27.

SUR LES ACTIONS MUTUELLES DES FORMES INVARIANTIVES DÉRIVÉES.

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Je comprends les invariants, les covariants, les contravariants et toutes les formes qui dérivent dans le même sens d'un système donné de Quantics sous le nom général de *dérivées invariantives*, et je vais établir un principe qui rend ces formes fécondes et donne à deux quelconques d'entre elles la faculté de produire, par l'action de l'une sur l'autre, de nouvelles formes invariantives. Si l'on se borne aux invariants d'un seul Quantic ou d'un système de Quantics, la manière de procéder pour cette génération est presque évidente d'elle-même. Car soient $F(a, b, c, \ldots)$, $G(a, b, c, \ldots)$ deux invariants du même Quantic, ou du même système de Quantics, et écrivons à la place de $a, b, c, \ldots$ dans l'une de ces deux fonctions $\frac{d}{da}, \frac{d}{db}, \frac{d}{dc}, \ldots$: si l'on opère avec la fonction ainsi modifiée sur l'autre, le résultat restera invariantif (sauf le cas dans lequel le résultat se réduit à zéro ou à une autre constante numérique). En effet, désignons par $\dot{a}, \dot{b}, \ldots$ les opérations $\frac{d}{da}, \frac{d}{db}, \ldots$ et admettons qu'une substitution quelconque appliquée aux variables des Quantics donnés dont F et G sont les dérivées, change en $a', b', c', \ldots$ les éléments donnés $a, b, c, \ldots$; alors on aura d'après un principe élémentaire du calcul différentiel

$$\dot{a}' = \dot{a}\frac{da}{da'} + \dot{b}\frac{db}{da'} + \ldots,$$
$$\dot{b}' = \dot{a}\frac{da}{db'} + \dot{b}\frac{db}{db'} + \ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots;$$

donc la substitution contraire à la substitution en question changera $\dot{a}, \dot{b}, \dot{c}, \ldots$ en $\dot{a}', \dot{b}', \dot{c}', \ldots$. Or $F(\dot{a}, \dot{b}, \dot{c}, \ldots)$ et $G(a, b, c, \ldots)$ ne subissant aucun changement, le résultat de l'opération de la première sur la seconde ne subira non plus de changement par une substitution quelconque opérée sur les variables.

Ce raisonnement reste bon dans le cas où l'on substitue à l'invariant G un *contravariant* quelconque. Mais dans le cas général, dans lequel les variables entrent en même temps dans F et dans G, on a besoin de s'appuyer sur des considérations additionnelles d'un genre nouveau.

Or je remarque que la formation d'un Quantic quelconque se compose de trois genres des quantités—des variables,—des parties littérales des coefficients et enfin—des multiplicateurs numériques qui les affectent et qui forment, pour ainsi dire, l'équipement arithmétique de la forme. Dans la vieille algèbre ces multiplicateurs numériques ont été réduits à l'unité, dans l'algèbre moderne on les égale aux nombres binômes ou polynômes.—Dans la théorie que je vais produire on aura besoin de se servir d'un équipement qui tient pour ainsi dire la moyenne entre les deux dont je viens de parler, c. à d. que le multiplicateur d'un élément quelconque sera la racine carrée du nombre binôme ou polynôme qui lui serait égalé dans la notation ordinaire des Quantics. Quand les multiplicateurs numériques sont mis sous cette forme, je dirai que le Quantic est un Quantic *préparé*.

Remarquons que, quel que soit l'équipement numérique d'un Quantic, une substitution quelconque opérée sur les variables *induit* une substitution corrélative opérée sur les éléments, c. à d. que si dans le Quantic $(\lambda a, \mu b, \nu c, \ldots \between x, y, z, \ldots)^i$ on écrit pour $x, y, z, \ldots$ des fonctions linéaires de $x, y, z, \ldots$, le Quantic se changera en un autre $(\lambda A, \mu B, \nu C, \ldots \between x, y, z, \ldots)^i$ où $A, B, C, \ldots$ seront des fonctions linéaires de $a, b, c, \ldots$. Qu'on sépare les coefficients qui entrent dans les fonctions linéaires données de $x, y, z, \ldots$: on obtient une matrice;—qu'on sépare les coefficients qui entrent dans les fonctions linéaires de $a, b, c, \ldots$: on aura une autre matrice,—et je dirai que cette seconde matrice est *induite* par la première. Puisque les changements peuvent être effectués par des altérations insensibles on pourrait même sans trop d'incorrection affirmer que le mouvement des variables *induit* ou entraîne avec lui un mouvement dans les éléments d'une forme donnée. L'ordre du déterminant inducteur ne dépend que du nombre des variables, celui du déterminant induit dépend en même temps du nombre des variables et du degré de la forme par rapport aux variables.

On comprend le sens qu'il faut attribuer à la désignation de matrices contraires. La matrice dont les termes servent à exprimer le système des variables $\xi, \eta, \zeta, \ldots$ en fonctions linéaires des variables $x, y, z, \ldots$ et celle dont les termes servent à exprimer les $x, y, z, \ldots$ en fonctions linéaires des $\xi, \eta, \zeta, \ldots$ seront appelées matrices contraires*, et de même les deux substitutions dont les coefficients sont données par deux matrices contraires seront appelées substitutions contraires.

Cela posé, je suis en état d'énoncer le suivant théorème fondamental. *Dans un Quantic préparé deux substitutions contraires opérées sur les variables induisent deux substitutions contraires opérées sur les éléments.* La préparation indiquée ci-dessus est la condition nécessaire pour la validité

* Mot plus précis que celui d'*inverse* ou de *réciproque* dont on se sert quelquefois dans un cas plus vague quant à la grandeur *absolue* des termes dont on parle.

du théorème: si le Quantic n'était pas dûment *préparé* de la manière indiquée, des substitutions contraires appliquées aux variables n'induiraient point des substitutions contraires opérées sur les éléments. Avant d'entrer dans la démonstration de ce théorème je rappellerai que la matrice produite par la multiplication de deux et conséquemment d'un nombre quelconque de matrices et la matrice produite par la multiplication des matrices respectivement contraires aux premières, sont encore des matrices contraires entre elles—cet énoncé suppose toutefois que les multiplications des matrices soient effectuées de manière que leurs éléments soient combinés ligne par colonne ou colonne par ligne, mais non ligne par ligne ou colonne par colonne.

Considérons pour un instant une matrice de la forme

$$\begin{array}{cccccccc}
1 & r' & & & & & & \\
r & 1 & s' & & & & & \\
 & s & 1 & t' & & & & \\
 & & t & 1 & \cdot & & & \\
 & & & \cdot & \cdot & \cdot & & \\
 & & & & \cdot & 1 & \tau' & \\
 & & & & & \tau & 1 & \sigma' \\
 & & & & & & \sigma & 1 & \rho' \\
 & & & & & & & \rho & 1
\end{array}$$

où toutes les places vacantes dans le carré dont la ligne des unités est une des diagonales, doivent être remplies par des zéros.

En se bornant aux termes de la première dimension en $r, s, t, \ldots$, la valeur du déterminant est $1-rr'-ss'-tt'\ldots$. Autant qu'il est permis de se servir de cette valeur approchée, les dérivées du déterminant par rapport à $r, s, t, \ldots$ auront les valeurs $-r', -s', -t', \ldots$ et vice versâ, donc en attribuant des valeurs infiniment petites à $r, s, t, \ldots$ et faisant évanouir chacune des lettres accentuées, la matrice

$$[\mathrm{A.}]\ \left\{\begin{array}{ccccccccc}
1 & & & & & & & & \\
r & 1 & & & & & & & \\
 & s & 1 & & & & & & \\
 & & t & 1 & & & & & \\
 & & & \cdot & \cdot & & & & \\
 & & & & \cdot & \cdot & & & \\
 & & & & & \cdot & 1 & & \\
 & & & & & & \tau & 1 & \\
 & & & & & & & \sigma & 1 \\
 & & & & & & & & \rho \quad 1
\end{array}\right.$$

aura pour sa contraire la matrice

$$[\mathrm{B.}]\ \left\{\begin{array}{ccccccccc}
1 & \bar{r} & & & & & & & \\
 & 1 & \bar{s} & & & & & & \\
 & & 1 & \bar{t} & & & & & \\
 & & & 1 & \cdot & & & & \\
 & & & & \cdot & \cdot & & & \\
 & & & & & \cdot & \cdot & & \\
 & & & & & & 1 & \bar{\tau} & \\
 & & & & & & & 1 & \bar{\sigma} \\
 & & & & & & & & 1 \quad \bar{\rho} \\
 & & & & & & & & \quad\ \ 1
\end{array}\right.$$

où $\bar{r}, \bar{s}, \bar{t}, \ldots$ doivent être remplacées par $-r, -s, -t, \ldots$.

J'établirai d'abord la loi de l'induction des contraires dans le cas d'un Quantic binôme dûment préparé, x et y étant les variables et $a, b, c, \ldots$ les éléments.

Si la loi des contraires est vraie pour des substitutions dont la matrice a l'unité pour valeur de son déterminant, elle sera vraie pour des substitutions quelconques. De plus on démontre aisément que toute substitution au déterminant $=1$ opérée sur x, y peut être effectuée par trois substitutions *simples* successives, c. à d. par les substitutions successives de $x+hy$ pour x, de $y+kx$ pour y et de $x+ly$ pour x. Donc en vertu d'une remarque faite précédemment, la démonstration cherchée se réduit à la démonstration dans le cas d'une substitution *simple*, c. à d. de la substitution de $x+hy$ pour x. Mais cette substitution même peut être effectuée par une succession infinie de substitutions de la forme $x+\epsilon y$, où ϵ est une quantité infiniment petite: donc en vertu de la même remarque il ne nous reste qu'à établir la loi des contraires dans le cas où l'on substitue $x+\epsilon y$ au lieu de x dans la forme préparée

$$ax^i+\sqrt{(i)}\,bx^{i-1}y+\sqrt{\left\{\frac{i(i-1)}{2}\right\}}\,cx^{i-2}y^2+\ldots$$
$$+\sqrt{\left\{\frac{i(i-1)}{2}\right\}}\,hx^2y^{i-2}+\sqrt{(i)}\,kxy^{i-1}+ly^i.$$

Soit $$a'x^i+\sqrt{(i)}\,b'x^{i-1}y+\ldots+\sqrt{(i)}\,k'xy^{i-1}+l'y^i$$

ce que devient la forme après cette substitution, de sorte que

$$a'=a,\ b'=b+\sqrt{(i)}\,\epsilon a,\ c'=c+\sqrt{\{2(i-1)\}}\,\epsilon b,\ \ldots,$$
$$k'=k+\sqrt{\{2(i-1)\}}\,\epsilon h,\ l'=l+\sqrt{(i)}\,\epsilon k,$$

et posons

$$\sqrt{(i)}\,\epsilon=r,\ \sqrt{\{2(i-1)\}}\,\epsilon=s,\ \sqrt{\{3(i-2)\}}\,\epsilon=t,\ \ldots,\ \sqrt{\{2(i-1)\}}\,\epsilon=s',\ \sqrt{(i)}\,\epsilon=r',$$

la matrice de la substitution par laquelle $a, b, c, \ldots$ se transforment en $a', b', c', \ldots$ formera le cas particulier contenu dans la matrice [A.] lorsque le système des quantités $r, s, t, \ldots, \tau, \sigma, \rho$, que je désigne ici par $r, s, t, \ldots, t', s', r'$, est réversible, c. à d. que l'on a

$$r=r',\ s=s',\ t=t',\ \ldots.$$

Or au lieu de la substitution $\begin{matrix} 1 & \epsilon \\ 0 & 1 \end{matrix}$ opérée sur x, y prenons la substitution contraire $\begin{matrix} 1 & 0 \\ -\epsilon & 1 \end{matrix}$, c. à d. la substitution simple de $y-\epsilon x$ pour y: il est évident que la forme de substitution induite sera donnée par la matrice [B.] *avec les mêmes valeurs* de $r, s, t, \ldots, t', s', r'$ qu'auparavant. Mais ces deux matrices sont contraires. Donc le théorème est démontré dans le cas des formes binaires. On voit la nécessité de la condition que le Quantic soit *préparé* quant à son équipement numérique. Car sans cela les deux matrices induites qui se trouvent toujours sous les deux formes [A.] et [B.] ne seraient plus contraires, car le système des quantités $r, s, t, \ldots, t', s', r'$ étant renversé dans ces deux formes et les lettres accentuées et non-accentuées ne conservant plus des valeurs identiques, les deux matrices [A.] et [B.] cesseraient d'être corrélatives.

Comme exemple du théorème qui vient d'être établi, considérons le cas très-simple de la forme préparée $ax^2+\sqrt{(2)}\,bxy+cy^2$.

Opérons sur x, y la substitution $fx+gy$ pour x et $hx+ky$ pour y (où pour plus de simplicité je supposerai que $fk-gh=1$); les valeurs induites en a, b, c répondront à la matrice

$$\begin{matrix} f^2, & \sqrt{(2)}fh, & h^2, \\ \sqrt{(2)}fg, & fk+gh, & \sqrt{(2)}hk, \\ g^2, & \sqrt{(2)}gk, & k^2, \end{matrix}$$

dont l'inverse, en négligeant le facteur commun $fk-gh$, sera

$$\begin{matrix} k^2, & -\sqrt{(2)}gk, & g^2, \\ -\sqrt{(2)}kh, & gh+fk, & -\sqrt{(2)}fg, \\ h^2, & -\sqrt{(2)}fh, & f^2, \end{matrix}$$

qui est évidemment la matrice d'induction qui répond à la substitution de $kx-hy$ pour x et de $-gx+fy$ pour y: c. à d. que les deux substitutions contraires $\begin{matrix} f & g \\ h & k \end{matrix}$, $\begin{matrix} k & -h \\ -g & f \end{matrix}$, opérées sur les variables induisent des substitutions contraires opérées sur les éléments a, b, c.

J'ajouterai deux observations dont la première trouvera son application dans la démonstration générale et dont la seconde facilitera l'application du principe que je vais fonder sur la loi des contraires.

1°. Il est évident que pour *préparer* un Quantic, il n'est pas nécessaire que les multiplicateurs numériques soient les nombres binômes eux-mêmes;

il suffit que les rapports entre ces multiplicateurs soient les mêmes qu'entre les nombres binômes.

2°. Si l'on applique aux variables deux substitutions contraires dans deux Quantics ayant les mêmes éléments mais des multiplicateurs numériques distincts, les substitutions induites seront contraires pourvu que les produits des multiplicateurs qui affectent le même élément dans les deux Quantics, suivent la loi des multiplicateurs dans un Quantic préparé: ainsi comme cas particulier, si l'on introduit deux substitutions contraires, l'une dans un Quantic de la forme normale $(a, b, c, \ldots \between x, y)^i$, l'autre dans un Quantic où les éléments sont les mêmes mais dépourvus de tout multiplicateur numérique (c. à d. dans deux Quantics avec les mêmes éléments, l'une écrite selon la méthode ancienne, l'autre selon la méthode moderne) les deux substitutions induites sur les éléments seront contraires. A l'aide de cette remarque on évite l'inconvénient d'introduire des racines carrées qui doivent nécessairement disparaître dans les résultats.

Passons à l'application du théorème sur les substitutions contraires. Soit $F(a, b, c, \ldots : x, y)$ un covariant d'un Quantic ou d'un système de Quantics: écrivons comme auparavant $\dot{a}, \dot{b}, \dot{c}, \ldots$ pour $\frac{d}{da}, \frac{d}{db}, \frac{d}{dc}, \ldots$; je dis que $F(\dot{a}, \dot{b}, \dot{c}, \ldots : x, y)$ possédera toutes les propriétés d'un contravariant, c. à d. que si $G(a, b, c, \ldots : x, y)$ est un contravariant quelconque du même Quantic ou du même système, $F(\dot{a}, \dot{b}, \dot{c}, \ldots : x, y)$ appliqué comme opérateur à la forme $G(a, b, c, \ldots : x, y)$ conduira à un contravariant. Dans cet énoncé on suppose toutefois que les Quantics soient exprimés chacun dans leurs formes préparées ou bien (ce qui revient au même) que des deux formes F et G l'une appartienne à un système de Quantics pleins (c. à d. à éléments affectés de nombres binômes) et l'autre à un système de Quantics vides (c. à d. à éléments dépourvus de multiplicateurs binômes). Sous cette condition la forme $F \divideontimes G$, c. à d. le résultat de l'opération du covariant F sur le contravariant G, sera un contravariant du système auquel G appartient. Si au contraire F est un contravariant et G un covariant le résultat $F \divideontimes G$ de l'opération de F sur G sera un covariant. Dans ce qui suit je supposerai pour plus de simplicité que les formes dont il est question soient présentées dans leur forme préparée.

Je vais passer maintenant à des générations de formes dérivées que l'on obtient, si dans la forme dérivée $F(a, b, c, \ldots : x, y)$, qui pourra être covariant ou contravariant, on remplace non seulement les éléments $a, b, c, \ldots$ par leurs inverses symboliques, c. à d. par $\dot{a}=\frac{d}{da}, \dot{b}=\frac{d}{db}, \dot{c}=\frac{d}{dc}, \ldots$, mais en même temps les variables x, y par leurs inverses symboliques, c. à d. par

$$\dot{x}=\frac{d}{dx},\quad \dot{y}=\frac{d}{dy}.$$

Soit Φ ce que devient F après ce remplacement, de sorte que

$$\Phi = F(\dot{a}, \dot{b}, \dot{c}, \ldots : \dot{x}, \dot{y}),$$

et soit $G(a, b, c, \ldots : x, y)$ une seconde forme dérivée du même système, qui pourra être covariant ou contravariant; cela posé, suivant que le produit $F.G$ (c. à d. $F(a, b, c, \ldots : x, y)$ multiplié par $G(a, b, c, \ldots : x, y)$) est un covariant ou un contravariant, le résultat $\Phi \times G$ de l'opération de Φ sur G sera également un covariant ou un contravariant.

Considérons encore l'opération Ψ qui résulte d'une forme dérivée F lorsque, sans altérer les éléments, on remplace seulement les variables x, y par leurs inverses symboliques $\dot{x} = \frac{d}{dx}$, $\dot{y} = \frac{d}{dy}$. Dans ce cas comme dans celui que nous avons considéré en premier lieu et dans lequel on remplaçait seulement les éléments (et non les variables) par leurs inverses symboliques, le caractère de F est renversé, de covariant il devient contravariant et vice versâ. En un mot: une seule inversion symbolique renverse, deux inversions simultanées reproduisent le caractère de F. Ces propositions n'ont pas besoin d'être démontrées formellement, elles découlent comme conséquences des deux principes:

1°. que la marche du mouvement d'un système quelconque de lettres et de leurs inverses symboliques est contraire,

2°. que les mouvements induits dans les éléments* d'un Quantic *préparé* par deux mouvements contraires des variables sont eux-mêmes contraires.

Donnons le nom de différentiant-en-x à une fonction D' des éléments d'un Quantic binaire ou d'un système de plusieurs Quantics binaires, qui ait la propriété de rester la même après la substitution de $x + hy$ au lieu de x, c. à d. qui dans la notation pleine d'éléments affectés de multiplicateurs binômes satisfasse à l'identité

$$\Sigma (a\dot{b} + 2b\dot{c} + 3c\dot{d} + \ldots + ik\dot{l})\, D' = 0.$$

De même soit $'D$ un différentiant-en-y, c. à d. une fonction des éléments qui satisfasse à l'identité

$$\Sigma \{ib\dot{a} + (i-1)\, c\dot{b} + \ldots + l\dot{k}\}\, 'D = 0.$$

Pour les Quantics préparés ces équations prennent la forme

$$(1) \quad \Sigma [\sqrt{(n)}\, a\dot{b} + \sqrt{\{2(n-1)\}}\, b\dot{c} + \ldots + \sqrt{\{2(n-1)\}}\, h\dot{k} + \sqrt{(n)}\, k\dot{l}]\, D' = 0,$$

$$(2) \quad \Sigma [\sqrt{(n)}\, b\dot{a} + \sqrt{\{2(n-1)\}}\, c\dot{b} + \ldots + \sqrt{\{2(n-1)\}}\, k\dot{h} + \sqrt{(n)}\, l\dot{k}]\, 'D = 0.$$

On sait que D' sera toujours le coefficient de la plus haute puissance de x dans quelque covariant du système et $'D$ celui de la plus haute puissance de y

* De la combinaison de ces deux principes il résulte que le second principe peut être énoncé non seulement pour les éléments mais également pour leurs inverses symboliques.

dans quelque contravariant; et puisque en substituant au lieu des éléments leurs inverses symboliques le résultat de son action sur le covariant sera en vertu de notre dernier théorème un covariant et le coefficient de la plus haute puissance en x un différentiant-en-x, on en tire la conséquence que l'action d'un différentiant-en-y *rendu opératif* (par inversion symbolique) sur un différentiant-en-x donnera naissance à un différentiant-en-x: ce qui revient à dire que si $\dot{\Phi}$ est une fonction (des systèmes de $\dot{a}, \dot{b}, \dot{c}, \ldots$) qui satisfait à l'équation (2) quand on y remplace $a, b, c, \ldots$ par $\dot{a}, \dot{b}, \dot{c}, \ldots$ et $\dot{a}, \dot{b}, \dot{c}, \ldots$ par $\frac{d}{d\dot{a}}, \frac{d}{d\dot{b}}, \frac{d}{d\dot{c}}, \ldots$ c. à d. par $\ddot{a}, \ddot{b}, \ddot{c}, \ldots$ et que si D' satisfait à l'équation (1), alors $\dot{\Phi} D'$ doit satisfaire à la même équation (1).

Pour donner une démonstration indépendante de cette conclusion, je nommerai Ω' l'opérateur qui reduit D' à zéro, $'\Omega$ celui qui réduit $'D$ à zéro. La démonstration restant essentiellement la même dans le cas d'un système et dans celui d'un seul Quantic, on se bornera pour plus de simplicité à ce dernier cas, ce qui permet de supprimer les signes de sommation (Σ). Evidemment la proposition qu'on veut établir sera vraie si les deux opérations $\dot{\Phi}\Omega'$ et $\Omega'\dot{\Phi}$ que l'on obtient en appliquant l'opération $\dot{\Phi}$ et l'opération Ω' l'une après l'autre dans un ordre différent, ne diffèrent pas entre elles, ou ce qui est la même chose, si $\dot{\Phi}\Omega'$ et $\Omega'\dot{\Phi}$ ne diffèrent pas en puissance opérative.

Or bornons-nous pour le moment à un seul terme quelconque, p. e. au terme $\lambda p\dot{q}$ contenu dans Ω' (λ étant un nombre), et considérons la différence entre l'opération de $p\dot{q}\dot{\Phi}$ et de $\dot{\Phi}p\dot{q}$. Comme ce n'est que l'existence de $\dot{p}$ en $\dot{\Phi}$ qui produit cette différence, étudions l'effet de chaque terme $M\dot{p}^i$ séparément où M ne contient pas p. D'après le théorème de Leibnitz la différence entre l'effet de $\dot{p}^i(p\dot{q})$ et de $p\dot{q}(\dot{p}^i)$ sera $\dot{q}i\dot{p}^{i-1}$, c. à d. $\dot{q}\frac{d}{d\dot{p}}(\dot{p}^i)$ ou bien $\dot{q}\ddot{p}(\dot{p}^i)$.

Donc la valeur de la différence opérative entre $p\dot{q}\dot{\Phi}$ et $\dot{\Phi}p\dot{q}$ sera $\dot{q}\ddot{p}\dot{\Phi}$, et conséquemment la valeur totale de la différence entre $\Omega'\dot{\Phi}$ et $\dot{\Phi}\Omega'$ sera $\Sigma(\lambda\dot{q}\ddot{p}\dot{\Phi})$, c. à d. elle sera ce que Ω' devient quand après avoir renversé l'ordre des lettres dans chaque conjonction $a\dot{b}$, $b\dot{c}$, $c\dot{d}$ qui s'y trouve, on remplace les lettres non-accentuées par les lettres une fois accentuées et ces dernières par les lettres deux fois accentuées, ce qui fait voir que la différence opérative entre $\dot{\Phi}\Omega'$ et $\Omega'\dot{\Phi}$ sera nulle, vu que par hypothèse $\dot{\Phi}(\dot{a}, \dot{b}, \dot{c}, \ldots)$ est un différentiant-en-y de l'expression dans laquelle se change le Quantic donné (ou bien les Quantics simultanés donnés) quand on y remplace les éléments $a, b, c, \ldots$ par leurs inverses $\dot{a}, \dot{b}, \dot{c}, \ldots$, et que Ω' se change en $'\Omega$ quand on renverse l'ordre des éléments. J'ajouterai un seul exemple pour illustrer ce résultat, et pour éviter l'emploi des racines carrées je me servirai de la forme pleine pour les opérandes et de la forme vide pour les opérateurs.

Soit donné le Quantic $(x, y)^3$ et choisissons-en le discriminant

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd.$$

Differentiant par rapport à a cet invariant que je regarde pour l'instant comme un différentiant-en-y, j'obtiens le nouveau différentiant-en-y

$$ad^2 - 3bcd + 2c^3.$$

Pour obtenir l'opérateur qui y répond par rapport à la forme vide, il faut écrire $\frac{b}{3}, \frac{c}{3}$ au lieu de b, c, ce qui donne l'opérateur

$$27\dot{a}\dot{d}^2 - 9\dot{b}\dot{c}\dot{d} + 2\dot{c}^3.$$

Appliquons cet opérateur au différentiant-en-x, que l'on obtient en multipliant le discriminant par $ac - b^2$ et qui est

$$4a^2c^4 - 7ab^2c^3 - (6a^2bd + 3b^4)c^2 + (a^3d^2 + 10ab^3d)c - a^2b^2d^2 - 4b^5d.$$

Le résultat des différentiations indiquées sera

$$192a^2c - 84ab^2 - 270ab^2 + 108a^2c - 108ab^2 + 162a^2c = 462a(ac - b^2),$$

ce qui est évidemment un différentiant-en-x, comme il doit être. Passons rapidement à l'établissement des théorèmes analogues relatifs aux formes dérivées d'un nombre quelconque de variables.

La loi des mouvements contraires étant vraie pour les Quantics binaires dûment préparés, sera également vraie pour les Quantics ternaires pareillement préparés. Car soit i le degré d'un Quantic dans ses variables x, y, z; qu'on le range suivant les puissances ascendantes de z, évidemment chacun des Quantics binaires qui multiplient ces puissances sera dûment préparé. Le premier aura pour son équipement numérique les racines carrées des nombres binômes de l'ordre i, le second les racines carrées des nombres binômes de l'ordre $i-1$ multipliés chacun par $\sqrt{i}$, le troisième les racines carrées des nombres binômes de l'ordre $i-2$ multipliés chacun par $\sqrt{\tfrac{1}{2}i(i-1)}$, et ainsi de suite. Or il est facile de voir comme auparavant que le théorème sera vrai pour des substitutions quelconques s'il est vrai pour les substitutions pour lesquelles le déterminant est l'unité, et chaque substitution de ce dernier genre peut être effectuée par une succession de substitutions simples de la forme $x + hy$; $y + kz$, etc. Donc on n'a besoin que de démontrer le théorème pour une seule substitution de ce genre comme $x + hy$: mais pour cette substitution, tous les Quantics en x, y dont j'ai parlé étant dûment préparés, on a déjà démontré que le théorème est vrai. Donc le théorème est vrai pour chaque Quantic ternaire. De la même façon on passe des Quantics ternaires aux Quantics quaternaires et de même progressivement aux Quantics d'un nombre quelconque de variables. De plus il est facile de voir que la démonstration peut être étendue sans difficulté à des Quantics multipartites, c. à d. contenant un nombre quelconque de systèmes de

variables: car chacun de ces systèmes étant assujetti à une substitution à part, le système des éléments subira une substitution composée des substitutions induites par chacune des substitutions partielles relatives à un système isolé de variables. De plus deux substitutions contraires appliquées à un quelconque des systèmes de variables induira deux substitutions contraires appliquées aux éléments.

Si pour donner plus de simplicité aux énoncés, on se borne au cas de Quantics unipartites, on peut résumer les conséquences qui découlent des principes établis en affirmant qu'une dérivée invariantive d'un système quelconque de Quantics unipartites préparés reste une dérivée invariantive, quand on substitue pour les variables ou pour les éléments ou pour les unes et les autres simultanément, leurs inverses symboliques avec la distinction que sous la première supposition le caractère est changé dans son opposé et sous la dernière il reste le même.—Dans mes premiers mémoires sur ce sujet dans le *Quarterly Journal of Mathematics* j'ai déjà donné substantiellement cette loi en me servant de la forme pleine pour les opérandes et de la forme vide pour les opérateurs—mais je crois que personne n'en a jamais donné la preuve.—C'est l'idée lumineuse et très-inattendue de la loi des mouvements contraires relative aux Quantics *préparés* qui simplifie la théorie et en rend la démonstration presque intuitive.

Cependant ce n'est que par exception qu'on doit se servir de la forme *préparée* pour désigner les Quantics.—Parmi les autres avantages de la notation ordinaire on peut citer la *permanence* de chaque expression d'un différentiant, c. à d. qu'un différentiant qui appartient à un Quantic d'un degré quelconque restera un différentiant de tout Quantic contenant le même nombre de variables d'un degré supérieur. Car soit

$$\Omega = a\dot{b} + 2b\dot{c} + \ldots + ik\dot{l} \quad \text{et} \quad \Omega F(a, b, c, \ldots, l) = 0,$$

il est évident que si l'on augmente Ω par des termes additionnels

$$(i+1)\, l\dot{m} + \text{etc.}$$

et que l'on désigne par Ω_1 l'opération Ω augmenté, on aura

$$\Omega_1 F(a, b, c, \ldots, l) = 0.$$

Dans cette notation un covariant ou contravariant qui appartient à un Quantic quelconque donné, appartiendra donc également à tout autre Quantic composé du même nombre de variables et qui, en dépendant des mêmes éléments, s'élève pourtant à un degré supérieur*.

* Il peut arriver qu'un différentiant qui est irréductible pour un degré donné de son Quantic cesse de l'être pour un degré supérieur. Cela a lieu, par exemple, dans le cas du discriminant de la forme binaire du troisième ou du cinquième degré (il va sans dire qu'en élevant le degré, on augmente en même temps le nombre des éléments). Il y a donc des différentiants qui sont absolument irréductibles et d'autres qui ne le sont que conditionellement. Ainsi $a^2d - 3abc + 2b^3$

De plus il y a dans cette notation des moyens qui permettent de donner à un différentiant unique la faculté de propager, pour ainsi dire, son espèce, sans agir sur une autre forme du même genre et sans en subir l'action.— Voici un exemple de ce genre de propagation. Soit $F(a, b, c, \dots, l)$ un différentiant-en-x d'un Quantic binaire donné $(a, b, c, \dots, l \between x, y)^j$, de l'ordre j dans les éléments; remplaçons les éléments $a, b, c, \dots, l$ de F par $s_0, s_1, s_2, \dots, s_j$, où s_q signifie la somme des puissances $q^{\text{èmes}}$ des racines $\frac{x}{y}$ du Quantic donné; alors $a^\mu F(s_0, s_1, s_2, \dots, s_j)$ restera encore un différentiant-en-x du même Quantic, μ étant un nombre égal à l'ordre du différentiant transformé. En effet les formules connues du calcul différentiel pour passer d'un système donné de variables indépendantes à un autre, étant appliquées à la transformation de l'expression $\frac{d}{d\alpha_1} + \frac{d}{d\alpha_2} + \dots + \frac{d}{d\alpha_j}$, où $\alpha_1, \alpha_2, \dots, \alpha_j$ désignent les valeurs de $\frac{x}{y}$ qui annulent le Quantic donné, cette expression se transformera dans l'une et l'autre des deux expressions

$$-\frac{1}{a}[a\dot{b} + 2b\dot{c} + 3c\dot{d} + \dots], \quad [s_0\dot{s}_1 + 2s_1\dot{s}_2 + 3s_2\dot{s}_3 + \dots].$$

Par conséquent l'identité $(a\dot{b} + 2b\dot{c} + \dots)\, D = 0$ étant satisfaite, l'identité corrélative $(s_0\dot{s}_1 + 2s_1\dot{s}_2 + \dots)\, \Delta = 0$ le sera également, Δ désignant la transformée de D selon la règle donnée; et comme à chaque différentiant appartient un seul covariant dont il constitue un coefficient principal, on a le moyen de passer par une substitution facile d'un invariant ou covariant à un autre covariant qui sera en général d'un degré différent par rapport aux variables. Ainsi par exemple si l'on regarde l'invariant $ae - 4bd + 3c^2$ appartenant au Quantic $(a, b, c, d, e \between x, y)^4$ comme un différentiant-en-x, on voit que $\alpha, \beta, \gamma, \delta$ étant les quatre valeurs de $\frac{x}{y}$ qui annulent le Quantic donné,

$$a^4 \{(\alpha^4 + \beta^4 + \gamma^4 + \delta^4) - 4(\alpha + \beta + \gamma + \delta)(\alpha^3 + \beta^3 + \gamma^3 + \delta^3) + 3(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)^2\}$$

sera aussi un différentiant-en-x du Quantic donné. On vérifie aisément que cette expression est:

$$a^4 [\tfrac{1}{3} \{\Sigma(\alpha - \beta)^2\}^2 - \tfrac{8}{3}\Sigma(\alpha\gamma - \beta\delta)^2] = -b(ac - b^2)^2 - \tfrac{3}{2}a^2(ae - 4bd + 3c^2);$$

ainsi l'invariant $ae - 4bd + 3c^2$ donne naissance au différentiant $ac - b^2$ dont le poids est égal à $\frac{2.4}{2} - 2$ et qui par conséquent sert à déterminer un

appartient à la première catégorie, $a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd$ à la seconde; car en l'attribuant au Quantic $(a, b, c, d, e \between x, y)^4$, il peut être exprimé sous la forme

$$(ac - b^2)(ae - 4bd + 3c^2) - a(ace - ad^2 + 2bcd - c^3 - b^2e),$$

c. à d. il devient une fonction entière des quatre différentiants

$$a, \quad ac - b^2, \quad ae - 4bd + 3c^2, \quad ace - ad^2 + 2bcd - c^3 - b^2e.$$

covariant quadratique de l'ordre 2 dans les éléments. Ainsi un invariant a servi à produire un covariant. La double représentation des différentiants au moyen des éléments et au moyen des racines, fournit une démonstration d'un théorème assez important dans la théorie des partitions qu'il serait peut-être difficile d'établir par une démonstration directe. Voici en quoi consiste ce théorème. Désignons le nombre de manières de composer un nombre n avec j des nombres $0, 1, 2, \dots, i$ par $(n : i, j)$: on sait que $(n : i, j) = (n : j, i)$ et l'on voit sans peine que si

$$m + \mu = ij, \quad (m : i, j) = (\mu : i, j).$$

Cela posé, le théorème en question affirme que pour les valeurs de m qui n'excèdent pas $\frac{1}{2}ij$, $(m : i, j)$ peut être égal à $(m - 1 : i, j)$ ou plus grand que ce nombre, mais non plus petit que ce nombre, ou, ce qui revient au même, que si m est plus grand que $\frac{1}{2}ij$, $(m : i, j)$ ne peut pas être plus grand que $(m - 1 : i, j)$. L'une de ces propositions implique l'autre en vertu de l'égalité $(m : i, j) = (ij - m : i, j)$. C'est la première que je veux établir et je l'établirai au moyen de la seconde. Cette démonstration étant accomplie je ferai une généralisation facile et du théorème et de la démonstration qui y conduit, pour en faire l'extension aux systèmes de couples i, j.

En vertu de l'équation $\Omega D = 0$, on sait selon l'observation précieuse que M. Cayley a fait le premier, que le nombre de différentiants linéairement indépendants de l'ordre j dans les éléments, qui appartiennent à un Quantic binaire donné du degré i, dont le poids par rapport à x est w, doit être égal à $(w : i, j) - (w - 1 : i, j)$.

Sans même se servir de cette proposition qui est certainement vraie mais qui exige la vérification de l'indépendance des équations qu'on obtient en satisfaisant à l'identité $\Omega D = 0$, on peut affirmer avec une certitude absolue que le nombre des différentiants dont il s'agit *ne peut pas être inférieur à* $(w : i, j) - (w - 1 : i, j)$, ce qui suffit pour la démonstration proposée. Or je dis qu'il ne peut pas exister de différentiants pour lesquels w est plus grand que $\frac{1}{2}ij$. Car en vertu de l'identité $\Omega = a\Sigma \frac{d}{d\alpha}$ un différentiant quelconque doit être de la forme $a^j\Sigma(\alpha - \alpha')(\alpha'' - \alpha''')(\alpha^{\text{IV}} - \alpha^{\text{V}}) \dots$ où le nombre des facteurs est w et chaque α une des i valeurs de $\frac{x}{y}$ qui annulent le Quantic donné du degré i, bien entendu qu'il n'y a nulle restriction sur la répétition des mêmes racines. L'ordre de cette fonction symétrique relatif aux coefficients étant j, on en conclura d'après un théorème connu de l'algèbre ordinaire qu'aucune racine α ne peut se présenter plus de j fois, mais dans chacun des w facteurs il paraîtra deux lettres; donc le poids w est la moitié du nombre total de ces apparitions. Or puisque nulle lettre ne paraît plus de j fois, le nombre total de ces apparitions aura ij pour son maximum et conséquemment la valeur maximum de w est $\frac{1}{2}ij$, c. à d. qu'il n'existe pas de différentiants pour lesquels w excède $\frac{1}{2}ij$; donc comme il existe toujours $(w:i,j)-(w-1:i,j)$ *au moins* (je dis *au moins* pour ne pas m'appuyer sur la vérification de l'indépendance citée plus haut), il s'ensuit que, pour $w>\frac{1}{2}ij$, $(w:i,j)$ ne peut pas excéder $(w-1;i,j)$ et par conséquent que pour $w=\frac{1}{2}ij$ ou pour $w<\frac{1}{2}ij$, $(w:i,j)$ ne peut pas être plus petit que $(w-1:i,j)$, ce qu'il fallait démontrer. On peut étendre ce raisonnement en se fondant sur la proposition que pour un nombre quelconque de Quantics, p. e. pour deux Quantics $(a, b, c, \ldots, l)(x, y)^i$, $(a', b', \ldots, f')(x, y)^{i'}$, le nombre des différentiants du poids w en x, de l'ordre j dans $a, b, c, \ldots$ et de l'ordre j' dans $a', b', \ldots$ a pour expression ou au moins pour valeur maximum* la différence entre deux dénumérants dont l'un est le nombre de solutions en nombres positifs entiers du système

$$x_0+x_1+\ldots+x_i=j, \quad y_0+y_1+\ldots+y_{i'}=j',$$
$$x_1+2x_2+\ldots+ix_i+y_1+2y_2+\ldots+i'y_{i'}=w,$$

et l'autre le dénumérant du système qui en résulte lorsqu'on y remplace w par $w-1$. En suivant cette voie et après avoir démontré par la même méthode dont on s'est servi ci-dessus et à l'aide des fonctions symétriques des racines des deux Quantics donnés que la valeur maximum de w est $\frac{ij+i'j'}{2}$, on arrivera à cette conclusion analogue que la valeur de la différence entre ces deux dénumérants ne peut jamais être négative, conclusion qui reste vraie en général. A ce résultat on peut donner l'énoncé remarquable: Que l'on développe suivant les puissances de $a, b, c, \ldots$ le produit d'un nombre quelconque de fonctions

$$[(1-a)(1-at)(1-at^2)\ldots(1-at^i)]^{-1}\times$$
$$[(1-b)(1-bt)(1-bt^2)\ldots(1-bt^k)]^{-1}\times$$
$$[(1-c)(1-ct)(1-ct^2)\ldots(1-ct^l)]^{-1}\times$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

que l'on cherche dans ce développement la fonction de t qui multiplie un produit quelconque donné de puissances de $a, b, c, \ldots$; cette fonction ordonné suivant les puissances ascendantes de t présentera une série de coefficients numériques distribués symétriquement autour de son milieu et ayant des valeurs non décroissantes depuis l'une des extrémités jusqu'au terme unique ou jusqu'aux deux termes qui forment le milieu de la série.

L'importance de cette proposition pour la théorie des invariants consiste dans le fait qu'elle énonce et d'après lequel *l'expression analytique* du nombre des différentiants linéairement indépendants d'un système de

* Abstraction faite de l'indépendance non démontrée des équations données par l'application de l'opérateur $\left(a\frac{d}{db}+2b\frac{d}{dc}+\ldots\right)+\left(a'\frac{d}{db'}+2b'\frac{d}{dc'}+\ldots\right)$.

Quantics binaires donné est toujours un nombre positif ou nul, pourvu que ces différentiants soient d'une forme concevable, c. à d. que le poids donné w n'excède pas le maximum dont il est susceptible. Au contraire, pour les différentiants inconcevables, c. à d. dont le poids donné excède le maximum, l'expression analytique du nombre est toujours négatif ou zéro.

La loi de l'accroissement et du décroissement citée ci-dessus peut être exprimée au moyen d'une transformation facile à vérifier. En se bornant au cas d'un seul Quantic on a l'énoncé qu'en représentant par $\chi(\theta)$ un des deux produits

$$(1-2a\cos\theta+a^2)(1-2a\cos 3\theta+a^2)\ldots[1-2a\cos(2q+1)\theta+a^2],$$
$$(1-a)(1-2a\cos 2\theta+a^2)(1-2a\cos 4\theta+a^2)\ldots(1-2a\cos 2q\theta+a^2),$$

la fraction $\frac{\sin\theta}{\chi(\theta)}$ développée selon les puissances positives de a et les sinus des multiples de θ sera *omnipositive*, c. à d. ne contiendra que des coefficients numériques positifs.—La même conclusion aura lieu quand on multiplie ensemble plusieurs fonctions de l'une ou l'autre forme de χ, savoir

$$\chi(a, q, \theta)\,\chi(a', q', \theta)\ldots.$$

En désignant par $\Pi\chi(\theta)$ ce produit, le développement de $\frac{\sin\theta}{\Pi\chi(\theta)}$ suivant les puissances de a, a', a'', ... et leurs combinaisons et suivant les sinus des multiples de θ sera omnipositif.

Il paraît que ce théorème reste vrai quand on considère la fonction entière $\sin\theta\Pi\chi(\theta)$ au lieu de la fonction fractionnaire, mais je n'en possède point de preuve. Dans le cas simple de $\sin\theta\chi(\theta)$ cela reviendrait à l'énoncé que les coefficients des puissances de t dans le développement de

$$(1+a)(1+at)(1+at^2)\ldots(1+at^i),$$

qui forme évidemment une série symétrique, jouissent de la même propriété que les coefficients du développement de la fonction réciproque, c. à d. que les valeurs des coefficients peuvent augmenter ou rester stationnaires en passent de l'une ou l'autre extrémité de la série vers le milieu, mais qu'elles ne peuvent jamais décroître.—Il paraît qu'une proposition analogue peut être avancée pour le produit $\phi(1)\phi(t)\phi(t^2)\ldots\phi(t^i)$, où $\phi(x)$ signifie

$$1+ax+a^2x^2+\ldots+a^jx^j,$$

et pour des formes encore plus générales.

Dans les recherches précédents je suis tombé sur une démonstration exacte du théorème fondamental de la théorie des invariants, théorème qui a été accepté comme vrai par son illustre auteur M. Cayley sur la foi d'une induction *à posteriori* purement empirique et dont l'exactitude a été révoquée en doute par un écrivain distingué sur les formes binaires, apparemment en conséquence d'une méprise relative à l'explication donnée par M. Cayley sur la source de la conclusion erronée qu'il avait énoncée sur le nombre des invariants fondamentaux pour les degrés supérieurs.

On démontre facilement que si $D(w:i,j)$ est le nombre des différentiants linéairement indépendants* de l'ordre j, du poids w, et qui appartiennent à un Quantic binaire du degré i,

$$D(w:i,j) = \text{ou} > (w:i,j) - (w-1:i,j),$$

différence que je dénoterai désormais par $\Delta(w:i,j)$. Cette conclusion est une conséquence immédiate de l'identité $\Omega D = 0$, D étant un différentiant quelconque du Quantic $(a, b, c, \ldots, l \between x, y)^i$ et Ω l'opérateur

$$a\delta_b + 2b\delta_c + 3c\delta_d + \ldots.$$

Mais pour établir le théorème en question, c. à d. l'équation

$$D(w:i,j) = \Delta(w:i,j),$$

il faudrait avoir prouvé l'indépendance de toutes les équations entre les constantes indéterminées, que l'identité $\Omega D = 0$ fournit (en regardant D comme une fonction composée des combinaisons des $a, b, c, \ldots$ multipliées chacune par une telle constante)—ce qui n'a jamais été fait et offre des difficultés presque insurmontables si l'on se propose de résoudre la question par escalade.—Je suivrai une marche différente—commençant par l'alternative d'égalité ou de supériorité entre D et Δ, je démontre que la dernière est inadmissible—l'indépendance dont j'ai parlé est donc une conséquence et non la clef de la démonstration.—Lorsqu'un opérateur quelconque Φ satisfait à l'équation $\Phi F = G$, je dirai dans ce qui suit que Φ transforme F en G, et lorsqu'on a identiquement $\Phi F = 0$, je dirai que Φ annule F.

Je remarque que $D(0:i,j) = 1$ parce que dans tous les cas il y a un seul différentiant-en-x libre du poids zéro, à savoir une puissance de a: d'autre part $(0:i,j)$, c. à d. le nombre de manières de composer zéro avec $0, 1, 2, 3, \ldots, i$ prises j à j, est aussi $= 1$, par conséquent la relation $D(w:i,j) \geqq \Delta(w:i,j)$ fournit

$$D(w:i,j) + D(w-1:i,j) + D(w-2:i,j) + \ldots + D(0:i,j) \geqq (w:i,j),$$

où le symbole $\geqq$ signifie "est égal à ou plus grand que."

* Pour plus de commodité je dirai différentiants *libres* au lieu de différentiants linéairement indépendants.

De plus on aura

$$D(w:i,j) + 2D(w-1:i,j) + 3D(w-2:i,j) + \ldots$$
$$\geqq (w:i,j) + (w-1:i,j) + (w-2:i,j) + \ldots,$$

ce qui est vrai pour toutes les valeurs de w. Je supposerai à présent que w ait la valeur $\frac{1}{2}ij$ pour ij pair et la valeur $\frac{1}{2}(ij-1)$ pour ij impair. Dans ce cas la somme qui forme la seconde partie de la dernière relation devient évidemment égale au nombre des combinaisons j à j (avec répétitions) formées des $(i+1)$ chiffres $0, 1, 2, 3, \ldots, i$, et assujetties à la restriction que la somme des chiffres d'une combinaison n'excède pas $\frac{1}{2}ij$: nombre de combinaisons que je dénoterai par $P(i,j)$.

Remplaçons chaque différentiant qui fait partie du groupe dont le nombre est $D(w:i,j)$, de même du groupe dont le nombre est $D(w-1:i,j)$, etc. par le covariant qui y correspond.—Le degré de ces covariants par rapport aux variables étant, pour une valeur quelconque de w, $ij - 2w$, les degrés des covariants dans les groupes successifs seront $1, 3, 5, \ldots$ dans le cas de ij impair, et $0, 2, 4, \ldots$ dans le cas de ij pair. Imaginons que chaque coefficient de chacun de ces covariants soit représenté par une dame d'un damier, qu'on se borne à prendre le premier coefficient des covariants du premier groupe, les deux premiers coefficients des covariants du second groupe, les trois premiers du troisième groupe, etc., on peut alors former un triangle rectangulaire de piles des dames. La pile au sommet contiendra $D(w:i,j)$, les deux piles qui suivent $D(w-1:i,j)$, les trois piles qui suivent $D(w-2:i,j)$ chacune, et ainsi de suite. Le nombre total des dames sera la fonction qui est $\geqq P(i,j)$.

Pour donner plus de précision à cette image je remarque que les dames dans la première colonne verticale représentent des différentiants et que chaque pile à la base se réduit nécessairement à une seul dame, dont la partie essentielle (abstraction faite de la partie numérique qui n'a pas d'influence sur le raisonnement et que je négligerai dans tout ce qui suit) n'est autre chose qu'un coefficient du Quantic du degré i élevé à la puissance j.

Remarquons que, d'après une propriété bien connue des covariants, chaque quantité dans la première colonne sera annulée par Ω, dans la seconde par $(\Omega)^2$, en général dans la $q^{\text{ème}}$ par $(\Omega)^q$ et que l'opérateur $(\Omega)^{q-1}$ appliqué à un terme de la $q^{\text{ème}}$ colonne produit le différentiant qui se trouve à la première place de la même horizontale avec ce terme.

Remarquons encore qu'en avançant de gauche à droite dans la même horizontale, les poids des quantités augmentent d'une unité de l'une à l'autre, qu'au contraire, en avançant de haut en bas dans la même verticale, les poids des quantités diminuent d'une unité de l'une à l'autre, de sorte que dans une ligne diagonale descendante (de gauche à droite), ou ce qui est la même chose, parallèle à l'hypoténuse, tous les termes sont de poids égal.

Or j'affirme que nulle liaison linéaire ne peut exister entre les quantités du triangle en question. Evidemment ce n'est qu'entre les quantités isobariques qu'une telle liaison serait imaginable. Prenons une ligne quelconque parallèle à l'hypoténuse ou bien l'hypoténuse elle-même.

1°. Je dis que nulle équation linéaire ne peut lier des quantités qui se trouvent exclusivement dans une seule pile. Car si cette pile se trouvait dans la $q^{\text{ème}}$ colonne, en vertu du fait que l'opérateur $(\Omega)^{q-1}$ fait naître de chacune d'elle le différentiant qui se trouve à la première place de la même ligne horizontale, la liaison supposée subsisterait encore entre des différentiants d'une même pile, ce qui est contraire à l'hypothèse de la construction.

2°. Je dis que nulle équation linéaire ne peut lier les quantités qui se trouvent dans des piles distinctes. En effet, supposons donnée une relation de ce genre, d'après la condition du poids égal il ne peut y avoir dans chaque colonne qu'une seule pile comprenant des quantités qui entrent dans l'équation supposée. Soit q le rang de la colonne la plus avancée qui renferme une pile comprenant des quantités liées entre elles par l'équation linéaire. L'opérateur Ω, appliqué au premier membre de l'équation qui exprime cette liaison un nombre de fois inférieur à $q-1$, produira une équation d'une forme analogue. Mais lorsqu'on applique l'opérateur $(\Omega)^{q-1}$, toutes les quantités comprises dans des colonnes d'un rang inférieur à q seront annulées, tandis que celles qui sont comprises dans la pile de la $q^{\text{ème}}$ colonne seront transformées en des différentiants appartenant à la même ligne, c. à d. qu'il y aurait une liaison linéaire entre les différentiants d'une même pile, ce qui est contraire à l'hypothèse de la construction.

On a donc démontré que nulle équation linéaire ne subsiste entre les quantités du triangle.—De plus il est évident que le poids d'un coefficient quelconque qui se trouve dans le triangle ne peut excéder $\frac{1}{2}ij$. Donc les quantités comprises dans le triangle sont des fonctions linéaires et homogènes sans liaison linéaire entre elles de $P(i,j)$ quantités. Donc le nombre de ces quantités ne peut pas excéder $P(i,j)$, c. à d. que

$$D(w:i,j)+2D(w-1:i,j)+3D(w-2:i,j)+\dots$$

ne peut pas excéder $P(i,j)$.

Mais si dans une seule des relations $D(w:i,j) \geqq \Delta(w:i,j)$, pour des valeurs de w quelconques, le signe applicable était $>$ et non $=$, la somme en question serait $>P(i,j)$.

Donc on a toujours $D(w:i,j)=\Delta(w:i,j)$, ce qu'il fallait démontrer. Comme corollaire il s'ensuit que l'indépendance des équations données par l'identité $\Omega D=0$ est établie. Précisément la même méthode peut être suivie pour démontrer l'égalité

$$D(w:i,j:i',j':\text{etc.})=\Delta(w:i,j:i',j':\text{etc.})$$

où D dénote le nombre des différentiants libres d'un système de Quantics binaires, $i, i', \dots$ désignant les degrés des Quantics, $j, j', \dots$ l'ordre des différentiants par rapport aux coefficients de chacun des Quantics, et où Δ dénote la différence entre deux dénumérants, l'un désignant le nombre des solutions en nombres entiers et positifs du système des équations simultanées

$$x_0+x_1+x_2+\dots+x_i=j,\quad x'_0+x'_1+\dots+x'_{i'}=j', \text{ etc.}$$
$$x_1+2x_2+\dots+ix_i+x'_1+2x'_2+\dots+i'x'_{i'}+\dots=w,$$

et l'autre le dénumérant du système qui en résulte lorsqu'on y remplace w par $w-1$.

Un autre corollaire que le théorème contient comme cas particulier est la proposition déjà démontrée, que $\Delta(w:i,j)$ ne peut jamais devenir négatif pour des valeurs de w qui n'excèdent pas $\frac{1}{2}ij$. En effet si cette assertion n'était pas vraie, il devrait exister une valeur de w qui n'excède pas $\frac{1}{2}ij$ et pour laquelle $D(w:i,j)>\Delta(w:i,j)$, ce qui a été prouvé impossible.

En dernier lieu je remarque qu'en démontrant inadmissible le signe de supériorité, on a établi pour $w=\frac{1}{2}ij$ quand ij est pair et pour $w=\frac{1}{2}(ij-1)$ quand ij est impair, l'équation

$$D(w:i,j)+2D(w-1:i,j)+3D(w-2:i,j)+\dots=P(i,j).$$

Soit ij impair, en vertu de l'équation $(x:i,j)=(ij-x:i,j)$ le nombre $P(i,j)$ sera évidemment la moitié du nombre total des combinaisons j à j des $i+1$ éléments $0, 1, 2, 3, \dots, i$. Donc pour ij impair $P(i,j)=\frac{1}{2}\frac{\Pi(i+j)}{\Pi i\Pi j}$.

Soit au contraire ij pair, on aura

$$\begin{aligned} P(i,j) &= \tfrac{1}{2}\left\{\frac{\Pi(i+j)}{\Pi i\Pi j}+(w:i,j)\right\} \\ &= \tfrac{1}{2}\frac{\Pi(i+j)}{\Pi i\Pi j}+\tfrac{1}{2}\{D(w:i,j)+D(w-1:i,j)+D(w-2:i,j)+\dots\}. \end{aligned}$$

Le degré des covariants qui correspondent un à un aux différentiants dont le nombre est $D(x:i,j)$ étant $ij-2x$, on peut substituer pour $D(x:i,j)$ le nombre $K(i,j:ij-2x)$ où i est le degré du Quantic donné, j l'ordre par rapport aux coefficients, $ij-2x$ le degré relatif aux variables des covariants dont K exprime le nombre total.

Par conséquent quand ij est impair, on aura

$$2K(i,j:1)+4K(i,j:3)+6K(i,j:5)+\ldots=\frac{\Pi(i+j)}{\Pi i\,\Pi j}$$

et quand ij est pair

$$K(i,j:0)+3K(i,j:2)+5K(i,j:4)+\ldots=\frac{\Pi(i+j)}{\Pi i\,\Pi j}.$$

En remarquant que pour ij impair il n'existe pas de covariants de degré pair, et pour ij pair il n'en existe pas de degré impair, on peut réunir ces deux formules dans une seule formule remarquable, qui assujettit les quantités transcendantes K à une loi algébrique et qui pourrait même être très-utile dans certains cas comme formule de vérification :

$$K(i,j:0)+2K(i,j:1)+3K(i,j:2)+\ldots=\frac{\Pi(i+j)}{\Pi i\,\Pi j}.$$

J'en donnerai quelques exemples.

Soit $i=4,\quad j=2.$

On trouve

$$K(4,\,2:0)=1\,;\ K(4,\,2:2)=0\,;\ K(4,\,2:4)=1\,;\ K(4,\,2:6)=0\,;$$
$$K(4,\,2:8)=1$$

et de là $1+5+9=15=\dfrac{\Pi 6}{\Pi 2\,\Pi 4}.$

Soit $i=3,\quad j=3.$

En se rappelant l'échelle fondamentale pour les cubiques

$$3.1,\quad 4.0,\quad 2.2,\quad 3.3$$

on trouve

$$K(3,\,3:1)=0,\ \ K(3,\,3:3)=1,\ \ K(3,\,3:5)=1,\ \ K(3,\,3:7)=0,$$
$$K(3,\,3:9)=1$$

et de là $4+6+10=20=\dfrac{\Pi 6}{\Pi 3\,\Pi 3}.$

Soit $i=3,\quad j=4.$

On trouve

$$K(3,\,4:0)=1,\ \ K(3,\,4:2)=0,\ \ K(3,\,4:4)=1,\ \ K(3,\,4:6)=1,$$
$$K(3,\,4:8)=1,\ \ K(3,\,4:10)=0,\ \ K(3,\,4:12)=1$$

et de là $1+5+7+9+13=35=\dfrac{\Pi 7}{\Pi 3\,\Pi 4}.$

Le théorème que j'ai vérifié par ces exemples peut être résumé dans les termes suivants. Chaque covariant d'un ordre donné j par rapport aux coefficients d'un Quantic binaire de degré donné i, étant répété autant de fois qu'il y a de chiffres dans la série qui commence par zéro et se termine

par le degré du covariant, relatif aux variables qui y entrent, le nombre total de ces expressions, chacune comptée autant de fois qu'elle est répétée, est égal au nombre binôme symétrique par rapport aux nombres i et j, c. à d. égal à $\dfrac{\Pi(i+j)}{\Pi i\,\Pi j}.$

La règle des nombres binômes s'applique avec une modification légère au cas de plusieurs Quantics binaires de degrés donnés et de covariants d'ordres donnés relatifs aux coefficients de ces Quantics. Dans ce cas général on substituera au nombre binôme unique qui se présente dans le cas d'un seul Quantic, le produit de plusieurs nombres binômes dont chacun est symétrique par rapport au degré i de l'un des Quantics et à l'ordre j du covariant relatif aux coefficients du même Quantic.

Considérons comme exemple le cas de deux quadratiques binaires. Dans ce cas qui correspond à $i=2,\ i'=2$ il y a trois covariants de l'ordre $j=1$ par rapport aux coefficients de chacune, savoir :

1° le produit des deux Quantics,

2° leur *Hessien,*

3° leur *Connectif.*

Les degrés de ces trois expressions relatifs aux variables étant respectivement 4, 2, 0, on aura

$$5+3+1=\left(\frac{\Pi 3}{\Pi 1\,\Pi 2}\right)^2=9,$$

ce qui s'accorde avec la règle énoncée ci-dessus.

A l'énumération que j'ai faite des propriétés essentielles du triangle de piles, j'ajoute la remarque que le poids maximum d'une quelconque des quantités qui s'y trouvent, est évidemment celui de la quantité qui appartient à l'hypoténuse et se trouve au sommet du triangle. Ce poids maximum est $\frac{1}{2}ij$ ou $\frac{1}{2}(ij-1)$ et par conséquent n'excède jamais $\frac{1}{2}ij$. C'est ainsi qu'on voit que les quantités comprises dans le triangle ne sont autre chose que des fonctions linéaires des combinaisons de l'ordre j par rapport aux coefficients du Quantic proposé, combinaisons dont le nombre est $P(i,j)$.

Postscriptum 1. La démonstration donnée du théorème fondamental $D(w:i,j)=\Delta(w:i,j)$ peut être abrégée et simplifiée comme il suit.

Au lieu de se servir de la condition

$$D(w:i,j)+2D(w-1:i,j)+\ldots \gtreqless P(i,j)$$

il suffit de considérer l'équation préalable

$$\Sigma D(w:i,j)=(w:i,j).$$

Pour un différentiant quelconque que je désignerai par $[w-\delta]$ et dont le poids soit $w-\delta$ substituons l'expression $('\Omega)^{\delta}[w-\delta]$, expression qui résulte de $[w-\delta]$ en y appliquant l'opérateur* $('\Omega)^{\delta}$ et qui jouit de la propriété qu'en opérant sur elle avec $(\Omega')^{\delta}$, le résultat est un multiple numérique de $[w-\delta]$. Toutes les expressions ainsi obtenues seront du même poids w et par conséquent des fonctions linéaires des $(w:i, j)$ combinaisons qui sont du poids w et de l'ordre j dans les $i+1$ coefficients du Quantic donné.

On démontre comme auparavant que ces expressions sont linéairement indépendantes entre elles et que par conséquent leur nombre ne peut pas excéder $(w:i, j)$; donc leur nombre est égal à $(w:i, j)$, et la proposition est établie.

Pour démontrer que $(\Omega')^{\delta}('\Omega)^{\delta}[w-\delta]$ est, à un facteur numérique près, égal à $[w-\delta]$, on n'a pas besoin de sortir de la sphère des différentiants et de faire appel aux propriétés des covariants. On établit aisément que pour une quantité quelconque D du poids w et de l'ordre j dans les coefficients d'un Quantic binaire du degré i, on aura

$$(\Omega'\,'\Omega - '\Omega\,\Omega')\,D = (ij-2w)\,D.$$

En partant de là et supposant que $\Omega' D = 0$, on trouve par une induction algébrique facile que

$$(\Omega')^{k}('\Omega)^{k} D = 1\,.\,2 \ldots k\,(ij-w)(ij-w-1)\ldots(ij-w-k+1)\,.\,D,$$

où le facteur numérique ne s'évanouit que lorsque k est plus grand que $ij-w$.

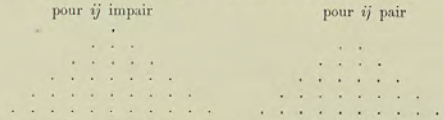
Considérons le système complet des expressions

$$[w-\delta],\quad '\Omega\,[w-\delta],\quad ('\Omega)^{2}[w-\delta],\quad \ldots,\quad ('\Omega)^{ij-2w}[w-\delta],$$

dont la dernière, qui résulte de l'opération $'\Omega$ répétée $ij-2w$ fois, se réduit à un différentiant-en-y, tandis que les suivantes produites par la même opération répétée $ij-2w+1$ ou un plus grand nombre de fois s'évanouissent identiquement.

En représentant toujours par des piles l'ensemble de toutes les expressions $('\Omega)^{m}[w-\delta]$ pour les mêmes valeurs de m et de δ, distinguant les deux cas de ij pair ou impair, et commençant par la plus grande valeur de w qui est $\frac{1}{2}ij$ pour ij pair et $\frac{1}{2}(ij-1)$ pour ij impair, on arrivera aux deux tableaux† suivants de points, qui donnent une image du système en question de piles

pour ij impair pour ij pair



* Le signe $('\Omega)^{\delta}$ exprime l'opération $'\Omega$ répétée δ fois.

† Le point au sommet du premier tableau correspond à des invariants, les points à gauche se rapportent aux différentiants-en-x, ceux à droite aux différentiants-en-y, ceux de la base aux coefficients successifs du Quantic élevé à la puissance j (pour l'un et l'autre des deux tableaux).

Prenons l'ensemble de toutes les combinaisons des coefficients du Quantic proposé du degré i, qui sont de l'ordre j dans les coefficients et des poids $0, 1, 2, \ldots, ij$ quant à x, alors, dans l'un cas comme dans l'autre, les quantités qui se trouvent dans chaque colonne verticale, seront du même nombre que l'ensemble correspondant des combinaisons des coefficients, elles seront en même temps des fonctions homogènes et linéaires des combinaisons qui y appartiennent.

Les piles qui se trouvent dans une ligne horizontale quelconque peuvent se réduire à une seule quantité, cas qui se présente toujours pour la dernière ligne horizontale: elles peuvent même s'évanouir identiquement, ce qui arrive pour certaines valeurs de w, i, j pour lesquelles il n'existe point de différentiant.

Postscriptum 2. Je suppléerai dans ce qui suit à une lacune qui se trouve dans les recherches précédentes, en donnant la démonstration de la proposition suivante :

Dans un Quantic préparé les inverses symboliques des éléments subissent par une substitution quelconque des variables une substitution induite qui est identique avec celle que les éléments eux-mêmes subiraient par la substitution contraire des variables.

Les mêmes raisonnements dont on s'est déjà servi plusieurs fois, font voir que pour la démonstration générale de cette proposition il suffit de la vérifier dans le cas spécial dans lequel le Quantic est binaire et $x+\epsilon y$ la valeur que l'on substitue pour x, ϵ étant infiniment petit. Soit i le degré du Quantic donné, soient $a, b, c, \ldots, h, k, l$ ses éléments et

$$1,\quad \sqrt{i},\quad \sqrt{\left\{\frac{i(i-1)}{2}\right\}},\quad \ldots,\quad \sqrt{i},\quad 1,$$

les multiplicateurs numériques des éléments, soient $a', b', c', \ldots$ les valeurs dans lesquelles se changent les éléments donnés après la substitution de $x+\epsilon y$ au lieu de x, ϵ étant infiniment petit; cela posé et en faisant

$$\lambda = \sqrt{i},\quad \mu = \sqrt{\{2(i-1)\}},\quad \nu = \sqrt{\{3(i-2)\}},\quad \ldots$$

les nouveaux éléments et les éléments primitifs s'exprimeront les uns par les autres au moyen des relations linéaires

$$a' = a,\quad b' = b + \lambda\epsilon a,\quad c' = c + \mu\epsilon b,\quad \ldots,\quad k' = k + \mu\epsilon h,\quad l' = l + \lambda\epsilon k,$$
$$a = a',\quad b = b' - \lambda\epsilon a',\quad c = c' - \mu\epsilon b',\quad \ldots,\quad k = k' - \mu\epsilon h',\quad l = l' - \lambda\epsilon k'.$$

D'autre part les inverses symboliques de ces deux systèmes d'éléments

$$\dot{a} = \frac{d}{da},\quad \dot{b} = \frac{d}{db},\quad \ldots,$$
$$\dot{a}' = \frac{d}{da'},\quad \dot{b}' = \frac{d}{db'},\quad \ldots$$

satisfont aux équations

$$\begin{aligned}
\dot a' &= \dot a \frac{da}{da'} + \dot b \frac{db}{da'} = \dot a - \lambda\epsilon \dot b,\\
\dot b' &= \dot b \frac{db}{db'} + \dot c \frac{dc}{db'} = \dot b - \mu\epsilon \dot c,\\
&\vdots \qquad\qquad\qquad\qquad \vdots\\
\dot k' &= \dot k \frac{dk}{dk'} + \dot l \frac{dl}{dk'} = \dot k - \lambda\epsilon \dot l,\\
\dot l' &= \dot l \frac{dl}{dl'} \qquad\qquad = \dot l,
\end{aligned}$$

ce qui fait voir que la substitution des inverses symboliques $\dot a, \dot b, \ldots, \dot l$ induite par la substitution de $x + \epsilon y$ au lieu de x, y restant inaltéré, est précisément la même que la substitution contraire de $y - \epsilon x$ au lieu de y, x restant inaltéré, induirait dans les éléments mêmes $a, b, \ldots, l$.

Je terminerai ces additions par l'énoncé d'un théorème général sur les formes invariantives dérivées qui montre d'une manière frappante le parti avantageux que l'on tire de la forme préparée sous laquelle je présente les Quantics.

Soit $F(a, b, c, \ldots : x, y, \ldots)$ un contravariant et $\Phi(a, b, c, \ldots : x, y, \ldots)$ un covariant du même Quantic donné; on connaît depuis longtemps le théorème que la nouvelle forme

$$F\left(a, b, c, \ldots : \frac{d\Phi}{dx}, \frac{d\Phi}{dy}, \ldots\right)$$

est un covariant du même Quantic. Or j'ajoute que si le Quantic proposé est présenté dans la forme préparée, alors la nouvelle forme

$$F\left(\frac{d\Phi}{da}, \frac{d\Phi}{db}, \frac{d\Phi}{dc}, \ldots : x, y, \ldots\right)$$

sera également un covariant du même Quantic. Si le Quantic proposé est présenté dans la forme ordinaire (pleine), cette dernière expression se change en

$$F\left(\frac{1}{m}\frac{d\Phi}{da}, \frac{1}{n}\frac{d\Phi}{db}, \frac{1}{p}\frac{d\Phi}{dc}, \ldots : x, y, \ldots\right),$$

$m, n, p, \ldots$ désignant les nombres binômes ou polynômes qui multiplient les éléments $a, b, c, \ldots$, elle se change au contraire en

$$F\left(m\frac{d\Phi}{da}, n\frac{d\Phi}{db}, p\frac{d\Phi}{dc}, \ldots : x, y, \ldots\right),$$

si le Quantic est présenté dans la forme vide. La démonstration de ce théorème se fait immédiatement à l'aide des principes exposés dans ce mémoire.

28.

ON A RULE FOR ABBREVIATING THE CALCULATION OF THE NUMBER OF IN- OR CO-VARIANTS OF A GIVEN ORDER AND WEIGHT IN THE COEFFICIENTS OF A BINARY QUANTIC OF A GIVEN DEGREE.

[*Messenger of Mathematics*, VIII. (1879), pp. 1—8.]

If i is the degree of a quantic we know now by *apodictic* reasoning that the number of its in- or co-variants of order j and of weight w in the coefficients is $(w : i, j) - [(w-1) : i, j]$, where in general $(x : i, j)$ denotes the number of modes of composing x with j numbers each having any value from 0 to i (both inclusive) or (what is the same thing) with i numbers each having any value from 0 to j. The object of this note is to show how to calculate the *difference* between the two denumerants above given without calculating each of them separately, whereby the actual amount of calculation required will be reduced to a small fraction of what it would otherwise be. I shall not stop to draw theoretical consequences from this theorem, but present it to the readers of the *Messenger* in the way it has occurred to me as a rule for abbreviating labour.

It is founded on the exhaustive method of representing partition systems by following a dictionary order of sequence, and it will be best understood by beginning with an example.

Suppose then that $w = 7$, $i = 5$, $j = 4$, we may find $(7 : 5, 4)$ by setting out and counting the arrangements where 4 is the number of parts and 5 the limit to each part, namely, 5 . 2, 5 . 1 . 1, 4 . 3, 4 . 2 . 1, 4 . 1 . 1 . 1, 3 . 3 . 1, 3 . 2 . 2, 3 . 2 . 1 . 1, 2 . 2 . 2 . 1.

For brevity the zeros required to fill up the number of parts to 4 are omitted in this table.

To find (6 : 5, 4) we may consider

(1) Those arrangements which begin with 5.

(2) Those arrangements which begin with a number less than 5.

To obtain the latter also arranged in dictionary order of sequence, we may (subject to an exception to be stated immediately below) proceed by diminishing each initial number in the above table by unity.

The exception to be made is where 2 initial numbers are alike, as in 3 . 3 . 1; 2 . 2 . 2 . 1. These arrangements must not be counted in, as the arrangements 2 . 3 . 1; 1 . 2 . 2 . 1 will already have been obtained from 4 . 2 . 1; 3 . 2 . 1 . 1 respectively.

Hence the number of arrangements in the above table to be preserved is less by 2 than the total number.

On the other hand we shall have the arrangement 5 . 1, to which there is nothing corresponding in the table for (7 : 5, 4). Hence the difference required is

$$2-1, \text{ that is, } (7:\ 5, 4)-(6:\ 5, 4)=1.$$

Let us take as a second example w (the weight) 12, i (the limit to each part) 6, and j (the number of parts) 4.

Let A be the table for (12 : 6, 4) in dictionary order, and let A' be the part of the table for (11 : 6, 4), also arranged in dictionary order, for which 6 is nowhere the initial term. Let A_1 be what A becomes when each initial number is diminished by unity.

Then, by the same reasoning as above, we must have $A'-A_1=$ 6 . 6, 5 . 5 . 2, 5 . 5 . 1 . 1, 4 . 4 . 4, 4 . 4 . 3 . 1, 4 . 4 . 2 . 2, 3 . 3 . 3 . 3, 7 in number.

Also calling B the part of the table for (11 : 6, 4), beginning with 6 we have $B=$ 6 . 5, 6 . 4 . 1, 6 . 3 . 2, 6 . 3 . 1 . 1, 6 . 2 . 2 . 1, 5 in number.

Hence $(12:\ 6, 4)-(11:\ 6, 4)=7-5, =2$.

To verify this, let us interchange the values 6 and 4, this by a well-known theorem leaves the value of each denumerant unaltered.

We have now $A'-A_1=$ 4 . 4 . 4, 4 . 4 . 3 . 1, 4 . 4 . 2 . 2, 4 . 4 . 2 . 1 . 1, 4 . 4 . 1 . 1 . 1 . 1, 3 . 3 . 3 . 3, 3 . 3 . 3 . 2 . 1, 3.3.3.1.1.1, 3.3.2.2.2, 3.3.2.2.1.1, 2 . 2 . 2 . 2 . 2 . 2, number is 11.

Also $B=$ 4 . 4 . 3, 4 . 4 . 2 . 1, 4 . 4 . 1 . 1 . 1, 4.3.3.1, 4.3.2.2, 4.3.2.1.1, 4.3.1.1.1.1, 4.2.2.2.1, 4.2.2.1.1.1, number is 9, and thus

$$(12:\ 4, 6)-(11:\ 4, 6)=11-9=2$$

as before. Evidently this identity between the two forms of

$$(w:\ i, 5)-\{(w-1):\ i, 5\},$$

given by this method, and also the incapability of this difference becoming negative when w is not greater than $\frac{1}{2}ij$, which I have elsewhere demonstrated, may be made to yield arithmetical properties of a new kind, and not unlikely to prove very valuable in certain parts of the theory of numbers; but what has impressed itself on my mind is the *enormous saving* of labour in the actual business of calculating invariantive formulæ, which this method confers. The existence of a perfectly definite table exhibiting an exhaustive arrangement of *ruled partitions* (as I call partitions subject to the two indices i, j) in itself constitutes a theorem (however simple), and the method above given is a further and more recondite theorem deduced from it, combined of course with other *intuitional* propositions.

Let us take as another example $w=20$, $i=13$, $j=3$.

Here $A'-A_1=$ 10 . 10, 9 . 9 . 2, 8 . 8 . 4, 7 . 7 . 6. $B=$ 13 . 6, 13 . 5 . 1, 13 . 4 . 2, 13 . 3 . 3. Therefore $(20:\ 13, 3)-(19:\ 13, 3)=0$.

Again let us calculate $(40:20, 4)-(39:20, 4)$.

Here $A'-A_1=$ 20 . 20, 19 . 19 . 2, 19 . 19 . 1 . 1, 18 . 18 . 4, 18 . 18 . 3 . 1, 18 . 18 . 2 . 2, 17 . 17 . 6, 17 . 17 . 5 . 1, 17 . 17 . 4 . 2, 17 . 17 . 3 . 3, and similarly 16 . 16 with 5 duads, 15 . 15 with 6 duads, 14 . 14 with 7 duads. Also 13 . 13 with 13 . 1, 12 . 2, 11 . 3, 10 . 4, 9 . 5, 8 . 6, 7 . 7, 12 . 12 with 12 . 4, 11 . 5, 10 . 6, 9 . 7, 8 . 8, 11 . 11 with 11 . 7, 10 . 8, 9 . 9, 10 . 10 . 10 . 10. Thus the number of terms in $A'-A_1$ is

$$(1+2+3+4+5+6+7)+(7+5+3+1)=44.$$

And B is composed of arrangements containing 20, together with the number of triads into which $39-20$, that is, 19 can be decomposed, none greater than 20, that is, the number of terms in B is 19 : 20, 3, which is the same as the absolute number of modes of resolving 19 into 3 parts or fewer, which is

$$1+1+2+2+3+3+4+4+(10:9, 2)+(9:10, 2)$$
$$+(8:11, 2)+(7:12, 2)=25+5+5+3+2=40.$$

Thus $$(40:20, 4)-(39:20, 4)=44-40=4,$$

which is easily verified, for the difference between the above two denumerants is the number of linearly independent invariants of the 20th order to a quartic, that is, is the number of ways of composing 20 with 2 and 3 (the orders of the fundamental invariants) which is 4 as found above.

The method thus simply and almost intuitively deduced, may be expressed in the form of a theorem as follows:

$$\Sigma_{q=0}^{q=i}(w-2q:q, j-2)-(w-i-1:i, j-1)=(w:i, j)-(w-1:i, j)$$
$$=\Sigma_{q=0}^{q=j}(w-2q:q, i-2)-(w-j-1:j, i-1).$$

The inferior unit is taken zero for the purpose of theoretical simplicity. Let the effective value of this limit be called $[q]$, and consider the first of the above three equals.

The value of $[q]$ is given by the condition that

$$w - 2\,[q] \text{ shall be not greater than } (j-2)\,[q],$$

that is, $$[q] \text{ not less than } \frac{w}{j},$$

that is, $$[q] \text{ is } \frac{w}{j} \text{ if } \frac{w}{j} \text{ is an integer, } \frac{w}{j}+1 \text{ if } \frac{w}{j} \text{ is fractional,}$$

that is, $$[q] = E\,\frac{w+j-1}{j};$$

(E standing as usual for the integer part of the quantity which it precedes).

The number of actual terms differing from zero under the sign of summation is therefore

$$i + 1 - E\,\frac{w+j-1}{j}, \text{ that is } 1 + E\,\frac{ij-w}{j},$$

similarly the number of terms under the sign of summation in the conjugate form will be $1 + E\,\dfrac{ij-w}{i}$.

Thus the first or second expression will be the best to employ, according as j is greater or less than i.

Again, since $$(w : i, j) = (ij - w : i, j),$$

we may in place of

$$(w : i, j) - (w - 1 : i, j),$$

employ $$(w' : i, j) - (w' + 1 : i, j),$$

which is $$-[(w' + 1 : i, j) - (w' : i, j)].$$

Hence, we may always secure in the application of this method, that the numerator in $E\,\dfrac{ij-w}{i}$ or in $E\,\dfrac{ij-w}{j}$ shall not be greater than $\frac{1}{2}ij$. Supposing j to be greater or not less than i, so that the first formula is applied, it will be found most convenient, so long as q is less or not greater than $j-2$, to consider q the number of the parts in any of the quantities

$$(w - 2q : q, j - 2),$$

and $j-2$ the limit to the magnitude of each part, and until q becomes equal to $i-1$, this hypothesis will always be the case. When $q = i$ or when $q = i$ and $q = i - 1$ in the respective cases of j being only one unit greater than i or equal to i, the two indices $q : j - 2$ may with advantage be reversed. For any other values of $j - i$, the order of the indices need not be disturbed. It may be worth while to call attention to the *two* independent theorems

of reciprocity made use of in the preceding discussion, indicated by the equations

$$\begin{aligned} &(w : i, j) \\ &= (w : j, i) \\ &= (ij - w : i, j) \\ &= (ij - w : j, i), \end{aligned}$$

both of them of importance in the theory of invariants after the English method.

ADDITION.

Notwithstanding what has been stated above as to the choice between the two formulæ representing $\Delta\,(w : i, j)$, the advantage of diminishing the smaller of the two indices i, j, will simplify the calculations to a degree that far more than outweighs the disadvantage of increasing the number of terms under the sign of summation. Let us suppose then that j is less than w, and that $\Delta\,(w : i, j)$ is positive, representing in fact indifferently the number of linearly independent covariants of order i to a quantic of degree j, or of order j to a quantic of degree i. Then, unless these covariants are invariants, we must have $w < \frac{1}{2}ij$.

Consequently, the best formula to apply in such case will be obtained by writing

$$\begin{aligned} \Delta\,(w : i, j) &= (ij - w : i, j) - (ij - w + 1 : i, j) \\ &= -\Delta\,(ij - w + 1 : i, j) \\ &= (ij - i - w : i, j - 1) - \Sigma_{q=0}^{q=i}\,(ij - w + 1 - 2q : q, j - 2). \end{aligned}$$

The number of terms other than zero under the sign of summation will then be $1 + E\,\dfrac{w}{j}$.

For the case of invariants we may with at least equal advantage use the formula

$$\Sigma_{q=0}^{q=i}\,(\tfrac{1}{2}ij - 2q : q, j - 2) - (\tfrac{1}{2}ij - 1 - i : i, j - 1).$$

Let us apply this to the case of finding

$$\Delta\left(\frac{18 \,.\, 5}{2} : 18, 5\right), \text{ that is } (45 : 18, 5).$$

In the work below I use, whenever useful, the formula of transformation

$$(x : i, j) = (ij - x : i, j),$$

and employ $\dfrac{\mu}{3}$ to denote the number of ways of breaking up μ into three or fewer parts, which we know is the nearest integer to $\frac{(\mu+3)^2}{12}$; and in like manner $\frac{\nu}{2}$ for the number of ways of breaking up ν into two parts: also in place of $(x : k, 3)$, whenever k is at least as great as x, I use the obviously equivalent value $\frac{x}{3}$.

Let us then first calculate

$$\Sigma_{q=9}^{q=18}\,[45-2q : q, 3], \text{ say } S.$$

The values of q inferior to 9 will give quantities in which $3q < 45 - 2q$, and which will therefore be zero.

We have thus

$$\begin{aligned} S &= (9:18, 3)+(11:17, 3)+(13:16, 3)+(15:15, 3) \\ &\qquad +(17:14, 3)+(19:13, 3)+(21:12, 3)+(23:11, 3) \\ &\qquad +(25:10, 3)+(27:9, 3) \\ &= \tfrac{9}{3}+\tfrac{11}{3}+\tfrac{13}{3}+\tfrac{15}{3}+(17:14, 3)+(19:13, 3)+(15:12, 3) \\ &\qquad +(10:11, 3)+(5:10, 3)+(0:9, 3). \end{aligned}$$

Also

$$\begin{aligned} (17:14, 3) &= (17:17, 3)-\tfrac{1}{2}-\tfrac{2}{2}-\tfrac{3}{2}=\tfrac{17}{3}-1-2-2=\tfrac{17}{3}-5, \\ (19:13, 3) &= (19:19, 3)-\tfrac{1}{2}-\tfrac{2}{2}-\tfrac{3}{2}-\tfrac{4}{2}-\tfrac{5}{2}-\tfrac{6}{2}=(19:19, 3)-15, \\ (15:12, 3) &= (15:15, 3)-\tfrac{1}{2}-\tfrac{2}{2}-\tfrac{3}{2}=\tfrac{15}{3}-5. \end{aligned}$$

Thus

$$\begin{aligned} S &= \tfrac{9}{3}+\tfrac{11}{3}+\tfrac{13}{3}+\tfrac{15}{3}+\tfrac{17}{3}+\tfrac{19}{3}+\tfrac{15}{3}+\tfrac{10}{3}+\tfrac{5}{3}+\tfrac{0}{3}-25 \\ &= \tfrac{0}{3}+\tfrac{5}{3}+\tfrac{9}{3}+\tfrac{10}{3}+\tfrac{11}{3}+\tfrac{13}{3}+2\,.\,\tfrac{15}{3}+\tfrac{17}{3}+\tfrac{19}{3}-25. \end{aligned}$$

Again let $S' = (44-18:18, 4) = (26:18, 4)$.

Then

$$\begin{aligned} S' &= (8:18, 3)+(9:17, 3)+(10:16, 3)+(11:15, 3) \\ &\qquad +(12:14, 3)+(13:13, 3)+(14:12, 3)+(15:11, 3) \\ &\qquad +(16:10, 3)+(17:9, 3)+(18:8, 3)+(19:7, 3) \\ &= \tfrac{8}{3}+\tfrac{9}{3}+\tfrac{10}{3}+\tfrac{11}{3}+\tfrac{12}{3}+\tfrac{13}{3}+(\tfrac{14}{3}-3)+(\tfrac{15}{3}-8) \\ &\qquad +(\tfrac{14}{3}-8)+(\tfrac{10}{3}-1)+\tfrac{6}{3}+\tfrac{2}{3}-20 \\ &= \tfrac{2}{3}+\tfrac{6}{3}+\tfrac{8}{3}+\tfrac{9}{3}+2\,.\,\tfrac{10}{3}+\tfrac{11}{3}+\tfrac{12}{3}+\tfrac{13}{3}+2\,.\,\tfrac{14}{3}+\tfrac{15}{3}-20; \end{aligned}$$

therefore

$$\begin{aligned} S-S' &= \tfrac{0}{3}-\tfrac{2}{3}+\tfrac{5}{3}-\tfrac{6}{3}-\tfrac{8}{3}-\tfrac{10}{3}-\tfrac{12}{3}-2\,.\,\tfrac{14}{3}+\tfrac{15}{3}+\tfrac{17}{3}+\tfrac{19}{3}-5 \\ &= 1-2+5-7-10-14-19-48+27+33+40-5 \\ &= 106-105=1, \end{aligned}$$

which is right, there being just one invariant to the quantic of the eighteenth order in the coefficients, so that $\Delta(45:18, 5)=1$.

It appears from the tables given in M. Faà de Bruno's valuable *Théorie des Formes Binaires*, Turin, 1877, that this invariant contains 848 terms. Therefore the value of $(18:8, 5)$ is very considerably greater* than 848.

Thus, by the direct method of calculating $\Delta(45:18, 5)$, many more than 1695 terms would have required setting out.

There is one case which deserves special consideration, namely, when one of the indices i or j becomes infinite.

The function $\Delta(w:\mu, \infty)$ then represents the total number of in- and co-variants of weight w of any given order not less than w to a quantic of the μth degree.

The two formulæ for this case become respectively

$$\Sigma_{q=0}^{q=\infty}\,[w-2q : q, \mu],$$

and

$$\Sigma_{q=0}^{q=\mu}\,[w-2q : q, \infty]-[w-\mu-1 : \mu, \infty],$$

or if we agree to understand in all cases by $\frac{n}{m}$ the number of ways of making up n with the integers $0, 1, 2, 3 \ldots m$, or, what is the same, the number of ways of breaking up n into m or fewer parts, the second formula becomes

$$\Sigma_{q=0}^{q=\mu}\frac{w-2q}{q}-\frac{w-\mu-1}{\mu};$$

of these two the first is by far the most expeditious.

Let us take as an example $\Delta(20:6, \infty)$, that is $\tfrac{20}{6}-\tfrac{13}{6}$.

The first formula {neglecting the values of q which make $w-2q$ negative and those which make $4q<(w-2q)$}, will give for the value of Δ

$$\begin{array}{ll} (0:10, 4) & = \ (0) \\ +(2:9, 4) & +(2) \\ +(4:8, 4) & +(4) \\ +(6:7, 4) & +\tfrac{6}{4} \\ +(8:6, 4) & +(8:6, 4) \\ +(10:5, 4) & +(10:5, 4) \\ +(12:4, 4) & +(4:4, 4), \text{ that is } (4), \end{array}$$

* I say very considerably greater than, because only a certain number of the terms which satisfy the required conditions of order and weight actually appear in the octodecimal invariant in question. Thus, for example, there is no f^9, no f^8, and of the $(10:11, 5)$ that is $\tfrac{10}{5}$ terms which might contain f^7, only six, namely the terms contained in $a\,(ac-b^2)^5$ actually make their appearance in it.

where in general (m) means *all* the modes of breaking up m into parts. The value of $(10:5,\ 4)$ will be easily found to be 9, of $(8:6,\ 4)$ 12 and of $\frac{6}{4}$, 9, also of (4) is 5. The value of $\frac{20}{6}-\frac{19}{6}$ thus becomes

$$1+2+5+9+12+9+5=43.$$

By the second formula the value of the same quantity would be

$$\tfrac{8}{6}+\tfrac{10}{5}+\tfrac{12}{4}+\tfrac{14}{3}+\tfrac{16}{2}+\tfrac{18}{1}-\tfrac{13}{5},$$

which would be exceedingly tedious to calculate.

In like manner if w is odd we shall have a series of denumerants of the form

$$\left(1:\frac{w-1}{2},\mu\right),\quad \left(3:\frac{w-3}{2},\mu\right),\quad \left(5:\frac{w-5}{2},\mu\right),\ \&c.$$

Thus, for example, $\frac{11}{6}-\frac{10}{6}$ (that is, the number of in- and co-variants to a sextic of weight 11 and of any given order not inferior to 11, or, if we please to vary the expression, the number of in- and co-variants of weight 11 and the sixth order to any quantic of a degree not inferior to 11) will be

$$\left.\begin{array}{r}(1:\ 5,\ 4)\\ +(3:\ 4,\ 4)\\ +(5:\ 3,\ 4)\\ +(7:\ 2,\ 4)\end{array}\right\}=\left\{\begin{array}{l}\quad(1)\\ +(3)\\ +(5:3,\ 4)\\ +(1:2,\ 4)\text{ that is }(1)\end{array}\right.$$

$$=1+3+4+1=9.$$

29.

NOTE ON CONTINUANTS.

[*Messenger of Mathematics*, VIII. (1879), pp. 187—189.]

To find the number of terms in the cumulant or continuant $(a_1, a_2, \ldots a_n)$, we may proceed as follows:

(1) There is the term $a_1a_2a_3 \ldots a_n$.

(2) The number of terms of the first order of degradation, that is, obtained by leaving out any pair of consecutive elements, is $n-1$, say $u_{n,1}$.

(3) The number of terms of the second order of degradation obtained by leaving out any two pairs of such, that is, by leaving out the first and second and some other pair of those that follow the second, the second and third and a pair of those that follow the third, the third and fourth and a pair of those that follow the fourth and so on, is

$$u_{n-2,1}+u_{n-3,1}+u_{n-4,1}+\ldots,$$

and, consequently,

$$(n-3)+(n-4)+(n-5)+\ldots$$

$$=\frac{(n-2)(n-3)}{2}=\text{, say, } u_{n,2}.$$

(4) The number of the third order of degradation is in like manner

$$u_{n-2,2}+u_{n-3,2}+u_{n-4,2}+\ldots,$$

that is

$$=\frac{(n-4)(n-5)}{1.2}+\frac{(n-5)(n-6)}{1.2}+\ldots$$

$$=\frac{(n-3)(n-4)(n-5)}{1.2.3},$$

and so in general

$$u_{n,r} = u_{n-2,r-1} + u_{n-3,r-1} + u_{n-4,r-1} + \dots$$
$$= \frac{(n-r)(n-r-1)\dots(n-2r-1)}{1.2\dots r}.$$

Hence, the total number is

$$1 + (n-1) + \frac{(n-2)(n-3)}{1.2} + \frac{(n-3)(n-4)(n-5)}{1.2.3} + \dots .$$

Verification. In general

$$(i\sin\theta + \cos\theta)^n - (i\sin\theta - \cos\theta)^n$$
$$= 2\cos\theta\,\{(2i\sin\theta)^{n-1} + (n-2)(2i\sin\theta)^{n-3} + \frac{(n-3)(n-4)}{2}(2i\sin\theta)^{n-5} + \dots\},$$

for we know that

$$\cos\theta\,\{(2\sin\theta)^{n-1} - (n-2)(2\sin\theta)^{n-3} + \frac{(n-3)(n-4)}{2}(2\sin\theta)^{n-5} - \dots\}$$

$= (-1)^{\frac{1}{2}(n-1)}\cos n\theta$, or $(-1)^{\frac{1}{2}(n-2)}\sin n\theta$, according as n is odd or even.

