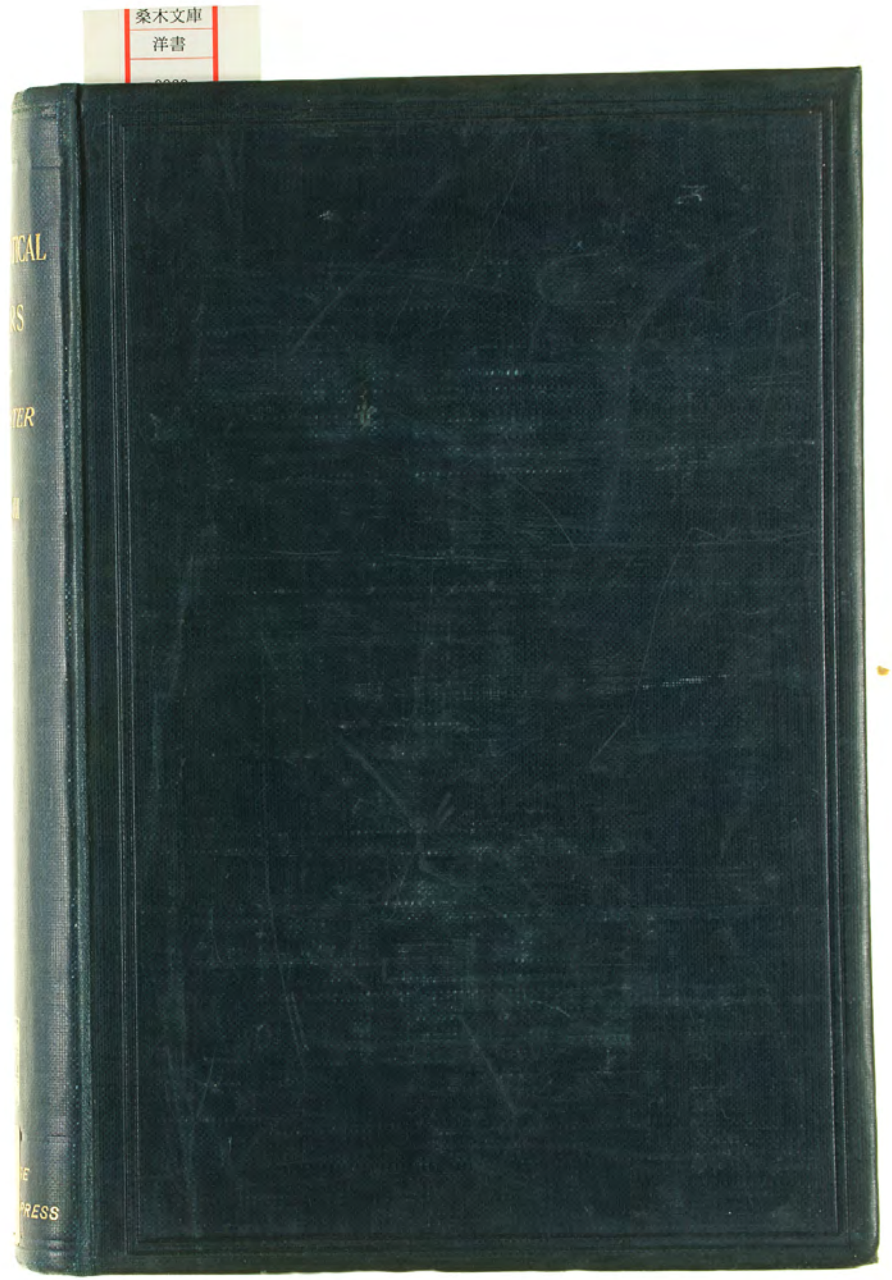
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MATHEMATICAL PAPERS

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THE COLLECTED MATHEMATICAL PAPERS

OF

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PREFATORY NOTE.

THE present volume deals very largely with the Author's enumerative method of obtaining the complete system of concomitants of a system of quantics, with the help of generating functions; the brief but very luminous papers here reprinted, at the end of the volume, from the *Johns Hopkins University Circulars* shew the Author preparing his memoir on the Constructive Theory of Partitions, which begins the next, and last, volume of his Mathematical Works. The previous volume included the period of the Author's activity at the Military Academy, Woolwich; this volume nearly covers the time of that surely most interesting experiment in educational method when, at Baltimore, unhindered by traditional routine, and encouraged to give full rein to his invention, he was able, nay obliged, as he tells us (p. 76), to yield to the inquisitive student who would have the New Algebra, that or nothing; with results that are imperishable. The matter is seen so well from the Author's point of view in his Commemoration day Address at Johns Hopkins University (1877), that, after some hesitation, a reprint of this is included in the present volume (No. 10). The Remarks on Research, in *Nature*, vol. XVI. (1877), are from this Address. The present volume also includes the Author's investigations on Chemistry and Algebra (No. 24), the paper on Certain Ternary Cubic-Form Equations (No. 39), and the paper on Subinvariants and Perpetuants (No. 67). In connection with the enumerative methods in this volume the reader's attention may be directed to a paper, by F. Franklin, "On the Calculation of the Generating Functions and Tables of Groundforms for Binary Quantics," in the *American Journal of Mathematics*, III. (1880), pp. 128—153, to which, as to one or two other memoirs referring to matters dealt with in the text, I have ventured to add a reference at the appropriate place.

H. F. BAKER.

ST JOHN'S COLLEGE, CAMBRIDGE.
24 *November* 1909.

TABLE OF CONTENTS

		PAGES
1.	*On the rotation of a rigid body* . . (Nature 1870)	1—6
2.	*On recent discoveries in mechanical conversion of motion* . . . . . . (Proceedings of the Royal Institution of Great Britain 1873—75) (La Revue Scientifique 1874—75) (Van Nostrand's Engineering Magazine 1875)	7—25
3.	*On the plagiograph* aliter *the skew pantigraph* . . . . (Nature 1875, Archivo de Mat. I.)	26—34
4.	*On a lady's fan, on parallel motion, and on an orthogonal web of jointed rods* . (Proceedings of the London Mathematical Society 1875)	35, 36
5.	*Note on spherical harmonics* . . . (Philosophical Magazine 1876)	37—51
6.	*Sur les invariants fondamentaux de la forme binaire du huitième degré* . . . (Comptes Rendus de l'Académie des Sciences 1877)	52—57
7.	*Sur une méthode algébrique pour obtenir l'ensemble des invariants et des covariants fondamentaux d'une forme binaire et d'une combinaison quelconque de formes binaires* . . . . (Comptes Rendus de l'Académie des Sciences 1877)	58—62
8.	*Sur le vrai nombre des covariants élémentaires d'un système de deux formes biquadratiques binaires* . . . . (Comptes Rendus de l'Académie des Sciences 1877)	63—66

