 upwards (0 itself included) if the residue of λ in respect to k is μ, and if $f'r$ and fr are the frequencies of r in respect to the two series 1, 2, ... j, 1, 2, ... i, then when r belongs to the range whose number is λ, and the residue of λ in respect to k is (λ), it will be found that $f'r - kfr = (\lambda)$.

By way of example suppose $k = 3$, then writing

$$\vartheta j = \left(Tj + T\frac{j}{2} - 2T\frac{j}{3}\right) + \left(T\frac{j}{4} + T\frac{j}{5} - 2T\frac{j}{6}\right) + \ldots$$

it may readily be proved that according as $j = 3i$, or $3i + 1$, or $3i + 2$, ϑj will equal $3i^2$, or $3i^2 + 3i + 1$, or $3i^2 + 6i + 3$, and similarly if

$$\vartheta j = \left(\vartheta j + \vartheta\frac{j}{2} \ldots + \vartheta\frac{j}{k-1} - k\vartheta\frac{j}{k}\right) + \left(\vartheta\frac{j}{k+1} \ldots + \vartheta\frac{j}{2k-1} - k\vartheta\frac{j}{2k}\right) + \ldots$$

then ϑj according as $j = ki$, or $ki + 1$, ... or $ki + (k-1)$ will have k distinct and perfectly determinate values of which the first will be $\frac{k-1}{2k} j^2$ and the last $\frac{k-1}{2k}(j+1)^2$.

More general formulæ still may be obtained by supposing

$$j = ki + r,\quad j' = k'i + r',$$

where k, k' are relative prime numbers and r, r' less than k, k' respectively.

Let $P = kk'i + R$, R being less than kk' and congruent to r in respect to modulus k and to r' in respect to modulus k', then if we divide P into harmonic ranges and call fr, $f'r$ the frequencies of r in respect to the two series $1, 2, \ldots j$; $1, 2, \ldots j'$, and call ν the number of the range to which r belongs, and δ, δ' the residues of ν in respect to k, k' respectively, it will be found that $kf'r - k'fr = \delta' - \delta$.

Take as an example $i = 20$, $j = 41$, $j' = 62$, so that $k = 2$, $k' = 3$, then $P = 125$ and for

$$\begin{aligned} \nu &= 0,\ 1,\ 2,\ 3,\ 4,\ 5, \\ \delta &= 0,\ 1,\ 0,\ 1,\ 0,\ 1, \\ \delta' &= 0,\ 1,\ 2,\ 0,\ 1,\ 2, \\ \delta' - \delta &= 0,\ 0,\ 2,\ \bar{1},\ 1,\ 1, \end{aligned}$$

the harmonic ranges of P beginning with Range No. 2 will be seen to be

$125 - 63$, $62 - 42$, $41 - 32$, $31 - 24$, $23 - 21$, $20 - 18$, $17 - 16$, $15 - 14$, 13, etc.,

and the corresponding frequencies of the numbers in those ranges in regard to the series $1, 2, \ldots 41, 1, 2, \ldots 62$, will be seen to be

$$1, 0;\ 1, 1;\ 2, 1;\ 2, 1;\ 3, 2;\ 3, 2;\ 4, 2;\ 4, 3;\ \ldots \text{ respectively,}$$

and we have

$$2.1 - 3.0 = 2,\ 2.1 - 3.1 = \bar{1},\ 2.2 - 3.1 = 1,\ 2.2 - 3.1 = 1,\ 2.3 - 3.2 = 0,$$
$$2.3 - 3.2 = 0,\ 2.4 - 3.2 = 2,\ 2.4 - 3.3 = \bar{1}\ \ldots,$$

in which the recurring period is as it ought to be, $2, \bar{1}, 1, 1, 0, 0$.

By means of this division a still wider latitude could be won were it worth while, for the expression of the functional equation to the sum-totient. Another statement and further extensions of the theory are contained in a Note intended for publication in the *Comptes Rendus* of the Institute of France. I may add that I have had a table constructed of the values of Tx for all values of x up to 500 inclusive, and that Tx is always intermediate within this range between $3/\pi^2 x^2$ and $3/\pi^2 (x+1)^2$—a very noteworthy result: and which I have little doubt remains true for all values of x.

85.

PROOF OF A WELL-KNOWN DEVELOPMENT OF A CONTINUED PRODUCT IN A SERIES.

[*Johns Hopkins University Circulars*, II. (1883), p. 46.]