Hence, putting

$$i\sin\theta + \cos\theta = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{5},$$

so that

$$i\sin\theta - \cos\theta = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{5},$$
$$2i\sin\theta = 1,$$

and

$$2\cos\theta = \sqrt{5},$$

$$1 + (n-1) + \frac{(n-2)(n-3)}{1.2} + \frac{(n-3)(n-4)(n-5)}{1.2.3} + \dots$$
$$= \frac{(\frac{1}{2} + \frac{1}{2}\sqrt{5})^{n+1} - (\frac{1}{2} - \frac{1}{2}\sqrt{5})^{n+1}}{\sqrt{5}}.$$

But because

$$(a_1, a_2, \dots a_n) = a_n(a_1, a_2, \dots a_{n-1}) + (a_1, a_2, \dots a_{n-2}),$$

if u_n is the number of terms in $(a_1, a_2, \dots a_n)$,

$$u_n = u_{n-1} + u_{n-2},$$

with the initial conditions

$$u_0 = 1, \quad u_1 = 1.$$

Solving this difference-equation, we shall obtain

$$u_n = \frac{1}{\sqrt{5}}\{(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5})^{n+1} - (\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5})^{n+1}\},$$

agreeing with the preceding result.

Corollary 1. The value of the continuant of the nth order $(x, x, \dots x)$, is

$$x^n + (n-1)x^{n-2} + \frac{(n-2)(n-3)}{1.2}x^{n-4} + \dots,$$

which admits also of the clumsy representation

$$[\{\tfrac{1}{2}x + \tfrac{1}{2}\sqrt{(x^2+4)}\}^{n+1} - \{\tfrac{1}{2}x - \tfrac{1}{2}\sqrt{(x^2+4)}\}^{n+1}] \div \sqrt{(x^2+4)}.$$

Corollary 2. The value of the pro-continuant of the nth order

$$(2\cos\theta, 2\cos\theta, \dots 2\cos\theta),$$

is

$$\frac{\sin(n+1)\theta}{\sin\theta}.$$

By the pro-continuant is to be understood what a continuant becomes, when in its representative determinant, the oblique lines of negative units are all changed into positive units so that the matrix has two precisely similar bands of units one above and one below the diagonal line and in opposition with it.

Corollary 3. The integral of the partial-difference equation

$$u_{x+1,y} - u_{x,y} - u_{x-1,y-1} = 0,$$

limited by the conditions

$$u_{x,0} = 1, \quad u_{x+1,x+1} = 0,$$

is

$$u_{x,y} = \frac{\Pi(x-y)}{\Pi(x-2y)\,\Pi y}.$$

30.

SUR UNE PROPRIÉTÉ ARITHMÉTIQUE D'UNE CERTAINE SÉRIE DE NOMBRES ENTIERS.

[*Comptes Rendus*, LXXXVIII. (1879), pp. 1297, 1298.]

NOMMONS le nombre de termes distincts qui figurent dans le développement d'un déterminant gauche son *dénumérant*. Soit

$$[1 \,.\, 3 \,.\, 5 \ldots (2n-1)]\, u_n$$

le dénumérant d'un déterminant gauche de l'ordre $2n$. On aura pour u_1, u_2, u_3, u_4, u_5, u_6, … les valeurs successives

$$1,\ 2,\ 8,\ 50,\ 418,\ 4348,\ \ldots$$

et en général

$$u_x = (2x-1)\, u_{x-1} - (x-1)\, u_{x-2}.$$

Soit $\theta\left(\frac{2x+1}{8}\right)$ l'entier le plus proche (en excès ou en défaut) de $\frac{2x+1}{8}$. Alors je dis que *le plus grand diviseur commun à u_x, u_{x+1} est égal au nombre 2 élevé à la puissance $\theta\left(\frac{2x+1}{8}\right)$*.

Ce théorème se déduit des deux propositions suivantes :

1°. On démontre que u_x et x ne peuvent avoir un facteur commun impair pour aucune valeur de x; c'est une conséquence immédiate de cette loi que deux u consécutifs ne peuvent avoir non plus un facteur commun impair.

2°. On démontre que $\frac{u_{4x-2}}{2^x}$, $\frac{u_{4x-1}}{2^x}$, $\frac{u_{4x}}{2^x}$, $\frac{u_{4x+2}}{2^x}$ sont tous les quatre des nombres entiers, dont le premier et le troisième sont des nombres impairs; cela suffit pour établir le théorème. Mais j'ajoute, comme corollaire, que la quatrième de ces quantités est aussi un nombre impair et la seconde un nombre pair, qui est toujours divisible par 4.

Le fondement du raisonnement au moyen duquel on établit cette proposition remarquable est l'identité que j'ai donnée dans* l'*American Journal of Mathematics*

$$\frac{e^{\frac{t}{4}}}{\sqrt[4]{(1-t)}} = 1 + u_1 \frac{t}{2} + u_2 \frac{t^2}{2 \,.\, 4} + u_3 \frac{t^3}{2 \,.\, 4 \,.\, 6} + \ldots.$$

[* p. 272 below.]

31.

SUR LA VALEUR MOYENNE DES COEFFICIENTS DANS LE DÉVELOPPEMENT D'UN DÉTERMINANT GAUCHE OU SYMÉTRIQUE D'UN ORDRE INFINIMENT GRAND ET SUR LES DÉTERMINANTS DOUBLEMENT GAUCHES*.

[*Comptes Rendus*, LXXXIX. (1879), pp. 24—26.]

DANS un déterminant ou gauche ou symétrique, j'ai fait voir ailleurs que tous les coefficients qui ne sont pas des unités seront des puissances de 2. J'ajoute que, dans le dernier cas, si n est l'ordre du déterminant, la plus haute puissance de 2 qui entre comme coefficient sera la partie entière de $\frac{n}{3}$ et dans le premier cas $\frac{n}{4}$ (n dans ce cas étant un nombre pair).

M. Cayley a le premier démontré que, si le nombre des termes distincts dans le développement d'un déterminant symétrique de l'ordre x est $(1 \,.\, 2 \,.\, 3 \ldots x)\, \Omega_x$, Ω_x aura pour sa fonction génératrice $\frac{e^{\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{(1-t)}}$; et, de ma part, j'ai démontré que, si le nombre des termes distincts dans un déterminant gauche de l'ordre $2x$ est $1 \,.\, 3 \,.\, 5 \ldots (2x-1)\, \omega_x$, ω_x aura pour sa fonction génératrice $\sqrt[4]{\left(\frac{e^t}{1-t}\right)}$.

Ces deux formules suffisent pour la solution du problème proposé. Commençons par le déterminant gauche. En vertu de la formule donnée, on aura

$$\omega_x = \left[1 + x + 1 \,.\, 5x + 1 \,.\, 5 \,.\, 9 \frac{x(x-1)}{2} + 1 \,.\, 5 \,.\, 9 \,.\, 13 \frac{x(x-1)(x-2)}{2 \,.\, 3} + \ldots + 1 \,.\, 5 \,.\, 9 \ldots (4x-3)\right] \frac{1}{2^x},$$

nombre qui est toujours entier, car ω_x est assujetti à satisfaire à l'équation $\omega_x = (2x-1)\, \omega_{x-1} - (x-1)\, \omega_{x-2}$; de sorte que ω_0, ω_1 étant 1, 1, tous les ω

[* See below, p. 257.]

seront des nombres entiers. En posant $1 \,.\, 3 \,.\, 5 \ldots (2x-1)\, \omega_x = u_{2x}$, on trouve facilement, à l'aide de cette expression, que, pour $x = \infty$,

$$\frac{u_{2x}}{1 \,.\, 2 \,.\, 3 \ldots 2x} = e^{\frac{1}{4}} \frac{1 \,.\, 5 \,.\, 9 \ldots (4x-3)}{4 \,.\, 8 \,.\, 12 \ldots 4x}.$$

De plus, par une méthode bien connue, on trouve

$$\begin{aligned} \log \{1 \,.\, 5 \,.\, 9 \ldots (4x-3)\} &= C - x + \frac{3}{4} + \frac{4x-1}{4} \log (4x-3) \\ &\quad + \frac{1}{12} \frac{d}{dx} \log (4x-3) - \frac{1}{720} \frac{d^3}{dx^3} \log (4x-3) + \ldots \\ &= \left(C - \frac{\log 4}{4}\right) - x + \log 4x + x \log x + \frac{A}{x} \ldots. \end{aligned}$$

On a aussi

$$\log (4 \,.\, 8 \ldots 4x) = x \log 4 + \log \sqrt{(2\pi)} + x \log x - x + \frac{1}{2} \log x + \frac{A'}{x} \ldots.$$

On aura donc

$$\frac{1 \,.\, 5 \ldots (4x-3)}{4 \,.\, 8 \ldots 4x} = \frac{e^C}{2\sqrt{(\pi)}} \frac{1}{x^{\frac{3}{4}}},$$

et, puisque la somme des coefficients pris tous positivement en u_{2x} est égale à $\{1 \,.\, 3 \,.\, 5 \ldots (2x-1)\}^2$ et $\dfrac{\{1 \,.\, 3 \,.\, 5 \ldots (2x-1)\}^2}{1 \,.\, 2 \ldots 2x} = \dfrac{1}{\sqrt{(\pi x)}}$, on a finalement la valeur moyenne des coefficients, c'est-à-dire

$$\frac{\{1 \,.\, 3 \,.\, 5 \ldots (2x-1)\}^2}{u_{2x}} = \frac{2}{e^{\frac{1}{4}+C}} x^{\frac{1}{4}}.$$

Pour trouver C je me sers de la formule

$$\begin{aligned} C = \log \{1 \,.\, 5 \,.\, 9 \ldots (4x-3)\} - \frac{3}{4} \\ + x - \frac{4x-1}{4} \log (4x-3) - \frac{1}{3} \frac{1}{4x-3} + \frac{8}{45} \frac{1}{(4x-3)^3} \cdots \end{aligned}$$

et, en mettant $4x - 3 = 125$, on trouve, à l'aide des Tables ordinaires de logarithmes,

$$C = -\,0{,}022508\ldots,$$

ce qui donne pour la valeur moyenne cherchée $(1{,}593\ldots)\, x^{\frac{1}{4}}$.

Comme vérification, j'ai fait calculer u_4, u_8, u_{12}, u_{16}, par le moyen des formules

$$u_{2x} = 1 \,.\, 3 \,.\, 5 \ldots (2x-1)\, \omega_x,$$

$$\omega_x = (2x-1)\, \omega_{x-1} - (x-1)\, \omega_{x-2},$$

et, en posant

$$\frac{1 \,.\, 3 \,.\, 5 \ldots (2x-1)}{\omega_x} = \rho_x x^{\frac{1}{4}},$$

j'ai trouvé

$$\rho_4 = 1{,}262\ldots, \quad \rho_8 = 1{,}485\ldots, \quad \rho_{12} = 1{,}528\ldots, \quad \rho_{16} = 1{,}551\ldots,$$

ce qui s'accorde très bien avec la valeur $\rho_\infty = 1{,}593\ldots$.

Pour le déterminant symétrique, en vertu de la formule de M. Cayley, on sait que la valeur moyenne cherchée est le coefficient de t^x dans $\dfrac{e^{\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{(1-t)}}$, qui sera le même, quand $x = \infty$, que dans $\dfrac{e^{\frac{3t}{4}}}{\sqrt{(1-t)}}$, et l'on trouve facilement que cette valeur est égale à $e^{-\frac{3}{4}} \sqrt{(\pi x)}$.

J'ajoute quelques mots sur les déterminants *doublement* gauches, c'est-à-dire gauches par rapport à l'une et à l'autre des deux diagonales.

1°. Je trouve que, pour que ces déterminants ne s'évanouissent pas, l'ordre doit être divisible par 4.

2°. Considérons la *racine carrée* d'un déterminant doublement gauche de l'ordre $4x$. Je trouve que la somme de ses coefficients pris tous positivement est égale à

$$1 \,.\, 2 \,.\, 5 \,.\, 6 \,.\, 9 \,.\, 10 \ldots (4x-3)(4x-2).$$

3°. Soit ϕ_x le nombre des termes *distincts* dans cette racine carrée. Je trouve qu'en posant $\phi_x = 2 \,.\, 4 \,.\, 6 \ldots (4x-2)\, \psi_x$, ψ_x sera toujours un nombre entier défini par l'équation

$$\psi_x = (4x-3)\, \psi_{x-1} - 2x \psi_{x-2}, \quad \psi_0 = 1, \quad \psi_1 = 1,$$

et que la fonction génératrice de ψ_x sera $\sqrt[8]{\left(\dfrac{e^t}{1-t}\right)}$, de sorte que

$$\begin{aligned} \psi_x = \Big[1 + x + 1 \,.\, 9 \frac{x(x-1)}{2} + 1 \,.\, 9 \,.\, 17 \frac{x(x-1)(x-2)}{1 \,.\, 2 \,.\, 3} \ldots \\ + 1 \,.\, 9 \,.\, 17 \ldots (8x-7)\Big] \div 2^x. \end{aligned}$$

4°. On démontre facilement que deux des ψ consécutifs quelconques seront toujours premiers entre eux et que tous les coefficients dans la racine carrée du déterminant doublement gauche de l'ordre $4x$ sont des puissances de 2, dont la plus haute sera désignée par la partie entière de $\dfrac{4x}{8}$, c'est-à-dire de $\dfrac{x}{2}$.

32.

TABLE DES NOMBRES DE DÉRIVÉES INVARIANTIVES D'ORDRE ET DE DEGRÉ DONNÉS, APPARTENANT À LA FORME BINAIRE DU DIXIÈME ORDRE.

[*Comptes Rendus*, LXXXIX. (1879), pp. 395, 396.]

Degré dans les coefficients.	Ordre dans les variables.													
	0	2	4	6	8	10	12	14	16	18	20	22	24	26
1.........						1								
2.........	1		1		1		1		1					
3.........		1		2	1	1	2	1	1	1	1		1	
4.........	1		3	1	3	3	2	3	1	2	1	1		1
5.........		3	3	4	5	4	5	2	4		1			
6.........	4	2	5	8	6	8	2	3						
7.........		7	10	8	12	2	3							
8.........	5	8	11	15	4	5								
9.........	5	13	19	8	4									
10.........	8	20	12	10										
11.........	8	18	21											
12.........	12	30												
13.........	15	16												
14.........	13	17												
15.........	19													
16.........	5													
17.........	3													

Pour trouver par cette Table le nombre d'invariants ou covariants fondamentaux de l'ordre ω et du degré δ, on cherche dans la colonne numérotée ω et dans la ligne numérotée δ; le chiffre qui se trouve au point de concours de cette colonne et de cette ligne est le nombre en question. S'il n'existe aucune combinaison de colonne et de ligne numérotées ω et δ respectivement, il n'y aura aucun covariant (ou invariant) du degré δ et de l'ordre ω.

Cette Table a été construite sous ma direction par M. Franklin, de Baltimore, avec l'aide des fonds que l'Association britannique pour l'avancement de la Science, dans sa dernière session à Dublin, a eu la bonté de mettre à ma disposition pour effectuer des calculs de ce genre.

Les Tables analogues pour la forme binaire de l'ordre 7 et de l'ordre 8 ont déjà paru* dans ces *Comptes rendus*, et celle pour l'ordre† 9 dans l'*American Journal of Mathematics* de cette année, de sorte qu'aujourd'hui on connaît toutes les dérivées invariantives fondamentales ayant rapport à des formes uniques binaires de chaque ordre, depuis 2 jusqu'à 10 inclusivement.

[* pp. 146, 115 above.] [† p. 281 below.]

33.

SUR LA VALEUR MOYENNE DES COEFFICIENTS NUMÉRIQUES DANS UN DÉTERMINANT GAUCHE D'UN ORDRE INFINIMENT GRAND.

[*Comptes Rendus*, LXXXIX. (1879), pp. 497, 498.]

PAR une inadvertance regrettable, j'ai omis* de donner la valeur moyenne des coefficients numériques dans un déterminant gauche d'un ordre infini sous sa forme exacte. Pour cela, on n'a besoin que de se servir de la formule

$$\frac{a(a+\delta)(a+2\delta)\dots(a+x\delta)}{b(b+\delta)(b+2\delta)\dots(b+x\delta)}=\frac{\Gamma\dfrac{b}{\delta}}{\Gamma\dfrac{a}{\delta}}x^{\frac{a-b}{\delta}},$$

où l'on suppose que x est infiniment grand.

Or la somme des coefficients, tous pris positivement dans le déterminant gauche de l'ordre x, est

$$[1\,.\,3\,.\,5\dots(x-1)]^2,$$

et le nombre des termes distincts (x étant supposé infiniment grand) est

$$e^{\frac{1}{4}}(1\,.\,2\,.\,3\dots 2x)\frac{1\,.\,5\,.\,9\dots(4x-3)}{4\,.\,8\,.\,12\dots 4x};$$

en conséquence, la valeur moyenne cherchée sera

$$\frac{1}{e^{\frac{1}{4}}}\frac{1\,.\,3\,.\,5\dots(2x-1)}{2\,.\,4\,.\,6\dots 2x}\frac{4\,.\,8\,.\,12\dots 4x}{1\,.\,5\,.\,9\dots(4x-3)}=\frac{1}{e^{\frac{1}{4}}}\frac{\Gamma 1}{\Gamma\frac{1}{2}}\frac{\Gamma\frac{1}{4}}{\Gamma 1}x^{\frac{3}{4}-\frac{1}{2}}=\frac{\Gamma\frac{1}{4}}{\Gamma\frac{1}{2}}\left(\frac{x}{e}\right)^{\frac{1}{4}}.$$

Si l'on écrit cette valeur sous la forme $Cx^{\frac{1}{4}}$, on aura

$$\log C=\log\Gamma\tfrac{1}{4}+\log 2-\log\Gamma\tfrac{1}{2}-\tfrac{1}{4}\log e$$
$$=9573211+3010300-9475449-1085711=2022351.$$

On a donc

$$C=1{,}59307,$$

expression dont les quatre premiers chiffres avaient été précédemment trouvés; mais l'expression exacte $\dfrac{\Gamma\frac{1}{4}}{e^{\frac{1}{4}}\sqrt{(\pi)}}x^{\frac{1}{4}}$, qui me paraît remarquable, est ici donnée pour la première fois.

[* above, p. 253.]

34.

SUR LE VRAI NOMBRE DES COVARIANTS FONDAMENTAUX D'UN SYSTÈME DE DEUX CUBIQUES.

[*Comptes Rendus*, LXXXIX. (1879), pp. 828—832.]

L'ÉNUMÉRATION des invariants et covariants pour un système de deux cubiques binaires, donnée par M. Salmon (*Modern Higher Algebra*, p. 186) et attribuée par lui à MM. Clebsch et Gordan, comprend huit covariants linéaires, dont deux sont du degré 3 par rapport aux coefficients de l'une des cubiques, et l'autre du degré 4. Par ma méthode, j'avais trouvé précisément les mêmes invariants et covariants fondamentaux que MM. Clebsch et Gordan; mais tout récemment, en refaisant mes calculs, M. Franklin, de Baltimore, a découvert qu'il y avait une faute d'arithmétique commise dans mon tamisage, et que les deux covariants linéaires dont j'ai parlé plus haut ne doivent pas figurer dans ma Table. Je vais donc démontrer qu'en effet ces covariants, supposés fondamentaux également par MM. Clebsch et Gordan et moi-même, ne le sont pas; de sorte que le nombre total des *Grundformen*, pour un système de deux cubiques, est 26 et non pas 28, comme on avait pensé jusqu'à ce jour.

En démontrant une chose pareille dans le cas d'un système de deux biquadratiques, je me suis servi de la méthode pour ainsi dire positive, c'est-à-dire j'ai donné la décomposition de deux des formes supposées fondamentales par M. Gordan. Dans le cas beaucoup plus difficile du système traité par M. Gundelfinger d'une cubique et une biquadratique, je me suis servi de la méthode négative en prouvant *à priori* l'impossibilité de l'existence de formes fondamentales ayant le type (c'est-à-dire les degrés et l'ordre) qu'avaient trois des *Grundformen* imaginées par cet auteur distingué.

Je vais me servir de cette dernière méthode comme étant la plus courte dans le cas actuel, en démontrant qu'un covariant linéaire du type 3, 4 ou du type gémeau 4, 3 appartenant à un système de deux cubiques ne peut pas être indécomposable.

Je commence avec la détermination du nombre des covariants du type 4, 3 : 1 (ou bien, ce qui est absolument le même, du type 3, 4 : 1), linéairement indépendants, appartenant à un système de deux cubiques. Pour cela, par le théorème que j'ai démontré avec le dernier degré de rigueur dans le *Journal de M. Borchardt** et dans le *Philosophical Magazine*†, on sait, puisque $\frac{4\,.\,3+3\,.\,3-1}{2}=10$, que le nombre cherché sera

$$(10:3,4:3,3)-(9:3,4:3,3),$$

en se servant, en général, de la notation $(w: i, j: i', j')$ pour signifier le nombre des représentations de w par la somme bifide

$$x_1+2x_2+3x_3+\ldots+ix_i+y_1+2y_2+3y_3+\ldots+i'y_{i'},$$

où les x peuvent être chacun 0, 1, 2, 3, ... ou j, et les y, 0, 1, 2, 3, ... ou j'. Le nombre de partitions, sans exclusion des zéros, en trois parties, dont aucune n'excède 4, est respectivement pour les chiffres

0	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	4	5	4	4	3	2
1	1	2	3	3	3	3	2	1	1	0

quand, le nombre des parties restant 3, la limite supérieure de chaque partie, au lieu de 4, devient 3. Conséquemment on aura

$$\begin{aligned}(10:3,4:3,3)&=1.1+2.1+3.2+4.3+4.3+5.3+4.3+4.2+3.1+2.1\\&=1+2+6+12+12+15+12+8+3+2=73,\\(9:3,4:3,3)&=1.1+1.1+2.2+3.3+4.3+4.3+5.3+4.2+4.1+3.1\\&=1+1+4+9+12+12+15+8+4+3=69;\end{aligned}$$

c'est-à-dire que le nombre des covariants des degrés 3, 4 pour les coefficients et de l'ordre 1 pour les variables linéairement indépendants sera $73-69$ ou 4.

Je vais démontrer qu'il y a, en effet, exactement quatre covariants de ce type non irréductibles, mais linéairement indépendants; de sorte qu'il n'y aura pas place dans la nature des choses pour des covariants irréductibles, c'est-à-dire non composés ou fondamentaux, de ce même type.

Prenons les deux formes $(a, b, c, d \between x, y)^3$, $(\alpha, \beta, \gamma, \delta \between x, y)^3$. Je me servirai de la notation $p\,.\,q\,.\,i$ qui signifiera un covariant du degré p pour les coefficients a, b, c, d; q pour $\alpha, \beta, \gamma, \delta$; et i pour les variables. On connaît les invariants fondamentaux 1 . 1 . 0, 2 . 2 . 0, 3 . 1 . 0, disons A, B, C, et les covariants linéaires 2 . 1 . 1, 1 . 2 . 1, 3 . 2 . 1, disons U, V, W, avec l'aide desquels on peut former les quatre covariants décomposables A^2U, BU, CV, AW, du type 4 . 3 . 1.

[* p. 232 above.] [† p. 117 above.]

3.1.0 et 2.2.0 seront les valeurs des deux émanants, $E\Delta$, $E^2\Delta$, où

$$E = \alpha\frac{d}{da} + \beta\frac{d}{db} + \gamma\frac{d}{dc} + \delta\frac{d}{dd}$$

et $$\Delta = a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd.$$

1.1.0 sera le combinant $a\delta - 3b\gamma + 3c\beta - d\alpha$; 2.1.1 sera*

$$\begin{vmatrix} a & b & c \\ b & c & d \\ \alpha x + \beta y & \beta x + \gamma y & \gamma x + \delta y \end{vmatrix}$$

et 3.2.1 sera le produit de l'opération du hessien de $(\alpha, \beta, \gamma, \delta)\left(\frac{d}{dy}, -\frac{d}{dx}\right)^3$ sur le covariant cubique de $(a, b, c, d)(x, y)^3$. Pour plus de facilité, faisons $b = 0$, $d = 0$, $\alpha = 0$, $\gamma = 0$; alors on voit que 3.1.0 s'évanouit et que 2.2.0 et 1.1.0 deviennent (en omettant dans le premier le coefficient numérique 2) $a^2\delta^2 - 6ac\beta\delta - 3c^2\beta^2$ et $a\delta + 3c\beta$ respectivement.

Bornons-nous aux coefficients de y dans 2.1.1 et 3.2.1; le dernier devient $ac\delta - c^2\beta$, et, puisque le hessien écrit plus haut devient

$$\beta\delta\left(\frac{d}{dx}\right)^2 - \beta^2\left(\frac{d}{dy}\right)^2,$$

si l'on nomme le covariant cubique dont j'ai parlé

$$Lx^3 + Mx^2y + Nxy^2 + Py^3,$$

le coefficient de y dans 3.2.1 deviendra $2\beta\delta M - 6\beta^2 P$, ou

$$M = 3abd - 6ac^2 + 3b^2c = -6ac^2,$$
$$P = -ad^2 + 3bcd - 2c^3 = -2c^3,$$

de sorte que ce coefficient, en omettant le coefficient numérique -12, devient $ac^2\beta\delta - c^3\beta^2$.

* Cela est une conséquence immédiate du fait connu qu'aux deux formes $(a, b, c, d)(x, y)^3$, $(\lambda, \mu, \nu)(x, y)^2$ appartient un déterminant invariantif

$$\begin{vmatrix} a & b & c \\ b & c & d \\ \lambda & \mu & \nu \end{vmatrix};$$

de même, pour deux biquadratiques, il y aura un déterminant invariantif

$$\begin{vmatrix} a & b & c & d \\ b & c & d & e \\ \alpha & \beta & \gamma & \delta \\ \beta & \gamma & \delta & \epsilon \end{vmatrix};$$

et, en général, à un système de i formes binaires des degrés $n_1, n_2, \ldots, n_i$, en faisant $\frac{\Sigma(n) - 2}{i+1} = \mu$, pourvu que μ soit entier et moindre qu'un quelconque des n, on peut toujours former avec les coefficients des i formes un déterminant de l'ordre $\mu + 2$, analogue à ceux que j'ai écrits plus haut, qui sera un invariant du système. Cet invariant est, en effet, l'analogue pour un système de l'invariant bien connu nommé *catalecticant* dans le cas d'une seule forme.

Si donc une équation linéaire telle que $\lambda A^2U + \mu BU + \nu CV + \rho AW = 0$ lie ensemble les quatre covariants composés dans leur forme générale, on aura

$$\lambda(a\delta + 3c\beta)^2(ac\delta - c^2\beta) + \mu(a^2\delta^2 - 6ac\beta\delta - 3c^2\beta^2)(ac\delta - c^2\beta)$$
$$+ \rho(a\delta + 3c\beta)(ac^2\beta\delta - c^3\beta^2)$$

identiquement égal à zéro; c'est-à-dire

$$\lambda(a\delta + 3c\beta)^2 + \mu(a^2\delta^2 - 6ac\beta\delta - 3c^2\beta^2) + \rho c\beta(a\delta + 3c\beta) = 0.$$

En égalant à zéro les coefficients de $a^2\delta^2$, $ac\beta\delta$, $c^2\beta^2$, dans cette identité, on obtient trois équations linéaires et homogènes en λ, μ, ρ auxquelles (vu que leur déterminant

$$\begin{vmatrix} 1 & 1 & 0 \\ 6 & -6 & 1 \\ 9 & -3 & 3 \end{vmatrix}$$

n'est pas zéro) on ne peut pas satisfaire simultanément sans poser

$$\lambda = 0, \ \mu = 0, \ \rho = 0.$$

Conséquemment nulle liaison linéaire ne peut exister entre les quatre covariants composés qu'on a formés du type 4.3.1; en sorte que ces quatre covariants étant linéairement indépendants, en dehors d'eux ne peut exister nul covariant indécomposable de ce même type: ce qui était à démontrer.

Ainsi, pour la troisième fois, l'exactitude de mon *postulatum* fondamental s'est trouvée en contradiction avec les résultats obtenus par les géomètres allemands, et pour la troisième fois elle est sortie victorieuse du conflit. C'est à la précision, qu'on ne peut trop louer, de M. Franklin comme calculateur et à sa passion pour ne laisser échapper aucune erreur, que la Science est redevable de cette troisième correction, bien remarquable et tout à fait inattendue.

Tous mes autres résultats, qui, avec ces trois exceptions, sont en conformité avec ceux de MM. Clebsch, Gordan et Gundelfinger, et y ajoutent un caractère de certitude qu'auparavant ils étaient très loin de posséder, ont été pleinement confirmés par les calculs indépendants exécutés par M. Franklin. Quelques erreurs typographiques, dont il est bon d'avertir, existent dans les Tables que j'ai publiées; elles seront corrigées dans la collection complète de Tables qui va prochainement* paraître dans l'*American Journal of Mathematics*.

[* p. 283 below.]

35.

NOTE ON AN EQUATION IN FINITE DIFFERENCES.

[*Philosophical Magazine*, VIII. (1879), pp. 120, 121.]

I GAVE* a great many years ago in this Magazine the integral of the equation in differences

$$u_x = \frac{u_{x-1}}{x} + u_{x-2},$$

which I obtained by observing that the equation could be solved by supposing each u of an odd order to be equal to the u of the order immediately superior, and also by supposing it to be equal to the u of the order immediately inferior. The upshot of the investigation expressed in the simplest language was to furnish two particular integrals of which one gives rise to the series

$$u_0 = 1,\quad u_1 = 1,\quad u_2 = \tfrac{1}{2},\quad u_3 = \tfrac{1}{2},\quad u_4 = \frac{1.3}{2.4},\quad u_5 = \frac{1.3}{2.4}\ldots,$$

the other

$$u_0 = 1,\quad u_1 = 2,\quad u_2 = 2,\quad u_3 = \frac{2.4}{1.3},\quad u_4 = \frac{2.4}{1.3},\quad u_5 = \frac{2.4.6}{1.3.5}\ldots.$$

See also Boole's *Finite Differences*, 2nd Edition (edited by Mr Moulton), p. 235.

Now let ϕ, a function of any letter t, be the generating function of u_x. Then, since

$$xu_x - (x-2)\,u_{x-2} - u_{x-1} - 2u_{x-2} = 0,$$

we shall have

$$(1-t^2)\frac{d\phi}{dt} + (-1-2t)\,\phi = C;$$

and integrating we find

$$(1-t)^{\frac{3}{2}}(1+t)^{\frac{1}{2}}\phi = C\int dt\,\sqrt{\left(\frac{1-t}{1+t}\right)},$$

or

$$\phi = C'\frac{1+t}{(1-t^2)^{\frac{3}{2}}} + C\frac{\sin^{-1}t + \sqrt{(1-t^2)}}{(1-t)^{\frac{3}{2}}(1+t)^{\frac{1}{2}}}.$$

[* Vol. II. of this Reprint, p. 690.]

$\dfrac{1+t}{(1-t^2)^{\frac{3}{2}}}$ we see at a glance gives the values of u_x corresponding to the first particular integral; and since the two first terms of the function multiplied by C are $1+2t$, it follows that this function is the generatrix of the second particular integral—in other words, that

$$\frac{\sin^{-1}t + \sqrt{(1-t^2)}}{(1-t)^{\frac{3}{2}}(1+t)^{\frac{1}{2}}} = 1 + 2t + 2t^2 + \frac{2.4}{1.3}t^3 + \frac{2.4}{1.3}t^4 + \frac{2.4.6}{1.3.5}t^5 + \ldots.$$

Hence

$$\frac{t\sin^{-1}t + \sqrt{(1-t^2)}}{(1-t^2)^{\frac{3}{2}}} = \frac{1}{1+t}\left\{t\left(\frac{\sin^{-1}t + \sqrt{(1-t^2)}}{(1-t)^{\frac{3}{2}}(1+t)^{\frac{1}{2}}}\right) + 1\right\}$$

$$= 1 + \frac{2}{1}t^2 + \frac{2.4}{1.3}t^4 + \frac{2.4.6}{1.3.5}t^6 + \ldots;$$

and integrating

$$\frac{\sin^{-1}t}{\sqrt{(1-t^2)}} = t + \frac{2}{1}\cdot\frac{t^3}{3} + \frac{2.4}{1.3}\cdot\frac{t^5}{5} + \frac{2.4.6}{1.3.5}\frac{t^7}{7} + \ldots.$$

Thus we have the remarkable identity

$$\left(1 + \frac{1}{2}\tau + \frac{1.3}{2.4}\tau^2 + \frac{1.3.5}{2.4.6}\tau^3 \ldots\right)$$

$$\times\left(1 + \frac{1}{2}\frac{\tau}{3} + \frac{1.3}{2.4}\frac{\tau^2}{5} + \frac{1.3.5}{2.4.6}\frac{\tau^3}{7}\ldots\right)$$

$$= 1 + \frac{2}{1}\frac{\tau}{3} + \frac{2.4}{1.3}\frac{\tau^2}{5} + \frac{2.4.6}{1.3.5}\frac{\tau^3}{7}\ldots.$$

I do not recollect ever having met with these remarkable series before I discovered them by the preceding method; but on showing them to Dr Story of this University, he ascertained that they had been stated not long ago by Mr Glaisher in a paper in the *Mathematical Messenger*, and made the foundation there of various summations for calculating π; but where Mr Glaisher found these series, which are not given in the ordinary books on the Calculus, or (if new) how he lighted upon them, he has not stated, and it is desirable that he should do so.

36.

NOTE ON DETERMINANTS AND DUADIC DISYNTHEMES.

[*American Journal of Mathematics*, II. (1879), pp. 89—96, 214—222.]

A GENERAL algebraical determinant in its developed form (viewed in relation to any one arbitrarily selected term) may be likened to a mixture of liquids seemingly homogeneous, but which being of differing boiling points, admit of being separated by the process of fractional distillation. Thus, for example, suppose a general determinant of the 6th order. The 720 terms which make it up will fall, in relation to the leading diagonal product, into as many classes (most of which comprise several similarly constituted families) as there are unlimited partitions of 6. These, 11 in number, are

6; 5, 1; 4, 2; 4, 1, 1; 3, 3; 3, 2, 1; 3, 1, 1, 1; 2, 2, 2; 2, 2, 1, 1; 2, 1, 1, 1, 1;
1, 1, 1, 1, 1, 1.

Let the determinant be represented, in the umbral notation, by

$$\begin{matrix} a & b & c & d & e & f \\ a & b & c & d & e & f \end{matrix}.*$$

Let us, by way of illustration, consider the class corresponding to 6; this will consist of the 1 . 2 . 3 . 4 . 5 (120) terms obtained by forming the 120 distinct circular arrangements that belong to $a\ b\ c\ d\ e\ f$. Thus:

$$\begin{array}{ccc} & \longrightarrow & \\ & a \quad c & \\ b & & e \\ & f \quad d & \\ & \longleftarrow & \end{array}$$

* The cyclical method of the text shows what was not previously apparent, that the umbral notation $\begin{matrix} ab \dots l \\ ab \dots l \end{matrix}$ possesses an essential advantage over $\begin{matrix} ab \dots l \\ \alpha\beta \dots \lambda \end{matrix}$ even for unsymmetrical determinants. This mode of notation of course implies some ground of preference for one diagonal group over all others and thus virtually regards a general determinant as related to a lineo-linear as a symmetrical one is to a quadratic form. For instance the general determinant of the second order is to be regarded as appurtenant to the lineo-linear form $aaxx' + abxy' + bayx' + bbyy'$.

will signify $ac \times ce \times ed \times df \times fb \times ba$, which will be one of the 120 in question. So, again, 3, 3 will denote, in the first place, the 10 sets of double triads of the general form $abc : def$, and, as each triad will give two cyclical orders, there will in all be 10×2^2, that is, 40, terms of the form $ab . bc . ca . de . ef . fd$. So, again, there will be $15 . 1^3$, that is, 15, corresponding to 2, 2, 2. So 3, 2, 1 will give 10 groupings of the form $abc : de : f$, and each of these will give rise to two terms, namely,

$$ab . bc . ca . de . ed . ff, \quad ac . cb . ba . de . ed . ff,$$

the number of cycles corresponding to two elements de being 1, and to one element f also 1.

This simple theory affords us a direct means of calculating the number of distinct terms in a symmetrical determinant, that is, one in which $i . j$ and $j . i$ are identical. It enables us to see at once that the coefficient of every term is unity or a power of 2; the rule being that plus or minus terms* of the class corresponding to $m_1, m_2, m_3, \dots$ will take the coefficient 2^ν, ν being the number of the quantities m which are neither 1 nor 2, for, in every other case, the total number of cycles in each partial group will arrange themselves in pairs which give the same result, thus, for example,

$$\begin{array}{ccc} & a & \\ d & & b \\ & c & \end{array} \quad \text{and} \quad \begin{array}{ccc} & a & \\ b & & d \\ & c & \end{array}$$

will give the equal products $ab . bc . cd . da$ and $ad . dc . cb . ba$.

As an example of the direct method of computation, take a symmetrical determinant of the 5th order. Write

5   4 . 1   3 . 2   3 . 1 . 1   2 . 2 . 1   2 . 1 . 1 . 1   1 . 1 . 1 . 1 . 1.

To these 7 classes there will belong respectively

1 . 12	with the	coefficient	2
5 . 3	,,	,,	2
10 . 1	,,	,,	2
10 . 1	,,	,,	2
15	,,	,,	1
10	,,	,,	1
1	,,	,,	1.

Thus the number of distinct terms will be

$$12 + 15 + 10 + 10 + 15 + 10 + 1 = 73,$$

and the sum of the coefficients

$$24 + 30 + 20 + 20 + 15 + 10 + 1 = 120,$$

both of which are right.

* The complete value of the coefficient is $(-)^\mu 2^\nu$, ν being the number of elements in the partition other than 1 or 2, and μ the number of even elements.

Again, if we have a skew determinant of an even order, it will easily be seen that any partition embracing one or more odd numbers will give rise to pairs of terms that mutually cancel, but when all the parts into which the exponent of the order is divided are even, the coefficient will be given by the same rule as for symmetrical determinants, that is, its arithmetical value will be 2^ν, where ν is the number of parts exceeding 2. Thus, for example, for a skew determinant of the order 6 we have

$$6 \quad 4.2 \quad 2.2.2.$$

The number of terms corresponding to these partitions being 60 with coefficient 2, 15×3 also with coefficient 2, and 15 with coefficient 1, making 120 distinct terms in all, the sum of the coefficients will be

$$120 + 90 + 15 = (1.3.5)^2,$$

which is right, because the result is the square of the sum of 15 synthemes of the form $1.2 \times 3.4 \times 5.6$. It may be observed that 120 is $\frac{15.16}{2}$, as it ought to be, because, until we reach the order 8, the same *double duadic syntheme* can only be made up in one way of two simple ones, but this ceases to be the case from and after 8. Thus, for example, the pair of synthemes

$$1.2 \quad 3.4 \quad 5.6 \quad 7.8 \quad \text{and} \quad 1.3 \quad 2.4 \quad 5.7 \quad 6.8$$

combined will produce the same double syntheme as the pair

$$1.2 \quad 3.4 \quad 5.7 \quad 6.8 \quad \text{and} \quad 1.3 \quad 2.4 \quad 5.6 \quad 7.8,$$

and accordingly for 8 we have the partitions

$$8 \quad 6.2 \quad 4.4 \quad 4.2.2 \quad 2.2.2.2,$$

giving rise to

2520	with	coefficient	2
28.60	„	„	2
35.3^2	„	„	4
210.3	„	„	2
105	„	„	1,

making in all $2520 + 1680 + 315 + 630 + 105$, that is, 5250, distinct terms, whereas

$$\frac{(1.3.5.7)^2 + (1.3.5.7)}{2} = 5565,$$

the difference, 315, being due to the fact that there are that number of double synthemes which admit of a twofold resolution into two single synthemes.

I will not stop to prove, but any person conversant with the subject will see at once that this method gives an intuitive and direct proof of the theorem that a pure skew determinant for an even order is a perfect square*. Having

* That a skew determinant of an odd order vanishes is apparent from the fact that an odd number cannot be made up of a set of even ones. I use the term skew determinant in its strict sense as referring to a matrix for which $ij = -ji$ and $ii = 0$.

only a limited space at my command, I will pass on at once to forming the equation in differences for the case of a symmetrical, a skew, and one or two other special forms of determinants.

For a symmetrical determinant, taking as a diagram, to fix the ideas, the matrix of the 6th order

$$\begin{matrix} a & b & c & d & e & f \\ b & g & h & k & l & m \\ c & h & n & p & q & r \\ d & k & p & s & t & u \\ e & l & q & t & v & w \\ f & m & r & u & w & \omega \end{matrix},$$

calling u_m the number of distinct terms in a symmetrical matrix of the mth order, and, resolving the entire determinant into a sum of determinants of the order $(m-1)$ multiplied by the letters in the top line, we shall obviously get u_{m-1} together with $(m-1)$ quantities, positive or negative (and we know, by what precedes, that there can be no cancelling, so that the sign, for the object in view, may be entirely neglected) of the form

$$b \times \begin{matrix} b & h & k & l & m \\ c & n & p & q & r \\ d & p & s & t & u \\ e & q & t & v & w \\ f & r & u & w & \omega \end{matrix}.$$

Among these $(m-1)$ quantities all the terms containing bc, bd, be, bf will occur twice over, but those containing b^2 do not recur. Hence, to find the number of distinct terms we may reckon each of such distinct terms as contain bc, bd, be, bf worth only $\frac{1}{2}$, the others counting as 1. But if, instead of the column (which I write as a line) *bcdef*, we had the column *bhklm*, the rule for calculating the number of distinct terms might be calculated by this very same rule, except that the terms multiplied by hc, kd, le, mf ought to count as *units* instead of *halves*. Hence obviously

$$u_m + (m-1)(m-2)\, u_{m-3} \times \tfrac{1}{2} = u_{m-1} + (m-1)\, u_{m-1} = m u_{m-1},$$

or

$$u_m = m u_{m-1} - \tfrac{1}{2}(m-1)(m-2)\, u_{m-3},$$

which is Mr Cayley's equation, but obtained by a much more expeditious process (see Salmon's *Higher Algebra*, 3rd edition, pp. 40—42); writing $u_m = (1.2 \ldots m)\, v_m$ we obtain the equation in differences, linear in regard to the independent variable,

$$m v_m - m v_{m-1} + \tfrac{1}{2} v_{m-3} = 0,$$

and this, treated by the general method applicable to all such, gives rise to a linear differential equation in which, on account of the particular initial values of u_0, u_1, u_2, the third term is wanting, and finally v_m is found to be the coefficient of t^m in

$$\frac{e^{\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{(1-t)}}.$$

If we apply a similar method to the case of a symmetrical determinant in which the diagonal of symmetry is filled out with zeros (an invertebrate symmetrical or symmetrical bialar determinant, as we may call it) we shall easily obtain the equation in differences

$$u_m = (m-1)[u_{m-1} + u_{m-2}] - \tfrac{1}{2}(m-1)(m-2)\,u_{m-3},$$

and, making $u_m = 1.2 \dots m v_m$,

$$mv_m - (m-1)\,v_{m-1} - v_{m-2} + \tfrac{1}{2}v_{m-3} = 0,$$

from which, calling $y = v_0 + v_1 t + v_2 t^2 + \dots$ and having regard to the initial values v_0, v_1, v_2, we obtain

$$2\frac{dy}{y} = \frac{2t-t^2}{1-t}\,dt,$$

and

$$y = \frac{e^{-\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{(1-t)}}.$$

By way of distinction, using u' and v' for this case, and u, v for the preceding one, the slightest consideration shows that

$$u_m = u'_m + mu'_{m-1} + \frac{m(m-1)}{2}u'_{m-2} + \frac{m(m-1)(m-2)}{2.3}u'_{m-3} + \dots,$$

or

$$v_m = v'_m + v'_{m-1} + \frac{v'_{m-2}}{1.2} + \frac{v'_{m-3}}{1.2.3} + \dots.$$

Hence the generating function for v_m ought to be that for u_m multiplied by e^t, as we see is the case.

So, in like manner, the generating function for v_m, that is, $\frac{u_m}{1.2\dots m}$, in the case of a general determinant being $\frac{1}{1-t}$, that of v_m for an invertebrate or zero-axial but otherwise general determinant we see must be $\frac{e^{-t}}{1-t}$*, that is,

$$v_m = 1 - 1 + \frac{1}{1.2} - \frac{1}{1.2.3} + \dots \pm \frac{1}{1.2\dots m},$$

* It may easily be proved that the difference between the numbers of positive and negative combinations in the development of an invertebrate determinant of the mth order is $(-)^{m-1}(m-1)$ in favour of the former. From this it is easy to prove that the generating function for $\frac{\text{number of positive terms in such determinant}}{1.2.3\dots m}$ is

$$\frac{1}{2}\left\{\frac{e^{-t}}{1-t} - (1+t)e^{-t}\right\}, \text{ or } \frac{t^2e^{-t}}{2(1-t)}.$$

the well known value $\left(\text{ultimately equal to } \frac{1}{e}\right)$, as it ought obviously to be, of the chance of two cards of the same name not coming together when one pack of m distinct cards is laid card for card under another precisely similar pack.

Returning to the case of the invertebrate symmetrical determinant, it will readily be seen, by virtue of the prolegomena, that the number of terms (the u_m) for such a determinant of the mth order is the same thing as the total number of duadic disynthemes that can be formed with m things, meaning by a duadic disyntheme any combination of duads with or without repetition, in which each element occurs twice and no oftener. Thus, when $m=6$, 1.2 2.3 1.3 4.5 4.6 5.6 and 1.2 2.3 3.4 5.6 6.1 and 1.2 2.3 3.4 1.4 5.6 5.6 are all three of them disynthemes. But the two latter ones are each resoluble into single synthemes, whereas the first one is not. It is clear that, when a disyntheme is formed by means of cycles all of an even order, it will be resoluble into a pair of single synthemes, and in no other case. The problem, then, of finding the number of distinct double synthemes with m elements is one and the same as that of finding the number of distinct terms in a *proper* (that is, invertebrate) skew determinant, which I proceed to consider.

Following a method (not identical with but) analogous to that adopted for the symmetrical cases, we shall find, by a process which the terms below written will sufficiently suggest

$$u_m + \frac{(m-1)(m-2)(m-3)}{2}u_{m-4} = (m-1)\,u_{m-2} + (m-1)(m-2)\,u_{m-2},$$

or

$$u_m = (m-1)^2 u_{m-2} - \frac{(m-1)(m-2)(m-3)}{2}u_{m-4}.$$

Of course, when m is odd $u_m = 0$. From this it is readily seen that $\frac{u_{2m}}{1.3.5\dots.2m-1}$, say ω_m, is an integer; for we shall have

$$\omega_m = (2m-1)\,\omega_{m-1} - (m-1)\,\omega_{m-2},$$

also,

$$\omega_1 = 1, \quad \omega_2 = 2,$$

Whence it follows that the number of positive terms in a general invertebrate determinant of the mth order is $m\frac{m-1}{2}$ times the total number of the terms in one of the $(m-2)$th order. The equation of differences for U_m, the total number, is of course

$$U_m = (m-1)(U_{m-1} + U_{m-2}),$$

and the successive values of

U_m for 1, 2, 3, 4, 5, 6, 7, 8, ...,
are 0, 1, 2, 9, 44, 265, 1854, 14833, ...

so that

$$\begin{aligned} \omega_3 &= 5\,.\,2 - 2\,.\,1 = 8, \\ \omega_4 &= 7\,.\,8 - 3\,.\,2 = 50, \\ \omega_5 &= 9\,.\,50 - 4\,.\,8 = 418, \\ \omega_6 &= 11\,.\,418 - 5\,.\,50 = 4348, \end{aligned}$$

and the conventional $\omega_0 = 3\omega_1 - \omega_2 = 1$.

By the above formula u_m can be calculated with prodigious rapidity. If, however, we wish to obtain a generating function for u_m, the differential equation obtained from the above equation in differences does not lead to a simple explicit integral, but if we make $u_{2m} = (1\,.\,2\,.\,3 \ldots 2m)\,v_m$, as in the preceding cases, or, which is the same thing, $\omega_m = 2^m\,(1\,.\,2 \ldots m)\,v_m$, we get

$$4mv_m - 4\,(m-1)\,v_{m-1} - 2v_{m-1} + v_{m-2} = 0,$$

and, writing as before $\quad y = v_0 + v_1 t + v_2 t^2 + \ldots,$

$$4\frac{dy}{dt} - 4t\frac{dy}{dt} - 2y + ty$$

will be found to be equal to zero. [This vanishing of the 3rd term in the differential equation being a feature common to all the cases we have considered, and due to the initial values of the v series in each case.] We have thus

$$\frac{4y'}{y} = \frac{1}{1-t} + 1, \quad y = \frac{e^{\frac{t}{4}}}{(1-t)^{\frac{1}{4}}}.$$

By way of verification, we may observe that

$$v_0 = 1, \quad v_1 = \frac{1}{2}, \quad v_2 = \frac{1}{4}, \quad v_3 = \frac{1}{6}, \ldots,$$

$$y = \left(1 + \frac{t}{4} + \frac{t^2}{32} + \frac{t^3}{384} + \ldots\right)\left(1 + \frac{t}{4} + \frac{5t^2}{32} + \frac{45t^3}{384} + \ldots\right),$$

and $\quad \frac{1}{4} + \frac{1}{4} = \frac{1}{2}, \quad \frac{1}{32} + \frac{1}{16} + \frac{5}{32} = \frac{1}{4}, \quad \frac{45}{384} + \frac{5}{128} + \frac{1}{128} + \frac{1}{384} = \frac{1}{6}$*.

We may now proceed to calculate the number of distinct terms in an improper or vertebrated skew-determinant, which is interesting on account of its connection with the theory of orthogonal transformations. Using v_{2m}, instead of v_m, the generating function for the case last considered becomes $\dfrac{e^{\frac{t^2}{4}}}{\sqrt[4]{(1-t^2)}}$. Let $(1\,.\,2\,.\,3 \ldots m)\,V_m = U_m$ in general be used to denote the number of distinct terms in a vertebrate skew-determinant of the mth order. Then obviously

$$U_{2m} = u_{2m} + m\,.\,\frac{m-1}{2}\,u_{2m-2} + m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}\,.\,\frac{m-3}{4}\,u_{2m-4} + \ldots,$$

* The values of $v_1, v_2, v_3 \ldots$ are $\frac{1}{2}, \frac{2}{2\,.\,4}, \frac{8}{2\,.\,4\,.\,6}, \frac{50}{2\,.\,4\,.\,6\,.\,8}, \ldots$; that is, $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{25}{192}, \ldots$

or $$V_{2m} = v_{2m} + \frac{v_{2m-2}}{1\,.\,2} + \frac{v_{2m-4}}{1\,.\,2\,.\,3\,.\,4} + \ldots.$$

Hence the generating function for V_{2m}

$$= \frac{e^{\frac{t^2}{4}}}{(1-t^2)^{\frac{1}{4}}}\left\{1 + \frac{t^2}{1\,.\,2} + \frac{t^4}{1\,.\,2\,.\,3\,.\,4} + \ldots\right\} = \frac{1}{2}\left\{\frac{e^{t+\frac{t^2}{4}} + e^{-t+\frac{t^2}{4}}}{(1-t^2)^{\frac{1}{4}}}\right\},$$

and in like manner, since

$$U_{2m-1} = mu_{2m-2} + \frac{m\,(m-1)\,(m-2)}{1\,.\,2\,.\,3}\,u_{2m-4} + \ldots,$$

the generating function for V_{2m-1} will be

$$\frac{1}{2}\left\{\frac{e^{t+\frac{t^2}{4}} - e^{-t+\frac{t^2}{4}}}{(1-t^2)^{\frac{1}{4}}}\right\}.$$

Hence the number of distinct cross-products in the development of an orthogonal transformation-matrix of the mth order is

$$(1\,.\,2\,.\,3 \ldots m) \times \text{coefficient of } t^m \text{ in } \frac{e^{t+\frac{t^2}{4}}}{(1-t^2)^{\frac{1}{4}}}.$$

Postscript.—Let us consider the case of $2m$ elements; call the number of ways in which any disyntheme composed with them may be resolved into a pair of single synthemes one in each hand* its weight; furthermore, call the aggregate of those which appertain to an odd number of cycles the first class, and the other the second class. The entire sum of the weights we know is $1^2\,.\,2^2\,.\,3^2 \ldots (2m-1)^2$, but, furthermore, I find that the excess of the total weight of the first class over that of the second is

$$1^2\,.\,2^2\,.\,3^2 \ldots (2m-3)^2\,(2m-1);$$

or, in other words, the weights of the two classes are in the ratio of m to $m-1$.

The expressions for the sum and for the difference may, of course, by the *prolegomena* be translated into two theorems on the partition of numbers, neither of which, as far as I can see, is obvious upon the face of it†.

* The two hands are introduced in order to double, by the effect of permutation, what the weight otherwise would be, except when the two component synthemes are identical, in which case the weight remains unity.

† Remark [by F. Franklin].—The equation in differences for the number of double duadic synthemes may be obtained without recourse to determinants, as follows: Single out any element, 1; it may be paired in each of the component synthemes with any one of the remaining elements 2, 3, 4, ..., and there are two cases to be distinguished, namely, 1 may be paired either with the same element (2) or with two different elements (2, 3), in the two synthemes. The former may be done in $(m-1)$ ways, and, after having made our choice, we have still the choice of all the double synthemes that can be formed from 3, 4, ... m; 3, 4, ... m. The choice of two *different* elements may be made in $\frac{(m-1)(m-2)}{2}$ ways, and having chosen, we have still the choice of all the double

The properties of the ω series 1, 1, 2, 8, 50, ... [see p. 269] present some features of interest. These are the numbers of distinct terms in pure skew determinants of the order $2n$ divided by the product of the odd integers inferior to $2n$. Such numbers themselves may be termed the denumerants, and the quotients, when they are so divided, the reduced denumerants of the corresponding determinants; or for greater brevity we may provisionally call these reduced denumerants *skew numbers*. We have found, in what precedes, that

$$\frac{e^{\frac{t}{4}}}{\sqrt[4]{(1-t)}} = \omega_0 + \omega_1 \frac{t}{2} + \omega_2 \frac{t^2}{2.4} + \omega_3 \frac{t^3}{2.4.6} + \dots.$$

From this we may easily obtain

$$\omega_x = \frac{Fx}{2^x},$$

where

$$Fx = 1 + 1.x + 1.5\frac{x(x-1)}{1.2} + 1.5.9\frac{x(x-1)(x-2)}{1.2.3} + \dots$$
$$+ \dots + 1.5.9\dots(4x-3),$$

which shows that Fx, for all values of x, contains 2^x as a factor, and that if we take x greater than unity, 2^{x+1} will be a factor of Fx. In general, it follows from the fundamental equation $\omega_x = (2x-1)\,\omega_{x-1} - (x-1)\,\omega_{x-2}$ that if two consecutive skew numbers ω_c, ω_{c+1} have a common factor, all those of superior orders, and consequently $\frac{Fx}{2^x}$, for all values of x from c upwards, will contain such factor. It becomes then a matter of interest to assign, if possible, a general expression for the greatest common measure of ω_x, ω_{x+1}.

In the first place I say these can have no common odd factor other than unity.

Lemma. It is well known that, in the development of $(1+a)^x$, all the coefficients except the first and last will contain x when it is a prime number. More generally it may easily be shown (and the mode of proof* is too obvious to need setting out) that whatever x may be, any prime number contained in it must either divide any number r, or else the coefficient of a^r in the binomial expression above referred to. Hence we may prove that ω_x and x cannot have a common odd factor other than unity. For if possible, let $x = qp$, where p is a prime number contained in ω_x. Let the qp terms in Fx subsequent to the first term be divided into q groups, each containing p terms. Each of the terms in any one group (except the last) contains a binomial coefficient, which, by virtue of the lemma, will contain p. Moreover, the last term in the kth group will contain the factor

$$1.5.9\dots(4kp-1).$$

If p is of the form $4n-3$, the nth term of the series 1, 5, 9, ... will be p, and if it is of the form $4n-1$, the $(3n)$th term will be $3p$; and as $\frac{p+3}{4}$ and $3\frac{p+1}{4}$ are each not greater than p (and *à fortiori* not greater than kp) when p is greater than 1, it follows that the last coefficient, as well as all the others in any group, contains p. Hence $Fx = pP + 1$, and therefore ω_x, that is, $\frac{Fx}{2^x}$, cannot contain p. Hence the greatest common measure of ω_x and ω_{x+1} is a power of 2.

It will presently be shown by induction (waiting a strict proof)* that $\frac{\omega_{4x-2}}{2^x}$, $\frac{\omega_{4x-1}}{2^x}$, $\frac{\omega_{4x}}{2^x}$, $\frac{\omega_{4x+1}}{2^x}$ are all of them integers, and the first, third and fourth, odd integers; from this it will easily be seen that the greatest common measure of ω_x, ω_{x+1} is $2^{\theta\left(\frac{2x+1}{8}\right)}$, where, in general, $\theta(\mu)$ means the integer nearest to μ. Let us call the above fractions q_{4x-2}, q_{4x-1}, q_{4x}, q_{4x+1}, to which we may give the name of simplified skew numbers. In the subjoined table I have calculated the values of the residues of these numbers by a regular algorithm in respect to *moduli* beginning with 2^{22} and regularly decreasing according to the descending powers of 2. R stands for the words *residue of*.

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synthemes that can be formed from 3, 4, ... m; 3, 4, ... m. Now it is plain that the number of these can be obtained from the number of double synthemes that can be formed from 3, 4, ... m; 3, 4, ... m, by counting twice all except those in which 3 is paired twice with the same element; and is equal, therefore, from what precedes, to

$$2u_{m-2} - (m-3)\,u_{m-4}.$$

We have, therefore,

$$u_m = (m-1)\,u_{m-2} + \frac{(m-1)(m-2)}{2}\left[2u_{m-2} - (m-3)\,u_{m-4}\right]$$
$$= (m-1)^2 u_{m-2} - \frac{(m-1)(m-2)(m-3)}{2}\,u_{m-4}.$$

* Some of the prolixity of the more obvious mode of proof of this lemma may be avoided by the substitution of the following method:

Call $(1+t)^n = 1 + A_1 t + A_2 t^2 + A_3 t^3 + \dots$, so that

$$n(1+t)^{n-1} = A_1 + 2A_2 t + 3A_3 t^2 + \dots$$
$$= B_0 + B_1 t + B_2 t^2 + \dots = \phi t.$$

Suppose $n = qp$: then designating the qth roots of unity by $\rho_1, \rho_2 \dots \rho_q$, we have

$$\frac{1}{q}\Sigma \rho^{q-k}\phi(\rho t) = B_k t^k + B_{k+q} t^{k+q} + B_{k+2q} t^{k+2q} + \dots + B_{k+(p-1)q} t^{k+(p-1)q},$$

and the left hand side of the equation is obviously a multiple of p. Hence, putting t successively equal to 0, 1, 2, 3, ... $(p-1)$, we obtain, by a well-known theorem of determinants,

$$\Delta B_{k+\lambda q} \equiv 0 \ [\text{mod } p],$$

where Δ, being the product of the differences of 0, 1, 2, ... $(p-1)$, cannot contain p. Hence $B_{k+\lambda q} \equiv 0$ [mod p], and consequently giving k all values from 0 to $(q-1)$, and λ all values from 0 to $(p-1)$, we see that all the B's, from B_0 to B_{pq-1}, must contain p as a factor as was to be proved.

* Since the above was set up in print, I have found an easy proof, for which see *Postscript* [p. 279 below].

Modulus	x	Rq_{4x-2}	Rq_{4x-1}	Rq_{4x}	Rq_{4x+1}
8,388,608	0			1	1
4,194,304	1	1	4	25	209
2,097,152	2	1,087	13,504	194,951	1,088,983
1,048,576	3	929,451	442,068	992,179	576,715
524,288	4	287,913	118,168	393,089	71,201
262,144	5	201,913	14,228	126,417	179,945
131,072	6	51,071	56,656	46,407	127,767
65,536	7	56,531	24,452	15,131	46,739
32,768	8	12,521	29,928	22,753	29,729
16,384	9	14,289	5,412	15,209	14,305
8,192	10	1,119	2,784	4,063	4,751
4,096	11	3,283	3,156	2,331	3,059
2,048	12	1,721	1,632	425	1,801
1,024	13	913	84	1,001	385
512	14	215	240	479	239
256	15	91	132	99	219
128	16	81	8	9	9
64	17	41	36	1	57
32	18	23	0	31	15
16	19	3	4	11	3
8	20	1	0	1	1
4	21	1	0	1	1
2	22	1	0	1	1

From this table it appears that q_{8i-5} is 4 times an odd number, and that q_{8i-1} is 8 times a number which may be odd or even; thus we know the exact number of times that 2 will divide out all the skew numbers other than those whose orders are of the form $8i-1$, and an inferior limit to that number for that case.

It will further be noticed that, when x is of the form $4i$, or $4i+1$, the simplified skew numbers q_{4x-2}, q_{4x}, q_{4x+1} are all of the form $8\lambda+1$, that when x is of the form $4i+2$ the above named simplified skew numbers are of the form $8\lambda+7$, and when x is of the form $4i+3$, they are of the form $8\lambda+3$.

Before quitting this subject, I think it desirable briefly to refer to other series of integers closely connected with those which I have called *skew numbers*. To this end we may write, in general,

$$e^{\frac{t}{4}}(1-t)^{\frac{4\mu-1}{4}} = 1 + \omega_{1,\mu}\frac{t}{2} + \omega_{2,\mu}\frac{t^2}{2\,.\,4} + \omega_{3,\mu}\frac{t^3}{2\,.\,4\,.\,6} + \dots,$$

μ being any positive or negative integer, so that $\omega_{x,0}$ is the same as I have called hitherto ω_x. It may then easily be shown that $\omega_{x,\mu+1} = \dfrac{2\omega_{x+1,\mu} - \omega_{x,\mu}}{4\mu+1}$, that $\omega_{x,\mu-1} = \omega_{x,\mu} - 2x\omega_{x-1,\mu}$, and that the equation in differences for $\omega_{x,\mu}$, for μ constant, becomes

$$\omega_{x,\mu} = (2x+2\mu-1)\,\omega_{x-1,\mu} - (x-1)\,\omega_{x-2,\mu},$$

with the initial conditions $\omega_{0,\mu}=1$, $\omega_{1,\mu}=2\mu+1$. Also, it is clear from the definition, that the explicit value of $\omega_{x,\mu}$ in a series becomes

$$\frac{1}{2^x}\left\{1 + (4\mu+1)x + (4\mu+1)(4\mu+5)\,x\frac{x-1}{2}\right.$$
$$\left. + (4\mu+1)(4\mu+5)(4\mu+9)\,x\frac{x-1}{2}\,.\,\frac{x-2}{3} + \dots\right\},$$

which is easily seen to verify the equation

$$2\omega_{x,\mu} - \omega_{x-1,\mu} = (4\mu+1)\,\omega_{x-1,\mu+1}.^*$$

We might call the $\omega_{x,\mu}$ series skew numbers of the μth degree, and, as for the case of $\mu=0$, so it may be shown in general that two consecutive skew numbers of the same degree can have no common odd factor. Also, it remains true that the greatest common factor of any two consecutive skew numbers of the same degree and the orders x, $x+1$, is $2^{\theta\left(\frac{2x+1}{8}\right)}$; $\omega_{4x-2,\mu}$, $\omega_{4x-1,\mu}$, $\omega_{4x,\mu}$, $\omega_{4x+1,\mu}$ being all divisible by 2^x, and the resulting quotients being, the first, third and fourth of them, always odd integers, and the second divisible by 4 or some higher power of 2 when μ is even, but only by the first power of 2 when μ is odd. But it would carry me too far away from the original object of this note, and from other investigations of more pressing moment to myself, to pursue further the theory of general skew numbers, which, however, seems to me to be well worthy of the study of arithmeticians.

I will only stop to point out that the rule for the greatest common measure of ω_x and ω_{x+1}, serves to prove the rule for the general case of $\omega_{x,\mu}$ and $\omega_{x+1,\mu}$. Thus suppose μ to be positive. Then since $\omega_{k,1} = 2\omega_{k+1} - \omega_k$, and $\omega_{4k-2} = 2^k(2\lambda+1)$, $\omega_{4k-1} = 2^{k+1}\tau$, $\omega_{4k} = 2^k(2\nu+1)$, $\omega_{4k+1} = 2^k(2\pi+1)$, $\omega_{4k+2} = 2^{k+1}(2\rho+1)$; it follows that

$$\omega_{4k-2,1} = 2^k(2\lambda'+1),\ \omega_{4k-1,1} = 2^{k+1}\tau',\ \omega_{4k,1} = 2^k(2\nu'+1), \text{ and } \omega_{4k+1,1} = 2^k(2\pi'+1).$$

It is obvious further that, τ being even, τ' is odd. So again from these results we may, in like manner, deduce

$$\omega_{4k-2,2} = 2^k(2\lambda''+1),\ \omega_{4k-1,2} = 2^{k+1}\tau'',\ \omega_{4k,2} = 2^k(2\nu''+1),\ \omega_{4k+1,2} = 2^k(2\pi''+1),$$

* And of course, in general, the equation

$$\lambda u_{x,y} - u_{x-1,y} + \phi y u_{x-1,y+\delta} = 0,$$

with the condition that $u_{0,y}$ is constant, has for its integral

$$u_{x,y} = \frac{c}{\lambda^x}\left\{1 - \phi y x + \phi y \phi(y+\delta)\,x\frac{x-1}{2} - \phi y \phi(y+\delta)\,\phi(y+2\delta)\,x\frac{x-1}{2}\,\frac{x-2}{3} + \dots\right\}.$$

subject also to the remark that, τ' being odd, τ'' is even, and so on continually, τ being alternately even and odd. Again if μ is negative, we may, in like manner, by means of the formula $\omega_{k,\mu-1} = \omega_{k,\mu} - 2k\omega_{k-1,\mu}$, pass successively from the case of ω_k to that of $\omega_{k,-1} : \omega_{k,-2} : \dots \omega_{k,-\mu}$, and establish precisely the same conclusion in regard to powers of 2 as for the case of μ positive, and it will be remembered that I have already shown how to establish that $\omega_{k,\mu}$ and $\omega_{k+1,\mu}$ have no common *odd* factor.

In the first note on this subject (Vol. II, No. 1, of the *Journal**) I showed how a general determinant could be completely represented by means of systems of cycles and that accordingly the terms in the total development would split up into families, as many in number as there are indefinite partitions of the index of the order of the determinant—the particular mode of aggregation depending upon the term chosen to represent the product of the elements in the principal diagonal, so that for the order n there would be $1.2.3 \dots n$ distinct modes of distribution into families. This gives rise to a theory of transformation of cycles, corresponding to a transposition of the rows or columns of the matrix. Thus, for example, suppose the *umbrae* to be $1, 2, 3, \dots n : r, s$ signifying the element in the rth row and sth column. Then if we interchange the mth and nth columns, this will have the effect of changing pm into pn and pn into pm.

Suppose now that a term of the developed determinant is expressed by a system of cycles such that m and n lie in two distinct cycles, say Xm and nY, where X, Y are each of them single elements, or aggregates of single elements; then the effect of the interchange will be to bring these cycles into the single cycle $XnYm$. If Xm, nY were both odd ordered or both even ordered cycles, their sum will be even ordered, and the number of *even* cycles will be increased or diminished by unity; so if one was of odd and the other of even order, their sum will be of odd order, and the number of even cycles will be diminished by unity. In either case, therefore, the sign, which depends on the *parity* of the number of even cycles, is reversed.

Again, suppose m and n to lie in the same cycle $mXnY$. Then the effect of the interchange will be to break this up into two cycles mX, nY, and for the same reason as above the sign will be reversed. Thus the sign of every term in the development will, we see, be reversed, as we know *à priori* ought to be the case.

[* p. 264 above.]

I shall conclude with applying the formula $\omega_x = \dfrac{Fx}{2^x}$ to determining the *asymptotic* mean value of the coefficients in a skew determinant of the order $2x$, that is, the function of x to which the mean value of the coefficients converges when x is taken indefinitely great. We know that all the coefficients, both in this case and in that of a symmetrical determinant, are different powers of 2; to find the mean of the indices of these powers would be seemingly an investigation of considerable difficulty, but there will be little or none in finding the ultimate expression for the mean of the coefficients themselves, or, which is the same thing, the first term in the function which expresses this mean in terms of descending powers of x. We shall find that, for symmetrical determinants, this is a certain multiple of the square root and, for skew determinants, of the fourth root of x, as I proceed to show.

From the equation

$$2^x \omega_x = 1 + x + 5x\frac{x-1}{2} + \dots + \{1.5 \dots (4x-3)\},$$

we have, when $x = \infty$,

$$2^x \omega_x = 1.5.9 \dots (4x-3)\left\{1 + \frac{x}{4x-3} + \frac{1}{2}\frac{x(x-1)}{(4x-3)(4x-7)} + \dots\right\}$$
$$= e^{\frac{1}{4}}.1.5.9 \dots (4x-3).$$

The number of terms in the Pfaffian (the square root of the determinant taken with suitable algebraical sign) being $1.3.5 \dots (2x-1)$ and—as follows from what was shown in the first note—cancelling being out of the question, the sum of the coefficients all taken positively in the determinant itself will be $\{1.3.5 \dots (2x-1)\}^2$. Hence the mean value required is $\{1.3.5 \dots (2x-1)\}^2$ divided by $1.3.5 \dots (2x-1)\,\omega_x$, to express which quotient in exact terms we may make use of the formula

$$\frac{a(a+\delta)(a+2\delta)\dots(a+x\delta)}{b(b+\delta)(b+2\delta)\dots(b+x\delta)} = \frac{\Gamma\frac{b}{\delta}}{\Gamma\frac{a}{\delta}}\,x^{\frac{a-b}{\delta}}.$$

For the mean value is

$$\frac{1}{e^{\frac{1}{4}}}.\frac{1.3.5\dots(2x-1)}{2.4.6\dots(2x)}.\frac{4.8.12\dots(4x)}{1.5.9\dots(4x-3)} = \frac{1}{e^{\frac{1}{4}}}.\frac{1}{\Gamma\frac{1}{2}}\,x^{-\frac{1}{2}}.\Gamma\tfrac{1}{4}\,x^{\frac{3}{4}} = \frac{\Gamma\frac{1}{4}}{e^{\frac{1}{4}}\sqrt{\pi}}\,x^{\frac{1}{4}}.$$

If we write this under the form $Qx^{\frac{1}{4}}$, we have

$$Q = \frac{\Gamma\frac{1}{4}}{e^{\frac{1}{4}}\Gamma\frac{1}{2}},$$

$$\log Q = \log \Gamma \tfrac{5}{4} + \log 2 - \log \Gamma \tfrac{3}{2} - \tfrac{1}{4} \log e$$
$$= 9{\cdot}9573211 + {\cdot}3010300 - 9{\cdot}9475449 - {\cdot}1085736$$
$$= {\cdot}2022326,$$

or $$Q = 1{\cdot}59306.$$

This result as may easily be seen remains unaffected when, instead of a pure skew determinant, one is taken in which the diagonal terms retain general values. The effect of this change will be to increase the numerator and denominator of the fraction which expresses the mean value, in the proportion of $\frac{e^2+1}{2e}$ to 1.

Finally, as regards the ultimate mean value of the coefficients of symmetrical determinants. This, for one of the order x, by virtue of Professor Cayley's formula previously given, will be the reciprocal of the coefficient of t^x in $\frac{e^{\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{(1-t)}}$. It may readily be shown in general that, ϕt being any series of integer powers of t, the coefficient of t^x (when x becomes infinite) in $\frac{e^{\phi t}}{\sqrt{(1-t)}}$ is in a ratio of equality to the coefficient of t^x in $\frac{e^{(\phi 1)t}}{\sqrt{(1-t)}}$, so that in the present case this coefficient is the same as the coefficient of t^x in $\frac{e^{\frac{3}{4}t}}{\sqrt{(1-t)}}$, that is, in

$$\left\{1 + \frac{1}{2}t + \frac{1\,.\,3}{2\,.\,4}t^2 + \ldots + \frac{1\,.\,3\,.\,5\ldots(2x-1)}{2\,.\,4\,.\,6\ldots 2x}t^x + \ldots\right\}$$
$$\times\left\{1 + \frac{3}{4}t + \left(\frac{3}{4}\right)^2\frac{t^2}{2} + \ldots + \left(\frac{3}{4}\right)^x\frac{t^x}{1\,.\,2\ldots x} + \ldots\right\},$$

which is obviously, when x is infinite, equal to $\frac{1\,.\,3\,.\,5\ldots(2x-1)}{2\,.\,4\,.\,6\ldots 2x}e^{\frac{3}{4}}$. Hence the ultimate mean value of the coefficients is $\frac{1}{e^{\frac{3}{4}}}\frac{2\,.\,4\,.\,6\ldots 2x}{1\,.\,3\,.\,5\ldots(2x-1)}$, or $\frac{\pi^{\frac{1}{2}}}{e^{\frac{3}{4}}}\sqrt{x}$.

For a symmetrical determinant in which all the diagonal terms are wanting, the numerator of the fraction giving the mean value becomes $e^{-1}(1\,.\,2\,.\,3\ldots x)$ and the denominator is $(1\,.\,2\,.\,3\ldots x)$ into the coefficient of t^x in $\frac{e^{-\frac{t}{2}+\frac{t^2}{4}}}{\sqrt{(1-t)}}$, which is the same as in $\frac{e^{-\frac{1}{4}t}}{\sqrt{(1-t)}}$. The result then is $\frac{\pi^{\frac{1}{2}}e^{\frac{1}{4}}}{e}\sqrt{x}$, or $\frac{\pi^{\frac{1}{2}}}{e^{\frac{3}{4}}}\sqrt{x}$ as before. It may perhaps be just worth while to notice that the *skew numbers* (the ω's of the text) may be put under the form of a determinant, the nature of which is sufficiently indicated by the annexed diagram.

1	1	0	0	0	0	0
1	3	2	0	0	0	0
0	1	5	3	0	0	0
0	0	1	7	4	0	0
0	0	0	1	9	5	0
0	0	0	0	1	11	6
0	0	0	0	0	1	13

The successive principal minors in this matrix represent the successive skew numbers of all orders from 1 to 6 inclusive.

Postscript. [See p. 273 above, footnote.]

Since $\omega_{x+1} = (2x+1)\,\omega_x - x\omega_{x-1}$, we have

$$\omega_{x+2} = (4x^2 + 7x + 2)\,\omega_x - (2x^2 + 3x)\,\omega_{x-1},$$
$$\omega_{x+3} = (8x^3 + 32x^2 + 34x + 8)\,\omega_x - (4x^3 + 15x^2 + 13x)\,\omega_{x-1},$$
$$\omega_{x+4} = (16x^4 + 116x^3 + 273x^2 + 231x + 50)\,\omega_x - (8x^4 + 56x^3 + 122x^2 + 82x)\,\omega_{x-1}.$$

Suppose now that, for a given value of i, $q_{4i-2} = \frac{\omega_{4i-2}}{2^i} = 2\lambda + 1$, $q_{4i-1} = \frac{\omega_{4i-1}}{2^i} = 4\mu$, $q_{4i} = \frac{\omega_{4i}}{2^i} = 2\nu + 1$ and $q_{4i+1} = \frac{\omega_{4i+1}}{2^i} = 2\rho + 1$. Call $\omega_{x+4} = E_x\omega_x - F_x\omega_{x-1}$. Then when $x \equiv \pm 2$, $F_x \equiv 4$ [mod 8], and therefore, assuming that $q_{4i-3} = \frac{\omega_{4i-3}}{2^{i-1}}$ is odd, $\frac{F_{4i-2}\omega_{4i-3}}{2^{i+1}}$ is odd. Also, $E_{4i-2} \equiv 462 + 50 \equiv 0$ [mod 4], and consequently $\frac{E_{4i-2}\omega_{4i-2}}{2^{i+1}}$ is even; hence $q_{4i+2} = \frac{\omega_{4i+2}}{2^{i+1}}$ is integer and odd. Again when $x = 4i - 1$, $E_x \equiv 1 - 3 + 50 \equiv 0$ [mod 4], and $F_x \equiv 122 - 82 \equiv 0$ [mod 8]; hence $q_{4i+3} = \frac{\omega_{4i+3}}{2^{i+1}}$ is an integer divisible by 4. Again, when $x = 4i$, $E_{4i} \equiv 2$ and $F_{4i} \equiv 0$ [mod 4]; hence $q_{4i+4} = \frac{\omega_{4i+4}}{2^{i+1}}$ is integer and odd; and when $x = 4i + 1$, $E_{4i+1} \equiv 2$ and $F_{4i+1} \equiv 0$ [mod 4]; hence $q_{4i+5} = \frac{\omega_{4i+5}}{2^{i+1}}$ is integer and odd.

Thus it has been shown that if it be true up to $\lambda = i$ that $\frac{\omega_{4\lambda-2}}{2^\lambda}$, $\frac{\omega_{4\lambda-1}}{2^{\lambda+2}}$, $\frac{\omega_{4\lambda}}{2^\lambda}$, $\frac{\omega_{4\lambda+1}}{2^\lambda}$ are all integer, and the first, third and fourth odd integers, the same proposition can be affirmed for all superior values of i, and being true for $\omega_0, \omega_1, \omega_2, \omega_3$, the quotients corresponding to which are 1, 1, 1, 1, the theorem is true universally. It is inconceivable that it could have occurred to any human being to lay down so singular a train of induction as the one above employed, unless previously prompted to do so by an *à priori* perception of the law to be established, acquired through a preliminary study and direct inspection of the earlier terms in the series of numbers to which it applies. Here then we have a salient example (if any were needed) of the importance of the part played by the *faculty of observation* in the discovery and establishment of pure mathematical laws.

37.

ON THE COMPLETE SYSTEM OF THE "GRUNDFORMEN" OF THE BINARY QUANTIC OF THE NINTH ORDER.

[*American Journal of Mathematics*, II. (1879), pp. 98, 99.]

ENUMERATION OF THE IRREDUCIBLE INVARIANTS AND COVARIANTS OF THE BINARY QUANTIC OF THE NINTH ORDER.

Degree in the Coefficients	Order in the Variables. 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	21	22
1										1											
2			1				1				1				1						
3				1		1		1		2		1		1		1		1		1	
4	2				2		2		8		2		2		2		1		1		1
5		1		3		4		4		3		4		2		2					
6			4		4		6		6		3		3								
7		4		7		8		7		5											
8	5		8		10		10		2												
9		9		14		10		2													
10	5		15		14																
11		17		16																	
12	14		23																		
13		25																			
14	17		9																		
15		26																			
16	21																				
17		5																			
18	25																				

The foregoing table has been calculated, out of the funds voted by the British Association, under my superintendence, by Mr Franklin, Fellow of Johns Hopkins University. A statement of the method employed will be given in a future number of the *Journal*.

The total number of irreducible forms will be seen from the table to be 415. The highest degree in the coefficients is 18, and the highest order in the variables 22. The *representative* generating function in this case (as in all others which have been hitherto treated, with the sole exception of the seventhic) has a *finite* numerator.

The total number of groundforms for the orders 0, 2, 4, 6 respectively (counting, as one ought to do, the absolute constant as one of them) is 1, 3, 6, 27, which becomes a regular series on increasing 6, which corresponds to a square index 4, in the proportion of 2 : 3. In like manner, for the orders 1, 3, 5, 7, 9, the series is 2, 5, 24, 125, 416, which, on increasing the last term corresponding to the square index 9 in the ratio 2 : 3, forms an almost regular progression 2, 5, 24, 125, 624, highly suggestive of the geometrical series 1, 5, 25, 125, 625. It seems then to be a not altogether improbable conjecture, that the number of groundforms for 10, which I hope very soon to get completely worked out, will be in the neighbourhood of a ratio of equality to 243*, and for 11, which there is not much prospect of calculating for some time to come, a number not very far out from a ratio of equality to 3125. In the next, or next but one, number of the *Journal* I hope to set out a synoptical table of the groundforms for all orders up to 10 inclusive, with their reduced and representative generating functions, as also for combinations of the orders: 2, 3; 2, 4; 3, 3; 3, 4; 4, 4; all the materials for which, with the exception of what pertains to the covariants *proper* of the tenthic, are already in existence.

* The number of groundforms for the Octavic (I quote from memory) is 70, not more inferior to 81 than might have been anticipated, when the composite form of the number 8 is taken into account. It seems likely that for 10, 243 is at all events a superior limit. [See below, p. 307.]

38.

TABLES OF THE GENERATING FUNCTIONS AND GROUNDFORMS FOR THE BINARY QUANTICS OF THE FIRST TEN ORDERS.

[*American Journal of Mathematics*, II. (1879), pp. 223—251.]

IN what follows, "G. F." stands for the words *Generating Function*. In the Generating Functions, the exponents of the letter a refer to degree in the coefficients, and the exponents of the letter x to order in the variables. The Generating Functions for differentiants take account only of degree in the coefficients, without regard to the order in the variables of the covariant of which the differentiant is the "*source*." In the *tabulated* numerators of the Generating Functions, the *minus* sign is placed *over* instead of *to the left of* the number which it affects.

QUADRIC.

G. F. for differentiants, $\dfrac{1}{(1-a)(1-a^2)}$.

G. F. for covariants, $\dfrac{1}{(1-a^2)(1-ax^2)}$.

Groundforms: 1 of deg. 1, ord. 2; 1 of deg. 2, ord. 0.

CUBIC.

G. F. for differentiants, $\dfrac{1+a^3}{(1-a)(1-a^2)(1-a^4)}$.

G. F. for covariants, reduced form, $\dfrac{1-ax+a^2x^2}{(1-a^4)(1-ax)(1-ax^3)}$.

G. F. for covariants, representative form, $\dfrac{1+a^3x^3}{(1-a^4)(1-a^2x^2)(1-ax^3)}$.

Groundforms: 1 of deg. 1, ord. 3; 1 of deg. 2, ord. 2; 1 of deg. 3, ord. 3; 1 of deg. 4, ord. 0.

QUARTIC.

G. F. for differentiants, $\dfrac{1+a^3}{(1-a)(1-a^2)^2(1-a^3)}$.

G. F. for covariants, reduced form, $\dfrac{1-ax^2+a^2x^4}{(1-a^2)(1-a^3)(1-ax^2)(1-ax^4)}$.

G. F. for covariants, representative form, $\dfrac{1+a^3x^6}{(1-a^2)(1-a^3)(1-a^2x^4)(1-ax^4)}$.

Groundforms: 1 of deg. 1, ord. 4; 1 of deg. 2, ord. 0; 1 of deg. 2, ord. 4; 1 of deg. 3, ord. 0; 1 of deg. 3, ord. 6.

Quintic.

G. F. for differentiants,

$$\frac{1+a^2+3a^3+3a^4+5a^5+4a^6+6a^7+6a^8+4a^9+5a^{10}+3a^{11}+3a^{12}+a^{13}+a^{15}}{(1-a)(1-a^2)(1-a^4)(1-a^6)(1-a^8)}.$$

G. F. for covariants, reduced form,

Denominator: $(1-a^4)(1-a^6)(1-a^8)(1-ax)(1-ax^3)(1-ax^5)$.

Numerator: $1+a(-x-x^3)+a^2(x^2+x^4+x^6)-a^3x^7+a^4x^4+a^5(x+x^3-x^5)$
$+a^6(-1-x^4)+a^7(2x+x^3+x^5)+a^8(-x^2-x^4-2x^6)$
$+a^9(x^3+x^7)+a^{10}(x^2-x^4-x^6)-a^{11}x^3+a^{12}+a^{13}(-x-x^3-x^5)$
$+a^{14}(x^4+x^6)-a^{15}x^7$.

G. F. for covariants, representative form,

Denominator: $(1-a^4)(1-a^8)(1-a^{12})(1-a^2x^2)(1-a^2x^6)(1-ax^5)$.

Numerator: $1+a^3(x^3+x^5+x^9)+a^4(x^4+x^6)+a^5(x+x^3+x^7-x^{11})$
$+a^6(x^2+x^4)+a^7(x+x^5-x^9)+a^8(x^2+x^4)+a^9(x^3+x^5-x^7)$
$+a^{10}(x^2+x^4-x^{10})+a^{11}(x+x^3-x^9)+a^{12}(x^2-x^8-x^{10})$
$+a^{13}(x-x^7-x^9)+a^{14}(x^4-x^6-x^8)+a^{15}(-x^7-x^9)$
$+a^{16}(x^2-x^6-x^{10})+a^{17}(-x^7-x^9)+a^{18}(1-x^4-x^8-x^{10})$
$+a^{19}(-x^5-x^7)+a^{20}(-x^2-x^6-x^8)-a^{23}x^{11}$.

Table of Groundforms.

Degree in the Coefficients.	Order in the Variables. 0	1	2	3	4	5	6	7	9
1						1			
2			1				1		
3				1		1			1
4	1				1		1		
5		1		1				1	
6			1		1				
7		1				1			
8	1		1						
9				1					
11		1							
12	1								
13		1							
18	1								

Sextic.

G. F. for differentiants, $\dfrac{1+a^2+3a^3+4a^4+4a^5+4a^6+3a^7+a^8+a^{10}}{(1-a)(1-a^2)^2(1-a^3)(1-a^4)(1-a^5)}$.

G. F. for covariants, reduced form,*

Denominator: $(1-a^2)^2(1-a^3)(1-a^4)(1-a^5)(1-ax^2)(1-ax^4)(1-ax^6)$.

Numerator: $1+a(-x^2-x^4)+a^2(-1+x^4+x^6+x^8)+a^3(-1+2x^2+x^4-x^{10})$
$+a^4(x^2-x^6-x^8)+a^5(-x^6-x^8+x^{10})+a^6(1-x^2-x^8+x^{10})$
$+a^7(1-x^2-x^4)+a^8(-x^2-x^4+x^8)+a^9(-1+x^6+2x^8-x^{10})$
$+a^{10}(x^2+x^4+x^6-x^{10})+a^{11}(-x^6-x^8)+a^{12}x^{10}$.

G. F. for covariants, representative form,

Denominator: $(1-a^2)(1-a^4)(1-a^6)(1-a^{10})(1-a^2x^4)(1-a^3x^8)(1-ax^6)$.

Numerator: $1+a^3(x^2+x^6+x^8+x^{12})+a^4(x^4+x^8+x^{10})+a^5(x^2+x^4+x^8-x^{16})$
$+a^6(x^4+2x^6)+a^7(x^2+x^4+x^8-x^{12})+a^8(x^2+x^4+x^8-x^{14})$
$+a^9(x^4+x^6-x^{10}-x^{12})+a^{10}(x^2+x^4-x^{12}-x^{14})+a^{11}(x^4$
$+x^6-x^{10}-x^{12})+a^{12}(x^2-x^{10}-x^{12}-x^{14})+a^{13}(x^4-x^8-x^{12}-x^{14})$
$+a^{14}(-2x^{10}-x^{12})+a^{15}(1-x^8-x^{12}-x^{14})+a^{16}(-x^6-x^{10}-x^{12})$
$+a^{17}(-x^4-x^8-x^{10}-x^{14})-a^{20}x^{16}$.

Table of Groundforms.

Degree in the Coefficients.	Order in the Variables. 0	2	4	6	8	10	12
1				1			
2	1		1		1		
3		1		1	1		1
4	1		1	1		1	
5		1	1		1		
6	1			2			
7		1	1				
8		1					
9			1				
10	1	1					
12		1					
15	1						

* This is not strictly the minimum form, its numerator and denominator being divisible by $1-a$; it is, however, the lowest form to which the fraction can be reduced when the factors of the denominator are all of the forms $1-a^r$, $1-a^rx^s$. The same remark applies to the "reduced form" in the case of the decimic.

SEPTIMIC.

G. F. for differentiants,

Denominator: $(1-a)(1-a^2)(1-a^4)(1-a^6)(1-a^8)(1-a^{10})(1-a^{12})$.

Numerator: $1+2a^2+6a^3+10a^4+19a^5+28a^6+44a^7+61a^8+79a^9$
$+102a^{10}+129a^{11}+156a^{12}+173a^{13}+196a^{14}+215a^{15}$
$+230a^{16}+231a^{17}+231a^{18}+230a^{19}+215a^{20}+196a^{21}$
$+173a^{22}+156a^{23}+129a^{24}+102a^{25}+79a^{26}+61a^{27}+44a^{28}$
$+28a^{29}+19a^{30}+10a^{31}+6a^{32}+2a^{33}+a^{35}$.

G. F. for covariants, reduced form,

Denominator: $(1-a^4)(1-a^6)(1-a^8)(1-a^{10})(1-a^{12})(1-ax)(1-ax^3)$
$(1-ax^5)(1-ax^7)$.

Numerator:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}
a^0	1														
a^1		$\bar{1}$		$\bar{1}$		$\bar{1}$									
a^2			1		1		2		1		1				
a^3								$\bar{1}$		$\bar{1}$		$\bar{1}$		$\bar{1}$	
a^4					2				1						1
a^5		1		2						$\bar{1}$		$\bar{1}$			
a^6	$\bar{1}$		2		$\bar{1}$				$\bar{1}$		$\bar{1}$		1		
a^7		4		1		3				$\bar{1}$		1			
a^8	2		$\bar{1}$				$\bar{3}$		$\bar{3}$		$\bar{1}$		$\bar{1}$		
a^9		1		3		1		$\bar{1}$		2				2	
a^{10}	$\bar{1}$		4				$\bar{1}$		$\bar{2}$		$\bar{2}$				$\bar{1}$
a^{11}		5		3		2		$\bar{1}$		$\bar{2}$		$\bar{1}$		1	
a^{12}	5		1				$\bar{4}$		$\bar{6}$		$\bar{4}$		$\bar{1}$		2
a^{13}		1				$\bar{4}$		$\bar{4}$		$\bar{1}$		1		4	
a^{14}	2		5		1		1		$\bar{2}$				3		$\bar{1}$
a^{15}		8		$\bar{1}$		$\bar{1}$		$\bar{7}$		$\bar{5}$		$\bar{1}$		$\bar{1}$	
a^{16}	6		3		3		$\bar{4}$		$\bar{8}$				$\bar{1}$		5
a^{17}		$\bar{1}$		$\bar{2}$		$\bar{9}$		$\bar{8}$		$\bar{4}$		$\bar{3}$		4	

Numerator—(*Continued.*)

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}
a^{18}	2		6		1		2		2		1		6		2
a^{19}		4		$\bar{3}$		$\bar{4}$		$\bar{8}$		$\bar{9}$		$\bar{2}$		$\bar{1}$	
a^{20}	5		$\bar{1}$				$\bar{3}$		$\bar{4}$		3		3		6
a^{21}		$\bar{1}$		$\bar{1}$		$\bar{5}$		$\bar{7}$		$\bar{1}$		$\bar{1}$		3	
a^{22}	$\bar{1}$		3				$\bar{2}$		1		1		5		2
a^{23}		4		1		$\bar{1}$		$\bar{4}$		$\bar{4}$				1	
a^{24}	2		$\bar{1}$		$\bar{4}$		$\bar{6}$		$\bar{4}$				1		5
a^{25}		1		$\bar{1}$		$\bar{2}$		$\bar{1}$		2		3		5	
a^{26}	$\bar{1}$				$\bar{2}$		$\bar{2}$		$\bar{1}$				4		$\bar{1}$
a^{27}		2				2		$\bar{1}$		1		3		1	
a^{28}			$\bar{1}$		$\bar{1}$		$\bar{3}$		$\bar{3}$				$\bar{1}$		2
a^{29}				1		$\bar{1}$				3		1		4	
a^{30}			1		$\bar{1}$		$\bar{1}$				$\bar{1}$		2		$\bar{1}$
a^{31}				$\bar{1}$		$\bar{1}$						2		1	
a^{32}	1						1				2				
a^{33}		$\bar{1}$		$\bar{1}$		$\bar{1}$		$\bar{1}$							
a^{34}					1		1		2		1		1		
a^{35}										$\bar{1}$		$\bar{1}$		$\bar{1}$	
a^{36}															1

Owing to the non-existence of an irreducible invariant whose degree is 10, or any multiple of 10, no representative generating function with a *finite* numerator can be obtained for the septimic; the factor $1-a^{10}$ in the denominator has to be got rid of by dividing numerator and denominator by it, or, in other words, by striking it out of the denominator and multiplying the numerator by the infinite series $1+a^{10}+a^{20}+\ldots$. We thus obtain:

G. F. for covariants, representative form, (with infinite numerator),

Denominator: $(1-a^4)(1-a^8)(1-a^{12})^2(1-a^2x^2)(1-a^2x^6)(1-a^2x^{10})(1-ax^7)$.