		PAGES
9.	*Théorie pour trouver le nombre des covariants et des contrevariants d'ordre et de degré donnés linéairement indépendants d'un système quelconque de formes simultanées contenant un nombre quelconque de variables* (Comptes Rendus de l'Académie des Sciences 1877)	67—71
10.	*Address on Commemoration Day at Johns Hopkins University, 22 February* 1877 (Cushings and Bailey, Baltimore)	72—87
11.	*On a generalization of Taylor's theorem* (Philosophical Magazine 1877)	88—92
12.	*Sur les invariants* (Comptes Rendus de l'Académie des Sciences 1877)	93—100
13.	*On the limits to the order and degree of the fundamental invariants of binary quantics* (Proceedings of the Royal Society of London 1878)	101, 102
14.	*Chemistry and Algebra* (Nature 1877—78)	103, 104
15.	*Sur la loi de réciprocité pour les invariants et covariants des quantics binaires* (Comptes Rendus de l'Académie des Sciences 1878)	105—107
16.	*Sur la théorie des formes associées de MM. Clebsch et Gordan* (Comptes Rendus de l'Académie des Sciences 1878)	108, 109
17.	*Détermination d'une limite supérieure au nombre total des invariants et covariants irréductibles des formes binaires* (Comptes Rendus de l'Académie des Sciences 1878)	110—116
18.	*Proof of the hitherto undemonstrated fundamental theorem of invariants* (Philosophical Magazine 1878)	117—126
19.	*Sur les covariants fondamentaux d'un système cubo-biquadratique binaire* (Comptes Rendus de l'Académie des Sciences 1878)	127—131

		PAGES
20.	*Sur le vrai nombre des formes irréductibles du système cubo-biquadratique* (Comptes Rendus de l'Académie des Sciences 1878)	132—135
21.	*Détermination du nombre exact des covariants irréductibles du système cubo-biquadratique binaire* (Comptes Rendus de l'Académie des Sciences 1878)	136—139
22.	*Sur les covariants irréductibles du quantic du septième ordre* (Comptes Rendus de l'Académie des Sciences 1878)	140—143
23.	*Sur la forme binaire du septième ordre* (Comptes Rendus de l'Académie des Sciences 1878)	144—147
24.	*On an application of the new atomic theory to the graphical representation of the invariants and covariants of binary quantics,—with three appendices* (American Journal of Mathematics 1878)	148—206
25.	*Note on the theorem contained in Professor Lipschitz's paper* (American Journal of Mathematics 1878)	207—209
26.	*A synoptical table of the irreducible invariants and covariants to a binary quintic, with a scholium on a theorem in conditional hyper-determinants* (American Journal of Mathematics 1878)	210—217
27.	*Sur les actions mutuelles des formes invariantives dérivées* (Crelle's Journal für die reine und angewandte Mathematik 1878)	218—240
28.	*On a rule for abbreviating the calculation of the number of in- or co-variants of a given order and weight in the coefficients of a binary quantic of a given degree* (Messenger of Mathematics 1879)	241—248
29.	*Note on continuants* (Messenger of Mathematics 1879)	249—251

		PAGES
30.	*Sur une propriété arithmétique d'une certaine série de nombres entiers* . . (Comptes Rendus de l'Académie des Sciences 1879)	252
31.	*Sur la valeur moyenne des coefficients dans le développement d'un déterminant gauche ou symétrique d'un ordre infiniment grand et sur les déterminants doublement gauches* . . . . (Comptes Rendus de l'Académie des Sciences 1879)	253—255
32.	*Table des nombres de dérivées invariantives d'ordre et de degré donnés, appartenant à la forme binaire du dixième ordre* . (Comptes Rendus de l'Académie des Sciences 1879)	256
33.	*Sur la valeur moyenne des coefficients numériques dans un déterminant gauche d'un ordre infiniment grand* . . (Comptes Rendus de l'Académie des Sciences 1879)	257
34.	*Sur le vrai nombre des covariants fondamentaux d'un système de deux cubiques* . (Comptes Rendus de l'Académie des Sciences 1879)	258—261
35.	*Note on an equation in finite differences* . (Philosophical Magazine 1879)	262, 263
36.	*Note on determinants and duadic disynthemes* . . . . (American Journal of Mathematics 1879)	264—280
37.	*On the complete system of the "Grundformen" of the binary quantic of the ninth order* . . . . . . (American Journal of Mathematics 1879)	281, 282
38.	*Tables of the generating functions and groundforms for the binary quantics of the first ten orders* . . . . (American Journal of Mathematics 1879)	283—311
39.	*On certain ternary cubic-form equations* . (American Journal of Mathematics 1879—80)	312—391

		PAGES
40.	*Tables of the generating functions and groundforms for simultaneous binary quantics of the first four orders, taken two and two together* . . . . (American Journal of Mathematics 1879)	392—410
41.	*Note sur une propriété des équations dont toutes les racines sont réelles* . . . (Crelle's Journal für die reine und angewandte Mathematik 1879)	411—413
42.	*On the theorem connected with Newton's rule for the discovery of imaginary roots of equations* . . . . (Messenger of Mathematics 1880)	414—425
43.	*On the exact relation which resultants and discriminants bear to the product of differences of roots of equations* . . (Messenger of Mathematics 1880)	426, 427
44.	*Sur les diviseurs des fonctions cyclotomiques* (Comptes Rendus de l'Académie des Sciences 1880)	428—432
45.	*Sur la loi de réciprocité dans la théorie des nombres* . . . . (Comptes Rendus de l'Académie des Sciences 1880)	433—437
46.	*Sur les équations à 3 et à 4 périodes des racines de l'unité* . . . . . (Compte Rendu de la Association Française, Reims, 1880)	438, 439
47.	*On a point in the theory of vulgar fractions*. (American Journal of Mathematics 1880)	440—445
48.	*Instantaneous proof of a theorem of Lagrange on the divisors of the form* $Ax^2+By^2+Cz^2$, *with a postscript on the divisors of the functions which multisect the primitive roots of unity* . . . (American Journal of Mathematics 1880)	446—448
49.	*Sur l'entrelacement d'une fonction par rapport à une autre* . . . . . (Crelle's Journal für die reine und angewandte Mathematik 1880)	449, 450

PAGES

50. *Preuve instantanée d'après la méthode de Fourier, de la réalité des racines de l'équation séculaire* 451, 452
(Crelle's Journal für die reine und angewandte Mathematik 1880)