To prove that the general term in the development in a series of powers of a of the reciprocal of

$$(1 - a)(1 - ax) \ldots (1 - ax^i)$$

(say of Fx) is

$$(1 - x^{j+1})(1 - x^{j+2}) \ldots (1 - x^{j+i}) \div (1 - x)(1 - x^2) \ldots (1 - x^i) \,.\, a^j$$

say $X_j a^j$, I proceed as follows.

I call the coefficient of a^j in the development, J_x, and show that every linear factor of X_j is contained in J_x.

Any such factor is a primitive factor of $x^r - 1$, where r is any integer such that

$$E\frac{i+j}{r} - E\frac{i}{r} - E\frac{j}{r} = 1,$$

and it is unrepeated.

Let $x = \rho$, and let the negative minimum residue of $i + 1$ in respect to r be $-\delta$.

Then $F\rho$ is equal to the product of δ linear functions of a divided by a power of $(1 - a^r)$, and consequently the only powers of a (say a^θ) which appear in its development will be those for which the residue of θ in respect to r, is $0, 1, 2, \ldots \delta$, and consequently

$$E\frac{i+\theta}{r} - E\frac{i}{r} - E\frac{\theta}{r} = 0.$$

Hence a^j will not appear therein: so that J_x vanishes when any factor of X_j is zero, and consequently since every such factor is unrepeated, J_x contains X_j.

But J_x is obviously of the degree ij in x, and X_j which is the sum of the j-ary homogeneous products of $1, x, x^2, \ldots, x^i$ is of the same degree. Hence the two functions of x can only differ by a constant factor. On making $x=1$, Fx becomes $(1-a)^{-(i+1)}$; so that X_j becomes

$$\frac{(j+1)(j+2)\ldots(j+i)}{2\ldots i}$$

and J_x becomes the product of vanishing fractions

$$\frac{1-x^{j+1}}{1-x}, \frac{1-x^{j+2}}{1-x^2}, \ldots \frac{1-x^{j+i}}{1-x^i}, \text{ that is, } (j+1), \frac{j+2}{2}, \ldots \frac{j+i}{i}.$$

Hence $X_j = J_x$. Q.E.D.

The expansion of

$$(1-ax)(1-ax^2)\ldots(1-ax^i)$$

in terms of powers of a may be verified in like manner.

It is not without interest to observe (if the remark has not been made before) how this development is connected by the principle of correspondence with the preceding one.

Throwing out by multiplication the factor $(1-a)$ in the denominator of Fx we obtain the reciprocal of

$$(1-ax)(1-ax^2)\ldots(1-ax^i),$$

say $\frac{1}{Gx}$, under the form

$$1+\ldots+\frac{(1-x^{j+1})(1-x^{j+2})\ldots(1-x^{j+i-1})}{(1-x)(1-x^2)\ldots(1-x^{i-1})}x^j a^j+\ldots.$$

Consequently the number of ways in which n can be divided into exactly j parts $1, 2, \ldots i$ (repetitions admissible) is the coefficient of x^n in the expansion according to ascending powers in x of the above multiplier of a^j.

But if any such partition be arranged in ascending order, and $0, 1, 2, \ldots (j-1)$ be added (each to each) to its components, it is converted into a partition without repetitions, and by a converse process of subtraction each such partition is convertible into one of the former, but in which either repetition or non-repetition is allowable. Hence the free partitions of $n-\frac{j^2-j}{2}$ into j parts limited not to exceed $i-j+1$, have a one-to-one correspondence with the unrepetitional partitions of n into j parts limited not to exceed i, and must be equal to them in number. Hence the coefficient of a^j in $G(-x)$ may be deduced from that of a^j in $(Gx)^{-1}$ by multiplying the latter

by $x^{\frac{1}{2}(j^2-j)}$ and changing i into $i-j+1$. Hence the general term in $G(-x)$ will be

$$\frac{(1-x^{j+1})(1-x^{j+2})\ldots(1-x^i)}{(1-x)(1-x^2)\ldots(1-x^{i-j})}x^{\frac{j^2+j}{2}}a^j$$

which is right.

When $i=\infty$ each of these developments (like a multitude of others, including the Theta-functions) may be obtained intuitively by the graphical method of points given in my communication to the Johns Hopkins Scientific Association at its last meeting; it remains a desideratum to apply the same method to the above two developments, or either of them, for the case of i*.

In the Ferrers, Franklin, Durfee-Sylvester and other conjugate systems of partitions, the partible number is the same for the corresponding partitions; in this last example (and the like will be shown to be the case in the graphical development of the Theta-function, and its generalizations), the one-to-one correspondence is between partitions of two different numbers.

* Not so—the result derived springs from the immediate application of a general logical principle as will hereafter be shown.

86.