Numerator: (Given to the terms containing the 45th power of a, inclusive; after which, each column can be continued by repeating *the last five coefficients* occurring in it, *ad inf.*)

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}	x^{20}	x^{21}	x^{22}	x^{23}
a^0	1																							
a^3				1		1		1		1		1				1								
a^4					2		1		2		1				1									
a^5		1		2		2		2		2								$\bar{1}$				$\bar{1}$		
a^6			3		2		3		3				2		$\bar{1}$		$\bar{1}$							
a^7		3		2		4		4				1				$\bar{2}$				1				1
a^8	2		3		4		6		$\bar{1}$		3		$\bar{1}$		$\bar{2}$				$\bar{1}$					
a^9		3		5		7		1		4				$\bar{2}$		$\bar{1}$		$\bar{2}$				1		
a^{10}			5		8		6		4		1		$\bar{4}$				$\bar{8}$		$\bar{1}$					
a^{11}		5		8		8		8		4		$\bar{4}$		$\bar{1}$		$\bar{5}$		$\bar{1}$						
a^{12}	4		9		9		12		4		$\bar{1}$		$\bar{3}$		$\bar{5}$		$\bar{6}$				$\bar{1}$		1	
a^{13}		9		9		12		6		$\bar{1}$		$\bar{3}$		$\bar{8}$		$\bar{9}$		$\bar{3}$		$\bar{1}$		1		
a^{14}	4		9		13		11		$\bar{1}$		$\bar{3}$		$\bar{9}$		$\bar{10}$		$\bar{7}$		$\bar{2}$				3	
a^{15}		9		12		16		3		2		$\bar{10}$		$\bar{11}$		$\bar{8}$		$\bar{3}$				3		2
a^{16}	5		14		15		12		1		$\bar{5}$		$\bar{16}$		$\bar{9}$		$\bar{9}$		$\bar{1}$		8		3	
a^{17}		12		15		16		6		$\bar{3}$		$\bar{17}$		$\bar{13}$		$\bar{15}$		$\bar{5}$		2		3		
a^{18}	9		14		15		14		$\bar{3}$		$\bar{13}$		$\bar{20}$		$\bar{15}$		$\bar{15}$		2		2		5	
a^{19}		15		16		18				$\bar{8}$		$\bar{18}$		$\bar{20}$		$\bar{19}$		$\bar{3}$		3		5		4
a^{20}	7		14		18		12		$\bar{10}$		$\bar{16}$		$\bar{25}$		$\bar{19}$		$\bar{12}$		2		5		9	
a^{21}		14		17		19		$\bar{1}$		$\bar{8}$		$\bar{27}$		$\bar{25}$		$\bar{16}$		$\bar{2}$		4		8		4
a^{22}	9		17		19		11		$\bar{8}$		$\bar{18}$		$\bar{31}$		$\bar{17}$		$\bar{15}$		6		9		9	
a^{23}		17		19		18		$\bar{3}$		$\bar{13}$		$\bar{31}$		$\bar{25}$		$\bar{21}$		$\bar{4}$		9		9		5
a^{24}	8		17		17		10		$\bar{12}$		$\bar{27}$		$\bar{32}$		$\bar{22}$		$\bar{16}$		9		9		12	
a^{25}		18		17		19		$\bar{6}$		$\bar{17}$		$\bar{31}$		$\bar{28}$		$\bar{22}$		8		10		12		9
a^{26}	9		18		18		11		$\bar{17}$		$\bar{23}$		$\bar{34}$		$\bar{21}$		$\bar{10}$		10		14		15	

Numerator—(*Continued.*)

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}	x^{20}	x^{21}	x^{22}	x^{23}
a^{27}		17		17		19		$\bar{9}$		$\bar{16}$		$\bar{36}$		$\bar{29}$		$\bar{19}$		3		13		14		7
a^{28}	8		17		18		9		$\bar{16}$		$\bar{26}$		$\bar{38}$		$\bar{18}$		$\bar{13}$		14		15		14	
a^{29}		18		19		17		$\bar{8}$		$\bar{16}$		$\bar{36}$		$\bar{25}$		$\bar{21}$		6		16		16		9
a^{30}	9		18		18		10		$\bar{18}$		$\bar{27}$		$\bar{35}$		$\bar{19}$		$\bar{11}$		16		15		17	
a^{31}		17		17		17		$\bar{8}$		$\bar{19}$		$\bar{36}$		$\bar{29}$		$\bar{19}$		8		15		17		8
a^{32}	9		18		18		8		$\bar{18}$		$\bar{26}$		$\bar{35}$		$\bar{19}$		$\bar{10}$		17		17		18	
a^{33}		18		18		18		$\bar{9}$		$\bar{18}$		$\bar{34}$		$\bar{26}$		$\bar{17}$		8		18		18		9
a^{34}	8		17		17		9		$\bar{17}$		$\bar{28}$		$\bar{36}$		$\bar{18}$		$\bar{8}$		17		17		17	
a^{35}		18		17		18		$\bar{9}$		$\bar{17}$		$\bar{35}$		$\bar{27}$		$\bar{18}$		9		18		17		8
a^{36}	9		19		18		9		$\bar{18}$		$\bar{25}$		$\bar{34}$		$\bar{17}$		$\bar{9}$		17		19		18	
a^{37}		17		17		18		$\bar{9}$		$\bar{18}$		$\bar{37}$		$\bar{26}$		$\bar{18}$		9		17		17		9
a^{38}	9		17		17		9		$\bar{18}$		$\bar{26}$		$\bar{37}$		$\bar{18}$		$\bar{9}$		18		17		17	
a^{39}		18		19		17		$\bar{9}$		$\bar{17}$		$\bar{34}$		$\bar{25}$		$\bar{18}$		9		18		19		9
a^{40}	9		17		18		9		$\bar{18}$		$\bar{27}$		$\bar{35}$		$\bar{17}$		$\bar{9}$		18		17		18	
a^{41}		17		17		17		$\bar{8}$		$\bar{18}$		$\bar{36}$		$\bar{28}$		$\bar{17}$		9		17		17		8
a^{42}	9		18		18		8		$\bar{17}$		$\bar{26}$		$\bar{34}$		$\bar{18}$		$\bar{9}$		18		18		18	
a^{43}		18		18		18		$\bar{9}$		$\bar{18}$		$\bar{34}$		$\bar{26}$		$\bar{17}$		8		18		18		9
a^{44}	8		17		17		9		$\bar{17}$		$\bar{28}$		$\bar{36}$		$\bar{18}$		$\bar{8}$		17		17		17	
a^{45}		18		17		18		$\bar{9}$		$\bar{17}$		$\bar{35}$		$\bar{27}$		$\bar{18}$		9		18		17		9

etc. etc. etc.

Table of Groundforms.

ORDER IN THE VARIABLES.

DEGREE IN THE COEFFICIENTS.	0	1	2	3	4	5	6	7	8	9	10	11	14	15
1								1						
2			1				1				1			
3				1		1		1		1		1		1
4	1				2		1		2		1		1	
5		1		2		2		2		2				
6			3		2		2		2					
7		3		2		4		2						
8	3		3		3		3							
9		3		5		2								
10			4		3									
11		5		3										
12	6		6											
13		7												
14	4													
15		3												
16	2													
17		2												
18	9													
22	1													

OCTAVIC.

G. F. for differentiants,

Denominator: $(1-a)(1-a^2)^2(1-a^3)^2(1-a^4)(1-a^5)(1-a^7)$.

Numerator: $1+2a^2+6a^3+12a^4+19a^5+25a^6+31a^7+36a^8+38a^9+36a^{10}$
$+31a^{11}+25a^{12}+19a^{13}+12a^{14}+6a^{15}+2a^{16}+a^{18}$.

G. F. for covariants, reduced form,

Denominator: $(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)$
$(1-ax^2)(1-ax^4)(1-ax^6)(1-ax^8)$.

Numerator:

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}
a^0	1									
a^1		$\bar{1}$	$\bar{1}$	$\bar{1}$						
a^2			1	1	2	1	1			
a^3			1			$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	
a^4			2							1
a^5		1	2		$\bar{1}$		$\bar{1}$			
a^6		1	1		$\bar{1}$	$\bar{1}$	$\bar{1}$	1		
a^7		2	1	1	$\bar{1}$	$\bar{1}$	$\bar{1}$	1		
a^8	1	2			$\bar{2}$	$\bar{2}$	$\bar{2}$	1		
a^9	1	2	$\bar{2}$		$\bar{2}$	$\bar{2}$	$\bar{1}$	1	1	
a^{10}	1	1	$\bar{2}$		$\bar{2}$	$\bar{1}$			1	
a^{11}		1	$\bar{1}$		$\bar{1}$	$\bar{1}$		$\bar{1}$	1	
a^{12}		1			$\bar{1}$	$\bar{2}$		$\bar{2}$	1	1
a^{13}		1	1	$\bar{1}$	$\bar{2}$	$\bar{2}$		$\bar{2}$	2	1
a^{14}			1	$\bar{2}$	$\bar{2}$	$\bar{2}$			2	1
a^{15}			1	$\bar{1}$	$\bar{1}$	$\bar{1}$	1	1	2	
a^{16}			1	$\bar{1}$	$\bar{1}$	$\bar{1}$		1	1	
a^{17}				$\bar{1}$		$\bar{1}$		2	1	
a^{18}	1							2		
a^{19}		$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$			1		
a^{20}				1	1	2	1	1		
a^{21}							$\bar{1}$	$\bar{1}$	$\bar{1}$	
a^{22}										1

G. F. for covariants, representative form,

Denominator: $(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^2x^4)$
$(1-a^2x^8)(1-a^2x^{12})(1-ax^8)$.

Numerator:

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}
a^0	1															
a^3			1	1	1	1	1	1		1						
a^4			2	1	1	2	1	1		1						
a^5		1	2	2	1	3		1		$\bar{1}$		$\bar{1}$		$\bar{1}$		
a^6		1	2	3	2	2		1		$\bar{1}$		$\bar{1}$				
a^7		2	2	3	2	2		$\bar{1}$		$\bar{2}$		$\bar{1}$				1
a^8	1	2	2	3	3	1	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{3}$	$\bar{1}$					
a^9	1	3	1	3	2	$\bar{1}$	$\bar{1}$	$\bar{3}$	$\bar{2}$	$\bar{4}$	$\bar{1}$	$\bar{1}$		1		
a^{10}	1	2	1	2	1	$\bar{2}$	$\bar{2}$	$\bar{5}$	$\bar{4}$	$\bar{4}$	$\bar{2}$	$\bar{1}$		1		
a^{11}		2	1	1	1	$\bar{4}$	$\bar{2}$	$\bar{6}$	$\bar{4}$	$\bar{4}$	$\bar{3}$			2		
a^{12}		1	1	1	1	$\bar{4}$	$\bar{2}$	$\bar{6}$	$\bar{6}$	$\bar{2}$	$\bar{3}$			2	1	
a^{13}		1	2			$\bar{3}$	$\bar{2}$	$\bar{6}$	$\bar{6}$	$\bar{2}$	$\bar{4}$	1	1	1	1	
a^{14}			2			$\bar{3}$	$\bar{4}$	$\bar{4}$	$\bar{6}$	$\bar{2}$	$\bar{4}$	1	1	1	2	
a^{15}			1		$\bar{1}$	$\bar{2}$	$\bar{4}$	$\bar{4}$	$\bar{5}$	$\bar{2}$	$\bar{2}$	1	2	1	2	1
a^{16}			1		$\bar{1}$	$\bar{1}$	$\bar{4}$	$\bar{2}$	$\bar{3}$	$\bar{1}$	$\bar{1}$	2	3	1	3	1
a^{17}						$\bar{1}$	$\bar{3}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	1	3	3	2	2	1
a^{18}	1				$\bar{1}$		$\bar{2}$		$\bar{1}$		2	2	3	2	2	
a^{19}					$\bar{1}$		$\bar{1}$		1		2	2	3	2	1	
a^{20}			$\bar{1}$		$\bar{1}$		$\bar{1}$		1		3	1	2	2	1	
a^{21}							1		1	1	2	1	1	2		
a^{22}							1		1	1	1	1	1	1		
a^{25}																1

Table of Groundforms.

Degree in the Coefficients.	Order in the Variables. 0	2	4	6	8	10	12	14	18
1					1				
2	1		1		1		1		
3	1		1	1	1	1	1	1	1
4	1		2	1	1	2	1	1	1
5	1	1	2	2	1	3		1	
6	1	1	2	3	1	1			
7	1	2	2	3					
8	1	2	2	2					
9	1	3	1						
10	1	2							
11		2							
12		1							

Nonic.

G. F. for differentiants,

Denominator: $(1-a)(1-a^2)(1-a^4)(1-a^6)(1-a^8)(1-a^{10})(1-a^{12})$
$(1-a^{14})(1-a^{16})$.

Numerator: $1+3a^2+10a^3+23a^4+49a^5+93a^6+172a^7+289a^8+457a^9$
$+701a^{10}+1036a^{11}+1477a^{12}+2023a^{13}+2720a^{14}+3568a^{15}$
$+4573a^{16}+5702a^{17}+7013a^{18}+8466a^{19}+10043a^{20}+11672a^{21}$
$+13400a^{22}+15155a^{23}+16880a^{24}+18487a^{25}+20013a^{26}$
$+21392a^{27}+22539a^{28}+23398a^{29}+24013a^{30}+24355a^{31}$
$+24355a^{32}+24013a^{33}+23398a^{34}+22539a^{35}+21392a^{36}$
$+20013a^{37}+18487a^{38}+16880a^{39}+15155a^{40}+13400a^{41}$
$+11672a^{42}+10043a^{43}+8466a^{44}+7013a^{45}+5702a^{46}+4573a^{47}$
$+3568a^{48}+2720a^{49}+2023a^{50}+1477a^{51}+1036a^{52}+701a^{53}$
$+457a^{54}+289a^{55}+172a^{56}+93a^{57}+49a^{58}+23a^{59}+10a^{60}$
$+3a^{61}+a^{63}$.

G. F. for covariants, reduced form,

Denominator: $(1-a^4)(1-a^6)(1-a^8)(1-a^{10})(1-a^{12})(1-a^{14})(1-a^{16})$
$(1-ax)(1-ax^3)(1-ax^5)(1-ax^7)(1-ax^9)$.

Numerator:

	x^{0}	x^{1}	x^{2}	x^{3}	x^{4}	x^{5}	x^{6}	x^{7}	x^{8}	x^{9}	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}	x^{20}	x^{21}	x^{22}	x^{23}
a^{0}	1																							
a^{1}		$\overline{1}$		$\overline{1}$		$\overline{1}$		$\overline{1}$																
a^{2}			1		1		2		2		2		1		1									
a^{3}								$\overline{1}$		$\overline{1}$		$\overline{2}$		$\overline{2}$		$\overline{2}$		$\overline{1}$		$\overline{1}$				
a^{4}	1				2		1		2				1		1		1		1		1		1	
a^{5}				2		1				$\overline{2}$		$\overline{1}$		$\overline{2}$		$\overline{1}$		$\overline{1}$						$\overline{1}$
a^{6}	$\overline{1}$		4		1		3										1		1		1			
a^{7}		5		5		5		1		$\overline{1}$		$\overline{3}$		$\overline{2}$		$\overline{2}$						$\overline{1}$		
a^{8}	5		3		4		$\overline{2}$		$\overline{3}$		$\overline{7}$		$\overline{5}$		$\overline{3}$		1		2				1	
a^{9}		5		8		2		1		$\overline{4}$		$\overline{2}$		$\overline{1}$		3		3		2		1		$\overline{1}$
a^{10}	3		15		5		5		$\overline{3}$		$\overline{7}$		$\overline{7}$		$\overline{2}$		1		$\overline{1}$		1		$\overline{2}$	
a^{11}		17		11		9		$\overline{2}$		$\overline{10}$		$\overline{16}$		$\overline{5}$		$\overline{3}$		2		4		$\overline{1}$		1
a^{12}	18		14		15		$\overline{2}$		$\overline{11}$		$\overline{24}$		$\overline{14}$		$\overline{3}$		3		8		3		1	
a^{13}		17		17		2		$\overline{12}$		$\overline{27}$		$\overline{21}$		$\overline{6}$		8		11		9		3		$\overline{5}$
a^{14}	15		39		21		6		$\overline{13}$		$\overline{26}$		$\overline{13}$		2		13		10		8		$\overline{7}$	
a^{15}		42		24		10		$\overline{28}$		$\overline{45}$		$\overline{52}$		$\overline{17}$		$\overline{5}$		13		11		$\overline{5}$		$\overline{3}$
a^{16}	44		41		31		$\overline{15}$		$\overline{33}$		$\overline{59}$		$\overline{26}$		8		28		31		13		$\overline{2}$	
a^{17}		44		28		$\overline{14}$		$\overline{52}$		$\overline{78}$		$\overline{63}$		$\overline{9}$		15		34		18		1		$\overline{18}$
a^{18}	43		77		33		$\overline{5}$		$\overline{85}$		$\overline{63}$		$\overline{11}$		28		51		34		20		$\overline{20}$	
a^{19}		79		32		$\overline{6}$		$\overline{82}$		$\overline{113}$		$\overline{108}$		$\overline{20}$		8		36		19		$\overline{17}$		$\overline{11}$
a^{20}	82		76		43		$\overline{89}$		$\overline{70}$		$\overline{109}$		$\overline{22}$		48		80		69		29		$\overline{13}$	
a^{21}		76		37		$\overline{43}$		$\overline{121}$		$\overline{159}$		$\overline{117}$				36		70		29		$\overline{10}$		$\overline{44}$
a^{22}	76		122		41		$\overline{35}$		$\overline{75}$		$\overline{112}$		6		83		118		76		38		$\overline{45}$	
a^{23}		120		37		$\overline{41}$		$\overline{163}$		$\overline{201}$		$\overline{165}$		$\overline{5}$		31		75		33		$\overline{43}$		$\overline{47}$
a^{24}	122		112		37		$\overline{86}$		$\overline{121}$		$\overline{161}$		2		120		160		123		40		$\overline{40}$	
a^{25}		109		31		$\overline{92}$		$\overline{205}$		$\overline{242}$		$\overline{154}$		39		88		120		37		$\overline{40}$		$\overline{82}$
a^{26}	107		151		25		$\overline{82}$		$\overline{116}$		$\overline{147}$		52		166		203		117		39		$\overline{82}$	
a^{27}		148		25		$\overline{85}$		$\overline{239}$		$\overline{267}$		$\overline{190}$		44		79		113		83		$\overline{84}$		$\overline{76}$
a^{28}	147		125		13		$\overline{136}$		$\overline{161}$		$\overline{188}$		50		206		237		158		37		$\overline{74}$	
a^{29}		121		14		$\overline{137}$		$\overline{265}$		$\overline{286}$		$\overline{152}$		107		135		157		35		$\overline{77}$		$\overline{122}$
a^{30}	119		153		$\overline{1}$		$\overline{123}$		$\overline{141}$		$\overline{151}$		111		243		263		137		28		$\overline{124}$	
a^{31}		149		1		$\overline{123}$		$\overline{281}$		$\overline{286}$		$\overline{165}$		108		123		138		27		$\overline{127}$		$\overline{107}$
a^{32}	147		112		$\overline{15}$		$\overline{167}$		$\overline{169}$		$\overline{164}$		109		270		280		166		13		$\overline{108}$	

	x^{0}	x^{1}	x^{2}	x^{3}	x^{4}	x^{5}	x^{6}	x^{7}	x^{8}	x^{9}	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}	x^{20}	x^{21}	x^{22}	x^{23}
a^{33}		108		$\overline{13}$		$\overline{166}$		$\overline{280}$		$\overline{270}$		$\overline{109}$		164		169		167		15		$\overline{112}$		$\overline{147}$
a^{34}	107		127		$\overline{27}$		$\overline{138}$		$\overline{123}$		$\overline{108}$		165		286		281		123		$\overline{1}$		$\overline{149}$	
a^{35}		124		$\overline{28}$		$\overline{137}$		$\overline{263}$		$\overline{243}$		$\overline{111}$		151		141		123		1		$\overline{153}$		$\overline{119}$
a^{36}	122		77		$\overline{35}$		$\overline{157}$		$\overline{135}$		$\overline{107}$		152		286		265		137		$\overline{14}$		$\overline{121}$	
a^{37}		74		$\overline{37}$		$\overline{158}$		$\overline{237}$		$\overline{206}$		$\overline{50}$		188		161		136		$\overline{13}$		$\overline{125}$		$\overline{147}$
a^{38}	76		84		$\overline{33}$		$\overline{113}$		$\overline{79}$		$\overline{44}$		190		267		239		85		$\overline{25}$		$\overline{148}$	
a^{39}		82		$\overline{39}$		$\overline{117}$		$\overline{203}$		$\overline{166}$		$\overline{52}$		147		116		82		$\overline{25}$		$\overline{151}$		$\overline{107}$
a^{40}	82		40		$\overline{37}$		$\overline{120}$		$\overline{83}$		$\overline{39}$		154		242		205		92		$\overline{31}$		$\overline{109}$	
a^{41}		40		$\overline{40}$		$\overline{123}$		$\overline{160}$		$\overline{120}$		$\overline{2}$		161		121		86		$\overline{37}$		$\overline{112}$		$\overline{122}$
a^{42}	43		43		$\overline{33}$		$\overline{75}$		$\overline{31}$		5		165		201		163		41		$\overline{37}$		$\overline{120}$	
a^{43}		45		$\overline{38}$		$\overline{76}$		$\overline{118}$		$\overline{83}$		$\overline{6}$		112		75		35		$\overline{41}$		$\overline{122}$		$\overline{76}$
a^{44}	44		10		$\overline{29}$		$\overline{70}$		$\overline{36}$				117		159		121		43		$\overline{37}$		$\overline{76}$	
a^{45}		13		$\overline{29}$		$\overline{69}$		$\overline{80}$		$\overline{48}$		22		109		70		39		$\overline{43}$		$\overline{76}$		$\overline{82}$
a^{46}	15		17		$\overline{19}$		$\overline{36}$		$\overline{3}$		20		108		113		82		6		$\overline{32}$		$\overline{79}$	
a^{47}		20		$\overline{20}$		$\overline{34}$		$\overline{51}$		$\overline{28}$		11		63		35		5		$\overline{33}$		$\overline{77}$		$\overline{43}$
a^{48}	18		$\overline{1}$		$\overline{18}$		$\overline{34}$		$\overline{15}$		9		63		78		52		14		$\overline{28}$		$\overline{44}$	
a^{49}		2		$\overline{13}$		$\overline{31}$		$\overline{28}$		$\overline{8}$		26		59		33		15		$\overline{31}$		$\overline{41}$		$\overline{44}$
a^{50}	3		5		$\overline{11}$		$\overline{13}$		5		17		52		45		28		$\overline{10}$		$\overline{24}$		$\overline{42}$	
a^{51}		7		$\overline{8}$		$\overline{10}$		$\overline{13}$		$\overline{2}$		18		26		13		$\overline{6}$		21		$\overline{39}$		$\overline{15}$
a^{52}	5		$\overline{3}$		$\overline{9}$		$\overline{11}$		$\overline{8}$		6		21		27		12		$\overline{2}$		$\overline{17}$		$\overline{17}$	
a^{53}		$\overline{1}$		$\overline{3}$		$\overline{8}$		$\overline{3}$		3		14		24		11		2		$\overline{15}$		$\overline{14}$		$\overline{18}$
a^{54}	$\overline{1}$		1		$\overline{4}$		$\overline{2}$		3		5		16		10		2		$\overline{9}$		$\overline{11}$		$\overline{17}$	
a^{55}		2		$\overline{1}$		1		$\overline{1}$		2		7		7		3		$\overline{5}$		$\overline{5}$		$\overline{15}$		$\overline{3}$
a^{56}	1		$\overline{1}$		$\overline{2}$		$\overline{3}$		$\overline{3}$		1		2		4		$\overline{1}$		$\overline{2}$		$\overline{8}$		$\overline{5}$	
a^{57}		$\overline{1}$				$\overline{2}$		$\overline{1}$		3		5		7		3		2		$\overline{4}$		$\overline{3}$		$\overline{5}$
a^{58}			1						2		2		3		1		$\overline{1}$		$\overline{5}$		$\overline{5}$		$\overline{5}$	
a^{59}				$\overline{1}$		$\overline{1}$		$\overline{1}$										$\overline{3}$		$\overline{1}$		$\overline{4}$		1
a^{60}	1						1		1		2		1		2				$\overline{1}$		$\overline{2}$			
a^{61}		$\overline{1}$		$\overline{1}$		$\overline{1}$		$\overline{1}$		$\overline{1}$		$\overline{1}$				$\overline{2}$		$\overline{1}$		$\overline{2}$				$\overline{1}$
a^{62}					1		1		2		2		2		1		1							
a^{63}										$\overline{1}$		$\overline{1}$		$\overline{2}$		$\overline{2}$		$\overline{2}$		$\overline{1}$		$\overline{1}$		
a^{64}																	1		1		1		1	
a^{65}																								$\overline{1}$

G. F. for covariants, representative form,

Denominator: $(1-a^4)(1-a^8)(1-a^{10})(1-a^{12})^2(1-a^{14})(1-a^{16})(1-a^2x^6)$
$(1-a^2x^{10})(1-a^2x^{14})(1-ax^9)$.

Numerator:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}
a^0	1																			
a^3				1		1		1		2		1		1		1		1		
a^4	1				2		2		3		2		2		2		1		1	
a^5		1		3		4		4		3		4		2		2				
a^6			4		4		7		7		5		6		1		2			
a^7		4		8		9		10		11		7		6		2				
a^8	5		8		13		16		16		14		7		6		1		$\bar{1}$	
a^9		10		17		20		22		19		15		7		1		$\bar{3}$		$\bar{7}$
a^{10}	4		20		25		30		33		20		13		2		$\bar{3}$		$\bar{10}$	
a^{11}		21		32		41		43		40		20		11		$\bar{4}$		$\bar{14}$		$\bar{13}$
a^{12}	17		35		50		60		57		37		16				$\bar{18}$		$\bar{25}$	
a^{13}		39		57		75		71		57		28		6		$\bar{29}$		$\bar{34}$		$\bar{41}$
a^{14}	20		64		86		90		92		44		13		$\bar{31}$		$\bar{46}$		$\bar{59}$	
a^{15}		67		94		121		108		96		23		$\bar{11}$		$\bar{63}$		$\bar{73}$		$\bar{79}$
a^{16}	47		103		135		143		135		57		$\bar{7}$		$\bar{65}$		$\bar{91}$		$\bar{117}$	
a^{17}		108		142		181		154		116		3		$\bar{45}$		$\bar{139}$		$\bar{136}$		$\bar{148}$
a^{18}	61		152		195		191		181		37		$\bar{43}$		$\bar{149}$		$\bar{176}$		$\bar{198}$	
a^{19}		157		201		257		199		149		$\bar{38}$		$\bar{104}$		$\bar{239}$		$\bar{221}$		$\bar{222}$
a^{20}	97		211		270		260		225		21		$\bar{107}$		$\bar{252}$		$\bar{271}$		$\bar{302}$	
a^{21}		215		273		339		239		157		$\bar{108}$		$\bar{200}$		$\bar{391}$		$\bar{330}$		$\bar{338}$
a^{22}	120		281		348		308		262		$\bar{42}$		$\bar{206}$		$\bar{412}$		$\bar{410}$		$\bar{434}$	
a^{23}		284		348		418		269		159		$\bar{215}$		$\bar{327}$		$\bar{562}$		$\bar{462}$		$\bar{440}$

x^{20}	x^{21}	x^{22}	x^{23}	x^{24}	x^{25}	x^{26}	x^{27}	x^{28}	x^{29}	x^{30}	x^{31}	x^{32}	x^{33}	x^{34}	x^{35}	x^{36}	x^{37}	x^{38}	x^{39}	
																				a^0
	1																			a^3
		1																		a^4
			$\bar{2}$				$\bar{1}$				$\bar{1}$									a^5
		$\bar{1}$		$\bar{1}$				$\bar{1}$												a^6
	$\bar{3}$				$\bar{1}$								1				1			a^7
$\bar{3}$		$\bar{4}$		$\bar{1}$		$\bar{2}$				1										a^8
	$\bar{4}$		$\bar{6}$		$\bar{3}$		$\bar{1}$				1				1				$\bar{1}$	a^9
$\bar{11}$		$\bar{9}$		$\bar{7}$		$\bar{2}$						1		1						a^{10}
	$\bar{16}$		$\bar{11}$		$\bar{6}$				2		2		3							a^{11}
$\bar{23}$		$\bar{24}$		$\bar{9}$		$\bar{4}$		1		3		5		2						a^{12}
	$\bar{36}$		$\bar{29}$		$\bar{9}$				4		7		2		2				$\bar{1}$	a^{13}
$\bar{55}$		$\bar{46}$		$\bar{20}$		$\bar{4}$		7		9		11		4		1		$\bar{1}$		a^{14}
	$\bar{65}$		$\bar{40}$		$\bar{9}$		8		20		15		12		4					a^{15}
$\bar{89}$		$\bar{78}$		$\bar{20}$				27		24		23		9		1		$\bar{4}$		a^{16}
	$\bar{102}$		$\bar{74}$		$\bar{5}$		25		38		30		17		7		$\bar{4}$		$\bar{5}$	a^{17}
$\bar{147}$		$\bar{121}$		$\bar{23}$		19		57		41		45		13				$\bar{10}$		a^{18}
	$\bar{150}$		$\bar{87}$		25		57		83		55		39		6		$\bar{8}$		$\bar{4}$	a^{19}
$\bar{202}$		$\bar{164}$		9		50		112		83		74		16		$\bar{3}$		$\bar{21}$		a^{20}
	$\bar{194}$		$\bar{113}$		63		109		137		86		48		6		$\bar{19}$		$\bar{17}$	a^{21}
$\bar{276}$		$\bar{202}$		43		107		194		121		112		16		$\bar{11}$		$\bar{39}$		a^{22}
	$\bar{230}$		$\bar{102}$		149		194		232		126		81		$\bar{2}$		$\bar{34}$		$\bar{20}$	a^{23}

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}
a^{24}	165		353		419		366		278		$\overline{122}$		$\overline{338}$		$\overline{586}$		$\overline{555}$		$\overline{569}$	
a^{25}		353		417		490		275		115		$\overline{356}$		$\overline{481}$		$\overline{777}$		$\overline{593}$		$\overline{551}$
a^{26}	189		415		484		386		269		$\overline{247}$		$\overline{496}$		$\overline{800}$		$\overline{716}$		$\overline{692}$	
a^{27}		413		478		544		254		68		$\overline{519}$		$\overline{652}$		$\overline{976}$		$\overline{708}$		$\overline{622}$
a^{28}	223		471		529		403		235		$\overline{374}$		$\overline{669}$		$\overline{996}$		$\overline{839}$		$\overline{794}$	
a^{29}		464		521		570		211		$\overline{22}$		$\overline{694}$		$\overline{821}$		$\overline{1181}$		$\overline{795}$		$\overline{671}$
a^{30}	241		506		551		375		171		$\overline{530}$		$\overline{840}$		$\overline{1186}$		$\overline{959}$		$\overline{844}$	
a^{31}		499		538		568		139		$\overline{120}$		$\overline{859}$		$\overline{978}$		$\overline{1326}$		$\overline{832}$		$\overline{649}$
a^{32}	254		521		541		332		87		$\overline{669}$		$\overline{988}$		$\overline{1327}$		$\overline{998}$		$\overline{839}$	
a^{33}		510		529		534		49		$\overline{224}$		$\overline{1007}$		$\overline{1088}$		$\overline{1420}$		$\overline{809}$		$\overline{584}$
a^{34}	254		508		508		260		$\overline{5}$		$\overline{792}$		$\overline{1098}$		$\overline{1401}$		$\overline{991}$		$\overline{773}$	
a^{35}		499		492		474		$\overline{42}$		$\overline{322}$		$\overline{1104}$		$\overline{1144}$		$\overline{1432}$		$\overline{729}$		$\overline{459}$
a^{36}	241		475		449		183		$\overline{101}$		$\overline{877}$		$\overline{1143}$		$\overline{1406}$		$\overline{915}$		$\overline{650}$	
a^{37}		464		435		399		$\overline{132}$		$\overline{398}$		$\overline{1144}$		$\overline{1137}$		$\overline{1376}$		$\overline{593}$		$\overline{297}$
a^{38}	223		419		380		97		$\overline{184}$		$\overline{905}$		$\overline{1133}$		$\overline{1335}$		$\overline{788}$		$\overline{483}$	
a^{39}		413		367		311		$\overline{205}$		$\overline{446}$		$\overline{1122}$		$\overline{1076}$		$\overline{1240}$		$\overline{423}$		$\overline{128}$
a^{40}	189		357		297		16		$\overline{240}$		$\overline{891}$		$\overline{1062}$		$\overline{1203}$		$\overline{619}$		$\overline{306}$	
a^{41}		353		288		222		$\overline{251}$		$\overline{456}$		$\overline{1049}$		$\overline{956}$		$\overline{1051}$		$\overline{250}$		47
a^{42}	165		284		217		$\overline{40}$		$\overline{272}$		$\overline{825}$		$\overline{940}$		$\overline{1011}$		$\overline{441}$		$\overline{121}$	
a^{43}		284		210		147		$\overline{274}$		$\overline{446}$		$\overline{923}$		$\overline{801}$		$\overline{844}$		$\overline{80}$		191
a^{44}	120		213		147		$\overline{88}$		$\overline{278}$		$\overline{728}$		$\overline{780}$		$\overline{818}$		$\overline{264}$		34	
a^{45}		215		146		85		$\overline{270}$		$\overline{386}$		$\overline{769}$		$\overline{630}$		$\overline{619}$		65		297
a^{46}	97		152		91		$\overline{101}$		$\overline{256}$		$\overline{588}$		$\overline{615}$		$\overline{599}$		$\overline{107}$		145	
a^{47}		157		94		36		$\overline{242}$		$\overline{333}$		$\overline{604}$		$\overline{465}$		$\overline{427}$		158		338
a^{48}	61		102		46		$\overline{112}$		$\overline{219}$		$\overline{468}$		$\overline{452}$		$\overline{422}$		$\overline{7}$		215	
a^{49}		108		52		5		$\overline{203}$		$\overline{255}$		$\overline{446}$		$\overline{317}$		$\overline{253}$		209		359
a^{50}	47		62		17		$\overline{91}$		$\overline{175}$		$\overline{333}$		$\overline{309}$		$\overline{258}$		76		243	
a^{51}		67		25		$\overline{10}$		$\overline{158}$		$\overline{192}$		$\overline{307}$		$\overline{196}$		$\overline{136}$		224		321

x^{20}	x^{21}	x^{22}	x^{23}	x^{24}	x^{25}	x^{26}	x^{27}	x^{28}	x^{29}	x^{30}	x^{31}	x^{32}	x^{33}	x^{34}	x^{35}	x^{36}	x^{37}	x^{38}	x^{39}	
$\overline{321}$		$\overline{224}$		136		196		307		192		158		10		$\overline{25}$		$\overline{67}$		a^{24}
	$\overline{243}$		$\overline{76}$		258		309		333		175		91		$\overline{17}$		$\overline{62}$		$\overline{47}$	a^{25}
$\overline{359}$		$\overline{209}$		253		317		446		255		203		$\overline{5}$		$\overline{52}$		$\overline{108}$		a^{26}
	$\overline{215}$		7		422		452		468		219		112		$\overline{46}$		$\overline{102}$		$\overline{61}$	a^{27}
$\overline{338}$		$\overline{158}$		427		465		604		333		242		$\overline{36}$		$\overline{94}$		$\overline{157}$		a^{28}
	$\overline{145}$		107		599		615		588		256		101		$\overline{91}$		$\overline{152}$		$\overline{97}$	a^{29}
$\overline{297}$		$\overline{65}$		619		630		769		386		270		$\overline{85}$		$\overline{146}$		$\overline{215}$		a^{30}
	$\overline{34}$		264		818		780		728		278		88		$\overline{147}$		$\overline{213}$		$\overline{120}$	a^{31}
$\overline{191}$		80		844		801		923		446		274		$\overline{147}$		$\overline{210}$		$\overline{284}$		a^{32}
	121		441		1011		940		825		272		40		$\overline{217}$		$\overline{284}$		$\overline{165}$	a^{33}
$\overline{47}$		250		1051		956		1049		456		251		$\overline{222}$		$\overline{288}$		$\overline{353}$		a^{34}
	306		619		1203		1062		891		240		$\overline{16}$		$\overline{297}$		$\overline{357}$		$\overline{189}$	a^{35}
128		423		1240		1076		1122		446		205		$\overline{311}$		$\overline{367}$		$\overline{413}$		a^{36}
	483		788		1335		1133		905		184		$\overline{97}$		$\overline{380}$		$\overline{419}$		$\overline{223}$	a^{37}
297		593		1376		1137		1144		398		132		$\overline{399}$		$\overline{435}$		$\overline{464}$		a^{38}
	650		915		1406		1143		877		101		$\overline{183}$		$\overline{449}$		$\overline{475}$		$\overline{241}$	a^{39}
459		729		1432		1144		1104		322		42		$\overline{474}$		$\overline{492}$		$\overline{499}$		a^{40}
	773		991		1401		1098		792		5		$\overline{260}$		$\overline{508}$		$\overline{508}$		$\overline{254}$	a^{41}
584		809		1420		1088		1007		224		$\overline{49}$		$\overline{534}$		$\overline{529}$		$\overline{510}$		a^{42}
	839		998		1327		988		669		$\overline{87}$		$\overline{332}$		$\overline{541}$		$\overline{521}$		$\overline{254}$	a^{43}
649		832		1326		978		859		120		$\overline{139}$		$\overline{568}$		$\overline{538}$		$\overline{499}$		a^{44}
	844		959		1186		840		530		$\overline{171}$		$\overline{375}$		$\overline{551}$		$\overline{506}$		$\overline{241}$	a^{45}
671		795		1181		821		694		22		$\overline{211}$		$\overline{570}$		$\overline{521}$		$\overline{464}$		a^{46}
	794		839		996		669		374		$\overline{235}$		$\overline{403}$		$\overline{529}$		$\overline{471}$		$\overline{223}$	a^{47}
622		708		976		652		519		$\overline{68}$		$\overline{254}$		$\overline{544}$		$\overline{478}$		$\overline{413}$		a^{48}
	692		716		800		496		247		$\overline{269}$		$\overline{386}$		$\overline{484}$		$\overline{415}$		$\overline{189}$	a^{49}
551		593		777		481		356		$\overline{115}$		$\overline{275}$		$\overline{490}$		$\overline{417}$		$\overline{353}$		a^{50}
	569		555		586		338		122		$\overline{278}$		$\overline{366}$		$\overline{419}$		$\overline{353}$		$\overline{165}$	a^{51}

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}
a^{52}	20		34		2		$\overline{81}$		$\overline{126}$		$\overline{232}$		$\overline{194}$		$\overline{149}$		102		230	
a^{53}		39		11		$\overline{16}$		$\overline{112}$		$\overline{121}$		$\overline{194}$		$\overline{107}$		$\overline{43}$		202		276
a^{54}	17		19		$\overline{6}$		$\overline{48}$		$\overline{86}$		$\overline{137}$		$\overline{109}$		$\overline{63}$		113		194	
a^{55}		21		3		$\overline{16}$		$\overline{74}$		$\overline{83}$		$\overline{112}$		$\overline{50}$		$\overline{9}$		164		202
a^{56}	4		8		$\overline{6}$		$\overline{39}$		$\overline{55}$		$\overline{83}$		$\overline{57}$		$\overline{25}$		87		150	
a^{57}		10				$\overline{13}$		$\overline{45}$		$\overline{41}$		$\overline{57}$		$\overline{19}$		23		121		147
a^{58}	5		4		$\overline{7}$		$\overline{17}$		$\overline{30}$		$\overline{38}$		$\overline{25}$		5		74		102	
a^{59}		4		$\overline{1}$		$\overline{9}$		$\overline{23}$		$\overline{24}$		$\overline{27}$				20		78		89
a^{60}					$\overline{4}$		$\overline{12}$		$\overline{15}$		$\overline{20}$		$\overline{8}$		9		40		65	
a^{61}		1		$\overline{1}$		$\overline{4}$		$\overline{11}$		$\overline{9}$		$\overline{7}$		4		20		46		55
a^{62}	1				$\overline{2}$		$\overline{2}$		$\overline{7}$		$\overline{4}$				9		29		36	
a^{63}						$\overline{2}$		$\overline{5}$		$\overline{3}$		$\overline{1}$		4		9		24		23
a^{64}							$\overline{3}$		$\overline{2}$		$\overline{2}$				6		11		16	
a^{65}						$\overline{1}$		$\overline{1}$						2		7		9		11
a^{66}	1				$\overline{1}$				$\overline{1}$				1		3		6		4	
a^{67}										$\overline{1}$				2		1		4		3
a^{68}			$\overline{1}$				$\overline{1}$								1				3	
a^{69}												1				1		1		
a^{70}									1				1				2			
a^{71}																		$\overline{1}$		
a^{72}																			$\overline{1}$	
a^{73}																				

x^{20}	x^{21}	x^{22}	x^{23}	x^{24}	x^{25}	x^{26}	x^{27}	x^{28}	x^{29}	x^{30}	x^{31}	x^{32}	x^{33}	x^{34}	x^{35}	x^{36}	x^{37}	x^{38}	x^{39}	
440		462		562		327		215		$\overline{159}$		$\overline{269}$		$\overline{418}$		$\overline{348}$		$\overline{284}$		a^{52}
	434		410		412		206		42		$\overline{262}$		$\overline{308}$		$\overline{348}$		$\overline{281}$		$\overline{120}$	a^{53}
338		330		391		200		108		$\overline{157}$		$\overline{239}$		$\overline{339}$		$\overline{273}$		$\overline{215}$		a^{54}
	302		271		252		107		$\overline{21}$		$\overline{225}$		$\overline{260}$		$\overline{270}$		$\overline{211}$		$\overline{97}$	a^{55}
222		221		239		104		38		$\overline{149}$		$\overline{199}$		$\overline{257}$		$\overline{201}$		$\overline{157}$		a^{56}
	198		176		149		43		$\overline{37}$		$\overline{181}$		$\overline{191}$		$\overline{195}$		$\overline{152}$		$\overline{61}$	a^{57}
148		136		139		45		$\overline{3}$		$\overline{116}$		$\overline{154}$		$\overline{181}$		$\overline{142}$		$\overline{108}$		a^{58}
	117		91		65		7		$\overline{57}$		$\overline{135}$		$\overline{143}$		$\overline{135}$		$\overline{103}$		$\overline{47}$	a^{59}
79		73		63		11		$\overline{23}$		$\overline{96}$		$\overline{108}$		$\overline{121}$		$\overline{94}$		$\overline{67}$		a^{60}
	59		46		31		$\overline{13}$		$\overline{44}$		$\overline{92}$		$\overline{90}$		$\overline{86}$		$\overline{64}$		$\overline{20}$	a^{61}
41		34		29		$\overline{6}$		$\overline{28}$		$\overline{57}$		$\overline{71}$		$\overline{75}$		$\overline{57}$		$\overline{39}$		a^{62}
	25		18				$\overline{16}$		$\overline{37}$		$\overline{57}$		$\overline{60}$		$\overline{50}$		$\overline{35}$		$\overline{17}$	a^{63}
13		14		4		$\overline{11}$		$\overline{20}$		$\overline{40}$		$\overline{43}$		$\overline{41}$		$\overline{32}$		$\overline{21}$		a^{64}
	10		3		$\overline{2}$		$\overline{13}$		$\overline{20}$		$\overline{33}$		$\overline{30}$		$\overline{25}$		$\overline{20}$		$\overline{4}$	a^{65}
7		3		$\overline{1}$		$\overline{7}$		$\overline{15}$		$\overline{19}$		$\overline{22}$		$\overline{20}$		$\overline{17}$		$\overline{10}$		a^{66}
	1		$\overline{1}$		$\overline{6}$		$\overline{7}$		$\overline{14}$		$\overline{16}$		$\overline{16}$		$\overline{13}$		$\overline{8}$		$\overline{5}$	a^{67}
				$\overline{2}$		$\overline{6}$		$\overline{7}$		$\overline{11}$		$\overline{10}$		$\overline{9}$		$\overline{8}$		$\overline{4}$		a^{68}
			$\overline{2}$		$\overline{1}$		$\overline{6}$		$\overline{5}$		$\overline{7}$		$\overline{7}$		$\overline{4}$		$\overline{4}$			a^{69}
				$\overline{2}$		$\overline{2}$		$\overline{4}$		$\overline{3}$		$\overline{4}$		$\overline{4}$		$\overline{3}$		$\overline{1}$		a^{70}
	$\overline{1}$		$\overline{1}$		$\overline{2}$		$\overline{2}$		$\overline{2}$		$\overline{3}$		$\overline{2}$		$\overline{2}$				$\overline{1}$	a^{71}
		$\overline{1}$		$\overline{1}$		$\overline{1}$		$\overline{1}$		$\overline{2}$		$\overline{1}$		$\overline{1}$		$\overline{1}$				a^{72}
																			$\overline{1}$	a^{73}

Table of Groundforms.

Degree in the Coefficients	Order in the Variables: 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	21	22
1										1											
2			1				1				1				1						
3				1		1		1		2		1		1		1		1		1	
4	2				2		2		3		2		2		2		1		1		1
5		1		3		4		4		3		4		2		2					
6			4		4		6		6		3		3								
7		4		7		8		7		5											
8	5		8		10		10		2												
9		9		14		10		2													
10	5		15		14																
11		17		16																	
12	14		23																		
13		25																			
14	17		9																		
15		26																			
16	21																				
17		5																			
18	25																				

Decimic.

G. F. for differentiants,

Denominator: $(1-a)(1-a^2)^2(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)$
$(1-a^8)(1-a^9)$.

Numerator: $1 + 3a^2 + 11a^3 + 27a^4 + 58a^5 + 112a^6 + 193a^7 + 318a^8 + 485a^9$
$+ 699a^{10} + 951a^{11} + 1245a^{12} + 1541a^{13} + 1842a^{14} + 2108a^{15}$
$+ 2321a^{16} + 2451a^{17} + 2506a^{18} + 2451a^{19} + 2321a^{20} + 2108a^{21}$
$+ 1842a^{22} + 1541a^{23} + 1245a^{24} + 951a^{25} + 699a^{26} + 485a^{27}$
$+ 318a^{28} + 193a^{29} + 112a^{30} + 58a^{31} + 27a^{32} + 11a^{33} + 3a^{34}$
$+ a^{36}$.

G. F. for covariants, reduced form,*

Denominator: $(1-a^2)^2(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^8)$
$(1-a^9)(1-ax^2)(1-ax^4)(1-ax^6)(1-ax^8)(1-ax^{10})$.

Numerator:

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}
a^0	1														
a^1		$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$										
a^2	$\bar{1}$		1	1	2	2	2	1	1						
a^3	$\bar{1}$	2	1	2	1	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{1}$			
a^4		1	2			$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{1}$	1	1	1	1	1	
a^5		2	2			$\bar{1}$	$\bar{2}$		1	1	1	1			$\bar{1}$
a^6	3	1	1	1	$\bar{1}$	$\bar{2}$	$\bar{2}$			1		1	$\bar{1}$	$\bar{1}$	
a^7		1		1	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{1}$		1	1		$\bar{1}$	$\bar{1}$	1
a^8	2	3	4	2	1	2	$\bar{2}$	$\bar{1}$		2				$\bar{1}$	1
a^9	2	5		1	$\bar{2}$	$\bar{6}$	$\bar{7}$	$\bar{7}$	$\bar{3}$	$\bar{2}$					
a^{10}	4	3	3		$\bar{4}$	$\bar{6}$	$\bar{6}$	$\bar{3}$		4	5	4	2	1	
a^{11}		4	2	$\bar{3}$	$\bar{6}$	$\bar{7}$	$\bar{7}$	$\bar{4}$	2	2	5	1	1	$\bar{1}$	$\bar{3}$
a^{12}	6	5	4	1	$\bar{2}$	$\bar{5}$	$\bar{7}$	$\bar{2}$	2	5	4	3	1	$\bar{2}$	
a^{13}	$\bar{1}$	3	$\bar{1}$	$\bar{5}$	$\bar{11}$	$\bar{17}$	$\bar{12}$	$\bar{9}$		2	6	3	1	$\bar{1}$	$\bar{2}$
a^{14}	1	4	7	$\bar{1}$	$\bar{3}$	$\bar{6}$	$\bar{5}$	5	10	14	11	7	4	$\bar{3}$	$\bar{2}$
a^{15}		5	1	$\bar{4}$	$\bar{9}$	$\bar{17}$	$\bar{12}$	$\bar{6}$	3	3	5	1	$\bar{4}$	$\bar{5}$	$\bar{4}$
a^{16}	3	1	2	$\bar{5}$	$\bar{11}$	$\bar{11}$	$\bar{6}$	3	10	17	13	8		$\bar{2}$	
a^{17}	$\bar{4}$	1	$\bar{3}$	$\bar{9}$	$\bar{10}$	$\bar{10}$	$\bar{2}$	4	15	13	8	1	$\bar{4}$	$\bar{6}$	$\bar{6}$
a^{18}	4		$\bar{1}$	$\bar{1}$	$\bar{1}$	2	3	13	13	14	4	1	$\bar{6}$	$\bar{8}$	1
a^{19}	$\bar{3}$	$\bar{5}$	$\bar{8}$	$\bar{8}$	$\bar{8}$	$\bar{7}$	1	2	4	4	1	$\bar{3}$	$\bar{9}$	$\bar{3}$	$\bar{1}$
a^{20}		$\bar{3}$	$\bar{1}$		4	14	13	16	13	14	4		$\bar{1}$	$\bar{3}$	

* Numerator and denominator divisible by $1-a$; see foot-note to reduced form for sextic.

Numerator—(*Continued.*)

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}
a^{21}	$\bar{1}$	$\bar{3}$	$\bar{9}$	$\bar{3}$	1	4	4	2	1	$\bar{7}$	$\bar{8}$	$\bar{8}$	$\bar{8}$	$\bar{5}$	$\bar{3}$
a^{22}	1	$\bar{8}$	$\bar{6}$	1	4	14	13	13	3	2	$\bar{1}$	$\bar{1}$	$\bar{1}$		4
a^{23}	$\bar{6}$	$\bar{6}$	$\bar{4}$	1	8	13	15	4	$\bar{2}$	$\bar{10}$	$\bar{10}$	$\bar{9}$	$\bar{3}$	1	$\bar{4}$
a^{24}		$\bar{2}$		8	13	17	10	3	$\bar{6}$	$\bar{11}$	$\bar{11}$	$\bar{5}$	2	1	3
a^{25}	$\bar{4}$	$\bar{5}$	$\bar{4}$	1	5	3	3	$\bar{6}$	$\bar{12}$	$\bar{17}$	$\bar{9}$	$\bar{4}$	1	5	
a^{26}	$\bar{2}$	$\bar{3}$	4	7	11	14	10	5	$\bar{5}$	$\bar{6}$	$\bar{3}$	$\bar{1}$	7	4	1
a^{27}	$\bar{2}$	$\bar{1}$	1	3	6	2		$\bar{9}$	$\bar{12}$	$\bar{17}$	$\bar{11}$	$\bar{5}$	$\bar{1}$	3	$\bar{1}$
a^{28}		$\bar{2}$	1	3	4	5	2	$\bar{2}$	$\bar{7}$	$\bar{5}$	$\bar{2}$	1	4	5	6
a^{29}	$\bar{3}$	$\bar{1}$	1	1	5	2	2	$\bar{4}$	$\bar{7}$	$\bar{7}$	$\bar{6}$	$\bar{3}$	2	4	
a^{30}		1	2	4	5	4		$\bar{3}$	$\bar{6}$	$\bar{6}$	$\bar{4}$		3	3	4
a^{31}						$\bar{2}$	$\bar{3}$	$\bar{7}$	$\bar{7}$	$\bar{6}$	$\bar{2}$	1		5	2
a^{32}	1	$\bar{1}$				2		$\bar{1}$	$\bar{2}$	2	1	2	4	3	2
a^{33}	1	$\bar{1}$	$\bar{1}$		1	1		$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	1		1	
a^{34}		$\bar{1}$	$\bar{1}$	1		1			$\bar{2}$	$\bar{2}$	$\bar{1}$	1	1	1	3
a^{35}	$\bar{1}$			1	1	1	1		$\bar{2}$	$\bar{1}$			2	2	
a^{36}		1	1	1	1	1	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$			2	1	
a^{37}				$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{1}$	1	2	1	2	$\bar{1}$
a^{38}							1	1	2	2	2	1	1		$\bar{1}$
a^{39}											$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	
a^{40}															1

G. F. for covariants, representative form,

Denominator: $(1-a^2)(1-a^4)(1-a^6)^2(1-a^8)(1-a^9)(1-a^{10})(1-a^{14})$
$(1-a^2x^4)(1-a^2x^8)(1-a^2x^{12})(1-a^2x^{16})(1-ax^{10})$.

Numerator:

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}	x^{36}	x^{38}	x^{40}	x^{42}	x^{44}	x^{46}	x^{48}
a^0	1																								
a^3		1		2	1	1	2	1	1	1	1		1												
a^4			3	1	3	3	2	3	1	2	1	1		1											
a^5		3	3	4	5	4	5	2	4		1		$\bar{1}$		$\bar{2}$		$\bar{1}$		$\bar{1}$						
a^6	2	2	6	8	8	9	6	7	2	4				$\bar{2}$		$\bar{1}$		$\bar{1}$							
a^7		7	10	11	13	11	11	7	6	1			$\bar{4}$		$\bar{2}$				1		1		1		
a^8	4	8	14	18	20	22	12	11	4	2	$\bar{2}$	$\bar{4}$	$\bar{3}$	$\bar{6}$	$\bar{1}$	$\bar{3}$				1					
a^9	4	15	21	27	30	24	23	12	7	$\bar{1}$	$\bar{8}$	$\bar{6}$	$\bar{9}$	$\bar{5}$	$\bar{5}$	$\bar{1}$	$\bar{1}$		2		1				$\bar{1}$
a^{10}	7	20	31	37	39	39	22	15	2	$\bar{8}$	$\bar{11}$	$\bar{18}$	$\bar{14}$	$\bar{15}$	$\bar{5}$	$\bar{4}$		2	1	2		1			
a^{11}	8	28	41	50	56	46	31	12	$\bar{2}$	$\bar{17}$	$\bar{28}$	$\bar{25}$	$\bar{26}$	$\bar{18}$	$\bar{13}$	$\bar{5}$	1	2	5	1	2		$\bar{1}$		
a^{12}	15	38	54	67	69	60	33	11	$\bar{12}$	$\bar{33}$	$\bar{41}$	$\bar{45}$	$\bar{36}$	$\bar{31}$	$\bar{12}$	$\bar{3}$	2	7	6	7	2	1		$\bar{1}$	
a^{13}	15	49	72	84	90	70	37	3	$\bar{26}$	$\bar{54}$	$\bar{66}$	$\bar{62}$	$\bar{56}$	$\bar{39}$	$\bar{21}$	$\bar{2}$	8	12	14	7	4	1	$\bar{1}$		
a^{14}	20	61	87	104	106	82	32	$\bar{9}$	$\bar{48}$	$\bar{86}$	$\bar{95}$	$\bar{93}$	$\bar{73}$	$\bar{55}$	$\bar{20}$	2	14	20	18	16	5	2	$\bar{3}$	$\bar{3}$	
a^{15}	27	75	108	127	128	92	32	$\bar{26}$	$\bar{76}$	$\bar{120}$	$\bar{134}$	$\bar{119}$	$\bar{100}$	$\bar{66}$	$\bar{25}$	11	27	32	32	18	9		$\bar{2}$	$\bar{2}$	$\bar{2}$
a^{16}	29	90	129	147	146	100	22	$\bar{49}$	$\bar{110}$	$\bar{165}$	$\bar{172}$	$\bar{157}$	$\bar{120}$	$\bar{77}$	$\bar{18}$	26	41	52	44	32	9	1	$\bar{6}$	$\bar{7}$	
a^{17}	35	105	148	168	164	103	5	$\bar{81}$	$\bar{153}$	$\bar{218}$	$\bar{227}$	$\bar{195}$	$\bar{150}$	$\bar{88}$	$\bar{15}$	44	67	70	63	37	14		$\bar{9}$	$\bar{8}$	$\bar{4}$
a^{18}	40	119	168	191	179	105	$\bar{11}$	$\bar{115}$	$\bar{201}$	$\bar{272}$	$\bar{274}$	$\bar{232}$	$\bar{169}$	$\bar{94}$	1	71	93	101	82	51	13	$\bar{6}$	$\bar{13}$	$\bar{15}$	$\bar{4}$
a^{19}	44	132	189	204	192	101	$\bar{36}$	$\bar{154}$	$\bar{254}$	$\bar{330}$	$\bar{330}$	$\bar{267}$	$\bar{190}$	$\bar{88}$	24	108	132	133	112	62	17	$\bar{7}$	$\bar{20}$	$\bar{20}$	$\bar{7}$
a^{20}	47	147	202	221	200	94	$\bar{64}$	$\bar{202}$	$\bar{305}$	$\bar{395}$	$\bar{379}$	$\bar{303}$	$\bar{203}$	$\bar{85}$	48	150	172	171	133	74	14	$\bar{16}$	$\bar{30}$	$\bar{28}$	$\bar{8}$
a^{21}	55	154	216	232	203	83	$\bar{98}$	$\bar{241}$	$\bar{365}$	$\bar{447}$	$\bar{431}$	$\bar{327}$	$\bar{208}$	$\bar{70}$	92	196	222	208	166	85	11	$\bar{26}$	$\bar{41}$	$\bar{38}$	$\bar{15}$
a^{22}	52	164	226	236	202	63	$\bar{127}$	$\bar{292}$	$\bar{413}$	$\bar{506}$	$\bar{470}$	$\bar{346}$	$\bar{210}$	$\bar{42}$	130	257	272	255	194	93	7	$\bar{37}$	$\bar{53}$	$\bar{49}$	$\bar{15}$
a^{23}	57	166	229	237	194	50	$\bar{168}$	$\bar{333}$	$\bar{465}$	$\bar{550}$	$\bar{502}$	$\bar{359}$	$\bar{193}$	17	186	310	327	296	220	103	$\bar{4}$	$\bar{52}$	$\bar{73}$	$\bar{61}$	$\bar{20}$
a^{24}	56	172	228	236	187	22	$\bar{191}$	$\bar{372}$	$\bar{499}$	$\bar{585}$	$\bar{527}$	$\bar{353}$	$\bar{176}$	28	238	375	380	336	247	104	$\bar{13}$	$\bar{71}$	$\bar{88}$	$\bar{75}$	$\bar{27}$
a^{25}	57	166	227	225	168	7	$\bar{229}$	$\bar{401}$	$\bar{536}$	$\bar{610}$	$\bar{529}$	$\bar{347}$	$\bar{143}$	66	298	433	430	376	266	105	$\bar{31}$	$\bar{89}$	$\bar{109}$	$\bar{90}$	$\bar{29}$
a^{26}	52	164	217	211	155	$\bar{24}$	$\bar{249}$	$\bar{431}$	$\bar{551}$	$\bar{624}$	$\bar{536}$	$\bar{323}$	$\bar{114}$	119	346	487	474	403	281	98	$\bar{46}$	$\bar{111}$	$\bar{131}$	$\bar{105}$	$\bar{35}$
a^{27}	55	154	203	198	130	$\bar{38}$	$\bar{273}$	$\bar{442}$	$\bar{562}$	$\bar{620}$	$\bar{512}$	$\bar{296}$	$\bar{65}$	160	407	537	512	430	286	93	$\bar{68}$	$\bar{134}$	$\bar{150}$	$\bar{119}$	$\bar{40}$
a^{28}	47	147	190	176	112	$\bar{64}$	$\bar{281}$	$\bar{448}$	$\bar{556}$	$\bar{603}$	$\bar{490}$	$\bar{252}$	$\bar{26}$	216	448	578	541	443	296	76	$\bar{87}$	$\bar{155}$	$\bar{169}$	$\bar{132}$	$\bar{44}$

Numerator—(*Continued.*)

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}	x^{36}	x^{38}	x^{40}	x^{42}	x^{44}	x^{46}
a^{29}	44	132	169	155	87	$\overline{76}$	$\overline{296}$	$\overline{443}$	$\overline{541}$	$\overline{578}$	$\overline{448}$	$\overline{216}$	26	252	490	603	556	448	281	64	$\overline{112}$	$\overline{176}$	$\overline{190}$	$\overline{147}$
a^{30}	40	119	150	134	68	$\overline{93}$	$\overline{286}$	$\overline{430}$	$\overline{512}$	$\overline{537}$	$\overline{407}$	$\overline{160}$	65	296	512	620	562	442	273	38	$\overline{130}$	$\overline{198}$	$\overline{203}$	$\overline{154}$
a^{31}	35	105	131	111	46	$\overline{98}$	$\overline{281}$	$\overline{403}$	$\overline{474}$	$\overline{487}$	$\overline{346}$	$\overline{119}$	114	323	536	624	551	431	249	24	$\overline{155}$	$\overline{211}$	$\overline{217}$	$\overline{164}$
a^{32}	29	90	109	89	31	$\overline{105}$	$\overline{266}$	$\overline{376}$	$\overline{430}$	$\overline{433}$	$\overline{298}$	$\overline{66}$	143	347	529	610	536	401	229	$\overline{7}$	$\overline{168}$	$\overline{225}$	$\overline{227}$	$\overline{166}$
a^{33}	27	75	88	71	13	$\overline{104}$	$\overline{247}$	$\overline{336}$	$\overline{380}$	$\overline{375}$	$\overline{238}$	$\overline{28}$	176	353	527	585	499	372	191	$\overline{22}$	$\overline{187}$	$\overline{236}$	$\overline{228}$	$\overline{172}$
a^{34}	20	61	73	52	4	$\overline{103}$	$\overline{220}$	$\overline{296}$	$\overline{327}$	$\overline{310}$	$\overline{186}$	17	193	359	502	550	465	333	168	$\overline{50}$	$\overline{194}$	$\overline{237}$	$\overline{229}$	$\overline{166}$
a^{35}	15	49	53	37	$\overline{7}$	$\overline{93}$	$\overline{194}$	$\overline{255}$	$\overline{272}$	$\overline{257}$	$\overline{130}$	42	210	346	470	506	413	292	127	$\overline{63}$	$\overline{202}$	$\overline{236}$	$\overline{226}$	$\overline{164}$
a^{36}	15	38	41	26	$\overline{11}$	$\overline{85}$	$\overline{166}$	$\overline{208}$	$\overline{222}$	$\overline{196}$	$\overline{92}$	70	208	327	431	447	365	241	98	$\overline{83}$	$\overline{203}$	$\overline{232}$	$\overline{216}$	$\overline{154}$
a^{37}	8	28	30	16	$\overline{14}$	$\overline{74}$	$\overline{133}$	$\overline{171}$	$\overline{172}$	$\overline{150}$	$\overline{48}$	85	203	303	379	395	305	202	64	$\overline{94}$	$\overline{200}$	$\overline{221}$	$\overline{202}$	$\overline{147}$
a^{38}	7	20	20	7	$\overline{17}$	$\overline{62}$	$\overline{112}$	$\overline{133}$	$\overline{132}$	$\overline{108}$	$\overline{24}$	88	190	267	330	330	254	154	36	$\overline{101}$	$\overline{192}$	$\overline{204}$	$\overline{189}$	$\overline{132}$
a^{39}	4	15	18	6	$\overline{13}$	$\overline{51}$	$\overline{82}$	$\overline{101}$	$\overline{93}$	$\overline{71}$	$\overline{1}$	94	169	232	274	272	201	115	11	$\overline{105}$	$\overline{179}$	$\overline{191}$	$\overline{168}$	$\overline{113}$
a^{40}	4	8	9		$\overline{14}$	$\overline{37}$	$\overline{63}$	$\overline{70}$	$\overline{67}$	$\overline{44}$	15	88	160	195	227	218	153	81	$\overline{5}$	$\overline{163}$	$\overline{164}$	$\overline{168}$	$\overline{148}$	$\overline{105}$
a^{41}		7	6	$\overline{1}$	$\overline{9}$	$\overline{32}$	$\overline{44}$	$\overline{52}$	$\overline{41}$	$\overline{26}$	18	77	120	157	172	165	110	49	$\overline{22}$	$\overline{100}$	$\overline{146}$	$\overline{147}$	$\overline{129}$	$\overline{90}$
a^{42}	2	2	2		$\overline{9}$	$\overline{18}$	$\overline{32}$	$\overline{32}$	$\overline{27}$	$\overline{11}$	25	66	100	119	134	120	76	26	$\overline{32}$	$\overline{92}$	$\overline{128}$	$\overline{127}$	$\overline{108}$	$\overline{75}$
a^{43}		3	3	$\overline{2}$	$\overline{5}$	$\overline{16}$	$\overline{18}$	$\overline{20}$	$\overline{14}$	$\overline{2}$	20	55	73	93	95	86	48	9	$\overline{32}$	$\overline{82}$	$\overline{106}$	$\overline{104}$	$\overline{87}$	$\overline{61}$
a^{44}			1	$\overline{1}$	$\overline{4}$	$\overline{7}$	$\overline{14}$	$\overline{12}$	$\overline{8}$	2	21	39	56	62	66	54	26	$\overline{3}$	$\overline{37}$	$\overline{70}$	$\overline{90}$	$\overline{84}$	$\overline{72}$	$\overline{49}$
a^{45}		1		$\overline{1}$	$\overline{2}$	$\overline{7}$	$\overline{6}$	$\overline{7}$	$\overline{2}$	3	12	31	36	45	41	33	12	$\overline{11}$	$\overline{33}$	$\overline{60}$	$\overline{69}$	$\overline{67}$	$\overline{54}$	$\overline{38}$
a^{46}			1		$\overline{2}$	$\overline{1}$	$\overline{5}$	$\overline{2}$	$\overline{1}$	5	13	18	26	25	28	17	2	$\overline{12}$	$\overline{31}$	$\overline{46}$	$\overline{56}$	$\overline{50}$	$\overline{41}$	$\overline{28}$
a^{47}				$\overline{1}$		$\overline{2}$	$\overline{1}$	$\overline{2}$		4	5	15	14	18	11	8	$\overline{2}$	$\overline{15}$	$\overline{22}$	$\overline{39}$	$\overline{39}$	$\overline{37}$	$\overline{31}$	$\overline{20}$
a^{48}	1				$\overline{1}$		$\overline{2}$		1	1	5	5	9	6	8	1	$\overline{7}$	$\overline{12}$	$\overline{23}$	$\overline{24}$	$\overline{30}$	$\overline{27}$	$\overline{21}$	$\overline{15}$
a^{49}						$\overline{1}$				3	1	6	3	4	2	$\overline{2}$	$\overline{4}$	$\overline{11}$	$\overline{12}$	$\overline{22}$	$\overline{20}$	$\overline{18}$	$\overline{14}$	$\overline{8}$
a^{50}			$\overline{1}$		$\overline{1}$		$\overline{1}$				2		4			$\overline{1}$	$\overline{6}$	$\overline{7}$	$\overline{11}$	$\overline{11}$	$\overline{13}$	$\overline{11}$	$\overline{10}$	$\overline{7}$
a^{51}								1		1		2				$\overline{4}$	$\overline{2}$	$\overline{7}$	$\overline{6}$	$\overline{9}$	$\overline{8}$	$\overline{8}$	$\overline{6}$	$\overline{2}$
a^{52}							1		1		2		1		$\overline{1}$		$\overline{4}$	$\overline{2}$	$\overline{5}$	$\overline{4}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{3}$
a^{53}												$\overline{1}$		$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{1}$	$\overline{3}$	$\overline{2}$	$\overline{3}$	$\overline{3}$	$\overline{1}$	$\overline{3}$	
a^{54}													$\overline{1}$		$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{1}$	$\overline{1}$	$\overline{2}$		$\overline{1}$
a^{55}																								

Table of Groundforms.

Degree in the Coefficients.	Order in the Variables. 0	2	4	6	8	10	12	14	16	18	20	22	24	26
1						1								
2	1		1		1		1		1					
3		1		2	1	1	2	1	1	1	1		1	
4	1		3	1	3	3	2	3	1	2	1	1		1
5		3	3	4	5	4	5	2	4		1			
6	4	2	5	8	6	8	2	3						
7		7	10	8	12	2	3							
8	5	8	11	15	4	5								
9	5	13	19	8	4									
10	8	20	12	10										
11	8	18	21											
12	12	30												
13	15	16												
14	13	17												
15	19													
16	5													
17	3													

The total number of irreducible invariants and covariants for the first 10 orders (counting in the absolute constant and the quantic itself), it appears from what precedes, is as follows:

Order of Quantic: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

Number of Groundforms: 1, 2, 3, 5, 6, 24, 27, 125, 70, 416, 476.

For the benefit of those new to the subject, it may be well to recall the immediate algebraical meaning of either form of the generating function to a binary quantic $(x, y)^n$.

Suppose n an odd number, say 5, then if

$$\frac{1 - x^{-2}}{(1 - ax^{-5})(1 - ax^{-3})(1 - ax^{-1})(1 - ax)(1 - ax^3)(1 - ax^5)}$$

is expanded in a *bivergent* series, (that is, one going, as regards the powers of x, in two directions towards infinity), either generating function of the tables for the quintic is the sum of the terms which contain no negative powers of x. So if n be an even number, say 6,

$$\frac{1-x^{-2}}{(1-ax^{-6})(1-ax^{-4})(1-ax^{-2})(1-a)(1-ax^{2})(1-ax^{4})(1-ax^{6})}$$

being similarly expanded, either generating function of the tables for the sextic is, as before, the sum of the terms which contain only positive or zero powers of x. And so in general, for $(x, y)^n$, the numerator of the so-called *crude* generating function, being always $1-x^{-2}$ and its denominator a product of factors of the form $1-ax^{n-2i}$ (where i takes all values from nought up to n inclusive). Either generating function of the tables for the n^{ic} is the algebraic equivalent of the *positive* branch of the corresponding bivergent series, (that in which only positive powers of x appear), *plus* the *neutral* branch or term, namely, that which contains neither positive nor negative powers of x, or, which is the same thing, is a function only of a.

I subjoin a few reflexions which appear to me to be desirable on the foregoing tables.

It is scarcely necessary to state, that, in the development of the generating function, whether reduced or representative, the coefficient of $a^m x^\mu$ is the total number of linearly independent covariants of the degree m in the coefficients and the order μ in the variables.

Mr Franklin will probably, in a future number of the *Journal*, draw up a statement of the mode in which the tables have been calculated and the precautions taken to insure accuracy *; as regards the reduced form, three methods have been employed in calculating it, namely, Mr Sylvester's first method, Professor Cayley's method, fully explained in a preceding number of the *Journal* by its eminent author, and Mr Sylvester's second method, much briefer than his other, but, in general, not so brief as Professor Cayley's, which last, however, involves a delicate point in the expansion of series, the assumed principle of which, although its validity on moral grounds of evidence is unquestionable, cannot be regarded as *a priori* self-evident †.

The theory of the generating function, alike for single and simultaneous forms, depends on the law for determining the number of linearly indepen-

* In especial I wish to single out an ingenious device of Mr Franklin to check the operation of tamisage by introducing a common superfluous factor into the numerator and denominator of the representative generating function so selected as that the augmented denominator shall not cease to be representative; the effect of this will be to cause the groundforms obtained by tamisage of the augmented numerator to be the same as before, except that the groundform represented by the additional factor will not be found among them.

† In Prof. Cayley's method the crude generating function is regarded as a function of a; in my two methods as a function of x.

dent in- and co-variants of given order and degree or degrees belonging to a given quantic or system of quantics, a proof of which will be found at the end of a memoir by Mr Sylvester in *Borchardt's Journal* *, and also in the *London and Edinburgh Philosophical Magazine* †, that leaves nothing to be desired as regards rigour of demonstration. The law itself for the case of a single quantic was first stated by Professor Cayley whilst the theory was still in its infancy.

But besides this fundamental theorem, in order to deduce the tables of groundforms, a *fundamental postulate* still awaiting demonstration is necessary, which is, that no more linear relations between in- or co-variants are to be supposed to exist than are necessary in order to satisfy the *fundamental theorem*. The application of this principle in such a mode as to substitute a finite for an infinite process, leads to the use of representative generating functions and the simplified method of *tamisage*. The validity of the fundamental-postulate which is in accord with the law of parcimony is verified by its conducting to results which have been proved to be accurate for single binary quantics up to the sixth order inclusive, for pairs of binary quantics up to the fourth order inclusive, and also for systems of an indefinite number of linear and quadratic binary forms ‡.

The application of this principle discloses the remarkable singularity that for the quantic of the seventh order, there exists no finite representative generating function as shown in what follows.

The invariantive part of the numerator of the reduced form for the seventhic is

$$1-a^6+2a^8-a^{10}+5a^{12}+2a^{14}+6a^{16}+2a^{18}+5a^{20}-a^{22}+2a^{24}-a^{26}+a^{32},$$

and the invariantive part of the denominator is $(1-a^4)(1-a^6)(1-a^8)(1-a^{10})$. Multiplying numerator and denominator by $(1+a^6)$, their invariantive portions§ become, respectively,

$$1+2a^8-a^{10}+4a^{12}+4a^{14}+5a^{16}+7a^{18}+7a^{20}+5a^{22}+4a^{24}+4a^{26}-a^{28}+2a^{30}+a^{38},$$

and $$(1-a^4)(1-a^8)(1-a^{10})(1-a^{12}).$$

[* p. 232 above.] [† p. 117 above.]

‡ If the *fundamental postulate* were called into question, this (it may be proved) would not affect the fact of the existence of the groundforms obtained by its aid, but only the possibility of the existence of other groundforms over and above those so obtained. Thus my tables of groundforms could only err (were that possible, which I do not believe it to be) in defect; and as those found by the German method can only err in excess, it follows that, whenever the tables coincide, both must be correct. The tables of groundforms here given, up to the sixth order, inclusive, and all those that follow, coincide exactly with those obtained by Clebsch, Gordan and Gundelfinger, when these latter are rectified by the omission of certain supposed groundforms which, in the *Comptes Rendus*, I have conclusively proved to be composite.

§ The factors in the denominator which involve x never offer any difficulty, as they represent the given quantic along with the complete system of covariants of the second degree, the several orders of which follow a well known rule.

The factors of the denominator are now, with the exception of $1-a^{10}$, representative factors; $1-a^{10}$ is not such, as a^{10} occurs in the numerator with the coefficient -1. If we multiply numerator and denominator by $1+a^{10}$, the factor $1-a^{20}$ will take the place of $1-a^{10}$ in the denominator, and the numerator will become

$$1+2a^{8}+4a^{12}+4a^{14}+5a^{16}+9a^{18}+6a^{20}+\ldots.$$

Here the coefficient of a^{20} is not negative, but it is less than the number (8) obtained by composition from the terms $2a^{8}$ and $4a^{12}$; hence, by the fundamental postulate there is no irreducible invariant of the degree 20. If, instead of multiplying numerator and denominator by $1+a^{10}$, we multiply them by the infinite series $1+a^{10}+a^{20}+\ldots$, the denominator becomes representative and the invariantive part of the numerator becomes the *recurrent* series given in the table (p. [288]), in which the coefficient of a^{30}, a^{40} and, in general, all powers of a whose exponents are multiples of and greater than 20, is 9; but 9 is less than the number obtained in the composition of a^{30}, a^{40} (and *a fortiori* of a^{50}, a^{60}, ...) out of the preceding terms; therefore, by the fundamental postulate, there is no irreducible invariant whose degree is any multiple of 10. It is a remarkable and significant fact that in this case the erroneous assumption of $1-a^{10}$ being a representative factor in the denominator of the complete generating function will be found to lead to no subsequent further error in the determination of the other groundforms of the seventhic.

A chorographical law obtains in the numerical tables of the numerators of the representative forms, which plays a considerable part in the complete theory of tamisage, and is too important to be passed over without notice, namely, it will be seen that all these tables consist of a small number of irregular but continuous bands or blocks of alternately positive and negative coefficients which can be drawn asunder without tearing or leaving any hole in the paper*. For the first four orders there is but one such block, for the quintic and the sextic two, for the seventhic five, for the octavic three, and for the 9^{ic} and 10^{ic} four. A similar law obtains for systems of quantics, as for instance in the case of two simultaneous quantics, the corresponding tables consist of detachable solid blocks, alternately positive and negative, and small in number in comparison with the number of terms which they contain, as will be seen in the tables to appear in the next number of the *Journal* which will contain a complete set of them for all the systems that can be formed of two binary quantics of orders, m, n where neither m nor n exceeds 4.

It is my duty to state that the expense of calculating the tables for quantics of the 7th, 8th, 9th and 10th orders, has been defrayed out of a grant made by the British Association for the Advancement of Science, and I have pleasure in returning my thanks to that distinguished body for this act of aid in enabling me to bring to a successful issue an undertaking of such unusual magnitude and of such pith and moment to the progress of Algebraical Theory.

* In the operation of tamisage on the numerator of the representative groundforms the terms of the negative blocks are disregarded. In every case treated in these tables, and those to follow in the next number of the *Journal*, the only surviving terms will be found to be comprised in the first block. Had it turned out otherwise it would have been necessary to ascertain whether the surviving terms belonging to the other odd-numbered blocks would survive the operation of tamisage performed on the infinite aggregate of terms obtained by the development of the generating function; if not, they would have to be rejected. This is what I have found actually happens in a system of quadratic or linear forms when a sufficient number of such forms is employed. In that case, terms not confined to the first block emerge from the tamisage of the numerator of the representative groundforms, but disappear when the tamisage is performed on the infinite aggregate of terms of which the groundform is the sum. Such aggregate, it may be noticed, (I have proved elsewhere), consists exclusively of positive terms, the coefficients corresponding to non-existing types being always zero and never negative. It is very likely to be found true hereafter that in no case need any, except the first block of terms in the numerator of the representative groundforms, be submitted to tamisage in order to obtain the groundforms not represented in the denominator, and so in like manner that, in order to obtain the ground-syzygies of the first kind, that is, those that concern the groundforms, only the first positive and the first negative block need be considered, and so on for syzygies of the higher orders, each time a new block being taken into account until all are exhausted, it being quite conceivable that the number of blocks may designate the highest order of syzygy that occurs in any case, subject in the case of a linear or quadratic form (for which the block reduces to a single term, namely, unity) to the obvious exception that, for them, the syzygies become abortive.

To explain what is meant by syzygies of successive orders, suppose Z to be a rational and integral function of groundforms which, regarded as a function of the coefficients, is identically zero, then $Z=0$ is a syzygy and Z may be termed a syzygant of the first order and, if incapable of being resolved into a sum of products of syzygants multiplied respectively by rational algebraic functions of the groundforms, will be an irreducible or ground-syzygy of the first order. In like manner, if Z' is a function of ground-syzygants which, regarded as a function of the groundforms, vanishes identically $Z'=0$ is a syzygy and Z' is a counter-syzygant or a syzygant of the second order, and, if incapable of representation as a sum of products of other syzygants of the second order multiplied respectively by rational integral functions of syzygants of the first order, is a ground-syzygant of the second order; and so on indefinitely.

39.

ON CERTAIN TERNARY CUBIC-FORM EQUATIONS.

[*American Journal of Mathematics*, II. (1879), pp. 280—285, 357—393; III. (1880), pp. 58—88, 179—189.]

CHAPTER I. *On the Resolution of Numbers into the sums or differences of Two Cubes.*

SECTION 1.

M. LUCAS has written to inform me that in some one or more of a series of memoirs commencing with 1870, or elsewhere, the Reverend Father Pépin has made considerable additions to my published theorems* on the classes of numbers irresoluble into the *sum or difference*† of two rational cubes.

Using p, q to denote primes of the forms $18n+5$, $18n+11$, besides the 6 forms published by me, M. Pépin has found 10 other general classes of irresoluble numbers, the total number (as I understand from M. Lucas) known to the Reverend Father being as follows:

$$p,\quad q^2,\quad p^2,\quad q,\quad 2p,\quad 2q^2,\quad 4p^2,\quad 4q,$$
$$9p,\quad 9q^2,\quad 9p^2,\quad 9q,\quad 25p,\quad 25q^2,\quad 5p^2,\quad 5q,$$

but the last four of these classes are special cases only, of three out of the four more general irresoluble classes pq, p^2q^2, $p_1p_2^2$, $q_1q_2^2$, where p_1, p_2 are any two numbers of the p class and q_1, q_2 any two of the q class. On making $p=5$ in the first two of these, and $p_1=5$, $p_2=p$, or $p_2=5$, $p_1=p$, in the third, Father Pépin's last four classes result. It is also true that the numbers in my four additional general classes respectively multiplied by 9 are still irresoluble. Hence the number of known classes of numbers (depending on p and q) irresoluble into the sum or difference of cubes may be arranged as follows:

$$p,\quad q,\quad p^2,\quad q^2,\quad pq,\quad p^2q^2,\quad p_1p_2^2,\quad q_1q_2^2,$$
$$9p,\quad 9q,\quad 9p^2,\quad 9q^2,\quad 9pq,\quad 9p^2q^2,\quad 9p_1p_2^2,\quad 9q_1q_2^2,$$
$$2p,\quad 4q,\quad 4p^2,\quad 2q^2.$$

[* See Vol. I. of this Reprint, pp. 107—118, and Vol. II. pp. 63, 107.]

† It is well to understand that a number resoluble into the sum is necessarily also resoluble into the difference of two positive cubes and *vice versâ*.

Moreover, I have ascertained the truth of the following two theorems of a somewhat different character:

1st. Let ρ, ψ, ϕ denote prime numbers respectively of the forms $18n+1$, $18n+7$, $18n+13$ and suppose ρ, ψ, ϕ to be *not* of the form f^2+27g^2 and consequently *not* to possess the cubic residue 2, then I say that all the numbers comprised in any one of the eight classes

$$2\rho,\quad 4\rho,\quad 2\rho^2,\quad 4\rho^2,\quad 2\psi,\quad 4\psi^2,\quad 4\phi,\quad 2\phi^2$$

are irresoluble into the sum of two cubes*.

2nd. Provided 3 is not a cubic residue to ν† [where ν, any $6n+1$ prime, is the same as ρ, ϕ, ψ taken collectively], 3ν and $3\nu^2$ are similarly irresoluble.

With the aid of these theorems and certain special cases of irresolubility noticed by Father Pépin, communicated to me by M. Lucas, supplemented by calculations of M. Lucas and my own as regards the non-excluded numbers, it follows (*mirabile dictu*) that of the first 100 of the natural order of numbers, there is only a single one, namely, 66, of which it cannot at present be affirmed with certitude either that it is or is not resoluble into the sum of two cubes, and of which, in the former case, the resolution cannot be exhibited.

The proof of these statements, and the resolutions into cubes in their lowest terms, when they exist, will be given in the next number of the *Journal*. For the present I limit myself to noticing (what I much regret not to have done before the paper was printed) a statement of M. Lucas which is capable of being misunderstood and might give rise to an erroneous conception.

It is where this distinguished contributor to our *Journal* speaks of deriving from one rational point on a cubic curve (defined by a cubic equation with integer coefficients) another by means of its intersections with a

* The exclusion of 2 as a cubic residue blocks out the possibility of the "distribution of the amplitude"; the form p^2+27q^2 blocks out the possibility of a solution in which x^2-xy+y^2 has a common factor with the amplitude, and thereby imposes upon the equation containing x, y, z (were it soluble in integers) the necessity of repeating itself perpetually with smaller numbers, which of course is impossible. But the two conditions thus separately stated are in fact mutually implicative, every number of the form f^2+27g^2 having 2 for a cubic residue and *vice versâ* every number of the form $6n+1$ to which 2 is a cubic residue being of the form f^2+27g^2. The sole condition, therefore, in order that a number coming under any of the eight categories in the text shall be known at sight to be irresoluble into the sum of two cubes, is that its variable part shall not be of the form p^2+27q^2, that is, shall not be 31, 43, 91, 109, 127, 157, 223, 229, 247, etc.

† If I am not mistaken this is tantamount to the proviso that ν shall not be of the form $f^2\pm 9fg+81g^2$. It is worth noticing that the above quantity multiplied by 3, say $3N$, is equal to $\frac{(9g\mp f)^3+(18g\pm f)^3}{27g}$, so that when g is a cube number N is immediately resoluble. The initial values of N will be found to be 61, 67, 73, 103, 151, 193, 271, 367, 547, etc., for each of which, up to 367 inclusive, $g=1$ or $g=-1$, so that their products by 3 are immediately resoluble.

conic drawn through five consecutive points situated at the given rational one; but, in fact, it follows from my theory of *residuation* that this point is collinear with the given point and its second tangential: just as a ninth point in which the cubic would be met by any other cubic passing through *eight* consecutive points situated at the given point would be the third tangential to the latter *.

Hence M. Lucas' third method amounts only to a combination of the other two; and in fact there is *but one single scale* of rational derivatives from any given point in a general cubic, the successive terms of which expressed in terms of the coordinates of the primitive are of the orders 1, 4, 16, 25, 49, ... the squares of the natural numbers with the multiples of 3 omitted †.

Scholium.

I term lmn the *amplitude* of the equation $lx^3 + my^3 + nz^3 = 0$, and if A cannot be broken up in any way into factors l, m, n, such that

$$lx^3 + my^3 + nz^3 = 0$$

shall be soluble in integers, I call the amplitude A of the equation

$$x^3 + y^3 + Az^3 = 0$$

undistributable.

When A is of the form $\dfrac{x^3 - 3x^2y + y^3}{3z^3}$, the equation $x^3 + y^3 + Az^3 = 0$ is always soluble, and when this equation is soluble, then, provided that its amplitude is undistributable and contains no prime factor of the form $6i + 1$, the equation $x^3 - 3x^2y + y^3 = 3Az^3$ must be soluble in integers, which cannot be the case when A contains any factor other than 3, or of the form $18i \pm 1$, inasmuch as *the cubic form* $x^3 - 3x \pm 1$ *contains no factors other than* 3 *or of the form* $18i \pm 1$.

* I make the important additional remark that at those special points of the cubic where this ninth point (sometimes elegantly called the subosculatrix) coincides with the point osculated, the scheme of rational derivatives returns upon itself, and instead of an infinite number there will be only two rational derivatives to such point. That is to say the infinite scheme becomes a system of 3 continually recurring points. The general theory of the special points which have only a finite number of rational derivatives will be given in the next number of the *Journal*.

† When the cubic is of the form $Ax^3 + Ay^3 + Cz^3 + Mxyz = 0$, where A, C, M are integers, then a rational point of inflection $x = 1$, $y = -1$, $z = 0$ is known and, in that case, from any other rational point *besides the ordinary ones* derivative rational points of the missing orders 9, 36, 81 can be found, but no others, and so universally if in the general cubic a rational point of inflection and a rational point (a, b, c) are given the scale of rational derivatives will be of the orders 1, 4, 9, 16, ... in a, b, c. This scale will of course be duplex, consisting of a series of points and a second series in which the radii drawn through the points of the first series and the point of inflection again meet the cubic.

This last theorem is a particular case of the following: If k be any integer and $F(x, y)$, the product of factors of the form $\left(x - 2\cos\dfrac{2\lambda\pi}{k}\, y\right)$, where λ is every number prime to k up to $\frac{1}{2}(k-1)$, then Fx $[= F(x, 1)]$ contains no prime factors excepting such as are contained in k or else are of the form $ki \pm 1$ *.

If it could be shown, in analogy with what holds for the quadratic forms Fx which result from making $k = 8, 10, 12$, that the cubic form $x^3 - 3xy^2 \pm y^3$ which results from making $k = 18$ may always be made to represent any prime number of the form $18n \pm 1$ itself, or else its treble (and for our purpose rational numbers would be as efficient as integers), we should then be able to affirm that any prime $18n \pm 1$ or else its nonuple could be resolved into the sum of two cubes. As a matter of fact I have ascertained that every prime number $18n \pm 1$ as far as 537 inclusive (and have no ground for supposing that the law fails at that point) can be represented by

$$x^3 - 3xy^2 \pm y^3$$

or else by its third part with *integer* values of x, y. Moreover, I find that the same thing is true of 17^2, $17 . 19$, 19^2, $17 . 37$, $19 . 37$, 37^2, $17 . 53$, $19 . 53$, $37 . 53$, that is, in fact for all the binary combinations of the natural progression of "r, ρ" numbers 17, 19, 37, 53, 71, 73, 89 (21 in all), as also 17^3, 19^3, 37^3 †. The number of *consecutive* r, ρ primes for which the law has been verified, that is, the number of those not exceeding 537 will be found to be 39, namely, 17, 19, 37, 53, 71, 73, 89, 107, 109, 127, 163, 179, 181, 197, 199, 233, 251, 269, 271, 307, 323, 341, 359, 361, 377, 379, 397, 413, 431, 433, 449, 451, 467, 469, 487, 503, 521, 523, 541, which according to the usual canons of induction would, I presume, be considered almost sufficient to establish the theorem for the case of $k = 9$.