51. *Sur un déterminant symétrique qui comprend comme cas particulier la première partie de l'équation séculaire* . . . 453—455
(Crelle's Journal für die reine und angewandte Mathematik 1880)

52. *Sur les déterminants composés* . . . . 456—473
(Crelle's Journal für die reine und angewandte Mathematik 1880)

53. *On the triangles in- and ex-scribable to a general cubic curve* 474
(Johns Hopkins University Circulars 1880)

54. *On the resultant of two congruences* . . . 475
(Johns Hopkins University Circulars 1881)

55. *On the prerogative of a ternary denominational system of coinage* 476
(Johns Hopkins University Circulars 1881)

56. *On the multisection of the roots of unity* . 477, 478
(Johns Hopkins University Circulars 1881)

57. *Sur les diviseurs des fonctions des périodes des racines primitives de l'unité* . . . 479, 480
(Comptes Rendus de l'Académie des Sciences 1881)

58. *Sur les covariants irréductibles du quantic binaire du huitième ordre* . . . . 481—488
(Comptes Rendus de l'Académie des Sciences 1881)

59. *Tables of the generating functions and groundforms of the binary duodecimic, with some general remarks, and tables of the irreducible syzygies of certain quantics* . . . . 489—508
(American Journal of Mathematics 1881)

60. *A demonstration of the impossibility of the binary octavic possessing any groundform of deg-order* 10.4 . . . . . 509—529
(American Journal of Mathematics 1881)

PAGES

61. *On Tchebycheff's theory of the totality of the prime numbers comprised within given limits* . . . . 530—545
(American Journal of Mathematics 1881)

62. *On the solution of a certain class of difference or differential equations* . . 546—550
(American Journal of Mathematics 1881)

63. *Note on the theory of simultaneous linear differential or difference equations with constant coefficients* . . . . . 551—556
(American Journal of Mathematics 1881)

64. *Note on mechanical involution* . . . 557—561
(American Journal of Mathematics 1881)

65. *Sur les puissances et les racines de substitutions linéaires* . . . . . . 562—564
(Comptes Rendus de l'Académie des Sciences 1882)

66. *Sur les racines des matrices unitaires* . . 565—567
(Comptes Rendus de l'Académie des Sciences 1882)

67. *On subinvariants, that is, semi-invariants to binary quantics of an unlimited order* . 568—622
(American Journal of Mathematics 1882)

68. *Tables of generating functions, reduced and representative, for certain ternary systems of binary forms* . . . . . 623—632
(American Journal of Mathematics 1882)

69. *On a certain integrable class of differential and finite difference equations* . . 633
(Johns Hopkins University Circulars 1882)

70. *On a question of partitions* 634
(Johns Hopkins University Circulars 1882)

71. *On a geometrical proof of a theorem in numbers* . . . . 635—639
(Johns Hopkins University Circulars 1882)

72. *On the geometrical forms called trees* . . 640, 641
(Johns Hopkins University Circulars 1882)

		PAGES
73.	*On the 8-square imaginaries* . . . . (Johns Hopkins University Circulars 1882)	642, 643
74.	*On a geometrical treatment of a theorem in numbers* . . . . (Johns Hopkins University Circulars 1882)	644
75.	*On the properties of a split matrix* . . (Johns Hopkins University Circulars 1882)	645, 646
76.	*A word on nonions* . . . . . . (Johns Hopkins University Circulars 1882—3)	647—650
77.	*On mechanical involution* . . . . . (Johns Hopkins University Circulars 1882)	651, 652
78.	*On Crocchi's theorem* . . . . . . (Johns Hopkins University Circulars 1883)	653—655
79.	*On certain successions of integers that cannot be indefinitely continued* . . . . (Johns Hopkins University Circulars 1883)	656, 657
80.	*On the fundamental theorem in the new method of partitions* . . . . . (Johns Hopkins University Circulars 1883)	658—660
81.	*Note on the paper of Mr Durfee's* . . (Johns Hopkins University Circulars 1883)	661—663
82.	*On Dr F. Franklin's proof of Euler's theorem concerning the form of the infinite product* $(1-x)(1-x^2)(1-x^3)\ldots$ (Johns Hopkins University Circulars 1883)	664—666
83.	*On the use of cross-gratings to obtain certain developments connected with the theory of elliptic functions* . . . . . (Johns Hopkins University Circulars 1883)	667—671
84.	*On the number of fractions in their lowest terms whose numerators and denominators are limited not to exceed a certain number* . . . . (Johns Hopkins University Circulars 1883)	672—676

		PAGES
85.	*Proof of a well-known development of a continued product in a series* . . (Johns Hopkins University Circulars 1883)	677—679
86.	*On a new theorem in partitions* . . . (Johns Hopkins University Circulars 1883)	680—682
87.	*Note on the graphical method in partitions* . (Johns Hopkins University Circulars 1883)	683, 684
88.	*An instantaneous graphical proof of Euler's theorem on the partitions of pentagonal and non-pentagonal numbers.* . . (Johns Hopkins University Circulars 1883)	685, 686
89.	*On Farey Series* . . . . (Johns Hopkins University Circulars 1883)	687, 688

1.

ON THE ROTATION OF A RIGID BODY.

[Three letters to *Nature*, Vol. I. (1870), pp. 482, 532, 582.]

The Motion of a Free rotating Body.

I SHALL feel obliged if, through the medium of your widely-circulated journal, you will allow me to point out an extraordinary mistake into which M. Radau has fallen, in a memoir inserted in the *Annales Scientifiques de l'École Normale Supérieure*, tom. VI. 1869, in which he criticises certain of my conclusions about the representation of the motion of a free rotating body contained in a paper published by me in the *Philosophical Transactions* for 1866*. In his preamble, M. Radau says, speaking of the theory of rotation in connection with the names of Poinsot, Rueb, Jacobi, and Richelot:—"Tout récemment M. Sylvester a essayé d'appliquer au même sujet des considérations nouvelles qui l'ont conduite à des résultats intéressants, à côté d'autres dont l'exactitude peut être contestée."