ON A NEW THEOREM IN PARTITIONS.

[*Johns Hopkins University Circulars*, II. (1883), p. 70.]

IT is a well-known theorem that the number of partitions of n into odd numbers is equal to the number of its partitions into unequal numbers. This equality was seen by Euler to result from the identity

$$(1+x)(1+x^2)(1+x^3)\ldots = \frac{1}{(1-x)(1-x^3)(1-x^5)\ldots}.$$

It may also be proved easily by the method of correspondence. For if we call the partitions of n into odd numbers (repeated or not) the U, and into unequal numbers the V system, any V will be of the form $[V_1, V_2, V_3, \ldots]$, where V_i is of the form

$$q_i \,.\, 2^{a_i}, \quad q_i \,.\, 2^{a_i'}, \quad q_i \,.\, 2^{a_i''}, \quad \ldots,$$

each q being an odd number and all the q's unlike.

Hence writing

$$2^{a_i} + 2^{a_i'} + 2^{a_i''} \ldots = A_i,$$

V is transformable into $A_1q_1, A_2q_2, A_3q_3, \ldots$, which is a member of the U system.

And conversely any U as $A_1q_1, A_2q_2, A_3q_3, \ldots$, will be transformable into a V by decomposing each U into a sum of products of its largest odd divisor into distinct powers of 2 which can be effected in one and only one manner; so that there is a one-to-one correspondence between the U's and V's, and the number of the one set is therefore the same as the number of the other. The theorem which is now to be explained is, so to say, a differentiation (in the Herbert Spencer sense) of this theorem.

It regards the U and V systems each broken up into classes and affirms the equality between the numbers of U's in any class and of the V's in the homonymous class. The proof of this by an analytic identity remains to be

discovered—it is effected without great difficulty by the method of correspondence: but what is very worthy of notice is that the V which corresponds to a U, in the more refined construction about to be explained, is in general (and it may be universally) different from the V which corresponds to it, when the preceding method of conjugation is adopted.

Every U which contains i distinct parts is said to be a U of the ith class, and every V which contains i distinct sequences (not running together) of consecutive numbers is said to be a V of the ith class—and my theorem may be expressed by saying that there exists a one-to-one correspondence (and therefore equality of content) between the U's of any class and the V's of the same class. I ought perhaps rather to say that a correspondence can be *instituted* than that a correspondence *exists*, for the fact that two absolutely unlike bonds of correspondence connect the totality of the U and that of the V system seems to indicate that such correspondence should rather be regarded as something put into the two systems by the human intelligence than an absolute property inherent in the relation between the two. Kant makes a similar remark upon the elementary conceptions (such as the circle), which form the groundwork of geometry.

As an example of the numerical part of the theorem consider the 3rd class of the U's and V's for $n = 16$.

The U's of this class will be

$$11.3.1^2;\ 9.5.1^2;\ 9.3^2.1;\ 9.3.1^4;\ 7.5.1^4;\ 7.3^2.1^3;$$
$$7.3.1^6;\ 5^2.3.1^3;\ 5.3^3.1^2;\ 5.3^2.1^5;\ 5.3.1^8;$$

and the V's which are somewhat more difficult to calculate by an exhaustive process will be found to be

1.6.9; 1.2.5.8; 2.6.8; 1.5.10; 1.2.4.9; 2.5.9;
1.4.11; 1.3.4.8; 3.5.8; 2.4.10; 1.3.12.

So again of the 4th class there is only one U and one V, namely, 1.3.5.7, which is common to the two systems—and of the first class owing to 16 containing only one odd divisor, namely, unity, there is also but one U and one V, namely, the undivided 16 for each alike. In general for the first class the number of U's is obviously the number of odd divisors of the partible number n and the number of single sequences is easily seen to be the same. Thus, for example, for 15 there exist the sequences 1.2.3.4.5; 4.5.6; 7.8; 15; and for 9 the sequences 2.3.4; 4.5; 9.

I will now indicate the mode of proof, the particulars of which will be found set out in full in the forthcoming number of the *American Journal of Mathematics* [Vol. IV. of this Reprint].

The partible number n being given, I take any U belonging to it and form two graphs, one whose rows represent the major halves of each part of U and the other its minor halves [$q+1$ is the major and q the minor half of $2q+1$]. I then dissect each of these graphs into its component angles and take the content of each; it is easily seen that beginning with the major and passing from it to the minor and back again to the major and so on continually in alternate succession, the readings will form a continually decreasing series of numbers whose sum will be the same as of the parts of the U, and thus U will be transformed into V. The number of parts in V, if we agree to consider that number as always *even* by supplying a zero at the end if it should happen to be *actually* odd, will be $2i$ where i is the number of points in the side of the Durfee-square appertaining to the major graph.