* Thus, by making $k = 8$ we learn that $x^2 - 2$ contains no factors except 2 and $8i \pm 1$ and by making $k = 16$, that $y^4 - 4y^2 + 2$, none except 2 or $16i \pm 1$, by making $k = 9$ that $x^3 - 3x + 1$, by making $k = 18$, that $x^3 - 3x - 1$ contain no other factors but 3, or numbers of the form $18n \pm 1$. The theorem, I am aware, is well known for the case where k is a prime number and possibly is so for the general case. The proof of the irresolubility into two cubes of the 20 classes of numbers involving p's and q's, given at page [312], is an instantaneous consequence of the theorem for the case of $k = 9$, for which case also there is no shadow of doubt of the theorem being true.

† 53^2 has not yet made its appearance. All the primes of that form themselves occurring in the first six hundred numbers have already occurred in my calculations except 557 and 593. I have worked with the formula $x^3 - 3xy^2 \pm y^3$ [x and y relative primes], giving to x and to y all the values possible from 1 to 36, and intend to extend the table to the limit of 50 or 100. The longer a moderate-sized number is in making its appearance, the longer it is likely to be before it appears, inasmuch as the large numbers of which it is the residuum or balance are becoming continually greater. It may very well then happen that the missing numbers alluded to may transcend all practicable limits of calculation to find them just as would be the case, for certain values of A, with finding values of x, y to satisfy the Pellian equation $x^2 - Ay^2 = 1$, were there not a theoretical method of arriving at them.

The table of "*special* cases" of irresoluble numbers found by Father Pépin (according to the information most kindly communicated to me by M. Lucas) comprises the numbers

14, 21, 31, 38, 39, 52, 57, 60, 67, 76, 77, 93, 95 *,

all of which I have verified as irresoluble except the number 60, which I accept as such on the erudite and sagacious Father's authority.

Reverting to F, it is hardly necessary to recall that $F(x^2+y^2, xy)$ is the primitive factor of x^k-y^k, and that it is capable of very easy demonstration that this primitive factor contains no prime factors except such as are divisors of k or of the form $ki+1$, the linear divisor $ki-1$ being here excluded. It seems to be very probable that for $k=9$, $F(x, y)$ or else $3F(x, y)$ does represent any prime of the form $18n \pm 1$, and consequently that every such form of prime or else 9 times the same is the sum of two rational cubes †.

This last conjectural theorem, it will be noticed, is not in any real analogy to the theorem that every product of primes of the form $4n+1$, and also the double thereof, is the sum of two *integer* squares; the real analogy is between the fact, of which this theorem is a consequence, that $x^3-3xy^2 \pm y^3$ or its third part represents every number which is a product of primes of the form $18n \pm 1$, and each one of the facts that x^2-2y^2, x^2-5y^2 represent all numbers of the form $8i \pm 1$, $10i \pm 1$ respectively, and that x^2-3y^2 or its third part represents all numbers of the form $12i \pm 1$. On account of its importance to this theory it seems desirable to give a name to the law which governs the prime factors of $F(x, y)$, and I take advantage of the circumstance that $F(x^2+y^2, xy)$ contains prime factors of the form $ki+1$, but not of the form $ki-1$, whilst $F(x, y)$ contains prime factors of either of these forms indifferently, to characterize it as the Law of Twin Prime Factors. Let us suppose the circumference of a circle divided by points into k equal parts, and agree to designate the shorter arc between any two of the points a *primitive* division of the circle in respect to k, provided that no number less than k would be adequate to give rise to an equal length of arc, so that $\frac{2\lambda\pi}{k}$, when λ is prime to k and less than $\frac{k}{2}$, will serve to represent any such division. The assumed Law of Twin Factors (well known, I repeat, for the case of k a prime number and possibly in its extended form likewise) may then be enunciated as follows:

* Of these numbers all except 60, 31, 67, 77, 95 belong to some one or other of the general classes of irresoluble numbers given in the text.

† It may be and probably is true also that $x^3-3xy^2 \pm y^3$ will represent the product or else three times the product of any two primes each of which is of the form r or ρ, and possibly the square or else three times the square of any r or ρ; it cannot possibly represent three times *any cube*, for if it did we should be able to infer that a cube was resoluble into two cubes, which we know is not true.

That function of x whose first coefficient is unity and whose roots are the doubled cosines of all the primitive divisions of the circle in respect to k contains no prime factors except such as are divisors of, or else when increased or diminished by unity, are divisible by k. This may be called again the *Exclusional or Negative Theorem of Twin Factors;* and on the other hand the more extraordinary theorem which asserts (on evidence not yet conclusive) that the function of x above defined, when made homogeneous in x, y, will represent (at all events for the case of $k=9$) every prime number of the form $ki \pm 1$, or else certain specific multiples of any such number, may be called the *Inclusional or Representational Theorem of Twin Factors.*

Excursus A. *On the Divisors of Cyclotomic Functions.*

Title 1. *Cyclotomic Functions of the* 1*st Species.* In the preceding section which should have been termed and will be hereafter referred to as the *Proem* of Chapter I., I stated that the proof of the first batch of theorems on the irresoluble cases of equations in numbers of the form $x^3+y^3+Az^3=0$, or, as we might say, of the forms of numbers A irresoluble into a pair of rational cubes, depends on the demonstration of the form of the numerical linear divisors of the function x^3-3x+1. At the time when this proem went to press I had reduced to a certainty the law of the divisors by numerical verifications without end, but had not obtained a rational demonstration of it, nor was I able to find such or even a statement of the law itself in any of the current text-books, such as Gauss, Legendre, Bachmann, Lejeune-Dirichlet or Serret. I was therefore compelled to seek out a demonstration for myself, and in so doing was unavoidably led to consider the general theory of the species of *cyclotomic* (*Kreistheilung*) functions of which the cubic function above written is an example of what may be called the second species and incidentally also the theory of the simpler or first species which, although discussed ever since the time of Euler, appears to me to remain still in a somewhat cloudy and incomplete condition. As this inquiry extends beyond the strict needs of the subject which called it forth, I entitle it an *excursus*. It will be necessary for me eventually to introduce another and still more important excursus or lateral digression on certain consequences of the Geometrical Theory of Residuation, which theory itself also took its rise in and is required for the purposes of the arithmetical theory which forms the subject of the entire memoir.

If fx is any rational integral function of the order ω in its variable, we know that in respect to a prime number p as modulus fx regarded as the subject of a congruence cannot have more than ω distinct real roots. If we take p^j as modulus, certain conditions increasing in number with the value of j, will have to be satisfied in order that fx may have a superfluity (that is, more than ω) of real roots.

One condition, the universal *sine quâ non*, will serve for the object I have in view, so that it will be sufficient to make $j=2$. Obviously when this superfluity exists two of the roots must differ by a multiple of p since otherwise there would be a superfluity of roots *quâ* the first power of p as modulus. If then a and $a+\lambda p$ where $\lambda<p$ be two of the roots, we have $fa\equiv 0$ and $fa+\lambda f'a \,.\, p+Rp^2\equiv 0$ mod. p^2. Hence $fa\equiv 0$ and $f'a\equiv 0$ mod. p, so that $fa+\lambda p=0$ and $f'a+\mu p=0$.

Applying the dialytic method to eliminate a it is obvious that the resultant of these two equations will differ only by a multiple of p from that of fa and $f'a$, that is, from the arithmetical discriminant of fa (I use the term arithmetical to distinguish it from the algebraical discriminant in obtaining which latter fx is supposed to be affected with binomial numerical coefficients ω, $\frac{1}{2}\omega(\omega-1)$, ... and the factor ω to be struck out from each of the two equations $\dfrac{df(x,1)}{dx}=0$, $\dfrac{df(x,1)}{d1}=0$).

We see then that a rational integer function (the subject of a congruence) cannot have a superfluity of roots in respect to the power of a prime p^j as modulus, unless the strict (arithmetical) discriminant of the function contains p.

It is necessary for the purpose I have in view to express the strict relation between the arithmetical discriminant of a function, Δfx, and the product of the squares of the differences of its roots, $\zeta^2 fx$. I shall for greater simplicity suppose that the initial coefficient of fx is unity, as it is in the cases with which we shall have to deal.

We know that $\Delta f=\mu\zeta^2 f$ where μ is a function of n the order of f, so that to determine μ we may specialize f in any manner we please, provided the order is maintained. Let $fx=x^n-1$. Then it is easily proved that, making $\rho=\cos\dfrac{2\pi}{n}+i\sin\dfrac{2\pi}{n}$,

$$(-)^{n\frac{n-1}{2}}\zeta^2 f=(\rho^{n-1})^{(n-1)\frac{n}{2}} \,.\, n^n,$$

so that

$$\zeta^2 f=(-)^{\frac{(n-1)(n-2)}{2}} \,.\, n^n,$$

and

$$\Delta f=(-)^{n-1} \,.\, n^{2n-2}.$$

Hence

$$\Delta f=(-)^{(n-1)\frac{n}{2}} \,.\, n^{n-2}\zeta^2 f\,^*$$

expresses the universal relation between the arithmetical discriminant and the squared product of the root-difference of a function. If we had been

* As regards the application to be made of this result it was of course not necessary to determine the index of the power to which $(-)$ is raised, but it was hardly worth while to leave it undetermined.

dealing with the algebraical discriminant, it would have been necessary to replace n^{n-2} by n^{-n} in the above equation. It is furthermore to be observed that the discriminant is fixed in its sign by the condition that the term containing the highest power of the product of the expressed coefficients is to be taken positively.

So again it will be seen presently to be necessary to ascertain the strict relation between the resultant of two functions of degrees r, s and the product of the differences between the several roots ρ of the one and the several roots σ of the other of them, or, as we may say, between $R_{r,s}$ and $D_{\rho,\sigma}$, where if we choose to pay attention to algebraical signs that of $R_{r,s}$ may be understood to mean the resultant so taken that the term containing the highest power of the coefficient in the r-degreed function is positive and $D_{\rho,\sigma}$ to mean the product of the rs differences $(\rho-\sigma)$.

I shall again, for greater simplicity, suppose the initial coefficients of each of the two functions to be unity.

We know that $R_{r,s}=\mu D_{\rho,\sigma}$ where μ is a function of r and s exclusively. To determine it we may take x^r and x^s+1 as the two functions, it will be found without difficulty that

$$R_{r,s}=1^* \text{ and } D_{\rho,\sigma}=\{-(-1)^{\frac{1}{s}}\}^{rs}=(-)^{rs+r}.$$

Hence we have universally $R_{r,s}=(-)^{rs+r}D_{\rho,\sigma}$.

This seems to be the proper place to ascertain (what will be needed for future purposes) how far or under what qualifications the reciprocal connexion of the two facts: 1. Of two functions in x having a common root. 2. Of their resultant being zero, admits of being extended to roots of congruences in respect to a prime-number modulus.

Suppose fx, gx to be two in all respects (numerically† as well as algebraically) integer rational functions of the degrees i, j in x, then by eliminating dialytically $(i+j-1)$ powers of x between

$$fx,\ xfx,\ x^2fx\ \dots\ x^{j-1}fx,\quad gx,\ xgx,\ \dots\ x^{i-1}gx,$$

we may obtain the equation $\lambda xfx+\mu xgx=Rx^q$ (q having any integer value from 0 to $i+j-1$) where R is the resultant of f, g and λx, μx are in all respects integer functions of x of degrees $j-1$ and $i-1$ in x whose values

* Thus, for example, let $r=4$, $s=2$. Then $R_{r,s}$ is the dialytic resultant of

$$\begin{array}{cccccc}
x^5 & & & & & \\
 & x^4 & & & & \\
x^5 & & +x^3 & & & \\
 & x^4 & & +x^2 & & \\
 & & x^3 & & +x & \\
 & & & x^2 & & +1
\end{array}$$

which is obviously equal to unity.

† By which I mean that the coefficients are exclusively integer numbers.

depend on the value of q. If, then, fx and gx are simultaneously zero for some value of x, we must universally have $R=0$ even if x should be zero, for thus we might make $q=0$.

But this equation will not suffice to show that fx and gx will simultaneously vanish for some value of x, provided that $R=0$; for every value of x which makes fx vanish, might, as far as this equation discloses (and for all values of g), have the effect of making μx vanish*. We may, however, prove the fact in question, on a certain hypothesis to be presently stated, by availing ourselves of the knowledge that R is, to a *numerical factor près*, the product of the differences between the roots of f and those of g.

The hypothesis I make is that $fx \equiv 0$ mod. p is a congruence *all whose roots are real;* in this case I shall show that if the resultant R of fx and gx satisfies the congruence $R \equiv 0$ mod. p (that is, if R contains p) then gx must have at least one real root in common with fx *quâ* modulus p.

From the congruence of $fx \equiv 0$ mod. p we may, by a well known principle, infer the existence of an equation $Fx = fx + p\phi x = 0$ whose roots are the same as those of the congruence above written, and the dialytic method of elimination renders it self-evident that the resultant of Fx and gx will differ only by a multiple of p from that of fx and gx, and will, therefore, be a multiple of p.

If, then, we call the roots of Fx (all real by hypothesis) $a_1, a_2, \dots a_i$, we shall have $ga_1 . ga_2 . ga_3 \dots ga_i \equiv 0$ mod. p, and, as all the factors on the left hand side of the equation are real, one of them must contain p. Hence, if $R\,(fx, gx) \equiv 0$ mod. p, and $fx \equiv 0$ mod. p *has all its roots real*, one of these roots must belong also to the congruence $gx \equiv 0$ mod. p.

Going back now to what precedes this investigation, let us determine strictly the relation between the arithmetical discriminants and resultant of two functions in x and the discriminant of their product.

Let ω, ω_1 be the degrees in x of two altogether integer functions fx, f_1x, and suppose $Fx = fx . f_1x$. Then obviously $\zeta^2 Fx = \zeta^2 fx . \zeta^2 f_1x . (D\,(fx, f_1x))^2$. Hence $\omega^{\omega-2} . \omega_1^{\omega_1-2} \Delta Fx = (\omega + \omega_1)^{\omega+\omega_1-2} \Delta fx . \Delta f_1x\,(R\,(fx, f_1x))^2$.

If, then, p any prime number is contained in Δfx, and ω, ω_1 are each less than p, p will necessarily be contained in ΔFx. And as a particular case of this theorem, if p were contained in the discriminant of any factor of $x^{p-1}-1$ it would be contained in the discriminant of $x^{p-1}-1$, that is, in a power of $(p-1)$, which is impossible. Hence, by a preceding theorem, no factor of $x^{p-1}-1$, regarded as the subject of a congruence, can contain *a superfluity* of real roots (that is, more real roots than there are units in its degree) in respect to the modulus p^j.

* I think it would not be incorrect to say that *in all cases* the fact of the resultant of two functions of x containing a prime number raises a strong presumption that the functions have a common congruence root in respect to that number.

It is easy to show, although I do not find it distinctly stated in any of the current text-books, that $x^{p-1} - 1 \equiv 0$ mod. p^j has $p-1$ real roots.

For let $x = y^{p^{j-1}}$. Then the congruence becomes

$$y^{p^{j-1}.(p-1)} - 1 \equiv 0 \text{ mod. } p^j,$$

where $p^{j-1}.(p-1)$ is what is commonly designated as the ϕ function of p^j, the number of numbers less than p^j and prime to it, (the so-called ϕ function of any number I shall here and hereafter designate as its τ function and call its Totient). This last congruence by Fermat's extended theorem has all its roots real. It is easy to see that they will consist of $(p-1)$ groups, each group containing p^{j-1} numbers for which the value of x *quâ* modulus p^j will be the same, but different for numbers belonging to two different groups. For let y_1 be any of the y roots, and $y_2^{p^{j-1}} - y_1^{p^{j-1}} \equiv 0$ mod. p^j. Then *quâ* mod. p, $y_2^{p^{j-1}} \equiv y_2$ and $y_1^{p^{j-1}} \equiv y_1$, because $p^{j-1}-1$ contains $p-1$.

All the values of y_2 will, therefore, be comprised in the series

$$y_1,\ y_1 + p,\ y_1 + 2p,\ \dots\ y_1 + (p^{j-1}-1)\,p,$$

and $$(y_1 + \lambda p)^{p^{j-1}} = y_1^{p^{j-1}} + p^{p^j} . Q.$$

Hence the p^j terms of the series (and no other values of z) all satisfy the congruence

$$z^{p^{j-1}} - y_1^{p^{j-1}} \equiv 0 \text{ mod. } p^j.$$

Hence $x = y^{p^{j-1}}$ has $(p-1)$ distinct real values *quâ* p^j or there are $(p-1)$ real roots to the congruence $x^{p-1} - 1 \equiv 0$ mod. p^j. Hence, if fx is any factor of $x^{p-1}-1$, $fx \equiv 0$ [mod. p^j] will have all its roots real.

For let $fx . f_1x = x^{p-1} - 1$.

Then since $x^{p-1} - 1 \equiv 0$ mod. p^j has all its roots real, and fx and f_1x have no congruence root *quâ* mod. p in common*, if $fx \equiv 0$ to the modulus p^j has not *its full quota*, f_1x will have a *superfluity* of roots, but this has been shown to be impossible.

Now, let $p = mk + 1$. Then $x^k - 1$ is a factor of $x^{p-1} - 1$. Let $\chi_k x$ be the factor of $x^k - 1$, which contains all its primitive roots; this is what I term a *cyclotomic function of the first species* to the index k. $\chi_k x$ being a factor of $x^k - 1$ is a factor of $x^{p-1} - 1$, and will therefore, by what has just been shown, have all its roots real *quâ* the modulus p^j.

Hence a cyclotomic function of the 1st species to the index k contains, as a divisor, any power of any prime number of the form $mk+1$, and, moreover, if ω is its degree, (where ω represents the *totient* of k), $(mk+1)^j$ will be an ω-fold divisor of the function, that is, will be a divisor thereof corresponding to ω distinct values of the variable of the function, that is, values incongruent with one another *quâ* the modulus p^j.

* For if this were the case two factors of $x^{p-1} - 1$ *quâ* mod. p having two roots in common $x^{p-1} - 1$ would not have its full quota of roots.

The divisors of the cyclotomic function to index h may be divided into two classes, namely, divisors which do not divide the index, which may be called superior or extrinsic divisors, and divisors which divide at the same time the function and its index which may be termed inferior or intrinsic divisors. I shall begin with showing that any prime number extrinsic divisor diminished by unity must contain the index, that is, that if p is an extrinsic divisor and k the index, we must have $p = mk + 1$ which is a reciprocal proposition to the one just established.

If possible let p, any prime such that $p-1$ does not contain k nor k contain p, be a divisor of the cyclotomic function of the first species $\chi_k x$. And let δ be the greatest common divisor of $p-1$ and k. Then we shall have $x^\delta - 1 \equiv 0$ mod. p. But we have also $\chi_k x \equiv 0$ mod. p. Hence the resultant of $x^\delta - 1$ and $\chi_k x$ must contain p, but $\dfrac{x^k - 1}{x^\delta - 1}$ contains $\chi_k x$; *à fortiori* therefore the resultant of this and $x^\delta - 1$ will contain p. But this resultant is evidently equal to the value of $\dfrac{x^k - 1}{x^\delta - 1}$ (where $x^\delta = 1$) raised to the power δ, that is, $= \left(\dfrac{k}{\delta}\right)^\delta$ and therefore, *ex hypothesi*, does not contain p.

It has thus been proved that every extrinsic divisor of $\chi_k x$ can only be of the form $mk+1$.

Next let $k = k_1 p^j$ (k_1 being prime to p) and suppose p to be a divisor of $\chi_k x$.

Then p is a divisor of $(x^{p^j})^{k_1} - 1$ and, therefore, by what has been shown, must be of the form $mk_1 + 1$, unless $x^{p^j} - 1$ contained p in which case since $p^j - 1$ is divisible by $p-1$, $x-1$ must contain p and consequently p will be a divisor of $\chi_k 1$.

To find the value of $\chi_k 1$ we may proceed as follows:

Let $k = a^\alpha . b^\beta . c^\gamma . d^\delta . e^\epsilon$. Then the totient of k is

$$a^{\alpha-1} . b^{\beta-1} . c^{\gamma-1} . d^{\delta-1} . e^{\epsilon-1} \begin{Bmatrix} \alpha\beta\gamma\delta\epsilon + \Sigma\alpha\beta\gamma + \Sigma\alpha \\ -\Sigma\alpha\beta\gamma\delta - \Sigma\alpha\beta - 1 \end{Bmatrix},$$

and if we write this $L + M + N - P - Q - R$

$$\chi_k x = \frac{(x^L - 1)(x^M - 1)(x^N - 1)}{(x^P - 1)(x^Q - 1)(x^R - 1)},$$

and so in general the expression for $\chi_k x$, however many the distinct prime factors of k, imitates and follows *pari passu* the expression for the totient of k; and if $L, M, N, \ldots$ be the positive terms and $P, Q, R, \ldots$ be the negative ones in the algebraical representation of that totient, the common theory of vanishing fractions shows that $\chi_k 1 = \dfrac{L . M . N \ldots}{P . Q . R \ldots}$. There are two cases:

(1) When k contains i distinct prime factors, where $i > 1$. In that case supposing a to be one of them and α its index, the index of a in $L . M . N \ldots$ will be

$$\alpha \left\{1 + \frac{(i-1)(i-2)}{1 . 2} + \frac{(i-1)(i-2)(i-3)(i-4)}{1 . 2 . 3 . 4} + \ldots\right\}$$

and in $P . Q . R \ldots$

$$\alpha \left\{(i-1) + \frac{(i-1)(i-2)(i-3)}{1 . 2 . 3} \ldots\right\},$$

so that the index in the quotient is $\alpha(1-1)^{i-1}$, that is, is zero. And so for $b, c, \ldots$. Hence $\chi_k 1 = 1$.

(2) When $i = 1$ and $k = a^\alpha$, the value of $\chi_k x = \dfrac{x^{a^\alpha} - 1}{x^{a^{\alpha-1}} - 1}$, and consequently $\chi_k 1 = a$. Hence, when $k = k_1 p^j$, and k_1 is not unity, p, if a divisor of $\chi_k x$, must be of the form $mk_1 + 1$. Moreover, the case of $k_1 = 1$ offers no exception to this conclusion, inasmuch as when $k_1 = 1$, p, (like every other number) comes under the form $mk_1 + 1$.

It now remains to show the converse that if $k = k_1 p^j$ and $p = mk_1 + 1$, p will be a divisor of $\chi_k x$.

For the sake of greater simplicity, we may consider apart the case where $k = p^j$. Here $\chi_k x = \dfrac{x^{p^j} - 1}{x^{p^{j-1}} - 1} = 1 + x^{p^{j-1}} + x^{2p^{j-1}} + \ldots + x^{(p-1)p^{j-1}}$, which, (to modulus p) $\equiv 1 + x + x^2 + \ldots + x^{p-1} \equiv \dfrac{x^p - 1}{x - 1}$, and, therefore, can only contain p, if $x^p - 1$, and, consequently, $x - 1$ contains it. Hence, the only root of $\chi_k x \equiv 0$ [mod. p], for this case is $x = 1$.

Moreover, only p itself, and no higher power of p, can be a divisor of the cyclotomic function in question, because

$$\frac{(1+\lambda p)^{p^j} - 1}{(1+\lambda p)^{p^{j-1}} - 1} = \frac{\lambda p^{j+1} + \ldots}{\lambda p^j + \ldots} = p + Bp^2 + Cp^3 + \ldots + Lp^{(p-1)p^{j-1}}$$

does not contain p^2*.

To save unnecessary fatigue of attention, about a small matter, to my readers and myself, I will take, as a representative of the general case, $k = k_1 p$, $k_1 = abc$, $p = mk_1 + 1$; it will easily be verified that the increase of the number of distinct prime factors a, b, c, or the affection of them or of p with indices, will in no manner affect the course of the demonstration or the validity of the conclusion.

* When $p = 2$ and $j = 1$ the third term will not be of a higher power in p than the second term in the development of the numerator, so that the conclusion ceases to hold; as ought to be the case for the cyclotomic of the 1st species to the index 2, namely, $x + 1$ will obviously contain every power of 2 as a divisor.

In the above *special case*

$$\chi_k x = \frac{(x^{abcp}-1)(x^{ab}-1)(x^{ac}-1)(x^{bc}-1)(x^{ap}-1)(x^{bp}-1)(x^{cp}-1)(x-1)}{(x^{abc}-1)(x^{abp}-1)(x^{acp}-1)(x^{bcp}-1)(x^{a}-1)(x^{b}-1)(x^{c}-1)(x^{p}-1)}.$$

Let now $x^{k_1}-1=0$, so that $x^p=x$. Then obviously $\chi_k x = \dfrac{x^{abcp}-1}{x^{abc}-1}=p$. Hence the resultant of $\chi_{k_1}x$ and $\chi_k x$ is $p^{\tau(k_1)}$ ($\tau(k_1)$ meaning the totient of k_1). Consequently since $\chi_{k_1}x\equiv 0$ [mod. p] has all its roots real, one root at least of $\chi_k x\equiv 0$ [mod. p] will be a root of the preceding congruence.

It will be noticed that if instead of $\chi_{k_1}x$ we took $\chi_{k'_1}x$ where k'_1 is a factor of k_1 it would not be true that the resultant of it and $\chi_k x$ would contain p.

For example, if $k'_1=ab$ and $x^{k'_1}-1=0$ we should have

$$\chi_k x = \frac{x^{abcp+1}-1}{x^{abc}-1}\cdot\frac{x^{ab}-1}{x^{abp}-1}=\frac{p}{p}=1.$$

Or again, if $k'_1=a$ and $x^{k'_1}-1=0$ we should have

$$\chi_k x = \frac{x^{abcp}-1}{x^{abc}-1}\cdot\frac{x^{ab}-1}{x^{abp}-1}\cdot\frac{x^{ac}-1}{x^{acp}-1}\cdot\frac{x^{ap}-1}{x^{a}-1}=p\cdot\frac{1}{p}\cdot\frac{1}{p}\cdot p=1$$

as before. So that the resultant instead of being p would, in each case, be 1, and consequently $x^k-1\equiv 0$ [mod. p] and $x^{k'_1}-1\equiv 0$ [mod. p] could not have a root in common. And so in general it may be shown that if $k=k_1p^j$ and $k'_1=\dfrac{k_1}{\delta}$ the resultant of $x^{k'_1}-1$ and $\chi_k x$ is 1, except when $\delta=1$ in which case it is p.

Hence the roots of $\chi_k x\equiv 0$ [mod. p] are to be sought not among all the roots of $x^{k_1}-1\equiv 0$ [mod. p], but exclusively among only such of them as belong to the congruence $\chi_{k_1}x\equiv 0$ [mod. p].

We have seen that if p, a prime number, is an extrinsic divisor of a cyclotomic function to the index k, any power of p is also a divisor of the function. On the contrary, if p is an intrinsic divisor it will appear that p^2 cannot (and consequently no higher power of p than the 1st, can) be a divisor. For if x satisfies the congruence $\chi_{k_1}x\equiv 0$ [mod. p] we must have $x^{k_1}=1+\lambda p$ and $x^p=x^{mk_1}.\,x=(1+mp)x$, where m represents a series of ascending powers of p. Hence

$$\chi_k x = \frac{x^{k_1p}-1}{x^{k_1}-1}\cdot\frac{x^{ab}-1}{x^{abp}-1}\cdot\frac{x^{ac}-1}{x^{acp}-1}\cdot\frac{x^{bc}-1}{x^{bcp}-1}\cdot\frac{x^{ap}-1}{x^{a}-1}\cdots,$$

where the first factor, being equal to $x^{k_1(p-1)}+x^{k_1(p-2)}+\ldots+1$, will be of the form $p(1+Pp)$, P being a series containing only positive powers of p. Again,

$$\frac{x^{ab}-1}{(1+Qp)x^{ab}-1}=1+\frac{Qpx^{ab}}{1-x^{ab}}+\frac{Q^2p^2x^{2ab}}{(1+x^{ab})^2}+\ldots=1+Q_1p$$

where Q_1 is an infinite series containing positive powers only of p and x.

In like manner $\dfrac{x^{ap}-1}{x^a-1}=\dfrac{(1+Rp)x^a-1}{x^a-1}=1+R_1p$ where R_1 (like R) is an infinite series of positive powers of p and x, and so for each separate factor.

On multiplying the product of these infinite series by $p(1+Pp)$, we shall necessarily obtain a finite series of the form $p(1+Gp)$. Consequently, the cyclotomic function will divide by p but not by p^2. And we might have used this method exclusively to have established the fact of the first power of p, under the conditions presupposed, being a divisor of the function. This method serves also to establish directly that *every* root of $\chi_{k_1}x\equiv 0$ is a root of the congruence $\chi_k x\equiv 0$ [mod. p]. And we thus see that the intrinsic divisor, when it exists, is a $\tau(k_1)$-fold divisor of the cyclotomic function.

When k is the index to a cyclotomic function, and $k=k_1p^j$, where p is a prime not contained in k, let us agree to call k_1 the sub-index to p. Then, from what precedes, we may draw the conclusion that a cyclotomic function of the first species has never more than one intrinsic divisor, which, if it exists, is the greatest prime number contained in the index, but is such only in the case when diminished by unity, it contains its own sub-index, (a conclusion necessarily satisfied when the index is a prime, for then its sub-index is unity), and, moreover, that the first power only of such intrinsic divisor, when it exists, is a divisor of the function.

It being true and capable of easy demonstration, that when a rational integer function contains, as a divisor, each of two numbers prime to one another, their product will also be a divisor of the function, it follows that any number, each of whose prime factors, diminished by unity, contains the index and also every such number multiplied by the highest prime number which is contained in the index (provided that when diminished by unity that prime contains its own sub-index) is a divisor of a cyclotomic function of the first species. This, as I have said, is only another name for that irreducible factor of a binomial x^k-1 whose degree in x is the *totient* of k.

When the cyclotomic function of any species is made homogeneous by the introduction of a second variable y, relatively prime to x, it becomes a form, (in the technical sense of the word), and may then very conveniently be designated a *cyclo-quantic*.

Title 2. *Cyclotomic Functions of the Second Species* (*Conjugate Class*)*. I pass on to the theory of the divisors of the function which has for roots the sum of the binomial (*zweigliedrig*) groups of the primitive roots of x^k-1,

* When, in the matter comprehended under this title, by inadvertence, cyclotomic functions of the second species are spoken of without a qualification annexed, it is to be understood, in all cases, that only those of the conjugate class or, in other words, those whose roots are all real, are intended. For brevity I shall usually call this class of functions cyclotomics of the second sort.

or, in other words, all the distinct values, $\frac{1}{2}\tau(k)$ in number, of $2\cos\frac{2\lambda\pi}{k}$ where λ is any number less than $\frac{1}{2}k$ and prime to k.

Such a function, in which the coefficient of the highest power of the variable is supposed to be unity, I call a cyclotomic function, or simply a cyclotomic, of the second species and conjugate class to the index k. It may be found most readily by dividing the corresponding one of the first species, whose variable say is x, by $x^{\frac{1}{2}\tau(k)}$, substituting u for $x+\frac{1}{x}$, and applying for successive values of m the trigonometrical formula for expressing $\cos m\theta$ in terms of powers of $\cos\theta$, except when the index is a prime number, in which case the function in u is given more expeditiously at once by the well-known formula

$$u^m+u^{m-1}-\frac{m-1}{1}u^{m-2}-\frac{m-2}{1}u^{m-3}+\frac{(m-2)(m-3)}{1\,.\,2}u^{m-4}$$
$$+\frac{(m-3)(m-4)}{1\,.\,2}u^{m-5}-\ldots,$$

which last coefficient, in the French edition of the *Disq. Arith.*, 1807, it may be worth noting, is written erroneously $\frac{(m-1)(m-4)}{1\,.\,2}$.

I have thought it would be useful and convenient for many of my readers to be able to see before them the functions of the two sorts, and I accordingly annex a table of their values for all indices up to 36 inclusive.

To the index 1 or 2, the cyclotomic of the second species has no existence. Those of the first species to the index 1 or 2, and of the second to the index 3, 4 or 6 are linear, and of course as forms, have no arithmetical properties, but contain every number as a divisor, linear forms being, as it were, the protoplasm out of which the higher forms are organized.

Table of Cyclotomic Functions of the first species and the conjugate class of the second species for all values of the index from 1 to 36 inclusive.

Index	1st Species	2nd Species, Conjugate Class
1	$x-1$	
2	$x+1$	
3	x^2+x+1	$u+1$
4	x^2+1	u
5	$x^4+x^3+x^2+x+1$	u^2+u-1
6	x^2-x+1	$u-1$
7	$x^6+x^5+x^4+x^3+x^2+x+1$	u^3+u^2-2u-1
8	x^4+1	u^2-2
9	x^6+x^3+1	u^3-3u-1
10	$x^4-x^3+x^2-x+1$	u^2-u+1

Index	1st Species	2nd Species, Conjugate Class
11	$x^{10}+x^9+\ldots+x+1$	$u^5+u^4-4u^3-3u^2+3u+1$
12	x^4-x^2+1	u^2-3
13	$x^{12}+x^{11}+\ldots+x+1$	$u^6+u^5-5u^4-4u^3+6u^2+3u-1$
14	$x^6-x^5+x^4-x^3+x^2-x+1$	u^3-u^2+2u+1
15	$x^8-x^7+x^5-x^4+x^3-x+1$	$u^4-u^3-4u^2+4u+1$
16	x^8+1	u^4-4u^2+2
17	$x^{16}+x^{15}+\ldots+x+1$	$u^8+u^7-7u^6-6u^5+15u^4+10u^3-10u^2-4u+1$
18	x^6-x^3+1	u^3-3u+1
19	$x^{18}+x^{17}+\ldots+x+1$	$u^9+u^8-8u^7-7u^6+21u^5+15u^4$ $+10u^3-10u^2+5u+1$
20	$x^8-x^6+x^4-x^2+1$	u^4-5u^2+5
21	$x^{12}-x^{11}+x^9-x^8+x^6-x^4$ $+x^3-x+1$	$u^6-u^5-6u^4+6u^3+8u^2-8u+1$
22	$x^{10}-x^9+\ldots-x+1$	$u^5-u^4-4u^3+3u^2-3u+1$
23	$x^{22}+x^{21}+\ldots+x+1$	$u^{11}+u^{10}-10u^9-9u^8+36u^7+28u^6-56u^5$ $-35u^4+35u^3+15u^2-6u-1$
24	x^8-x^4+1	u^4-4u^2+1
25	$x^{20}+x^{15}+x^{10}+x^5+1$	$u^{10}-10u^8+35u^6+u^5-50u^4-5u^3+25u^2-5u-1$
26	$x^{12}-x^{11}+\ldots-x+1$	$u^6-u^5-5u^4+4u^3+6u^2-3u-1$
27	$x^{18}-x^9+1$	$u^9-9u^7+27u^5-30u^3+9u-1$
28	$x^{12}-x^{10}+x^8-x^6+x^4-x^2+1$	$u^6-7u^4+14u^2-7$
29	$x^{28}+x^{27}+\ldots+x+1$	$u^{14}+u^{13}-13u^{12}-12u^{11}+66u^{10}+55u^9-165u^8$ $-120u^7+210u^6+126u^5-126u^4-56u^3+28u^2$ $+7u-1$
30	$x^{16}-x^{14}+x^{10}-x^8+x^6-x^2+1$	$u^8-9u^6+26u^4-26u^2+1$
31	$x^{30}+x^{29}+\ldots+x+1$	$u^{15}+u^{14}-14u^{13}-13u^{12}+78u^{11}+66u^{10}-220u^9$ $-165u^8+330u^7+210u^6-252u^5-126u^4+84u^3$ $+28u^2-4u-1$
32	$x^{16}+1$	$u^8-8u^6+20u^4-16u^2+2$
33	$x^{20}-x^{19}+x^{17}-x^{16}+x^{14}-x^{13}$ $+x^{11}-x^{10}+x^9-x^7+x^6-x^4$ $+x^3-x+1$	$u^{10}-u^9-10u^8+10u^7+34u^6-34u^5$ $-43u^4+43u^3+12u^2-12u-1$
34	$x^{16}-x^{15}+x^{14}-\ldots+x^2-x+1$	$u^8-u^7-7u^6+6u^5+15u^4-10u^3-10u^2+4u+1$
35	$x^{24}-x^{23}+x^{19}-x^{18}+x^{17}-x^{16}$ $+x^{14}-x^{13}+x^{12}-x^{11}+x^{10}-x^8$ $+x^7-x^6+x^5-x+1$	$u^{12}-u^{11}-12u^{10}+11u^9+54u^8-43u^7-113u^6$ $+71u^5+110u^4-46u^3-40u^2+8u+1$
36	$x^{12}-x^6+1$	$u^6-6u^4+9u^2-3$

A very good test (or, in most cases, pair of tests) of the correctness of the figures is to write $u=\pm 2$* corresponding to $x=\pm 1$ and see if the values for the same index agree. Our interest will presently be concentrated on the single entry in the right hand column, that which expresses the conjugate class of the second species of cyclotomic to the index 9, but the function for the neighbouring case of the index 8 is worthy of arresting the reader's attention for a moment, inasmuch as the general theory of cyclotomic divisors applied to it will be seen to supply an instantaneous proof that all prime

* And a further double test is given by taking $u=0$, $x=i$, as we ought to find $\chi i=\pm i^{\frac{1}{2}\tau k}\psi 0$.

numbers of the form $8n \pm 1$, and no other prime numbers have 2 for a quadratic residue*.

It is hardly necessary to observe that, when the index is a prime number, it may be duplicated without affecting the character of either set of functions, the only effect produced thereby being the entirely unimportant one of a change in the sign of the variable.

The formula which I have employed for computing $\cos n\theta$ is that which, beginning with the *highest* power of $\cos\theta$, admits of a uniform scheme of setting down the work, which is not the case when the series is started from the other end. It, and the series used for $\dfrac{\sin\frac{p\theta}{2}}{\sin\frac{\theta}{2}}$, also required for my purposes, may be obtained by a much simpler method than any I have seen given in the text-books as follows.

In general, the denominator of $\dfrac{1}{a_1 -}\,\dfrac{1}{a_2 -}\ldots\dfrac{1}{a_n}$, say the procumulant $[a_1, a_2, \ldots a_n] = A_0 - A_1 + A_2$ etc., where A_0 is $a_1 . a_2 . \ldots a_n$, A_1 is the sum of the quotients of A_0 by any pair of consecutive elements $a_i . a_{i+1}$, A_2 of the quotients of A_0 by the product of any two such pairs as $a_i . a_{i+1} . a_j . a_{j+1}$, and so on. If we call the *number* of such quotients in A_i, $D_i n$, it is obvious that

$$D_{i+1}n = \Sigma_{t=0}^{t=n-2} D_i t.$$

Hence $D_0 n = 1$, $D_1 n = n-1$, $D_2 n = (n-2)\dfrac{n-3}{2}$, $D_3 n = \dfrac{(n-3)(n-4)(n-5)}{1.2.3}$, and so on.

On making $a_1 = a_2 = \ldots = a_n = 2\cos\theta$, it will immediately be seen that the procumulant $[2\cos\theta,\ 2\cos\theta \ldots$ to n terms$]$ expresses $\dfrac{\sin(n+1)\theta}{\sin\theta}$, because, calling this u_n, the equation in difference for finding it is

$$u_{n+1} = 2\cos u_n - u_{n-1} \text{ and } u_0 = 1.$$

Consequently

$$\frac{\sin(n+1)\theta}{\sin\theta} = (2\cos\theta)^n - n(2\cos\theta)^{n-2} + \frac{(n-1)(n-2)}{2}(2\cos\theta)^{n-4}\ldots.$$

Hence $2\cos n\theta = 2\left(\dfrac{\sin(n+1)\theta}{\sin\theta} - \cos\theta\dfrac{\sin n\theta}{\sin\theta}\right) = (2\cos\theta)^n - n(2\cos\theta)^{n-2}$ $+ n\dfrac{n-3}{2}(2\cos\theta)^{n-4}\ldots.$ Also, $\dfrac{\sin\frac{2n+1}{2}\theta}{\sin\frac{\theta}{2}} = \dfrac{\sin(n+1)\theta}{\sin\theta} + \dfrac{\sin n\theta}{\sin\theta} = (2\cos\theta)^n$ $+ (2\cos\theta)^{n-1} - n(2\cos\theta)^{n-2} - (n-1)(2\cos\theta)^{n-4} + \ldots.$†

* So, under the third Title, it will be found that $u^2 + 2$ is a *non-conjugate* cyclotomic of the second species to the index 8, of which, according to the general cyclotomic law, the odd prime divisors are of the form $8m+1$ or $8m+3$.

† This expansion Gauss (*Rech. Arith.*, Paris, 1757, p. 431) suggests deriving by means of the exceedingly awkward and unmanageable process indicated by the formula $\dfrac{\sqrt{(1-\cos n\theta)}}{1-\cos\theta}$, $\cos n\theta$ being previously supposed to be expanded in terms of powers of $\cos\theta$. *Quandoque bonus dormitat Homerus.*

Writing u in place of $2\cos\theta$ these are the two expansions which I have used to express $x^n + \dfrac{1}{x^n}$ and $\dfrac{x^{\frac{p-1}{2}} - x^{-\frac{p-1}{2}}}{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}$ in terms of powers of $x + \dfrac{1}{x}$ in calculating the cyclotomics of the 2nd sort whose values are given in the preceding table.

Since $(x^{p-1} - 1)(x^{p+1} - 1) = x^{2p} - x^{p+1} - x^{p-1} + 1$, if, for convenience, we write $x + \dfrac{1}{x} = u = 2\cos\theta$, it is evident that $\cos p\theta - \cos\theta$, regarded as an algebraical function of $\cos\theta$, will contain all the cyclotomic functions of the second species (conjugate class) whose indices are divisors of $p-1$ or $p+1$ and in addition to these $\left(x - \dfrac{1}{x}\right)^2$ or $u^2 - 4$ derived from the factor $x^2 - 1$ which is common to $x^{p-1} - 1$ and $x^{p+1} - 1$, but does not give rise to a cyclotomic of this sort until it is squared; $\cos p\theta - \cos\theta$ is thus a product exclusively of cyclotomics of the second sort.

It is well known that $\cos p\theta - \cos\theta \equiv 0 \pmod{p}$ regarded as a congruence in $\cos\theta$ has the p roots $\cos\theta = 0, 1, 2, 3, \ldots (p-1)$, p being supposed to be a prime number.

But more generally the congruence $\cos p^j\theta - \cos p^{j-1}\theta \equiv 0$ mod. p^j has its full complement of p^j real roots—a theorem, this, which is the analogue of the theorem of Fermat extended to powers of prime numbers put under the form of affirming that $x^{p^j} - x^{p^{j-1}} \equiv 0$ mod. p^j has its full complement of real roots; but, as I do not recall seeing the *cosine* theorem for modulus p^j anywhere stated, and as it is wanted for the theory I am developing, and its truth is not obvious, I shall proceed to prove it. For greater simplicity of notation let us begin with the case where $j = 2$. We have then

$$\cos p^2\theta = (\cos\theta)^{p^2} - p^2\frac{p^2-1}{2}(\cos\theta)^{p^2-2}.(\sin\theta)^2 + \frac{p^2(p^2-1)(p^2-2)(p^2-3)}{1.2.3.4}(\cos\theta)^{p^2-4}.(\sin\theta)^4\ldots$$

and $$\cos p\theta = (\cos\theta)^p - p\frac{p-1}{2}(\cos\theta)^{p-2}.(\sin\theta)^2 + \frac{p.(p-1)(p-2)(p-3)}{1.2.3.4}(\cos\theta)^{p-4}.(\sin\theta)^4\ldots$$

where of course all the powers of $(\sin\theta)^2$ are regarded as functions of $\cos\theta$. It will easily be recognized that every coefficient in the first series will be divisible by p^2 with the exception of those terms in which a new multiple of p first makes its appearance among the factors of the denominator, which will lose one power of p; the next coefficient to any such as last named taking up a new factor of p into the numerator, the fraction to which it belongs will recover the lost p and be again divisible by p^2.

The difference, therefore, between the two series *quâ mod.* p^2 will be

$$(\cos\theta)^{p^2} - (\cos\theta)^p$$

$$+\frac{p^2(p^2-1)\dots(p^2-2p+1)}{1.2\dots 2p}(\cos\theta)^{p^2-2p}.(\sin\theta)^{2p} - p\frac{p-1}{2}(\cos\theta)^{p-2}(\sin\theta)^2$$

$$+\frac{p^2(p^2-1)\dots(p^2-4p+1)}{1.2\dots 4p}(\cos\theta)^{p^2-4p}.(\sin\theta)^{4p}$$

$$-\frac{p(p-1)(p-2)(p-3)}{1.2.3.4}(\cos\theta)^{p-4}(\sin\theta)^4$$

....

It may be shown that every pair of terms in the above is divisible by p^2 for all real values of $\cos\theta$.

(1) $(\cos\theta)^{p^2} - (\cos\theta)^p$ contains p^2 by Fermat's extended theorem.

(2) *Quâ* p, $(\cos\theta)^{p^2-2p} \equiv (\cos\theta)^{p-2}$ and $(\sin\theta)^{2p} \equiv (\sin\theta)^2$.

Hence *quâ* p^2, the sum of the second pair of terms

$$\equiv p\frac{p-1}{2}\left\{\frac{(p+1)(p-2)(p-3)\dots(p^2-2p+1)}{2.3\dots(2p-1)} - 1\right\} \equiv 0$$

$$\equiv p\frac{p-1}{2}\left\{\frac{2.3\dots(2p-1)}{2.3\dots(2p-1)} - 1\right\} \equiv 0.$$

(3) *Quâ* p, inasmuch as

$p^2-5p+4=(p-1)(p-4)$, $(\cos\theta)^{p^2-4p} \equiv (\cos\theta)^{p-4}$ and $(\sin\theta)^{4p} \equiv (\sin\theta)^4$.

Also, $\quad p^2-1 \equiv p-1,\ p^2-2 \equiv p-2$ and $p^2-3 \equiv p-3$.

Hence the sum of the 3rd pair of terms *quâ* p^2

$$\equiv \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}\left\{\frac{(p^2-4)(p^2-5)\dots(p^2-4p+1)}{4.5\dots(4p-1)}\right\} \equiv 0.$$

And so each pair of terms may be proved to be congruous to zero *quâ* p^2.

The same form of demonstration may be shown to apply to the case of the modulus p^j*, and we may regard as proved the important theorem that $\cos p^j\theta - \cos p^{j-1}\theta \equiv 0$ [mod. p^j] contains the maximum number of roots p. It follows that $\cos p\theta - \cos\theta \equiv 0$ mod. p^j will contain p distinct roots. For, if we make $\theta = p^{j-1}\phi$, the congruence becomes $\cos p^j\phi - \cos p^{j-1}\phi \equiv 0$ mod. p^j,

* The reader will please bear in mind that in the expansion of $(a+b)^{p^j}$ the number of coefficients in which p enters to the power $j, j-1, \dots 2, 1, 0$ respectively is $p^j - p^{j-1}, p^{j-1}-p^{j-2}, \dots p^2-p, p-1, 2$.

which has p^j roots. These roots will separate into p groups of p^{j-1} each, such $\cos(p^{j-1}\phi)$ will be the same for all the $(\cos\phi)$'s in the same group, but different (*quâ* mod. p^j) for any two belonging to distinct groups. For if $\cos\phi_1$ be one of the values regarded as given, and $\cos(p^{j-1}\phi_2) \equiv \cos(p^{j-1}\phi_1)$ mod. p^j,

$$\left.\begin{aligned}&\cos(p^{j-1}\phi_2) \equiv \cos\phi_2\\ \text{and}\qquad &\cos(p^{j-1}\phi_1) \equiv \cos\phi_1\end{aligned}\right\}\ \text{mod. } p.$$

If, then, we form the series

$$\cos\phi_1,\ \cos\phi_1+p,\ \cos\phi_1+2p,\ \dots\ \cos\phi_1+(p^{j-1}-1)p,$$

all the values of $\cos\phi_2$ must be included among the terms of this series. Conversely, if we make $\cos\phi_2 = \cos\phi_1 + \lambda p$, we shall have

$$\cos p^{j-1}\phi_2 - \cos p^{j-1}\phi_1 \equiv 0 \text{ mod. } p^j.$$

For, writing q for p^{j-1},

$$\cos q\phi_2 = (\cos\phi_2)^q - q\frac{q-1}{2}(\cos\phi_2)^{q-2}.(\sin\phi_2)^2 + \dots.$$

If in this development we take the term containing $(\cos\phi_2)^{q-2t}.(\sin\phi_2)^{2t}$, its coefficient will contain q, except in the case where t contains p^i, in which case the coefficient will contain $\frac{q}{p^i}$ but not q, and the index of $(\cos\phi_2)$ and $(\sin\phi_2)^2$ will each contain p^i. Hence, since $\cos\phi_2 = \cos\phi_1 + \lambda p$, and consequently $(\sin\phi_2)^2$ is of the form $(\sin\phi_1)^2 + \Delta p$, it follows that the difference between this term and the corresponding one in the development of $\cos q\phi_1$ will in the one case contain qp and in the other $\frac{q}{p^i}p^{i+1}$, in either case therefore it contains $p.q$, that is, p^j, and consequently making $\cos\phi_2$ equal to any of the p^{j-1} terms of the series, we shall have $\cos(p^{j-1}\phi_2) \equiv \cos(p^{j-1}\phi_1)$ mod. p^j as was to be shown. Hence $\cos p\theta - \cos\theta \equiv 0$ mod. p^j will have p real roots.

Again no algebraical factor of $\cos p\theta - \cos\theta$ can have a *superfluity* of real roots *quâ* mod. p^j, for if it had then by the same reasoning as applied to the cyclotomics of the first species, it would be necessary for p to be contained in the discriminant of $\cos p\theta - \cos\theta$ regarded as a function of $\cos\theta$, but *quâ* mod. p, this is the same as the discriminant of $(\cos\theta)^p - \cos\theta$ in regard to $\cos\theta$ or of $x^p - x$ in regard to x which is the discriminant of $x^{p-1}-1$ multiplied by the squared resultant of x and $x^{p-1}-1$, and is therefore a power of $(p-1)$. Hence every algebraical factor of $\cos p\theta - \cos\theta$ *quâ mod.* p^j contains *its full quota* of real roots, that is, as many roots as there are units in its degree.

If then $p = mk + \epsilon$, where $\epsilon = \pm 1$, since $\cos p\theta - \cos\theta$ will contain the cyclotomic of the second sort to the index k, such cyclotomic equivalented to zero [mod. p^j] will have all its roots real, so that $(mk \pm 1)^j$ will be a $\frac{1}{2}\tau(k)$-fold divisor of such function.

As in the case of cyclotomics of the 1st species we may separate the divisors of those of the 2nd sort into intrinsic and extrinsic, according as they are or are not divisors of the index.

First, as regards the extrinsic divisors, we may prove that no other prime numbers except those of the form $k \pm 1$ can be divisors of the 2nd species of cyclotomics to the index k.

To show this I proceed as follows: $\psi_k u$ is contained algebraically in $\dfrac{\sin \frac{k}{2}\theta}{\sin \frac{\theta}{2}}$, and *à fortiori* in its square, that is, in $\dfrac{1-\cos k\theta}{1-\cos\theta}$, so that if $2\cos\theta$ is a value of u, which makes $\psi_k u$ contain p,

$$\cos k\theta \equiv 1 \text{ mod. } p,$$

but also $\cos p\theta \equiv \cos\theta$ mod. p, and if $\dfrac{\sin p\theta}{\sin\theta} \equiv a + bp$,

$$1 = (\cos\theta)^2 + a^2(1-\cos\theta)^2 + cp,$$

and $(1-a^2)(1-\cos\theta)^2 = cp$, and, therefore, $a = \pm 1$ mod. p, for $\dfrac{1-\cos k\theta}{1-\cos\theta}$ does not contain $(1-\cos\theta)$, and if $(1-\cos k\theta)$ contains $1-(\cos\theta)^2$, which is only the case when k is even, $\dfrac{1-\cos k\theta}{1-(\cos\theta)^2}$, does not contain either $1-\cos\theta$ or $1+\cos\theta$, and, therefore, $\psi_k u$, which, in that case, is contained in $\dfrac{1-\cos k\theta}{1-(\cos\theta)^2}$, will not contain either $1-\cos\theta$ or $1+\cos\theta$.

Hence $1-(\cos\theta)^2$ is not zero, and, consequently, $a \equiv \pm 1$, and, therefore, $\dfrac{\sin p\theta}{\sin\theta} \equiv \pm 1$ mod. p.

Hence, either

$$\left.\begin{array}{l} \cos(p-1)\theta = \cos p\theta \,.\, \cos\theta + \dfrac{\sin p\theta}{\sin\theta}(\sin\theta)^2 \equiv (\cos\theta)^2 + (\sin\theta)^2 \equiv 1 \\ \text{or} \\ \cos(p+1)\theta = \cos p\theta \,.\, \cos\theta - \dfrac{\sin p\theta}{\sin\theta}(\sin\theta)^2 \equiv (\cos\theta)^2 + (\sin\theta)^2 \equiv 1 \end{array}\right\} \text{mod. } p,$$

and writing $\epsilon = \pm 1$, we must have

$$\cos(p-\epsilon)\theta \equiv 1 \text{ mod. } p.$$

If possible, let $(p-\epsilon)$ not contain k, and δ (less than k) be the greatest common measure of k and $(p-\epsilon)$.

Let $\lambda(p-\epsilon) - \mu k = \delta$. Then

$$\left.\begin{array}{ll} \cos\lambda(p-\epsilon)\theta \equiv 1 & \dfrac{\sin\lambda(p-\epsilon)\theta}{\sin\theta} \equiv 0 \\ \cos\mu k\theta \equiv 1 & \dfrac{\sin\mu k\theta}{\sin\theta} \equiv 0 \end{array}\right\} \text{mod. } p.$$

Hence $\cos\delta\theta \equiv 1$ mod. p, and, consequently, the resultant of $\psi_k u$ and $\cos\delta\theta - 1$ in respect to $\cos\theta$ must contain p. But $\psi_k u$, when δ is any divisor of k other than k itself, is an algebraical factor of $\dfrac{\cos k\theta - 1}{\cos\delta\theta - 1}$ *à fortiori*, therefore, the resultant of this last named function of $\cos\theta$ and of $\cos\delta\theta - 1$ must contain p.

This resultant will be the product of the values of $\dfrac{\cos k\theta - 1}{\cos\delta\theta - 1}$ for every root of $\cos\delta\theta - 1$, it is therefore the δth power of the value of the vanishing fraction $\dfrac{\cos\mu\phi - 1}{\cos\phi - 1}$ $\left[\text{where } \mu = \dfrac{k}{\delta}\right]$ when $\cos\phi = 1$, that is, of $\left(\dfrac{\sin\frac{\mu}{2}\phi}{\sin\frac{\phi}{2}}\right)^2$ when $\phi = 0$. The resultant is, therefore, $\left(\dfrac{k}{\delta}\right)^{2\delta}$, which cannot contain p, since, by hypothesis, p is not contained in k. Hence $p - \epsilon = mk$, or $p = mk \pm 1$. So that, for the extrinsic divisors, the law, both as regards what numbers are and what are not such divisors, is precisely the same as for the cyclotomics of the first species, except that $mk \pm 1$ takes the place of $mk + 1$.

Next, for the intrinsic divisors. Suppose p to be any such, and that $k = k_1 p^j$, where k_1 is prime to p. Then p is a divisor of $\cos k_1 p^j\theta - 1$, and, therefore, by what has been shown, must be of the form $mk_1 \pm 1$, unless $(\cos p^j\theta - 1)$ contains p, in which case, since

$$\cos p^j\theta = (\cos p^j\theta - \cos p^{j-1}\theta) + (\cos p^{j-1}\theta - \cos p^{j-2}\theta) + \ldots + \cos\theta,$$

$\cos\theta - 1$ must contain p, and, consequently, p must be a divisor of $\psi_k 2$, that is, of $\chi_k 1$, which we have seen is equal to 1, except when $k_1 = 1$. Hence, p must be of the form $mk_1 \pm 1$. To show the converse, that when $k = k_1 p^j$ and $p = mk_1 \pm 1$, p will be a divisor of $\psi_k u$. Taking, first, the case of $k_1 = 1$ or $k = p^j$, $\psi_k u$, for $u = 2$ will be equal to $\chi_k 1$, which, as we have seen, will divide by p, and not by p^2.

To ascertain if there is any other value of u which will make the function divisible by p, I observe that, for this case, $(\psi_k u)^2 = \dfrac{\cos p^j\theta - 1}{\cos p^{j-1}\theta - 1}$, which is of the form $\dfrac{\cos\theta - 1 + Lp}{\cos\theta - 1 + lp}$, and if this function contains p, we must obviously have $\cos\theta \equiv 1$ mod. p.

Proceeding to the more general case where $k = k_1 p^j$ and k_1 is other than unity, taking as I did for the first species the specimen case $k = k_1 p$, $k_1 = abc$, $p = mk_1 \pm 1$, we shall have

$$(\psi_k u) = \frac{(\cos abcp\theta - 1)(\cos ab\theta - 1)(\cos ac\theta - 1)(\cos bc\theta - 1)(\cos ap\theta - 1)(\cos bp\theta - 1)(\cos cp\theta - 1)(\cos\theta - 1)}{(\cos abc\theta - 1)(\cos abp\theta - 1)(\cos acp\theta - 1)(\cos bcp\theta - 1)(\cos a\theta - 1)(\cos b\theta - 1)(\cos c\theta - 1)}$$

If, now, $\cos k_1\theta - 1 = 0$, and we suppose $\cos\theta$ to be a root of $\psi_k u = 0$, $\cos p\theta = \cos(\pm\theta) = \cos\theta$, $(\psi_k u)^2$ becomes equal to $\dfrac{\cos pk_1\theta - 1}{\cos k_1\theta - 1} = p$, and paying no attention to the algebraical sign which is immaterial to our object, we shall have $\psi_k u = p$, and the resultant of $\psi_{k_1} u$ and $\chi_k u$ will be $p^{\frac{1}{2}\tau k_1}$, and, consequently, since $\chi_{k_1} u \equiv 0$ mod. p has all its roots real, one of them, at all events, will belong to $\chi_k u \equiv 0$ mod. p, and precisely in like manner, as in the case for cyclotomics of the 1st species, it may be shown that this reasoning ceases to apply if $\cos\theta$, although satisfying $\cos k_1\theta - 1 = 0$, does not satisfy $\chi_{k_1} u = 0$, in which case the resultant, instead of being a power of p, would become unity, so that the value of $\cos\theta$, satisfying $\cos k_1\theta - 1 \equiv 0$ mod. p, could not be a congruence root of $\chi_k u \equiv 0$ mod. p. Finally, as for the case of the 1st species, it may be shown that *every* congruence root of $\chi_{k_1} u \equiv 0$ [when $k = k_1 p^j$ and $p = mk_1 \pm 1$] will satisfy the congruence $\chi_k u \equiv 0$ mod. p, and that only p, and not p^2, will be a divisor of $\chi_k u$, subject, however, to an exception for the case of $p = 2$, when $k = 2$ or $k = 4$, and also for the case of $p = 2$ and $p = 3$ when $k = 6$*. As regards these intrinsic divisors, it is clear that any root must be the highest prime factor of the index unless its sub-index is 3, in which case it may be 2. It is obvious, then, that except the index is 6 or 12, the second cyclotomic function can have only one intrinsic divisor. When the index is 6, the function is simply $u - 1$, and contains of course *every power* of 2 and 3, as well as every power of $6i \pm 1$ as a divisor.

Leaving out of consideration the three known cyclotomics, whose indices are 3, 4, 6, and the one just referred to, $u^2 - 3$, whose index is 12, we may combine the results obtained into the statement that any number, each of whose factors, diminished or increased by unity, contains the index, and any such number, multiplied by the highest prime number in the index, provided that that number, when increased or diminished by unity, contains its sub-index, and no other numbers but such as satisfy one or the other of these two descriptions, will be a divisor of a non-linear cyclotomic function of the conjugate class of the second species whose index is other than 12. As regards the index 12, any number, whose factors are all of the form $12m \pm 1$, as also the double, treble and sextuple of any such number, will be a divisor of the function.

By way of example let us consider the indices 15, 21, 35.

$\chi_{15}x$ will contain neither 3 nor 5, $\psi_{15}x$ will contain 5 but not 3.

$\chi_{21}x$ will contain 7 but not 3, $\psi_{21}x$ will contain 7 but not 3.

$\chi_{35}x$ will contain neither 5 nor 7, $\psi_{35}x$ will contain neither 5 nor 7.

* I may probably show this in full in a future note. But since the vast and dazzling theory for cyclotomics of all species, with an indefinite number of classes to each species, has loomed into view, I must confess to a certain feeling of impatience as regards working out these small details for a single class of a single species. The inordinately augmented amplitude of the subject calls for some broader method of treatment.

To find a value of x which makes $\psi_{15}x$ contain 5, write $\psi_3 u = u + 1 \equiv 0$ mod. 5, then $u \equiv -1$.

To find values of x which make $\psi_{21}x$ contain 7, write $u + 1 \equiv 0$ mod. 7, then $u \equiv 6$; and to find values of x which make $\chi_{21}x$ contain 7, write $x^2 + x + 1 \equiv 0$ mod. 7, then $x \equiv 2$ or $x \equiv 4$.

On turning to the table p. [327] it will be seen that

$\psi_{15}(-1) = 1 + 1 - 4 - 4 + 1 = -5$,

$\psi_{21}(-1) = 1 + 1 - 6 - 6 + 8 + 8 + 1 = 7$,

$$\left.\begin{aligned}\psi_{21}2 = 4096 + 512 + 64 + 8 + 1\\ -2048 - 256 - 16 - 2\end{aligned}\right\} = 4681 - 2322 = 2359 = 7\,.\,(16\,.\,21 + 1),$$

and of course since $\chi_{21}x^2$ contains $\chi_{21}x$ as an algebraical factor, $\chi_{21}4$ will also contain the intrinsic divisor 7 on the general principle that if λ be any number prime to k, $\chi_k x^\lambda$ must contain $\chi_k x$ as an algebraical factor, as admits of easy demonstration.

Also $\psi_{21}6 \equiv \psi_{21}\left(2 + \frac{1}{2}\right) \equiv \chi_{21}2$ [mod. 7] will also contain 7. Lastly, to mod. 5, for $x = 0, 1, 2, 3, 4$

$$\chi_{35}(x) \equiv 1, 1, 1, 1, 1; \quad \psi_{35}(x) \equiv 1, 1, 1, 1, 1;$$

and to mod. 7, for $x = 0, 1, 2, 3, 4, 5, 6$,

$$\chi_{35}(x) \equiv 1, 1, 1, 1, 1, 1, 1; \quad \psi_{35}(x) \equiv 1, 2, 1, 3, 3, 1, 2;$$

so that neither 5 nor 7 is a divisor of either function to index 35.

Title 3. *On Cyclotomic Functions of Any Species and Class.* The cyclotomic functions, called by me, of the second *sort* or conjugate class of the second species discussed under the preceding title, constitute the leading class of a much more general kind of binomial (*zweigliedrig*) cyclotomics, which it would mislead were I to suppress all allusion to.

Suppose k to contain θ distinct odd prime factors, then we know that the number of square roots of unity to the modulus k is 2^θ, except when k is divisible by 4, in which case it is $2^{\theta+1}$, or $2^{\theta+2}$, according as $\frac{k}{8}$ is fractional or integer, or, setting apart unity, the number remaining is $2^\theta - 1$, $2^{\theta+1} - 1$, $2^{\theta+2} - 1$ in the three cases respectively. Let $\sqrt{1}$ (one of the totitives to k) denote any *specific one* of these square roots. Then, if we call ρ any primary kth root of unity and make $x = \rho + \rho^{\sqrt{1}}$, we shall obtain a completely integer function of the degree $\frac{1}{2}\tau k$ in x, which may be called a binomial cyclotomic. When k is divisible by 4, one value of $\sqrt{1}$ will be $\frac{k}{2} + 1$, and the value of $\rho + \rho^{1+\frac{k}{2}}$ being zero, the cyclotomic function that ought to be, degenerates into a power of x. Hence, when k is not divisible by 4, the number of binomial cyclotomics is $2^\theta - 1$, when it is divisible by 4, $2^{\theta+1} - 2$, or the double of the former value, and when by 8, $2^{\theta+2} - 2$.

All these binomial cyclotomics will be found to possess similar properties to those which have been demonstrated under Title 2 concerning their leading class, as the annexed examples will serve to demonstrate, where the odd prime extrinsic factors it will be seen are of the form $mk+1$ or $mk+\sqrt{1}$; that is to say, in respect to the index, are congruous to one or the other of the *primordinal* totitives 1 and $\sqrt{1}$ where the latter quantity has a definite value for each of the cyclotomics in question.

Thus, suppose $k=15$, the square roots of unity (*quâ* 15) are $\pm 1, \pm 4$. Let $\sqrt{1}=4$, and make $x=\rho+\rho^4$, then it will be found that $x^4 - x^3 + 2x^2 + x + 1$ will contain the four roots of x and all the odd prime divisors of this function are of the form $15m+1, 4$.

Or, again, let $\alpha=\rho+\rho^{11}$, then it will be found that x is a root of the function $x^4+x^3+x^2+x+1$, the prime factors of which, other than 5, are of the form $15m+1, 11$, which is, in effect, the same as the form $5m+1$.

Again, let $k=20$. The values of $\sqrt{1}$ [mod. 20] are $\pm 1, \pm 9$. If we were to put $x=\rho+\rho^{11}$, its value would be zero, but writing $x=\rho+\rho^9$, we shall find it will be the root of x^4+3x^2+1, all the prime factors of which, other than the intrinsic one 5, are of the form $20m+1, 9$*.

We may now proceed to generalize these results by considering cyclotomics of every possible numerosity of grouping for a given index, and of every possible order of conjunction for a given numerosity—in a word, we are brought face to face with the most general theory of ν-nomial cyclotomic functions†.

I have accordingly calculated cyclotomic functions for the cases following:

$k=15$	$\mu=2$	$\nu=4$
$k=21$	$\mu=4$	$\nu=3$
	$\mu=3$	$\nu=4$
	$\mu=2$	$\nu=6$
$k=26$	$\mu=4$	$\nu=3$
	$\mu=2$	$\nu=6$
$k=28$	$\mu=4$	$\nu=12$
	$\mu=2$	$\nu=6$
$k=25$	$\mu=5$	$\nu=4$
$k=33$	$\mu=5$	$\nu=4$
	$\mu=4$	$\nu=5$
	$\mu=2$	$\nu=10$

* If $k=8$ and we take $x=\rho+\rho^3$ it will be a root of x^2+2 of which the odd extrinsic factors will be of the form $8m+1, 3$.

† All the species with their several classes here referred to form but a single genus of cyclotomic functions. The second genus will arise from the subdivision of groups into smaller groups and so on continually.

Understanding by the "totitives" of k the numbers less than k and prime to it, these totitives may be arranged in (among others) the natural groups hereunder written.

Totitives to 15 for $\mu=2$, $\nu=4$

1	4	11	14
2	7	8	13

" to 21 for $\mu=4$, $\nu=3$

1	4	16
2	8	11
5	17	20
10	13	19

" " for $\mu=3$, $\nu=4$

1	8	13	20
2	5	16	19
4	10	11	17

" " for $\mu=2$, $\nu=6$

1	4	5	16	17	20
2	8	10	11	13	19

" to 26 for $\mu=4$, $\nu=3$

1	3	9
5	15	19
7	11	21
17	23	25

" " for $\mu=3$, $\nu=4$

1	5	21	25
3	11	15	23
7	9	17	19

" to 28 for $\mu=4$, $\nu=3$

1	9	25
3	27	19
5	17	13
11	15	23

" " for $\mu=2$, $\nu=6$

1	3	9	19	25	27
8	10	11	17	18	23

" to 25 for $\mu=5$, $\nu=4$

1	7	18	24
2	11	14	23
3	4	21	22
6	8	17	19
9	12	13	16

To save space, I omit the groupings to $k=33$.

If, in any of the above tables, we call the totitives of the several rows,

$$\begin{array}{c} \tau_{1,1},\ \tau_{1,2} \cdots \tau_{1,\nu} \\ \tau_{2,1},\ \tau_{2,2} \cdots \tau_{2,\nu} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \tau_{\mu,1},\ \tau_{\mu,2} \cdots \tau_{\mu,\nu} \end{array}$$

and if ρ be a primitive root of x^k-1, and we write $R_\theta = \rho^{\tau_{\theta,1}} + \rho^{\tau_{\theta,2}} + \ldots \rho^{\tau_{\theta,\nu}}$, $R_1, R_2, \ldots R_\mu$ will be the roots of a cyclotomic of the νth species to the index k, or, as we may say, the index k and *nome* ν.

The values of the cyclotomic functions may be found most easily by calculating all the values of σ_i (the sum of the ith powers of its roots), from $i=1$ to $i=\mu$ where $\mu=\frac{\tau(k)}{\nu}$.

The value of $X_{k,\nu}$ will then be the sum of the terms not containing negative powers of x in the development of $x^\mu \left\{e^{-\frac{\sigma_1}{x}-\frac{\sigma_2}{2x^2}-\ldots-\frac{\sigma_\mu}{\mu x^\mu}}\right\}$.

It will, of course, be recognized that the first row of numbers (the primordinal totitives, as we may term them) in any of the foregoing natural schemes of decomposition of the kth primitive roots of unity into groups are νth roots (not necessarily comprising any primitive root) of unity in respect to the index k as modulus.

The values of the cyclotomics are exhibited in the annexed table.

Index	Nome	Cyclotomic function	Primordinal Totitives
15	4	x^2-x-1	1, 4, 11, 14
21	3	$x^4-x^3-x^2-2x+4$	1, 4, 6
"	4	x^3-x^2-2x+1	1, 8, 13, 20
"	6	x^2-x-5	1, 4, 5, 16, 17, 20
26	3	$x^4-x^3+2x^2+4x+3$	1, 3, 9
"	4	x^3-x^2-4x-1	1, 5, 21, 25
28	3	x^4-3x^2+4	1, 9, 25
"	6	x^2-7	1, 3, 9, 19, 25, 27
25	4	$x^5-10x^3+5x^2+10x+1$	1, 7, 18, 24
33	4	$x^5-x^4-4x^3+3x^2+3x-1$ *	±1, ±10
"	5	$x^4-x^3-2x^2-3x+9$	1, -2, 4, -8, 16
"	10	x^2-x-8	±1, ±2, ±4, ±8, ±16

In each of the above cases calling the index k, its totient $\mu\nu$, the nome ν and the primordinal totitives $\theta_1, \theta_2 \ldots \theta_\nu$ it will be found that all the *odd* extrinsic prime number divisors (that is, primes dividing the function but not its index) are of the form $mk+\theta_1, \theta_2, \ldots \theta_\nu$.

* The values of $\sigma_2, \sigma_3, \sigma_4, \sigma_5$ in this case follow the noticeable progression 9, 4, 25, 16.

Here, for the present, I must be content to leave this great theory, or I should be in danger of never finding my way back from it to the original object of the memoir which, although its parent, it transcends in importance; for Bachmann's work, as it seems to me, gives proof, that Cyclotomy is to be regarded not as an incidental application, but as the natural and inherent centre and core of the arithmetic of the future.

Remark on the intrinsic divisors of cyclotomic functions of the 1*st species.*

It has been seen that if $k=\frac{p-1}{m}p^{j-1}=k_1p^{j-1}$, $\chi_k x \equiv 0$ *mod.* p^j has all its roots the same as those of $\chi_{k_1}x \equiv 0$ mod. p and does not contain p^2. If, then, we make j successively 0, 1, 2 ... $j-1$ it will follow that

$$\chi_{k_1},\ \chi_{k_1p},\ \chi_{k_1p^2},\ \cdots\ \chi_{k_1p^{j-1}}$$

will each contain p, but only in the first power for the same τk_1 values of x. Hence $x^{\frac{(p-1)p^{j-1}}{m}}-1$, which contains all the above written cyclotomics, will contain p^j, so that $x^{\frac{\tau p^j}{m}}-1\equiv 0$ mod. p^j will have $\tau\left(\frac{p-1}{m}\right)$ primitive roots, and it is easy to see that $x^{\frac{k}{m}}-1$ will not have any congruence root in common with $x^{k_1}-1$ in respect to the modulus p^j.

The theory of intrinsic divisors, it will thus be seen, contains within itself the whole theory of primitive roots, which I notice because it induces me to withdraw the remark made in a previous footnote that the exact determination of the properties of the intrinsic cyclotomic divisors is a matter of comparatively small importance.

Notes to Proem.

1. *On the rational in- and- exscribed triangle to the cubic curve*

$$x^3-3xy^2-y^3+3z^3=0.$$

In the proem it was, under another form of expression, intimated in advance of what will be shown in the second section of this chapter, that the curve $x^3+y^3+Az^3=0$ has a correspondence with the curve

$$x^3-3xy^2-y^3+3Az^3=0,$$

of such a kind that whenever the second equation has a rational solution, the same must be true of the first, so that, for example, on making $A=1$, the solubility of $x^3-3xy^2-y^3+3z^3=0$ in integers implies the like of the equation $x^3+y^3+z^3=0$. Hence it might, at first sight, be rashly inferred (which is what happened to me when writing the 2nd footnote to page [316] from a sick bed) that since a cube number cannot be broken up into the sum of two others, the former of these last written equations is insoluble in integers. But the fact stares one in the face that it has three solutions in integers, namely,

$$x:y:z::\ 1:\ 1:1$$
$$x:y:z::-2:\ 1:1$$
$$x:y:z::\ 1:-2:1.$$

In general, (except at points of inflexion or at points whose ith tangentials are points of inflexion *), one rational point in a cubic gives rise to an infinite series of rational derivatives, but in this case the three points $1:1:1$, $-2:1:1$, $1:-2:1$ are the angles of a triangle in- and- exscribed to the curve $x^3-3xy^2-y^3+3z^3$, and are the only rational points on the curve. Each of them is its own third tangential, so that, at any one of the three, an infinite number of cubic curves can be made to pass having plethoric, or, so to say, pluperfect contact with each other (9-point contact) and accordingly will not intersect each other in any other point.

To these three points will be found to correspond (as will presently be shown in § 2) points for which x or y is zero in the curve $x^3+y^3+z^3=0$. This perfectly explains the seeming paradox.

The sides of the rational in- and- exscribed triangle are easily seen to be $y-z=0$, $x+y+z=0$, $x-z=0$.

In general, if any cubic be thrown into the form $x^2y+y^2z+z^2x+\lambda xyz$, it will obviously be in- and- exscribed to the triangle x, y, z†. In the present instance, if we write $x-z=u$, $y-z=v$, $x+y+z=-w$, it will be found that the curve $x^3-3xy^2-y^3+3z^3$ becomes simply $uv^2+vw^2+wu^2$, of which the Hessian is the three straight lines $u^3+v^3+w^3-3uvw$. If we take the sides of an equilateral triangle whose area is $\frac{1}{2}\Delta$ for the axes of u, v, w, we shall have $u+v+w=\Delta$, and the three real points of inflexion being in the line $u+v+w$, will pass off to infinity, so that the curve will possess three infinite branches. Writing $\omega=\frac{2\pi}{9}$, each asymptote will cut the sides of the angles of reference in three pairs of segments abutting at the several angles, such that the ratio to each other of the segments in the several pairs, taken in regular order, will be (for the three asymptotes respectively),

$$\frac{\cos\omega}{\cos 2\omega},\ \frac{\cos 2\omega}{\cos 4\omega},\ \frac{\cos 4\omega}{\cos\omega},$$
$$\frac{\cos 2\omega}{\cos 4\omega},\ \frac{\cos 4\omega}{\cos\omega},\ \frac{\cos\omega}{\cos 2\omega},$$
$$\frac{\cos 4\omega}{\cos\omega},\ \frac{\cos\omega}{\cos 2\omega},\ \frac{\cos 2\omega}{\cos\omega}.$$

* Thus we have the following distinction of cases as regards the algebraically rational derivatives of any point on a cubic curve: (1) An infinite succession of links. (2) A finite open chain reducing in the case of inflexions to a single point. (3) A closed chain with a finite number of links.

† For x will touch the cubic at x, y; y at y, z; z at z, x.

These ratios, of course, remain the same, for the conjugate cubic $u^2v+v^2w+w^2u$, except that the order of the readings has to be reversed.

According to my departed friend, (of cherished memory), Otto Hesse's dictum, I suppose it may almost be taken for granted without proof, which would obviously be easy, that the two sets of real asymptotes for the conjugate cubics will envelop one and the same conic.

In a future excursus I propose to demonstrate the existence of an infinite number of polygons in- and- exscribable about any given cubic, and to determine the number of such polygons for any existent number of sides. Since $wv^2+vu^2+uw^2=0$ is equivalent to $(2uw+v^2)^2+(4u^3v-v^4)=0$, we are able to deduce, from the fact that one cube cannot be the sum of two others, the theorem that the equation $v^4-4u^3v=t^2$ has no solution in integers*, (zeros excluded) which seems to me (the way in which it is got, I mean, not the theorem itself) a very surprising inference.

Scholium. *On triangles and polygons in- and- exscribable to a general cubic.*

The apices of any such triangle must be points which are their own 3rd tangentials. Any such point, it may be shown, is completely defined by the condition that two right lines, drawn, the first through it and any one chosen at will, of the 9 points of inflexion, the second through its tangential and some other point of inflexion, shall meet the curve in the same point.

If, then, the cubic be written under its canonical form, and we select the point of inflexion (I), for which $x=1$, $y=1$, and through the point $P(x, y, z)$, which is to be its own 3rd tangential, and I draw a ray meeting the curve in P', and through P' and Q, the tangential to P, [that is, the point whose coordinates are $x(y^3-z^3)$, $y(z^3-x^3)$, $z(x^3-y^3)$] draw a ray, the point (X, Y, Z), where that ray meets the curve, must be a point of inflexion, and, *vice versâ*, if the condition is fulfilled, P is its own 3rd tangential.

* Suppose the equation $u^2v+v^2w+w^2u=0$ is resoluble in non-zero integers. We may regard u, v, w as having no common measure, as any such, if it existed, could be driven out of the equation by division. Suppose p to be any prime number entering exactly α times into u and β times into v; then writing $u=p^{\alpha}u_1$, $v=p^{\beta}v_1$, since w^2u contains p^{α}, and v^2w, $p^{2\beta}$, we must have $\alpha=2\beta$ and $p^{3\beta}u_1^2v_1+v_1^2w+w^2u_1=0$, and proceeding similarly with each prime common measure of u, v, of v, w and of w, u, it is obvious that, calling the greatest common measure of these three pairs δ, ϵ, θ, we must have $\delta^3u'^2v'+\epsilon^3v'^2w'+\theta^3w'^2u'=0$, where u', v', w' have no two of them any common measure. Hence, apart from algebraical sign u', v', w' must be each of them unity, and the above equation may be written $\delta_1^3+\epsilon_1^3+\theta_1^3=0$, the same in form as that which gave birth to the equation $\xi^3-3\xi\eta^2+\eta^3=0$, of which $u^2v+v^2w+w^2u=0$ is a transformation. It is worthy also of remark that the two equations $u^2v+v^2w+w^2u=0$ and $x^3+y^3+z^3=0$ pass into one another through the medium of the self-reciprocal substitution-matrix

$$\begin{matrix} 1 & 1 & 1 \\ \rho^{\frac{1}{3}} & \rho^{\frac{4}{3}} & \rho^{\frac{7}{3}} \\ \rho^{\frac{2}{3}} & \rho^{\frac{5}{3}} & \rho^{\frac{8}{3}} \end{matrix}$$

where ρ is a primitive cube root of unity.

It will be found that

$$X : -x^6y^3 - y^6z^3 - z^6x^3 + 3x^3y^3z^3$$
$$:: Y : -x^3y^6 - y^3z^6 - z^3x^6 + 3x^3y^3z^3$$
$$:: Z : \quad xyz\,(x^6 + y^6 + z^6 - x^3y^3 - y^3z^3 - z^3x^3),$$

and we must have $X = 0$ or $Y = 0$ or $\dfrac{Z}{xyz} = 0$, the factor which figures in Z being disregarded, because it would lead to the 9 points of inflexion, which may be thrown out of account, as for each of them the in- and- exscribed triangle reduces to a point.

Combining each of the above equations taken separately with the equation to the cubic, we see that there will be $3 \times (9 + 9 + 6)$, that is 72 points forming the apices of 24 in- and- exscribed triangles to the cubic. It may be shown further that these 24 triangles consist of 12 pairs of conjugate triangles, every pair being so situated that each is a threefold perspective representation of the other, the three perspective centres being some one of the 12 sets of 3 collinear points of inflexion*.

The 24 in- and- exscribed triangles may therefore be distributed into 4 groups, each containing 3 pairs of conjugate triangles. This theory and the general one of in- and- exscribed polygons with any number of sides to a cubic curve will be treated more fully in a future excursus. It may, however, be remarked here that the equation $\dfrac{Z}{xyz} = 0$ is equivalent to the two $x^3 + \rho y^3 + \rho^2 z^3 = 0$, and $x^3 + \rho^2 y^3 + \rho z^3 = 0$, so that 18 of the points xyz may be found by solving two cubic equations between x^3, y^3 or y^3, z^3 or z^3, x^3. The

* ABC, LMN are in threefold perspective when AL, BM, CN; AM, BN, CL; AN, BL, CM meet in three several points. If ABC be taken as the triangle of reference and the coordinates of L, M, N are a, b, c; a', b', c'; a'', b'', c'' respectively, the triple "perspectivische lage" requires only the satisfaction of two conditions, namely, $ab'c'' = bc'a'' = ca'b''$, so that there is nothing between single and triple perspective relation. This statement constitutes a porism. The double condition $ba'c'' = cb'a'' = ac'b''$ of course corresponds to the contrary relation of triple perspective where AM, BL, CN; AL, BN, CM; AN, BM, CL meet in three several points.

Let $I, I', I'', J, J', J'', K, K', K''$ denote three points of collinear inflexions and P, Q the 3rd point collinear with P and Q, any two points on the cubic. If Q is the tangential to P, one of the vertices in question, it may be proved that any inflexion I, being assumed, another J may be found such that $IP = JQ$. From this it follows that PQ will satisfy the 10 equations

$$\begin{array}{lll} & PP = Q & \\ IP = JQ & JP = KQ & KP = IQ \\ I'P = J'Q & J'P = K'Q & K'P = I'Q \\ I''P = J''Q & J''P = K''Q & K''P = I''Q. \end{array}$$

These will necessarily continue to be satisfied when I and J are interchanged, provided that $4P$, Q be written KP and KQ or $K'P$ and $K'Q$ or $K''P$ and $K''Q$, and, consequently, to P, Q, R one in- and- exscript, will correspond another denotable indifferently by KP, KQ, KR, $K'P$, $K'Q$, $K'R$, $K''P$, $K''Q$, $K''R$, which will obviously therefore be in triple *perspectivische lage* with the first named one.

remaining 54 may be found by substituting for x, y, z respectively (in the simple equations which express their ratios)

$$\begin{array}{llll} 1^\circ. & x + y + z & x + \rho y + \rho^2 z & x + \rho^2 y + \rho z \\ 2^\circ. & x + y + \rho z & x + \rho y + z & \rho x + y + z \\ 3^\circ. & x + y + \rho^2 z & x + \rho^2 y + z & \rho^2 x + y + z \end{array}$$

(these substituted values, together with the original values of x, y, z, representing the sides of the 4 triangles which contain 3 points of inflexion on each side)*.

We may thus neglect altogether the equations $X = 0$, $Y = 0$, the values of x, y, z, to which they would lead, being comprised among those resulting from the above method†.

In like manner, as we have found the number of in- and- exscribable triangles, it may be shown that the number of quadrilaterals in- and- exscribable to a cubic is 54, and of p-laterals, when p is a prime number, $8\,(2^{p-1} - 1)(2^{p-2} + 1)$. For a k-sided polygon, where k is any number whatever, the rule is as follows. Let

$$\phi x = 8\,(2^{x-1} - (\bar{1})^{x-1})\,(2^{x-2} - (\bar{1})^{x-2}),$$

and let the totient of k, (supposed to contain i distinct prime factors) be expressed in the usual manner as the sum of 2^{i-1} positive terms P and the like number 2^{i-1} negative terms Q.

Then it may be proved (for it requires proof) that $\Sigma\phi P - \Sigma\phi Q$ will contain k; the quotient will contain the number of k-sided polygons in- and- exscribable about a cubic.

This theorem does not accord with the formula given by Professor Cayley in the *Phil. Tr.* for 1871, as quoted in the *Math. Fortschr.*, Vol. III.

* When the cubic is $x^3 + y^3 + z^3$, X, Y, Z become $x^9 + 6x^6y^3 + 3x^3y^6 - y^9$, ..., $xyz\,(x^6 + x^3y^3 + y^6)$ $X = 0$ then gives $\dfrac{x^3}{y^3} = t - t^2$ if $t^3 - 3t + 1 = 0$, that is, $t = 2\cos\dfrac{2\pi}{9}$, $2\cos\dfrac{4\pi}{9}$, $2\cos\dfrac{8\pi}{9}$; calling the three values of $\dfrac{x^3}{y^3}$ thus obtained τ_1, τ_2, τ_4, one of the two real in- and- exscribed triangles will have at its vertices $\dfrac{x}{y}$, $\dfrac{y}{z}$, $\dfrac{z}{x} = \tau_1^{\frac13}$, $\tau_2^{\frac13}$, $\tau_4^{\frac13} = \tau_2^{\frac13}$, $\tau_4^{\frac13}$, $\tau_1^{\frac13} = \tau_4^{\frac13}$, $\tau_1^{\frac13}$, $\tau_2^{\frac13}$ respectively, and the triangle conjugate to it will have at its vertices $\dfrac{x}{y}$, $\dfrac{y}{z}$, $\dfrac{z}{x}$ equal to the same three systems of ratios.

† If $x^3 + y^3 + z^3 + 3mxyz$ be the given cubic, one set of 9 points will be found from the equation

$$[(1 - \rho)\,y^3 + (1 - \rho^2)\,z^3]^3 + 27m^3\,(\rho y^6 z^3 + \rho^2 y^3 z^6) = 0,$$

or

$$y^9 - 3\,\{(1 - \rho^2)\,m^3 - \rho^2\}\,y^6 z^3 - \{(1 - \rho)\,m^3 - \rho\}\,y^3 z^6 + z^9 = 0,$$

and the fellow set by interchanging y and z. The disadvantage of this method consists in its leading to equations with imaginary coefficients for finding *inter alia* real roots which the equations $Y = 0$ or $Z = 0$, being of odd degrees, show must necessarily always exist.

The number of triangles in- and- exscribable to a curve whose order is x, whose class is X and whose number of cusps + three times its class is ξ, is there stated to be

$$X^4 + (2x^3 - 18x^2 + 52x - 46) X^3 + (\overline{18}x^3 + 162x^2 - 420x + 221) X^2$$
$$+ (52x^3 - 420x^2 + 704x + 172) X + (x^4 - 46x^3 + 221x^2 + 172x)$$
$$+ \xi \{\overline{9}X^2 + (\overline{12}x + 135) X + (\overline{9}x^2 + 135x - 600)\}.$$

On making $x = 3$, $X = 6$ and $\xi = 18$ we ought to have 24 the number of in- and- exscribable triangles to a general cubic, but on making these substitutions the result will be found to be zero. It is *quite certain*, therefore, that this formula requires some correction which has been overlooked by its illustrious author. For I have actually, in the text, given a cubic and a triangle in- and- exscribable to it, not to add that it is manifestly impossible for a general cubic to refuse to pass under the form $xy^2 + yz^2 + zx^2 + mxyz$.

Before quitting this subject I wish to call attention to the fact that the formula above given for composite numbers is a form deduced from the form ϕk precisely as in the excursus, the expression for $\log \chi_k x$ was deduced from $\log (x^k - 1)$*. It is clear from general logical considerations that this sort of deduction must be continually liable to occur and a name is imperatively called for to express it as much as one was formerly wanted to express the kind of deduction which leads from an algebraical form to its Hessian. Here the deduction depends on the arithmetical constitution of the subject of the form, and it is a great impediment to the free course of ratiocination not to be able to pass at once, in language and in thought, from the form to its deduct. I intend then in future to call such deduct the *functional totient* of the form, say ϕk, from which it is derived, and to denote it by $(\phi \tau) k$. This constitutes a very important gain to arithmetical nomenclature.

I would further call attention to the fact of an arithmetical theorem, of some considerable difficulty to demonstrate (by means of Fermat's extended theorem) in the general case, as any one, who goes through the process of the proof for the single case of k = the product of two primes, will easily satisfy himself, (I mean the theorem that the *functional totient* of $8 (2^{k-1} - (\overline{1})^{k-1}) (2^{k-2} - (\overline{1})^{k-2})$ is always divisible by k) should admit of an intuitional proof through the intervention of a pure property of cubic curves without any recourse to concepts drawn from reticulated arrangements, as in the applications of geometry to arithmetic made by Dirichlet and Eisenstein. This example of the possibility of such application (akin to that whereby the binomial theorem is made to prove that $\dfrac{\pi (m + m')}{\pi m \,.\, \pi m'}$ is an integer) is, as far as I can recall, without a precedent in mathematical history.

* The expression actually there given is for $\chi_k x$ and not its logarithm; using the notation explained above, and calling $\phi k = \log (x^k - 1)$ the cyclotomic of the 1st species to the index k, is $e^{(\phi\tau)k}$.

Postscriptum.

Mr Franklin obtains my result as follows: The condition that the $(i-1)$th tangential shall lie on the first polar is of the degree $2 \,.\, 4^{i-1} + 1$; the number of points on the cubic (exclusive of inflexions) satisfying this condition is $3 (2 \,.\, 4^{i-1} + 1) - 27 = 24 (4^{i-2} - 1)$. But the $(i-1)$th tangential will be on the first polar, not only when it is a true antitangential, but also when it is the original point itself or the consecutive point; so that we have to deduct from the above number twice the number of points (exclusive of inflexions) whose $(i-1)$th tangentials are the points themselves; that is, denoting by u_i the number of vertices of in- and- exscribed i-laterals, we have

$$\begin{aligned} u_i &= 24 (4^{i-2} - 1) - 2u_{i-1} \\ &= 24 \{2^{2i-4} - 2^{2i-5} + \ldots + (-2)^{i-1} - (1 - 2 + 2^2 - \ldots + (-2)^{i-3})\} \\ &= 8 (2^{i-1} + (-1)^{i-2}) (2^{i-2} - (-1)^{i-2}), \end{aligned}$$

which will be the number of the vertices, not only of true i-laterals, but also of all the $\dfrac{i}{\delta}$-laterals, (δ being any divisor of i except i itself) as well. Mr Franklin further suggests that the discrepancy between this result for $i = 3$ and Prof. Cayley's formula may be due to the latter not taking account of the peculiar kind of in- and- exscription in which the curve is in- and- exscribed at the same points. Finally, let us call the *summant* of a number k of the form $a^\lambda . b^\mu . c^\nu$ (a, b, c being primes) the well-known quantity consisting of $(1 + \lambda)(1 + \mu)(1 + \nu) \ldots$ terms which represents the sum of the divisors of k. We may speak of a *functional summant* to ϕk obtained by prefixing ϕ to each monomial term in the *development* of the summant and denote it by $(\phi\sigma) k$. The equation $(\phi\sigma) k = \omega (k)$ has for its solution $fk = (\omega\tau) k$. My method gives at once, for the *functional summant* of u^k (*without exclusion* of inflexions) $(2^k - \tau^k)^2$, and accordingly, the functional totient to this form divided by k is the simplest expression for the number of ex- and- inscribed k-laterals to the cubic. Thus, for $k = 1, 2, 3, 4, 5, 6$, that number is 9, 0, 24, 54, 216, 648 respectively.