Later on in his memoir M. Radau points out, and accompanies with very biting (albeit toothless) criticism, the nature of his objection, which is, in short, that I suppose Poinsot's ellipsoid, under the influence of an original impulse, to roll without slipping by virtue of its friction against the plane with which it is in contact. My answer is, that of course I do. And why not? when I suppose the plane "indefinitely rough" (see p. 761 of *Philosophical Transactions*, 1866†), and have actually determined the friction and pressure at each point of the motion, so that by solving a maximum and minimum problem of one variable, the extreme value of the ratio of one of these forces to the other, or if we please to say so, the limiting angle of friction, or, in other words, the necessary degree of roughness of the plane, may be analytically determined for every given case. M. Radau falls into the school-boy blunder of making the *ratio between the friction and pressure constant throughout the motion*, confounding the actual friction with its limiting maximum value! It is, indeed, surprising that such a perversion

[* Vol. II. of this Reprint, p. 577.] [† *ibid.* p. 582.]

of the facts of the case should have found insertion in a serious journal, such as that published by the École Normale Supérieure, and I might fairly have expected from M. Radau the courtesy habitual with his adopted countrymen, of applying to me for information on anything in my paper which might have appeared to him obscure or erroneous, before rushing into print with such a *mare's nest*.

But out of evil cometh good. M. Radau says:—"Mais M. Sylvester va plus loin; il pense que le problème pourrait se résoudre par l'observation directe du mouvement d'un ellipsoïde matériel tournant sur un plan fixe en même temps qu'il tournerait autour de son centre également fixe. On ne se figure pas facilement par quel artifice on fixerait le centre d'un ellipsoïde matériel."

In a future number of your esteemed journal (as time at present fails me) I propose to show how, by the simplest contrivance in the world, a downright material top of ellipsoidal form may be actually made to roll, with its centre fixed, on a fixed plane and so exhibit to the eye the surprising spectacle of a motion precisely identical *in time*, as well as in its successive displacements of *position*, with that of a body, turning round a fixed centre, but otherwise absolutely unconstrained.

This mode of representation, which flashed upon my mind almost instantaneously when my eye first lighted upon M. Radau's objections, is the compensating good to the evil of being made the victim (to the temporary disturbance of my beloved tranquillity) of so hasty and futile a criticism as has been allowed insertion in the "Scientific Annals" of so great an institution as the École Normale of Paris.

The *bureau de rédaction* must surely have been nodding when they allowed such observations, so easily refuted by turning to the original memoir, to pass unchallenged. It was only within the last few days that I received M. Radau's paper.

Rotation of a Rigid Body.

My previous communication about the rotating ellipsoid to this journal, has attracted the attention of M. Radau. "One touch of *Nature* makes the whole world kin." In a note addressed to me full of true dignity, this gentleman has made much more than sufficient reparation for his previous trifling act of inadvertence, and states that to his great regret he had misunderstood my meaning, in the passage of my memoir in question, and that "sa critique n'est pas fondée." I, on my part, deeply lament the unnecessary tone of acerbity in which my reference to this criticism was couched, and wish I could recall every ungracious expression which it contains. "When I spoke that, I was ill-tempered too."

I will pass over this, to me, painful topic, to say two or three words on the mode in which the rotating ellipsoid may be supposed to roll or *wobble* on a rough plane, with its centre fixed. My solution may remind the reader of Columbus' mode of supporting an egg on its point—or, rather, of a fairer mode which Columbus might have employed, and which would not have necessitated the breaking of the shell, namely, by resting the blade of a knife or rough plate on the upper end of his egg.

So, to make an ellipsoidal or spheroidal top roll, with its centre fixed—say, upon a rough horizontal plane—imagine a second horizontal plane in contact with the upper portion of its surface; then the line joining the two points of contact will pass through the centre of the top. We may conceive a slight perforation in either or each plane at its initial point of contact with the top, and a screw wire introduced through this, and inserted into a female screw in the body to be set rolling (a mode of spinning which Sir C. Wheatstone recommends as the most elegant in any case, and in this case evidently the most eligible). On withdrawing the wire with a jerk, the top may be set in motion about its centre, in such a direction as to remain in contact with the two planes, and if these be sufficiently rough the motion will eventually be reduced to one of pure rolling between them, the axis (that is, the line joining the two points of contact), continually shifting, but the centre remaining absolutely stationary: for, vertical motion this point cannot have, so long as the top continues to touch both planes, and any slight horizontal motion (if it should chance to take on such at the outset) would be checked and ultimately destroyed by the friction, which would also keep the two points of contact stationary (like the single point of contact of a wheel rolling on a rail), in each successive atom of time. Thus the motion upon the lower plane would in the end be precisely the same as if the upper plane were withdrawn, and the centre of the top kept fixed by some mechanical adjustment. If the spin were not sufficiently vigorous, after a time the rolling top might quit the upper plane, and of course sooner or later by the diminution of the *vis-viva* due to adhesion, resistance of the air, imperfection or deformation of the surfaces, and other disturbing causes, this would take place, but abstracting from these circumstances the principal axes of the spheroidal or ellipsoidal top would move precisely in place and time like the "axes of spontaneous rotation" of any free body of which the top was the "Kinematic Exponent."

I do not pretend to offer an opinion what materials for the planes and rolling body (ground glass and ebony or roughened ebonite have been suggested to me) it would be best to employ, or whether the "wobbling top" could easily be made to exhibit its evolutions. It is enough for a non-effective, unpractical man (as unfortunately I must confess to being) to have shown that there is no intrinsic impossibility in the execution of the conception.

With regard to the friction and pressure: if W be the weight of the body, F and P the friction and pressure in the case of a single plane (the values of which are set out in my memoir, pp. 764—766, *Philosophical Transactions*, 1866*), it may easily be proved that eventually the friction at each point of contact will be $\frac{F}{2}$, the pressure upwards at the lower point $\frac{P+W}{2}$, and downwards at the upper one $\frac{P-W}{2}$, so that if P should become equal to W the top would quit the upper plane and the experiment come to an end. At p. 766 of my memoir the factor $\sqrt{M\Lambda}$ has accidently dropped out of the expression for P which I mention here, in case any one should feel inclined to consult the memoir in consequence of this note. Mr Ferrers has taken up my investigations, and given more compendious expressions than mine for F and P; with the aid of these it would probably be not difficult to determine the maximum value of $\frac{F}{P}$ so as to assign the necessary degree of roughness of the confining planes, and also to ascertain under what circumstances $P-W$ would become zero, but I do not feel sufficient interest in the question, nor have I the courage to undertake these calculations with the complicated forms of P and F contained in my memoir. Mr Ferrers' results are contained in a memoir ordered to be printed in the *Philosophical Transactions*, and will shortly appear.