Conversely, if any V be given containing or made to contain $2i$ parts, it is easy to construct a system of $2i$ *linear* equations between the contents of the first i lines and the first i columns of an assumed U having a Durfee-square containing i^2 points, which shall transform into the given V, and to prove that these contents will be all of them greater than $2i$: hence *one* and *only one* U corresponds to a V, and consequently there is a one-to-one correspondence between the entire U and entire V systems. It remains to show that any U_i (a U of the ith class) by the prescribed process of transformation becomes a V_i (a V of the ith class).

This is effected as follows: suppose the first exterior angle to be removed simultaneously from a given major graph and its accompanying minor: begin with supposing that U_i becomes V_j: i is the number of unequal lines in either graph and it is easily proved that $i-j$ remains unaltered by the contraction of the graphs in the manner above indicated: that is, it can be shown that the effect of the contractions is to diminish i and j simultaneously each of them by 0, each of them by 1, or each of them by 2.

Continuing this process of *stripping* the graphs of their outside angles we must come at last to a graph consisting of one line and one column or of only one line, or only one column, or only a point. In the first of these four cases i and j are each equal to 2, and in the last three each equal to 1, hence $i-j$ is always zero and every U_i corresponds to a V_i. This establishes the very remarkable theorem that was to be proved.

87.

NOTE ON THE GRAPHICAL METHOD IN PARTITIONS.

[*Johns Hopkins University Circulars*, II. (1883), pp. 70, 71.]

It is well I think to draw attention to the fact that the graphical method introduces two new processes into Arithmetic as elementary and fundamental as those contained in the well-known four rules—which may be called Transversion and Apocopation.

Transversion is the operation of passing from a partition to its conjugate or transverse, and is identical with that which borrowing from the vernacular of the American Stock Exchange I have elsewhere denominated "calling."

The elements of a partition may be regarded as *Sellers* each holding a certain number of shares in the same stock. On the numbers 1, 2, 3 ... being successively called out each seller who holds at least that number of shares declares himself, and the number of those so responding each time being set down, a new partition is formed with numbers whose sum is identical with the total number of shares on sale.

The discovery of this process is due to Dr Ferrers, who informs me that he himself never published it but left it to me to do so in his name in the *London and Edinburgh Philosophical Magazine* for 1853*. I may mention that I have never missed an opportunity of expressing my sense of the great importance of the discovery and bringing it under the notice of my pupils, to one of whom, Mr Durfee (Fellow of this University), is due the discovery (after the lapse of 30 years) which leads to the second process, namely, Apocopation, which institutes a fixed relation between any partition and its transverse.

Apocopation is a process applied to a partition whose parts are arranged in descending order and consists in cutting off from its beginning, all those terms whose magnitude exceeds the number which denotes their place (reckoning from the highest term) in the arrangement. We have then

[* Vol. I. of this Reprint, p. 597; Vol. II., p. 120.]

this important theorem—*The number of terms subject to apocopation is the same for any partition and its transverse.*

Scaling or *co-summating* a partition consists in adding together each-to-each the apocopated terms in a partition and in its transverse, and diminishing these sums by the several numbers 1, 3, 5 .... In this way a new partition is formed which may be called the associate-sum of the original partition, so that to every partition there is a transverse and an associate-sum; and the content of each of these three partitions is identical.

The process of *scaling* or co-summation may be indefinitely continued and it is a curious question to determine how often the scaling process must be continued in order for a given partition to be eventually converted into a single term after which of course it remains unaltered by any further application of the process—this problem is naturally suggested by the practice of scaling and rescaling an inconveniently large public debt which is sometimes practised in the Old World and is not unknown in the New; but the analogy fails in this respect that in the one case the amount of the debt has a tendency to converge to zero, whereas in the other the content of the partition remains constant throughout.

The passage from a partition into odd numbers to the corresponding partition into unequal numbers, is effected by a co-summation operated simultaneously but independently upon two partitions, one of which has for its parts the major-halves and the other the minor-halves of the parts of the given partition.

88.

AN INSTANTANEOUS GRAPHICAL PROOF OF EULER'S THEOREM ON THE PARTITIONS OF PENTAGONAL AND NON-PENTAGONAL NUMBERS.

[*Johns Hopkins University Circulars*, II. (1883), p. 71.]