2. *On 2 and 3 as cubic residues.*

For the benefit of those among my readers in this country who may not have access to the later works on arithmetic, it may be as well to point out how with the aid of their Gauss or Legendre they may verify the conditions which, later on, I shall have need to employ of 2 or 3 being cubic residues to k, a prime of the form $6i + 1$. The cyclotomic function of the third degree in the variable, to the index k, if we make $4k = m^2 + 27n^2$, is known to be $x^3 + x^2 - \dfrac{k-1}{2} x - \dfrac{3k - 1 + \epsilon mk}{27}$, where $\epsilon^2 = \pm 1$ and $m - \epsilon$ contains 3. Connecting this with the same function formed in the manner in which the cyclotomics in the Excursus under Title 3 have been calculated, calling U the number of solutions of the congruence $1+\beta+\gamma\equiv 0$ (mod. k), where β, γ are any two unequal cubic residues to k, and θ the number of solutions (1 or 0) of the congruence $1+2\beta\equiv 0$ (mod. k), it will easily be found, by comparing the constant terms in the two expressions, that

$$U+\frac{3\theta}{2}=\frac{k-8+\epsilon m}{18}.$$

Hence, when $\theta=1$, that is when 2 is a cubic residue, m (and therefore also n) must be even, and consequently when $\theta=0$, or 2 is not a cubic residue, m must be odd, and *vice versâ*.

Again, if we compare the values of the sum of the 4th powers of the roots of the cyclotomic as found by the general method with that deducible from the given function, we shall find

$$V+\frac{2}{3}\vartheta=\frac{k^2+3k-66-4m\epsilon k}{162},$$

where V is the number of solutions of the congruence $1+\beta+\gamma+\delta\equiv 0$, plus the number of solutions of the congruence $1+\beta+2\gamma\equiv 0$ (β, γ, δ being cubic residues to k) and ϑ the number of solutions of the congruence $1+3\beta\equiv 0$ (mod. k), that is 1 or 0, according as 3 is, or is not, a cubic residue to k.

The numerator is necessarily divisible by 54, but the criterion of ϑ being 0 or 1 depends on its being divisible or not by 81. On substituting for k its value in terms of m and n, it will be found that 16 times the numerator to modulus 81 is congruous with 54 times $(n^2-1)+\epsilon\left\{\left(\frac{m-\epsilon}{3}\right)^3-\frac{m-\epsilon}{3}\right\}$, and consequently is divisible or not by 81 according as n is not, or is, divisible by 3. Hence $\vartheta=1$ when n is divisible by 3 and otherwise is 0.

The joint effect of these two results may be translated into the following statement, which is better adapted than the more complete* form of enunciation would be to the purposes of this memoir.

If $k=f^2+3g^2$, when $(f\pm g)$ contains 9, 3 is, and 2 is not, a cubic residue; when g contains 3, but not 9, 2 is, and 3 is not, a cubic residue; when g contains 9, 2 and 3 are each of them cubic residues, and in any other case neither 2 nor 3 is a cubic residue to k†.

The equation $U+\frac{3\theta}{2}=\frac{3k-1+\epsilon mk}{18}$ contains a complete solution of the interesting question, "How many times, if the cubic residues to a given

* I mean more complete in the sense of fixing the cubic character in the case of 3 being a non-residue, which is unimportant to the matter in hand.

† In other words, if $4p=m^2+27n^2$ [an equation always possible when $p=6i+1$], n divisible by 2 is the necessary and sufficient condition of 2, and n divisible by 3 is the necessary and sufficient condition of 3, being a cubic residue to p.

modulus are set out in a regular ascending series, will consecutive terms differ from one another by a single unit?" When 2 is not a cubic residue, the answer is obviously $2U$, for $1+\alpha+\beta=n$ gives two sequences, α, $n-\beta$ and β, $n-\alpha$, differing by units. But when 2 is a cubic residue, there will be three extra sequences not contained among the $2U$ just spoken of, namely,

$$1,\ 2;\ \frac{k-1}{2},\ \frac{k+1}{2};\ k-2,\ k-1.$$

Hence, in each case, the number is $2U+3\theta$, that is $\frac{k-8+\epsilon m}{9}$, or, if we count in 0 as a residue, $\frac{k+\epsilon m+1}{9}$.

Section 2.

On certain numbers and classes of numbers that cannot be resolved into the sum or difference of two rational cubes.

Title 1. *Theorem on irresoluble numbers whose prime factors other than* 2 *or* 3 *are of the form* $18n+5$ *or* $18n+11$*. I propose to prove the following collective theorem. If A represents any one of the numbers 1, 2, 3, 4, 18, 36 or any number of the form

$$\begin{array}{llll} p, & q, & p^2, & q^2, \\ 9p, & 9q, & 9p^2, & 9q^2, \\ 2p, & 4q, & 4p^2, & 2q^2, \\ pq, & p_1p_2^2, & q_1q_2^2, & p^2q^2, \end{array}$$

(where any p means a prime number of the form $18n+5$, and any q a prime of the form $18n+11$) A will be irresoluble into the sum of two unequal rational cubes.

Lemma. If we decompose A (when it is not a prime) into any factors f, g, h, prime to each other, other than 1, 1, A, the equation $fx^3+gy^3+hz^3=0$ will be irresoluble in integers.

I prove this by showing that the above equation converted into a congruence to *modulus* 9 is irresoluble in integers.

x^3, y^3, z^3, each of them to this modulus is equivalent to one or the other of the three numbers $\bar{1}$, 0, 1.

$$\begin{array}{llllll} p,\ p_1,\ p_2 & \text{to this} & \text{modulus} & \text{is equivalent to} & \bar{4} \\ q,\ q_1,\ q_2 & ,, & ,, & ,, & 2 \\ p^2,\ p_1^2,\ p_2^2 & ,, & ,, & ,, & \bar{2} \\ q^2,\ q_1^2,\ q_2^2 & ,, & ,, & ,, & 4, \end{array}$$

* This theorem includes and transcends all the cases of irresolubility that had been discovered prior to the date of publication of the Proem in the last number of the *Journal*, with the exception of certain specific numbers whose irresolubility had been determined by the Abbé Pépin.

and on inspection, it will easily be verified that the limited linear congruence $f\lambda + g\mu + h\nu \equiv 0$ [mod. 9], where λ, μ, ν must each be picked out of the three numbers $\bar{1}$, 0, 1, has no solution.

Hence, if $fx^3 + gy^3 + hz^3 = 0$ and $f.g.h = A$, and x, y, z are supposed to be prime to each other, two of the quantities f, g, h will be unities and the third equal to A.

Let, now, $x^3 + y^3 + Az^3 = 0$ be supposed soluble in integers. Then, since A contains no $6n+1$ prime, we must have

$$\left.\begin{aligned} x + y &= A\zeta^3 \\ x^2 - xy + y^2 &= \omega^3 \\ z &= -\zeta\omega \end{aligned}\right\} \text{when } x + y \text{ does not contain 3,}$$

and

$$\left.\begin{aligned} x + y &= 9A\zeta^3 \\ x^2 - xy + y^2 &= 3\omega^3 \\ z &= -3\zeta\omega \end{aligned}\right\} \text{when } x + y \text{ contains 3.}$$

If $x + y$ is even, since $x^2 - xy + y^2 = \left(\frac{x+y}{2}\right)^2 + 3\left(\frac{x-y}{2}\right)^2$, we must have $\frac{x+y}{2} + \sqrt{(-3)}\frac{x-y}{2} = \{\xi + \sqrt{(-3)}\eta\}^3$, when $x + y$ does not contain 3, and $\frac{x-y}{2} + \sqrt{(-3)}\frac{x+y}{6} = \{\xi + \sqrt{(-3)}\eta\}^3$, when $x + y$ contains 3. In the one case $\frac{x+y}{2} = \xi^3 - 9\eta^3$, $\frac{x-y}{2} = 3\xi^2\eta - 3\eta^3$, and in the other $\frac{x-y}{2} = \xi^3 - 9\eta^2\xi$, $\frac{x+y}{6} = 3\xi^2\eta - 3\eta^3$.

In the one case, then, $2\xi(\xi - 3\eta)(\xi + 3\eta) = A\zeta^3$, and in the other $2\eta(\xi - \eta)(\xi + \eta) = A\zeta^3$. In either case, therefore, there is an equation-system of the form $\rho\sigma\tau = -A\zeta^3$, $\rho + \sigma + \tau = 0$, to be satisfied; therefore, disregarding permutations of ρ, σ, τ, we must have

$$\rho = fx_1^3, \quad \sigma = gy_1^3, \quad \tau = hz_1^3$$
$$f.g.h = A, \quad x_1y_1z_1 = -\zeta$$
$$fx_1^3 + gy_1^3 + hz_1^3 = 0,$$

and consequently by the Lemma $x_1^3 + y_1^3 + Az_1^3 = 0$ (or the same equation with x_1, y_1, z_1 interchanged) where $x_1y_1z_1$ is a factor of z.

Continuing the same process perpetually, as long as the new x and y have the same parity, each new x, y, z being contained in the immediately preceding z, must perpetually decrease, and if the process could be indefinitely continued, x and y must each evidently become unity, since otherwise z could go on decreasing without limit. This could only happen when $A = 2$, and even then is excluded by the condition that the cubes are to be unequal

as well as rational*. Hence, if the proposed equation is soluble at all, it must contain solutions in which x and y are one even and the other odd.

On this hypothesis, let us consider separately case (1), where $x + y$ does not, and case (2) where $x + y$ does contain 3.

Case (1). Here $(x + y)^2 + 3(x - y)^2 = 4(L^2 + 3M^2) = 4\omega^3$, and all the solutions of this equation are necessarily included in those of the system $L^2 + 3M^2 = \omega^3$, $x + y = L + 3M$, $x - y = L - M$.

Hence $x + y = \xi_1^3 + 9\xi_1^2\eta_1 - 9\eta_1^2\xi_1 - 9\eta_1^3 = A\zeta^3$. On making $\xi_1 = \xi - 3\eta_1$, this becomes $\xi^3 - 36\xi\eta_1^2 + 72\eta_1^3 = A\zeta^3$, or, making $\eta' = 6\eta_1$, $3\xi^3 - 3\xi\eta'^2 + \eta'^3 = 3A\zeta^3$, which, on writing $\eta' = \eta + \xi$, becomes $\eta^3 - 3\eta\xi^2 + \xi^3 = 3A\zeta^3$, where A unless it is unity contains at least one factor that is not of the form $18n \pm 1$, or else (in the case when $A = 3$) the square of 3. Hence, by virtue of the cyclotomic law for index 9, species 2 (conjugate class) (see Table, p. [327]), the above equation is insoluble in integers†.

Case (2). Here, using L and M in the same sense as above, $\frac{x+y}{3} = L - M$ and $x - y = L + 3M$ or $\xi_1^3 - 3\xi_1^2\eta_1 - 9\xi_1\eta_1^2 + 3\eta_1^3 = 3A\zeta^3$. Here writing $2\eta_1 = -\xi$, $\xi_1 = \eta + 2\xi$, the equation becomes $\eta^3 - 3\eta\xi^2 + \xi^3 = 3A\zeta^3$, and is insoluble in integers as before. Hence, since by hypothesis $x + y$ is not even, and it has been shown that it cannot be odd, *the number A when not unity is irresoluble into the sum or difference of two unequal rational cubes*‡.

When A is unity the equation above written becomes $\eta^3 - 3\eta\xi^2 + \xi^3 = 3\zeta^3$, the necessity for discussing which may be avoided by choosing the x, y out of x, y, z (which in this case are indistinguishable) so as to make $x + y$ always

* To prove this, let ξ, η, ζ be the system of variables, for which $\xi = 1$, $\eta = 1$ and x, y, z the system immediately preceding it. Then we have $A = 2$, $\xi = 1$, $\eta = 1$, $\zeta = -1$, and either $x - y = 0$, or $x + y = 0$. The latter of these equations would imply $z = 0$ and the former $x : y : z :: 1 : 1 : -1$, and so continually until we fall back on the original equation in x, y, z. Hence the only possible resolution of 2, if $x + y$ is even, is into two equal cubes.

† $3A$ not containing any cube, ξ and $3A$ must be prime to each other, since otherwise η, ξ, ζ would have a common measure. Hence we may make $\eta = \xi\mu - 3A\lambda$, and, consequently, $(\mu^3 - 3\mu + 1)\xi^3 \equiv 0$ mod. $3A$, and, therefore, $\mu^3 - 3\mu + 1$ must contain $3A$.

This conclusion would not hold if $3A$ were of the form A_1B^3 where A_1 contained no cube. We could then only infer $\mu^3 - 3\mu + 1 \equiv 0$ mod. A_1. Thus, in the case of $A = 9$, $3A = B^3$, and our inference would become $\mu^3 - 3\mu + 1 \equiv 0$ mod. 1, which, of course, is satisfied, and, accordingly, 9 ought to be resoluble into two cubes, as it obviously is, namely, into 1 and 8. Thus, the equation $x^3 - 3xy^2 + y^3 = 3Az^3$, when $A = 9$ has an infinite number of solutions, when $A = 3$ has no solution, and when $A = 1$ has just 3 solutions.

It may be worth noting that, in general, if $(x, y)^n = Az^n$, and $A = A_1B^n$, where A_1 contains no nth power of a number, $(x, 1)^n$ will contain A_1 as a divisor, provided that the coefficient of x^n in $(x, y)^n$ is a prime to A_1. Cases of this inference being drawn of course frequently occur, but the general principle, obvious as it is, I do not recollect to have seen formulated in the text books. It may be made more precise by the statement that any factor of A_1, prime to the coefficient of x^n, will be a divisor of $(x, 1)^n$.

‡ The equations of substitution are: for case 1, $\xi = \xi_1 + 3\eta_1$, $\eta = -\xi_1 + 3\eta_1$; and for case 2, $\xi = -2\eta_1$, $\eta = \xi_1 - \eta_1$.

even, which is the ordinary and easier method; but it is not without interest to show how the desired conclusion may be arrived at by keeping $x+y$ always odd. This may be done as follows: The equation between ξ, η, ζ, on writing $\eta+\zeta=u$, $\zeta-\xi=v$, $-\eta+\xi+\zeta=w$* becomes $uv^2+vw^2+wu^2=0$ which, as shown in footnote to p. [341], involves the relations $u=y'^2z'$, $v=z'^2x'$, $w=x'^2y'$ and consequently $x'^3+y'^3+z'^3=0$ where $x'y'z'=\sqrt[3]{(uvw)}$.

Let us use in general two or more separate letters enclosed within a parenthesis to denote the absolute value of the *greatest one of them* (their *dominant* as I am wont to call it).

When $x+y$ does not contain 3, $x+y=\zeta^3$, $x^2-xy+y^2=(\xi_1^2+3\eta_1^2)^3$. Hence $\zeta<2^{\frac{1}{3}}(x^{\frac{1}{3}}, y^{\frac{1}{3}})$ $(\xi_1, \eta_1)<3^{\frac{1}{3}}(x^{\frac{1}{3}}, y^{\frac{1}{3}})$. Therefore $(\xi_1, \eta_1, \zeta)<3^{\frac{1}{3}}(x, y, z)^{\frac{1}{3}}$, and consequently since $\xi=\xi_1+3\eta_1$ and $\eta=-\xi_1+3\eta_1$, $(\xi, \eta, \zeta)<4 \,.\, 3^{\frac{1}{3}}(x, y, z)^{\frac{1}{3}}$ and therefore $(u, v, w)<4 \,.\, 3^{\frac{4}{3}}(x, y, z)^{\frac{1}{3}}$. Hence $x' \,.\, y' \,.\, z'<(u, v, w)<4 \,.\, 3^{\frac{4}{3}}$ $(x, y, z)^{\frac{1}{3}}$.

In like manner when $x+y$ does contain 3, from the equations $\xi=-2\eta_1$, $\eta=\xi_1-\eta_1$, $x+y=9\zeta^3$, $x^2-xy+y^2=3(\xi_1^2+3\eta_1^2)^3$, follow $\zeta<\left(\frac{1}{3}\right)^{\frac{2}{3}}(x, y)^{\frac{1}{3}}$ $(\xi_1, \eta_1)<(x, y)^{\frac{1}{3}}$, $(\xi_1, \eta_1, \zeta)<(x, y, z)^{\frac{1}{3}}$, $(\xi, \eta, \zeta)<(x, y, z)^{\frac{1}{3}}$, $x' \,.\, y' \,.\, z'<(u, v, w)$ $<3(x, y, z)^{\frac{1}{3}}$.

In any case therefore $x' \,.\, y' \,.\, z'<4 \,.\, 3^{\frac{4}{3}}(x, y, z)^{\frac{1}{3}}<18(x, y, z)^{\frac{1}{3}}$. But the difference between any two cubes except 8 and 1 being greater than 8, the smallest of the numbers x', y', z' cannot be less than 3, and, since neither 3^3+4^3 nor 3^3+5^3 is a cube, it follows that $\dfrac{x' \,.\, y' \,.\, z'}{(x', y', z')}>18$, and therefore $(x', y', z')<(x, y, z)^{\frac{1}{3}}$, or the dominant of the quantities x, y, z which satisfy $x^3+y^3+z^3=0$ is continually replaced by another similar dominant less than the cube root of its predecessor, which is impossible.

Hence $x^3+y^3+z^3=0$ is insoluble. Let us see how this is reconcilable with the existence of the 3 rational solutions of $\eta^3-3\eta\xi^2+\xi^3+3\zeta^3=0$, namely, $\xi, \eta, \zeta=\bar{1}, 1, 1$ or $2, 1, 1$ or $1, 2, \bar{1}$ respectively.

In case (1) $\xi=\xi_1+3\eta_1$ $\eta=-\xi_1+3\eta_1$ $\xi, \eta=\bar{1}, 1$ gives $\eta_1=0$ $\xi, \eta=2, 1$ gives $\eta_1=-\xi_1$ $\xi, \eta=1, 2$ gives $\eta_1=\xi_1$. In each instance therefore $M=3\eta_1(\xi_1^2-\eta_1^2)=0$ and consequently $x+y=L=x-y$ and $y=0$.

In case (2) $\xi=-2\eta_1$ $\eta=\xi_1-\eta_1$ $\xi, \eta=\bar{1}, 1$ gives $\xi_1=3\eta_1$ $\xi, \eta=2, 1$ gives $\xi_1=-3\eta_1$ and $\xi, \eta=1, 2$ gives $\xi_1=0$.

In each instance therefore $L=\xi_1(\xi_1^2-9\eta_1^2)=0$ and therefore $x=0$. Thus the rational solutions of the equation in ξ, η, ζ in both cases correspond to rational but futile solutions of the equation in x, y, z.

* From these equations it is obvious that the dominant, that is, the arithmetically greatest of the quantities u, v, w, is less than 3 times the dominant of ξ, η, ζ.

CHAPTER I.

Excursus B.—On the Chain Rule of Cubic Rational Derivation.

I think it desirable, while the colours, so to say, are still wet on the palette, and my mind is still dwelling upon the subject which has been casually introduced in the note to the proem contained in the last number of the *Journal* (and there made use of to determine the number of in-and-exscribed k-laterals to a cubic), without waiting to put forth the titles which in natural order of sequence, perhaps, should immediately follow Title 1 of Section 2, to proceed at once to develop the theory of derivation which, irrespective of the casual use of it alluded to, will be found to be of essential importance when I reach that part of my proposed task which deals with soluble cubic-form equations, nor less so when, in Chapter II., I have to treat of insoluble cases of certain classes of cubic-form equations with four or more terms.

Title 1.—On the Natural or Discontinuously Numbered Scale of Rational Derivatives to a Point on a Cubic Curve.

Let us take any point on a cubic curve along with its successive tangentials *ad infinitum*. We may, by drawing straight lines through any two of these points, either contiguous or apart, to meet the curve, obtain an additional set of points, and thus form an enlarged system which may again be subjected to a like process of collineation or tangentialization, and such method of augmentation and amplification may be continued indefinitely. Every point thus obtained will obviously be a rational derivative of the original point (that is, its co-ordinates will be rational integral functions of those of that point), and, at first sight, it would seem as if we might in this way obtain a network, or spread*, of rational derivatives; but I shall proceed to show that such is not the case, but that only a line or chain of points will be thus obtained, usually infinite in extent, although for certain positions of the initial point coming to a stop, and in other cases winding round and round upon itself so as still to include only a finite number of distinct points. It will be shown subsequently that, in order to complete the theory of the chain for the purposes of this memoir, it will be necessary to take into account the rational derivatives not merely from a single arbitrary point, but from such points, *combined with a point of inflexion*, and that this additional element will not alter the surprising fact of the absence of reticulation or spread, but merely bring about the insertion into the chain of

* Spread, as a noun (scarcely to be found in the dictionaries), I employ in the sense in which it occurs in the phrase *spread of foliage*. On this continent the word *spread* is also used to denote a thick coverlet or padded woollen quilt, laid over the bedclothes in winter to keep out the cold; also on both continents as a familiar name for a college banquet.

points corresponding to missing numbers in it as first described, and to the duplication of the chain so completed, owing to every point in it having an opposite point also situated on the curve and collinear with it in respect to the given inflexion. This duplication will be of little importance in general to the arithmetical theory with which we shall be occupied, inasmuch as opposite points will correspond to the same arithmetical values, with merely a change of name between two out of the three variables which denote the co-ordinates of any point. First, let us consider the chain law of derivation when a point on the cubic curve alone is given. I shall call the original point 1, and its first and second tangentials 2 and 4 respectively, and in general use (m, n) to denote the point on a given cubic collinear with two points m, n also situated upon it*. Obviously, then, we shall have $(1, 1) = 2$ $(2, 2) = 4$, using $(1, 1)$ $(2, 2)$ to denote, in either case, two consecutive points upon the cubic. It is also obvious that if $(m, n) = p$ then $(m, p) = n$ and $(n, p) = m$, so that $(1, 2) = 1$ $(2, 4) = 2$.

Let us call $(1, 4) = 5$ $(2, 5) = 7$ $(1, 7) = 8$ $(2, 8) = 10$ $(1, 10) = 11$ $(2, 11) = 13$ and so on. It will be seen that no number which is a multiple of 3 is brought into existence by this process. Supposing a, b to be any two integers, neither of them divisible by 3, let us agree to signify by $a \ddagger b$ that of the two values $a + b$, $a - b$ which is not divisible by 3. The theorem to be established is that the point (m, n) collinear to m and n will have for its value $m \ddagger n$; as, for instance, $(4, 4)$, or the third tangential to 1, will have for its value 8, that is, will be identical with $(1, 7)$, that is to say, with $\{1, [2, (1, 4)]\}$, where 2 and 4 are the first and second tangentials to 1, which amounts to a rule for obtaining the third tangential, when a point on a cubic and its first and second tangentials are given, by collineation alone. The *theory of residuation*, in its simplest form (see Salmon's *Higher Plane Curves*, 3rd ed., p. 134)† teaches us that the rule of the older chemistry known by the name of double decomposition, namely that $\{(a, b), (c, d)\} = \{(a, c), (b, d)\}$ is applicable to the same symbols regarded as points on a cubic curve. This rule of double decomposition is all that is required to prove the theorem in question.

Thus, for example, in order to prove that $(1, 7) = (4, 4)$, I write $(1, 7) = \{(1, 2), (2, 5)\} = \{(2, 2), (1, 5)\} = (4, 4)$. Q. E. D.

So, to prove in general that $(r, s) = r \ddagger s$ I proceed as follows:

* Sometimes, however, it will be found more convenient to use $P_1, P_2 \ldots P_n$; $P_1', P_2', \ldots P_n'$ in lieu of $1, 2, \ldots n$; $1', 2', \ldots n'$.

† The theory of residuation was originally brought by me before the Mathematical Society of London, and subsequently, in the form of questions, in the *Educational Times*. Dr Salmon makes no allusion to the fact of my applying the theory to curves of all orders: in the case of the quartic, the residual becomes a system of three points; of a quintic, a system of six points, and so on. I understood Professor H. S. Smith to say that he made use of my theory for the quartic in his memoir which gained half the prize for the subject set by the Academy of Sciences of Berlin, but which I have never seen.

(1) Suppose $r = 3i + 1$; $s = 3j + 1$, where $j - i$ is positive. Then

$$(r, s) = \{(3i - 1, 2), (3j + 2, 1)\} = \{(3i - 1, 1), (3j + 2, 2)\}$$
$$= (3i - 2, 3j + 4) = (r - 3, s + 3).$$

Hence $(r, s) = (r - 3i, s + 3i) = (1, s + r - 1) = s + r$.

(2) Suppose $r = 3i - 1$; $s = 3j - 1$. Then $(r, s) = \{(3i - 2, 1), (3j + 1, 2)\}$

$$= \{(3i - 2, 2), (3j + 1, 1)\} = (3i - 4, 3j + 2) = (r - 3, s + 3),$$

as before. Hence $(r, s) = \{r - 3(i - 1), s + 3(i - 1)\} = (2, s + r - 2) = s + r$.

(3) Suppose $r = 3i - 1$; $s = 3j + 1$. Then $(r, s) = \{(3i - 2, 1), (3j - 1, 2)\}$

$$= \{(3i - 2, 2), (3j - 1, 1)\} = (3i - 4, 3j - 2) = (r - 3, s - 3).$$

Hence $(r, s) = (r - 3i + 3, s - 3i + 3) = (2, s - r + 2) = s - r$.

(4) Suppose $r = 3i + 1$; $s = 3j - 1$. Then $(r, s) = \{(3i - 1, 2), (3j - 2, 1)\}$

$$= \{(3i - 1, 1), (3j - 2, 2)\} = (3i - 2, 3j - 4) = (r - 3, s - 3).$$

Hence $(r, s) = (r - 3i, s - 3i) = (1, s - r + 1) = s - r$.

Collecting the four cases, it will be seen that I have proved, for all values of the points r, s in the chain, that $(r, s) = r \ddagger s$. Q.E.D.

The points 2^i correspond to tangentials of the ith order to the point 1. It is obvious from the above theorem that no process of continued collineation or tangentialization performed upon these points can lead to any points extraneous to the series of points 1, 2, 4, 5, 7, 8 ... which form a simple chain extending in general to infinity. Moreover, as it follows from the theory of residuation that any single point reached through the intervention of curves drawn through any number of points on a cubic can be reached by simple linear constructions, it follows that by no conceivable geometrical process can any rational point be reached not included in the numbered chain, and the inference becomes in the highest degree probable, and, as a matter of fact, is undoubtedly true (although the reasoning upon which it is here made to rest is not absolutely conclusive), that no rational deducts from a *general* point on a *general* cubic exist save those that belong to the numbered chain, the points upon which constitute what may properly be termed a self-contained *group*, infinite or finite (as the case may be) in regard to the number of terms which it contains. I shall presently determine the order of each successive derivative, meaning thereby the order in the co-ordinates of the initial point of any one of the three functions which express the co-ordinates of the derived one*.

* There is a further question, but which, as not material to the object of this memoir, I shall not discuss here, namely, the *degree* in the coefficients of each such derivative. For the tangential, the degree-order (being that of the minor determinants of the matrix made up of the differential derivatives of the function and its Hessian) we know to be 4, 4. If x, y, z, be the original co-ordinates, and X, Y, Z, those of the tangential, we know that $F(X, Y, Z)$ being zero when $F(x, y, z)$ (the given cubic) is zero, must be divisible by $F(x, y, z)$. The quotient will be of the degree-order 13, 12 - 1, 3, that is, 12, 9, and is in fact the skew covariant of F.

The case in which the chain forms a closed polygon, which can only happen when for some number i the ith tangential coincides with the initial point, has already been discussed in the note to the proem.

If the chain is an open but finite one, it is necessary that a tangential of some order shall fall upon a point of inflexion, in which case the succeeding tangentials remain fixed at that point, but otherwise continual new tangentials could be drawn. These are obviously necessary conditions of the chain being finite, whether it be an open chain or winding round upon itself; it remains to show that they are sufficient as well as necessary, but that will best appear after the theory of derivation from a general point combined with a point of inflexion has been discussed.

I shall begin with finding the co-ordinates X, Y, Z of a point on the cubic curve collinear with any two given points x, y, z; ξ, η, ζ. Let

$$X = \lambda x + \mu\xi,\ Y = \lambda y + \mu\eta,\ Z = \lambda z + \mu\zeta;$$

then

$$F(X, Y, Z) = \lambda^3 F(x, y, z) + \lambda^2\mu\left(\xi\frac{d}{dx} + \eta\frac{d}{dy} + \zeta\frac{d}{dz}\right)F(x, y, z)$$
$$+ \mu^3 F(\xi, \eta, \zeta) + \lambda\mu^2\left(x\frac{d}{d\xi} + y\frac{d}{d\eta} + z\frac{d}{d\zeta}\right)F(\xi, \eta, \zeta).$$

Hence X, Y, Z will be the collineal to (x, y, z), (ξ, η, ζ) if

$$\lambda : \bar{\mu} :: \left(x\frac{d}{d\xi} + y\frac{d}{d\eta} + z\frac{d}{d\zeta}\right)F(\xi, \eta, \zeta) : \left(\xi\frac{d}{dx} + \eta\frac{d}{dy} + \zeta\frac{d}{dz}\right)F(x, y, z).$$

If now we write $F(x, y, z)$ under its canonical form $x^3 + y^3 + z^3 + Kxyz$, it will be found, on substituting for λ and $\bar{\mu}$ the quantities to which they are proportional, that

$$X = (y^2\eta\xi - y\eta^2 x + z^2\zeta\xi - z\zeta^2 x) + K(yz\xi^2 - \eta\zeta x^2)$$
$$Y = (z^2\zeta\eta - z\zeta^2 y + x^2\xi\eta - x\xi^2 y) + K(zx\eta^2 - \zeta\xi y^2)$$
$$Z = (x^2\xi\zeta - x\xi^2 z + y^2\eta\zeta - y\eta^2 z) + K(xy\zeta^2 - \xi\eta z^2).$$

But these expressions admit of a surprising simplification, namely, we may neglect the terms not containing K, for it will be found that the quantities affected with the coefficient K are to each other in the same ratios as the other three corresponding groups in the values of X, Y, Z. Thus, for example

$$(yz\xi^2 - \eta\zeta x^2)(z^2\zeta\eta - z\zeta^2 y + x^2\xi\eta - x\xi^2 y)$$
$$- (zx\eta^2 - \zeta\xi y^2)(y^2\eta\xi - y\eta^2 x + z^2\zeta\xi - z\zeta^2 x)$$
$$= (\xi y - x\eta)\{\xi\eta\zeta(x^3 + y^3 + z^3) - xyz(\xi^3 + \eta^3 + \zeta^3)\}$$

hence $\quad X : Y : Z :: yz\xi^2 - \eta\zeta x^2 : zx\eta^2 - \zeta\xi y^2 : xy\zeta^2 - \xi\eta z^2.$

We might, instead of these simple expressions, take for X, Y, Z the other three groups and (using $x_1y_1z_1$; $x_2y_2z_2$ instead of x, y, z; ξ, η, ζ and (pq) to

denote the determinant $p_1q_2 - p_2q_1$) say that X, Y, Z are the minor determinants of

$$\begin{array}{ccc} x_1 \cdot x_2 & y_1 \cdot y_2 & z_1 \cdot z_2 \\ (yz) & (zx) & (xy), \end{array}$$

and these are actually the expressions found by Cauchy, and given by him in his *Exercices de Mathématiques*, Paris, 1826, p. 256, ll. 18—21, pp. 257—60. I take this reference from a loose page of an article by M. Lucas, but have not access either to that article or to Cauchy's.

It is remarkable that Cauchy should have given quadrinomial expressions for the collineal to two given points on a cubic curve, or their connective, as I shall hereafter term it, when, as shown above, binomial ones fulfil the same purpose. The correctness of these remarkable formulæ admits of easy verification, as follows:

For greater simplicity denote x^3, y^3, z^3, xyz by u, v, w, μ; and $\xi^3, \eta^3, \zeta^3, \xi\eta\zeta$ by u', v', w', μ' respectively. Then

$$\Sigma(yz\xi^2 - \eta\zeta x^2)^3 = \Sigma(vwu'^2 - v'w'u^2) - 3\mu\mu'\{(u' + v' + w')\mu - (u + v + w)\mu'\}$$
$$= \Sigma(vwu'^2 - v'w'u^2).$$

Also $\quad K(yz\xi^2 - \eta\zeta x^2)(zx\eta^2 - \zeta\xi y^2)(xy\zeta^2 - \xi\eta z^2)$

$$= -Kxyz(\xi^3\eta^3 z^3 + \eta^3\zeta^3 x^3 + \zeta^3\xi^3 y^3) + K\xi\eta\zeta(x^3y^3\zeta^3 + y^3z^3\xi^3 + z^3x^3\eta^3)$$
$$= (u + v + w)(uv'w' + vw'u' + wu'v') - (u' + v' + w')(u'vw + v'wu + w'uv)$$
$$= \Sigma(u^2v'w' - u'^2vw).$$

Hence, giving X, Y, Z the values indicated by the formula, we find

$$X^3 + Y^3 + Z^3 + KXYZ = 0,$$

which equation depends, as seen, and as we know *a priori* must be the case, on the pure algebraical fact that $X^3 + Y^3 + Z^3 + KXYZ$ is a syzygetic function of $x^3 + y^3 + z^3 + Kxyz$ and $\xi^3 + \eta^3 + \zeta^3 + K\xi\eta\zeta$, taking no account of the function $\xi\eta\zeta(x^3 + y^3 + z^3) - xyz(\xi^3 + \eta^3 + \zeta^3)$, as that is itself a syzygetic function of the two others. If we call the syzygetic multipliers of those two Φ and F respectively, it will at once be seen from what precedes that

$$\Phi = 3\xi^2\eta^2\zeta^2 xyz - \xi^3\eta^3 z^3 - \eta^3\zeta^3 x^3 - \zeta^3\xi^3 y^3$$
$$F = 3x^2y^2z^2\xi\eta\zeta - x^3y^3\zeta^3 - y^3z^3\xi^3 - z^3x^3\eta^3\,*.$$

I now proceed to apply the foregoing results to the problem of determining the order in the co-ordinates of any derivative numbered j (where $j = 3i \pm 1$),

* Thus $\quad F = -(yz\xi + zx\eta + xy\zeta)(yz\xi + \rho zx\eta + \rho^2 xy\zeta)(yz\xi + \rho^2 zx\eta + \rho xy\zeta)$
$\quad \Phi = -(\eta\zeta x + \zeta\xi y + \xi\eta z)(\eta\zeta x + \rho\zeta\xi y + \rho^2\xi\eta z)(\eta\zeta x + \rho^2\zeta\xi y + \rho\xi\eta z),$

and it is worthy of notice that we have incidentally solved with quantic values for F, Φ, U, V, W the simultaneous algebraico-diophantine equations

$$U^3 + V^3 + W^3 = (a^3 + b^3 + c^3)\Phi - (\alpha^3 + \beta^3 + \gamma^3)F$$
$$UVW = abc\Phi - \alpha\beta\gamma F.$$

which may be called its index, and shall prove that *the order of any derivative is the square of its index**. It will also be shown that each of the derivatives above referred to will be of the form xU, yV, zW, where U, V, W are quantics in x^3, y^3, z^3 as variables, since these quantities satisfy the equation

$$(xU)^3+(yV)^3+(zW)^3+KxyzUVW=0,$$

where
$$Kxyz=-x^3-y^3-z^3.$$

From this it follows that, calling x^3, y^3, z^3; a, b, c respectively, the scheme of derivatives contains the various solutions of the algebraico-diophantine equation

$$aU^3+bV^3+cW^3-(a+b+c)\,UVW=0,$$

and that, supposing the law of the squares to be demonstrated, U, V, W will be of the order $\frac{1}{3}\{(3i\pm 1)^2-1\}$, that is, $3i^2\pm 2i$ in a, b, c, where i is any integer. We thus see that the above equation admits of solutions in which U, V, W are of the orders 1, 5, 8, 16, 21, 33, 40 ... respectively. It will hereafter be shown, in like manner, that the missing derivatives, whose indices are multiples of 3 (belonging to the arbitrary point and point of inflexion combined), will satisfy the equation

$$U^3+V^3+abcW^3-(a+b+c)\,UVW=0,$$

where U, V, W will be necessarily of the orders $3i^2\pm 2i$, $3i^2\pm 2i$, $(i\pm 1)(3i\pm 1)$ respectively, i, as before, representing any integer. Thus we see that, if $a+b+c=0$, the equations

$$aU^3+bV^3+cW^3=0 \text{ and } U^3+V^3+abcW^3=0$$

will admit of an infinite number of solutions in integers, when a, b, c are integer. This fact, as regards the latter equation, has been already pointed out by M. Lucas in this *Journal*, and previously by the Abbé Pépin in his memoir in *Liouville's Journal*, 2nd series, Tome XV.

Let us begin with applying the formulæ to obtaining the co-ordinates of the tangential.

Let
$$x^3+y^3+z^3+3kxyz=0$$

be the equation to the cubic. If we take x, y, z; $x+\delta x$, $y+\delta y$, $z+\delta z$ two consecutive points, their connective will be the tangential.

Applying the formulæ just obtained, we shall obtain for its co-ordinates expressions each of the form $P\delta x+Q\delta y+R\delta z$ with only one relation between $\delta x, \delta y, \delta z$. Hence, if we write $\delta z=\lambda\delta x+\mu\delta y$ the resulting ratios must be

* The proof here supplied is sufficiently exact to dispel any reasonable doubt as to the truth of the law; but an exact proof which does not assume but demonstrates the non-existence of latent common measures to the reduced values of the co-ordinates of the connective to any two derivatives will be furnished under Title 5—one of the most surprising feats of demonstration which it has ever fallen to the author's lot to accomplish.

independent of λ and μ. Consequently we may make $\delta z=0$. The two connectives then become

$$x,\ y,\ z$$
$$x+\delta x,\ y+\delta y,\ z,$$

and the co-ordinates of the tangential will therefore be proportional to

$$yz(x+\delta x)^2-z(y+\delta y)x^2 : zx(y+\delta y)^2-z(x+\delta x)y^2 : z^2\{xy-(x+\delta x)(y+\delta y)\}$$

that is, to
$$x(2y\delta x-x\delta y) : y(2x\delta y-y\delta x) : z(x\delta y+y\delta x)$$

where
$$\delta x : \delta y :: y^2+kxz : x^2+kyz.$$

Hence the co-ordinates required are as

$$x\{2y^3+x^3+3kxyz\} : y\{-2x^3-y^3-3kxyz\} : z(x^3-y^3),$$

that is, as
$$x(y^3-z^3) : y(z^3-x^3) : z(x^3-y^3),$$

a result which appears to have been first found by Cauchy for the general form, but previously by Euler, and before him by Fermat, for the case $k=0$.

If we write a, b, c, instead of x, y, z, and call the co-ordinates of the tangential x, y, z, we might find their values by virtue of the condition that the connective of a, b, c and x, y, z is a, b, c over again. This furnishes the equations

$$bcx^2-a^2yz=am$$
$$cay^2-b^2zx=bm$$
$$abz^2-c^2xy=cm,$$

which may be satisfied by writing

$$x=a(b^3-c^3)\rho;\quad y=b(c^3-a^3)\rho;\quad z=c(a^3-b^3)\rho;$$
$$(a^6+b^6+c^6-a^3b^3-b^3c^3-a^3c^3)\rho^2=m;$$

but whether or not the above is necessarily the only possible solution is not quite clear *a priori*, and *a posteriori* it looks as if the solutions might be manifold.

The co-ordinates of the point whose index is 4, that is, of the second tangential, will be those of the first tangential to the point

$$x(y^3-z^3) : y(z^3-x^3) : z(x^3-y^3),$$

namely,

$$x(y^3-z^3)\{y^3(x^3-z^3)^3+z^3(x^3-y^3)^3\} : y(z^3-x^3)\{z^3(y^3-x^3)^3+x^3(y^3-z^3)^3\}$$
$$: z(x^3-y^3)\{x^3(z^3-y^3)^3+y^3(z^3-x^3)^3\},$$

and are of the order 16.

To find the co-ordinates of the point whose index is 5, we may take the connective of the one last found, and of x, y, z, that is, of 4 and 1. Let us call them xU, yV, zW, and, for greater simplicity, denote x^3, y^3, z^3, by u, v, w. Then, omitting the common factor xyz,

$$U = (v-w)^2 \{v(u-w)^2 + w(u-v)^2\}^2$$
$$- (w-u)(u-v)\{w(v-u)^2 + u(v-w)^2\}\{u(w-v)^2 + v(w-u)^2\},$$

with similar quantities (*mutatis mutandis*) set against V and W.

These quantities will have the common measure

$$u^2 + v^2 + w^2 - uv - uw - vw.$$

To prove this let either one of its factors, as $u + \rho v + \rho^2 w, = 0$.

Then $$v - u = \rho^2 (w - v) \text{ and } u - w = \rho (w - v),$$

and the representative of U above written becomes

$$\{(v-w)^2 - (w-u)(u-v)\}(w-v)^8 = (v^2 + w^2 + u^2 - vw - uw - uv)(w-v)^8 = 0.$$

Hence the representative of U vanishes with, and therefore contains

$$u^2 + v^2 + w^2 - uv - uw - vw$$

as a factor, and the same must evidently be true for the representatives of V and W; hence, U, V, W, will be of the order $10 - 2$ or 8, in u, v, w, and the co-ordinates xU, yV, zW, of the order $3 . 8 + 1$, that is, of the order 25 in xyz.

The preceding demonstration depends essentially on the fact that my simplified formulæ for the co-ordinates of the connective of two points on a cubic fail, that is to say, become illusory, for a particular relation between the two points, as is easily seen; for let x, y, z; $x, \rho y, \rho^2 z$ be two points on a cubic, then the formulæ for X, Y, Z, the connective's co-ordinates, become

$$(\rho y . \rho^2 z - yz) x^2; \ (\rho^2 z . x - xz\rho^2) y^2; \ (x . \rho y - xy\rho^4) z^2,$$

that is, all vanish, whereas it may be remarked that the general expressions given at page [354],

$$X = (y^2\eta\xi - y\eta^2 x + z^2\zeta\xi - z\zeta^2 x) + K(yz\xi^2 - \eta\zeta x^2)$$
$$Y = (z^2\zeta\eta - z\zeta^2 y + x^2\xi\eta - x\xi^2 y) + K(zx\eta^2 - \zeta\xi y^2)$$
$$Z = (x^2\xi\zeta - x\xi^2 z + y^2\eta\zeta - y\eta^2 z) + K(xy\zeta^2 - \xi\eta z^2),$$

become the minors of

$$\begin{matrix} x^2 & \rho y^2 & \rho^2 z^2 \\ (\rho^2 - \rho) yz & (1 - \rho^2) zx & (\rho - 1) xy; \end{matrix}$$

that is, $(\rho^2 - \rho) x (y^3 - z^3), \ (\rho - 1) y (z^3 - x^3), \ (1 - \rho^2) z (x^3 - y^3)$,

which are the same as

$$x(y^3 - z^3), \quad \rho^2 y (z^3 - x^3), \quad \rho z (x^3 - y^3),$$

and remain perfectly valid.

This law of the failing case enables me to prove very easily the *Law of Squares*, as follows:

Suppose it proved that for all indices inferior to $6i$ the order of the derivative is equal to the square of its index; then, to prove that the same law is true up to $6(i+1)$, it is only necessary to consider the cases of $6i+1$, $6i+5$, for, as regards the indices $6i+2$ and $6i+4$, the derivatives may be regarded as the tangentials of the derivatives to indices $3i+1$ and $3i+2$, and will consequently be of the orders $4(3i+1)^2$ and $4(3i+2)^2$, that is, $(6i+2)^2$ and $(6i+4)^2$ respectively.

Let us further suppose that for derivatives whose indices are inferior to $6i$ the co-ordinates are of the form xU, yV, zW; U, V, W being quantics in x^3, y^3, z^3; then, obviously, from the mode of forming the tangential, this will be true for derivatives whose indices are $6i+2$, $6i+4$: for the tangential to xU, yV, zW is

$$xU(y^3V^3 - z^3W^3), \quad yV(z^3W^3 - x^3U^3), \quad zW(x^3U^3 - y^3V^3).$$

Let us consider the point (1) whose co-ordinates x, y, z satisfy the equation

$$x^3 + \rho y^3 + \rho^2 z^3 = 0.$$

For such a point $$y^3 - z^3 : z^3 - x^3 : x^3 - y^3 :: 1 : \rho : \rho^2,$$

and the point (2) becomes $x, \rho y, \rho^2 z$. Consequently the point (4) becomes $x(y^3 - z^3), \rho y (z^3 - x^3), \rho^2 z (x^3 - y^3)$, the same as $x, \rho^2 y, \rho z$; hence the point (5), the connective of (1, 4), becomes $x(y^3 - z^3), \rho y (z^3 - x^3), \rho^2 z (x^3 - y^3)$, the same as $x, \rho^2 y, \rho z$, so that, denoting the derivatives by their indices,

$$5 = 4 \qquad 7 = 1, \ 8 = 1, \ 1 = 2 \qquad 10 = 2, \ 8 = 2, \ 1 = 1$$
$$11 = 4, \ 7 = 4, \ 2 = 2 \qquad 13 = 2, \ 11 = 2, \ 2 = 4, \text{ etc.}$$

We have, thus, for all values of the point i

$$9i \pm 1, \ 2, \ \pm 4 = 1, \ 2, \ 4,$$

when 1 is the point for which $x^3 + \rho y^3 + \rho^2 z^3 = 0$.

Hence, if p, p' be any two points for which $p - p' = 3$, then p, p' will be respectively identical with some two out of the three points 1, 2, 4. And it will at once be seen that the simplified formulæ for the connective of any two of these three points become illusory.

Now the point $6i+1$ is the connective of $3i-1$ and $3i+2$, and the point $6i+5$ is the connective of $3i+1$ and $3i+4$.

Hence, in each of these cases, the simplified formulæ become illusory, that is, the expressions for each of the co-ordinates vanish when

$$x^6 + y^6 + z^6 - x^3y^3 - x^3z^3 - y^3z^3$$

vanishes, and must therefore contain it as a common measure. Moreover, the simplified formulæ for the connective co-ordinates for the points xU, yV, zW;

xU', yV', zW' will contain x^2yz, y^2zx, z^2xy, and will therefore have the common measure xyz. Hence the values of the co-ordinates when freed from these common measures will be of the order in x, y, z, $2(3i-1)^2+2(3i+2)^2-9$ for the point $6i+1$, and $2(3i+1)^2+2(3i+4)^2-9$ for the point $6i+5$, that is $(6i+1)^2$ and $(6i+5)^2$ respectively, and will obviously continue to be quantics in x^3, y^3, z^3 multiplied by x, y, z respectively. Hence the theorem being true for index inferior to 6 is true universally.

It will be observed that any co-ordinate X of the point k must contain the X co-ordinate of the point k' where k' is any factor of k; for if $k=\delta k'$ the point k may be obtained by forming the point δ to the point k', and it has been shown that the δ derivative to any point has co-ordinates which contain respectively those of the initial point. Consequently the X co-ordinate to any point k may be resolved into factors containing a primitive part of the order τk (the totient of k) in the variables, and a non-primitive part containing the primitive part of each power of a prime contained in k, and with the exception of the single factor x all the others will be quantics in x^3, y^3, z^3; and, of course, the same remark applies to the other two co-ordinates Y and Z. We might obtain the point $m \ddagger n$ as the connective of m, n. In that case the simplified formulæ would give expressions of the order $2(m^2+n^2)$ in x, y, z; and as the actual order of the co-ordinates in those variables is $(m \ddagger n)^2$, it follows that when $m-n\equiv 0$, mod. 3, there will be a common measure (a symmetrical function of x, y, z) of the order $(m-n)^2$, and when $m+n\equiv 0$, mod. 3, of the order $(m+n)^2$ running through those expressions, and it might be desirable to ascertain its form; but without waiting to solve this problem *, which is irrelevant to the matter in hand, I shall proceed at once to consider the derivatives corresponding to indices which are multiples of the number 3, to obtain which it is only necessary, as will be seen immediately, to combine one given point of inflexion with one arbitrary point of the curve. But, before doing so, it may be well to notice, that while the preceding investigation serves to show that the abridged formulæ for the connective co-ordinates possess the common measure

$$xyz(x^6+y^6+z^6-x^3y^3-x^3z^3-y^3z^3),$$

it does not demonstrate categorically that there is no other; or that some power of the second factor above written other than the first might not be a common measure. Consequently, what we have strictly proved, as will be evident on reviewing the argument, is that the order to a derivative of the index $3i\pm 1$ cannot *exceed* the square of that index; but before I come to an end of the discussion I trust to be able to establish with *Dirichletian* rigour that the order is actually equal to the square of the index †.

* It is completely solved in the corollary to Title 5.

† This anticipation (for it was only such when these words were written) will be found fully realised under Title 5.

Title 2.—On the Completed or Continuously Numbered Scale of Rational Derivatives to an Arbitrary Point on a Cubic, of which one Point of Inflexion is given.

Let I be the given point of inflexion, and let any point (or system of points) and another point (or system of points respectively) collinear with the former in respect to I be called opposites. It is obvious that $(I, I)=I$, or that the inflexion is its own opposite. It will be convenient to denote the opposite to any point by the same index, but accented.

We have, then, obviously,

$$(p', p)=I;\ (p')'=p \text{ and } (p', q)'=I,\ (p', q)=(I, I),\ (p', q)$$
$$=(I, p'),\ (I, q)=(p, q').$$

Let $(I', 2)=3$; $(I', 5)=6$; and in general $(I', 3i-1)=3i$. This is matter of definition. Let, now, the infinite system 1, 2, 3, 4, 5, 6, 7 ... and its opposite be regarded as a single group. I say, (1), that this will be a closed group, in the sense that a straight line drawn through any two points (contiguous or apart) of this double chain will cut the cubic in a third point included in the group, (2), that the new points will be rational in respect to the co-ordinates of the initial point and the given point of inflexion, and, (3), that the order in the variables for every point, without regard to its relation to the modulus 3, will be, as before, the square of its index.

I proceed to show that the connective of any two points in the double chain may be expressed as a single point therein. Several cases present themselves according to the form of each of the two connected points in respect to the modulus 3, except when the indices are congruent in respect to that modulus.

When the residues (r, r'), in respect to that modulus, are dissimilar, the result will in general be different according as one of them (as r) belongs to the higher or lower index.

In what follows it is to be understood that $i \lesssim j$.

Theorem 1. To prove that

$$3i+1,\ (3j+1)'=3j-3i$$

and

$$3i+2,\ (3j+2)'=(3j-3i)'.$$

[This will imply that

$$(3i+1)',\ 3j+1=(3j-3i)'$$

and

$$(3i+2)',\ 3j+2=3j-3i.]$$

We have

$$3i+1,\ (3j+1)'=(3i-1, 2),\ [(3j-1)', 2']=(2, 2'),\ [3i-1, (3j-1)']$$
$$=(3i-1)',\ 3j-1=[(3i-2)', 1'],\ (3j-2, 1)=(1, 1'),\ [(3i-2)', 3j-2]$$
$$=3i-2,\ (3j-2)'.$$

Hence, $3i+1,\ (3j+1)'=1,\ (3j-3i+1)'=(1,2),\ [(3j-3i-1)',2']$

$$=(2,2'),\ [1,(3j-3i-1)']=1',\ (3j-3i-1)=3j-3i$$

and $3i-1,\ (3j-1)'=I,\ [3i-2,(3j-2)']=(3j-3i)'.$

Theorem 2. To prove that

$$3i+1,\ (3j-1)'=(3i+3j)'$$

and

$$3i-1,\ (3j+1)'=3i+3j.$$

[This will imply that

$$(3i+1)',\ 3j-1=3i+3j$$

and

$$(3i-1)',\ 3j+1=(3i+3j)'.]$$

We have $3i+1,\ (3j-1)'=3i-1,\ 2;\ (3j+1)',\ 2'=(3i-1)',\ 3j+1$

$$=[(3i-2)',1'],\ (3j+2,1)=3i-2,\ (3j+2)'.$$

Therefore, $3i+1,\ (3j-1)'=1,\ (3j+3i-1)'=(3i+3j)'$

and $3i-1,\ (3j+1)'=I,\ [(3i-1)',3j+1]=3i+3j.$

Collecting the results of these two theorems, we see that

$$\left.\begin{array}{l} 3i\pm 1,\ (3j+1)'=3j\mp 3i=(3i\mp 1)',\ 3j-1 \\ 3i\pm 1,\ (3j-1)'=(3j\pm 3i)'=(3i\mp 1)',\ 3j+1 \end{array}\right\}, \qquad \text{(A)}$$

so that, using $p \overset{\circ}{-} q$ (where neither p nor q contains 3), to denote that one of the two numbers $p+q,\ p\sim q$, which is divisible by 3, (p, q') is always either $p \overset{\circ}{-} q$ or $(p \overset{\circ}{-} q)'$. Also

$$3i+1,\ (3j)'=(3i-1,2),\ [1,(3j-1)']=(1,2),\ [3i-1,(3j-1)']$$

$$=(3j-3i)',\ 1=(1',3j-3i+1),\ (2,1)=[(1',1),(3j-3i+1,2)]$$

$$=(3j-3i-1)';$$

again $3i,\ (3j+1)'=(3i-1,1'),\ [(3j-1)',2']=1',\ (3j-3i)'$

$$=(1',2'),\ [1,(3j-3i-1)']=(1,1'),\ [(3j-3i-1)',2']=3j+1-3i;$$

and lastly $3i,\ (3i+1)'=(3i-1,1'),\ [(3i-1)',2']=I,\ 1'=1.$

Hence, collecting the results, $3\iota,\ (3i+1)'=(3i+1)\sim 3\iota$, whatever the relation of magnitude may be between i and ι.

Similarly,

$$3i-1,\ (3j)'=(3i+1,2),\ [1,(3j-1)']=(1,2),\ [3i+1,(3j-1)']$$

$$=1,\ (3i+3j)'=(3i+3j-1)';$$

$$(3i)',\ 3j-1=[(3i-1)',1],\ (3j+1,2)=1,\ (3i+3j)'=(3i+3j-1)';$$

and $(3i)',\ 3i-1=[(3i-1)',1],\ (3i+1,2)=1,\ (6i)'=(6i-1)'.$

Hence, collecting the results, $3i-1,\ (3\iota)'=(3i+3\iota-1)'$, and we have

$$\left.\begin{array}{l} 3\iota,\ (3i+1)'=(3i+1)\sim 3\iota;\ \ (3\iota)',\ 3i+1=[(3i+1)\sim 3\iota]' \\ 3\iota,\ (3i-1)'=3i-1+3\iota;\ \ (3\iota)',\ 3i-1=(3i-1+3\iota)'. \end{array}\right\} \qquad \text{(B)}$$

Also,

$$\left.\begin{array}{l} 3\iota,\ 3i-1=(3\iota-1,1'),\ (3i-2,1)=(3\iota-1,3i-2)'=[(3i-1)\sim 3\iota]' \\ 3\iota,\ 3i+1=(3\iota-1,1'),\ (3i+2,1)=(3\iota-1,3i+2)'=(3\iota+3i+1)'. \end{array}\right\} \qquad \text{(B}'\text{)}$$

It remains only to determine the connectives of $3i,\ 3\iota$ and of $3i,\ (3j)'$ or $(3i)',\ 3j$, which is easily done, for

$$3i,\ 3\iota=(3i-1,1'),\ (3\iota-1,1')=(1',1'),\ (3i-1,3\iota-1)=2',\ 3i+3\iota-2.$$

Hence (by A) $3i,\ 3j=(3i+3j)'$ and consequently $(3i)',\ (3\iota)'=3i+3\iota$.

Again

$3i,\ (3j)'=(3i-1,1'),\ [(3j-1)',1]=(1,1'),\ [3i-1,(3j-1)']=$ (by theorem A) $I,\ (3j-3i)'=3j-3i$. Hence also $3j,\ (3i)'=(3j-3i)'$.

These three results may be designated theorem C; and theorems A, B, B′, C collectively prove that the original scale 1, 2, 4, 5, 7, 8 ..., which formed a closed system (so to say "group"), remains still closed when we complete it by insertion of multiples of 3, provided that we join on to the completed system 1, 2, 3, 4, 5, 6, 7 ... the opposite system 1′, 2′, 3′, 4′, 5′, 6′, 7′ ....

In every case it will be observed the connective of two indices (disregarding the accent) is either their sum or their difference.

The double scale may be formed by alternate addition of 1 and 1′ in the manner following:

$$1,1=2 \qquad 1',2=3 \qquad 1,3=4' \qquad 1',4'=5' \qquad 1,5'=6' \qquad 1',6'=7$$

$$1,7=8 \qquad 1',8=9 \qquad 1,9=10' \qquad 1',10'=11' \qquad 1,11'=12' \ldots$$

which gives the numbers 1, 2, 3, 4′, 5′, 6′, 7, 8, 9, 10′, 11′, 12′, etc.; and, in like manner, by interchanging 1, 1′, we may obtain 1′, 2′, 3′, 4, 5, 6, 7′, 8′, 9′, 10, 11, 12, etc.

The new points 3, 6, 9 ...; 3′, 6′, 9′ ... belong to the natural scales 1, 2, 5 ...; 1′, 2′, 5′ ... collectively and not respectively; and the accented and unaccented multiples of 3 might have had their significations interchanged without any impropriety. It is now necessary to extend the law of the order in the variables to these inserted points, and to prove that for them, as for the points in the natural scale, the order of any point, in the variables of the initial point, is the square of its index.

If the cubic be thrown into the canonical form $x^3+y^3+z^3+kxyz$, the point $x=1,\ y=-1,\ z=0$ may be taken to represent I, and if x, y, z be the initial point 1, the co-ordinates of 1′ (the connective of 1 and I) become by the general formula $yz,\ zx,\ z^2$, or, more simply, y, x, z.

To find 3, then, we have to take the connective of y, x, z and $x(y^3 - z^3)$, $y(z^3 - x^3)$, $z(x^3 - y^3)$; its co-ordinates, accordingly, by the general formula, are

$$yz\,(z^3 - x^3)\,(x^3 - y^3)\,y^2 - x^2z\,(y^3 - z^3)^2$$
$$xz\,(x^3 - y^3)\,(y^3 - z^3)\,x^2 - y^2z\,(z^3 - x^3)^2$$
$$xy\,(y^3 - z^3)\,(z^3 - x^3)\,z^2 - yxz^2\,(x^3 - y^3)^2;$$

or, neglecting the common factor z, the co-ordinates of 3 are

$$y^3\,(x^3 - y^3)\,(x^3 - z^3) + x^3\,(y^3 - z^3)^2$$
$$x^3\,(y^3 - x^3)\,(y^3 - z^3) + y^3\,(z^3 - x^3)^2$$

and $$xyz\,(z^3 - x^3)\,(z^3 - y^3) + xyz\,(x^3 - y^3)^2;$$

or $$y^3x^6 + z^3y^6 + x^3z^6 - 3x^3y^3z^3$$
$$x^3y^6 + z^3x^6 + y^3z^6 - 3x^3y^3z^3$$

and $$xyz\,(z^6 + y^6 + x^6 - x^3y^3 - z^3x^3 - y^3z^3).$$

In the particular case where $x^3 + y^3 + z^3 = 0$, these expressions (writing for greater brevity L, M, N for x^3, y^3, z^3) become

$$ML^2 - (L + M)\,M^2 + L\,(L + M)^2 + 3LM\,(L + M)$$
$$LM^2 - (L + M)\,L^2 + M\,(L + M)^2 + 3LM\,(L + M)$$
$$xyz\,[(L + M)^2 + L^2 + M^2 - LM + (L + M)^2]$$

or $$L^3 + 6L^2M + 3LM^2 - M^3$$
$$M^3 + 6M^2L + 3ML^2 - L^3$$
$$3xyz\,(L^2 + LM + M^2);$$

which remain equally good, as co-ordinates of the point 3 to the initial point x, y, z, when the cubic is $x^3 + y^3 + Cz^3$, as is easily seen by writing $C^{\frac{1}{3}}z = \zeta$.

The point 3, it follows from what precedes, is of the order 9 in the variables x, y, z, and the same will be true for $3'$, which is obtained from 3 by the interchange of x and y; but in order that these points may be arithmetically as well as algebraically rational, it is of course necessary that the given cubic may admit of being expressed under the form

$$Ax^3 + Ay^3 + Cz^3 + Kxyz,$$

where A, C and K are integers.

Again, since $6 = 3', 3'$, 6 is the 2 of $3'$, and similarly $6'$ is the 2 of 3; since $9 = 3', 6'$ and $6'$ is the 2 of 3, 9 is the 3 of 3. So again, since $12 = 3', 9'$ and $9'$ is the 3 of $3'$, 12 is the (1, 3) of $3'$, that is, the $4'$ of $3'$ or 4 of 3; and similarly $12'$ is the 4 of $3'$. So again,

$$15 = (3', 12') = (1, 4) \text{ of } 3' = 5 \text{ of } 3', \text{ and } 15' = 5 \text{ of } 3$$
$$18 = (3', 15') = (1, 5') \text{ of } 3' = 6' \text{ of } 3' = 6 \text{ of } 3, \text{ and } 18' = 6 \text{ of } 3'$$
$$21 = (3', 18') = (1, 6) \text{ of } 3' = 7' \text{ of } 3' = 7 \text{ of } 3, \text{ and } 21' = 7 \text{ of } 3'$$
$$24 = (3', 21') = (1, 7) \text{ of } 3' = 8 \text{ of } 3', \text{ and } 24' = 8 \text{ of } 3;$$
$$27 = (3', 24') = (1, 8') \text{ of } 3' = 9' \text{ of } 3' = 9 \text{ of } 3 \ldots.$$

Hence, in general,

$$9i + 3 = (3i + 1) \text{ of } 3;\; 9i + 6 = (3i + 2)' \text{ of } 3; \text{ and } 9i = 3i \text{ of } 3.$$

Consequently

$$3^q\,(3i + 1) = (3i + 1) \text{ of } 3 \text{ of } 3 \text{ of } 3 \ldots (q \text{ times repeated}),$$

and $$3^q\,(3i + 2) = (3i + 2)' \text{ of } 3 \text{ of } 3 \text{ of } 3 \ldots (q \text{ times repeated}).$$

From this it follows, obviously, that $3^q\,(3i \pm 1)$ and $[3^q\,(3i \pm 1)]'$ are each of the order $[3^q\,(3i \pm 1)]^2$ in the variables, and thus the law of the squares extends to all points alike in the completed scale.

Title 3.—On Compound Derivation.

The object of what follows is to show that any derivative of a derivative has for its index (due regard being paid to the accents) the product of the numerical values of the indices of the operator and operand derivatives, that is to say, the $i^?$ of $j^? = ij^?$; the mark of interrogation denoting either a blank or an accent, as the case may be. Thus, while connection involves addition or subtraction, composition involves a process of multiplication.

(1) Let us consider the i of j when neither i nor j contains 3. Then

$3k + 1$ of $j = (2 \text{ of } j), (3k - 1 \text{ of } j)$ and $3k + 2$ of $j = (1 \text{ of } j), (3k + 1 \text{ of } j)$.

Suppose the theorem proved up to $3k - 1$. Then

$$3k + 1 \text{ of } j = 2j,\, 3kj - j = (3k + 1)\,j$$
$$3k + 2 \text{ of } j = j,\, 3kj + j = (3k + 2)\,j.$$

Hence it is true up to $3\,(k + 1) - 1$, and, being true when $k = 1$ (since 1 of $j = j$ and 2 of $j = j, j = 2j$), it is true universally.

In like manner, since 1 of $j' = j'$ and 2 of $j' = j', j' = I, (j, j) = (2j)'$, it may be shown that i of $j' = (ij)'$. Moreover

$$1' \text{ of } j = j', \text{ and therefore } 2' \text{ of } j = (1' \text{ of } j), (1' \text{ of } j) = j', j' = 2j'$$

and $$(3k + 1)' \text{ of } j = (2' \text{ of } j), [(3k - 1)' \text{ of } j]$$
$$(3k + 2)' \text{ of } j = (1' \text{ of } j), [(3k + 1)' \text{ of } j];$$

so that, if the equation i' of $j = (ij)'$ holds good up to $i = 3k - 1$,

$$(3k + 1)' \text{ of } j = [(3k + 1)\,j]', \text{ and } (3k + 2)' \text{ of } j = [(3k + 2)\,j]';$$

so that the equation i' of $j = (ij)'$ will hold good up to $3\,(k + 1) - 1$, and, being true for $k = 1$, is true universally.

In like manner, since $1'$ of $j' = j$, it will follow that i' of $j' = ij$.

It remains to obtain the corresponding equations when i, j are one or both of them multiples of 3.

Since 3 of $3^q = (3^q, 3^q), (3^q)' = (2 \,.\, 3^q)', (3^q)' = 3^{q+1}$,

$$9 \text{ of } 3^q = 3 \text{ of } 3 \text{ of } 3^q = 3 \text{ of } 3^{q+1} = 3^{q+2},$$

$$27 \text{ of } 3^q = 3 \text{ of } 9 \text{ of } 3^q = 3 \text{ of } 3^{q+2} = 3^{q+3}, \text{ and so on.}$$

Hence $$3^p \text{ of } 3^q = 3^{p+q}.$$

Again, $$3 \text{ of } 3j+1 = (3j+1, 3j+1), (3j+1)'$$
$$= 6j+2, (3j+1)' = 9j+3 \text{ by A.}$$

Hence 3^2 of $3j+1 = 3$ of $9j+3 = (18j+6)', (9j+3)' = 27j+9$ by C,

3^3 of $3j+1 = 3$ of $27j+9 = (54j+18)', (27j+9)' = 81j+27$ by C,

and so on. Hence $$3^p \text{ of } 3j+1 = 3^p(3j+1).$$

Again, $$3 \text{ of } 3j+2 = (3j+2, 3j+2), (3j+2)'$$
$$= 6j+4, (3j+2)' = (9j+6)' \text{ by A.}$$

Hence 3^2 of $3j+2 = 3$ of $(9j+6)' = 18j+12, 9j+6 = (27j+18)'$ by C,

and so on. Hence $$3^p \text{ of } 3j+2 = [3^p(3j+2)]'.$$

Again, $3j+1$ of $3^p = (2 \text{ of } 3^p), (3j-1 \text{ of } 3^p) = (3^p, 3^p), (3j-1 \text{ of } 3^p)$
$$= (2 \,.\, 3^p)', (3j-1 \text{ of } 3^p)$$

and $$3j-1 \text{ of } 3^p = (1 \text{ of } 3^p), (3j-2 \text{ of } 3^p).$$

Suppose it true that $3j-2$ of $3^p = (3j-2)\,3^p$ for a certain value of j.

Then $$3j-1 \text{ of } 3^p = 3^p, (3j-2)\,3^p = [(3j-1)\,3^p]'$$

and $$3j+1 \text{ of } 3^p = (2 \,.\, 3^p)', [(3j-1)\,3^p]' = (3j+1)\,3^p.$$

But 1 of $3^p = 1 \,.\, 3^p$; hence, for all values of j,

$$3j+1 \text{ of } 3^p = (3j+1)\,3^p = 3^p \text{ of } 3j+1$$

$$3j-1 \text{ of } 3^p = [(3j-1)\,3^p]' = 3^p \text{ of } 3j-1.$$

Hence, by the well-known method of successive transformation, we obtain the following results:

When neither m nor n contains 3, when both contain 3, and when one of them contains 3 and the other is of the form $3j+1$, we have

$$m \text{ of } n = n \text{ of } m = m' \text{ of } n' = n' \text{ of } m' = mn$$

$$m \text{ of } n' = n' \text{ of } m = m' \text{ of } n = n \text{ of } m' = (mn)'.$$

In the remaining case (namely when of m and n, one contains 3 and the other is of the form $3j-1$), we have

$$m \text{ of } n = n \text{ of } m = m' \text{ of } n' = n' \text{ of } m' = (mn)'$$

$$m \text{ of } n' = n' \text{ of } m = m' \text{ of } n = n \text{ of } m' = mn.$$

This completes the algorithm of rational derivation.

Title 4.—On Pertactile or Periodic Points on a Cubic Curve.

A pertactile point, or point of pluperfect tactility, on a general cubic is a point at which the cubic admits of a higher order of contact with another curve than is in general possible. Thus the points of inflexion are pertactile points, because a tangent at one of them will meet the curve in three consecutive points. The same is the case with Plücker's twenty-seven points, because at each of them a conic of closest contact will pass through six consecutive points, the sixth point in which any conic passed through five consecutive points cuts the curve coinciding, in this case, with the point of contact. So, in general, a curve of the ith order can only be made to pass through $3i-1$ consecutive points situated at P; but if the ith derivative of P is a point of inflexion, then the $3i$th point common to all curves of the ith order passing through $3i-1$ consecutive points at P will coincide with P, so that such curves will pass through $3i$ consecutive points, and P may accordingly be termed a point of pluperfect tactility, or more briefly, a pertactile point.

To prove that this is the case, it is necessary, in the first place, to prove that, at a general point P in the cubic, the $3i$th point in which all curves of the ith order passing through $3i-1$ consecutive points at P intersect the cubic, is the $(3i-1)$th derivative of P, which may be done inductively as follows:

Suppose P_{3i-1} is the residual of $3i-1$ consecutive points at P. To find the residual of $3i+2$ consecutive points there, we may combine $3i-1$ giving the residual P_{3i-1}, two more of them giving the residual P_2, and one giving Q, R, any two points collinear with P. We then combine $(P_{3i-1}, P_2), (Q, R)$ and obtain P_{3i+1}, P_1 which gives P_{3i+2} as the required residual. Hence the theorem, being true for P_2 (the residual of two consecutive points at P) and true for $P_{3(i+1)-1}$ if true for P_{3i-1}, is true universally.

If, now, the residual of $3i-1$ points at P is to fall at P we must have $P_1 = P_{3i-1}$.

(1) Suppose $i = 3k-1$, then $P_1, P_{i-1} = P_{i-1}, P_{3i-1}$, that is $P_i = P_{2i}$.

Hence P_i is a point of inflexion I, or, as we may express it, P is an ith sub-derivative of such point, or $P = I_{\frac{1}{i}}$.

(2) Suppose $i = 3k+1$, then $P_1, P_2 = P_2, P_{3i-1}$, that is $P_1 = P_{3i+1}$.

Hence $P_1, P_{i+1} = P_{i+1}, P_{3i+1}$, that is $P_i = P_{2i}$, and, as before, $P = I_{\frac{1}{i}}$.

(3) Suppose $i = 3k$.

Then $1, (i-1)' = (i-1)', 3i-1$, that is $i' = 2i = i', i'$. Consequently i', and therefore also i, is a point of inflexion.

Hence, as in the other two cases, P is an ith sub-derivative of a point of inflexion*, which may either be the point used to form the scale, or any of the eight other inflexions†.

It may be well to notice here that whilst P_i, when i does not contain 3, is, as already shown, of the form xU, yV, zW, it follows from the law of compound derivation, since P_3 is of the form R, S, $xyz\Theta$ (where R, S, Θ, like U, V, W, are quantics in x^3, y^3, z^3) that P_i, when i is a multiple of 3 or any power of 3, will be of the form M, N, $xyz\Omega$ (where M, N, Ω are still quantics in x^3, y^3, z^3).

Calling X, Y, Z any ith derivative to $x^3+y^3+z^3+kxyz=0$, we must have $X^3+Y^3+Z^3+kXYZ=0$; and, in order for such derivative to be a point of inflexion, it is necessary and sufficient that $X=0$ or $Y=0$ or $Z=0$; combining these equations respectively with the given cubic, we shall obtain, in all, 3 times $3i^2$ or $9i^2$ points, sub-derivatives of the ith grade to one or other of the inflexions; but out of these, whether i be or be not divisible by 3, nine will correspond to $x=0$, $y=0$, or $z=0$ combined with the curve, that is, will be the points of inflexion themselves. Moreover, unless i be a prime number, it follows from the law of compound derivation, combined with the fact that x, y, z enter distributively or collectively into the derived co-ordinates X, Y, Z, that, if i' be any factor of i, and X', Y', Z' the co-ordinates of the i'th derivative, Z will contain Z' and X, Y or Y, X, will contain X', Y' respectively. There will thus be a *primitive* part to X, Y, Z which results from driving out all the factors corresponding to any factor of i (unity included), and, if we suppose $i=a^\alpha . b^\beta . c^\gamma \ldots$, the order of this primitive part in the variables x, y, z, it is easy to see, will be

$$a^{2(\alpha-1)} . b^{2(\beta-1)} . c^{2(\gamma-1)} \ldots \{(a^2-1)(b^2-1)(c^2-1)\ldots\},$$

which may be called the quadri-totient to i, and is the product of two factors, one the totient of i and the other what that totient becomes when $+1$ is substituted throughout for -1 in its expression, and which, if a name were needed for it, might be called the contra-totient.

The number of proper, or primitive, ith sub-derivatives of any point of inflexion will thus be the quadri-totient of i (just as the number of primitive ith roots of unity is the totient), and the total number of pertactile points of the ith grade, 9 times the quadri-totient of i.

It is easy to see that the points corresponding to the non-primitive factors of X, Y, Z satisfy, but in an *improper* manner, the conditions of the question. For, if i' is any sub-multiple of i $\left(\text{say } i'=\frac{i}{\delta}\right)$ and P' is an i'th sub-derivative

* A sub-derivative of an inflexion may conveniently be termed a sub-inflexion.

† The above formulæ show that i, $i'=3i=3i'$; hence $3i$ and $3i'$ coincide with the original point of inflexion, whereas i, i', $2i$, $2i'$ need not coincide with the original point of inflexion.

of a point of inflexion, through P' may be drawn δ curves each of the order i' (constituting an improper curve of the order i), each passing through $3i'$ consecutive points, and consequently their *ensemble* passes through $\delta . 3i'$ or $3i$ consecutive points. We have now obtained the generalization of the theorem of which the enumeration of the points of inflexion and Plücker's points constitute the two first steps, and it is very easy to calculate the number of pertactile points N of any given grade i. Thus for

$$i=1, 2, 3, \ 4, \ 5, \ 6, \ 7, \ 8, \ 9, 10, \ 11, 12\ldots$$

$$\frac{N}{9}=1, 3, 8, 12, 24, 24, 48, 48, 72, 72, 120, 96\ldots$$

The calculation is facilitated by the remark that if i, j are prime to each other, the number of (ij)th sub-derivatives to any one point of inflexion is the product of the number of ith by the number of jth sub-derivatives; the quadritotient obeying the same law as the totient in this particular.

If i is the grade of the pertactile point P, so that $P_1=P_{3i-1}$, then P_i is an inflexion, and P_{3i} is I, the original inflexion. Moreover

$$P_1=P_1, \ P_2=P_{3i-1}, \ P_2=P_{3i+1}$$

$$P_2=P_1, \ P_1=P_1, \ P_{3i-1}=P_{3i-2} \text{ and also} =P_2, \ P_4=P_{3i-2}, \ P_4=P_{3i+2}$$

$$P_4=P_2, \ P_2=P_2, \ P_{3i-2}=P_{3i-4} \text{ and also} =P_2, \ P_{3i+2}=P_{3i+4}, \text{ and so on.}$$

And again, $\quad P'_3=P_3, \ I=P_3, \ P_{3i}=P'_{3i+3}$, and therefore $P_3=P_{3i+3}$;

and $\quad P_{3i-3}=P'_{3i+3}, \ P_6=P'_3, \ P_6=P'_3$ whence $P_3=P'_{3i-1}$;

$$P_6=P'_3, \ P'_3=P'_{3i+3}, \ P'_{3i+3}=P_{6i+6} \text{ and also} =P_{3i-3}, \ P_{3i-3}=P'_{6i-6}.$$

Thus in general, $\quad P_{3r+1}=P_{3i\pm(3r+1)}$; $P_{3r-1}=P_{3r\pm(3r-1)}$

and $\quad P_{3r}=P_{3i+3r}=P'_{3i-3r}$.

Thus the natural scale $\quad P_1P_2P_4P_8\ldots$

and the completed scale $\quad \begin{cases} P_1P_2P_3P_4P_5P_6\ldots \\ P'_1P'_2P'_3P'_4P'_5P'_6\ldots \end{cases}$

are each of them periodic, the period of the indices being $3i$. We may, accordingly, describe pertactile by the simpler name of periodic points. Every complete set of periodic points forms a closed system. By a complete set is to be understood the $9i^2$ sub-derivatives of the 9 points of inflexion, and by a closed system is to be understood one such that every connective and tangential of the points which it contains is itself a point of the system. According to what law such closed system may be resolved into partial closed systems must form the subject of further inquiry. When $i=2$, the complete closed system of 36 points we know is resoluble into nine closed systems, each containing one point of inflexion and its three collinear anti-tangentials, and also, in four different ways, into three closed systems, each containing a collinear set of inflexions and their three sets of anti-tangentials.

We are now in a position to solve the problem of in-and-exscribed k-laterals.

Suppose $k=3$, then $2^3+1=3i$ where $i=3$, and the point P_1 will coincide with the point P_8, provided P_3 is a point of inflexion. So that the apices of the in-and-exscribed triangles are the 81 points which satisfy the equation $P_3=P'_8$, of which 9 will correspond to the points of inflexion and 72 remaining over will give 24 finite triangles. If we denote by p, p', p'' three consecutive points in a straight line at any point of inflexion, pp', $p'p''$, $p''p$ form an infinitesimal triangle degenerating into a straight line, and this furnishes an improper solution of the question.

Calling M, N, $xyz\Omega$ the co-ordinates of P_3 when $P_1=x, y, z$, the 72 points are given by combining the equation $MN\Omega=0$ with the equation to the curve.

If $k=4$, we make $2^4-1=3i$ where $i=5$, and if $P_1=P_{3i-1}$, we have also $P_1=P_{3i+1}$; and the apices of the quadrilateral are found by making P_i, that is P_5, a point of inflexion.

The general form of P_5 being xU, yV, zW, the proper sub-derivatives P_5 result from $UVW=0$ combined with the equation to the cubic, and there result $\frac{9(25-1)}{4}$, that is 54 in-and-exscribed quadrilaterals.

Each point of inflexion may still be regarded as yielding an improper solution of the question, since pp', $p'p''$, $p''p'$, $p'p$ may be viewed as a degenerate infinitesimal quadrilateral.

So when $k=5$, making $2^5+1=3i$, $i=11$; and there will result $\frac{9(11^2-1)}{5}=216$ in-and-exscribed pentagons.

Likewise, since $\frac{2^7+1}{3}=43$, there result $9\frac{43^2-1}{7}$, that is 9.264 or 2376 in-and-exscribed heptagons.

Let us now consider a case of k a composite number, and to fix the ideas, suppose $k=15$. Make $\frac{2^{15}+1}{3}=i$, then $i=10923$. $\frac{2^{15}+1}{3}$, *by virtue of its form*, contains the factors $\frac{2^3+1}{3}$ and $\frac{2^5+1}{3}$, that is 3 and 11, and is in fact equal to $3.11.331$. P_i will therefore be of the form $xU_3U_{11}\mathbf{U}$, $yV_3V_{11}\mathbf{V}$ $zW_3W_{11}\mathbf{W}$ (xU_3, yV_3, zW_3 corresponding to P_3, and xU_{11}, yV_{11}, zW_{11} to P_{11}).

Accordingly $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ will each be of the degree $(3.11.331)^2-3^2-11^2+1$, and the equation $\mathbf{UVW}=0$, combined with the equation to the curve, will give the apices of the in-and-exscribed quindecagons, not including the improper solutions due to the points of inflexion, nor those due to the apices of the in-and-exscribed triangles or pentagons, which, in a certain but improper

sense, each belong to the case of quindecagons. The number of apices of the proper quindecagons will therefore be $9[(3.11.331)^2-3^2-11^2+1]$, comprising sub-inflexions of several grades, as follows: $9(331^2-1)$ of the 331th grade, $9(3^2-1)(11^2-1)$ of the 33rd grade, $9(3^2-1)(331^2-1)$ of the 993rd grade, $9(11^2-1)(331^2-1)$ of the 3641th grade, and $9(3^2-1)(11^2-1)(331^2-1)$ of the 10923rd grade*. The above number of apices may be written $9[11^2.3^2(331^2-1)+(3^2-1)(11^2-1)]$, so that the number of quindecagons is $9[11^2.3^2.22.332+8^2]$.

It may be noticed that the primitive algebraical factor of $2^{15}+1$, namely 331, is a prime number. But the primitive part of 2^k-1 (k being even) or 2^k+1 (k being odd), that is 2^k-1 or 2^k+1 stripped of its obligatory factors dependent algebraically on the prime factors of k, may be a composite number.

Thus, let us suppose $k=9$, the problem being that of finding the nature and number of the in-and-exscribed nonagons. Here $i=\frac{2^9+1}{3}=171$, 2^9+1 having, besides the obligatory factor 2^3+1 due to its algebraical form, the *two* factors 3 and 19.

Taking each divisor of 171, namely 3, 9, 19, 57, 171, we see that the 3rd, 9th, 19th, 57th, and 171th sub-derivatives of the nine points of inflexion will each of them be an apex of an in-and-exscribed nonagon. Of these, the 3rd sub-derivatives, and they only, give improper solutions of the problem, they being the apices of the in-and-exscribed triangles. Hence the aggregate of proper apices and the corresponding nonagons separate into four distinct groups, corresponding to the primitive sub-derivatives of the 9th, 19th, 57th, and 171th grades respectively, of the inflexions. The number of the nonagons belonging to the several groups will be the quadritotients of 9, 19, 57, 171, that is 9^2-9, 19^2-1, $(19^2-1)(3^2-1)$, $(9^2-9)(19^2-1)$ respectively, that is 171^2-9, exactly the same as if 57 had been a prime number N, in which case the $(3N)^2$ sub-derivatives of an inflexion of the grade $3N$ would be subject to the deduction of $9-1$ for in-and-exscribed triangles, and 1 for the point itself.

To make more clear the distinct solutions of which the problem of in-and-exscription of a k-lateral in general admits, consider the case of $k=8$. Here

$$i=\frac{2^8-1}{3}=\frac{2^4-1}{3}(2^4+1)=85.$$

The first factor (the one algebraically contained in i) is 5 and the primitive algebraical factor is 17. The total number of octagonal apices

* It is obvious that any derivative of an inflexion is itself an inflexion. For instance, if J is an inflexion, J_2 is the same as J, and J_3 (namely, J', J_2) is either J, J_2, that is J, or (I, J), J_2, that is, (I, J), J, that is, I (I being some other point of inflexion). Hence if $P_{\frac{i}{\delta}}$ is an inflexion, P_i is also an inflexion.

will be $9(85^2-5^2)$, the number 5^2 corresponding to the points of inflexion and the in-and-exscribed quadrilaterals. These 255^2-15^2 apices will consist of points of the form $I_{\frac{1}{17}}$ and $I_{\frac{1}{85}}$, the number of the former being $9(17^2-1)$ and of the latter $9(17^2-1)(15^2-1)$.

It is easily seen that, in general, the number of apices of in-and-exscribed k-laterals is nine times the *functional totient* of $\left(\frac{2^k-\bar{1}^k}{3}\right)^2$, or, what is the same thing the number of apices is the functional totient of $(2^k-\bar{1}^k)^2$, as previously stated in Note to Proem in the last number of the *Journal**; the number of k-laterals is, of course, the number of apices divided by k. For instance, we thus have for the number of apices of quindecagons, nonagons, and octagons, respectively,

$$(2^{15}+1)^2-(2^5+1)^2-(2^3+1)^2+(2^1+1)^2,$$
$$(2^9+1)^2-(2^3+1)^2,\ (2^8-1)^2-(2^4-1)^2,$$

as found above.

Since i is odd, every divisor of i will necessarily be so too. Conversely, it is easy to prove that every odd sub-derivative of a point of inflexion is an apex of an in-and-exscribed polygon, and to determine the number of its sides. For let i, any odd number, be given, and let k be the least number which will satisfy the condition that $2^k-\bar{1}^k$ shall be a multiple of $3i$, then the sub-inflexions of the ith grade will be the apices of an in-and-exscribed k-lateral. I give, in the annexed table, the values of k corresponding to a given value of i, which, of course, are unique; whereas to a given value of k, in general, several values of i will correspond.

i	3	5	7	9	11	13	15	17	19	21	23	25	27
k	3	4	6	9	5	12	12	8	9	6	22	20	27

to which may be subjoined the reciprocal table

$k=3$	$i=3$
$k=4$	$i=5$
$k=5$	$i=11$
$k=6$	$i=7, 21$
$k=7$	$i=43$
$k=8$	$i=17, 85$
$k=9$	$i=9, 19, 57, 171$
$k=10$	$i=31, 341$
$k=11$	$i=683$
$k=12$	$i=13, 15, 35, 39, 65, 91, 195, 273, 455, 1365.$

[* See p. 345 above.]

To illustrate the way in which this table is formed, take the case of $k=12$; then $\frac{2^{12}-1}{3}=3.5.7\times 13$ where 3 belongs to $k=3$, 5 to $k=4$, 7 to $k=6$; the values of i are found by taking the divisors of 1365, except those which are found set against $k=3$, $k=4$, $k=6$, that is 3, 5, 7, 21.

The successive tangentials of any even-graded inflexional sub-derivative as $2^q i$, where i is odd, will evidently consist of a chain of q points attached to the ring formed by the apices of an in-and-exscribed polygon of k sides, where k is the least number which makes $2^k \pm 1$ divisible by $3i$.

In all cases (since k is to have the minimum value which makes $\frac{2^k \pm 1}{3}$ contain i) $2k$ must be $\tau(3i)$ or a submultiple of it, so that, if $i=3^q j$, k is either $3^q \tau j$ or a submultiple of it; when $i=3^q$, since the cyclotomic functions of the first species $\chi_3 2, \chi_{3^2} 2, \dots \chi_{3^q} 2$ can only contain the first power of the intrinsic divisor 3, it follows that $k=3^q=i$, as is seen in the table to be the case for $i=3, 9, 27$; or, in other words, a 3^qth sub-derivative of a point of inflexion is an apex of an in-and-exscribed polygon of 3^q sides.

It may be as well to mention again here, by way of a remind, that the number of in-and-exscribed k-laterals whose apices are ith sub-derivatives of the inflexions, is always the kth part of nine times the quadritotient of i; when $i=3^q$ this number will be $\frac{1}{3^{q-2}}\{3^{3q}-3^{3q-2}\}$, that is $3^{q+2}-3^q$, being thus 24, 72, 216, etc., for triangles, nonagons, eikosiheptagons, etc.

Title 5.—*An Exact Proof of the Scalar Law of Squares**.

I will now give an exact proof of the law that the order in the variables of P_n is n^2 in regard to the co-ordinates of P, and furthermore that the co-ordinates when $i=3m\pm 1$ are of the form xU, yV, zW, and when $i=3m$ are of the form M, N, $xyz\Omega$; x, y, z being the co-ordinates of the primitive P_1 and U, V, W, M, N, Ω quantics in x^3, y^3, z^3. Of course the order of a point means the order of its system of co-ordinates *expressed in its lowest terms*, that is to say when the values of the three co-ordinates have no common measure, and consequently the co-ordinates of any *two* of them are relatively prime in an algebraical sense, as follows from the equation

$$X^3+Y^3+Z^3+kXYZ=0.$$

The law to be established comprises, it will be seen, two elements,—one numerical, the *rule of squares*; the other formal, containing two rules, one regarding the *distribution* of x, y, z between the co-ordinates, the other the quanticity of the parts not multiplied by x, y, z or xyz in respect to x^3, y^3, z^3.

Let us suppose that the law is true up to n inclusive. I shall show that it is true up to $2n$ inclusive.

[* See below, p. 385.]

(1) For the case of $2i$ where $i \overline{\gtrless} n$.

Let X, Y, Z be the system of co-ordinates to P_i in its lowest terms; then, by the law of compound derivation, P_{2i} is

$$X(Y^3 - Z^3),\ Y(Z^3 - X^3),\ Z(X^3 - Y^3).$$

If these regarded as functions of X, Y, Z had any common measure X, Y or X, $Z^3 - X^3$ would have a common measure. Hence X, Y, Z would all have a common measure. Nor can they have any common factor F, a function of x, y, z. For in that case, when $F = 0$, we should have

$$Y^3 - Z^3 = 0,\ Z^3 - X^3 = 0 \text{ or } X^3 = Y^3 = Z^3,$$

and the arbitrary parameter k would be $-3 . 1^{\frac{1}{3}}$, so that the cubic would become a triplet of straight lines, a supposition which falls outside the pale of the question.

Hence P_{2i} will be of four times the order of P_i, and therefore, by hypothesis, of the order $4i^2$, that is, $(2i)^2$. Also, obviously, the form xU, yV, zW or M, N, $xyz\Omega$ (as the case may be) which exists for i is maintained for $2i$, which is or is not divisible by 3 according as i is or is not so divisible.

(2) Let the index be any odd number less than $2n$.

I shall first establish a Lemma concerning the co-ordinates given by my formulæ for the connectives of P, Q and P', Q, where P' is the opposite to P in respect to a given point of inflexion (say $x = 1$, $y = -1$), and

$$x^3 + y^3 + z^3 + kxyz = 0$$

is the equation to the cubic.

The connectives of $\begin{matrix}(u, v, w)\\(u', v', w')\end{matrix}$ and of $\begin{matrix}(v, u, w)\\(u', v', w')\end{matrix}$

are represented respectively by

$$\left.\begin{matrix} vwu'^2 - v'w'u^2 \\ wuv'^2 - w'u'v^2 \\ uvw'^2 - u'v'w^2 \end{matrix}\right\} \text{ and } \left\{\begin{matrix} uwu'^2 - v'w'v^2 \\ wvv'^2 - w'u'u^2 \\ vuw'^2 - u'v'w^2, \end{matrix}\right.$$

the 3rd co-ordinate being the same in both systems, which, of course, remain to be reduced to their simplest terms, being at present each of the order $2i^2 + 2j^2$.

I say that the same quantity F cannot divide each of the two sets of quantities when u, v, w; u', v', w' are derivatives, one of an even, the other of an odd grade of the same point on the cubic.

For, if so, let $F = 0$; then each quantity in the two systems becomes zero.