In my memoir will be found an exact kinematical method of reckoning the time of rotation by Poinsot's ellipsoid when the lower surface is made to roll on one fixed plane at the same time that its upper surface is sharpened off in a particular way (therein described) so as to roll upon a parallel plane which turns round a fixed axis; this upper plane is compelled to turn by the friction, and acts the part of a moveable dial in marking the time of the free body imaginarily associated with the ellipsoid. I have also shown there that the motion of any free body about a fixed centre may be regarded as compounded of a uniform motion of rotation and the motion of a disc, or, if one pleases, a pair of mutually bisecting cross-wires left to turn freely about their centre. But I fear that *Nature*, used to a more succulent diet, has had as much as it can bear upon so dry a topic, and, although having more to say, deem it wiser to bring these remarks to an end.

An after-dinner experiment.

Suppose in the experiment of an ellipsoid or spheroid, referred to in my last letter, rolling between two parallel horizontal planes, we were to scratch on the rolling body the two equal similar and opposite closed curves (the *polhods* so-called), traced upon it by the successive axes of instantaneous rotation; and suppose, further, that we were to cut away the two extreme

[* Vol. II., above, pp. 585, 587.]

segments marked off by those tracings, retaining only the barrel or middle portion, and were then to make this barrel roll under the action of friction upon its bounding curved edges between the two fixed planes as before, or more generally, imagine a body of any form whatever bounded by and rolling under the action of friction upon these two edges between two parallel fixed planes; it is easy to see that, provided the centre of gravity and direction of the principal axis be not displaced, the law of the motion will depend only on the relative values of the principal moments of inertia of the body so rolling, in comparison with the relative values of the axes of the ellipsoid or spheroid to which the *polhods* or rolling edges appertain; and consequently, that, when a certain condition is satisfied between these two sets of ratios, the motion will be similar in all respects to that of a free body about its centre of gravity.

That condition (as shown in my memoir in the *Philosophical Transactions**) is, that the nine-membered determinant formed by the principal moments of inertia of the rolling body, the inverse squares and the inverse fourth powers of the axes of the ellipsoid or spheroid shall be equal to zero—a condition manifestly satisfied in the case of the spheroid, provided that two out of the three principal moments of inertia of the rolling solid are equal to one another.

My friend Mr Froude, the well-known hydraulic engineer, with his wonted sagacity, lately drew my attention to the familiar experiment of making a wine-glass spin round and round on a table or table-cloth upon its base in a circle without slipping, believing that this phenomenon must have some connection with the motion referred to in my preceding letter to *Nature*: an intuitive anticipation perfectly well-founded on fact; for we need only to prevent the initial tendency of the centre of gravity to rise by pressing with a second fixed plane (say a rough plate or book-cover) on the top of the wine-glass, and we shall have an excellent representation of the free motion about their centre of gravity of that class of solids which have, so to say, a natural momental axis, that is (in the language of the schools) two of their principal moments of inertia equal. For greater brevity let me call solids of this class uniaxal solids. I suppose that the centre of gravity of the glass is midway between the top and bottom, and that the periphery of the base and of the rims are circles of equal radius. These circles will then correspond to *polhods* of a spheroid, conditioned by the angular magnitude and dip of the spinning glass; to determine from which two elements the ratio of the axes of the originally supposed but now superseded representative spheroid is a simple problem in conic sections; this being ascertained, the proportional values of the moments of inertia of the represented solid may be immediately inferred. The wine-glass

[* Vol. II., above, p. 583.]

itself belonging to the class of uniaxal bodies, the condition that ought to connect its moments of inertia with the axis of the representative spheroid (in order that the motion may proceed *pari passu* with that of a free body) is necessarily satisfied.

The conclusion which I draw from what precedes is briefly this—that a wine-glass equally wide at top and bottom, and with its centre of gravity midway down, spinning round upon its base and rim in an inclined position between two rough but level fixed horizontal surfaces, yields, so long as its *vis-viva* remains sensibly unaffected by disturbing causes, a perfect representation, both in space and time, of the motion of a free uniaxal solid, as for example, a prolate or oblate spheroid, or a square or equilateral prism or pyramid about its centre of gravity, and conversely that every possible free motion about its centre of gravity of every such solid admits of being so represented.

To revert for an instant to the general question of the representative rolling ellipsoid, I think it must be admitted that the addition of the time element to the theory and the substitution of a second fixed plane in lieu of a fixed centre, considerably enhance the value and give an unexpected roundness and completeness to Poinsot's image of the free motion of rotation of a rigid body, of which so much and not altogether undeservedly has been made. From an idea or shadow Poinsot's representation has now become a corporeal fact and reality, as if, so to say, Ixion's cloud, in a moment of fruition, had substantified into a living Juno. I heard the late Professor Donkin, of revered and ever-to-be-cherished memory, state that when as a referee of the Royal Society he first took in hand my paper on rotation, he did so with a conviction that all had already been said that could be said on the subject, and that it was a closed question; but that when he laid down the memoir he saw reason to change his opinion. I owe my thanks to M. Radau and the editors of the Annals of the École Normale Supérieure for having been at the pains to disentomb the little-known conclusions therein contained from their honourable place of sepulture in the *Philosophical Transactions*.

2.

ON RECENT DISCOVERIES IN MECHANICAL CONVERSION OF MOTION.

[*Proceedings of the Royal Institution of Great Britain*, VII. (1873—75), pp. 179—198. Also *La Revue Scientifique*, 1874—75, pp. 490—498, and *Van Nostrand's Engineering Magazine* (New York) XII. (1875), pp. 313—321.]

THE speaker stated that the subject he proposed to bring under the notice of the meeting related mainly to the discovery of a perfect parallel motion,—that is to say, of a mode of producing motion in a straight line by a system of pure link-work without the aid of grooves or wheel-work, or any other means of constraint than that due to fixed centres, and joints for attaching or connecting rigid bars. This important discovery was made by M. Peaucellier, an officer of Engineers in the French army*—and first published by him, in the form of a question, in the *Annales de Mathématique* in the year 1864, and subsequently formed the subject of two communications to the *Société Philomathique* of Paris by Captain Manheim, but seems not to have received the attention it deserved from that learned body, and may be said to have passed into oblivion; so much so, that when rediscovered by a young student of the University of St Petersburg, of the name of Lipkin, several years subsequently, the discovery was attributed to Lipkin instead of to Peaucellier even in works published in the French language, and so recently as 1873 by M. Colignan, in his *Traité de Cinématique*. The eminent Professor Tchebicheff had long occupied himself with the question, but with less than his usual success in overcoming difficulties insuperable to the rest of the world. Lipkin was a student in his class, and may thus have had his attention turned to the question; at all events, Professor Tchebicheff's warm interest in the subject was displayed by his bringing Lipkin's name before the Russian Government, and securing for him a substantial reward for his

* Now Colonel Peaucellier, and in command of the fortress of Toul; at the time of his discovery lieutenant and officier d'ordonnance on the staff of the "illustrious Marshal Niel."

supposed original discovery. Before Peaucellier's time all so-called parallel motions were imperfect, and gave merely approximate rectilinear motion*; in substance they will be without exception found to be merely modifications of Watt's original construction, and to depend on the motion of a point in, or rigidly connected with, a bar joining the extremities of two other bars rotating round fixed centres, which may be described briefly as three-bar motion. Peaucellier's exact parallel motion depends on a link-work of seven

* The late lamented Professor Rankine, in his treatise on Millwork, and elsewhere, mentions a so-called "exact parallel motion," the invention of which he dubiously assigns to Mr Scott Russell. In its *exact* form this is no parallel motion at all, for it works by means of a slide, and in its modified form it ceases to be *exact*, the motion produced being no longer truly rectilinear.