I START with the product

$$(1+ax)(1+ax^2)(1+ax^3)\ldots;$$

the coefficient of $x^n a^j$ in its development in a series according to powers of x and a is the number of partitions of n into j unequal parts; each such partition may be represented by a regular graph and these graphs classified according to the magnitude of the Durfee-square which they contain. Calling the side of any such square θ, two cases arise, namely, the vertical side of the square may either be completely covered or one point in it be left exposed: in the former case any number of the points in the base of the square, in the latter case not more than the first $\theta - 1$ points can be covered.

The first case contributes to the total number of partitions of n into j unequal parts the number of ways of distributing $n - \theta^2$ between two groups, one consisting of θ unequal parts unlimited, the other of j unequal parts not exceeding θ in magnitude.

The second case contributes the number of ways of distributing $n - \theta^2$ between two groups consisting one of $\theta - 1$ unequal parts unlimited, the other of $j - \theta$ unequal parts not exceeding $\theta - 1$ in magnitude.

Hence remembering that the number of ways of partitioning any number ν into θ parts is the coefficient of x^ν in

$$\frac{x^{\frac{\theta^2+\theta}{2}}}{1-x \,.\, 1-x^2 \ldots},$$

it is easily seen to follow that

$$(1+ax)(1+ax^2)(1+ax^3)\ldots$$

must be equal to the sum of the two series

$$1+\frac{1+xa}{1-x}x^2a\ldots+\frac{(1+xa)(1+x^2a)\ldots(1+x^\theta a)}{1-x\,.\,1-x^2\ldots1-x^\theta}x^{\theta^2+\frac{\theta^2+\theta}{2}}a^\theta+\ldots$$

and $$xa+\ldots+\frac{1+xa\,.\,1+x^2a\ldots1+x^{\theta-1}a}{1-x\,.\,1-x^2\ldots1-x^{\theta-1}}x^{\theta^2+\frac{\theta^2-\theta}{2}}a^\theta+\ldots;$$

on making $a=-1$ there results

$$(1-x)(1-x^2)(1-x^3)\ldots=1-x-x^2\ldots+(-)^\theta\left(x^{\frac{3\theta^2-\theta}{2}}+x^{\frac{3\theta^2+\theta}{2}}\right)+\ldots$$

which is the theorem to be proved.

In the Appendix or Exodion to a forthcoming paper in the *American Journal of Mathematics* [Vol. IV. of this Reprint] I give a proof by the method of correspondence of Jacobi's generalization of the above theorem, namely:

$$(1\pm x^{n-m})(1\pm x^{n+m})(1-x^{2n})(1\pm x^{3n-m})(1\pm x^{3n+m})(1-x^{4n})\ldots$$
$$=\sum_{-\infty}^{+\infty}(\pm)^i x^{ni^2+mi}.$$

89.

ON FAREY SERIES.

[*Johns Hopkins University Circulars*, II. (1883), p. 143.]

THE ordinary Farey Series is a succession of proper vulgar fractions arranged in order of magnitude, whereof the denominator does not exceed a given amount. The theory may be generalized and simplified by considering the terms of each fraction as the coordinates of a node in a "réseau." If a *simple* and *anautotomic* closed boundary drawn on a tessellation be called a scroll, and any node within it be assumed as origin, a radius of indefinite length rotating about that point as centre, will pass through a series of nearest nodes to it in succession, all lying within the scroll—and the coupled coordinates to those successive points, say (p, q), (p', q'), ... will form a certain series which in general but not universally will satisfy the equation $pq'-p'q=1$ or $=-1$ according as the order of magnitude is descending or ascending. The character of the series may be termed Farey if this law is satisfied throughout the entire succession and otherwise Non-Farey. The character obviously can only depend on the form, magnitude, position, and aspect of the scroll and the position of the assumed centre. The author of the paper showed that the position of the centre was indifferent except that it must be taken at some node within the scroll, and that the scroll might undergo uniform expansion and contraction about any internal node (and consequently also translation along any line of nodes) without change of character. Application of these principles was made to a triangular or rectangular scroll (including Mr Glaisher's extension of the theory of ordinary Farey Series to a two-fold constant limit), to the case potentially indicated by Dirichlet where the scroll is formed by two asymptotes to, and the branch of a hyperbola, and two other cases: the theory is deduced without the use of continued fractions or indeterminate linear equations or any other algebraical process whatever, from the well-known fact that all elementary triangles in a reticulation are of equal area and from the not very recondite theorem that a triangle may be divided into four equal and similar triangles by straight lines joining the bisections of its three sides; and with the aid of a solid reticulation may be extended to triplets and so on indefinitely. It will be found fully set forth in Note H, interact, part 2, Vol. v., No. 4, *American Journal of Mathematics* [Vol. IV. of this Reprint].

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