Call $\dfrac{u}{w}, \dfrac{v}{w}$; $\dfrac{u'}{w'}, \dfrac{v'}{w'}$, r, s; r', s' respectively.

Then
$$(1)\ldots\ldots sr'^2 - s'r^2 = 0 \qquad rr'^2 - s's^2 = 0\ldots\ldots(3)$$
$$(2)\ldots\ldots rs'^2 - r's^2 = 0 \qquad r'r^2 - ss'^2 = 0\ldots\ldots(4)$$
$$(5)\ldots\ldots rs = r's'.$$

Writing $r^3 = R$, $s^3 = S$, $r'^3 = R'$, $s'^3 = S'$; 5, (3, 4), (1, 2) respectively give $RS = R'S'$, $RR' = SS'$, $R'S = RS'$. The second and third of these combined give $R^2 = S^2$, $R'^2 = S'^2$ and the first and second combined give $R'^2 = S^2$. Hence, $R^2 = R'^2 = S^2 = S'^2$, and consequently the original equations (1), (2), (3) give $S = S'$, $R = R'$, $R = S'$ or $r^3 = s^3 = r'^3 = s'^3$.

Let $r = \alpha s$, $r' = \beta s'$, $s = \gamma s'$. Then $\alpha^3 = \beta^3 = \gamma^3 = 1$, and all the equations (1), (2), (3), (4), (5) will easily be found to be satisfied when (and only when) $\alpha = \beta\gamma$.

The equations $r^3 = s^3$, $r'^3 = s'^3$, that is, $u^3 = v^3$, $u'^3 = v'^3$, imply that the points P, Q are two either distinct or identical anti-tangentials to the same point of inflexion $x = 1$, $y = -1$. I say that this is impossible when P, Q are derivatives of the degrees i, j of the same point U on the curve, if $i + j$ is an odd number. It must be noticed that P and Q (two Plückerian points belonging to the same point of inflexion I) are identical with P' and Q' respectively.

Any even-degreed derivative of P or Q is I, and any odd-degreed derivative is the same point P or Q over again.

Let now $i\mu - j\nu = 1$. Then $U = U_{i\mu - j\nu}$ will be (without regard to the modulus 3) the connective of $U_{i\mu}$ and $U_{j\nu}$, because we may substitute at will U_i' for U_i and U_j' for U_j. But $U_{i\mu}$ and $U_{j\nu}$, if μ, ν be both odd, will be U_i and U_j over again, or if μ, ν be one odd and the other even, will be I and one of the two Plückerian points.

Hence U is the connective of I and a Plückerian point, or else of two Plückerians which are identical, or of two Plückerians (both appurtenant to I) which are distinct.

In the 1st and 3rd cases, then, U is a Plückerian, in the 2nd case a point of inflexion. But every derivative of a point of inflexion is a point of inflexion, and every even-degreed derivative of a Plückerian is also a point of inflexion; but by hypothesis (since one of the two numbers i, j is even) an even-degreed derivative of U is a Plückerian, which is self-contradictory. Hence, it follows that the expressions given by *my* formulæ for the connectives of P_i, P_j and P_i', P_j when $i + j$ is odd, say P, Q, R; P', Q', R', cannot have a common factor; so that if M is a common measure of P, Q, R and M' of P', Q', R', M is relatively prime to M'.

Let ϕ, ψ, ω be always understood to mean $\phi(x^3, y^3, z^3)$, $\psi(x^3, y^3, z^3)$, $\omega(x^3, y^3, z^3)$; let (μ), (ν) be understood to mean the prime systems of co-ordinates u, v, w; u', v', w' which represent μ, ν (μ and ν being numbers, accented or unaccented, representing derivatives to the indices μ and ν); let $[\mu, \nu]$ represent the unreduced system of the co-ordinates of the connective of μ, ν, namely, $v'w'u^2 - vwu'^2$, $w'u'v^2 - wuv'^2$, $u'v'w^2 - uvw'^2$; (μ, ν) the above system *reduced* by elimination of the greatest common measure of its terms.

If (μ), (ν) are each of the form $x\phi$, $y\psi$, $z\omega$, $[\mu, \nu]$ is of the form $x^2yz\phi_1$, $xy^2z\psi_1$, $xyz^2\omega_1$, but $[\mu', \nu]$, that is, the unreduced connective of $y\psi$, $x\phi$, $z\omega$; $x\phi'$, $y\psi'$, $z\omega'$, is of the form $z\phi_1$, $z\psi_1$, $xyz^2\omega_1$.

Again, if (μ) is of the form $x\phi$, $y\psi$, $z\omega$ and (ν) of the form ϕ_1, ψ_1, $xyz\omega_1$, $[\mu', \nu]$, the unreduced connective of the systems $y\psi$, $x\phi$, $z\omega$ and ϕ_1, ψ_1, $xyz\omega_1$, is easily seen to be of the form $zx\Phi$, $zy\Psi$, $z^2\Omega$.

Furthermore, the order in the variables of (p') is obviously the same as that of (p).

Now it has been shown under Title 2 that

$$6i - 1 = (3i - 1)', \; 3i \qquad 6i - 5 = (3i - 3)', \; (3i - 2)' \qquad 6i - 3 = (3i - 2)', \; 3i - 1.$$

If, then, $(3i)$ and $(3i - 3)$* are of the form ϕ, ψ, $xyz\omega$, and $(3i - 2)$, $(3i - 1)$ each of the form $x\phi$, $y\psi$, $z\omega$, it follows that $[6i - 1]$ and $[6i - 5]$ will be of the form $zx\phi$, $zy\psi$, $z^2\omega$, and $[6i - 3]$ of the form $z\phi$, $z\psi$, $xyz^2\omega$.

The above inference suffices to show that, if, for all values of $3\mu \pm 1$ and 3μ up to n inclusive, it be true that $(3\mu \pm 1)$ is of the form $x\phi$, $y\psi$, $z\omega$ and of the order $(3\mu \pm 1)^2$, and (3μ) is of the form ϕ, ψ, $xyz\omega$ and of the order $(3\mu)^2$; then the same will be true up to $2n$ inclusive.

That this is true for even values not exceeding $2n$ appears from what has been already shown. Confining, then, our attention to odd numbers less than $2n$; these must be representable by $6i - 5$, $6i - 3$ or $6i - 1$, and by hypothesis the form of each of the systems $(3i)$, $(3i - 1)$, $(3i - 2)$, $(3i - 3)$ fulfils the conditions of the last paragraph but one; consequently the form of $[6i - 5]$, $[6i - 3]$, $[6i - 1]$ will be $zx\phi$, $zy\psi$, $z^2\omega$; $z\phi$, $z\psi$, $xyz^2\omega$; $zx\phi$, $zy\psi$, $z^2\omega$, namely, in every case the factor z will be contained in each term of the system $[(\iota - 1)', \iota^?]$, which represents an unreduced system of co-ordinates of the point $2\iota - 1$, the mark of interrogation signifying a blank or an accent as the case may be.

But either the point 1 or the point 1′ will, in every case, correspond to the connective obtained by changing $(\iota - 1)'$ into $\iota - 1$†; moreover, the unreduced system of co-ordinates to that connective will have the third term, say π, in common with the unreduced system to $2\iota - 1$ above mentioned.

This contrary system we know must have the common factor $\frac{\pi}{z}$ because 1

* $(3i - 3)'$ will obviously be of the same form as $3i - 3$.

† For, on consulting Title 2, it will be found that *in every case*, if the arithmetical value of the index of P_i, P_j is $i \pm j$, that of P_i', P_j is $(\iota \mp j)^2$.

and 1′ are denoted by x, y, z; y, x, z respectively. Hence the unreduced system for $2\iota - 1$ can have no other common factor except z, which they have been shown to have; since, were it otherwise, the *two* contrary systems would have some quantity contained in $\frac{\pi}{z}$ for a joint common measure, which has been proved to be impossible.

Hence, the form of $(2\iota - 1)$ is $x\phi$, $y\psi$, $z\omega$ or ϕ, ψ, $xyz\omega$ according as $2\iota - 1$ is not or is divisible by 3, and its order is in all cases $2(\iota - 1)^2 + 2\iota^2 - 1$, that is, $(2\iota - 1)^2$.

Hence the form-law of distribution of the simple powers of the variables x, y, z and of the quanticity in x^3, y^3, z^3 of the multipliers of x, y, z or of 1, 1, xyz, as well as the numerical law that the order of any derivative is the square of its index, will be true up to $2n$ inclusive if true up to n inclusive; and being true for $n = 1$, is true universally.

As a corollary we may now do away with the restriction of $i + j$ being odd, and affirm that in all cases (the futile one of $i = j$ alone excepted), if the reduced system of co-ordinates to the connective of P_i, P_j be F, G, H, and to that of P_i', P_j be F', G', H', then the unreduced system expressing those connectives given by my formulæ of connection will be $H'F$, $H'G$, $H'H$; HF', HG', HH', respectively; for the two systems of unreduced co-ordinates (each of the order $2i^2 + 2j^2$) contain, one of them a common factor of the order $(2i^2 + 2j^2) - (i - j)^2$, that is, $(i + j)^2$, the other a common factor of the order $(2i^2 + 2j^2) - (i + j)^2$, that is, $(i - j)^2$, and these two factors being prime to each other, their product must be contained in the term common to the two systems, and being of the same order $(i + j)^2 + (i - j)^2$ as that common term, must be equal to it.

Hence, if π be the common unreduced term, and H, H' the two reduced terms, we must have $\pi = \frac{\pi}{H} \cdot \frac{\pi}{H'}$ or $\pi = HH'$, as was to be shown.

As a matter rather of curiosity than of real importance I will state the analogous law when the connective and cross-connective between two derivatives is expressed by Cauchy's formulæ instead of my own. These formulæ, it will be remembered, give for the co-ordinates of the connective of u, v, w; u_1, v_1, w_1 the minor determinants of the matrix

$$\begin{vmatrix} vw_1 - v_1w \; ; & w_1u - wu_1 \; ; & uv_1 - u_1v \\ uu_1 \quad ; & vv_1 \quad ; & ww_1 \end{vmatrix}$$

If, now, the prime system of co-ordinates to the connectives of P_i, P_j; P_i', P_j be denoted as before by F, G, H; F', G', H', I find by calculation that the Cauchian formulæ will present these two systems under the unreduced forms

$$(F' + G')F, \; (F' + G')G, \; (F' + G')H$$
$$(F + G)F', \; (F + G)G', \; (F + G)H',$$

between which there is no common term; and consequently, had I not discovered my own simpler formulæ, the method of proof of the Law of Squares which I have employed would have been inapplicable, and it is not easy to see what other strict method of proof could have taken its place.

I have thus accomplished the very difficult task of proving a negative, in this instance the non-existence of *latent* common factors to the co-ordinates of the connective of any two given derivatives. I might have founded a much easier proof of the Law of Squares upon Mr Franklin's geometrical solution of the problem of finding the number of in-and-exscribed k-laterals to a cubic (if one could feel quite assured *à priori* of the strict logic of the process*) as follows: He has virtually found (*vide* last number of the *Journal*) that the number of apices of the in-and-exscribed k-laterals of *every kind* [and not excluding the points of inflexion] is $(2^k - \bar{1}^k)^2$. If, then, $2^k - \bar{1}^k = 3i$, it follows from what has been shown in the preceding pages, that the order of P_i in the co-ordinates of P is $\frac{1}{9}(3i)^2$, that is, i^2.

Let now i' be any number whatever, and τ the totient of $3i'$; then τ is even, and, by Fermat's Theorem, $2^\tau - \bar{1}^\tau = 3i'i''$.

Hence, if μ', μ'' are the orders of $P_{i'}, P_{i''}$ respectively, the law of compound derivation will suffice to lead to the conclusion that $\mu'\mu''$ will be the order of $P_{i'i''}$, and accordingly $\mu'\mu'' = i'^2 i''^2$; but $\frac{\mu'}{i'^2}, \frac{\mu''}{i''^2}$, it has been proved under a preceding Title, are neither of them greater than unity: hence each of them is equal to unity, and i'^2 is the order of $P_{i'}$, as was to be shown.

Addendum on the Degorder of the Derivatives to a Point on a Cubic in the Natural Scale.

Let n be any number not divisible by 3. The nth derivative, it has been proved, is of the order n^2 in the variables. It remains to determine its *degree in the coefficients*.

When $n = 2$ we know that the degorder is $[4; 4]$, each new co-ordinate being one of the minors of the rectangular matrix

$$\begin{vmatrix} \frac{dU}{dx} & \frac{dU}{dy} & \frac{dU}{dz} \\ \frac{dH}{dx} & \frac{dH}{dy} & \frac{dH}{dz} \end{vmatrix},$$

where U is the cubic and H its Hessian.

* In that solution the apices are found as the intersections of the cubic with another curve. Certain of these intersections are seen from geometrical considerations to count twice, and others three times; but while we have no reason to suppose any further cause of reduction, the non-existence of such cause is not proved.—F. F.

Suppose ν to be the degree in the coefficients of the nth derivative. Then the degree of the $(2n)$th derivative regarded as the second of the nth will be $4\nu + 4$, and regarded as the nth of the second will be $n^2 . 4 + \nu$, and these two must be equal. Hence $3\nu = (n^2 - 1) 4$ or $\nu = \frac{4}{3}(n^2 - 1)$.

Hence the degorder of any nth derivative in the natural scale is $\left[\frac{4n^2 - 4}{3}; n^2\right]$. If we substitute the co-ordinates of this derivative in the given cubic U, the result must be of the form $U.R$ and will be of the degorder $[1 + 4n^2 - 4; 3n^2]$. Hence R is of the degorder $[4n^2 - 4; 3n^2 - 3]$. If the well-known covariant of the degorder $[12; 9]$ be called J, R is of the same degorder as $J^{\frac{n^2-1}{3}}$, and possibly may be found to be identical with it. To corroborate the validity of the determination of the degorder of the nth derivative, we may proceed as follows:

Imagine, at first, the cubic to be reduced to the canonical form $x^3 + y^3 + z^3 - 3kxyz$. The connective of P_1, P_2 in its reduced form is x, y, z; but in its unreduced form and prior to all simplification, will, by virtue of the theory (Titles 1 and 5), be of the form Mx, My, Mz where

$$M = x^3y^6 + y^3z^6 + z^3x^6 + x^6y^3 + y^6z^3 + z^6x^3 - 6x^3y^3z^3$$
$$+ kxyz(x^6 + y^6 + z^6 - y^3z^3 - z^3x^3 - x^3y^3)^*;$$

consequently M expressed (as I shall hereafter suppose) in terms of the original coefficients and variables, will be of the degorder $[9; 9]$: for Mx, My, Mz are of the degorder $[1 + 2.4; 2(1 + 4)]$, that is, $[9; 10]$†. Also the degorder of P_4 will be $[4 + 4.4; 16]$, that is, $[20; 16]$.

Suppose now we wish to find the degorder of P_5.

The unreduced connective of P_1, P_4 will be of the form MX, MY, MZ, where X, Y, Z are the reduced co-ordinates and M is exactly the same thing as before. The degorder of the unreduced co-ordinates will be $[1 + 2.20; 2(1 + 16)]$, that is, $[41; 34]$; and consequently, subtracting $[9; 9]$, the degorder of X, Y, Z will be $[32; 25]$, that is, $\left[4\frac{5^2 - 1}{3}; 5^2\right]$.

So, again, to find P_7 we may regard it as the connective of P_2, P_5. The unreduced degorder of P_7 will thus be seen to be $[1 + 2(4 + 32); 2(4 + 25)]$, that is, $[73; 58]$, and subtracting, as before, $[9; 9]$, the degorder of the

* It is worthy of remark that, if we make $U = 0$, so that $3kxyz$ becomes equal to $x^3 + y^3 + z^3$, the expression in the text for M gives $3M$ equal to the norm of $x + 1^{\frac{1}{3}}y + 1^{\frac{1}{3}}z$, namely,

$$(x^3 + y^3 + z^3)^3 - 27x^3y^3z^3.$$

† In fact, M, as may easily be shown, is the covariant $\left[\Sigma\left(\frac{dU}{dy}\cdot\frac{dH}{dz} - \frac{dU}{dz}\cdot\frac{dH}{dy}\right)\frac{d}{dx}\right]^2 U$, in other words the symmetrical determinant of the 5th order formed by double-bordering the Hessian matrix with the differential derivatives of the Hessian and of the original cubic.

reduced co-ordinates of P_7 becomes [64; 49], that is, $\left[4\dfrac{7^2-1}{3};\ 7^2\right]$, agreeable to what has been previously found; and so, in general, supposing the degrees of P_μ and $P_{\mu+3}$ in the coefficients to be $4\dfrac{\mu^2-1}{3}$ and $4\dfrac{(\mu+3)^2-1}{3}$, the unreduced degree of $P_{2\mu+3}$ will be $1+8\left\{\dfrac{\mu^2-1}{3}+\dfrac{(\mu+3)^2-1}{3}\right\}$, from which subtracting 9, the reduced degree becomes $8\left\{\dfrac{2\mu^2+6\mu+4}{3}\right\}$, which is the same thing as $4\left\{\dfrac{(2\mu+3)^2-1}{3}\right\}$, as ought to be the case. There is, therefore, no loophole for doubt left open as regards the degorder of any *natural* derivative to the index k (a number necessarily of the form $3i \pm 1$) being $[\frac{4}{3}(k^2-1);\ k^2]$, a notable result!

We are now in possession of a method for finding any natural derivative to the index n. If n is even, it may be derived immediately from the derivative to the index $\frac{n}{2}$. If n is odd, it must be of the form $2\mu+3$ where μ is not divisible by 3.

Taking P as the initial point, P_μ and $P_{\mu+3}$ may be considered as known. Calling their co-ordinates X, Y, Z; X_1, Y_1, Z_1 respectively, and substituting $\lambda X + \mu X_1$, $\lambda Y + \mu Y_1$, $\lambda Z + \mu Z_1$ in the equation to the cubic, we shall obtain an equation of the form $\lambda^2\mu B + \lambda\mu^2 C = 0$. The unreduced co-ordinates of $P_{2\mu+3}$ will then be $CX - BX_1$, $CY - BY_1$, $CZ - BZ_1$, which will contain a common measure M of the degorder [9; 9], and $\dfrac{CX-BX_1}{M}$, $\dfrac{CY-BY_1}{M}$, $\dfrac{CZ-BZ_1}{M}$ will be the expression for the point $P_{2\mu+3}$ in its simplest terms.

More generally, if $n = 2\mu + 3i$, we may obtain, in like manner as above, the unreduced co-ordinates of the connective to P_μ, $P_{\mu+3i}$, and, by an easy calculation, it will be found that the new common measure will be of the degorder $[12i^2 - 3;\ 9i^2]$, and will be constant, that is, independent of μ for any given value of i, and identical with the common measure to the unreduced co-ordinates of P_{3i+2} regarded as the connective to P and P_{3i+1}.

It is well worthy of remark that if X, Y, Z be the co-ordinates of any derivative, and ξ, η, ζ contragredient to x, y, z, $X\xi + Y\eta + Z\zeta$ will be an invariantive concomitant to the given cubic. This gives rise to a new series of reflexions, the development of which must be deferred to a more convenient occasion*.

* It is obviously a step towards the attainment of the desideratum of finding the general expression for any derivative in an explicit form, or, at all events, by explicit processes and without the necessity for division of the unreduced co-ordinates by a common measure. This latter, it should be observed however, by virtue of what is stated above, is always known *à priori*.

CHAPTER I.

Excursus C.—On the Trisection and Quartisection of the Roots of Unity to a Prime-Number Index.

What follows, so far as it relates to the trisection of the primitive roots of unity, may be regarded as auxiliary to Postscriptum 2, [p. 345, above], inasmuch as it establishes the equation in ω which, when $x = \dfrac{\omega - 1}{3}$, becomes the equation there assumed. The rest is episodical, except so far as it may be regarded as correlative to the subject matter of Titles 1 and 2 of Excursus A* [pp. 317 ff.].

It will be seen that the equations to a system of three and four periods, usually obtained by long and tedious processes, may, with the aid of one simple and well-known principle, be deduced by processes almost elementary in their character, and into which enter no algebraical calculations except of the very easiest kind.

A sketch of the method was laid by me before the Scientific Congress held at Rheims in the month of August last [p. 438, below].

The index p of the roots is, as usual, supposed to be a prime number; e is the number of the periods, f the number of roots whose sum forms a period, so that $ef = p - 1$; the periods themselves will be called η, namely, $\eta_1, \eta_2, \ldots \eta_e$.

Preliminaries.

1. I say, in the first place, that the sum of the ith powers of the periods will be congruous to $-f^{i-1}$ in respect to the modulus p.

For, were it not that in the development of the ith power of any one of the η's some of the combinations of the powers of the roots were unity, it is obvious that we should have $\Sigma\eta^i = -ef^i \div (p-1)$, that is, $-f^{i-1}$, and that we might regard every term in such development as equivalent to $-\dfrac{1}{p-1}$, without affecting this result. The existence of terms equal to unity will render it necessary to substitute for any such term 1 instead of $-\dfrac{1}{p-1}$, in order to obtain a correct result, and if there be N of them, the correction to be introduced will be $N\left(1 + \dfrac{1}{p-1}\right)$, that is, $\dfrac{N}{p-1}\cdot p$; but as it is obvious that the result must be an integer, it follows that N must be double by

* In any future redistribution of the contents of the entire memoir, it would be proper to incorporate the matter contained in Postscriptum 2, pp. [345—347], with this Excursus.

$(p-1)$, and consequently the value of $\Sigma\eta^i$ *to modulus* p will be $-f^{i-1}$, that is, $-\left(\frac{p-1}{e}\right)^{i-1}$, as was to be shown.

2. From the above it follows that *to modulus* p,

$$\Sigma(e\eta+1)^i \equiv (-1)^i + e(-1)^{i-1} + e\frac{e-1}{2}(-1)^{i-2} + \text{etc.}, \equiv (-1+1)^e \equiv 0,$$

or, in other words, $\Sigma(e\eta+1)^i$ is divisible by p.

But, if s_i and σ_i represent, respectively, the sum of the $i$ary combinations and $i$ary powers of the roots of an equation, we know that $(-)^i s_i =$ coefficient of x^i in $e^{-\sigma_1 x - \frac{\sigma_2}{2}x^2 - \frac{\sigma_3}{3}x^3 \cdots}$, so that s_i multiplied by numbers none exceeding i, is expressible as the sum of integer multiples of $\sigma_\lambda \sigma_\mu \sigma_\nu \ldots$ where

$$\lambda + \mu + \nu + \ldots = i.$$

3. Consequently, s_i multiplied by integers none greater than i, when the roots in question are the e values of $e\eta+1$ and $i>0$, will be divisible by p, and consequently, since e is less than p, all the coefficients of the equation to which those roots appertain will be divisible by p, the first, of course (which is unity), excepted.

Since $\Sigma(e\eta+1) = e\Sigma\eta + e = 0$, the equation whose roots are $\omega_1, \omega_2, \ldots \omega_e$ where $\omega = e\eta+1$ will be of the form $\omega^e + P\omega^{e-2} + Q\omega^{e-3} + \text{etc.}$, where P, Q, etc., each contain p; and I may remark, incidentally (although the fact is immaterial to the object in view), that, as may easily be seen, $\Sigma\omega^i$ will be divisible not only by p but also by e, and that consequently the coefficient of ω^{e-i}, in the above equation, will contain the greatest common divisor to e and i.

4. The coefficient P has one or the other of two determinate algebraical values according as f, that is, $\frac{p-1}{e}$, is even or odd.

In the former case, the congruence $x^f + 1 \equiv 0$ [mod p] is soluble, and in the latter, insoluble. Accordingly, in the latter case, we shall have $\Sigma\eta^2 = -f$, and in the former $\Sigma\eta^2 = p - f$, and in each case $\Sigma\eta^2$ will be an odd number. Also, when f is odd (which involves the necessity of e being even)

$$\Sigma\omega^2 = \Sigma(e\eta+1)^2 = -e^2\frac{p-1}{e} - 2e + e = -ep,$$

and when f is even $\Sigma\omega^2$ will be this result augmented by e^2p, that is, $(e^2-e)p$.

Consequently, $P = \frac{e}{2}p$, or $= -\frac{e^2-e}{2}p$, according as f is odd or even.

Thus, when $e=3$, f being necessarily even, $P=-3p$, and when $e=4$, $P=-6p$, or $=2p$, according as $\frac{p-1}{4}$ is even or odd*.

5. With regard to what immediately follows it will also be necessary to determine the *form* of Q in respect to certain moduli for the cases of e equal to 3 and e equal to 4. In the former case

$$\Sigma\omega^3 = \Sigma(e\eta+1)^3 = \Sigma(e^3\eta^3 + 3e^2\eta^2 + 3e\eta + 1) \equiv 3 \bmod 9,$$

and consequently, since $Q = -\frac{1}{3}\Sigma\omega^3$, $-3Q \equiv 3 \bmod 9$ and $-Q \equiv 1 \bmod 3$.

In the latter case, that is, when $e=4$, since $\Sigma\eta^2$ is always odd $\Sigma\omega^3$ [to mod 32] $\equiv 16 - 12 + 4$, that is, $\equiv 8$, and, consequently, $-3Q \equiv 8$ to that modulus.

These *preliminaries* being established, I will now proceed to state the principle referred to in the exordium.

Principle.

A rational integer function of any set of periods of the roots of unity whose coefficients are all whole numbers, which does not change its value for a circular substitution executed upon the periods, it is well-known, must be an integer number; but to this I add that if such function, without changing its arithmetical value, undergoes a change of sign when such a substitution is made, it must necessarily be an integer number multiplied by the difference of the two periods into which the entire sum of the roots may be divided, that is to say, will be a multiple of $\sqrt{p}$, when p is of the form $4K+1$ and of $\sqrt{(-p)}$, when p is of the form $4K-1$†.

As an example, the product of the differences of the roots of the equation in η will be an integer number when e, the number of the periods, is odd, and an integer number multiple of $\sqrt{p}$ or $\sqrt{(-p)}$ (according as $\frac{p-1}{2}$ is even or odd), when the number of periods is even. As another example, if $e = 2\epsilon$, the function

$$(\eta_0 - \eta_\epsilon)(\eta_1 - \eta_{\epsilon+1})(\eta_2 - \eta_{\epsilon+2}) \cdots (\eta_{\epsilon-1} - \eta_{2\epsilon-1}),$$

which changes its sign but not its quantitative value, when $0, 1, 2, 3, \ldots (2\epsilon-1)$ are replaced by $1, 2, 3, \ldots -1, 0$ will be an integer multiple of $\sqrt{p}$, or of $\sqrt{(-p)}$, according as ϵ is even or odd.

* When $e=2$, $P=p$ or $-p$ according as f is odd or even, so that the equation in ω takes the known form $\omega^2 \pm p = 0$.

† To put the matter more clearly, call the alternating function F and the difference spoken of Δ. Then ΔF is invariable in sign as well as in magnitude for the circular substitutions in question. Hence $F = \frac{\text{An Integer}}{\sqrt{(\pm p)}}$ but F^2 is an Integer; therefore $F = \text{An Integer}\sqrt{(\pm p)}$. Q.E.D.

We are now in a position to obtain without difficulty the well-known equivalent to the equation corresponding to $e = 3$, given at p. [345], and the corresponding pair of equations for the case of $e = 4$.

A. *Case of* $e = 3$.

The equation in ω, from what has been shown in the preliminaries, must be of the form $\omega^3 - 3px + pq = 0$, and it only remains to determine q.

The discriminant of the above equation being $q^2p^2 - 4p^3$, it follows that the product of the differences of its roots will be $27(4p^3 - q^2p^2)$. But this product is 3^6 into $(\eta_0 - \eta_1)^2 (\eta_0 - \eta_2)^2 (\eta_1 - \eta_2)^2$, which latter, by the *principle*, is of the form M^2. We have, therefore,

$$4p^3 - q^2p^2 = 27M^2 = 27m^2p^2.$$

Hence,

$$4p = q^2 + 27m^2,$$

which serves to determine the value of q^2 absolutely.

To find the value of q, it follows from the preliminaries that $qp \equiv -1 \bmod 3$, and, consequently, since $p \equiv 1 \bmod 3$, $q \equiv -1 \bmod 3$, so that q is perfectly determined.

B. *Case of* $e = 4$.

$\omega^2 - 2\sqrt{p}\omega + R = 0$, $\omega^2 + 2\sqrt{p}\omega + R' = 0$, will be the form of the equations containing, respectively, the pairs of roots ω_0, ω_2 and ω_1, ω_3; for

$$\omega_0 + \omega_2 = (4\eta_0 + 1) + (4\eta_2 + 1) = 2\{2(\eta_0 + \eta_2) + 1\} = 2(2\delta_0 + 1),$$

and, similarly,

$$\omega_1 + \omega_3 = 2\{2(\eta_1 + \eta_3) + 1\} = 2(2\delta_1 + 1)$$

where δ_0 and δ_1 are the two periods which make up together the sum of all the roots, so that $2\delta_0 + 1$ and $2\delta_1 + 1$ are the roots of the equation $\Omega^2 - p = 0$, the sign of the last term being fixed from the fact of $\frac{p-1}{2}$ being by hypothesis even.

Furthermore, R, R' must be of the form $Ap + B\sqrt{p}$, $Ap - B\sqrt{p}$; for $(R - R')\sqrt{p}$, being integer, requires that R, R' shall be of the form $A_1 + B\sqrt{p}$, $A_1 - B\sqrt{p}$, and then RR' being an integer multiple of p involves the necessity of A_1^2, and therefore of A_1 containing p.

The product $(\eta_0 - \eta_2)(\eta_1 - \eta_3)$ consequently becomes

$$\{(A-1)p + B\sqrt{p}\}\{(A-1)p - B\sqrt{p}\},$$

which by the principle must be of the form m^2p, and consequently,

$$(A-1)^2 p - B^2 = C^2 \text{ or } (A-1)^2 p = B^2 + C^2.$$

The coefficient of ω^2 becomes $-4p + 2Ap$ which, by the preliminaries, when $\frac{p-1}{4}$ is even must be equal to $-6p$, so that $A = -1$, and when $\frac{p-1}{4}$ is odd must be equal to $2p$, so that $A = 3$.

In each case, therefore, $(A-1)^2 = 4$ and $4p = B^2 + C^2$; consequently, if $p = g^2 + h^2$, $4g^2 = B^2$, and $4h^2 = C$, and the complete equation in ω containing the roots $\omega_0, \omega_1, \omega_2, \omega_3$, becomes $(\omega^2 - p)^2 - 4p(\omega + g)^2 = 0$ when $\frac{p-1}{4}$ is even and $(\omega^2 + 3p)^2 - 4p(\omega + g)^2 = 0$ when $\frac{p-1}{4}$ is odd. In either case g^2 is given, but the sign of g requires to be determined; alike, however, for one case as for the other, $-8pg$ being the 3rd coefficient after the first, we must have, as shown in the preliminaries, $24pg \equiv 8$ [mod 32], and consequently, since p is of the form $4K + 1$, $24g \equiv 8$ [mod 32]. Hence, $3g \equiv 1$ [mod 4], that is, $g \equiv -1$ [mod 4], which gives the required complete determination of g.

The quartisecting equations thus naturally arrived at are expressed in the form in which, according to Bachmann (*Kreistheilung*, p. 230), they were first presented by Lebesgue; the method here given for finding the equations for the trisection and quartisection of the roots of unity will be found on examination to be incomparably simpler, shorter, and more direct than any in common use, and as removing a serious stumbling-block from the path of the student, and, occurring, so far as regards trisection, in the natural course of the development of my subject, I have thought entitled to a place in this memoir. Why I require the trisecting equation is, as will be remembered, to enable me to obtain the conditions of 2 and of 3 being cubic residues to a given index. The condition for 2 being such, strange to say, is nowhere to be found in Bachmann's *Kreistheilung*, although the cubic character of 3 is there duly and fully made out.

The conditions of the one and of the other being cubic residues were, I am informed by M. Lucas, given for the first time in a letter from Gauss to Mlle. Sophie Germain.

EXCURSUS B.

Title 5 (bis).—On the Law of Squares.

There being errors and inaccuracies not a few in the matter printed under this title, owing to my absence abroad as it went through the press, I have thought it desirable to rewrite it, rectifying the errors, and supplying some steps which were wanting in the demonstrations*. I shall, in what follows,

* In the postscript [p. 378 above] which was thought out on board the transatlantic steamer, the *Bothnia*, and written out, as far as I can recollect, at a single sitting a day or two before

use throughout P_i to denote the ith derivative of P, and x_i, y_i, z_i to signify the reduced coordinates of P_i, so that P_1, x_1, y_1, z_1 will mean the same as P, x, y, z respectively. $x+y=0, z=0$ will be taken as the auxiliary point of inflexion, serving to complete the scale, and will be called I. In the natural scale it is easy to see that any derived co-ordinate, as z_i, must contain

posting it at Queenstown, I have not been able to detect any inaccuracy in the results, although some additional steps and explanations might advantageously have been supplied.

There is, perhaps, one slight exception to be made to this statement as regards the very important theorem, stated but not proved [p. 380], concerning the nature of the form $X\xi+Y\eta+Z\zeta$, where the coefficients of ξ, η, ζ are supposed to be the reduced co-ordinates of any derivative to x, y, z. If $U=0$ is the equation to the cubic in its general form, obviously X, Y, Z are indeterminate, as each may be augmented by an arbitrary multiple of U of suitable degree and order. Consequently, the theorem ought to have been stated in the following form. The co-ordinates X, Y, Z of any such derivative *may be* so expressed that $X\xi+Y\eta+Z\zeta$ shall be a mixed concomitant to U. The fundamental invariantive concomitants to a ternary cubic involving not more than one system of cogredients and a single linear system of contragredients are eleven in number and of the types underwritten:

4 . 0 . 0	4 . 4 . 1
6 . 0 . 0	5 . 4 . 1
1 . 3 . 0	7 . 4 . 1
3 . 3 . 0	9 . 7 . 1
8 . 6 . 0	11 . 7 . 1
12 . 9 . 0	

Hence the co-ordinates of every rational derivative in the natural scale to a point on a cubic curve may be expressed as the coefficients of the contragredient variables in a rational integer function of the above eleven quantities, linear in the latter five, and such that its degree and orders for the nth grade are $\frac{4(n^2-1)}{3}$; n^2, 1.

The particular forms of X, Y, Z which appertain to the concomitant $X\xi+Y\eta+Z\zeta$, and which may be called the *normal* forms, it may be added, are those which actually arise from the processes of *colligation* and *reduction* described in the excursus. By *colligation* I mean the determination of the *general* analytical connective of x, y, z; x', y', z' by the same method as that applied at pages [354, 355] to the canonical quadrinomial form of the cubic. The co-ordinates of such connective are absolutely determinate, inasmuch as the equation which each set of co-ordinates must satisfy is of the order 3, whereas the co-ordinates in question are of the second order only in each set of variables (and of course of the first degree in the coefficients of the cubic). By *reduction* I mean that when in the co-ordinates of the general connective for x, y, z; x', y', z' are substituted the *normal* forms of the co-ordinates for derivatives of the grades $\mu, \mu+3i$, their common factor of the degorder $(12i^2-3, 9i^2)$ is to be cast out.

This common factor, it may be noticed, is *always* a covariant of the cubic. When $i=1$, it is seen *a posteriori* that this is the case, for its value is expressible (see footnote, p. [379]) under the form of a known covariant, say Θ (which was obtained by means of using the canonical form of the cubic); that it must be true for all values of i may be deduced from the general algebraical theorem that if in a covariant to any given form, in place of the variables x, y, z be substituted $\frac{d\Omega}{d\xi}, \frac{d\Omega}{d\eta}, \frac{d\Omega}{d\zeta}$, where Ω is any invariantive concomitant to such form, and ξ, η, ζ are contragredient to x, y, z, the resulting expression will be itself an invariantive concomitant. To obtain now the reducing factor for the connective to $P_\mu, P_{\mu+3i}$ (p. [380]) it is only necessary to substitute in Θ x_i, y_i, z_i (the normal co-ordinates of the ith derivative) in lieu of x, y, z where $x_i\xi+y_i\eta+z_i\zeta$ is known to be an invariantive concomitant to the cubic. Hence, by the algebraical theorem above stated, the corresponding reducing factor (not containing ξ, η, ζ) is necessarily a covariant to the cubic, as was to be shown.

the original one, as z. For when $z=0$, P will be a point of inflexion and P_i identical with P, hence (x_i, y_i, z_i) will express the same point of inflexion, and consequently $z_i=0$; hence z_i must contain z. When we leave the rational scale, so that i is a multiple of 3, z must contain xyz. For when $z=0$, the ith derivative P will be one of the three points I, I', I'', expressed by $z=0, x^3+y^3=0$. If P is I, P_3 is obviously I; if P is I', P_2 is I', and P_3 will be the connective of P_2 and I''; consequently P_3 is I and $z=0$, and the same will be the case if P is I''; hence z_3' contains z.

Again, if $y=0$, P will be some inflexion J, and the connective to I, J being called K, P_3 will be the connective of J, K, that is I, as before; hence z_3 will contain y, and in like manner it will contain x. Also, since in each case P_3 is I, every derivative of P_3 will be I; hence, when $xyz=0$, $z_{3\nu}$ becomes 0; consequently z_i (if i is a multiple of 3) contains xyz.

Again, if x_i, y_i, z_i are the reduced co-ordinates of P_i, I say that $x_i(y_i^3-z_i^3)$; $y_i(z_i^3-x_i^3)$; $z_i(x_i^3-y_i^3)$ will be the *reduced* co-ordinates of x_{2i}, y_{2i}, z_{2i}.

For, if possible, let two of the above co-ordinates have a common factor F; then, since x_i, y_i, z_i have no common factor, $x_i^3-y_i^3$, $y_i^3-z_i^3$ have a common factor, and when $F=0$, $x_i^3=y_i^3=z_i^3$; but $x_i^3+y_i^3+z_i^3+Kx_iy_iz_i=0$. Hence, unless $x_i^3=y_i^3=z_i^3=0$, we must have $3+\sqrt[3]{1}K=0$, but K is arbitrary. Hence, F must be contained in x_i, y_i, z_i contrary to hypothesis.

Although it is a consequence of a general law* that z_i cannot contain z^2, for present purposes it will be sufficient to establish that z_i cannot, for each of two consecutive values of i, contain z^2. Thus, suppose z_{2i-1} and z_{2i} each contained z^2, then, because z_{2i} contains z^2, z_i must do so too; since, otherwise, $x_i^3-y_i^3$ must contain z. If that is possible, let $z=0$; then $x_i^3-y_i^3=0$; but P, and therefore P_i, becomes an inflexion, whereas $x_i^3=y_i^3$ is the necessary and sufficient condition that P_i is a Plückerian point, which is self-contradictory. But since z_i contains z^2, z_{i-1} must also contain z^2, for z_{2i-1} will be contained (see p. [374]) in $\frac{1}{z}(x_iy_iz^2_{i-1}-x_{i-1}y_{i-1}z_i^2)$, and therefore, if z_{i-1} does not contain z^2, z must be contained in x_i or y_i, which is impossible. In like manner, if z^2 is contained in z_{2i}, z_{2i+1}, it will be contained also in z_i and z_{i+1}. Hence it would be contained eventually in z, which is absurd.

Again, it may be shown that z will be the only common measure to z_{i-1} and z_i. For, if possible, let them have any other common measure F, and let F become zero. Then P_{i-1} and P_i both become points of inflexion belonging to the system previously designated as I, I', I'', and by a collineation process

* The law is that $x^i . y^i . z^i x_iy_iz_i$, cannot for any value of i contain a square algebraical factor, just as, and *en dernière analyse* for the same general kind of reason, the binomial exponential (a^i+b^i) can contain no such factor.

performed on these points alone or combined with I, P may be obtained. Hence P belongs to the same system of inflexions, that is $z=0$. Hence F would be contained in a power of z, contrary to hypothesis.

I will now show that if the two systems of unreduced co-ordinates obtained by the colligation of

$$\left.\begin{matrix} x_{i-1}, & y_{i-1}, & z_{i-1} \\ x_i\ , & y_i\ , & z_i \end{matrix}\right\} \text{ and of } \left\{\begin{matrix} y_{i-1}, & x_{i-1}, & z_{i-1} \\ x_i\ , & y_i\ , & z_i \end{matrix}\right.$$

be called F, G, H; F', G', H; respectively, the terms F, G, H; F', G', H can have no other measure common to all four than z, or, in other and more precise terms, z is the greatest common measure to the greatest common measures of F, G, H and of F', G', H. For brevity call the two sets of co-ordinates of P_{i-1} and P_i, u, v, w; u', v', w' respectively. Then the unreduced co-ordinates in question will be (p. [374])

$$\left.\begin{aligned} F &= vwu'^2 - v'w'u^2 \\ G &= wuv'^2 - w'u'v^2 \\ H &= uvw'^2 - u'v'w^2 \end{aligned}\right\} \text{ and } \left\{\begin{aligned} uwu'^2 - u'w'v^2 &= F' \\ wvv'^2 - w'v'u^2 &= G' \\ vuw'^2 - v'u'w^2 &= H \end{aligned}\right.$$

into each of which z necessarily enters as a factor, because w, w' have been proved each to contain z.

(u, v, it will be observed, cannot have a common factor, for then u, v, w would have a common factor contrary to hypothesis; and, in like manner, u', v' can have no common factor.)

I say, in the first place, that no indecomposable function of x, y, z, say M, not contained either in w or in w', can be common to F, G, F', G'. For, if so, let F vanish; then, calling $\frac{u}{w}, \frac{v}{w}; \frac{u'}{w'}, \frac{v'}{w'}$, r, s; r', s' respectively, we have

$$\begin{aligned} &(1)\quad sr'^2 - s'r^2 = 0, \qquad &(3)\quad rr'^2 - s's^2 = 0, \\ &(2)\quad rs'^2 - r's^2 = 0, \qquad &(4)\quad r'r^2 - ss'^2 = 0. \end{aligned}$$

Now, none of the terms r, s, r', s' can vanish: for example r cannot vanish, for, if so, from (1) it would follow that $s=0$, or $r'=0$, and from (3) that $s=0$, or $s'=0$, so that either $r=0$ and $s=0$, or $r'=0$ and $s'=0$, that is the general values of u and v or of u' and v' must have a common factor M, which is impossible. Hence, combining (1) and (2) or (3) and (4), we derive $rs=r's'$ (5), as might also be obtained immediately by equating to zero the term common to the two systems above.

From (5), from (3) and (4), and from (1) and (2) we obtain respectively

$$r^3s^3 = r'^3s'^3, \quad r^3r'^3 = s^3s'^3, \quad r'^3s^3 = r^3s'^3,$$

the second and third of which are equivalent to $r^6 = s^6$, $r'^6 = s'^6$, and the first and second combined give $r'^6 = s^6$. Hence $r^6 = r'^6 = s^6 = s'^6$, and consequently the original equations (1), (2), (3) give $r^3 = s^3 = r'^3 = s'^3$.

The equations $r^3=s^3$, $r'^3=s'^3$ imply that P_{i-1}, P_i are each of them distinct or identical antitangentials to one of the points of inflexion corresponding to $z=0$, that is are each of them a Plückerian point on the cubic, and P or (P, I) will be a residual either to P_{i-1}, P_i or to (P_{i-1}, I), P_i where I is the auxiliary inflexion used to complete the scale. Hence P is either a Plückerian or an inflexion point, and in either case P_2 will necessarily be an inflexion. Hence one at least of the derivatives P_{i-1}, P_i is an inflexion, but each is a Plückerian, which is absurd.

Thus M (an irresoluble factor common to F, G, F', G') must be contained either in w or in w'. Suppose it is not z and is contained in w, then it cannot be contained in w', for w, w' have no common measure except z, and consequently when $M=0$, $v'u^2=0$, $u'v^2=0$, $v'v^2=0$, and $u'u^2=0$, and either u and v or u' and v' each become zero, which is impossible seeing that neither the general values of u, v nor those of u', v' can have any common factor. In like manner, it follows that M cannot be contained in w'. Consequently, the two systems F, G, H; F', G', H can have no other common measure, except some power of z.

Finally, I say that the only common measure in question is z itself. (1) Suppose it were possible (which it is not) that one of the two terms w or w' (say w) contains z^2, then it has been proved that the other (w') cannot contain z^2. Hence, if $wvv'^2 - w'u'u^2$ contains z^2, u or u' must contain z, and in like manner, if w' and not w contained z, v or v' must contain z, none of which suppositions are admissible. (2) Suppose that neither w nor w' contains z^2. Then writing $w=\omega z$, $w'=\omega' z$, and writing for $\frac{u}{\omega}, \frac{v}{\omega}; \frac{u'}{\omega'}, \frac{v'}{\omega'}$, r, s; r', s' respectively, we shall obtain over again, as before, $r^3=s^3$, $r'^3=s'^3$, indicating as before that P_{i-1} and P_i are each of them Plückerian points when $z=0$, that is, when P is a point of inflexion, which is doubly absurd. Hence it follows that the common measures of F, G, H and of F', G', H have the common measure z, and no other.

We are now in a position to prove the *law of squares*. Suppose it is true for P_{i-1} and P_i, I say it will be true for P_{2i-1}. For consider the connectives of

$$\left.\begin{matrix} x_{i-1}, & y_{i-1}, & z_{i-1} \\ x_i\ , & y_i\ , & z_i \end{matrix}\right\} \text{ and of } \left\{\begin{matrix} y_{i-1}, & x_{i-1}, & z_{i-1} \\ x_i\ , & y_i\ , & z_i \end{matrix}\right.$$

as expressed by the formulas above employed. Let $z^2\Omega$ be the third term common to the unreduced systems of co-ordinates.

Allowing (as is the fact) that Ω does not contain z, the reducing factor common to the unreduced co-ordinates of P (or it may be its opposite in respect to I) must be $z\Omega$, and consequently to the other system corresponding to P_{2i-1} or its opposite, can only be z or z^2; but the latter is impossible, for then z_{2i-1} would not contain z.

Again, if Ω could be conceived equal to $z^q\Omega_1$, the reducing factor for P or its opposite would be $z^{1+q}\Omega_1$, and consequently that for P_{2i-1} or its opposite could not be z^2 and would be z as before. Hence the order of P_{2i-1} in the variables is necessarily $2(i-1)^2+2i^2-1$, that is, $4i^2-4i+1$ or $(2i-1)^2$.

Moreover, it has been shown that if x_i, y_i, z_i are the reduced co-ordinates for P_i, $x_i(y_i^3-z_i^3)$, $y_i(z_i^3-x_i^3)$, $z_i(x_i^3-y_i^3)$ are such for P_{2i}, and consequently, if the law is true for i, it is true for $2i$. Hence, being true for 1, it is true for 2, and therefore for 3, and therefore for 4 and 5 and 6, and therefore for $3+4$, that is, 7, and for 2.4, that is, 8, and for $4+5$, that is, 9, and for 2.5, that is, 10, and so on for every number, as was to be proved*. Thus, this negative proposition, as I have termed it (p. [356]), is completely established. There remains to prove the important proposition contained (but incorrectly proved) on p. [377], to wit, that the unreduced systems of co-ordinates arising from the colligation of

$$\left.\begin{matrix}(x_i, y_i, z_i)\\(x_j, y_j, z_j)\end{matrix}\right\} \text{ and of } \left\{\begin{matrix}(y_i, x_i, z_i)\\(x_j, y_j, z_j)\end{matrix}\right.$$

will be of the forms LN', MN', NN'; $L'N$, $M'N$, $N'N$, where L, M, N; L', M', N' are the reduced systems of the co-ordinates of the connectives of P_i, P_j, and P_i', P_j respectively.

To illustrate this proposition by an example, consider the connectives of P', P_3, that is, P_2 and of P, P_3, that is, P_4.

z_2 is $z(y^3-x^3)$ and z_4 is of the form $z(y^3-x^3)\,\Omega$, where Ω is of the order 12 in the variables.

Call X_4, Y_4, Z_4 the unreduced co-ordinates arising from the colligation of P, P_3. Suppose x^3-y^3 to become zero, then P becomes a Plückerian, and P_3 will be also such, namely, one of the nine appertaining to the inflexions given by $z=0$†. Hence $x_3^3-y_3^3$ becomes zero. Now X_4, Y_4 represent $yzx_3^2-y_3z_3x^2$,

* In other words, if the theorem is true up to i inclusive, any number between $i+1$ and $2i$ inclusive is either of the form $2j$ or $2j-1$, where j does not exceed i; and being true for j, it is true for $2j$, and being true for $j-1$ and j, it is true for $2j-1$. Hence, if true up to i it is true up to $2i$, but it is true for $i=1$ and therefore for all values of i. Q. E. D.

† The nine points of inflexion on a cubic curve form a closed group, but so also do any three of them which lie in a right line, and also any single one. In like manner, the nine inflexions with their antitangentials, any three of these lying in a right line with their antitangentials, and any one with its antitangentials, form closed groups containing 36, 12, and 4 points respectively. The ornamental-gardening problem of *alignement*, anglice *allineation*, which consists in so disposing a number of points on a plane as to obtain the maximum number or all the various possible numbers of right lines each containing three of the points, finds its systematic solution in the theory of groups of inflexional and sub-inflexional points of various grades.

$xzy_3^2-x_3z_3y^2$ respectively, and since

$$yzx_3^2 \,.\, x_3z_3y^2 - y_3z_3x^2 \,.\, xzy_3^2 = zz_3(x_3^3y_3^3 - y_3^3x_3^3) = 0,$$

$X_4 : Y_4 :: yx_3^2 : xy_3^2$, and consequently $X_4^3-Y_4^3=0$; but P_4 is a point of inflexion and not a Plückerian; hence X_4, Y_4 must each contain the factor x^3-y^3, and Z_4 must be of the form $z^2(x^3-y^3)^2\,\Omega$, for after division by $z(x^3-y^3)$ it must still contain that factor. Also X_4, Y_4, Z_4 can have no other common measure except $z(x^3-y^3)$, for after throwing out that factor the quotient is of the order 16, the order of z_4 given by the law of squares. Thus we see that the third unreduced coefficient common to (P, P_3) and (P', P_3) is equal to $z_2 \,.\, z_4$, as it ought to be according to the proposition in question.

In some very old numbers of the *Educational Times* will be found questions of the kind proposed by me (not reproduced in the Reprint), of which the solution depends on this order of considerations. In certain cases that had been studied, I ascertained the possible existence of a larger number of collineations than had previously been imagined by other writers on the subject, among whom Mr S. B. Woolhouse deserves special mention for the ingenuity of his constructions. As far as I am aware, the theory of allineation has never been treated by other writers than myself, except by empirical methods, and its dependence on the theory of the general cubic curve was not even suspected.

40.

TABLES OF THE GENERATING FUNCTIONS AND GROUNDFORMS FOR SIMULTANEOUS BINARY QUANTICS OF THE FIRST FOUR ORDERS, TAKEN TWO AND TWO TOGETHER.

[*American Journal of Mathematics*, II. (1879), pp. 293—306, 324—329.]

IN the Generating Functions given below, the exponents of the letters a, b, c, d, refer to degree in the coefficients of the quantics of the 1st, 2nd, 3rd and 4th orders respectively; the exponents of the letter x to order in the variables. Where the system consists of two quantics of the same order, the Latin letter and the corresponding Greek letter have been used. In the tabulated numerators, the *minus* sign has been placed *over* the number which it affects.

In each of the systems considered in this paper, with the exception of that consisting of a cubic and a quartic, it is found that there is never more than one groundform of any given type (that is, of a given order in the variables and given degrees in the coefficients of the quantics); where, therefore, in the enumeration of the groundforms, the *type* alone is given, the *number* of groundforms of the type is to be understood to be 1. The symbol (λ, μ) is used to indicate a form of the degrees λ and μ in the coefficients of the two quantics, the number placed first always relating to the quantic of lower order, when the orders are different. In the last three cases, the numbers, as well as the types, of the groundforms are given in tables, which require no explanation.

SYSTEM OF TWO LINEARS*.

G. F. for differentiants, $\dfrac{1}{(1-a)(1-\alpha)(1-a\alpha)}$.

G. F. for covariants, $\dfrac{1}{(1-a\alpha)(1-ax)(1-\alpha x)}$.

Groundforms:

Of order 0 .... (1, 1).
" " 1 .... (0, 1), (1, 0).

* "Linear" is here used as a noun, in conformity with the use of the words quadric, cubic, &c.

SYSTEM OF LINEAR AND QUADRIC.

G. F. for differentiants, $\dfrac{1+ab}{(1-a)(1-b)(1-b^2)(1-a^2b)}$.

G. F. for covariants, $\dfrac{1+abx}{(1-b^2)(1-a^2b)(1-ax)(1-bx^2)}$.

Groundforms:

Of order 0 .... (0, 2), (2, 1).
" " 1 .... (1, 0), (1, 1).
" " 2 .... (0, 1).

SYSTEM OF LINEAR AND CUBIC.

G. F. for differentiants, $\dfrac{1+a^2c+(a-a^3)c^2+(1-a^2)c^3-ac^4-a^3c^5}{(1-a)(1-c)(1-c^2)(1-c^4)(1-ac)(1-a^3c)}$.

G. F. for covariants, reduced form,

Denominator: $(1-c^4)(1-ac)(1-a^3c)(1-ax)(1-cx)(1-cx^3)$.

Numerator: $1-ac+a^2c^2+\{(-1+a^2)c+(2a-a^3)c^2-a^2c^3\}x$
$+\{ac+(1-2a^2)c^2+(-a+a^3)c^3\}x^2+\{-ac^2+a^2c^3-a^3c^4\}x^3$.

G. F. for covariants, representative form,

Denominator: $(1-c^4)(1-a^3c)(1-a^2c^2)(1-ax)(1-c^2x^2)(1-cx^3)$.

Numerator: $1+a^2c^2+\{a^2c+ac^2+(a^2-a^4)c^3\}x+\{ac+(a-a^3)c^3-a^3c^3\}x^2$
$+\{(1-a^2)c^3-a^3c^4-a^2c^5\}x^3+\{-ac^3-a^4c^5\}x^4$.

Groundforms:

Of order 0 .... (0, 4), (2, 2), (3, 1), (3, 3).
" " 1 .... (1, 0), (1, 2), (2, 1), (2, 3).
" " 2 .... (0, 2), (1, 1), (1, 3).
" " 3 .... (0, 1), (0, 3).

SYSTEM OF LINEAR AND QUARTIC.

G. F. for differentiants,

$$\frac{1+(a+a^3)d+(a+a^2-a^5)d^2+(1-a^3-a^4)d^3+(-a^2-a^4)d^4-a^5d^5}{(1-a)(1-d)(1-d^2)^2(1-d^3)(1-a^2d)(1-a^4d)}.$$

G. F. for covariants, reduced form,

Denominator: $(1-d^2)(1-d^3)(1-a^2d)(1-a^4d)(1-ax)(1-dx^2)(1-dx^4)$.

Numerator: $1-a^2d+a^4d^2+\{a^3d+(a^3-a^5)d^2\}x+\{(-1+a^2)d$
$+(2a^2-a^4)d^2-a^4d^3\}x^2+\{ad+(a-2a^3)d^2+(-a^3+a^5)d^3\}x^3$
$+\{(1-a^2)d^2-a^2d^3\}x^4+\{-ad^2+a^3d^3-a^5d^4\}x^5$.

G. F. for covariants, representative form,

Denominator: $(1-d^2)(1-d^3)(1-a^4d)(1-a^4d^2)(1-ax)(1-dx^4)(1-d^2x^4)$.

Numerator: $1+a^6d^3+\{a^3d+a^3d^2+(a^5-a^7)d^3\}x+\{a^2d+a^2d^2+(a^4-a^6)d^3\}x^2$
$+\{ad+ad^2+(a^3-a^5)d^3\}x^3+\{(a^2-a^4)d^3-a^6d^4-a^6d^5\}x^4$
$+\{(a-a^3)d^3-a^5d^4-a^5d^5\}x^5+\{(1-a^2)d^3-a^4d^4-a^4d^5\}x^6$
$+\{-ad^3-a^7d^6\}x^7.$

Groundforms:

Of order 0 .... (0, 2), (0, 3), (4, 1), (4, 2), (6, 3).
” ” 1 .... (1, 0), (3, 1), (3, 2), (5, 3).
” ” 2 .... (2, 1), (2, 2), (4, 3).
” ” 3 .... (1, 1), (1, 2), (3, 3).
” ” 4 .... (0, 1), (0, 2), (2, 3).
” ” 5 .... (1, 3).
” ” 6 .... (0, 3).

System of Two Quadrics.

G. F. for differentiants, $\dfrac{1+b\beta}{(1-b)(1-b^2)(1-\beta)(1-\beta^2)(1-\beta b)}.$

G. F. for covariants, $\dfrac{1+b\beta x^2}{(1-b^2)(1-\beta^2)(1-b\beta)(1-bx^2)(1-\beta x^2)}.$

Groundforms:

Of order 0 .... (0, 2), (1, 1), (2, 0).
” ” 2 .... (0, 1), (1, 0), (1, 1).

System of Quadric and Cubic.

G. F. for differentiants,

$$\frac{1+(2b+b^2)c+(b+b^2+b^3)c^2+c^3-b^4c^4+(-b-b^2-b^3)c^5+(-b^2-2b^3)c^6-b^4c^7}{(1-b)(1-b^2)(1-c)(1-c^2)(1-c^4)(1-bc^2)(1-b^3c^2)}.$$

G. F. for covariants, reduced form,

Denominator: $(1-b^2)(1-c^4)(1-bc^2)(1-b^3c^2)(1-bx^2)(1-cx)(1-cx^3).$

Numerator: $1+b^3c^4+\{(-1+b+b^2)c+(b+b^3)c^3-b^3c^5\}x$
$+\{(1+b^3)c^2+(-b-b^4)c^4\}x^2+\{bc+(-b^2-b^3)c^3$
$+(-b^2-b^3+b^4)c^5\}x^3+\{-bc^3-b^4c^6\}x^4.$

G. F. for covariants, representative form,

Denominator: $(1-b^2)(1-c^4)(1-bc^2)(1-b^3c^2)(1-bx^2)(1-c^2x^2)(1-cx^3).$

Numerator: $1+b^3c^4+\{(b+b^2)c+(b+b^2)c^3\}x+\{(b+b^2+b^3)c^2$
$+(b^2-b^4)c^4-b^3c^6\}x^2+\{bc+(1-b^2)c^3+(-b-b^2-b^3)c^5\}x^3$
$+\{(-b^2-b^3)c^4+(-b^2-b^3)c^6\}x^4+\{-bc^3-b^4c^7\}x^5.$

Groundforms:

Of order 0 .... (0, 4), (1, 2), (2, 0), (3, 2), (3, 4).
” ” 1 .... (1, 1), (1, 3), (2, 1), (2, 3).
” ” 2 .... (0, 2), (1, 0), (1, 2).
, ” 3 .... (0, 1), (0, 3), (1, 1).

System of Quadric and Quartic.

G. F. for differentiants,

$$\frac{1+(b+b^2)d+(2b-b^3)d^2+(1-2b^2)d^3+(-b-b^2)d^4-b^3d^5}{(1-b)(1-b^2)(1-d)(1-d^2)^2(1-d^3)(1-bd)(1-b^2d)}.$$

G. F. for covariants, reduced form,

Denominator: $(1-b^2)(1-d^2)(1-d^3)(1-bd)(1-b^2d)(1-bx^2)(1-dx^2)$
$(1-dx^4).$

Numerator: $1-bd+b^2d^2+\{(-1+b+b^2)d+(2b-b^3)d^2-b^2d^3\}x^2$
$+\{bd+(1-2b^2)d^2+(-b-b^2+b^3)d^3\}x^4$
$+\{-bd^2+b^2d^3-b^3d^4\}x^6.$

G. F. for covariants, representative form,

Denominator: $(1-b^2)(1-d^2)(1-d^3)(1-b^2d)(1-b^2d^2)(1-bx^2)(1-dx^4)$
$(1-d^2x^4).$

Numerator: $1+b^3d^3+\{(b+b^2)d+(b+b^2)d^2+(b^2-b^4)d^3\}x^2+\{bd+bd^2$
$+(b-b^3)d^3-b^3d^4-b^3d^5\}x^4+\{(1-b^2)d^3+(-b^2-b^3)d^4$
$+(-b^2-b^3)d^5\}x^6+\{-bd^3-b^4d^6\}x^8.$

Groundforms:

Of order 0 .... (0, 2), (0, 3), (2, 0), (2, 1), (2, 2), (3, 3).
” ” 2 .... (1, 0), (1, 1), (1, 2), (2, 1), (2, 2), (2, 3).
” ” 4 .... (0, 1), (0, 2), (1, 1), (1, 2), (1, 3).
” ” 6 .... (0, 3).

System of Two Cubics.

G. F. for differentiants,

Denominator: $(1-c)(1-c^2)(1-c^4)(1-\gamma)(1-\gamma^2)(1-\gamma^4)(1-c\gamma)$
$(1-c^3\gamma)(1-c\gamma^3).$

Numerator: $1+c^3+(2c+2c^2-c^5-c^6)\gamma+(2c+2c^2-c^4-c^5-c^6-c^7)\gamma^2$
$+(1+2c^3-c^4-2c^5-c^6-c^7)\gamma^3+(-c^2-c^3-c^5-c^6)\gamma^4$
$+(-c-c^2-2c^3-c^4+2c^5+c^8)\gamma^5+(-c-c^2-c^3-c^4+2c^6+2c^7)\gamma^6$
$+(-c^2-c^3+2c^6+2c^7)\gamma^7+(c^5+c^8)\gamma^8.$

G. F. for covariants, reduced form,

Denominator: $(1-c^4)(1-\gamma^4)(1-c\gamma)(1-c^3\gamma)(1-c\gamma^3)(1-cx)(1-cx^3)$
$(1-\gamma x)(1-\gamma x^3).$

Numerator:

		γ^0	γ^1	γ^2	γ^3	γ^4	γ^5	γ^6
x^0	c^0	1						
	c^2			1				
	c^3				1			
	c^5						1	
x^1	c^0		$\bar{1}$					
	c^1	$\bar{1}$		1		1		
	c^2		1					
	c^4		1					
	c^5							$\bar{1}$
	c^6						$\bar{1}$	
x^2	c^0			1				
	c^1		2				$\bar{1}$	
	c^2	1		$\bar{1}$		$\bar{1}$		
	c^4			$\bar{1}$				
	c^5		$\bar{1}$					
	c^6							1

		γ^1	γ^2	γ^3	γ^4	γ^5	γ^6	γ^7
x^6	c^2		1					
	c^4				1			
	c^5					1		
	c^7							1
x^5	c^1		$\bar{1}$					
	c^2	$\bar{1}$						
	c^3						1	
	c^5						1	
	c^6			1		1		$\bar{1}$
	c^7						$\bar{1}$	
x^4	c^1	1						
	c^2						$\bar{1}$	
	c^3					$\bar{1}$		
	c^5			$\bar{1}$		$\bar{1}$		1
	c^6		$\bar{1}$				2	
	c^7					1		

		γ^1	γ^2	γ^3	γ^4	γ^5	γ^6
x^3	c^1		$\bar{1}$		$\bar{1}$		
	c^2	$\bar{1}$				1	
	c^3				$\bar{1}$		$\bar{1}$
	c^4	$\bar{1}$		$\bar{1}$			
	c^5		1				$\bar{1}$
	c^6			$\bar{1}$		$\bar{1}$	

G. F. for covariants, representative form,

Denominator: $(1-c^4)(1-\gamma^4)(1-c\gamma)(1-c^3\gamma)(1-c\gamma^3)(1-c^2x^2)(1-cx^3)$
$(1-\gamma^2x^2)(1-\gamma x^3)$.

Numerator:

		γ^0	γ^1	γ^2	γ^3	γ^4	γ^5	γ^6	γ^7
x^0	c^0	1							
	c^2			1					
	c^3				1				
	c^5						1		
x^1	c^1			1		1			
	c^2		1		1				
	c^3			1		1			
	c^4		1		1				
x^2	c^1		1		1				
	c^2			1					
	c^3		1		1				
	c^4					1			
	c^5								$\bar{1}$
	c^7						$\bar{1}$		
x^3	c^0				1				
	c^1			1		$\bar{1}$		$\bar{1}$	
	c^2		1						
	c^3	1				$\bar{2}$		$\bar{1}$	
	c^4		$\bar{1}$		$\bar{2}$				
	c^5							$\bar{1}$	
	c^6		$\bar{1}$		$\bar{1}$		$\bar{1}$		

		γ^1	γ^2	γ^3	γ^4	γ^5	γ^6	γ^7	γ^8
x^8	c^3			1					
	c^5					1			
	c^6						1		
	c^8								1
x^7	c^4					1		1	
	c^5				1		1		
	c^6					1		1	
	c^7				1		1		
x^6	c^1			$\bar{1}$					
	c^3	$\bar{1}$							
	c^4				1				
	c^5					1		1	
	c^6						1		
	c^7					1		1	
x^5	c^2			$\bar{1}$		$\bar{1}$		$\bar{1}$	
	c^3		$\bar{1}$						
	c^4					$\bar{2}$		$\bar{1}$	
	c^5		$\bar{1}$		$\bar{2}$				1
	c^6							1	
	c^7		$\bar{1}$		$\bar{1}$		1		
	c^8					1			

		γ^1	γ^2	γ^3	γ^4	γ^5	γ^6	γ^7
x^4	c^1	1				$\bar{1}$		
	c^2				$\bar{1}$		$\bar{1}$	
	c^3			$\bar{1}$		$\bar{2}$		$\bar{1}$
	c^4		$\bar{1}$		$\bar{2}$		$\bar{1}$	
	c^5	$\bar{1}$		$\bar{2}$		$\bar{1}$		
	c^6		$\bar{1}$		$\bar{1}$			
	c^7			$\bar{1}$				1

*Table of Groundforms**.

Order in the Variables.	Deg. in coeff's of 2d cubic.	Deg. in coeff's of 1st cubic. 0	1	2	3	4
0	0					1
	1		1		1	
	2			1		
	3		1		1	
	4	1				
1	1			1		1
	2		1		1	
	3			1		
	4		1			

Order in the Variables	Deg. in coeff's of 2d cubic.	Deg. in coeff's of 1st cubic. 0	1	2	3
2	0			1	
	1		1		1
	2	1		1	
	3		1		
3	0		1		1
	1	1		1	
	2		1		
	3	1			
4	1		1		

SYSTEM OF CUBIC AND QUARTIC.

G. F. for differentiants,

Denominator: $(1-c)(1-c^2)(1-c^4)(1-d)(1-d^2)^2(1-d^3)(1-c^2d)$
$(1-c^4d)(1-c^2d^3)(1-c^4d^3)$.

Numerator: $1+c^3+(3c+2c^2+2c^3+c^4-2c^5-c^6-c^7)\,d$
$+(3c+5c^2+2c^3+2c^4-3c^5-4c^6-2c^7-2c^8+c^9)\,d^2$
$+(1+3c^2+3c^3+c^4-c^5-6c^6-5c^7-4c^8+2c^{10})\,d^3$
$+(-c^2+c^3-c^4-2c^5-5c^6-6c^7-3c^8-c^9+3c^{10}+2c^{11}+c^{12})\,d^4$
$+(-2c^2-3c^3-3c^4-3c^5-2c^6-2c^7-c^8+2c^{10}+4c^{11}+3c^{12}+c^{13})\,d^5$
$+(-c^2-3c^3-4c^4-2c^5+c^7+2c^8+2c^9+3c^{10}+3c^{11}+3c^{12}+2c^{13})\,d^6$
$+(-c^3-2c^4-3c^5+c^6+3c^7+6c^8+5c^9+2c^{10}+c^{11}-c^{12}+c^{13})\,d^7$
$+(-2c^5+4c^7+5c^8+6c^9+c^{10}-c^{11}-3c^{12}-3c^{13}-c^{15})\,d^8$
$+(-c^6+2c^7+2c^8+4c^9+3c^{10}-2c^{11}-2c^{12}-5c^{13}-3c^{14})\,d^9$
$+(c^8+c^9+2c^{10}-c^{11}-2c^{12}-2c^{13}-3c^{14})\,d^{10}+(-c^{12}-c^{15})\,d^{11}$.

G. F. for covariants, reduced form,

Denominator: $(1-c^4)(1-d^2)(1-d^3)(1-c^2d)(1-c^4d)(1-c^2d^3)(1-c^4d^3)$
$(1-cx)(1-cx^3)(1-dx^2)(1-dx^4)$.

* The forms of ord. 1, deg. 3, 4 and of ord. 1, deg. 4, 3 given by Clebsch and Gordan, do not appear in this table, and it has been proved by the author that no fundamental forms of either of these types exist. [See page 409, below.]

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9
x^0	c^0	1									
	c^2		$\bar{1}$								
	c^4			2	2	2	1				
	c^6			1	1		$\bar{1}$	$\bar{1}$			
	c^8				$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$			
	c^{10}								1		
	c^{12}									$\bar{1}$	
x^1	c^1	$\bar{1}$		1							
	c^3		3	2	1	1					
	c^5		1	$\bar{2}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	1			
	c^7			$\bar{2}$	$\bar{1}$	$\bar{1}$		1	$\bar{1}$		
	c^9				1	1	1		$\bar{2}$		
	c^{11}							$\bar{1}$	$\bar{2}$		
	c^{13}									1	
x^2	c^0		$\bar{1}$								
	c^2	1	1	3	2	1					
	c^4		$\bar{1}$		$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$			
	c^6		$\bar{1}$		$\bar{2}$	$\bar{2}$			1		
	c^8			1		1	1		2		
	c^{10}						$\bar{1}$	$\bar{1}$		$\bar{2}$	
	c^{12}							1	1		1
x^3	c^1		2								
	c^3			$\bar{3}$							
	c^5		$\bar{1}$	$\bar{2}$	1		1		$\bar{1}$		
	c^7					$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{1}$	1	
	c^9						$\bar{1}$			1	
	c^{11}					1	1	2	1	2	
	c^{13}									1	$\bar{1}$

		d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}
x^8	c^2		1								
	c^4			$\bar{1}$							
	c^6				2	2	2	1			
	c^8				1	1		$\bar{1}$	$\bar{1}$		
	c^{10}					$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$		
	c^{12}									1	
	c^{14}										$\bar{1}$
x^7	c^1		$\bar{1}$								
	c^3			2	1						
	c^5			2		$\bar{1}$	$\bar{1}$	$\bar{1}$			
	c^7			1	$\bar{1}$		1	1	2		
	c^9				$\bar{1}$	1	1	1	2	$\bar{1}$	
	c^{11}						$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	
	c^{13}								$\bar{1}$	$\bar{1}$	1
x^6	c^2	$\bar{1}$		$\bar{1}$	$\bar{1}$						
	c^4		2		1	1					
	c^6			$\bar{2}$		$\bar{1}$	$\bar{1}$		$\bar{1}$		
	c^8			$\bar{1}$			2	2		1	
	c^{10}				1	2	2	1		1	
	c^{12}						$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$	$\bar{1}$
	c^{14}									1	
x^5	c^1	1	$\bar{1}$								
	c^3		$\bar{2}$	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{1}$				
	c^5		$\bar{1}$			1					
	c^7		$\bar{1}$	1	2	1	1				
	c^9			1		$\bar{1}$		$\bar{1}$	2	1	
	c^{11}								3		
	c^{13}									$\bar{2}$	

Numerator—(*Continued.*)

		d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9
x^4	c^0		1							
	c^2	$\bar{1}$		$\bar{1}$	$\bar{1}$	$\bar{1}$				
	c^4	$\bar{1}$		$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^6		1	$\bar{2}$	$\bar{1}$			1		
	c^8			$\bar{1}$			1	2	$\bar{1}$	
	c^{10}					1	2	2		1
	c^{12}					1	1	1		1
	c^{14}								$\bar{1}$	

G. F. for covariants, representative form,

Denominator: $(1-c^4)(1-d^2)(1-d^3)(1-c^4d)(1-c^4d^2)(1-c^2d^3)(1-c^4d^3)$
$(1-cx^3)(1-c^2x^2)(1-dx^4)(1-d^2x^4)$.

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}
x^0	c^0	1											
	c^4			1	2	2	1						
	c^6			1	3	2	1						
	c^8					$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$				
	c^{10}					$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^{14}										$\bar{1}$		
x^1	c^1		1	1									
	c^3		2	3	2	1							
	c^5		1	2	3	2	1	1					
	c^9				$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$			
	c^{11}						$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$			
	c^{13}								$\bar{1}$	$\bar{1}$			

		d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}	d^{12}
x^{11}	c^3			1									
	c^7					1	2	2	1				
	c^9					1	3	2	1				
	c^{11}							$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$		
	c^{13}							$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$		
	c^{17}												$\bar{1}$
x^{10}	c^4				1	1							
	c^6				2	3	2	1					
	c^8				1	2	3	2	1	1			
	c^{12}						$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	
	c^{14}								$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	
	c^{16}										$\bar{1}$	$\bar{1}$	

Numerator—(*Continued.*)

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}
x^2	c^2		2	3	2	1							
	c^4		2	4	5	3	1						
	c^6			$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{2}$	$\bar{1}$				
	c^{10}						$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$			
	c^{12}					1	1		$\bar{2}$	$\bar{2}$	$\bar{1}$		
	c^{16}										1		
x^3	c^1		1	1	1								
	c^3	1	1	3	5	3	1						
	c^5		$\bar{1}$		2	3	2						
	c^7		$\bar{1}$	$\bar{2}$	$\bar{4}$	$\bar{5}$	$\bar{3}$	$\bar{3}$	$\bar{1}$				
	c^9					$\bar{2}$	$\bar{4}$	$\bar{3}$	$\bar{3}$	$\bar{1}$			
	c^{11}				1	1	1		$\bar{1}$	$\bar{1}$	$\bar{1}$		
	c^{13}						1	1	2	1	$\bar{1}$		
	c^{15}								1	1	1		
x^4	c^2		1	2	3	1							
	c^4		$\bar{1}$		2	1	$\bar{1}$	$\bar{2}$	$\bar{1}$				
	c^6		$\bar{1}$	$\bar{2}$	$\bar{4}$	$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{1}$				
	c^8			$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{5}$	$\bar{2}$	1	2	1		
	c^{10}					$\bar{1}$	$\bar{1}$	1	1	1			
	c^{12}					1	3	4	4	2			
	c^{14}						1	2	3	2	1	1	1
x^5	c^1		1	1									
	c^3				$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$					
	c^5		$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{2}$	$\bar{1}$			
	c^7			$\bar{2}$	$\bar{4}$	$\bar{5}$	$\bar{4}$	$\bar{2}$	$\bar{2}$	$\bar{1}$			
	c^9					$\bar{1}$	$\bar{1}$	$\bar{1}$	1	2	2	1	
	c^{11}					1	2	2	4	5	4	2	
	c^{13}						1	3	4	4	4	1	
	c^{15}							1	1	1	1		
	c^{17}										$\bar{1}$		

		d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}	d^{12}
x^9	c^1			$\bar{1}$									
	c^5			1	2	2		$\bar{1}$	$\bar{1}$				
	c^7				2	3	2	1					
	c^9					1	2	3	3	3	1		
	c^{13}							$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{4}$	$\bar{2}$	
	c^{15}								$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	
x^8	c^2			$\bar{1}$	$\bar{1}$	$\bar{1}$							
	c^4			1	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{1}$					
	c^6			1	1	1		$\bar{1}$	$\bar{1}$	$\bar{1}$			
	c^8				1	3	3	4	2				
	c^{10}					1	3	3	3	4	2	1	
	c^{12}							$\bar{2}$	$\bar{3}$	$\bar{2}$		1	
	c^{14}							$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{3}$	$\bar{1}$	$\bar{1}$
	c^{16}									$\bar{1}$	$\bar{1}$	$\bar{1}$	
x^7	c^3	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$					
	c^5				$\bar{2}$	$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{1}$				
	c^7				$\bar{1}$	$\bar{1}$	$\bar{1}$	1	1				
	c^9			$\bar{1}$	$\bar{2}$	$\bar{1}$	2	5	5	3	1		
	c^{11}					1	3	4	4	4	2	1	
	c^{13}					1	2	1	$\bar{1}$	$\bar{2}$		1	
	c^{15}								$\bar{1}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	
x^6	c^0			1									
	c^2			$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$						
	c^4		$\bar{1}$	$\bar{4}$	$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{1}$					
	c^6		$\bar{2}$	$\bar{4}$	$\bar{5}$	$\bar{4}$	$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^8		$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$	1	1	1				
	c^{10}				1	2	2	4	5	4	2		
	c^{12}				1	2	2	3	4	5	3	1	
	c^{14}						1	1	1	1			
	c^{16}										$\bar{1}$	$\bar{1}$	

Table of Groundforms *.

Order in the Variables.	Deg. in coeff's of cubic.	Deg. in coeff's of quartic. 0	1	2	3	4	5
0	0			1	1		
	2				1		
	4	1	1	2	3	2	1
	6			1	3	2	1
1	1		1	1			
	3		2	3	2	1	
	5		1	2	2		

Order in the Variables.	Deg. in coeff's of cubic.	Deg. in coeff's of quartic. 0	1	2	3
2	2	1	2	2	1
	4		2	2	
3	1	1	1	1	1
	3	1	1	1	1
4	0		1	1	
	2		1	1	1
5	1		1	1	
6	0				1

System of Two Quartics.

G. F. for differentiants,

Denominator: $(1-d)(1-d^2)^2(1-d^3)(1-\delta)(1-\delta^2)^2(1-\delta^3)(1-d\delta)(1-d^2\delta)$
$(1-d\delta^2)$.

Numerator: $1+d^3+(3d+3d^2-d^4-d^5)\delta+(3d+4d^2-d^3-3d^4-2d^5-d^6)\delta^2$
$+(1-d^2-2d^4-3d^5-d^6)\delta^3+(-d-3d^2-2d^3-d^5+d^7)\delta^4$
$+(-d-2d^2-3d^3-d^4+4d^5+3d^6)\delta^5+(-d^2-d^3+3d^5+3d^6)\delta^6$
$+(d^4+d^7)\delta^7$.

G. F. for covariants, reduced form,

Denominator: $(1-d^2)(1-d^3)(1-\delta^2)(1-\delta^3)(1-d\delta)(1-d^2\delta)(1-d\delta^2)$
$(1-dx^2)(1-dx^4)(1-\delta x^2)(1-\delta x^4)$.

* The form of ord. 1, deg. 5, 4, and the two forms of ord. 2, deg. 4, 3, given by Gundelfinger, do not appear in this table, and it has been proved by the author that no fundamental forms of either of these types exist. [See below, p. 409.]

Numerator:

		δ^0	δ^1	δ^2	δ^3	δ^4	δ^5
x^0	d^0	1					
	d^2			1			
	d^4					1	
x^2	d^0		$\bar{1}$				
	d^1	$\bar{1}$	1	1	1		
	d^2		1	1			
	d^3		1		1		
	d^4						$\bar{1}$
	d^5					$\bar{1}$	
x^4	d^0			1			
	d^1		2		$\bar{1}$	$\bar{1}$	
	d^2	1		$\bar{1}$	$\bar{2}$		
	d^3		$\bar{1}$	$\bar{2}$			
	d^4		$\bar{1}$				$\bar{1}$
	d^5					$\bar{1}$	1

		δ^1	δ^2	δ^3	δ^4	δ^5	δ^6
x^{10}	d^2		1				
	d^4				1		
	d^6						1
x^8	d^1		$\bar{1}$				
	d^2	$\bar{1}$					
	d^3			1		1	
	d^4				1	1	
	d^5			1	1	1	$\bar{1}$
	d^6					$\bar{1}$	
x^6	d^1	1	$\bar{1}$				
	d^2	$\bar{1}$				$\bar{1}$	
	d^3				$\bar{2}$	$\bar{1}$	
	d^4			$\bar{2}$	$\bar{1}$		1
	d^5		$\bar{1}$	$\bar{1}$		2	
	d^6				1		

G. F. for covariants, representative form,

Denominator: $(1-d^2)(1-d^3)(1-\delta^2)(1-\delta^3)(1-d\delta)(1-d^2\delta)(1-d\delta^2)$
$(1-dx^4)(1-d^2x^4)(1-\delta x^4)(1-\delta^2 x^4)$.

Numerator:

		δ^0	δ^1	δ^2	δ^3	δ^4	δ^5	δ^6
x^0	d^0	1						
	d^2			1				
	d^4					1		
x^2	d^1		1	1	1			
	d^2		1	1	1			
	d^3		1	1	1			
x^4	d^1		1	1				
	d^2		1	1				
	d^3				1	1		
	d^4				1		$\bar{1}$	$\bar{1}$
	d^5					$\bar{1}$		
	d^6					$\bar{1}$		
x^6	d^0				1			
	d^1		1	1		$\bar{1}$	$\bar{1}$	
	d^2		1	1	$\bar{1}$	$\bar{2}$	$\bar{1}$	
	d^3	1		$\bar{1}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	
	d^4		$\bar{1}$	$\bar{2}$	$\bar{2}$			
	d^5		$\bar{1}$	$\bar{1}$	$\bar{1}$			

		δ^1	δ^2	δ^3	δ^4	δ^5	δ^6	δ^7
x^{14}	d^3			1				
	d^5					1		
	d^7							1
x^{12}	d^4				1	1	1	
	d^5				1	1	1	
	d^6				1	1	1	
x^{10}	d^1			$\bar{1}$				
	d^2			$\bar{1}$				
	d^3	$\bar{1}$	$\bar{1}$		1			
	d^4			1	1			
	d^5					1	1	
	d^6					1	1	
x^8	d^2				$\bar{1}$	$\bar{1}$	$\bar{1}$	
	d^3				$\bar{2}$	$\bar{2}$	$\bar{1}$	
	d^4		$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$		1
	d^5		$\bar{1}$	$\bar{2}$	$\bar{1}$	1	1	
	d^6		$\bar{1}$	$\bar{1}$		1	1	
	d^7				1			

Table of Groundforms *.

Order in the Variables.	Deg. in coeff's of 2d quartic.	Deg. in coeff's of 1st quartic. 0	1	2	3
0	0			1	1
	1		1	1	
	2	1	1	1	
	3	1			
2	1		1	1	1
	2		1	1	1
	3		1	1	

Order in the Variables.	Deg. in coeff's of 2d quartic.	Deg. in coeff's of 1st quartic. 0	1	2	3
4	0		1	1	
	1	1	1	1	
	2	1	1		
6	0				1
	1		1	1	
	2		1		
	3	1			

The following table exhibits the total numbers of groundforms; the quantics themselves and the absolute constant are included in the numbers†.

		Order of Quantic. 0	1	2	3	4
Order of Quantic.	0	1	2	3	5	6
	1		4	6	14	21
	2			7	16	19
	3				27	62
	4					29

* The forms of ord. 4, deg. 2, 2, and of ord. 6, deg. 2, 2, given by Gordan, do not appear in this table, and have been proved by the author to be compound forms. [See below, p. 409.]

† Some remarks on the preceding tables (to save delay in going to press) have been made the subject of a separate article in this number. [p. 406, below.]

Remarks on the Tables for Binary Quantics.

The valuable idea of using different roman letters, a, b, c, d, to correspond to the coefficients of quantics of different orders, is due to Mr Franklin. Had it occurred previously I should have employed it in the tables of the generating functions and groundforms of single quantics. The nth letter of the alphabet, say θ, will in this way symbolize the $(n+1)$ coefficients $\theta_0, \theta_1, \theta_2, \ldots \theta_n$ and so x regarded as a new point of departure in the alphabet will symbolize x_0, x_1.

I pass on to a remark of greater importance referring to the separation of the Parallelopiped which may be imagined to represent the complete tabulation of the representative G.F. to a system of two simultaneous quantics, and its use in simplifying the process of tamisage.

To fix the ideas, let us take the case of a Cubic and Quartic. Then, to represent the collected signification of the rectangles at pp. [400, 401, above]*, we may suppose a parallelopiped 12 inches in length, 17 in breadth, and 11 in depth, 12, 17, 11 being the highest exponents which appear in such rectangles of d, c, x, respectively, and confine our attention to the sign proper to each of the 12.17.11 cubical spaces (inch cubes) which may be either + or − or vacancy, if sign that may be called where sign is none. We may, if we please, imagine these cubes or cells to be filled with positive, negative or neutral electricity.

According to the chorographical law (foot-note, p. [310, above]), it ought to and would be found that the occupied portions of this parallelopiped would separate into a certain number of distinct blocks of positive and negative signs. Let us limit our attention to the first of these blocks†. The tamisage, according to the principle laid down in the remarks at the end of the preceding paper, may be limited to this block, although, as a matter of fact (and for greater assurance) in deducing the tables of

* The vacant lines and columns suppressed in the rectangular tables referred to, are supposed to be supplied.

† Planes passing through that angle of the parallelopiped at which is situated the absolute constant, may be termed the planes of reference.

In order to determine whether or not a given space or cell (as we may term it) belongs to the first block, the following is the rule to be observed: (1) If its sign is negative, it is to be rejected. (2) If three lines be drawn through its centre parallel to the edges of the parallelopiped *towards* the planes of reference, and any of these passes through a negative cell, it is to be rejected. (3) In every other case, the cell (or term which occupies it) forms a part of the primary block. So to obtain the second block required for determining the syzygants of the first species, (and notice that under a general point of view groundforms may be regarded as syzygants of species zero or on the other hand and preferably syzygants of the ith may be regarded as groundforms of the $(i+1)$th species) we may take any negative cell such that the three lines drawn through it parallel to the edges and towards the plane of reference shall not pass through any positive one. The *ensemble* of such constitute the second block. Then for the third block we may take the

groundforms, it was actually applied to all the positive terms in the 11 rectangles.

An inspection of the rectangle affected with x^7 and x^8, p. [401, above] will show that they may be omitted as forming no part of the first positive block. In the rectangle affected with x^9, it will be found that the only terms subject to examination, that is, the only terms with positive coefficients which are not *preceded* vertically or horizontally by terms with negative coefficients, are

$$\begin{array}{ccc} 2c^5d^4x^9 & 2c^5d^5x^9 & \\ 2c^7d^4x^9 & 3c^7d^5x^9 & 2c^7d^6x^9 \\ & c^9d^5x^9 & 2c^9d^6x^9 \end{array}$$

Calling any one of these terms $kc^\lambda d^\mu x^9$, it will be found, on examining the preceding rectangles, that $c^\lambda d^\mu$ occurs in one or more of them affected with a negative numerical coefficient. Consequently, these terms do not belong to the primary block, and, in like manner, it will be found that the rectangles subsequent to x^9 form no part of it.

The tamisage may therefore be confined to the rectangles belonging to $x^0, x^1, x^2, x^3, x^4, x^5, x^6$ and the only terms to be retained will be seen to be those exhibited in the following table:

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ensemble of positive cells not included in the first block and such that the lines through any one of them drawn as before shall not pass through a negative cell, and so on until all the cells are distributed into their respective blocks.

It may not be out of place to observe here that groundforms and syzygants may be regarded as existences and privations of existence, and the Fundamental Postulate so often previously quoted (on which the legitimacy of tamisage depends) is analogous to the asertion that free electricities of the two kinds cannot coexist at the same time at the same point of a body. Are there not some phenomena in electricity (certain visible effects at the poles of an electrical machine or at the extremities of the electric arc) which seem to indicate that the two electricities, although mutually quelling, are not absolutely antithetical in the sense that they might be reversed throughout an environment without any change of effect of any kind resulting? Unless this is true the analogy of the relation of Groundforms and Syzygants to Positive and Negative Electricity halts on one foot. But if it be true we may perhaps see foreshadowed in the constitution of the generating function, the possibility of physical research hereafter bringing to light residual phenomena in which freer and rarer kinds of positive and negative electricity in succession will make their appearance.

Their supposed possible prototypes as yet, play no part in any developed algebraical theory, and indeed the consciousness of only a few algebraists is as yet fully awakened to a sense of their existence. If to any one the idea of physical being foreshadowed in algebraical laws should appear extravagant and visionary, let him reflect on the certain fact that the conception of chemical units as molecules composed of atoms and of the new theory of atomicity or *valence* in each essential particular might have been safely inferred as a possible hypothesis, from the ascertained laws of the constitution and mutual actions upon one another of invariantive forms. If we only allow that the so-called laws of nature have their origin in reason and are not merely arbitrary or *fiat* laws, we can very well understand how an unfailing parallelism should exist between the phenomena of the outer world and those phenomena of the pure intelligence with which algebraical science is concerned.

	c^4d^2	$2c^4d^3$	$2c^4d^4$	c^4d^5		
	c^5d^2	$3c^5d^3$	$2c^5d^4$	c^5d^5		
	cdx	cd^2x				
	$2c^3dx$	$3c^3d^2x$	$2c^3d^3x$	c^3d^4x		
	c^5dx	$2c^5d^2x$	$3c^5d^3x$	$2c^5d^4x$	c^5d^5x	c^5d^6x
	$2c^2dx^2$	$3c^2d^2x^2$	$2c^2d^3x^2$	$c^2d^4x^2$		
	$2c^4dx^2$	$4c^4d^2x^2$	$5c^4d^3x^2$	$3c^4d^4x^2$	$c^4d^5x^2$	
	cdx^3	cd^2x^3	cd^3x^3			
c^3x^3	c^3dx^3	$3c^3d^2x^3$	$5c^3d^3x^3$	$3c^3d^4x^3$	$c^3d^5x^3$	
	c^2dx^4	$2c^2d^2x^4$	$3c^2d^3x^4$	$c^2d^4x^4$		
	cdx^5	cd^2x^5				
	d^3x^6					

Thus, it is evident at a glance that the highest order in the variables, the highest degrees in the cubic and quartic coefficients respectively, of any groundform, are 6, 4 and 5 respectively. Prior to all tamisage, 6, 4, 5 are seen to be superior limits to such order and degrees, because no powers of x, d, c figure among the above terms higher than 6, 4, 5, and a slight examination shows that some terms, containing x^6, d^4, c^5, survive the operation of the tamisage.

The number of types submitted to tamisage, it will be seen, is 45, as previously stated.

The number of forms contained under these types is 83.

The number of types absolutely abolished by the operation is 10, bringing down the number to 35; and the reduction in the total number of forms is 33, bringing down the number to 50*.

These remarks have reference *solely to the groundforms* represented by the *numerator* of the Generating Function. The denominator yields 11 groundforms, thus raising the total number to 61, which is the right number when the absolute constant is not counted *in* as the representative of an invariant†.

Possibly, when I may be again able to secure the services of Mr Franklin, without whose intelligent cooperation I believe it would have been impracticable for me to have calculated the tables contained in this and the preceding

* There is every reason to believe that a calculating machine might be constructed without difficulty for performing mechanically the process of *tamisage* whether simple (involving only a single variable) as for invariants of single forms or compound (involving several variables) as for covariants or invariants of systems.