Mr Kaulbach, a mechanical draughtsman, resident in London, has shown the speaker a sketch of a very ingenious *quasi*-parallel motion, which he took the first steps to patent a year or two ago, but has not thought it worth his while to proceed with further. Its principle depends upon finding a curve made to rotate about a fixed point, and enjoying the property that the tangent to each point of it, as that point passes a given vertical line, shall take up a horizontal position. A piston-rod is guided in the direction of such vertical line, and the beam, which always presses on a friction wheel attached to the rod, is so shaped in its outward contour as to satisfy the above condition; the consequence is that the reaction on the piston-rod can only take effect vertically, that is, in the direction of its motion, and no lateral pressure is produced.

Peaucellier's invention effects the perfect conversion of circular into linear motion. An easy practical deduction from this is the conversion of spherical into plane motion, by aid of universal joints and other familiar modes of effecting free motion in space, of a shaft about a fixed point or round another shaft. The announcement of these facts has occasioned many persons unacquainted with the technical language of mechanism to suppose that the discovery of Peaucellier is connected with the quadrature of the circle or cubature of the sphere, and led to the idea that the speaker was in possession of some secret for flattening spheres and turning circles into right lines. Such a misconception was one (as indeed the wide extent of its prevalence demonstrates) quite likely to occur even to intelligent persons untrained in mathematical science. Technical names are a frequent occasion of traps to the uninitiated. A lady present at one of Mr Norman Lockyer's course of lectures on Spectral Analysis, near the close of it was overheard inquiring with some anxiety as to "when the spectres might be expected to make their appearance." Names are of course all-important to the progress of thought, and the invention of a really good name, of which the want, not previously perceived, is recognized, when supplied, as having ought to be felt, is entitled to rank on a level in importance with the discovery of a new scientific theory. Imagine *plane*, *straight*, *circle*, and you are potentially a geometer. Think the meaning of the one word *Syzygy*, and the logic of algebra has become part of your being. But, on the other hand, there are cases where over-naming does harm. The speaker has no doubt that if reading music on the piano with the fingers were taught without the intervention of learning the names of the notes, twice the velocity of execution (and quick reading is here the *sine-quâ-non* for the existence of every other kind of excellence) might be acquired in half the time required under the present system. The names of the notes of course would have to be learned at a later stage as a medium for discourse; but they should not be used as a vehicle for obtaining command of digitation, as such use amounts to throwing upon the brain the labour of going through two steps when one would suffice, and the passage of a direct nervous current from the eye to the touch in the act of reading, even at an advanced stage, becomes by force of habit interrupted and diverted into a broken channel. The new method for learning to read on the pianoforte here suggested may be distinguished as the abnominal or undenominational or tactile method. The writer is prepared to show in detail how it can be carried out in practice.

bars moving like Watt's, and the other imperfect parallel motions of the same class, round two fixed centres*.

To understand the principle of Peaucellier's link-work, it is convenient to consider previously certain properties of a linkage† (to coin a new and useful

* The perfect parallel motion of Peaucellier looks so simple and moves so easily that people who see it at work almost universally express astonishment that it waited so long to be discovered. The idea of the facility of the result by a natural mental illusion gets transferred to the process of conception, as if a healthy babe were to be accepted as proof of an easy act of parturition. No impression can be more erroneous. The speaker, on the contrary, the more he reflects upon the problem that was to be solved, and the nature of the solution (essentially a process of transformation operating on polar co-ordinates), wonders the more that it was ever found out, and can see no reason why it should have been discovered for a hundred years to come. Viewed *à priori* there was nothing to lead up to it. It bears not the remotest analogy (except in the fact of a double centring) to Watt's parallel motion or any of its progeny. In the three-bar motion the two fixed points are so to say one as good as the other, there is no distinction to be drawn between them; whereas the two fixed centres (hereafter designated as the fulcrum and pivot) in Peaucellier's seven-bar arrangement are absolutely dissimilar in position and function. Peaucellier's apparatus naturally resolves itself into a cell and a spare link; no such decomposition presents itself in the three-bar motion. Again, looking at the matter *à posteriori*, it occurs to many well-grounded mathematicians to suppose that, as the most general motion of a link-work of seven or any number of bars for each possible mode of conjunction and centring must be capable of being expressed by a general algebraical equation, the particular combination for rectilinear motion, when such motion is possible, ought to be contained therein and inferrible therefrom by studying under what conditions the characteristic of the general equation can degenerate into a power of a linear function or, as might perhaps happen (and would be sufficient if it did), into such power multiplied by a function incapable of changing its sign. But the answer to this is that *practically* there could be little or no hope of ever obtaining the general equation. In one-bar motion the general curve (that is, a circle) is of the 2nd order; in three-bar motion, as is well known, of the 6th order; very likely, therefore, in five-bar motion it would be of the 24th order at least; and in seven-bar motion, of the 120th order at least. The equation or system of equations of the 120th order, supposed to be applicable to seven-bar motion, one could hardly dream of obtaining, or of being able to manipulate if obtained. Written out at full length in a handwriting of moderate size, the area of a very large room might be insufficient to contain the whole of its terms, which would consist of 7381 groups, and might be tens or hundreds of thousands in number. No; it must either have been fallen upon in a chance or experimental way, and subsequently verified theoretically, or else hit off in some sudden glow of insight akin to but of a much intenser degree of illumination than that under which Professor Stokes was able to see that the hydrodynamical theorem of Lagrange before him, proved imperfectly by its author and others, and correctly but with great difficulty by Cauchy, was an immediate inference from the pretty nearly self-obvious fact of the complete time-derivatives of the three quantities to be proved *if ever then always zero*, being by virtue of the well-known general hydrodynamical equations, syzygetic functions of these quantities themselves. Dr Tchebicheff has informed the writer that he has succeeded in proving the non-existence of a five-bar link-work capable of producing a perfect parallel motion; he is probably therefore in possession of the actual numerical order of the general equation or system of equations applicable to this case. It is not proved, and may not be true, that Peaucellier's is the only seven-bar link-work that will solve the problem of a perfect parallel motion. Who shall say whether there may not exist some other combination of seven bars in which the same or an analogous zig-zag symmetry to that which exists in the three-bar arrangement may reappear! This is a point which should not be allowed to remain subject to doubt.