† It should be noticed that some of the entries in the Table of Groundforms, p. [402], are made up partly from the numerator and partly from the denominator, as for example the number 3 in the column headed 3 and in the line marked 4 for the order 0, is made up partly of the 2 in the surviving term $2d^3c^4$ of the numerator and partly of a unit taken from the term $1 - d^3c^4$ of the

number of the *Journal*, I shall be able to extend the limit to the order of the combined quantics. At all events, the labour of forming the tables of the combinations of 1, 2, 3, 4, 5, 6 with 6, would probably not exceed the amount which has been incurred in calculating the groundforms of a single quantic of the 9th order. The references to the *Comptes Rendus* made in the footnotes are to Vol. LXXXIV. 1ier semestre for 1877, p. 1285, for the disproof of the existence of the *two* forms given in the accepted tables belonging to a system of two binary quartics*; to Vol. LXXXVII. 2me semestre for 1878, p. 445, and again p. 477, for the disproof of the existence of the *three* accepted superfluous forms for a system of a binary cubic and quartic†, and to Vol. LXXXIX. 2me semestre for 1879, p. 828, for the disproof of the existence of the *two* superfluous accepted forms belonging to the system of two binary cubics‡. The proof of the Fundamental Theorem is given as a Postscriptum in a paper in *Borchardt's Journal* "Sur les actions mutuelles des formes invariantives," 1878 [p. 232, above], and in a paper entitled "Proof of the hitherto undemonstrated fundamental theorem for Invariants," in the *Philosophical Magazine* for the same year, 1878 [p. 117, above].

The term *Reduced Generating Function* being apt to lead to the erroneous impression that it is obtained by reducing the representative one, whereas the representative is in fact obtained from the reduced G.F. by multiplication of its numerator and denominator by a common factor, it may be well to explain that I use the appellation *reduced* with reference to the *crude form of the generating function*, the former representing that branch, or the totality of those branches, in the development of the crude form which contain no negative powers of x.

I add a few words respecting differentiants which are simply such symmetrical functions of the roots as are complete functions of the differences of the roots of the form or system of forms to which the several tables refer.

In the G.F. for differentiants for a single quantic, the coefficient of a^j represents the total number of linearly independent differentiants of the degree j belonging to a quantic of the order i; that is, the total number of covariants of the degree j in the coefficients and of *all* orders in the variables, belonging to that quantic. The G.F. for differentiants can therefore be obtained from the G.F. for covariants (although not in its simplest form) by putting $x=1$ in the latter. In like manner, for a system of quantics, the

denominator. It is an erroneous and misleading expression into which invariantists (myself included) have fallen of speaking of a definite number, say ν, of groundforms of a certain type. The true idea is that of an unique form of that type with ν parameters. It is, so to say, a single *form* of the νth degree of plasticity or deformability or of ν dimensions in the sense in which we speak of the dimensions of space. I mean that an elastic string, an india-rubber disk and an india-rubber ball may be regarded as symbols of a groundform with one, two and three parameters respectively.

[* p. 63, above.] [† pp. 132, 136, above.] [‡ p. 258, above.]

G. F. for differentiants (or to speak more precisely, its algebraical equivalent) can be obtained from the G. F. for covariants by putting $x=1$.

To obtain the G. F. for differentiants for a single form without previously having the G. F. for covariants, we may make use of the fact that the sum of the quantities

$$(w:\ i, j)-(w-1:\ i, j)^*$$

for *all* admissible values of w is equal to the value of $(w:\ i, j)$ for the *highest* admissible value of w. Now the *order* corresponding to the highest weight is 0 or 1†; hence the number of differentiants of the degree j belonging to a quantic of the order i is the coefficient of a^jx^0 or of a^jx^1 (according as ij is even or odd) in the development of

$$\frac{1}{(1-ax^i)(1-ax^{i-2})\dots(1-ax^{-i+2})(1-ax^{-i})}.$$

The generating function for differentiants is therefore the sum of the multipliers of x^0 and x^1 in the development of the above fraction. (When the quantic is of even order, x^1 does not appear in the development, and the G. F. for differentiants is simply the part independent of x in the development.)

In like manner, for a system of two quantics, the G. F. for differentiants is the sum of the multipliers of x^0 and x^1 in the development of

$$\frac{1}{(1-ax^i)(1-ax^{i-2})\dots(1-ax^{-i})(1-ax^{i'})(1-ax^{i'-2})\dots(1-ax^{-i'})}.$$

And we may proceed in an analogous manner when a system of forms is in question. I need hardly add that a differentiant in respect to either variable, say x, is only another name for any rational integral function of the coefficients of a quantic which, when the coefficient of the highest power of the selected variable (x) in the quantic is made equal to unity, becomes a function of the differences of its $\frac{x}{y}$ roots. Gordan's and Jordan's results concerning symbolical determinants are correlative and coextensive with theorems concerning root-differences, so that the method of differentiants when fully developed would lead to the substitution of actual differences or determinants for symbolical determinants in the Gordan theory, it being borne in mind that to determine the ground-covariants of a quantic or quantic system is the same question as that of determining its ground-differentiants, inasmuch as to every covariant corresponds a single differentiant, and *vice versâ*.

* w is the weight of any covariant, j its degree in the coefficients and i the order of the quantic in the variables; and $(w:\ i, j)$ denotes the number of modes of composing w with j of the elements 0, 1, 2, 3, ... i or *vice versâ* with i of the elements 0, 1, 2, 3, ... j each any number of times repeated.

† If e is the order of the covariant in the variables $2w=ij-e$.

41.

NOTE SUR UNE PROPRIÉTÉ DES ÉQUATIONS DONT TOUTES LES RACINES SONT RÉELLES.

[*Crelle's Journal für die reine und angewandte Mathematik*, LXXXVII. (1879), pp. 217—219.]

(1) SOIT f une forme binaire $(a, b, c, \dots l \between x, y)^i$, ϕ un covariant de f de l'ordre ϵ et $F(a, b, c, \dots l)$ le coefficient de x^ϵ dans ϕ. Supposons que si dans la forme f on remplace y par $y-\frac{Y}{X}x$, les coefficients $a, b, c, \dots l$ se changent en $a', b', c', \dots l'$. Cela posé, si dans le covariant ϕ on remplace y, x par Y, X, on sait que ϕ se change en $X^\epsilon F(a', b', c', \dots l')$.

(2) Soit $(a_0, a_1, \dots a_{2\epsilon} \between x, y)^{2\epsilon}$ une forme binaire qui a toutes ses racines $\alpha_1, \alpha_2, \dots \alpha_{2\epsilon}$ (c. à d. les valeurs de $\frac{y}{x}$, qui font évanouir la forme) réelles, et soit

$$(-1)^\epsilon\left\{a_0a_{2\epsilon}-2\epsilon\,.\,a_1a_{2\epsilon-1}+\frac{2\epsilon(2\epsilon-1)}{2}a_2a_{2\epsilon-2}-\dots\right\} \qquad (1)$$

son invariant quadratique dont le signe est fixé de sorte que son dernier terme proportionnel à a_ϵ^2 ait le signe positif. Cet invariant divisé par le carré de $a_{2\epsilon}$ peut d'ailleurs, comme on sait, être présenté (à un facteur numérique près) sous la forme d'une somme de produits tels que

$$(\alpha_1-\alpha_2)^2(\alpha_3-\alpha_4)^2\dots(\alpha_{2\epsilon-1}-\alpha_{2\epsilon})^2,$$

par conséquent cet invariant est positif pour les formes à racines réelles.

Considérons à present la forme binaire $(a_0, a_1, \dots a_{2\epsilon} \dots a_{2\epsilon+\eta} \between x, y)^{2\epsilon+\eta}$ de l'ordre $2\epsilon+\eta$, qui ait également toutes ses racines réelles, alors l'expression (1) formée par rapport aux coefficients de la nouvelle forme gardera son signe positif, car en différentiant η fois de suite la nouvelle forme on retombe sur la forme binaire de l'ordre 2ϵ d'où l'on est parti.

(3) Remplaçons dans f la variable y par $y - \frac{Y}{X} x$ et supposons que Y, X soient des quantités réelles. Cette substitution ne changera en rien le caractère de la forme f relatif à la réalité de ses racines. Donc en combinant les deux observations précédentes on en conclut le résultat suivant:

Soit f une forme binaire qui a toutes ses racines réelles et ϕ un de ses covariants du second degré dans les coefficients, ϕ sera d'un signe invariable, c. à d. *si toutes les racines de f sont réelles, toutes les racines de tous les quadricovariants* (c. à d. des covariants du second degré) *de f sont imaginaires.*

Postscriptum.

M. Schramm, dans un mémoire inséré dans les *Annali di Matematica*, année 1867, avait déjà remarqué la propriété démontrée plus haut pour le cas du Hessien en se servant des fonctions covariantives en x, y, qu'il a démontré pouvoir remplacer les fonctions de Sturm en ce qui regarde la détermination du *nombre total* des racines réelles d'une équation.

M. Schramm a obtenu ces formules par une certaine transformation opérée sur celles qui portent mon nom ; mais on peut les obtenir immédiatement en se servant de la loi d'inertie pour les formes quadratiques.

En supposant que $f(x) = 0$ est une équation algébrique dont les racines sont $e_1, e_2, \dots e_n$, en écrivant

$$\Phi = \sum_{i=n}^{i=1} (\phi_1 e_i u_0 + \phi_2 e_i u_1 + \dots + \phi_n e_i u_n)^2$$

où $\phi_1, \phi_2, \dots \phi_n$ sont des fonctions rationnelles quelconques, on voit très facilement que l'inertie de Φ est égale à $n - 2\nu$, ν étant le nombre de *paires* de racines imaginaires. Si l'on pose

$$\phi_1 e = 1, \ \phi_2 e = e, \ \dots \ \phi_n e = e^{n-1},$$

la fonction quadratique Φ aura pour déterminant l'expression

$$\Delta = \begin{vmatrix} s_0 & s_1 & s_2 & \dots & s_{n-1} \\ s_1 & s_2 & s_3 & \dots & s_n \\ \dots & \dots & \dots & \dots & \dots \\ s_{n-1} & s_{n-2} & s_{n-3} & \dots & s_{2n-2} \end{vmatrix}.$$

De plus en considérant les mineurs successifs

$$s_0 \quad \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix} \quad \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} \quad \dots \quad \Delta$$

on sait que le nombre de permanences de signes dans cette série exprimera l'inertie de ϕ, c. à d. sera égal à $n - 2\nu$.

Comme on a d'ailleurs

$$s_0 = n, \quad \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix} = \Sigma (e_1 - e_2)^2, \quad \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} = \Sigma (e_1 - e_2)^2 (e_1 - e_3)^2 (e_2 - e_3)^2, \quad \dots$$

on déduit immédiatement de là la règle de Sturm pour le nombre total des racines réelles et imaginaires.

Si au lieu de poser $\phi_i e = e^{i-1}$ on posait $\phi_i e = \frac{1}{(\lambda - e)^{i-1}}$, λ étant une constante réelle quelconque, on obtiendrait de même une série de termes où le nombre de permanences serait encore égal à $n - 2\nu$.

En multipliant ces termes respectivement par des puissances paires d'un degré convenable de $(\lambda - e_1)(\lambda - e_2) \dots (\lambda - e_n)$, c. à d. par des quantités réelles et positives, on obtient les fonctions de Schramm avec cette seule différence que x et y s'y trouvent remplacées par λ et 1. Mais on fera disparaître cette différence en posant $\frac{x}{y}$ à la place de λ et en multipliant par la puissance paire de y qui rend l'expression entière.

42.

ON THE THEOREM CONNECTED WITH NEWTON'S RULE FOR THE DISCOVERY OF IMAGINARY ROOTS OF EQUATIONS.

[*Messenger of Mathematics*, IX. (1880), pp. 71—84.]

To save needless repetition in what follows I beg to refer the reader to Mr Todhunter's section 26, p. 236, in the third edition of his *Treatise on the Theory of Equations*. It will there be seen that in order to provide against any loss of double permanences consequent upon any of the f's changing sign $\gamma_1, \gamma_2, \gamma_3, \ldots \gamma_{n-1}$ must all be positive; and in order to provide against the same thing happening consequent upon any of the G's changing sign we must have, from $i=2$ to $i=n-1$ inclusive, $2-\gamma_i = \dfrac{1}{\gamma_{i+1}}$; and, moreover, $2-\gamma_{n-1}$ [denoted by $\dfrac{1}{\gamma_n}$, although strictly there is no γ_n, since G_n is simply a positive absolute], as well as $\gamma_1, \gamma_2, \ldots \gamma_{n-1}$, must be positive.

The solution of the equation $2-\gamma_i = \dfrac{1}{\gamma_{i+1}}$ is $\gamma_i = \dfrac{C+i-1}{C+i}$; and, in order that $\gamma_1, \gamma_2, \ldots \gamma_{n-1}, \gamma_n$ may all be *positive*, it is necessary that C shall be either positive or, if negative, of greater absolute value than n.

If we put $C=0$, $\gamma_1 = 0$; if we put $C=-n$, $\gamma_n = \infty$, so that, the condition of γ_i being *positive*, from 1 to n, will not in either case be complied with, the signs of zero and of infinity being ambiguous. It is well known, however, that we may put $C=-n$; in fact, $-n$ is the value ordinarily attributed to C, for the corresponding value of γ_i, namely, $\dfrac{n-i+1}{n-i}$, it is which leads to that form of the theorem in which, when we put $\mu = \infty$ and $\lambda = 0$, or $\mu = 0$ and $\lambda = -\infty$ in the equation $pP(\mu) - pP(\lambda) =$ (the number of roots between λ and μ) $+ 2i$, gives Newton's rule as stated by Newton himself. Equally, we shall find it is lawful to put $C=0$, but each of these two suppositions requires to be subjected to a special examination before its validity can be admitted. Take the much more important case first, that where $C=-n$, we

have then $\gamma_{n-1} = 2$, and the only object of $2-\gamma_{n-1}$ being positive is to prevent mischief in the event of G_{n-1}, that is, $(f_{n-1}x)^2 - 2f_{n-2}xf_nx$, changing its sign. But in this case $\dfrac{dG_{n-1}}{dx} = 0$ by simple differentiation from $\dfrac{df_nx}{dx} = 0$: in other words, G_{n-1} *is a constant* and never can change its sign. Thus, then, all necessity for $2-\gamma_{n-1}$ being positive is abolished by the very fact of its being zero.

It is worth noticing that this critical value of C, which makes $\gamma_i = \dfrac{n-i+1}{n-i}$, has the effect of lowering the degree of each G by two units; for if $\lambda = n-i+1$, we may write $f_{i-1} = px^\lambda + qx^{\lambda-1} + \ldots$, and then

$$G_i = f_i^2 - \frac{\lambda}{\lambda-1} f_{i-1} f_{i+1} = \{p\lambda x^{\lambda-1} + q(\lambda-1)x^{\lambda-2} + \ldots\}^2$$

$$+ \frac{\lambda}{\lambda-1}(px^\lambda + qx^{\lambda-1} + \ldots)\{p\lambda(\lambda-1)x^{\lambda-2} + q(\lambda-1)(\lambda-2)x^{\lambda-3} + \ldots\};$$

so that the coefficient of $x^{2\lambda-2}$ becomes

$$p^2\left\{\lambda^2 - \frac{\lambda}{\lambda-1}(\lambda^2-\lambda)\right\} = 0,$$

and that of $x^{2\lambda-3}$ becomes

$$pq\left\{2\lambda(\lambda-1) + \frac{\lambda}{\lambda-1}\lambda(\lambda-1) + (\lambda-1)(\lambda-2)\right\} = 0.$$

So again it will be found that C may be taken at the other extremity of the chasm or gap, which it is not permitted to enter; for if $C=0$ so that $g_1 = 0$, $G_1 = (f'x)^2$.

Consider now the first three terms of the double series

$$\begin{matrix} fx, & f'x, & f''x, \\ I, & I, & G_2x, \end{matrix}$$

where the two I's denote absolute positive quantities; at the moment of $f'x$ becoming zero, G_2x becomes positive, so that the succession of double permanences of sign for this double series is the same as of single permanences for $fx, f'x, f''x$, and consequently no double permanences can be lost by $f'x$ changing its sign. Since, then, we have shown that values of C giving rise to no negative but to an ambiguous sign, either of γ_1 or of γ_n, are not prohibited, it might for a moment be imagined that any negative integer value of C, say $-\omega$, lying in the gap between 0 and $-n$ might also be admissible, seeing that such value would also *not introduce any negative value of* γ, but only two values of ambiguous signs, namely, for γ_ω and $\gamma_{\omega+1}$, ∞ and 0 respectively; all the other γ's will be positive. But it will be seen that this is inadmissible, for the course of the demonstration shows that every γ_i and $2-\gamma_i$ must both be positive, which conditions cannot be fulfilled for γ_ω, whether we consider it equal to plus or minus infinity.

As I have referred to Mr Todhunter's treatise, I may notice the omission therein of the equation

$$\nu P\lambda - \nu P\mu = (\mu, \lambda) + 2i',$$

where i' is any positive integer and (μ, λ) the number of real roots between λ and μ. This may be deduced *pari passu*, and in precisely the same way as the parallel equation

$$pP\mu - pP\lambda = (\mu, \lambda) + 2i,$$

or either of these may be deduced from the other as follows. Let $fx = \phi(-x)$, and using the same parameter γ_i for the G's belonging to f and for those belonging to ϕ, let f_i, G_i for f become ϕ_i, T_i for ϕ. Then obviously

$$T_i(-c) = G_i c \text{ and } \phi_i(-c) = (-)^{n-1} f_i(c).$$

Hence, using π, Π in regard to ϕ in the same sense as p, P in regard to f, $\pi\Pi(-c) = \nu Pc$; also $(-\lambda, -\mu)$ in regard to ϕ is the same as (μ, λ) in regard to f. But remembering that if μ is greater than λ, then $-\lambda$ is greater than $-\mu$, the second equation above written applied to ϕ becomes

$$\pi\Pi(-\lambda) - \pi\Pi(-\mu) = (-\lambda, -\mu) \text{ in regard to } \phi + 2i.$$

Hence $$\nu P(\lambda) - \nu P(\mu) = (\mu, \lambda) \text{ in regard to } f + 2i,$$

as was to be shown*.

One other point deserves mentioning. If any G, say G_i, becomes incapable of changing its sign (of which G_1 becoming f_1^2 when $C = 0$, offers a particular example), the necessity for the equation $2 - \gamma_i = \frac{1}{\gamma_{i+1}}$ is done away with for that value of i, so that γ_{i+1} becomes arbitrary (within limits), and we may start with a new definition of the values of the γ's lying beyond γ_i, namely, $\gamma_{i+i'} = \frac{C' - 1 + i}{C' + i'}$ and so on, *toties quoties*, whenever in passing from G_1 to G_{n-1}, any of the G's becomes incapable of changing its sign†.

* This equation is stated in the original memoir in the *Proceedings of the Mathematical Society of London*‡. Dr Julius Petersen, of Copenhagen, in his treatise on Algebraical Equations, not having had the opportunity, as he has since informed me, of consulting this, and taking Mr Todhunter's chapter on the subject as his authority, was led to lay the fault of the omission at my door.

† Thus we see that in the expression $\gamma_r = \frac{C-1+r}{C+r}$, C is not absolutely prohibited from entering the gap comprised between 0 and $-n$, but that C may be $-i$ where i is an integer, or any quantity between $-i$ and $-\infty$, provided that G_{i-1}, that is, $f^2_{i-1} - \gamma_{i-1} f_{i-2} f_i$ is incapable of changing its sign. If $C = -i$, $\gamma_{i-1} = 2$.

As an application of the same principle we may make the γ series begin with G_2, that is, make G a positive absolute so as to have two positive absolutes instead of one positive absolute at the beginning of the series of "the Quadratic elements," that is, we may make $\gamma_1 = 0$ and $\gamma_{1+r} = \frac{C-1+r}{C+r}$, and continuing this process, $1+k$ (any number) of the initial G's may be converted into positive absolutes; that is to say, we may make $\gamma_1 = 0$, $\gamma_2 = 0$, ... $\gamma_k = 0$, $\gamma_{k+r} = \frac{C-1+r}{C+r}$.

[‡ Vol. II. of this Reprint, p. 501, *footnote*.]

It will have been noticed in what precedes, that I have made no allusion to special forms of an equation, whether absolute or having reference to the assumed arbitrary parameter in G, but have confined myself to the general case where only one term in the double series can vanish for any given value of x. Nor is it necessary to do more than this in treating the theory; for (1) if f contains no equal roots, we may, by infinitesimal or infinitely small variations attributed to the coefficients, cause those relations between them to subsist which are necessary in order that two or more of the terms may vanish simultaneously, and cannot thereby alter the character of the roots, which can only make the passage from real to imaginary, or *vice versâ*, after one or more pairs of them have passed through the state of equality; (2) if f contains equal roots, we may vary the coefficients in such a manner as not to disturb the equalities which subsist between them, and shall have independent relations enough to spare to abolish as before the relations implied in the fact of the simultaneous evanescence above referred to.

Thus it seems to me that we need trouble ourselves with the discussion of the consequences of such simultaneous evanescence only if we wish to know what inferences to draw if we are unfortunate enough to find that event occurring at one or the other of the actual limits λ, μ we may be dealing with, and for no other purpose.

Postscript.

As I was on the point of despatching what precedes by post to England, it occurred to me, in consequence of the previously unnoticed depression of the degrees of the terms in the G series, to examine more closely their constitution for the critical case, that namely where $\gamma_i = \frac{n-i+1}{n-i}$, and I have had the satisfaction of finding that every such G is proportional to the

If we make $k = n$, all the G's become positive absolutes, and *the theorem passes into Fourier's.* In connexion with this fact, it should be noticed that my theorem in its form as hitherto given does not logically contain Fourier's as a consequence; for it is possible that for certain values of λ and μ, $pP(\mu) - pP(\lambda)$ may be greater than $p(\mu) - p(\lambda)$, so that Fourier's theorem may indicate the passage of a smaller number of roots than the seemingly more stringent one; hence in applying my theorem, Fourier's should always be employed simultaneously with it, a practical direction which has hitherto been overlooked. Of course when the question concerns the total number of roots, Descartes' rule *is* logically contained in Newton's, or my generalisation of it as previously given.

It may be well to mention here, that a more general form of my theorem introducing a second arbitrary parameter will be found in some far back number of the *Educational Times* as the solution of a question proposed in a previous number. It is founded, if I recollect right, on the principle that if for the equation of the nth degree in x, say $fx = 0$, we substitute $\epsilon x^{n+\nu} + fx = 0$, where ν is any positive integer (ϵ being an infinitesimal), no new real root is introduced if ν is even, provided ϵ be taken with the right sign, and only one (of infinite value) if ν is odd. [See below: Solutions contributed by the Author to the *Educational Times*.]

Hessian of the f antecedent to it, regarded as a homogeneous function of x and 1, being that Hessian multiplied by a negative number.

To prove this I have to show that if $F(x, y)$ is of the order λ, then

$$\lambda F \frac{d^2F}{dx^2} - (\lambda - 1)\left(\frac{dF}{dx}\right)^2$$

is a positive multiple of y^2 multiplied by the Hessian of F in regard to x, y.

Now $$\lambda F = x\frac{dF}{dx} + y\frac{dF}{dy},$$

and $$(\lambda - 1)\frac{dF}{dy} = x\frac{d}{dx}\frac{dF}{dy} + y\frac{d^2F}{dy^2}.$$

Hence $$y\frac{dF}{dy} = \lambda F - x\frac{dF}{dx},$$

$$y\frac{d^2F}{dxdy} = (\lambda - 1)\frac{dF}{dx} - x\frac{d^2F}{dx^2},$$

and $$y^2\frac{d^2F}{dy^2} = (\lambda - 1)\left(\lambda F - x\frac{dF}{dx}\right) - x\frac{d}{dx}\left(\lambda F - x\frac{dF}{dx}\right)$$

$$= (\lambda^2 - \lambda)F - (2\lambda - 2)x\frac{dF}{dx} + x^2\frac{d^2F}{dx^2}.$$

Hence $$y^2\left\{\frac{d^2F}{dx^2}\frac{d^2F}{dy^2} - \left(\frac{d^2F}{dxdy}\right)^2\right\}, \text{ that is, } -y^2H(F),$$

$$= (\lambda^2 - \lambda)\frac{d^2F}{dx^2}F - (2\lambda - 2)x\frac{d^2F}{dx^2}F + x^2\left(\frac{d^2F}{dx^2}\right)^2 - \left\{(\lambda - 1)\frac{dF}{dx} - x\frac{d^2F}{dx^2}\right\}^2$$

$$= (\lambda - 1)\left\{\lambda\frac{d^2F}{dx^2}F - (\lambda - 1)\left(\frac{dF}{dx}\right)^2\right\},$$

where the least value of λ is 2 so that $\lambda - 1$ is always positive.

Thus the f and G series may be put under the following form, where f_i of course means $\frac{d^if}{dx^i}$ and $H\phi x$ signifies the Hessian of ϕ regarded as a quantic in x and 1,

$$\begin{array}{c} f : f_1 : f_2 : f_3 : \ldots : f_{n-1} : f_n, \\ -1 : Hf : Hf_1 : Hf_2 : \ldots : Hf_{n-2} : -1. \end{array}$$

I anticipate that it will be found possible to extend the theorem by the addition of a third series for the case of $n = 4$ or 5, a third and fourth for that of $n = 6$ or 7, and, in general, by the use of $\frac{1}{2}(n+2)$ or $\frac{1}{2}(n+1)$ series according as n is even or odd. And possibly it may turn out that the maximum number of series available for any given value of n will by the reckoning of the gain of complete permanences of sign (that is, treble, quadruple... permanences for 3, 4... series) as x increases from λ to μ, afford not merely a superior limit to, but the actual number of, real roots passed over in the interval.

As I find that Mr Todhunter uses a single symbol ϖ for the pP employed in my memoir in the second number of the *Proceedings of the London Mathematical Society**, it may be well to advise my readers that I use p, P to signify permanences of sign, and v, V variations of sign in the f and G series respectively; so that double permanences, permanence variations, variation permanences and variation variations would be denoted by the compound symbols pP, pV, vP, vV respectively.

The theorem above given is, I find, only a particular case of the one subjoined.

Let f_i denote $(a_0, a_1, a_2 \ldots a_i \between x, y)^i$ and $H_\epsilon(f_{i+\epsilon})$ that covariant of $f_{i+\epsilon}$ whose highest power of x bears the coefficient

$$\begin{vmatrix} a_0, & a_1, & a_2, & \ldots & a_\epsilon \\ a_1, & a_2, & a_3, & \ldots & a_{\epsilon+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_\epsilon, & a_{\epsilon+1}, & a_{\epsilon+2}, & \ldots & a_{2\epsilon} \end{vmatrix};$$

then is

$$\begin{vmatrix} f_{i-\epsilon}, & f_{i-\epsilon+1}, & f_{i-\epsilon+2}, & \ldots & f_{i+\epsilon} \\ f_{i-\epsilon+1}, & f_{i-\epsilon+2}, & f_{i-\epsilon+3}, & \ldots & f_{i+\epsilon+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_{i+\epsilon}, & f_{i+\epsilon+1}, & f_{i+\epsilon+2}, & \ldots & f_{i+2\epsilon} \end{vmatrix}$$

equal to $y^{\epsilon^2+\epsilon}H_\epsilon(f_{i+\epsilon})$.

The order in (x, y) of $H_\epsilon f_{i+\epsilon}$, since the weight of its leading coefficient is $\epsilon^2 + \epsilon$ and its degree in the coefficients $\epsilon + 1$, will be $(\epsilon + 1)(i + \epsilon) - 2(\epsilon^2 + \epsilon)$, that is, $(\epsilon + 1)i - \epsilon^2 - \epsilon$, so that multiplied by $y^{\epsilon^2+\epsilon}$ the order becomes $(\epsilon + 1)i$, as it ought to be.

The theorem may be proved as follows:

Let ϕ be any homogeneous function of λ dimensions in x, y, and denote $\frac{d}{dx}, \frac{d}{dy}$ by X, Y.

(1) I shall show that in respect of ϕ,

$$y^i \cdot Y^i = {}^i\lambda - i\,{}^{i-1}(\lambda - 1)xX + \frac{i \cdot i - 1}{2}\,{}^{i-2}(\lambda - 2)x^2X^2 \ldots + (-)^i x^i X^i,$$

where ${}^im$ for any positive integer values of m and i denotes the factorial quantity $m(m-1)\ldots(m-i+1)$.

Suppose the equation to be true for any assigned value of i, it will be true for $i + 1$. For $Y^i\phi$, it will be observed, is of $\lambda - i$ dimensions in x, y; hence

$$y^{i+1}Y^{i+1} = (\lambda - i - xX) * y^iY^i$$

[* Vol. II. of this Reprint, p. 498.]

for $(\lambda - i)\, Y^i\phi = (xX + yY)\, Y^i\phi$ by Euler's well-known theorem on homogeneous functions.

The $(j+1)$th and $(j+2)$th terms in $y^i Y^i$ are respectively

$$\mp \frac{i(i-1)\dots(i-j+1)}{1.2\dots j}(\lambda-j)(\lambda-j-1)\dots(\lambda-i+1)\, x^j X^j,$$

say

$$-Ax^j X^j$$

and $$\pm \frac{i(i-1)\dots(i-j)}{1.2\dots(j+1)}(\lambda-j-1)(\lambda-j-2)\dots(\lambda-i+1)\, x^{j+1}X^{j+1},$$

say

$$Bx^{j+1}X^{j+1}.$$

Now $$xX * x^{j+1}X^{j+1} = x^{j+2}X^{j+2} + (j+1)\, x^{j+1}X^{j+1}.$$

Hence the $(j+2)$th term in $y^{i+1}Y^{i+1}$ will be

$$A + (\lambda - i - j - 1)\, B,$$

that is, putting $$\frac{i(i-1)\dots(i-j+1)}{1.2\dots j(j+1)} = B', \text{ is } \mu B',$$

where $$\begin{aligned}\mu &= (j+1)(\lambda-j) + (i-j)(\lambda-i-1-j)\\ &= \{-j^2 + (\lambda-1)j + \lambda\} + \{j^2 - (\lambda-1)j + \lambda i - i^2 - i\}\\ &= (\lambda - i)(i+1).\end{aligned}$$

Thus the $(j+2)$th term in $y^{i+1}Y^{i+1}$ will be

$$\pm \frac{(i+1)i\dots(i+1-j)}{1.2\dots(i+1)}(\lambda-j-1)(\lambda-j-2)\dots\{\lambda-(i+1)+1\};$$

and consequently the equation is true for $i+1$.

Hence, being true for $i = 1$, it is true universally.

(2) Consider a persymmetrical determinant of the order $\epsilon + 1$ formed with the distinct constituents $\phi_0, \phi_1, \phi_2, \dots \phi_{2\epsilon}$, where ϕ_0 is a constant and in general $\frac{d}{dx}\phi_k = \pm k\phi_{k-1}$; as, for example, suppose $\epsilon = 2$, and let the determinant be

$$\begin{vmatrix} a, & ax+b, & P \\ ax+b, & P, & Q \\ P, & Q, & R \end{vmatrix},$$

where P, Q, R stand for

$$ax^2 + 2bx + c,\ ax^3 + 3bx^2 + 3cx + d,\ ax^4 + 4bx^3 + 6cx^2 + 4dx + e,$$

and $\frac{d\phi_k}{dx} = k\phi_{k-1}$. If we made $\frac{d\phi_k}{dx} = -k\phi_{k-1}$ the effect would be to change the signs of all the odd-degreed functions, but the value of the determinant would not be altered by this change. Calling the columns P_0, P_1, P_2,

$$P_0,\ P_1 - xP_0,\ P_2 - 2xP_1 + x^2P_0$$

will represent a determinant equal to the given one, but of the form

$$\begin{vmatrix} a, & b, & c \\ ax+b, & bx+c, & cx+d \\ ax^2+2bx+c, & bx^2+2cx+d, & cx^2+2dx+e \end{vmatrix},$$

and now, calling the lines L_0, L_1, L_2, the equivalent determinant

$$L_0,\ L_1 - xL_0,\ L_2 - 2xL_1 + x^2L_0$$

becomes $$\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix},$$

which is the same as if in the original form we made $x = 0$.

So in general for the order $\epsilon + 1$, calling the ϵ columns $P_0, P_1, P_2 \dots P_\epsilon$, we may pass to a new determinant by means of the combinations represented by

$$P_0,\ P_1 - xP_0,\ P_2 - 2xP_1 + x^2P_0,\ \dots\ P_\epsilon - \epsilon xP_{\epsilon-1} + \frac{\epsilon(\epsilon-1)}{2}x^2P_{\epsilon-2} \dots + (-)^\epsilon x^\epsilon P_0,$$

and calling the lines of these new determinants

$$L_0,\ L_1,\ L_2 \dots L_\epsilon,$$

$$L_0,\ L_1 - xL_0,\ L_2 - 2xL_1 + x^2L_0,\ \dots\ L_\epsilon - \epsilon xL_{\epsilon-1} + \frac{\epsilon(\epsilon-1)}{2}x^2L_{\epsilon-2} \dots + (-)^\epsilon x^\epsilon L_0,$$

will produce a determinant containing no power of x, and which is what the original one becomes on making $\epsilon = 0$.

(3) If we take for our $2\epsilon + 1$ distinct elements of the persymmetrical matrix, the quantities

$$X^{2\epsilon}\phi,\ yYX^{2\epsilon-1}\phi,\ y^2Y^2X^{2\epsilon-2}\phi,\ \dots\ Y^{2\epsilon}\phi,$$

where ϕ is of λ dimensions in x, y, we shall find by virtue of (1) that they will be represented by

$$A_0,\ A_1 - A_0x,\ A_2 - 2A_1x + A_0x^2,\ \dots$$
$$A_{2\epsilon} - 2\epsilon A_{2\epsilon-1}x + \frac{2\epsilon(2\epsilon-1)}{2}A_{2\epsilon-2}x^2 \dots + A_0x^{2\epsilon},$$

(where $\phi_k = -k\phi_{k-1}$) on making

$$A_0 = X^{2\epsilon}\phi,\ A_1 = (\lambda - 2\epsilon + 1)\, X^{2\epsilon-1}\phi,$$
$$A_2 = (\lambda - 2\epsilon + 2)(\lambda - 2\epsilon + 1)\, X^{2\epsilon-2}\phi,$$
$$\dots\dots\dots\dots\dots\dots$$
$$A_{2\epsilon} = \lambda(\lambda-1)\dots(\lambda - 2\epsilon + 1).$$

Now obviously the persymmetrical determinant in question on striking out each power of y from its several constituents will be diminished in the proportion of 1 to $y^{2+4\dots+2\epsilon}$, that is, $y^{\epsilon^2+\epsilon}$.

Hence

$$\begin{vmatrix} A_0, & A_1, & A_2, & \dots A_\epsilon \\ A_1, & A_2, & A_3, & \dots A_{\epsilon+1} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ A_\epsilon, & A_{\epsilon+1}, & A_{\epsilon+2}, & \dots A_{2\epsilon} \end{vmatrix}$$

$$= y^{\epsilon^2+\epsilon} \begin{vmatrix} X^{2\epsilon}\phi, & X^{2\epsilon-1}Y\phi, & \dots X^\epsilon Y^\epsilon \phi \\ X^{2\epsilon-1}Y\phi, & X^{2\epsilon-2}Y^2\phi, & \dots X^{\epsilon-1}Y^{\epsilon+1}\phi \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ X^\epsilon Y^\epsilon \phi, & X^{\epsilon-1}Y^{\epsilon+1}\phi, & \dots Y^{2\epsilon}\phi \end{vmatrix}.$$

This is true for any function ϕ homogeneous in x and y*.

If ϕ is a rational integral function of x, y, say

$$(a_0, a_1, a_2, \dots \between x, y)^\lambda,$$

the last written determinant becomes a covariant whose leading coefficient is the persymmetrical determinant formed with the elements $a_0, a_1, a_2, \dots a_{2\epsilon}$ multiplied by

$$\{\lambda(\lambda-1)\dots(\lambda-2\epsilon+1)\}^{\epsilon+1},$$

and if we write

$$\begin{aligned} \lambda(\lambda-1)\dots(\lambda-2\epsilon+1)B_0 &= X^{2\epsilon}\ \phi, \\ \lambda(\lambda-1)\dots(\lambda-2\epsilon\quad)B_1 &= X^{2\epsilon-1}\phi, \\ \lambda(\lambda-1)\dots(\lambda-2\epsilon-1)B_2 &= X^{2\epsilon-2}\phi, \\ \dots\dots\dots\dots & \dots\dots\dots \\ B_{2\epsilon} &= \qquad \phi, \end{aligned}$$

we shall have the persymmetrical determinant formed with the elements $B_0, B_1, \dots B_{2\epsilon}$, equal to $y^{\epsilon^2+\epsilon}$ multiplied into the covariant of which the leading coefficient is the persymmetrical determinant formed with the elements $a_0, a_1, \dots a_{2\epsilon}$, as was to be proved.

* It seems to me very likely or almost certain that every covariant of $f(x, y)$, or what becomes such when f is a quantic, may in like manner be converted into a function of f and of its derivatives in respect to one of the variables alone divided by an appropriate power of the other; and, if true, as it can hardly help being, the proof ought not to be far to seek.

It is indeed virtually contained in a formula obvious from inspection of the expression for y^iY^i in (1), namely,

$$\left\{\frac{d}{dx}(y^iY^iX^j)\right\} * = -i\{y^{i-1}Y^{i-1}X^{j+1} *\},$$

whatever homogeneous function is supposed to follow the asterisk. In connexion with this it should be observed that the determinant in (2) is bound to vanish, from the mere fact that on putting $x=0$, it becomes an invariant of $(a_0, a_1, a_2, \dots a_{2\epsilon} \between \xi, \eta)^{2\epsilon}$, and that its several terms are what the a elements become when X becomes $\xi + x\eta$. We are thus led to view the whole subject of invariance under a somewhat broader aspect, as a theory not directly concerned with quantics, but with homogeneous functions in general.

Scholium. The theory of hyperdeterminants teaches us that every in- and co-variant has its source of being in a higher existence, namely, in a pure form typified by

$$F(X^\mu\phi,\ YX^{\mu-1}\phi,\ Y^2X^{\mu-2}\phi, \dots Y^\mu\phi),$$

ϕ being a perfectly general operand, or as we may phrase it, an operand absolute. This enables me to express the idea which was struggling into light when I wrote the antecedent footnote. It is this: Let ϕ now be made to do duty for any given homogeneous function of given order λ in x, y.

The value of F will remain unaltered when we write

$$\begin{aligned} &\frac{\lambda-\mu+1}{y} \text{ in place of } Y, \\ &\frac{(\lambda-\mu+2)(\lambda-\mu+1)}{y^2} \quad \text{,,} \quad Y^2, \\ &\dots\dots\dots\dots\dots\dots \\ &\frac{\lambda(\lambda-1)(\lambda-2)\dots(\lambda-\mu+1)}{y^\mu} \quad \text{,,} \quad Y^\mu. \end{aligned}$$

This is an immediate consequence of the invariantive property of F combined with the fact that

$$\frac{d}{dx}y^iY^i = -iy^{i-1}Y^{i-1}X,$$

previously shown. The numerators in the above expressions are the first terms in the expression for y^iY^i as a function of xX modified by writing successively $\lambda-\mu+1, \lambda-\mu+2, \dots \lambda$ in place of λ on account of the powers of X which precede $Y, Y^2, \dots Y^\mu$ in F and lowering the degrees of the operands in respect to these powers by $\mu-1, \mu-2, \dots 0$ units respectively.

Thus, for example, the pure invariant

$$(X^4:)(Y^4:) - 4(X^3Y:)(XY^3:) + 3(X^2Y^2:)^2$$

where the colon (:) does duty for an operand absolute is equivalent to

$$\begin{aligned} \frac{1}{y^4}\cdot \quad &(\lambda-3)(\lambda-2)(\lambda-1)\lambda\,(X^4:)(X^0:) \\ &-4(\lambda-3)^2(\lambda-2)(\lambda-1) \quad (X^3:)(X:) \\ &+3(\lambda-3)^2(\lambda-2)^2 \qquad (X^2:)(X^2:), \end{aligned}$$

the colon now representing a homogeneous function of order λ in x, y.

So in general we may say that a *pure invariant*, or it might be more correct to say the *Schema* of an invariant, *is a function of symbolic inverses* $(X, Y, \dots)$ *to any number of letters and of any number of unconditional absolutes, possessing the property that when those absolutes become conditioned to stand for homogeneous functions of the letters, of* SPECIFIED *orders, it becomes a function of any one of the letters, of the symbolic inverses to the rest and of the absolutes so conditional.*

This property, which is certainly *necessary*, is in all probability sufficient to define a pure invariant, for I presume (nay I think it is obvious) that when it is satisfied, the only part the arbitrarily selected letter can play is that of contributing a power of itself as a factor to the function in which it figures. This definition of invariance, although it may appear abstruse, is in reality the most complete and simplest, in the sense of exemption from foreign ingredients and unnecessary specifications, that can be given, and may of course be extended without difficulty to systems of sets of letters $(x, y, \ldots)$. Nor should it be overlooked that in our great art, the *ars magna excogitandi*, a gain in expression is a gain in power*.

Returning from this rather wide excursus to our original theme of Newton's theorem, it may be useful to give the values of the G† series as far as required for equations of the 5th order inclusive corresponding to the critical value of the arbitrary parameter, that is, for the case of $C = -n$.

The given form being supposed to be $(a, b, c, \ldots \between x, y)^n$,

when $n = 2$, $-G_1 = ac - b^2$,

when $n = 3$, $-G_1 = (ac - b^2)x^2 + (ad - bc)x + (bd - c^2)$,

$$-G_2 = ac - b^2,$$

when $n = 4$, $-G_1 = (ac - b^2)x^4 + 2(ad - bc)x^3$

$$+ (ae + 2bd - 3c^2)x^2 + 2(be - cd)x + (ce - d^2),$$

$$-G_2 = (ac - b^2)x^2 + (ad - bc)x + (bd - c^2),$$

$$-G_3 = ac - b^2,$$

when $n = 5$, $-G_1 = (ac - b^2)x^6 + 3(ad - bc)x^5 + 3(ae + bd - 2c^2)x^4$

$$+ (af + 7be - 8cd)x^3 + 3(bf + ce - 2d^2)x^2 + 3(cf - de)x + (df - e^2),$$

G_2, G_3, G_4 being the G_1, G_2, G_3 of the preceding case‡.

In applying the series of these G's combined with the f series to ascertain the maximum possible number of real roots passed over in going *up* from λ

* The object of pure Physic is the unfolding of the laws of the intelligible world. ["The unseen world" belongs to another province altogether.] The object of pure Mathematic (which is only another name for Algebra) that of unfolding the laws of the human intelligence. With Geometry it fares as it was thought to be probably about to fare with a certain distant land—it is "wiped out" between the two neighbouring powers. Algebra takes for its share Geometry in the abstract. Sensible or empirical Geometry (as, thanks to the Copernican genius of Lobatcheffsky and the sublimated practical sense of Helmholtz, is now beginning to be well understood) falls into the domain of Physic.

So already Logic is divided between Psychology and Algebra; and so eventually with Grammar, whilst Linguistic is handed over to History, Psychology and Physiology; its theoretical part, the laws of syntax, declension or conjugation, regimen and collocation, must be eventually absorbed into Algebra.

† [*In line* 12 *of* p. 415 *above, the first sign should be* $-$, *not* $+$.]

‡ It is thus seen that the G series is formed of the second alliances or "überschiebungen" of the given form (made homogeneous in x, y), and of its successive derivatives each with itself; and I have great reason to believe (as already hinted) that we may append a 3rd, 4th, ... series

to μ it is proper to use simultaneously the three *independent* superior limits (1) the gain of pP's, (2) the loss of vP's, (3) the gain of p's or loss of v's, which two latter numbers are of course identical.

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by substituting the 4th, 6th, ... of such alliances in lieu of the second, filling up the vacant spaces with positive absolutes, and always reckoning the gain of the permanence-permanence-permanence...s in going up from λ to μ as one superior limit, and, as a consequence thereof, the loss of the variation-permanence-permanence...s as another. Thus, for example, for the case of $n = 4$, the series would be three in number, namely,

$$\begin{matrix} f, & f_1, & f_2, & f_3, & f_4, \\ 1, & -Hf, & -Hf_1, & -Hf_2, & 1, \\ 1, & 1, & s, & 1, & 1, \end{matrix}$$

where $s = ae - 4bd + 3c^2$ (and it may be noticed that we know from the expression for s in terms of the roots that when they are real, s must be positive).

For $n = 5$ the series would be

$$\begin{matrix} f, & f_1, & f_2, & f_3, & f_4, & f_5, \\ 1, & -Hf, & -Hf_1, & -Hf_2, & -Hf_3, & 1, \\ 1, & 1, & s, & s', & 1, & 1, \end{matrix}$$

where $$s' = ae - 4bd + 3c^2,$$

and $$s = (ae - 4bd + 3c^2)x^2 + (af - 3be + 2cd)x + (bf - 4ce + 3d^2).$$

When $n = 6$ or $n = 7$ a new series would dawn into existence, and so on continually. Thus we set a number of sieves, as it were, successively under each other; it is certain, however, that by this method we can never be assured that no more than the actual number of real roots have fallen through; but there is another method which might be studied, and is, I think, not unworthy of investigation, that is, to take for our third series the covariants of f which have for their common leading coefficient the discriminant of the form $(a, b, c, d \between x, y)^3$, for the fourth series the covariants which have for their common leading coefficient the discriminant of $(a, b, c, d, e \between x, y)^4$, and so on indefinitely, always filling up the vacant spaces with positive absolutes.

In this way I think it not improbable that the gain of compound permanences may be found to give not merely a superior limit to, but the actual number of real roots passed over in any ascent from one value of x to another.

Such a theorem, however, would have no practical value as a method for separating the roots, as its application would entail much greater labour than the ordinary Sturmian process.

43.

ON THE EXACT RELATION WHICH RESULTANTS AND DISCRIMINANTS BEAR TO THE PRODUCT OF DIFFERENCES OF ROOTS OF EQUATIONS.

[*Messenger of Mathematics*, IX. (1880), pp. 164—166.]

First, for Resultants.

Let there be two rational integral functions in x of the degrees r, s respectively; and, for greater simplicity, let the coefficients of x^r, x^s in these functions be each made equal to unity. Call ρ the roots of the one, σ of the other; and denote the product of the differences found by subtracting each σ from each ρ by $D_{\rho,\sigma}$.

Also, by the resultant $R_{r,s}$ understand that irreducible rational integral function of the coefficients, vanishing when the functions have a root in common, in which the highest power of the last coefficient of the "s" equations enters with the positive sign.

We must then have $R_{r,s} = \mu D_{\rho,\sigma}$; and it only remains to determine μ as a function of r, s.

To do this let the r function become x^r, and the s function $x^s + 1$.

For greater distinctness, suppose $r = 4$, $s = 2$.

Then, obviously, $R_{r,s}$ becomes the dialytic resultant of

$$\begin{array}{cccccc} x^5 & & & & & \\ & x^4 & & & & \\ x^5 & & +x^3 & & & \\ & x^4 & & +x^2 & & \\ & & x^3 & & +x & \\ & & & x^2 & & +1 \end{array}$$

which is equal to 1.

And in like manner for all values of r, s,

$$R_{r,s} = 1.$$

Again, $$D_{\rho,\sigma} = \{0 - (-1)^{\frac{1}{s}}\}^{rs} = (-)^{rs+r}.$$

Hence μ, which is a function of r, s exclusively, $= (-)^{rs+r}$.

Next, for Discriminants.

By the discriminant of fx of the order n, and where, for greater simplicity, the coefficient of x^n is supposed to be unity, I mean the resultant of fx and $f'x$; or, which is the same thing, of $\frac{df(x, 1)}{dx}$ and $\frac{df(x, 1)}{d1}$, when the term in which the highest power of the last coefficient in fx appears is made positive. Let this be called R_n, and the product of the squared differences of the roots Z_n; we have then $R_n = \mu Z_n$, where μ is a function of n to be determined. To find it let us take $fx = x^n - 1$.

R_n is then the resultant of nx^{n-1}, $-ny^{n-1}$, that is, is equal to

$$(-)^{n-1} n^{2n-2}.$$

Again, $$Z_n = \left.\begin{array}{l} (1-\rho) \quad (1-\rho^2) \quad \ldots (1-\rho^{n-1}) \\ (\rho-\rho^2) \quad (\rho-\rho^3) \quad \ldots (\rho-1) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (\rho^{n-1}-1)\,(\rho^{n-2}-\rho) \ldots (\rho^{n-1}-\rho^{n-2}) \end{array}\right\} \div (-)^{\frac{1}{2}(n \cdot n-1)},$$

ρ representing $\cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}$.

Hence $$Z_n = n^n \,.\, (-)^{\frac{1}{2}n(n-1)} \,.\, \{\rho^{\frac{1}{2}n(n-1)}\}^{n-1}$$
$$= (-)^\theta n^n,$$

where $$\theta = -\tfrac{1}{2}\{n(n-1)\} + (n-1)^2 = \tfrac{1}{2}\{(n-1)(n-2)\},$$

and $$\theta + (n-1) = \tfrac{1}{2}\{(n-1)\,n\}.$$

Hence $R_n = (-)^{\frac{1}{2}(n-1)n} n^{n-2} Z_n$, or $\mu = (-)^{\frac{1}{2}(n-1)n}$, which was to be found.

For ordinary algebraical investigations the determination of μ has little importance, which may account for its value being omitted in the ordinary text books; but for certain investigations concerning the numerical divisors of cyclotomic functions, with which I am occupied, I found it necessary to pay attention to the numerical part at least of this factor, and I have thought that the publication of the result might save others some unnecessary trouble.

44.

SUR LES DIVISEURS DES FONCTIONS CYCLOTOMIQUES.

[*Comptes Rendus*, xc. (1880), pp. 287—289, 345—347.]

SOIT k un nombre quelconque ; formons la série

$$\cos\lambda_1\frac{2\pi}{k},\ \cos\lambda_2\frac{2\pi}{k},\ \ldots,\ \cos\lambda_i\frac{2\pi}{k},$$

$\lambda_1, \lambda_2, \ldots, \lambda_i$ étant les $\frac{1}{2}\phi(k)$ nombres premiers à k et moindres que $\frac{1}{2}k$. Le produit de tous les facteurs $x - 2\cos\lambda\frac{2\pi}{k}$ est ce que l'on nomme une *fonction cyclotomique*, et k sera nommé son indice. En effet, la fonction cyclotomique en x à l'indice k est ce que devient le facteur primitif de $t^k - 1$ quand on le divise par $t^{\frac{1}{2}\phi(k)}$ et que l'on écrit $t + t^{-1} = x$. A l'indice 1 ou 2 ne correspond aucune fonction cyclotomique, et pour les indices 3, 4, 6, la fonction cyclotomique est linéaire, et conséquemment ne peut posséder aucune propriété arithmétique.

Je distingue les diviseurs de ces fonctions en deux classes. Les nombres qui divisent la fonction sans diviser l'indice se nomment *diviseurs extérieurs* ou *extrinsèques*, ceux qui divisent en même temps une fonction et son indice se nomment *diviseurs intérieurs* ou *intrinsèques*.

Voici les théorèmes que j'ai réussis à établir concernant ces diviseurs.

Quant à la première classe, je démontre :

1°. Que tout nombre dont les facteurs premiers diminués ou augmentés de l'unité sont divisibles par l'indice d'une fonction cyclotomique est diviseur de cette fonction. Je fais dépendre la démonstration de cette proposition du théorème suivant, qui est, pour ainsi dire, la clef de la théorie entière :

En posant $\qquad J(\cos\vartheta) = \cos(p^i\vartheta) - \cos(p^{i-1}\vartheta),$

$J(\cos\vartheta)$, *regardé comme fonction algébrique de* $\cos\vartheta$, *est divisible par* p^i *pour toute valeur réelle et entière attribuée à* $\cos\vartheta$.

La proposition précédente est une conséquence immédiate de ce théorème, quand on met $2\cos\vartheta = x$ et qu'on substitue, pour la congruence

$$J(\cos\vartheta) \equiv 0 \quad [\text{mod. } p^i],$$

la congruence équivalente

$$(t^{p^i - p^{i-1}} - 1)(t^{p^i + p^{i-1}} - 1) \equiv 0 \quad [\text{mod. } p^i];$$

de sorte que, a étant un nombre réel quelconque, il faut que l'un ou l'autre des deux facteurs $a^{p^i - p^{i-1}} - 1$, $a^{p^i + p^{i-1}} - 1$ soit toujours divisible par p^i, car, si les deux facteurs contenaient p, on aurait $a^{2p^i} - 1$ divisible par p ; c'est-à-dire, puisque $2p^i = 2 + \left(2\frac{p^i - 1}{p - 1}\right)(p-1)$, $a^2 - 1$ serait divisible par p, et conséquemment $a = \pm 1 + \lambda p$, auquel cas $a^{p^{i-1}} \equiv (\pm 1)$ mod. p^i, et les deux facteurs deviennent respectivement congrus à $(\pm 1)^{p^i \pm p^{i-1}} - 1$, c'est-à-dire tous les deux congrus à zéro par rapport à ce module, et par conséquent tous les deux divisibles par p^i et congrus à zéro. Avec l'exception de ces valeurs de a, c'est toujours l'un des deux facteurs exclusivement qui s'évanouit pour une valeur donnée de a.

2°. Je démontre, à l'aide du même théorème de forme trigonométrique, mais en faisant $i = 1$, que si un diviseur extérieur d'une fonction cyclotomique, disons ψ_k, est de la forme $mk \pm e$, k étant son indice, la congruence

$$\psi_k \equiv 0 \quad [\text{mod. } mk \pm e]$$

aura deux racines congrues l'une à l'autre, à moins que $e = 1$. On prouve facilement que cette équivalence est impossible avec l'aide du petit principe additionnel que, si ψ est congru à zéro selon un module quelconque, $\frac{d\psi}{dx}$ sera congru à zéro selon le même module.

Quant à la seconde classe des diviseurs, je démontre que, laissant à part les fonctions cyclotomiques linéaires $x + 1$, x, $x - 1$ appartenant aux indices 3, 4, 6 et la fonction quadratique qui répond à l'indice 12, il n'y a *au plus* qu'*un seul* diviseur intérieur (un nombre premier) ; bien entendu, la première puissance seulement de ce nombre. J'ai déjà dit que, pour que p^j soit un diviseur extérieur, il faut et il suffit que $p = mk + \epsilon$, k étant l'indice et $\epsilon = \pm 1$. Or, pour que p soit diviseur intérieur de la fonction cyclotomique à l'indice k, je démontre qu'il faut et qu'il suffit que k soit de la forme

$$\frac{p - \epsilon}{m} p^j.$$

En général, il n'y a *au plus* qu'une seule manière de mettre un indice k, donné sous la forme qui met en évidence un diviseur intérieur ; mais, quand $k = 12$, on peut écrire $m = 1$, $j = 2$, $p = 2$, $\epsilon = -1$ ou bien $m = 1$, $j = 1$, $p = 3$, $\epsilon = -1$; c'est pourquoi ψ_{12} possède les *trois* diviseurs intérieurs 2, 3, 6. En démontrant que la condition donnée plus haut pour que p soit diviseur intérieur est nécessaire et que la première puissance seulement de p est un diviseur de la fonction, je me sers du même théorème trigonométrique qu'auparavant et en même temps de la seconde proposition sur les facteurs extérieurs. Pour démontrer que cette condition est suffisante, j'ai recours à un théorème purement algébrique, savoir, que si $k=k_1(mk_1 \pm 1)^j$, $mk_1 \pm 1$ étant un nombre premier p, le résultant des deux équations $\psi_k=0$, $\psi_{k_1}=0$ est égal à $p^{\frac{1}{2}\phi(k_1)}$, en me servant en même temps d'un second petit principe, qu'afin que deux congruences soient satisfaites simultanément par rapport au même module, le résultant algébrique de ces congruences transformées en équations doit être congru à zéro par rapport au module.

La fonction cyclotomique à l'indice 9, x^3-3x+1, m'a amené à faire cette recherche; car j'avais grandement besoin de démontrer apodictiquement (ce que j'avais établi par des épreuves numériques sans fin) que les diviseurs de cette fonction sont 3 et les nombres premiers de la forme $18n \pm 1$ exclusivement. C'est à l'aide de ce théorème que je démontre qu'aucun nombre A de la forme*

$$pq,\ p^2q^2,\ p_1p_2^2,\ q_1q_2^2;\quad 9pq,\ 9p^2q^2,\ 9p_1p_2^2,\ 9q_1q_2^2,$$

où chaque p désigne un nombre premier de la forme $18n-5$ et chaque q un nombre premier de la forme $18n+7$, ne peut être décomposé en une somme ou différence de deux cubes rationnels. En effet, je démontre facilement que, si cette décomposition était possible, l'équation

$$x^3-3xy^2+y^3=3Az^3$$

serait résoluble en nombres entiers, ce qui est impossible, puisque x^3-3x+1 ne contient aucun p ou q. La même équation, en mettant $A=3$, devrait avoir lieu aussi si 3 était décomposable en deux cubes rationnels; ainsi on voit (comme on sait déjà) que cette décomposition est impossible, puisque x^3-3x+1 ne contient pas le diviseur intérieur 9.

Tout ce que j'ai pu trouver sur la question qui a fait le sujet de ma première Communication † est contenu dans le livre classique du professeur Bachmann, *Die Lehre von der Kreistheilung*‡, Leipzig, 1872, pp. 242, 243; mais cela même ne me servait à rien, car cet excellent auteur s'est borné au cas où l'indice est un nombre premier, pour lequel cas il énonce et démontre "qu'en dehors des diviseurs premiers de la forme $2mp \pm 1$" la fonction cyclotomique à l'indice p "contient seulement le diviseur premier p"; mais M. Bachmann n'a nullement démontré ni même affirmé, ce qui cependant est vrai, que tout nombre premier de la forme $2mp \pm 1$, et même un tel nombre élevé à une puissance quelconque *, est diviseur de la fonction cyclotomique à l'indice p.

Reste une remarque à faire. Si l'on prend le produit des facteurs $x-2\cos\lambda\frac{2\pi}{k}y$, on obtient ce qu'on peut nommer une *forme* cyclotomique. Quand on prend l'indice égal à 5 ou à 10, à 8 ou à 12, de sorte que l'ordre de cette forme, disons $F(x, y)$, devient 2, si D est un diviseur quelconque de la fonction cyclotomique à ces indices, on sait, par la théorie ordinaire des

[* See below, p. 437.]

† *Comptes Rendus*, séance du 16 février [p. 428, above.]

‡ *Kreistheilung*=cyclotomie. La fonction à racines réelles qui sert à la division du cercle en parties égales est celle que j'ai nommée *fonction cyclotomique*. Il y a aussi des fonctions cyclotomiques à racines imaginaires; je parle des facteurs primitifs de x^k-1, qu'on pourrait nommer *fonctions cyclotomiques simples* ou *irréduites*, dont les diviseurs sont assujettis à des conditions parallèles, mais non identiques avec celles des fonctions cyclotomiques que j'ai traitées dans le texte. En effet, voici la règle pour les diviseurs des fonctions cyclotomiques non réduites. Afin qu'un nombre quelconque soit diviseur d'une fonction cyclotomique non réduite à l'indice k, il faut et il suffit que chaque facteur premier de ce diviseur soit de la forme $ki+1$, avec exception d'un seul facteur premier p qui peut figurer aussi comme facteur du diviseur dans le cas, et seulement dans le cas, que k admet de la représentation (nécessairement et sans exception unique) $\frac{p-1}{m}p^j$. Ainsi, si P, p désignent des nombres premiers, J, j des nombres indéfinis, et k l'indice d'une fonction cyclotomique de l'une ou de l'autre espèce, et si

$$P=mk+\epsilon \text{ et } k=\frac{p-\epsilon}{m}p^j,$$

P^J et p seront diviseurs de la fonction dans un cas et dans l'autre, avec la distinction que pour les fonctions cyclotomiques simples $\epsilon=1$, tandis que pour les fonctions cyclotomiques à racines réelles $\epsilon=\pm 1$. En effet, le cours de la démonstration est précisément le même dans les deux cas, avec la seule exception que pour la première proposition, celle qui affirme que, p étant un nombre premier de la forme $mk+\epsilon$, p^j est diviseur de la fonction à indice k, pour les fonctions cyclotomiques d'une classe on se sert du théorème que la congruence $\cos p^j\vartheta - \cos p\vartheta \equiv 0$ [mod. p^j] a toutes ses racines réelles; pour les fonctions cyclotomiques de l'autre classe on se sert du théorème (mieux connu) que *la congruence*

$$x^{p^j}-x^{p^{j-1}} \equiv 0 \quad [\text{mod. } p^j]$$

a toutes ses racines réelles. Pour tout ce qui suit cette proposition, la méthode de démonstration pour les deux cas est absolument identique. Peut-être serait-il mieux de nommer les fonctions dont je parle spécialement dans le texte *fonctions cyclotomiques de la seconde*, et celles qui sont simplement facteurs primitifs de la forme binôme *fonctions cyclotomiques de la première espèce.* Il y a une raison qui me paraît assez grave pour ce changement de nomenclature, vu qu'il suggère l'idée d'une théorie de diviseurs des fonctions cyclotomiques dont le *rang de l'espèce* sera un nombre q quelconque, où figureront les racines $q^{\text{ièmes}}$ de l'unité, par rapport à l'indice comme module, de laquelle théorie je crois entrevoir assez distinctement et la haute probabilité de son existence et sa nature. J'espère développer cette théorie dans quelque futur Mémoire.

* Il est à peine nécessaire d'observer que la fonction cyclotomique de l'ordre ω [où $\omega=\frac{1}{2}\phi(k)$] étant divisible pour ω valeurs a de la variable incongrues par rapport à p^α, et ω autres valeurs b de la même variable incongrues par rapport à q^β, par p^α, q^β respectivement, on n'a qu'à combiner un a quelconque avec un b quelconque, et, en écrivant $p^\alpha u-a=t=q^\beta v-b$, on obtiendra une valeur réelle de t (et conséquemment ω valeurs réelles de t), qui substituée pour la variable rendra la fonction divisible par $p^\alpha q^\beta$; et de même on déduit que la fonction admettra comme diviseur un nombre quelconque dont les facteurs sont les nombres premiers de la forme $mk \pm 1$ accompagnés ou non (au choix) par le facteur intrinsèque, quand il y en a un, et par l'un ou l'autre ou tous les deux facteurs intrinsèques 2, 3, dans le cas où l'indice est le nombre 12.

formes quadratiques, qu'en écrivant $F(x, y) = Dz^2$ (les valeurs de F étant $x^2 \pm xy - y^2$, $x^2 - 2y^2$ ou $x^2 - 3y^2$), une telle équation est résoluble en nombres entiers.

Or une étude empirique très étendue sur le cas où l'indice est 9, qui mène à l'équation $x^3 - 3xy^2 + y^3 = Dz^3$, m'a donné lieu de croire qu'il y a une probabilité très considérable que cette équation est aussi toujours résoluble en nombres entiers. Si cela était établi, il deviendrait plus que probable que le théorème analogue est vrai pour toutes les formes cyclotomiques, et du cas de l'indice 9, si seulement la résolubilité de l'equation qui y appartient était démontrée, on tirerait la belle conséquence que tout nombre dont les facteurs premiers sont de la forme $18n \pm 1$, accompagné ou non accompagné (au choix) par le facteur 9, est décomposable en une somme de cubes de deux nombres rationnels. Car on démontre facilement qu'en substituant pour X, Y, Z, respectivement, certaines fonctions rationnelles et entières qu'on connaît, du neuvième degré en x, y, z, la fonction $X^3 + Y^3 + AZ^3$ contiendra

$$x^3 - 3xy^2 + y^3 - 3Az^3$$

comme facteur algébrique.

Voici, en quelques mots, le résumé des lois actuellement démontrées :

Tout diviseur de la fonction cyclotomique à l'indice k est de la forme $ik \pm 1$, excepté dans le cas que $k = \frac{p \mp 1}{m} p^j$, dans lequel cas p aussi (mais non pas p^2) sera un diviseur. Et réciproquement tout nombre dont les facteurs sont des puissances arbitraires de nombres premiers de la forme $ik \pm 1$ est diviseur de la fonction cyclotomique à l'indice k.

On peut y ajouter que, si l'ordre de la fonction cyclotomique [c'est-à-dire $\frac{1}{2}\phi(k)$] est nommé ω, et N un nombre quelconque qui ne divise pas k, il n'y aura aucune valeur ou ω valeurs de la variable, incongrues par rapport à N, qui rendront la fonction divisible par N. Mais si p, nombre premier, est un diviseur de k, le nombre des valeurs de la variable qui rendent la fonction divisible par p sera ou nul ou le quotient de k par la plus haute puissance qu'il contient de p.

45.

SUR LA LOI DE RÉCIPROCITÉ DANS LA THÉORIE DES NOMBRES.

[*Comptes Rendus*, xc. (1880), pp. 1053—1057, 1104—1106.]

Soit $\left(\frac{Q}{P}\right)$ le symbole bien connu de Jacobi, généralisation du symbole $\left(\frac{Q}{p}\right)$ de Legendre. Selon que $\left(\frac{Q}{P}\right) = +1$ ou -1, je dirai que l'aspect quadratique ou simplement l'aspect de Q vers P est positif ou négatif. On accorde que Q et P peuvent l'un et l'autre être ou positifs ou négatifs, avec la convention que $\left(\frac{Q}{-P}\right) = \left(\frac{Q}{P}\right)$ et $\left(\frac{Q}{1}\right) = 1$. Alors il est plus ou moins distinctement reconnu que, Q, P étant tous les deux nombres impairs et relativement premiers, si Q et P ne sont pas tous les deux négatifs, $\left(\frac{Q}{P}\right)\left(\frac{P}{Q}\right) = 1$ quand Q et P *ne sont pas*, et $= -1$ quand Q et P *sont*, tous les deux de la forme $4m + 3$.

Mais, si Q et P sont tous les deux négatifs, $\left(\frac{Q}{P}\right)\left(\frac{P}{Q}\right) = -1$ quand Q et P *ne sont pas*, et $= 1$ quand Q et P *sont*, tous les deux de la forme $4m + 3$.

Servons-nous du mot *reste quaternaire* pour exprimer le reste minimum absolu d'un nombre impair par rapport au module 4. Ce reste sera ou $+1$ ou -1. Servons-nous aussi, en général, du symbole $\binom{m}{n}$ ou $\binom{n}{m}$ pour signifier un nombre qui est -1 quand m et n sont tous les deux négatifs et $+1$ dans le cas contraire. Soient a, b deux nombres quelconques positifs ou négatifs, impairs et relativement premiers, a' et b' leurs restes quaternaires ; alors, en vertu des théorèmes précédents, on aura

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = \binom{a}{b}\binom{a'}{b'},$$

formule qui constitue le véritable théorème de réciprocité et suffit à elle-même comme formule universelle de réduction, sans avoir besoin de supplément (*Ergänzung*) aucun.

Je nomme, en général, *chaîne réductive* une suite de chiffres positifs ou négatifs dont le dernier est l'unité positive ou négative et dont chaque terme intermédiaire est un diviseur de la différence de ses deux termes voisins; une telle suite se nomme *chaîne réductive impaire* quand tous les termes sont impairs. Il est évident qu'on peut toujours former une chaîne réductive impaire dont les deux premiers termes sont des nombres impairs donnés, car dès le second terme on peut trouver des termes continuellement décroissants qui rempliront les conditions imposées.

Or je dis que, pour trouver la valeur de $\left(\frac{b}{a}\right)$, on n'a qu'à former une chaîne réductive impaire commençant avec a, b et une chaîne auxiliaire dont les termes sont les résidus quaternaires des termes de la première; alors, selon que la somme des nombres des permanences des signes *moins* prises dans une suite et dans l'autre est paire ou impaire, l'*aspect* de b vers a sera positif ou négatif.

En voici la preuve. Soient

$$a,\ b,\ c,\ d,\ \ldots,\ h,\ k,\ l,$$
$$a',\ b',\ c',\ d',\ \ldots,\ h',\ k',\ l',$$

la première une suite réductive impaire et la seconde une suite auxiliaire formée avec les restes quaternaires de l'autre. Alors on aura

$$\left(\frac{b}{a}\right)=\binom{b}{a}\binom{b'}{a'}\left(\frac{a}{b}\right)=\binom{b}{a}\binom{b'}{a'}\left(\frac{c}{b}\right),$$
$$\left(\frac{c}{b}\right)=\binom{c}{b}\binom{c'}{b'}\left(\frac{b}{c}\right)=\binom{c}{b}\binom{c'}{b'}\left(\frac{d}{c}\right),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\left(\frac{k}{h}\right)=\binom{k}{h}\binom{k'}{h'}\left(\frac{h}{k}\right)=\binom{k}{h}\binom{k'}{h'}\left(\frac{l}{k}\right),$$
$$\left(\frac{l}{k}\right)=\binom{l}{k}\binom{l'}{k'}\left(\frac{k}{l}\right)=\binom{l}{k}\binom{l'}{k'}.$$

Donc

$$\begin{aligned}\left(\frac{b}{a}\right)&=\binom{b}{a}\binom{c}{b}\cdots\binom{k}{h}\binom{l}{k}\\&\times\binom{b'}{a'}\binom{c'}{b'}\cdots\binom{k'}{h'}\binom{l'}{k'}\\&=(-1)^{n+n'},\end{aligned}$$

n étant le nombre de fois que les successions a, b; b, c; ...; h, k; k, l contiennent deux signes $-$ et n' le nombre correspondant pour a', b'; b', c'; ...; h', k'; k', l'; c'est-à-dire l'aspect de b vers a sera positif ou négatif, selon que $n+n'$ (que je nommerai ν) est pair ou impair, ce qui était à démontrer.

Je ferai l'application de cette méthode de calculer le symbole $\left(\frac{b}{a}\right)$ à des exemples tirés du Traité (*Zahlentheorie*) de Lejeune-Dirichlet. Pour trouver $\left(\frac{195}{1901}\right)$, on forme la chaîne réductive

$$\begin{array}{cccc} + & + & - & - \\ 1901 & 195 & 49 & 1, \end{array}$$

qui donne la chaîne auxiliaire

$$\begin{array}{cccc} + & - & - & - \\ 1 & 1 & 1 & 1. \end{array}$$

On a donc $n=1$, $n'=2$, $\nu=n+n'=3$; conséquemment $\left(\frac{195}{1901}\right)=-1$, et, puisque 1901 est nombre premier, 195 est non-résidu quadratique de ce nombre. Pour trouver $\left(\frac{74}{101}\right)=\left(\frac{-27}{101}\right)$, on obtient les deux chaînes (omettant dans la seconde le chiffre constant 1)

$$\begin{array}{cccc} + & - & - & + \\ 101, & 27, & 7, & 1; \\ + & + & + & + \end{array}$$

$\nu=1+0=1$, et, comme auparavant, 74 est non-résidu au nombre premier 101.

Si $b>a$, les suites prendront la forme

$$a,\ b,\ a,\ d,\ \ldots,\ l,$$
$$a',\ b',\ a',\ d',\ \ldots,\ l',$$

et, puisque la somme des permanences négatives dans aba et $a'b'a'$ est évidemment 0, 2 ou 4, on peut faire abstraction de ces parties de la chaîne double dans le calcul. Ainsi, par exemple, on aura pour $\left(\frac{27}{103}\right)$

$$\begin{array}{ccccc} + & + & - & - & + \\ 103, & 27, & 5, & 3, & 1, \\ - & - & - & + & + \end{array}$$

et pour $\left(\frac{103}{27}\right)$

$$\begin{array}{cccc} + & - & - & + \\ 27, & 5, & 3, & 1. \\ - & - & + & + \end{array}$$

Comme dernier exemple, je trouverai la valeur générale de $\left(\frac{2}{k}\right)$, c'est-à-dire de $\left(\frac{2-k}{k}\right)$. Si l'on donne à n les valeurs 1, 3, 5, 7, on obtient les chaînes doubles

$$\begin{array}{c} + \\ 1; \\ + \end{array}\quad \begin{array}{cc} + & - \\ 3, & 1; \\ - & - \end{array}\quad \begin{array}{ccc} + & - & - \\ 5, & 3, & 1; \\ + & + & - \end{array}\quad \begin{array}{cccc} + & - & - & + \\ 7, & 5, & 3, & 1; \\ - & - & + & + \end{array}$$

et, en général, pour $n=2i+1$, 3, 5, 7, on trouvera très facilement que les valeurs des quatre chaînes doubles de signes qui y correspondent seront

$$\begin{pmatrix}+ & - & - & +\\ + & + & - & -\end{pmatrix}^{i}\begin{matrix}+\\+\end{matrix};\quad \begin{pmatrix}+ & - & - & +\\ - & - & + & +\end{pmatrix}^{i}\begin{matrix}+ & -\\ - & -\end{matrix};$$
$$\begin{pmatrix}+ & - & - & +\\ + & + & - & -\end{pmatrix}^{i}\begin{matrix}+ & - & -\\ + & + & -\end{matrix};\quad \begin{pmatrix}+ & - & - & +\\ - & - & + & +\end{pmatrix}^{i}\begin{matrix}+ & - & - & +\\ - & - & + & +\end{matrix},$$

où l'indice supérieur i signifie que les signes contenus dans les parenthèses doivent être i fois répétés. Il est à remarquer que dans ces suites répétées de quatre signes il n'arrive jamais que le premier et le dernier signe sont tous les deux négatifs; de sorte qu'on n'obtiendra aucune permanence négative à la jonction de deux de ces suites.

On aura donc la somme des permanences négatives pour ces quatre cas égale à

$$2i,\ 2i+1,\ 2i+1,\ 2i+2,$$

respectivement: de sorte que l'aspect de 2 vers $8i+1, 7$ est positif et vers $8i+3, 5$ négatif: résultat qu'on a ainsi déduit avec l'aide de la seule formule de réduction pour les nombres impairs.

Il est digne de remarque que, puisque $\left(\frac{b}{a}\right)=\left(\frac{b}{-a}\right)$, il s'ensuit que, si, dans une série réductive impaire quelconque et la série de ses restes quaternaires, on change simultanément le signe des termes alternés en commençant avec le premier terme en chacune, la somme des permanences des signes négatifs sera augmentée ou diminuée par un nombre pair.

Il y a tant d'analogie entre la méthode exposée dans un précédent article et celles qu'on emploie dans les théorèmes de Newton et Fourier sur les racines réelles des équations algébriques, qu'on se sent très porté à soupçonner que le nombre que j'ai nommé ν est la limite supérieure à quelque affection de a, b à laquelle elle reste toujours congrue par rapport au module 2; mais de la nature de cette affection, si toutefois elle existe, je n'ai nulle connaissance.

De même qu'on a trouvé une expression générale pour l'aspect de $2-k$ vers k, on peut, avec l'aide du théorème de la chaîne, construire, d'une infinité de manières, des fonctions algébriques de k, dont on saura d'avance les aspects des unes vers les autres. Ainsi, pour prendre un exemple très simple, formons la série

$$1,\ 2,\ 5,\ 12,\ 29,\ 70,\ 169,\ 408,\ 985,\ 2378,\ \ldots,$$

où

$$u_k=2u_{k-1}+u_{k-2},\quad u_1=2,\quad u_0=1,$$

et conséquemment

$$u_k=2^k+(k-1)\,2^{k-2}+\frac{(k-1)(k-3)}{1.2}\,2^{k-4}+\ldots.$$

On peut se demander l'expression générale pour l'*aspect quadratique* de u_{2i-1} vers u_{2i} pour une valeur quelconque de i.

On trouvera sans peine que les suites de signes qui donnent les valeurs de $(\frac{2}{5})$, $(\frac{12}{29})$, $(\frac{70}{169})$, $(\frac{408}{985})$ sont

$$\begin{matrix}+ - -\\ + + -\end{matrix};\quad \begin{matrix}+ - - + +\\ + - - - +\end{matrix};\quad \begin{matrix}+ - - + + - -\\ + + - + + + -\end{matrix},$$

$$\begin{matrix}+ - - + + - - + +\\ + - - - + - - - +\end{matrix};$$

et, en général, que $\left(\frac{u_{4i+1}}{u_{4i+2}}\right)$ donne naissance à la chaîne double

$$\begin{pmatrix}+ & - & -\\ + & + & -\end{pmatrix}\begin{pmatrix}+ & + & - & -\\ + & + & + & -\end{pmatrix}^i$$

et $\left(\frac{u_{4i-1}}{u_{4i}}\right)$ à

$$\begin{pmatrix}+\\ +\end{pmatrix}\begin{pmatrix}- & - & + & +\\ - & - & - & +\end{pmatrix}^i.$$

Dans le premier cas, ν est égal à $i+1$, et dans le second à $3i$; ainsi les valeurs successives de ν étant 1, 3, 2, 6, 3, 9, 4, 12, 5, ..., l'aspect de u_{8j+1} à u_{8j+2} et de u_{8j+3} à u_{8j+4} est positif, mais de u_{8j+5} à u_{8j+6} et de u_{8j+7} à u_{8j+8} négatif.

Dans le *Zahlentheorie* de Lejeune-Dirichlet, rédigé par M. Dedekind (3ᵉ édition, p. 110; Braunschweig, 1879), on rencontre cette phrase: "Es zeigt sich nun, dass die damals nothwendige Zerlegung in Primzahlfactoren (abgesehen von dem Factor 2) ganz überflüssig geworden." Ce qui précède ici rend évident (il me semble) que cette exclusion du nombre 2 (due probablement à quelque mésintelligence de la part des auditeurs de l'illustre Dirichlet) est elle-même (*überflüssig*) superflue.

Je profite de cette occasion pour corriger la liste que j'ai donnée dans une Note précédente des nombres qu'on démontre, par le moyen des diviseurs de x^3-3x+1, être indécomposables dans une somme de cubes rationnels. Dans cette liste*, $9pq$, $9p_1p_2^2$, $9q_1q_2^2$, $9p^2q^2$ étaient insérés par erreur; la démonstration, en un seul coup, de l'irrésolubilité des seize formes générales qui restent a paru† dans le dernier fascicule de l'*American Journal of Mathematics.*

Post-scriptum.—Dans les exemples très nombreux que j'ai calculés de l'application de mon algorithme pour déterminer l'aspect de Q vers P, j'ai toujours trouvé que la *différence* δ de n et n' (les nombres de permanences négatives dans les deux suites), prise positivement, est une *limite inférieure* au nombre de cas où q est non-résidu de p (q étant un facteur premier quelconque de Q et p de P).

Si cette remarque est démontrée de validité universelle, elle fournira un moyen de mettre à l'épreuve, d'une infinité de manières, si un nombre donné P est un nombre premier. Car, en combinant P avec un nombre premier arbitraire Q, si δ est plus grand que 1, P, devant contenir au moins δ facteurs auxquels Q est non-résidu, sera nécessairement un nombre composé. Au contraire, quand P est nombre premier, δ sera toujours ou 0 ou 1, selon la valeur de Q, ce qui constituerait un théorème nouveau sur le symbole $\left(\frac{q}{p}\right)$ de Legendre.

[* Above, p. 430.] [† Above, p. 347.]

46.

SUR LES ÉQUATIONS À 3 ET À 4 PÉRIODES DES RACINES DE L'UNITÉ.

[*Compte Rendu de la Association Française* (1880), *Reims*, pp. 96—98.]

DÉSIGNONS par p l'ordre des racines de l'unité,
„ „ e le nombre des périodes,
„ „ $\eta_0, \eta_1, \eta_2, \ldots, \eta_{e-1}$, les périodes elles-mêmes;
„ „ $\omega_0, \omega_1, \omega_2, \ldots, \omega_{e-1}$ les périodes que j'appelle affectées et définies par l'équation $\omega = e\eta + 1$.

§ 1.

I. On prouve facilement que l'on a

$$\Sigma\eta^i \equiv \left(\frac{p-1}{e}\right)^{e-1} \pmod{p},$$

i étant un nombre entier quelconque.

II. On en conclut $\qquad \Sigma\omega = 0,$

et aussi $\qquad \Sigma\omega^i \equiv 0 \pmod{p}.$

III. On démontre facilement que

$$\Sigma\omega_0\omega_1 = \tfrac{1}{2}\Sigma\omega^2 = -\frac{e}{2}\,p, \text{ lorsque } \frac{p-1}{2} \text{ est impair;}$$

$$= \frac{e^2-e}{2}\,p, \text{ lorsque } \frac{p-1}{2} \text{ est pair.}$$

IV. Enfin, on démontre encore les relations

$$3\Sigma\omega_0\omega_1\omega_2 = \Sigma\omega^3 \equiv 3 \pmod{9}, \quad \text{pour } e = 3;$$
$$3\Sigma\omega_0\omega_1\omega_2 \equiv 8 \qquad \pmod{24}, \quad \text{pour } e = 4.$$

§ 2.

Si une fonction rationnelle et entière des périodes ne change pas de valeur par une substitution circulaire, on sait que cette fonction est nécessairement un nombre réel. Il est également vrai et on démontre facilement que si une telle fonction change de signe, mais conserve sa valeur absolue par une substitution circulaire, elle est un multiple entier de $\sqrt{p}$ ou de $\sqrt{(-p)}$, selon que $\frac{p-1}{e}$ est pair ou impair.

Ainsi par exemple, le produit des différences des périodes est un nombre entier lorsque e est impair, et un multiple entier de $\sqrt{(\pm p)}$ lorsque e est pair. De même, dans le cas de 4 périodes, le produit $(\eta_0 - \eta_2)(\eta_1 - \eta_3)$ est un multiple de $\sqrt{p}$.

§ 3.

La forme de l'équation à 3 périodes affectées en vertu de (II) et de (III) sera

$$\omega^3 - 3p - Ap = 0$$

dont le discriminant est $4A^2p^3 - p^3$; donc, en vertu de § 2,

$$\frac{p^3 - 4A^2p^3}{27} = M^2 = B^2p^2;$$

donc $4p = A^2 + 27B^2$; ainsi A^2 est déterminé. De plus, $3Ap$ en vertu de (III) sera congru à 3 (mod. 9), c'est-à-dire Ap, et conséquemment A sera congru à $+1$ (mod. 3): donc A est parfaitement déterminé.

V. Pour $e = 4$ on voit facilement que les équations dont $\omega_0, \omega_2, \omega_1, \omega_3$ sont les racines seront de la forme

$$\omega^2 - 2\sqrt{(p)}\,\omega + Ap - B\sqrt{p} = 0,$$
$$\omega^2 + 2\sqrt{(p)}\,\omega + Ap + B\sqrt{p} = 0,$$

où A et B seront des nombres entiers, et en vertu du § 2 on aura

$$\{(A-1)\,p + B\sqrt{p}\}\,\{(A-1)\,p - B\sqrt{p}\},$$

c'est-à-dire

$$(A-1)^2p^2 - pB^2 = m^2p.$$

Selon que $\frac{p-1}{2}$ est pair ou impair

$$2A = -\frac{e}{2} = -2 \text{ ou } \frac{e^2-e}{2} = 6;$$

ainsi dans l'un et l'autre cas $(A-1)^2 = 4$, c'est-à-dire $4p = B^2 + m^2$, de sorte que si $p = f^2 + g^2$, $B = 2f$ et $m = 2g$. Donc, pour les deux cas respectivement, l'équation aux périodes affectées sera

$$(\omega^2 - p)^2 - 4p\,(\omega + 2f)^2 = 0, \quad (\omega^2 + 3p)^2 - 4p\,(\omega + 2f)^2 = 0,$$

et avec l'aide de (IV) on trouve facilement que $f \equiv -1$ (mod. 4), de sorte qu'en mettant $p = f^2 + g^2$ on sait que f doit être impair et congru à 1 (mod. 4). Donc, les équations sont parfaitement déterminées.

Ces formules s'accordent, comme il est nécessaire, avec les résultats connus depuis longtemps, mais qu'on n'obtient par les méthodes de Gauss et de Jacobi qu'avec des calculs pénibles, pour ainsi dire fortuits, ou des raisonnements un peu détournés.

47.

ON A POINT IN THE THEORY OF VULGAR FRACTIONS.

[*American Journal of Mathematics*, III. (1880), pp. 332—335, 388—389.]

THE reciprocal of an integer I call a simple fraction; any other fraction, whether rational or irrational, may be termed complex; but it is to be understood that only proper fractional quantities of either sort, that is, fractions greater than zero and less than unity, will be considered in what follows.

Suppose Q to represent any fractional quantity; if Q lies between $\frac{1}{u_0 - 1}$ and $\frac{1}{u_0}$, we may make $Q = \frac{1}{u_0 + \delta} + Q'$, where δ is zero or a positive integer, and Q' will continue a proper fraction, which in like manner may be resolved into $\frac{1}{u_1 + \delta_1} + Q''$, and so on continually.

But if we make δ_0, δ_1, ... each zero, the process of expansion becomes determinate. Any such determinate representation of a fractional quantity I shall term a *sorites*. It is obvious that in expanding a given fraction under the form of a sorites, the successive denominators, which I shall call the *elements*, may be obtained by a process of division; if the fraction to be expanded is rational, the real divisor will be an integer which continually decreases*, and consequently every complex rational fraction can be expanded (and only in one way) under the form of a finite sorites.

The elements of a sorites are analogous to the partial quotients of a regular continued fraction; but there is this difference between the two cases, that whilst the latter quantities are perfectly arbitrary, the elements in question are subject to a certain law which I shall proceed to examine.

* See examples of development of sorites, page [443].

Let $n, p, q, \ldots r, s, \ldots t, u$ be the elements of a sorites. It is clear that the last remainder being the reciprocal of $\frac{1}{t} + \frac{1}{u}$, we must have $\frac{1}{t} + \frac{1}{u} < \frac{1}{t-1}$, that is to say, u greater than $t^2 - t$, that is, u is equal to or greater than $t^2 - t + 1$. Again, if we look to the residue which gives birth to the element r, that must be of the form $\frac{1}{s - \epsilon}$, where ϵ is some fraction, and we must now have $\frac{1}{r} + \frac{1}{s-\epsilon} < \frac{1}{r-1}$, or $s - \epsilon$ equal to or greater than $r^2 - r$. Hence s is equal to or greater than $r^2 - r + 1$, so that the relation between any two contiguous elements is the same, whether they are or are not the final two; and if u_x, u_{x+1} be any two consecutive integers in a series, the one necessary and sufficient condition for the possibility of the existence of the sorites, of which those terms shall be elements, is that we must have for all values of x, u_{x+1} equal to or greater than $u_x^2 - u_x + 1$.

If u_{x+1} is throughout equal to $u_x^2 - u_x + 1$, we obtain a series which may be termed a limiting sorites.

It is obvious that any simple fraction $\frac{1}{u_0 - 1}$ may be expanded under the form of an infinite sorites, of which the elements are $u_0, u_1, u_2 \ldots$ subject to the above relation. An infinite sorites read in the limiting case is therefore expressible under the form of a finite fraction, and the same will be true for a sorites in which the right-hand branch beginning from any term u_i, namely, $\frac{1}{u_i} + \frac{1}{u_{i+1}} + \frac{1}{u_{i+2}} \ldots$, forms a limiting sorites.

But in every other case of a sorites the sum cannot be a finite fraction; for such fraction can be expanded in only one way under the form of a sorites, and such sorites is necessarily finite in the number of its terms.

Hence it is impossible that the sum of the reciprocals of an ascending series of positive integers, such that the square root of the difference between any one of them and its immediate antecedent is greater than the difference between that antecedent and unity, can represent a rational quantity; for if so, we have $u_{x+1} - u_x$ greater than $(u_{x-1} - 1)^2$, that is, $u_{x+1} > u_x^2 - u_x + 1$, and the series will form a sorites not belonging to the limiting class.

I proceed to examine some of the properties of the series of terms defined by the condition $u_{x+1} = u_x^2 - u_x + 1$.

In the first place, I observe that any term u_{x+i} may be expressed under the form $Pu_x + 1$: for suppose this to be true for one value of i; then, since $u_{x+i+1} - 1 = u_{x+i}(u_{x+i} - 1)$, it is obviously true for the next above; here the proposition, being true when i is unity, is true universally.

It follows from this that each element of a limiting sorites is prime to all that follow it, and consequently any two terms of the sorites are prime to one another.

Again, for greater simplicity, let $v_0, v_1, v_2, \ldots$ be used to represent $(u_0 - 1)$, $(u_1 - 1)$, $(u_2 - 1)$, ...; we have, then,

$$v_1 - v_0 = v_0^2, \quad v_2 - v_1 = v_1^2, \quad v_3 - v_2 = v_2^2, \ldots.$$

Hence $v_2 - v_0$, $v_3 - v_0$, ... $v_x - v_0$ (as is obvious from successive addition of the above equations) will each of them be of the form Pv_0^2, where P is a rational integral function of v_0, and v_x will be of the form $Pv_0^2 + v_0$. This conclusion leads to a representation of the sum of any given number of terms of a limiting sorites by a fraction in its lowest terms. For

$$\frac{1}{v_x} - \frac{1}{v_{x+1}} = \frac{v_{x+1} - v_x}{v_x v_{x+1}} = \frac{v_x^2}{v_x v_{x+1}} = \frac{v_x}{v_x + v_x^2} = \frac{1}{v_x + 1} = \frac{1}{u_x}.$$

Hence

$$\frac{1}{u_0} + \frac{1}{u_1} + \ldots + \frac{1}{u_x} = \frac{1}{v_0} - \frac{1}{v_{x+1}} = \frac{v_{x+1} - v_0}{v_0 v_{x+1}} = \frac{(v_{x+1} - v_0) \div v_0^2}{v_{x+1} \div v_0},$$

which is of the form $\dfrac{P}{Pv_0 + 1}$ and is consequently a fraction in its lowest terms.

Again, if we denote the product of the elements $u_0, u_1, u_2, \ldots, u_x$ by Πu_x and the sum of their $(x-1)$-ary combinations by $\Pi' u_x$, $\dfrac{\Pi' u_x}{\Pi u_x}$ will also be the same fraction in its lowest terms, because (as has been shown) all the elements of the sorites are prime to one another.

Hence we may deduce the equations

$$u_{x+1} = u_0 + (u_0 - 1)^2 \, \Pi' u_x,$$
$$u_{x+1} = 1 + (u_0 - 1) \, \Pi u_x.$$

The second of these serves to give an inferior limit to the rate of convergence of any sorites. For in the limiting case we have

$$u_1 > (u_0^2 - u_0),$$
$$u_2 > (u_0 - 1) \; u_0 u_1 \; > (u_0^2 - u_0)^2,$$
$$u_3 > (u_0 - 1) \, u_0 u_1 u_2 > (u_0^2 - u_0)^4,$$

....

and so in general $u_x > (u_0^2 - u_0)^{2^{x-1}}$, because the solution of the equation

$$\theta_i = \theta_{i-1} + \theta_{i-2} + \ldots + \theta_0 \text{ is } \theta_i = 2^{i-1}\theta_0.$$

In any other sorites in which the initial element remains u_0, the value of the element at x-places distant must be *à fortiori* greater than the value $(u_0^2 - u_0)^{2^{x-1}}$ last obtained for the limiting case.