† A link-work consists of an odd number of bars, a linkage of an even number. A linkage may be converted into a link-work *additively* by fixing one point of it as a fulcrum and attaching

word of general application), consisting of an arrangement of six links, obtained in the following manner:—first conceive a rhomb or diamond formed by four equal links joined to one another; and now suppose a pair of equal links to be joined on to two opposite angles of such figure and to each other. All six links are supposed to lie (and to be constrained by the nature of their attachments to remain) in the same plane. The point of junction of the last-named pair of links (which it will be found convenient to call the fulcrum), according as they are greater or smaller than the sides of the diamond, will lie outside or inside the diamond. The linkage consisting of the six links may be termed a positive *cell* in the one case and a negative *cell* in the other*. It is easily seen, as a geometrical necessity, that the fulcrum,

a second point disconnected from the first by a new link to another fulcrum, or *ablatively* by fixing two ends of a link, which may then be removed. When one point only of a linkage is fixed, any other point may be made to describe an arbitrary curve, but then the path of every other point becomes prescribed. In order for a combination of links to fulfil this so to say fatalistic condition, and to entitle it to the name of a linkage in the speaker's sense, which when greater precision is required may be distinguished as a *perfect* linkage, equivalent to the French *système de tiges à liaison complète*, a numerical relation must be satisfied between the number of links and the number of joints, namely, three times the number of links must be four greater than twice the number of joints. In applying this rule it must be understood that, if three links are jointed together, the junction counts for two joints; if four are jointed together, for three joints; and so on. A compass or a pair of scissors is the simplest kind of linkage; a set of lazy-tongs is another; a Peaucellier cell, subsequently described in the text, a third. If no three joints lie on the same link, the above numerical relation between joints and links may be stated in another form, namely, twice the number of joints is four greater than the number of links. But in applying the rule in this form all joints count alike as units, and for a simple compass the ends must be reckoned as joints.

* Mr Penrose, the eminent architect and surveyor to St Paul's Cathedral, the scientific expositor and elucidator in succession to Mr Pennethorne of the surprising law of curvilinearity in the temples of the Greeks, has put up a house-pump worked by a negative Peaucellier cell, to the great wonderment of the plumber employed, who could hardly believe his senses when he saw the sling attached to the piston-rod moving in a true vertical line, instead of wobbling as usual from side to side. There seems to be no reason why the perfect parallel motion should not be employed with equal advantage in the construction of ordinary water-closets. The author has been admitted to see the geometrical pump at work in Mr Penrose's kitchen at Wimbledon. A sister pump of the ordinary construction stands beside it. The former, although quite as compact as its neighbour, throws up a considerably larger head of water with the same sweep of the handle. Its elegance, and the frictionless ease with which it can be worked (beauty as usual the stamp and seal of perfection) have made it the pet of the household. Some circular steps outside St Paul's Cathedral very lately requiring repair, Mr Penrose employed a circulo-circularly-adjusted Peaucellier cell to cut out templets in zinc for the purpose. The radius of the steps is about 40 feet, but to the great comfort and delectation of his clerk of the works, they were able to operate with a radius of not more than 6 or 7 feet in length. General Sir H. James, R.E., lately gave a lecture on the subject at Southampton, and informs the writer that this has been the means of inducing a gentleman of fortune residing there, well known in the yachting world, to fit up a marine engine with a Peaucellier parallel motion to use on board a steam yacht.

A very good idea of the form and operation of a negative cell may be gained by putting together the fore-fingers and ring-fingers of the two hands, and placing one middle finger a little over the other so as to keep all six fingers in the same plane. The first Peaucellier cell constructed in this country was a positive one, made by the speaker's friend, the eminent musician

in whatever way the linkage is moved about, will always lie in a straight line with the two free angles of the diamond, which may be called its poles, and the distances of these poles from the fulcrum, or the ideal lines which represent those distances, may be called the arms of the cell. It is upon the geometrical relation between these arms that the remarkable mechanical properties of Peaucellier's cell depend. The cell may be made to change its

and inventor of the laryngoscope, Mr Manuel Garcia, Ph.D., who happened to visit him shortly after his memorable interview with Dr Tchebicheff, in which that great mathematician announced in answer to his inquiries after the progress of the disproof of the impossibility of the exact conversion of circular into rectilinear motion, which had so long occupied the attention of his illustrious guest, that it, the thing itself, not the proof of its impossibility, had been actually effected in France, and subsequently in Russia, by a freshman student in his own class. He showed Mr Garcia the drawing of the cell and mounting left by Tchebicheff, and the next day was gratified by receiving from him a model constructed with a few pieces of wood, fastened together with nails as pivots, which, rough as it was, worked perfectly, and drew forth the most lively expressions of admiration from some of the most distinguished members of the Philosophical Club of the Royal Society (not mathematicians, but naturalists, geologists, chemists, and physicists), when it was brought in with the dessert, to be seen by them after dinner, as is the laudable custom among the members of that eminent body in making known to each other the latest scientific novelties. Presently after the speaker exhibited the same model in the hall of the Athenæum Club to his brilliant friend Sir William Thomson, of Glasgow, who nursed it as if it had been his own child, and when a motion was made to relieve him of it, replied, "No! I have not had nearly enough of it—it is the most beautiful thing I have ever seen in my life." This rude but invaluable model ought to be preserved in some physical laboratory as a historical relic. It served as an instrument by which the speaker in every case where it was seen gained immediate converts to the belief of the importance of Peaucellier's great discovery, whereas a mere geometrical diagram would have been as little regarded as a figure of the celebrated asses' bridge in Euclid at last, so great is the difference of the impression produced on the practical English mind by the *esse* and the *posse*—being told how a thing ought to act, and seeing it actually going. Considering the extraordinary conversions worked with Mr Garcia's model, it would not be unsuitable to write in letters of gold on the board attached to it which gives support to the two frail centres, the famous motto of Constantine—"In hoc signo vinces."