The preceding matter was suggested to me by the chapter in Cantor's *Geschichte der Mathematik* which gives an account of the singular method in use among the ancient Egyptians for working with fractions. It was their curious custom to resolve every fraction into a sum of simple fractions according to a certain traditional method, not leading, I need hardly say, except in a few of the simplest cases, to the expansion under the special form to which I have, in what precedes, given the name of a fractional *sorites*.

I subjoin examples of development of a rational fraction under the form of a sorites.

Let $\dfrac{4699}{7320}$ be the fraction to be expanded. The work may be arranged as follows:—

(2)	(8)	(60)	(3660)
4699	2078	1984	1920
7320	14640	117120	7027200
9398	16624	119040	7027200

(2) is the number one unit greater than $E\dfrac{7320}{4699}$; 9398 is 2×4699; 2078 is $9398 - 7320$; 14640 is 2×7320.

One element (2) is now determined, and the fraction $\dfrac{2078}{14640}$ remains to be expanded.

(8) is the number one unit greater than $E\dfrac{14640}{2078}$; 16624 is 8×2078; 1984 is $16624 - 14640$; 117120 is 8×14640.

A second element (8) is now found, and $\dfrac{1984}{117120}$ remains over to be expanded. Proceeding in this manner, and with numerators 4699, 2078, 1984, 1920, necessarily diminishing at each step, we come at last to the element 3660 with a remainder zero. The required sorites is therefore

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{60} + \frac{1}{3660}.$$

As a second example take the fraction $\dfrac{335}{336}$.

The work may be arranged in a similar manner to that of the foregoing example, and will be as follows:—

(2)	(3)	(7)	(48)
335	334	330	294
336	672	2016	14112
670	1002	2310	14112

and accordingly it will be found that

$$\frac{335}{336} = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{48}.$$

Postscript.

Let ϕx represent $x^2 - x + 1$, $\phi^n c$ will then be the general term of the "limiting sorites" whose first term is c, for which, if we please, $1 - c$ may be substituted. The properties of the numbers $\phi^n c$ seem to be worthy of some attention. I confine my observations in what follows to the lowest of such series, namely, where $c = 2$ or -1.

The first five terms in such series then become $\bar{1}$ or 2, 3, 7, 43, 1807, 3263443, of which all but 1807, which $= 13 . 139$, are prime numbers. Every term in the series must contain only factors of the form $6i + 1$, and this, joined to the fact that a prime factor which has once appeared in any term can never reappear in any other, favours a tendency, so to say, of the numbers to remain primes, or at all events, to be of very limited frangibility into a product of primes.

It is easy to determine whether any proposed prime can occur as a factor of any term whatever in the series; for taking that number, say p, as a modulus, if r is a remainder of any term to that modulus, the remainder of the next term will be $r^2 - r + 1$, and as soon as any remainder reappears the series of remainders becomes periodic; so that necessarily in less than the number p of remainders, if p does divide any term of the sorites, we must arrive at a remainder zero, subsequent to which all the remainders are unity. I give the remainders and periods in the annexed table for all values of p of the form $6i + 1$ up to 139, from which it will be seen that, under that limit, 13 and 73 are the only prime numbers which are contained as factors in the terms of the series.

p	Remainders of $\phi^n(2)$ to modulus p
2	0.
3	2, 0.
7	2, 3, 0.
13	2, 3, 7, 4, 0.
19	2, 3, 7, 5; 2, 3, 7, 5; ...
31	2, 3, 7, 12, 9, 11, 18, 28, 13; 2, 3, 7, ..., 13; ...
37	2, 3, 7; 6, 31; 6, 31; ...
43	2, 3, 7, 0.
61	2, 3, 7, 43, 38, 4, 13, 35, 32; 17, 29, 20, 15, 28, 25, 52, 30; ...
67	2, 3; 7, 43, 65; 7, 43, 65; ...
73	2, 3, 7, 43, 55, 51, 69, 21, 56, 15, 65, 0.
79	2, 3, 7; 43, 69, 32, 45, 6, 31, 61, 27, 71, 73; ...
97	2, 3, 7, 43, 61; 72, 69, 37; 72, 69, 37; ...
103	2, 3; 7, 43, 56, 94, 91, 54, 82, 51, 79, 86, 101; 7, 43, ...; ....
109	2, 3, 7, 43, 63, 92, 89, 94, 23, 71, 66, 40, 35, 101, 73, 25, 56, 29, 50, 53; 32, 12, 24, 8; 32, 12, 24, 8; ...
127	2, 3, 7, 43, 29, 51, 11; 111, 19, 89, 86, 72, 33, 41, 117; ...
139	2, 3, 7, 43, 0.
151	2, 3, 7, 43, 146; 31, 25, 148, 13, 6; ...
157	2, 3; 7, 43, 80, 41, 71, 104, 37, 77, 44, 9, 73, 76, 49, 155; ...
163	2, 3; 7, 43, 14, 20, 55, 37, 29, 161; ...
181	2, 3, 7, 43, 178, 13, 157, 58, 49, 0.
193	2; 3, 7, 43, 70, 6, 31, 159, 33, 92, 74, 192; ...
199	2, 3; 7, 43, 16, 42, 131, 116, 8, 57, 9, 73, 83, 41, 49, 164, 67, 45, 190, 91, 32, 197; ...

48.

INSTANTANEOUS PROOF OF A THEOREM OF LAGRANGE ON THE DIVISORS OF THE FORM $Ax^2+By^2+Cz^2$, WITH A POSTSCRIPT ON THE DIVISORS OF THE FUNCTIONS WHICH MULTISECT THE PRIMITIVE ROOTS OF UNITY.

[*American Journal of Mathematics*, III. (1880), pp. 390—392.]

IF possible, let p be not a divisor of x^2+y^2+1, and consequently not of the form $4i+1$, since, if it were of that form, x^2+1 would contain it.

Let ρ be any primitive pth root of unity.

Call $R=\Sigma\rho^{x^2}$, where x^2 means any one of the quadratic residues of and inferiors to p, and let the period conjugate to R be called R'.

Let R^2 be expanded as a sum of powers of ρ. Then, because p is not of the form $4i+1$, we cannot have $x^2+y^2=p$, so that no pth power of ρ can occur in that expansion; again, because by hypothesis neither $2x^2$ nor x^2+y^2 can be congruous to -1 [mod. p], no such power as ρ^{p-1} which belongs to R', nor consequently any other term of R', can appear in R^2; and as each power of ρ in R^2 belonging to the same period must appear a like number of times, we must have

$$R^2=\frac{p-1}{2}R, \text{ that is, } R=0, \text{ or } R=\frac{p-1}{2},$$

each of which suppositions is in the highest degree absurd. Hence p is a divisor of x^2+y^2+1. Q.E.D.

Compare Legendre's *Théorie des Nombres*, Ed. 1830, Tom. I. pp. 211—213, and again Serret's *Cours d'Algèbre Supérieure*, Tom. II. pp. 94—99, for proofs of the more general similar theorem due to Lagrange, concerning u^2+Bt^2+C. These proofs are highly ingenious, but long and laboured in no slight degree; and as the sole apparent object of either author in proving the general theorem is to make use of the particular case of it to which this note refers as a foundation to the proof of Fermat's law of the four squares, I have

thought that an intuitive proof of so important a lemma might not be without interest to some of the junior readers of the *Journal**.

But in fact the general theorem may be proved with scarcely any greater trouble than the particular case disposed of.

For, supposing A, B, C to be all quadratic residues to p, we may write

$$A\equiv\alpha^2,\quad B\equiv\beta^2,\quad C\equiv\gamma^2\ [\text{mod. } p],$$
$$\alpha x=u,\quad \beta y=v,\quad \gamma z=w;$$

and the congruence $u^2+v^2+w^2\equiv 0$, as previously shown, being soluble, evidently

$$Ax^2+By^2+Cz^2\equiv 0$$

will be so too, since

$$\alpha x\equiv u,\quad \beta y\equiv v,\quad \gamma z\equiv w\ [\text{mod. } p],$$

give integer values for x, y, z; and as obviously the case of A, B, C being all non-residues falls into the previous case by multiplying the congruence by any non-residue, we have only to consider the case of two of the three coefficients being residues and the third a non-residue, or the converse case, which, however, by multiplication as above, may be reduced to the former one.

Suppose, then, $A=\alpha^2$, $B=\beta^2$, C a non-residue, and that

$$Ax^2+By^2+Cz^2\equiv 0\ [\text{mod. } p]$$

is insoluble. For simplicity, let $z=1$. Then $u^2+v^2+C=0$ must be insoluble; if p is of the form $4i+3$, we shall obtain, precisely as before,

$$R^2=\frac{p-1}{2}R,$$

and if p is of the form $4i+1$,

$$R^2=2\frac{p-1}{4}+\left\{\left(\frac{p-1}{2}\right)^2-\left(\frac{p-1}{2}\right)\right\}\div\frac{p-1}{2}.R,$$

or $$R^2-\frac{p-3}{2}R+\frac{p-1}{2}=0, \text{ that is, } R=\frac{p-1}{2}, \text{ or } R=-1,$$

any of which conclusions are eminently absurd.

* From this lemma there is scarcely more than a step to the theorem in question. If P is contained as a factor in the sum of four squares, it is easy to see that we may write

$$PQ=f^2+g^2+h^2+k^2,$$

where $Q<P$, and

$$QQ'=(f-\alpha Q')^2+(g-\beta Q')^2+(h-\gamma Q')^2+(k-\delta Q')^2,$$

where $Q'<Q$, and consequently, applying the Quaternion law of multiplication,

$$PQ'=f'^2+g'^2+h'^2+k'^2,$$

and so we may form a continually decreasing series of quantities Q, Q', Q'', ... any one of which multiplied by P is a sum of four squares. Hence any divisor of such sum is itself such a sum, but by the lemma any prime number is a divisor of the sum of three, which plays the same part for present purposes as a sum of four squares, and is therefore a sum of four squares; consequently any number whatever, by the rule of multiplication already alluded to in this note, will be a sum of four squares.

Hence $Ax^2 + By^2 + Cz^2 \equiv 0$ [mod. p] *cannot* be insoluble; that is, the left-hand side of the congruence must contain p as a divisor.

P.S. In a future communication I will prove very simply that if a prime number $p = ef + 1$, and e is itself a prime number such that $(e-1)$ contains no odd square number, then every divisor, without exception (other than p), of the function whose roots are the e periods of the primitive pth roots of unity, must be an eth power residue of p. If $(e-1)$ contains any square number, the proof still holds good, except as regards the factors of such square, and there is no reason at present for supposing that the theorem may not be extended to the case of these excepted factors*. The same kind of reasoning may be applied also to the theory of period-functions for which e (the number of the periods) is not a prime number, and I find for the case of $e = 4$, that, leaving out of account the number 2 (which is always a divisor of the four-period function to p when p is of the form $8i + 1$, but never when it is of the form $8i + 5$, and may be or not a biquadratic residue of p, according to a well-known law), the divisors of the four-period function (excepting p) which do not divide g (the even term in the equation $[f^2 + g^2 = p]$), are necessarily biquadratic residues of p; as is also true of the prime-number divisors of g which are of the form $4i + 1$; but the prime-number divisors of g (all of which are necessarily divisors of the four-period function), of the form $4i - 1$, are quadratic only, and not biquadratic residues of p when p is of the form $8i + 5$; whereas for the case of $p = 8i + 1$ *all* the odd divisors of the four-period function (not counting p) are biquadratic residues of p†. The same investigation leads to the remarkable conclusion that if $p = f^2 + 4\gamma^2$, where f and γ are both of them odd and p a prime number, every divisor of $\frac{f^2 + 3\gamma^2}{4}$ is a biquadratic residue of p,—a theorem which I imagine would be difficult to prove by any other method.

* Thus, for example, if e is a prime number of the form $2^{2^x} + 1$, I am able to prove that *every* divisor of the e-period function (not excepting 2, if 2 should happen to be such a divisor) is an eth-power residue of p. Thus for $e = 2, 3, 5, 7, 11, 17$ we may be certain that there are none but eth-power-residue divisors of the period-function.

† Of course in a certain sense p or zero is an any-power residue. But there is good reason for separating p from the residues proper, inasmuch as only the *first* power of p, but an *unlimited* power of any true eth-power residue is a divisor of the e-period function,—a most important fact, which I presume must have been known to Bachmann, but has not been stated by him (in his *Kreistheilung*, 1872). An exceedingly simple proof of this and of the corresponding theorem for any cyclotomic function was given by Mr Hathaway at a recent meeting of the Mathematical Seminarium, at the Johns Hopkins University.

49.

SUR L'ENTRELACEMENT D'UNE FONCTION PAR RAPPORT À UNE AUTRE.

[*Crelle's Journal für die reine und angewandte Mathematik*, LXXXVIII. (1880), pp. 1—3.]

SOIENT $f(\lambda)$ et $\phi(\lambda)$ deux fonctions rationnelles et entières de λ. Désignons par $\lambda = A$ les racines réelles de l'équation $f(\lambda) = 0$ et par $\lambda = B$ les racines réelles de l'équation $\phi(\lambda) = 0$, et concevons que ces racines soient représentées par des points situés sur l'axe réel du plan. Les racines A et les racines B se suivront sur cet axe d'une manière quelconque. Supposons que toutes les fois qu'un nombre pair de points A forment une suite non interrompue par des points B on supprime ces points A, et que toutes les fois qu'un nombre pair de points B forme une suite non interrompue par des points A on supprime ces points B, de sorte qu'à la fin de ces suppressions on arrive à une suite alternative de points A et de points B, c. à d. à une suite finale dans laquelle on ne trouve ni un A suivi d'un A, ni un B suivi d'un B. Cela posé, le nombre des points A qui restent dans la suite finale sera ce que l'on peut nommer l'indice de l'entrelacement effectif des racines de l'équation $f = 0$ par rapport aux racines de l'équation $\phi = 0$, mais que pour plus de brièveté je nommerai plutôt *l'entrelacement de f par ϕ**.

Construisons la courbe $y = f(x)$ et la courbe $y = \phi(x)$ et supposons que chacune de ces courbes soit représentée par un fil flexible infiniment mince et fixé en deux points assez éloignés de l'axe des x. Désignons les deux courbes par f et par ϕ, et supposons que dans les points de rencontre des deux courbes situés au nord de l'axe des x la courbe f passe au-dessus de la courbe ϕ, qu'au contraire dans les points de rencontre situés au midi de l'axe des x la courbe ϕ passe au-dessus de la courbe f. Cela posé, si deux points de rencontre consécutifs se trouvent tous les deux du même côté de l'axe des x, ils ne contribuent point au nombre que je viens de nommer l'entrelacement de f par ϕ, et l'on peut ôter ces deux points par une flexion convenable de l'une des deux courbes. Cette construction donne donc une signification géométrique et intuitive au nombre que j'ai défini comme l'entrelacement de f par ϕ. En effet, si f est d'ordre pair ou si f et ϕ sont tous les deux d'ordre

* De même que je viens de définir l'entrelacement total de f par ϕ, de même on pourra définir l'entrelacement de f par ϕ entre les deux limites p et q en se bornant aux racines réelles A et B situées entre les deux limites $\lambda = p$ et $\lambda = q$.

impair, ce nombre est en même temps le nombre des intersections permanentes des deux courbes f et ϕ. Si f est d'ordre impair et ϕ d'ordre pair, il faut selon les circonstances augmenter ou diminuer d'une unité le nombre des intersections permanentes des deux courbes pour obtenir le nombre analytique défini comme l'entrelacement de f par ϕ, ce que j'exposerai plus amplement dans la suite.

Pour donner plus de précision à la construction expliquée ci-dessus on peut faire l'hypothèse que dans toute l'étendue de l'axe des x les deux parties du plan des xy soient séparées par une fente (désignée dorénavant sous le nom de fente de X), que des fils flexibles ou rubans représentant les courbes f et ϕ soient assujettis à passer par cette fente toutes les fois qu'ils traversent l'axe des x et que le fil ou ruban ϕ soit collé aux deux faces du plan des xy. De cette hypothèse on tire comme conséquence que f se trouve au-dessus de ϕ pour des y positifs et au-dessous de ϕ pour des y négatifs comme on a supposé antérieurement.

Dans le cas où la fonction f est d'ordre impair et où la fonction ϕ est d'ordre pair on a déjà avancé que le nombre des intersections permanentes diffère d'une unité positive ou négative de l'entrelacement de f par ϕ. Pour décider de cette ambiguïté, changeons d'abord, s'il est nécessaire, les signes des fonctions f et ϕ, ce qui est permis dans cette recherche, de sorte qu'après le changement de signe dans l'une comme dans l'autre des deux fonctions la plus haute puissance de x se trouve multipliée par un coefficient positif. Poursuivons le cours des deux courbes f et ϕ en marchant du côté positif de l'axe des x vers le côté négatif et désignons sous le nom de nœud (knot en anglais) les intersections permanentes des deux courbes en nous rappelant que le nombre total de toutes leurs intersections est pair. Cela posé, si le premier nœud précède le premier passage des deux courbes par la fente de X ou, ce qui est la même chose, si le dernier nœud suit le dernier passage, le nombre des nœuds *diminué* d'une unité est égal à l'entrelacement de f par ϕ, dans le cas contraire le nombre des nœuds *augmenté* d'une unité est égal à l'entrelacement de f par ϕ. Il y a un criterium (physique) auquel on peut réduire la distinction des deux cas dont il s'agit. Imaginons que l'on réunit le bout positif de f (ruban libre) au même bout de ϕ (ruban collé au plan). De cette manière on formera une aire (loop) comprise entre les parties extrêmes positives des deux courbes. Dans le premier cas discuté ci-dessus ce loop est transitoire et peut être supprimé par une déformation convenable de f. Dans le cas contraire il est impossible de supprimer le loop sans rupture des rubans réunis. Cela posé, l'entrelacement analytique de f par ϕ est égal au nombre des nœuds qui se trouvent dans les deux rubans réunis, si l'on fait la convention de ne compter du tout le loop quand il est transitoire, mais de le compter comme équivalent à deux nœuds quand il est permanent.

50.

PREUVE INSTANTANÉE D'APRÈS LA MÉTHODE DE FOURIER, DE LA RÉALITÉ DES RACINES DE L'ÉQUATION SÉCULAIRE.

[*Crelle's Journal für die reine und angewandte Mathematik*, LXXXVIII. (1880), pp. 4, 5.]

Soit M un carré de termes dont le déterminant est Δ, m un carré mineur quelconque de M composé des éléments

$$\begin{matrix} \lambda, & \lambda_{1,2} \ldots \lambda_{1,\epsilon}, \\ \mu_{2,1}, \mu & \ldots \mu_{2,\epsilon}, \\ \cdots\cdots & \cdots\cdots \\ \rho_{\epsilon,1} & \rho_{\epsilon,2} \ldots \rho, \end{matrix}$$

et considérons les coefficients différentiels de Δ pris par rapport à chacun des ϵ^2 éléments qui se trouvent dans le carré écrit ci-dessus; d'après un théorème connu on sait que le déterminant de l'ordre ϵ, formé de tous ces coefficients différentiels, est égal à

$$\Delta^{\epsilon-1} \frac{d^\epsilon \Delta}{d\lambda d\mu \ldots d\rho}.$$

Soient $\lambda, \mu, \ldots \rho$ des éléments qui se trouvent dans la diagonale de M, et supposons que M soit symétrique par rapport à cette diagonale, faisons de plus $\epsilon = 2$, le théorème énoncé ci-dessus se change en la formule élémentaire

$$\Delta \frac{d^2\Delta}{d\lambda d\mu} = \frac{d\Delta}{d\lambda}\frac{d\Delta}{d\mu} - \left(\frac{d\Delta}{d\lambda_{1,2}}\right)^2,$$

où le carré qui forme le dernier terme de la seconde partie de l'equation est écrit au lieu du produit des deux expressions $\frac{d\Delta}{d\lambda_{1,2}}$, $\frac{d\Delta}{d\mu_{2,1}}$ devenues égales en vertu de la symétrie des éléments compris dans M.

De cette équation on tire les deux conclusions suivantes:

(θ) Quand $\Delta = 0$, $\frac{d\Delta}{d\lambda}$ et $\frac{d\Delta}{d\mu}$ ont le même signe.

(ϕ) Quand $\frac{d\Delta}{d\lambda} = 0$, Δ et $\frac{d^2\Delta}{d\lambda d\mu}$ ont des signes contraires.

Supposons à présent que $\alpha, \beta, \ldots \rho$ soient tous les éléments compris dans la diagonale de M, que $\alpha, \beta, \ldots \rho$ soient remplacés par $\alpha + x, \beta + x, \ldots \rho + x$

et que D soit la valeur correspondante du déterminant Δ. Cela posé, si r est une valeur de x pour laquelle $D=0$, toutes les expressions $\frac{dD}{d\alpha}, \frac{dD}{d\beta}, \dots \frac{dD}{d\rho}$ et par conséquent $\frac{dD}{dx} = \frac{dD}{d\alpha} + \frac{dD}{d\beta} + \dots + \frac{dD}{d\rho}$ auront le même signe en vertu de (θ).

Soit de plus h une quantité positive infiniment petite, la valeur de D pour $x=r+h$ aura le même signe que $\frac{dD}{d\alpha}$ et la valeur de D pour $x=r-h$ aura le signe contraire. Formons la suite

$$D, \frac{dD}{d\alpha}, \frac{d^2D}{d\beta d\alpha}, \dots \frac{d^nD}{d\rho \dots d\beta d\alpha},$$

n désignant l'ordre du déterminant D, et faisons croître la variable x depuis une limite inférieure quelconque en la faisant passer par toutes les valeurs successives jusqu'à une limite supérieure quelconque. Toutes les fois que x passe par la valeur d'une racine r de $D=0$, il y aura après le passage une permanence de signe de plus dans la suite écrite ci-dessus qu'il n'y en avait avant; si au contraire x passe par une valeur pour laquelle D ne s'évanouit point, il y aura avant et après le passage le même nombre de permanences. En effet prenons dans la suite dont il s'agit trois termes consécutifs quelconques, p. e.

$$\frac{d^2D}{d\beta d\alpha}, \frac{d^3D}{d\gamma d\beta d\alpha}, \frac{d^4D}{d\delta d\gamma d\beta d\alpha},$$

et soit

$$D' = \frac{d^2D}{d\beta d\alpha};$$

en vertu de (ϕ) les deux expressions D' et $\frac{d^2D'}{d\delta d\gamma}$ auront des signes contraires quand $\frac{\partial D'}{\partial \gamma}$ s'évanouit. Mais les deux premiers termes présenteront, comme l'on a vu ci-dessus, une variation de signe avant le passage d'une racine de l'equation $D=0$ et une permanence après ce passage. Conséquemment la série dont il s'agit gagnera depuis la limite inférieure jusqu'à la limite supérieure autant de permanences de signes qu'il y a entre ces limites de racines de l'équation $D=0$.

Soit $-\infty$ la limite inférieure, $+\infty$ la limite supérieure de la variable x, n sera le nombre des permanences que la série dont il s'agit aura gagné, donc les racines de l'équation $D=0$ sont toutes réelles, ce qu'il fallait démontrer.

Post-scriptum. Je dois remarquer que la preuve donnée par M. Salmon (*Lessons on Higher Algebra*, 3^{me} édition p. 43), que je n'avais pas remarquée précédemment, est encore plus simple que celle donnée en haut, cependant elle est tant soit peu moins directe: dans cette autre preuve on ne fait usage que de la conclusion (ϕ).

51.

SUR UN DÉTERMINANT SYMÉTRIQUE QUI COMPREND COMME CAS PARTICULIER LA PREMIÈRE PARTIE DE L'ÉQUATION SÉCULAIRE.

[*Crelle's Journal für die reine und angewandte Mathematik*, LXXXVIII. (1880), pp. 6—9.]

LA théorie de l'équation séculaire s'étend aisément à un déterminant symétrique beaucoup plus général. Dans le théorème de Sturm ou dans le théorème plus complet sur les intercalations que j'ai donné dans mon mémoire "Sur les rapports syzygétiques, etc.", inséré aux *Philosophical Transactions**, on considère une suite de fonctions telles que trois fonctions consécutives quelconques P, Q, R soient liées par l'équation

$$P = Q'Q - R.$$

Dans le cas présent je m'occuperai de même d'une suite de fonctions telles qu'entre trois fonctions consécutives quelconques P, Q, R on ait l'équation

$$PR = Q'Q - Q''^2,$$

équation qui présente avec la première cette circonstance commune que pour $Q=0$ le produit PR est négatif.

Soit D un déterminant symétrique quelconque dont les éléments sont des fonctions rationnelles et entières de λ, et désignons par $a, b, c, \dots l$ les termes constants, c. à d. indépendants de λ, des éléments situés dans la diagonale de symétrie et rangés dans un ordre quelconque. Formons, comme je l'ai fait dans la preuve instantanée [p. 451 above], la suite

$$D, \delta_a D, \delta_b \delta_a D, \dots \delta_l \delta_k \dots \delta_a D$$

et soient p, q, r trois quantités consécutives prises dans la série $a, b, \dots k, l$; cela posé, on aura

$$(\delta_r \delta_q \delta_p \dots \delta_a) D . (\delta_p \dots \delta_a) D = (\delta_r \delta_p \dots \delta_a) D . (\delta_q \delta_p \dots \delta_a) D - M^2,$$

M étant une fonction entière de λ. La loi du signe contraire de deux termes, voisins d'un terme qui s'évanouit, de la suite considérée ci-dessus ne subit donc aucun changement.

[* Vol. I. of this Reprint, p. 545.]

La méthode dont on s'est servi dans la preuve instantanée conduit par conséquent à ce résultat que l'entrelacement de D par $\delta_a D$ entre des limites quelconques p et q est égal à la valeur numérique absolue de la différence entre le nombre des permanences de signe que présente la suite considérée ci-dessus pour les deux valeurs $\lambda = p$ et $\lambda = q$.

Je dis de plus que l'entrelacement de D par $\delta_a D$ est égal à l'entrelacement de D par $\delta_b D$, a et b étant deux quelconques des quantités $a, b, \dots l$.

En effet le premier de ces deux nombres dépend uniquement du rapport des signes de D et de $\delta_a D$ dans le voisinage des valeurs de λ pour lesquelles D s'évanouit, et pour le second de ces deux nombres on n'a qu'à substituer $\delta_b D$ au lieu de $\delta_a D$. Mais on sait que le produit $\delta_a D \,.\, \delta_b D$ excède le produit $D \,.\, (\delta_a \delta_b)\, D$ d'un carré positif. Les deux quantités $\delta_a D$ et $\delta_b D$ sont donc du même signe dans le voisinage des valeurs de λ pour lesquelles $D = 0$, ce qui prouve l'assertion qu'il s'agissait de démontrer.

En se bornant pour plus de simplicité au cas des entrelacements absolus, c. à d. au cas où $-\infty$ et $+\infty$ sont les limites de λ, on en tire le théorème suivant:

Soient $\qquad \theta, \theta_1, \theta_2, \dots \theta_n$

les degrés et $\qquad \mu, \mu_1, \mu_2, \dots \mu_n$

les coefficients des plus hautes puissances de λ dans

$$D, \ \delta_a D, \ (\delta_b \delta_a)\, D, \dots (\delta_l \dots \delta_b \delta_a)\, D;$$

cela posé, l'entrelacement de D par une quelconque des quantités $\delta_a D$, $\delta_b D, \dots \delta_l D$ est la valeur numérique de la différence entre le nombre des permanences de signes que présentent les deux suites

$$\begin{array}{ccccc} \mu, & \mu_1, & \mu_2, & \dots & \mu_n \\ (-1)^{\theta} \mu, & (-1)^{\theta_1} \mu_1, & (-1)^{\theta_2} \mu_2, & \dots & (-1)^{\theta_n} \mu_n. \end{array}$$

Considérons le cas particulier dans lequel les éléments du déterminant D sont des fonctions linéaires de λ. Supposons de plus que tous les coefficients $\mu, \mu_1, \mu_2, \dots \mu_n$ ont le signe positif. Dans ce cas l'entrelacement de D par l'une quelconque des déterminants dérivés $\delta_a D, \delta_b D, \dots \delta_l D$ est égal à n, et par conséquent toutes les racines de l'équation $D = 0$ sont réelles, résultat que l'on verifie aisément.

En effet, soit D_1 le déterminant formé des parties constantes des éléments de D ou, ce qui est la même chose, la valeur de D pour $\lambda = 0$, soit de plus Δ le déterminant formé des coefficients de λ des éléments de D ou, ce qui est la même chose, le coefficient de λ^n dans le développement de D; cela posé, D peut être regardé comme l'invariant de la forme quadratique à n variables $\Phi + \lambda\Omega$, Φ et Ω étant des formes quadratiques dont les invariants sont D_1 et Δ. Représentons d'une manière symbolique par

$$(\alpha x + \beta y + \gamma z + \dots + \lambda u)^2$$

la forme quadratique Ω, et par

$$\alpha, \beta, \gamma, \dots \lambda * p, q, r, \dots t$$

le déterminant
$$\begin{vmatrix} \alpha p, & \alpha q, & \alpha r, \dots \alpha t \\ \beta p, & \beta q, & \beta r, \dots \beta t \\ \gamma p, & \gamma q, & \gamma r, \dots \gamma t \\ \vdots & \vdots & \vdots \quad\ \ \vdots \\ \lambda p, & \lambda q, & \lambda r, \dots \lambda t \end{vmatrix};$$

posons de plus

$$\begin{aligned} \xi &= \alpha * (\alpha x + \beta y + \gamma z + \dots + \lambda u) \\ \eta &= \alpha\beta * (\alpha\beta y + \alpha\gamma z + \dots + \alpha\lambda u) \\ \zeta &= \alpha\beta\gamma * (\alpha\beta\gamma z + \dots + \alpha\beta\lambda u) \\ &\vdots \\ v &= \alpha\beta\gamma \dots \lambda * (\alpha\beta\gamma \dots \lambda u) \end{aligned}$$

et enfin $\qquad \dfrac{1}{A} = \alpha^2, \ \dfrac{1}{B} = \alpha^2 (\alpha, \beta)^2,$

$$\frac{1}{C} = (\alpha, \beta)^2 (\alpha, \beta, \gamma)^2, \dots \frac{1}{L} = (\alpha, \beta, \dots \kappa)^2 \,.\, (\alpha, \beta, \dots \kappa, \lambda)^2,$$

où la notation $(\alpha, \beta, \dots \lambda)^2$ désigne le déterminant

$$\alpha, \beta, \dots \lambda * \alpha, \beta, \dots \lambda;$$

cela posé, on aura $\qquad \Omega = A\xi^2 + B\eta^2 + \dots + Lv^2.$

Les expressions désignées d'une manière symbolique par

$$\alpha^2, \ (\alpha, \beta)^2, \ (\alpha, \beta, \gamma)^2, \dots (\alpha, \beta, \gamma, \dots \lambda)^2$$

étant identiques aux quantités désignées antérieurement par

$$\mu_n, \ \mu_{n-1}, \ \mu_{n-2}, \dots \mu,$$

il est aisé de voir que l'invariant de $\Phi + \lambda\Omega$ est, à un facteur numérique près, égal à l'invariant de $\Phi_1 + \lambda\Omega_1$, Φ_1 représentant une fonction quadratique de fonctions linéaires à coefficients réels de $x, y, z, \dots u$ et Ω_1 une somme de carrés de ces mêmes fonctions linéaires.

En admettant les hypothèses faites ci-dessus on retombe donc sur le cas de l'équation séculaire.

Je me suis servi dans les calculs précédents de la notation $*$ pour représenter une multiplication symbolique entre $\alpha, \beta, \gamma, \dots \lambda$; $\alpha', \beta', \gamma', \dots \lambda'$, tandis que je préfère garder le signe simple $\times$ pour représenter l'espèce d'opération disjonctive dont on se sert dans la théorie ordinaire de la multiplication des déterminants et qui donne naissance aux produits

$$(\alpha\alpha', \ \beta\beta', \ \gamma\gamma', \dots \lambda\lambda').$$

52.

SUR LES DÉTERMINANTS COMPOSÉS.

[*Crelle's Journal für die reine und angewandte Mathematik*, LXXXVIII. (1880), pp. 49—67.]

DANS l'article qui va suivre je m'occuperai de la théorie des déterminants composés, regardée sous un point de vue très général. Comme on sait, les déterminants composés sont des déterminants dont les éléments sont eux-mêmes des déterminants puisés dans la même matrice ou, ce qui est la même chose*, des sous-déterminants ou déterminants-mineurs d'un même déterminant primitif.

Servons-nous en premier lieu de la notation ombrale ordinaire pour représenter un déterminant simple—dans cette notation *une ligne double* c. à d. une paire de lignes de n ombres

$$\begin{matrix} a_1 a_2 \dots a_n \\ \alpha_1 \alpha_2 \dots \alpha_n \end{matrix}$$

servira à représenter un déterminant du n^{me} ordre. On peut aussi se servir avantageusement de la notation

$$a_1 a_2 \dots a_n \oplus \alpha_1 \alpha_2 \dots \alpha_n$$

pour représenter la même chose.

Un système de n quantités a étant donné, on se sert ordinairement de la notation

$$\Sigma a_1;\ \Sigma a_1 a_2;\ \dots$$

pour signifier $a_1 + a_2 + \dots + a_n;\ a_1 a_2 + a_1 a_3 + \dots + a_{n-1} a_n;\ \dots$

En changeant un peu cette notation, les expressions

$$\Sigma a_1,\ \Sigma a_1 a_2,\ \dots$$

seront employées pour représenter l'ensemble des termes

$$(a_1, a_2, \dots a_n),\ (a_1 a_2, a_1 a_3, \dots a_{n-1} a_n),\ \dots$$

au lieu de leur somme.

* Autant que la matrice est quadratique et non rectangulaire.

Un déterminant composé qui se rapporte à une seule ligne double

$$\begin{matrix} a_1 a_2 \dots a_n \\ \alpha_1 \alpha_2 \dots \alpha_n \end{matrix}$$

sera représenté par la notation:

$$\Sigma a_1 a_2 \dots a_i,\ \oplus \Sigma \alpha_1 \alpha_2 \dots \alpha_i.$$

Soit p. e. $n = 4$, la notation

$$\Sigma a_1 a_2 a_3,\ \oplus \Sigma \alpha_1 \alpha_2 \alpha_3,$$

signifiera le déterminant composé

$$\begin{matrix}
a_1a_2a_3 & a_1a_2a_4 & a_1a_3a_4 & a_2a_3a_4 \\
\alpha_1\alpha_2\alpha_3 & \alpha_1\alpha_2\alpha_3 & \alpha_1\alpha_2\alpha_3 & \alpha_1\alpha_2\alpha_3 \\
a_1a_2a_3 & a_1a_2a_4 & a_1a_3a_4 & a_2a_3a_4 \\
\alpha_1\alpha_2\alpha_4 & \alpha_1\alpha_2\alpha_4 & \alpha_1\alpha_2\alpha_4 & \alpha_1\alpha_2\alpha_4 \\
a_1a_2a_3 & a_1a_2a_4 & a_1a_3a_4 & a_2a_3a_4 \\
\alpha_1\alpha_3\alpha_4 & \alpha_1\alpha_3\alpha_4 & \alpha_1\alpha_3\alpha_4 & \alpha_1\alpha_3\alpha_4 \\
a_1a_2a_3 & a_1a_2a_4 & a_1a_3a_4 & a_2a_3a_4 \\
\alpha_2\alpha_3\alpha_4 & \alpha_2\alpha_3\alpha_4 & \alpha_2\alpha_3\alpha_4 & \alpha_2\alpha_3\alpha_4.
\end{matrix}$$

Ce déterminant est du 4^{me} ordre par rapport aux lignes doubles qui forment ses éléments et qui sont elles-mêmes des déterminants simples du 3^{me} ordre. La notation

$$\Sigma a_1 a_2,\ \oplus \Sigma \alpha_1 \alpha_2,$$

signifiera de même un déterminant du 4^{me} ordre dont les éléments sont des déterminants simples du 2^{me} ordre. Enfin la notation

$$\Sigma a_1,\ \oplus \Sigma \alpha_1,$$

signifiera le déterminant

$$\begin{matrix}
a_1 & a_2 & a_3 & a_4 \\
\alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 \\
a_1 & a_2 & a_3 & a_4 \\
\alpha_2 & \alpha_2 & \alpha_2 & \alpha_2 \\
a_1 & a_2 & a_3 & a_4 \\
\alpha_3 & \alpha_3 & \alpha_3 & \alpha_3 \\
a_1 & a_2 & a_3 & a_4 \\
\alpha_4 & \alpha_4 & \alpha_4 & \alpha_4.
\end{matrix}$$

La dernière notation étant équivalente à

$$a_1, a_2, a_3, a_4 \oplus \alpha_1, \alpha_2, \alpha_3, \alpha_4$$

représente le même déterminant que l'on écrit plus simplement sous la forme

$$a_1 a_2 a_3 a_4 \oplus \alpha_1 \alpha_2 \alpha_3 \alpha_4.$$

L'identité des valeurs de

$$\Sigma a_1, \divideontimes \Sigma \alpha_1, \text{ et de } a_1 a_2 \ldots a_n \divideontimes \alpha_1 \alpha_2 \ldots \alpha_n$$

est un cas particulier (le cas extrême) d'un théorème général qui se rapporte aux déterminants composés à une seule ligne double (et que je nommerai déterminants composés à un seul argument). Dans le cas de n ombres a et d'autant d'ombres α le théorème relatif à cette classe (la plus simple qu'on puisse former) de déterminants composés s'énonce comme il suit :

$$\Sigma a_1 a_2 \ldots a_i, \divideontimes \Sigma \alpha_1 \alpha_2 \ldots \alpha_i, = \begin{pmatrix} a_1 a_2 \ldots a_n \\ \alpha_1 \alpha_2 \ldots \alpha_n \end{pmatrix}^{\frac{n-1 \,.\, n-2 \ldots n-i+1}{1 \,.\, 2 \ldots i-1}}.$$

Ainsi si $i + j = n + 1$ on aura

$$\Sigma a_1 a_2 \ldots a_i, \divideontimes \Sigma \alpha_1 \alpha_2 \ldots \alpha_i, = \Sigma a_1 a_2 \ldots a_j, \divideontimes \Sigma \alpha_1 \alpha_2 \ldots \alpha_j;$$

car l'indice de la puissance de

$$\begin{pmatrix} a_1 a_2 \ldots a_n \\ \alpha_1 \alpha_2 \ldots \alpha_n \end{pmatrix}$$

a la même valeur $\dfrac{\Pi(n-1)}{\Pi(i-1)\,\Pi(j-1)}$ dans les deux cas.

Un cas bien connu de ce théorème est celui où $i = n - 1$. Dans ce cas l'indice de la puissance

$$\begin{pmatrix} a_1 a_2 \ldots a_n \\ \alpha_1 \alpha_2 \ldots \alpha_n \end{pmatrix}$$

devient $n - 1$; c'est le théorème qui affirme qu'en désignant par D un déterminant du n^{me} ordre et par Δ le déterminant dont les éléments sont les dérivées de D par rapport à ses éléments, on aura $\Delta = D^{n-1}$ *.

Avant de passer au théorème plus général qui se rapporte aux déterminants composés à un nombre quelconque de lignes doubles (ou disons plutôt à un nombre quelconque d'arguments) considérons d'abord un cas spécial qui est d'un grand intérêt par les applications qu'il admet, à savoir le cas dans lequel on attache à chaque ligne double dans le développement de

$$\Sigma a_1 a_2 \ldots a_i, \divideontimes \Sigma \alpha_1 \alpha_2 \ldots \alpha_i,$$

une ligne double constante

$$\begin{matrix} b_1 b_2 \ldots b_p \\ \beta_1 \beta_2 \ldots \beta_p. \end{matrix}$$

La matrice ainsi modifiée peut être désignée par la notation

$$[b_1 b_2 \ldots b_p \Sigma a_1 a_2 \ldots a_i,] \divideontimes [\beta_1 \beta_2 \ldots \beta_p \Sigma \alpha_1 \alpha_2 \ldots \alpha_i,].$$

* Le théorème pour les valeurs générales de i a été retrouvé récemment et inséré dans les *Comptes Rendus de l'Académie des Sciences de Paris* par un auteur distingué ; depuis on a porté à sa connaissance que j'avais déjà publié le même théorème dans le *Philosophical Magazine* de 1851. [Vol. I. of this Reprint, p. 252.]

Pour $n = 2$ et $p = 2$ la notation

$$[b_1 b_2 \Sigma a_1,] \divideontimes [\beta_1 \beta_2 \Sigma \alpha_1,]$$

représentera p. e. le déterminant :

$$\begin{matrix} b_1 b_2 a_1 & b_1 b_2 a_2 \\ \beta_1 \beta_2 \alpha_1 & \beta_1 \beta_2 \alpha_1 \\[1ex] b_1 b_2 a_1 & b_1 b_2 a_2 \\ \beta_1 \beta_2 \alpha_2 & \beta_1 \beta_2 \alpha_2. \end{matrix}$$

Cette classe de déterminants composés peut être désignée sous le nom de déterminants composés *à deux arguments dont l'un est non-distribué.*

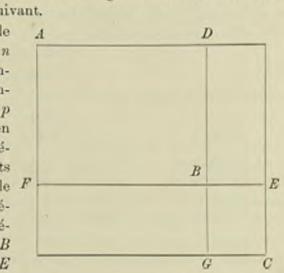
Je nommerai p et n les indices de l'étendue de B et de A respectivement et i l'indice de distribution de A ; je me servirai pour le déterminant de la notation $B_p \,.\, {}^i A_n$; cela posé, je dis que l'on a *

$$B_p \,.\, {}^i A_n = B^\sigma (AB)^\tau,$$

où $$\sigma = \frac{(n-1)(n-2)\ldots(n-i)}{1 \,.\, 2 \ldots i}, \quad \tau = \frac{(n-1)(n-2)\ldots(n-i+1)}{1 \,.\, 2 \ldots (i-1)}.$$

Il y a deux cas particuliers qui présentent un intérêt spécial, ce sont les cas de $i = n - 1$ et de $i = 1$.

Dans le premier cas on a $\sigma = 1$, $\tau = n - 1$, dans le second $\sigma = n - 1$, $\tau = 1$. Le premier cas de $i = n - 1$ donne le théorème auquel on est parvenu antérieurement. Le cas de $i = 1$ fournit une preuve immédiate de la règle connue pour trouver la valeur du déterminant de la matrice qui résulte de la multiplication de deux matrices carrées ou rectangulaires, comme on verra facilement à l'aide du diagramme suivant.



Imaginons que AB soit rempli de n lignes et de n colonnes, DE de n lignes et de p colonnes, FG de n colonnes et de p lignes d'éléments quelconques, enfin BC de p lignes et de p colonnes de zéros. Cela posé, et en se servant de AB, AC pour représenter les déterminants des éléments qui se trouvent dans ces carrés, le produit $(AB)^{n-1} AC$ sera égal au déterminant composé dont chaque élément est le déterminant du carré AB bordé par l'une des n colonnes de DE et par l'une des n lignes de FG.

Supposons que tous les éléments de AB s'évanouissent à l'exception des n éléments qui se trouvent dans la diagonale AB, et que ces derniers soient égaux à l'unité. Dans ce cas, si n est plus petit que p, le déterminant AC

[* As proved, pp. 249, 650 of Vol. I. of this Reprint.]

s'évanouira; de plus le carré AB bordé de la colonne $c_1c_2 \dots c_n$ et de la ligne $l_1l_2 \dots l_n$ produira le déterminant

$$(c_1, c_2, \dots c_n)(l_1, l_2, \dots l_n) \text{ c. à d. } c_1l_1 + c_2l_2 + \dots + c_nl_n:$$

de là on tire la conséquence que si l'on multiplie deux matrices rectangulaires selon la direction de leurs axes mineurs, le déterminant qui en résulte est égal à zéro. Dans le cas contraire où n est égal ou plus grand que p, AC devient la somme des produits de chaque déterminant de l'ordre p puisé dans DE multiplié par le déterminant correspondant puisé dans FG: et dans le cas particulier de $n = p$, c. à d. lorsque les matrices DE, FG deviennent des carrés, cette somme de produits se réduit au produit des déterminants DE, FG: ce qui est la règle élémentaire de multiplication des déterminants.

Voilà le point extrême jusqu'auquel j'avais précédemment avancé la théorie des déterminants composés—c. à d. jusqu'au cas dans lequel une ligne double non-distribuée s'attache comme une espèce de caput-mortuum aux lignes doubles puisées dans les combinaisons des ombres supérieures avec des ombres inférieures d'une seule ligne double: mais en y réfléchissant je me suis senti dans la nécessité morale d'étendre la théorie d'abord au cas où toutes les deux lignes doubles sont distribuées et puis au cas le plus général où l'on puise dans un nombre quelconque de lignes doubles les combinaisons des ombres supérieures et inférieures (chaque combinaison d'un ordre donné), et je vais donner l'expression générale des déterminants ainsi composés en termes des déterminants simples qui correspondent aux lignes doubles prises séparément ou combinées d'une manière quelconque entr'elles. Soient A, B, C, ... L, M, ... Z un nombre quelconque i de lignes doubles; soient a, b, c, ... l, m, ... z leurs indices d'étendue: ainsi par exemple A représentera une ligne double de la forme

$$\begin{array}{c} \theta_1 \theta_2 \dots \theta_a \\ \phi_1 \phi_2 \dots \phi_a. \end{array}$$

Construisons le déterminant composé

$${}^{\alpha}A^{\beta}B^{\gamma}C \dots {}^{\zeta}Z,$$

$\alpha, \beta, \gamma, \dots \zeta$ étant les indices de *distribution* de A, B, C, ... Z, c. à d. que l'on a un déterminant dont les éléments sont des déterminants représentés par des lignes doubles dans chacune desquelles la ligne supérieure est formée par la juxtaposition de α des ombres supérieures de A, β des ombres supérieures de B, ... ζ des ombres supérieures de Z, et de même la ligne inférieure par la juxtaposition des lignes inférieures correspondantes: par exemple ${}^{\alpha}A^{\beta}B^{\gamma}C$ signifiera un déterminant composé de la forme

$$\Sigma p_1p_2 \dots p_\alpha, \times \Sigma q_1q_2 \dots q_\beta, \times \Sigma r_1r_2 \dots r_\gamma$$
$$\div \Sigma p_1'p_2' \dots p_\alpha', \times \Sigma q_1'q_2' \dots q_\beta', \times \Sigma r_1'r_2' \dots r_\gamma'.$$

Or je dis que le déterminant ${}^{\alpha}A^{\beta}B^{\gamma}C \dots {}^{\zeta}Z$ sera* un produit des puissances de

[* See however, Borchardt: *Remarque relative au mémoire de M. Sylvester sur les déterminants composés*, *Crelle*, Bd. LXXXIX. (1880).]

A, de B, ... de Z, de AB, AC, BC, ... AZ, BZ, CZ et en général des lignes doubles en nombre $2^i - 1$ qu'on peut former par la juxtaposition de toutes les manières possibles d'un nombre quelconque des A, B, ... Z.

Pour obtenir les exposants de ces puissances voici la règle. Servons-nous en général de la notation (l, λ) pour réprésenter le nombre du binôme

$$\frac{l(l-1)\dots(l-\lambda+1)}{1.2\dots\lambda},$$

et posons $$(a-1, \alpha)(b-1, \beta)\dots(z-1, \zeta) = \varpi,$$

l'exposant de la puissance de $AB \dots L$ sera

$$\frac{(a-1, \alpha-1)(b-1, \beta-1)\dots(l-1, \lambda-1)}{(a-1, \alpha)(b-1, \beta)\dots(l-1, \lambda)}\varpi.$$

Comme vérification numérique je remarquerai que l'on a

$$\frac{(a-1, \alpha)}{(a-1, \alpha-1)} = \frac{a-\alpha}{\alpha}.$$

Or en regardant une puissance P^n comme répétition n-tuple de P, on verra facilement que dans le produit de puissances donné ci-dessus le nombre de fois que les combinaisons α^{mes} des ombres de A et les combinaisons β^{mes} des ombres de B etc. se trouvent répétées, sera le produit de toutes les quantités de la forme

$$(a-1, \alpha-1)\left(1 + \frac{(a-1, \alpha)}{(a-1, \alpha-1)}\right) = \frac{a}{\alpha}(a-1, \alpha-1) = (a, \alpha),$$

résultat qui s'accorde bien avec la remarque que l'ordre du déterminant composé dont il est question est évidemment égal au produit $(a, \alpha)(b, \beta)\dots(z, \zeta)$.

Je conclurai en appliquant à un exemple le théorème énoncé ci-dessus. Considérons le déterminant composé ${}^{m}A_m{}^{2}B_3{}^{1}C_3$ où m, 2, 1 sont les indices de distribution et m, 3, 3 les indices d'étendue de A, B, C. On forme les trois couples de nombres binômes consécutifs *

$$\begin{array}{cc} 1, & 0 \\ 2, & 1 \\ 1, & 2; \end{array}$$

alors en remarquant que

$$\begin{array}{lll} 1.1.2 = 2, & 1.2.2 = 4, & \\ 2.0.2 = 0, & 1.1.1 = 1, & 1.2.1 = 2, \\ 1.0.1 = 0, & 1.2.0 = 0, & \end{array}$$

on en déduit la conséquence que

$${}^{m}A_m{}^{2}B_3{}^{1}C_3 = A^2(AB)^4(AC)(ABC)^2.$$

Soient $$A = \begin{matrix} a_1a_2 \dots a_m \\ \alpha_1\alpha_2 \dots \alpha_m \end{matrix}, \quad B = \begin{matrix} b_1\,b_2\,b_3 \\ \beta_1\beta_2\beta_3 \end{matrix}, \quad C = \begin{matrix} c_1\,c_2\,c_3 \\ \gamma_1\gamma_2\gamma_3 \end{matrix},$$

* On remarquera que 1, 2 sont les coefficients de t^0, t^1 dans $(1+t)^{3-1}$, 2, 1 de t^1, t^2 dans $(1+t)^{3-1}$ et 1, 0 de t^{m-1}, t^m dans $(1+t)^{m-1}$.

on aura le déterminant composé du 9^me ordre dont la première ligne est

$$\begin{matrix} a_1\dots a_m\, b_1\, b_2\, c_1, & a_1\dots a_m\, b_1\, b_2\, c_2, & a_1\dots a_m\, b_1\, b_2\, c_3, & a_1\dots a_m\, b_1\, b_3\, c_1, & a_1\dots a_m\, b_1\, b_3\, c_2, \\ \alpha_1\dots\alpha_m\beta_1\beta_2\gamma_1, & \alpha_1\dots\alpha_m\beta_1\beta_2\gamma_1, & \alpha_1\dots\alpha_m\beta_1\beta_2\gamma_1, & \alpha_1\dots\alpha_m\beta_1\beta_2\gamma_1, & \alpha_1\dots\alpha_m\beta_1\beta_2\gamma_1, \end{matrix}$$

$$\begin{matrix} a_1\dots a_m\, b_1\, b_3\, c_3, & a_1\dots a_m\, b_2\, b_3\, c_1, & a_1\dots a_m\, b_2\, b_3\, c_2, & a_1\dots a_m\, b_2\, b_3\, c_3, \\ \alpha_1\dots\alpha_m\beta_1\beta_2\gamma_1, & \alpha_1\dots\alpha_m\beta_1\beta_2\gamma_1, & \alpha_1\dots\alpha_m\beta_1\beta_2\gamma_1, & \alpha_1\dots\alpha_m\beta_1\beta_2\gamma_1, \end{matrix}$$

et dont on forme les autres lignes en remplaçant $\alpha_1\dots\alpha_m\beta_1\beta_2\gamma_1$ successivement par

$$\begin{matrix} \alpha_1\dots\alpha_m\beta_1\beta_2\gamma_2, & \alpha_1\dots\alpha_m\beta_1\beta_3\gamma_1, & \alpha_1\dots\alpha_m\beta_2\beta_3\gamma_1, \\ \alpha_1\dots\alpha_m\beta_1\beta_2\gamma_3, & \alpha_1\dots\alpha_m\beta_1\beta_3\gamma_2, & \alpha_1\dots\alpha_m\beta_2\beta_3\gamma_2, \\ & \alpha_1\dots\alpha_m\beta_1\beta_3\gamma_3, & \alpha_1\dots\alpha_m\beta_2\beta_3\gamma_3. \end{matrix}$$

La valeur de ce déterminant est donnée par le produit:

$$\binom{a_1 a_2 \dots a_m}{\alpha_1\alpha_2\dots\alpha_m}^2 \binom{a_1 a_2\dots a_m b_1\, b_2\, b_3}{\alpha_1\alpha_2\dots\alpha_m\beta_1\beta_2\beta_3}^4 \binom{a_1 a_2\dots a_m c_1 c_2 c_3}{\alpha_1\alpha_2\dots\alpha_m\gamma_1\gamma_2\gamma_3} \binom{a_1 a_2\dots a_m b_1\, b_2\, b_3\, c_1\, c_2\, c_3}{\alpha_1\, \alpha_2\dots\alpha_m\beta_1\beta_2\beta_3\gamma_1\gamma_2\gamma_3}^2.$$

Le théorème général énoncé ci-dessus contient le résultat le plus général sur l'évaluation des déterminants composés du deuxième rang, c. à d. des déterminants dont les éléments sont des déterminants simples. On pourrait étendre ces recherches aux déterminants du 3^me rang, c. à d. dont les éléments sont des déterminants du 2^me rang, et même aux déterminants composés d'un rang quelconque.

J'introduirai une légère modification dans l'énoncé du théorème général. On peut toujours supposer qu'aux i arguments distribués désignés par $A, B, \dots Z$ soit associé un $i+1$^me argument non distribué Ω, bien entendu que dans un cas donné ce dernier peut disparaître. On n'a pas besoin de supposer l'existence de plus d'un seul argument non distribué, parce que, s'il y en avait plusieurs, on pourrait toujours les réunir en un seul.

Soient $a, b, \dots z$ les indices d'étendue et $\alpha, \beta, \dots \zeta$ les indices de distribution des arguments $A, B, \dots Z$, et désignons comme ci-dessus par (a, α) le nombre binôme

$$(a, \alpha) = \frac{a\,.\,a-1\dots a-\alpha+1}{1\,.\,2\dots\alpha},$$

par ϖ le produit

$$\varpi = (a-1, \alpha)(b-1, \beta)\dots(z-1, \zeta),$$

et par $a', b', \dots z'$ les quotients

$$a' = \frac{\alpha}{a-\alpha},\quad b' = \frac{\beta}{b-\beta},\quad \dots z' = \frac{\zeta}{z-\zeta}.$$

Cela posé, le théorème général pour l'évaluation des déterminants composés prendra la forme:

$$(\Omega\,{}^{\alpha}A_a\,{}^{\beta}B_b\dots{}^{\zeta}Z_z) = P^{\varpi},$$

P désignant le produit des 2^i facteurs

$$\begin{array}{l} (\Omega) \\ (\Omega A)^{a'}(\Omega B)^{b'}\dots(\Omega Z)^{z'} \\ (\Omega AB)^{a'b'}(\Omega AC)^{a'c'}\dots(\Omega BC)^{b'c'}\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ (\Omega AB\dots Z)^{a'b'\dots z'}. \end{array}$$

Quand Ω disparaît il faut remplacer (Ω) par l'unité.

Cette formule s'écrit plus aisément sous la forme logarithmique suivante:

$$\frac{1}{\varpi}.\log(\Omega\,{}^{\alpha}A_a\,{}^{\beta}B_b\dots{}^{\zeta}Z_z) = \log(\Omega) + \Sigma\frac{\alpha}{a-\alpha}\log(\Omega A)$$

$$+\Sigma\frac{\alpha}{a-\alpha}.\frac{\beta}{b-\beta}\log(\Omega AB) + \dots + \frac{\alpha}{a-\alpha}.\frac{\beta}{b-\beta}\cdots\frac{\zeta}{z-\zeta}\log(\Omega AB\dots Z),$$

les sommations s'étendant à tous les expressions semblables.

Il est bon de remarquer que la valeur du déterminant simple représenté par la juxtaposition d'un nombre quelconque d'arguments $A, B, \dots$ ne dépend, ni pour sa valeur, ni pour son signe algébrique, de l'ordre de la juxtaposition, car on a $(AB) = (BA)$ et de même pour un nombre quelconque d'arguments.

Pour donner une idée nette du théorème général, faisons disparaître Ω et considérons le cas de trois arguments A, B, C aux indices d'étendue a, b, c et aux indices de distribution α, β, γ. Dans la figure les carrés A, B, C correspondent aux déterminants (A), (B), (C); $AMBM'$, $AN'CN$, $BPCP'$ aux déterminants (AB), (AC) (BC); enfin le carré complet au déterminant (ABC).

A	M	N'
M'	B	P
N	P'	C

Les éléments du déterminant composé ${}^{\alpha}A_a\,{}^{\beta}B_b\,{}^{\gamma}C_c$ sont des déterminants simples de l'ordre $\alpha+\beta+\gamma$, formés avec

$$\alpha^2,\ \alpha\beta,\ \alpha\gamma,\ \beta\alpha,\ \beta^2,\ \beta\gamma,\ \gamma\alpha,\ \gamma\beta,\ \gamma^2,$$

éléments puisés respectivement dans

$$A,\ M,\ N',\ M',\ B,\ P,\ N,\ P',\ C.$$

Les éléments des déterminants simples se trouvent aux intersections de

α lignes qui passent par A, M, N',
β „ „ „ „ M', B, P,
γ „ „ „ „ N, P', C,

avec α colonnes qui passent par A, M', N,

β „ „ „ „ M, B, P',

γ „ „ „ „ N', P, C.

Posons $\alpha' = a - \alpha$, $\beta' = b - \beta$, … $\zeta' = z - \zeta$

et nommons déterminants complémentaires du déterminant $(\Omega^{\alpha}A_a{}^{\beta}B_b \dots {}^{\zeta}Z_z)$ tous ceux dans lesquels un nombre quelconque d'indices de distribution $\alpha, \beta, \dots \zeta$ sont remplacées par leurs indices complémentaires $\alpha', \beta', \dots \zeta'$. Cela posé, il existe une relation très-simple qui lie ensemble les 2^i déterminants complémentaires. En effet, en ajoutant les expressions que fournit le théorème général pour le logarithme de chacun de ces 2^i déterminants composés on trouve par un calcul facile que la somme de ces logarithmes est égale à

$$\frac{\Pi a \,.\, \Pi b \dots \Pi z}{\Pi \alpha \,.\, \Pi \alpha' \,.\, \Pi \beta \,.\, \Pi \beta' \dots \Pi \zeta \,.\, \Pi \zeta'} \{\log(\Omega) + \Sigma \log(\Omega A) + \Sigma \log(\Omega A B) + \dots + \log(\Omega A B \dots Z)\}.$$

La quantité, qui se trouve en parenthèse, étant indépendante de $\alpha, \beta, \dots \zeta$, on est arrivé à ce résultat remarquable: de quelque manière que l'on fasse la partition en deux nombres

$$\alpha, \alpha', \beta, \beta', \dots \zeta, \zeta'$$

des nombres donnés $a, b, \dots z$, la somme des logarithmes des 2^i déterminants mutuellement complémentaires sera toujours, à un facteur numérique près, la même.

En faisant disparaître dans le théorème général l'argument non-distribué Ω, on revient à la forme première dans laquelle le résultat a été énoncé, c. à d. à l'équation

$$\frac{\log({}^{\alpha}A_a{}^{\beta}B_b \dots {}^{\zeta}Z_z)}{(a-1,\alpha)(b-1,\beta)\dots(z-1,\zeta)} = \Sigma \frac{\alpha}{a-\alpha} \log(A) + \Sigma \frac{\alpha}{a-\alpha} \frac{\beta}{b-\beta} \log(AB) + \dots + \Sigma \frac{\alpha}{a-\alpha} \frac{\beta}{b-\beta} \cdots \frac{\zeta}{z-\zeta} \log(AB \dots Z).$$

L'indice de distribution α pouvant prendre toutes les valeurs depuis 0 jusqu'à a, considérons le cas particulier de $\alpha = a$. En multipliant par $(a-1, \alpha)$, nombre qui s'évanouit pour $\alpha = a$, et en posant $\alpha = a$, de tous les termes de la seconde partie de l'équation tous ceux qui ne contiennent pas la lettre A s'évanouissent et il vient

$$\frac{\log({}^{a}A_a{}^{\beta}B_b \dots {}^{\zeta}Z_z)}{(b-1,\beta)\dots(z-1,\zeta)} = \log(A) + \Sigma' \frac{\beta}{b-\beta} \log(AB) + \dots + \Sigma' \frac{\beta}{b-\beta} \cdots \frac{\zeta}{z-\zeta} \log(AB \dots Z)$$

où les sommations Σ' se rapportent seulement aux arguments $B \dots Z$ à l'exclusion de A.

Mais dans le cas général dans lequel la valeur de α est quelconque la seconde partie de l'équation qui donne la valeur de $\log({}^{\alpha}A_a{}^{\beta}B_b \dots {}^{\zeta}Z_z)$ peut être présentée sous la forme

$$\frac{\alpha}{a-\alpha}\left\{\log(A) + \Sigma' \frac{\beta}{b-\beta} \log(AB) + \dots + \Sigma' \frac{\beta}{b-\beta} \cdots \frac{\zeta}{z-\zeta} \log(AB \dots Z)\right\}$$
$$+ \Sigma' \frac{\beta}{b-\beta} \log(B) + \Sigma' \frac{\beta}{b-\beta} \frac{\gamma}{c-\gamma} \log(BC) + \dots$$
$$+ \Sigma' \frac{\beta}{b-\beta} \frac{\gamma}{c-\gamma} \cdots \frac{\zeta}{z-\zeta} \log(BC \dots Z)$$

équivalente à

$$\frac{\alpha}{a-\alpha} \cdot \frac{\log({}^{a}A_a{}^{\beta}B_b \dots {}^{\zeta}Z_z)}{(b-1,\beta)\dots(z-1,\zeta)} + \frac{\log({}^{\beta}B_b{}^{\gamma}C_c \dots {}^{\zeta}Z_z)}{(b-1,\beta)\dots(z-1,\zeta)}.$$

Donc puisque $\frac{\alpha}{a-\alpha}(a-1,\alpha) = (a-1,\alpha-1)$, on est conduit à la relation

$$(\vartheta.) \quad \log({}^{\alpha}A_a{}^{\beta}B_b \dots {}^{\zeta}Z_z) = (a-1,\alpha)\log({}^{\beta}B_b \dots {}^{\zeta}Z_z) + (a-1,\alpha-1)\log({}^{a}A_a{}^{\beta}B_b \dots {}^{\zeta}Z_z).$$

Je me bornerai au cas particulier de cette formule dans lequel le nombre des arguments ne s'élève qu'à deux, et je passerai à une formule plus générale qui forme une extension de l'équation (ϑ.) relative au cas de deux arguments. Concevons un carré composé contenant b^2 carrés simples chacun de κ^2 éléments. Rangeant ces b^2 carrés simples en b lignes et b colonnes et choisissant β de ces b lignes et β de ces b colonnes on est conduit à un déterminant composé de l'ordre (b, β) et que je désignerai par ${}^{\kappa^2.\beta}B_{\kappa^2.b}$. Les éléments de ce déterminant sont eux-mêmes des déterminants composés de l'ordre β dont chaque élément est un déterminant simple comprenant κ^2 éléments.

Ayant défini le sens de ${}^{\kappa^2.\beta}B_{\kappa^2.b}$, passons à l'interprétation de la notation ${}^{\alpha}A_a{}^{\kappa^2.\beta}B_{\kappa^2.b}$. Concevons un carré de $a + \kappa b$ lignes et d'autant de colonnes, divisé en deux carrés A et B_κ, de a^2 et de $\kappa^2 b^2$ termes respectivement, et en deux rectangles de $a \,.\, \kappa b$ termes. Choisissons α lignes et autant de colonnes qui passent par A, choisissons β groupes de κ lignes et autant de groupes de κ colonnes qui passent par B_κ. Les déterminants formés des $(\alpha + \kappa b)^2$ termes choisis formeront les éléments d'un déterminant composé, de l'ordre $(a, \alpha)(b, \beta)$ par rapport à ses éléments composés, et que je désignerai par ${}^{\alpha}A_a{}^{\kappa^2.\beta}B_{\kappa^2.b}$. Pour les déterminants ainsi définis on peut énoncer le théorème suivant

$$(\eta.) \quad \log({}^{\alpha}A_a{}^{\kappa^2.\beta}B_{\kappa^2.b}) = (a-1,\alpha)\log({}^{a}A_a{}^{\kappa^2.\beta}B_{\kappa^2.b}) + (a-1,\alpha-1)\log({}^{\kappa^2.\beta}B_{\kappa^2.b}).$$

Considérons l'exemple de

$$a=2,\quad b=2,\quad \varkappa=2,\quad \alpha=1,\quad \beta=1,$$

qui se rapporte aux déterminants mineurs de la matrice

$$\begin{array}{|cc|cccc|}
\hline
p & q & r & s & t & u \\
p' & q' & r' & s' & t' & u' \\
\hline
p'' & q'' & l & l' & m & m' \\
p''' & q''' & l'' & l''' & m'' & m''' \\
p^{\text{IV}} & q^{\text{IV}} & \lambda & \lambda' & \mu & \mu' \\
p^{\text{V}} & q^{\text{V}} & \lambda'' & \lambda''' & \mu'' & \mu''' \\
\hline
\end{array}\,.$$

Dans ce cas les déterminants

$$d=({}^{\varkappa^2.\beta}B_{\varkappa^2.b}),\quad D=({}^{a}A_{\alpha}{}^{\varkappa^2.\beta}B_{\varkappa^2.b}),\quad \Delta=({}^{a}A_{\alpha}{}^{\varkappa^2.\beta}B_{\varkappa^2.b})$$

ont les valeurs

$$d=\begin{array}{|cc|cc|}
\hline
l & l' & m & m' \\
l'' & l''' & m'' & m''' \\
\hline
\lambda & \lambda' & \mu & \mu' \\
\lambda'' & \lambda''' & \mu'' & \mu''' \\
\hline
\end{array}\,,$$

$$D=\begin{array}{|cccc|cccc|}
\hline
p & q & r & s & p & q & t & u \\
p' & q' & r' & s' & p' & q' & t' & u' \\
p'' & q'' & l & l' & p'' & q'' & m & m' \\
p''' & q''' & l'' & l''' & p''' & q''' & m'' & m''' \\
\hline
p & q & r & s & p & q & t & u \\
p' & q' & r' & s' & p' & q' & t' & u' \\
p^{\text{IV}} & q^{\text{IV}} & \lambda & \lambda' & p^{\text{IV}} & q^{\text{IV}} & \mu & \mu' \\
p^{\text{V}} & q^{\text{V}} & \lambda'' & \lambda''' & p^{\text{V}} & q^{\text{V}} & \mu'' & \mu''' \\
\hline
\end{array}\,,$$

$$\Delta=\begin{array}{|ccc|ccc|ccc|ccc|}
\hline
p & r & s & q & r & s & p & t & u & q & t & u \\
p'' & l & l' & q'' & l & l' & p'' & m & m' & q'' & m & m' \\
p''' & l'' & l''' & q''' & l'' & l''' & p''' & m'' & m''' & q''' & m'' & m''' \\
\hline
p' & r' & s' & q' & r' & s' & p' & t' & u' & q' & t' & u' \\
p'' & l & l' & q'' & l & l' & p'' & m & m' & q'' & m & m' \\
p''' & l'' & l''' & q''' & l'' & l''' & p''' & m'' & m''' & q''' & m'' & m''' \\
\hline
p & r & s & q & r & s & p & t & u & q & t & u \\
p^{\text{IV}} & \lambda & \lambda' & q^{\text{IV}} & \lambda & \lambda' & p^{\text{IV}} & \mu & \mu' & q^{\text{IV}} & \mu & \mu' \\
p^{\text{V}} & \lambda'' & \lambda''' & q^{\text{V}} & \lambda'' & \lambda''' & p^{\text{V}} & \mu'' & \mu''' & q^{\text{V}} & \mu'' & \mu''' \\
\hline
p' & r' & s' & q' & r' & s' & p' & t' & u' & q' & t' & u' \\
p^{\text{IV}} & \lambda & \lambda' & q^{\text{IV}} & \lambda & \lambda' & p^{\text{IV}} & \mu & \mu' & q^{\text{IV}} & \mu & \mu' \\
p^{\text{V}} & \lambda'' & \lambda''' & q^{\text{V}} & \lambda'' & \lambda''' & p^{\text{V}} & \mu'' & \mu''' & q^{\text{V}} & \mu'' & \mu''' \\
\hline
\end{array}\,;$$

et la relation qui existe entre eux se réduit à

$$\Delta=d\,.\,D.$$

Du théorème (η.) relatif à deux arguments dont l'un est complexe* on pourrait faire l'extension à un nombre quelconque d'arguments complexes, mais sans m'y arrêter je vais au contraire passer à un cas particulier du théorème (η.) et que je nommerai le théorème du gnomon†. Faisons coincider respectivement dans l'exemple donné ci-dessus les huit quantités

$$r,\quad s,\quad r',\quad s',\quad p'',\quad q'',\quad p''',\quad q'''$$

avec

$$t,\quad u,\quad t',\quad u',\quad p^{\text{IV}},\quad q^{\text{IV}},\quad p^{\text{V}},\quad q^{\text{V}}.$$

Dans ce cas chaque carré dont le déterminant forme un élément de d doit être enveloppé par le gnomon non distribué

$$(G.)\qquad \left\{\begin{array}{cccc}
p & q & r & s \\
p' & q' & r' & s' \\
p'' & q'' & & \\
p''' & q''' & &
\end{array}\right.$$

pour passer des éléments de d aux éléments de D.

* Je me sers ici du mot complexe dans un autre sens que dans celui qui est généralement en usage dans l'analyse.

† γνώμων (équerre), voyez le premier livre des éléments d'Euclide.

De même pour obtenir les éléments de Δ déduisons du gnomon $(G.)$ les gnomons partiels

$$\begin{array}{lllllll} p & r & s & \qquad & q & r & s \\ p'' & & & & q'' & & \\ p''' & & & & q''' & & \\ p' & r' & s' & & q' & r' & s' \\ p'' & & & & q'' & & \\ p''' & & & & q''' & & \end{array}$$

et enveloppons successivement chaque élément de d par ces quatre gnomons partiels, ce procédé que l'on peut nommer enveloppement distributif nous conduit aux éléments de Δ.

Les mêmes lois de formation existent pour un gnomon d'ordre quelconque que je désignerai par ${}^{a,\kappa}G_{a,\kappa}$ et que l'on combinera avec un carré de l'ordre b de carrés de l'ordre κ.

En dénotant pour plus de brièveté par B_κ le carré de b^2 carrés de κ^2 éléments on a donc l'équation

$$\log({}^{\alpha,\kappa}G_{a,\kappa}B_\kappa) = (a-1, \alpha)\log(B_\kappa) + (a-1, \alpha-1)\log({}^{a,\kappa}G_{a,\kappa}B_\kappa),$$

résultat que l'on peut énoncer de la manière suivante. Étant donné un carré de carrés simples et un gnomon ayant ses deux branches rectangulaires de longueur égale au côté des carrés simples: le déterminant composé du second rang dont les éléments sont les déterminants des carrés simples donnés enveloppés par un carré de gnomous partiels dérivés du gnomon donné sera égal au produit de puissances entières de deux déterminants composés. Les éléments de ces déterminants sont les déterminants des carrés simples donnés et les déterminants de ces carrés simples enveloppés par le gnomon donné.

Les exposants des puissances qui entrent dans la formule en question ne dépendent que de l'indice a d'étendue et de l'indice α de distribution de la partie carrée du gnomon.

Le théorème du gnomon contient comme cas particulier l'équation $(\vartheta.)$. Pour s'en convaincre on n'a qu'à former le carré composé qui correspond à ${}^\beta B_b \ldots {}^\zeta Z_z$, carré qui porte la notation $(b, \beta)\,(c, \gamma)\ldots(z, \zeta)$ et qui comprend des carrés simples de l'ordre $\beta+\gamma+\ldots+\zeta$. Ces deux nombres devront remplacer les nombres b et κ que l'on a considérés dans l'étude du gnomon. En appliquant à ce carré composé un gnomon dont l'étendue des branches est égale à $\beta+\gamma+\ldots+\zeta$ et l'étendue de la partie carrée égale à $(b, \beta)\,(c, \gamma)\ldots(z, \zeta)$ on retrouve l'équation $(\vartheta.)$.

En désignant par Φ le symbole

$${}^\beta B_b {}^\gamma C_c \ldots {}^\zeta Z_z$$

l'équation $(\vartheta.)$ prend la forme

$$\log({}^a A_a \Phi) = (a-1, \alpha)\log(\Phi) + (a-1, \alpha-1)\log({}^a A_a \Phi).$$

On peut énoncer le résultat plus général qu'entre les trois expressions

$$\log {}^{\alpha_1}A_a\Phi, \quad \log {}^{\alpha_2}A_a\Phi, \quad \log {}^{\alpha_3}A_a\Phi$$

il y a toujours une relation linéaire à coefficients indépendants de b, β, c, γ, ..., z, ζ. En effet soit

$$\alpha_1' = a - \alpha_1, \quad \alpha_2' = a - \alpha_2, \quad \alpha_3' = a - \alpha_3,$$

la relation dont il s'agit s'exprime par l'équation

$$\sum_1^3 \Pi\alpha_3 \,.\, \Pi\alpha_3'\,(\alpha_1\alpha_2' - \alpha_2\alpha_1')\log {}^{\alpha_3}A_a\Phi = 0,$$

le signe $\sum\limits_1^3$ exprimant la somme que l'on obtient en ajoutant au terme écrit les deux autres qui en dérivent par une permutation cyclique des trois indices 1, 2, 3. Cette équation ne change pas quand on échange α_1, α_2, α_3 avec α_1', α_2', α_3', de plus elle peut être présentée dans cette forme plus simple

$$\Sigma\Pi\alpha_3\Pi\alpha_3'(\alpha_1-\alpha_2)\log {}^{\alpha_3}A_a\Phi = 0.$$

Désignant par Ψ le symbole

$${}^\gamma C_c \ldots {}^\zeta Z_z$$

on trouvera aisément la relation

$$\log {}^a A_a {}^\beta B_b \Psi = (a-1, \alpha)(b-1, \beta)\log\Psi + (a-1, \alpha-1)(b-1, \beta)\log A\Psi$$
$$+ (a-1, \alpha)(b-1, \beta-1)\log B\Psi + (a-1, \alpha-1)(b-1, \beta-1)\log AB\Psi$$

dans laquelle A, B sont écrits au lieu de ${}^a A_a {}^b B_b$. On peut énoncer le résultat plus général qu'entre cinque expressions de la forme

$$\log {}^{\alpha_r}A_a {}^{\beta_r}B_b\Psi \qquad (r = 1, 2, 3, 4, 5)$$

il existe une relation linéaire. En effet, soit

$$\alpha_r + \alpha_r' = a, \quad \beta_r + \beta_r' = b,$$

la relation dont il s'agit prend la forme

$$\Sigma\Pi\alpha_5\Pi\alpha_5'\Pi\beta_5\Pi\beta_5' D_{1,2,3,4}\log {}^{\alpha_5}A_a {}^{\beta_5}B_b\Psi = 0.$$

Dans cette équation $D_{1,2,3,4}$ désigne le déterminant

$$\begin{array}{llll} \alpha_1\beta_1 & \alpha_1\beta_1' & \alpha_1'\beta_1 & \alpha_1'\beta_1' \\ \alpha_2\beta_2 & \alpha_2\beta_2' & \alpha_2'\beta_2 & \alpha_2'\beta_2' \\ \alpha_3\beta_3 & \alpha_3\beta_3' & \alpha_3'\beta_3 & \alpha_3'\beta_3' \\ \alpha_4\beta_4 & \alpha_4\beta_4' & \alpha_4'\beta_4 & \alpha_4'\beta_4'. \end{array}$$

En divisant par a^2b^2 il prend la forme plus simple

$$\begin{vmatrix} 1 & \alpha_1 & \beta_1 & \alpha_1\beta_1 \\ 1 & \alpha_2 & \beta_2 & \alpha_2\beta_2 \\ 1 & \alpha_3 & \beta_3 & \alpha_3\beta_3 \\ 1 & \alpha_4 & \beta_4 & \alpha_4\beta_4. \end{vmatrix}$$

Ce résultat peut être aisément étendu à un nombre quelconque d'arguments. En effet désignons par Θ le symbole

$${}^{a_{i+1}}A^{(i+1)}_{a_{i+1}}\,{}^{a_{i+2}}A^{(i+2)}_{a_{i+2}}\ldots{}^{a_m}A^{(m)}_{a_m}$$

et soit

$$j = 2^i;$$

cela posé, il y aura toujours entre les $j+1$ expressions

$$\log{}^{a_{1,q}}A^{(1)}_{a_1}\,{}^{a_{2,q}}A^{(2)}_{a_2}\ldots{}^{a_{i,q}}A^{(i)}_{a_i}\Theta \qquad (q = 1, 2, \ldots j+1)$$

une relation linéaire exprimée par l'équation

$$\Sigma\left\{\begin{matrix}\Pi\alpha_{1,j+1}\Pi\alpha'_{1,j+1}\Pi\alpha_{2,j+1}\Pi\alpha'_{2,j+1}\ldots\Pi\alpha_{i,j+1}\Pi\alpha'_{i,j+1}\\ \times D_{1,2,\ldots j}\log{}^{a_{1,j+1}}A^{(1)}_{a_1}\,{}^{a_{2,j+1}}A^{(2)}_{a_2}\ldots{}^{a_{i,j+1}}A^{(i)}_{a_i}\end{matrix}\right\} = 0$$

dans laquelle

$$\alpha'_{p,q} + \alpha_{p,q} = a_p$$

et où $D_{1,2,\ldots j}$ désigne le déterminant dialytique formé par les développements des expressions:

$$(1+\alpha_{1,1}x_1)(1+\alpha_{2,1}x_2)\ldots(1+\alpha_{i,1}x_i)$$
$$(1+\alpha_{1,2}x_1)(1+\alpha_{2,2}x_2)\ldots(1+\alpha_{i,2}x_i)$$
$$\vdots$$
$$(1+\alpha_{1,j}x_1)(1+\alpha_{2,j}x_2)\ldots(1+\alpha_{i,j}x_i)$$

dans lesquelles les j produits différents des variables sont considérés comme j variables indépendantes.

Remarquons encore que lorsque $i = 2$ ou > 2 on aura même une relation entre $j = 2^i$ logarithmes, pourvu qu'entre les nombres $\alpha_{p,q}$ une condition se trouve remplie.

Pour $i = 2$, p. e. si α_1, β_1, α_2, β_2, α_3, β_3, α_4, β_4 sont les coordonnées de quatre points d'une hyperbole dans un système de coordonnées parallèles aux asymptotes, on aura l'équation

$$\begin{vmatrix} U_1 & \alpha_1 & \beta_1 & 1 \\ U_2 & \alpha_2 & \beta_2 & 1 \\ U_3 & \alpha_3 & \beta_3 & 1 \\ U_4 & \alpha_4 & \beta_4 & 1 \end{vmatrix} = 0$$

U_r désignant l'expression

$$U_r = \Pi\alpha_r\Pi(a-\alpha_r)\,\Pi\beta_r\Pi(b-\beta_r)\log{}^{\alpha_r}A_a{}^{\beta_r}B_b\Psi.$$

Des valeurs particulières des α_r, β_r qui satisfont à la condition indiquée sont p. e. les suivantes:

$$\alpha_1 = 1+\lambda, \quad \alpha_2 = p+\lambda, \quad \alpha_3 = q+\lambda, \quad \alpha_4 = pq+\lambda,$$
$$\beta_1 = pq+\mu, \quad \beta_2 = q+\mu, \quad \beta_3 = p+\mu, \quad \beta_4 = 1+\mu;$$

pour ces valeurs la relation linéaire entre U_1, U_2, U_3, U_4 se réduit à la forme

$$\begin{vmatrix} U_1 & p & q & 1 \\ U_2 & q & p & 1 \\ U_3 & 1 & pq & 1 \\ U_4 & qp & 1 & 1 \end{vmatrix} = 0$$

que l'on obtient par la remarque que les coefficients des U ne changent point lorsqu'on fait coincider avec le centre de l'hyperbole l'origine des coordonnées. De plus en divisant par le facteur commun $(1-p)(1-q)$ l'équation devient

$$(pq-1)(U_2-U_1) = (q-p)(U_4-U_3),$$

ce qui fait voir que deux déterminants $({}^{\alpha_r}A_a{}^{\beta_r}B_b\Psi)$ élevés chacun à une puissance entière et multipliés ensemble donnent un produit égal à une expression semblable relative aux deux autres déterminants de la même forme.

Le cas de $i = 1$ fait exception. Dans ce cas une relation linéaire entre moins de trois expressions différentes est impossible.

Sans sortir de la sphère des déterminants composés du 2^{me} rang il reste à faire une grande généralisation de la théorie précédente.

Jusqu'ici on n'a considéré que des matrices à forme carrée ou, ce qui est la même chose, on ne s'est servi que d'arguments à un seul indice d'étendue et à un seul indice de distribution.—Mais il y a une théorie à construire relative aux arguments ayant chacun deux indices distincts et d'étendue et de distribution. Pour le moment je me borne au cas comparativement simple dans lequel il n'y a qu'un seul indice de distribution tandis que chaque argument a deux indices distincts d'étendue et se rapporte par conséquent à une matrice de forme rectangulaire. Comme il n'y a qu'un seul indice de distribution, les matrices que l'on forme au moyen des matrices rectangulaires données seront des matrices carrées représentant des déterminants comme dans le cas traité jusqu'à présent.

Il sera utile de se servir de l'expression 'déterminant virtuel' ou 'valeur virtuelle d'une matrice rectangulaire.' Cette dénomination ne définit point une quantité que l'on peut directement mettre en évidence, mais plutôt une valeur de nature ombrale ou idéale : cependant, comme je vais faire voir, on pourra établir des rapports actuels entre ces valeurs idéales.

Soient $\mathfrak{D}$ et D deux matrices rectangulaires qui contiennent en dernier lieu les mêmes éléments simples (réels ou ombrals). Accentuons les éléments primitifs et désignons par $\mathfrak{D}'$, D' ce que deviennent alors $\mathfrak{D}$, D.

Multiplions ensemble $\mathfrak{D}$ et $\mathfrak{D}'$, D et D', suivant la règle ordinaire pour la multiplication des matrices, appliquée dans la direction de leur plus grande étendue, et comparons les valeurs des déterminants $(\mathfrak{D}\,.\,\mathfrak{D}')$, $(D\,.\,D')$.

Supposons que ces déterminants remplissent identiquement l'équation

$$(\mathfrak{D}\,.\,\mathfrak{D}')^i = (D\,.\,D')^j,$$

comprenant comme cas particulier l'équation

$$(\mathfrak{D}^2)^i = (D^2)^j,$$

dans ce cas j'écrirai l'équation idéale

$$\mathfrak{D}^i = D^j.$$

Avec cette notion des valeurs virtuelles on peut donner avec un trait de plume une grande extension au théorème général établi dans le mémoire précédent.

En effet, considérons $\Omega, A, B, C, \dots Z$ comme des matrices non plus carrées mais rectangulaires avec cette convention que A_a représente une matrice aux indices d'étendue a et a' et *que a' ne soit pas moindre que a.* Alors ${}^{\alpha}A_a$ représentera une matrice dont les deux indices d'étendue sont (a, α), (a', α) respectivement.

En définissant de cette manière le sens des notations Ω, ${}^{\alpha}A_a$, ${}^{\beta}B_b$, … *je dis que la valeur fournie par le théorème général pour*

$$\frac{\log \Omega\, {}^{\alpha}A_a {}^{\beta}B_b \dots {}^{\zeta}Z_z}{\varpi}$$

ne subit point de changement, quand on remplace les déterminants réels par les déterminants virtuels dans le cas où les matrices carrées se changent en matrices rectangulaires et que l'on n'a pas besoin de tenir compte de l'excès de a' sur a, de b' sur b, etc.

Pour donner un exemple bien simple des déterminants virtuels j'énoncerai l'extension que l'on peut donner à l'équation élémentaire qui dit que le déterminant actuel

$$\mathfrak{C} = \begin{vmatrix} ab' - a'b & ac' - a'c & bc' - b'c \\ ab'' - a''b & ac'' - a''c & bc'' - b''c \\ a'b'' - a''b' & a'c'' - a''c' & b'c'' - b''c' \end{vmatrix}$$

est égal au carré du déterminant actuel

$$C = \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}.$$

Cette extension consiste à dire que le déterminant virtuel de

$$\mathfrak{D} = \begin{vmatrix} ab' - a'b & ac' - a'c & ad' - a'd & bc' - b'c & bd' - b'd & cd' - c'd \\ ab'' - a''b & ac'' - a''c & ad'' - a''d & bc'' - b''c & bd'' - b''d & cd'' - c''d \\ a'b'' - a''b' & a'c'' - a''c' & a'd'' - a''d' & b'c'' - b''c' & b'd'' - b''d' & c'd'' - c''d' \end{vmatrix}$$

est égal au carré du déterminant virtuel de

$$D = \begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \end{vmatrix}.$$

En effet désignons par $\mathfrak{D}_1$, D_1 ce que deviennent les matrices $\mathfrak{D}$, D lorsqu'on y remplace les a, b, c, d par des $\alpha, \beta, \gamma, \delta$; alors des déterminants actuels $(\mathfrak{D}\,.\,\mathfrak{D}_1)$ et $(D\,.\,D_1)$ sont liés par l'équation

$$(\mathfrak{D}\,.\,\mathfrak{D}_1) = (D\,.\,D_1)^2,$$

d'où l'on est en droit de tirer la conséquence énoncée ci-dessus relativement aux déterminants virtuels.

Comme second exemple je considère la ligne-couple

$$\begin{bmatrix} a_1 a_2 \dots a_m b_1 b_2 \dots b_m \\ \alpha_1 \alpha_2 \dots \alpha_m \end{bmatrix}$$

qui représente une matrice de longueur double de sa largeur. Formons le déterminant composé

$$[(\Sigma a_1 a_2 \dots a_i,) \times (\Sigma b_1 b_2 \dots b_j,)] \divideontimes (\alpha_1 \alpha_2 \dots \alpha_m)$$

où

$$i + j = m.$$

D'après un théorème bien connu ce déterminant est égal à

$$\begin{pmatrix} a_1 a_2 \dots a_m \\ \alpha_1 \alpha_2 \dots \alpha_m \end{pmatrix}^{(m-1,\, i)} . \begin{pmatrix} b_1 b_2 \dots b_m \\ \alpha_1 \alpha_2 \dots \alpha_m \end{pmatrix}^{(m-1,\, j)}.$$

Or considérons la ligne-couple

$$\begin{bmatrix} a_1 a_2 \dots a_\nu b_1 b_2 \dots b_\pi \\ \alpha_1 \alpha_2 \dots \alpha_m \end{bmatrix}$$

et supposons que m ne surpasse ni ν ni π, soit de plus comme auparavant

$$i + j = m,$$

cela posé, je dis qu'on aura toujours l'équation

$$[(\Sigma a_1 a_2 \dots a_i,) \times (\Sigma b_1 b_2 \dots b_j,)] \divideontimes (\alpha_1 \alpha_2 \dots \alpha_m)$$
$$= \begin{pmatrix} a_1 a_2 \dots a_\nu \\ \alpha_1 \alpha_2 \dots \alpha_m \end{pmatrix}^{(m-1,\, i)} . \begin{pmatrix} b_1 b_2 \dots b_\pi \\ \alpha_1 \alpha_2 \dots \alpha_m \end{pmatrix}^{(m-1,\, j)}.$$

Dans cette équation et relativement à chacune des trois matrices qu'elle contient il faut substituer lorsqu'il est nécessaire la valeur virtuelle du déterminant au déterminant même, lorsque la matrice est rectangulaire au lieu d'être carrée.

53.

ON THE TRIANGLES IN- AND EX-SCRIBABLE TO A GENERAL CUBIC CURVE.

[*Johns Hopkins University Circulars*, I. (1880), p. 49.]

THE general cubic being thrown into the form $xy^2 + yz^2 + zx^2 + mxyz = 0$, the lines $x = 0$, $y = 0$, $z = 0$ will constitute an in- and ex-scribable triangle to the curve. The number of such was stated to be 24; consisting of 12 pairs of conjugates, each pair being in triple-perspective position with respect to each other, and the centres of the perspective projection being three collinear points of inflexion. Accordingly, the twenty-four triangles will consist of four groups of three pairs of conjugate in- and ex-scripts. The law for the number of polygonal in- and ex-scripts, with any assigned number of sides, will be found stated in No. 4, Vol. II. of the *American Journal of Mathematics* [above, p. 341].

54.

ON THE RESULTANT OF TWO CONGRUENCES.

[*Johns Hopkins University Circulars*, I. (1881), p. 131.]

LET an integer function of a variable be understood to mean an integral rational function thereof whose coefficients are all of them positive or negative integers.

Suppose p to be a fixed prime number; any integer function which is contained in $Fx + p\psi x$, where ψ is an arbitrary integer form, may be termed a modular factor of Fx and all modular factors which are equivalent (quâ the fixed modulus) may be regarded as identical.

An integer function containing no modular factor (except itself) may be regarded as modularly irreducible, and as a very advantageous *façon de parler* may be affirmed to have as many modular roots as there are units in its degree. If linear, there is one modular root which is *actual*, in other cases the modular roots may be termed *hypothetic*, (words which seem preferable to *real* and *imaginary* for the purpose in view). The theorem of Galois, that the number of modular roots of any integer function is the same as the number of units in its degree, is then tantamount to the affirmation that just as an integer number is capable of being resolved in only one way into a product of prime integer factors, so an integer function can be resolved in only one way into a product of modularly irreducible factors.

If one integer root of an irreducible integer function is also a root of a second function, it is well known that all the roots of the first are roots of the second: from that it follows that, *If the resultant of two integer functions vanishes, they must have an irreducible factor in common.* This is analogous to, or, rather is, so to say, an exaltation of, the fact that if the resultant of two real functions of a variable vanishes, they must have a real factor, linear or quadratic, in common*; indissoluble association of pairs of imaginary roots in the world of real quantity being the analogue of indissoluble association of groups of hypothetic roots in the world of integer numbers. In what immediately precedes, the factors spoken of are ordinary algebraical factors. If now we pass from ordinary to modular factors or roots, the theorem above stated, on the introduction of the word 'modular', becomes the theorem referred to by Professor Smith, in the *British Association Report*, 1860, p. 162, and by Mr Hathaway at the last meeting, which may be thus expressed: "*If the resultant of two integer functions is modularly zero* (that is, contains the modulus), *they must have a modular factor in common.*"

* So as a particular exemplification, if one of two integral rational functions with only real coefficients has no real root and their resultant vanishes, they must have two roots in common.