Apropos of the mistaken impressions of great men. Did not Newton live and die in the belief of the incurability of chromatic dispersion; Cayley affirm the infinitude of the number of the asyzygetic invariants of binary quantics beyond the sixth order, thereby arresting for many years the progress of the triumphal car which he had played a principal part in setting in motion; Pontecoulant the possibility of the existence of a rotating fluid ellipsoid of equilibrium for other than forms of revolution?

And as regards the speaker himself, twenty years ago he emitted* in the *Philosophical Magazine* a conjectural criterion for distinguishing *à priori*, geometrical propositions capable only of indirect demonstration from those susceptible of direct, when, lo and behold! but a few days ago came over a seemingly incontrovertible refutation of his supposed law, addressed to the Vice-Chancellor of our University of Cambridge (as a sort of Patriarch of the West, and recognized Official Defender of the Faith (as it is in Euclid) for the British isles), by Miss Chart, of Oakland, California, U.S., which it is to be hoped will speedily appear in the same journal where the erroneous hypothetical dogma first saw the light. His sin, after so long a delay, and travelling half round the world in the interim, has found him out. It ought to be added that Miss Chart does not claim for herself the merit of the refutation, but represents herself as having received it some years ago from a gentleman bearing the, to geometrical ears, auspicious-sounding name of Hesse.

[* Vol. I. of this Reprint, p. 395.]

form like a set of lazy-tongs or any other kind of linkage, by closing or opening the diamond: as this is done evidently the lengths of the arms alter; but it will be found, and is capable of easy geometrical proof, that they remain subject to a very simple condition, namely, one increases just as much as the other decreases, so that their product remains invariable; this product is equal to the difference between the square of either of the links (called the connectors) proceeding to the fulcrum and the square of any side of the diamond, to which we may give the name of the modulus of the cell. The speaker illustrated this property experimentally, using a negative cell for the purpose. When the fulcrum was midway between the two poles each arm was 12 inches in length. When one arm was made 18 inches the other was found to be 8; when again it was stretched to the length of 24 inches the other was 6, and so on, the product of the two remaining always 144; or, reckoning in feet, to the lengths 1, $\frac{3}{2}$, 2, 3 of one arm corresponded the lengths 1, $\frac{2}{3}$, $\frac{1}{2}$, $\frac{1}{3}$ of the other; showing that the length of one arm was so governed by the length of the other as that the numbers denoting the two were always inverse or reciprocal to each other when the modulus was taken as unity. Hence a Peaucellier's cell may be conveniently termed a Reciprocator or Inverter. If we were to suppose the connectors at their free ends, instead of being attached to the side angles of the diamond, to be joined on to two adjoining sides in such a manner as to become parallel to the other pair of sides, this parallelism would continue to subsist for all positions of the linkage, and the arms or distances of the fulcrum from the opposite angles or poles of the diamond would still remain in the same right line, but the relation between them would now be one of direct instead of inverse proportion. Conceive the fulcrum in such an arrangement to become fixed. Since we can not only alter the angles of the diamond, but make the whole arrangement turn round the fixed point, we can make either pole describe any plane curve whatever: the other pole will then describe a curve precisely similar in shape, but drawn on a different scale, as in any ordinary pantigraph*.

But if we revert to the Peaucellier cell or Reciprocator, whether of the positive or negative form, and treat it in the same manner as the supposed pantigraphic arrangement, fixing the fulcrum, and making one of the poles—that is, an extremity of one of the arms—describe any plane curve, the other pole will no longer describe a similar curve, but what in the language of geometry

* Sometime, according to the authority of a questionist in the *Educational Times* for the current month, called a Pentegram. Theoretically two Peaucellier cells are equivalent to a Pentegram (for we may change r into $\frac{K}{r}$ by one, and that into Kr by the other), but whilst combinations of the former are adequate to the transformation of r into any algebraic function of r, the latter are absolutely sterile, leading only to the one single sort of transformation (if it may be called so), r into Kr. It seems then going too far to say (as does the writer alluded to above) that the germ of Peaucellier's invention is contained in the Pentegram.

is termed an inverse of the curve in question, the fulcrum being the origin of the inversion.

Suppose now one of the poles is made to describe a circle, the other will describe the inverse of a circle, which geometricians are well aware will in general be another circle, subject to the exception that if the arc described by one pole is part of a circle passing through the fulcrum, which is here the origin of the inversion, the path of the second pole will be no longer a circle, but a perfect straight line, which, under a mathematical point of view, may be regarded as a circle with an infinite radius. If then, in addition to fixing the fulcrum, we still further constrain the motion of the Peaucellier cell by attaching one of the poles to a centre (which for the sake of distinction from the other fixed point above defined we may term the *pivot*) round which it can revolve, situated at an equal distance from that pole and the fulcrum, the other pole will describe a perfect straight line perpendicular to the line joining the fulcrum and the pivot. We have thus a combination of seven radiating bars attached to two fixed centres, one point of which describes a true rectilinear path, and thus the long-sought-for problem of a perfect parallel motion meets for the first time its complete solution*.

* The centre above spoken of may be taken in the line itself, which joins the poles and the fulcrum. If it be taken not too far out of this position of symmetry it will in the course of the motion be brought into such position; but if it be taken at starting (as it may be), at a sufficiently great distance from the cell, the position of symmetry may never be attained throughout the whole possible course of the motion. This circumstance has been generally overlooked, and accordingly too narrow a rule has been given for the construction of a Peaucellier parallel motion, namely, it is laid down that the pivot is to be taken midway between the fulcrum and one of the poles for some certain position of the instrument. The position of the fulcrum relative to the two poles gives rise to the distinction between a negative and positive cell; but the preceding remark shows that there is a further subdivision of Peaucellier parallel motions depending on the length of the mounting radius, and that positive and negative mounted cells each of them embrace two radically different forms or genera, which may be distinguished as the symmetrical and non-symmetrical respectively; in the one form there exists a position where the *first* lies in the line containing the *fulcrum* and the two poles, in the other, no such position can be found. In the ordinary rule given for the construction of a P.P.M. only the former of these two genera is included which, as machines, differ between themselves as much as do the ellipse and hyperbola as curves.

It ought to be added that the motion of the *parallel-point* is always perpendicular to the line of centres, and in every position makes, with the line containing the fulcrum and the poles, an angle equal to the angle contained in the segment of the circle (of which one pole describes an arc), which lies between it and the fulcrum. If we join the two fixed centres by a new link, and then unfix them, we obtain a linkage of eight bars, possessed of very remarkable properties, of one of which Peaucellier has availed himself to obtain a mechanical description of the Limaçon of Pascal, which is the inverse of a conic in respect to the focus as the origin of inversion.

By a combination of such linkages it is possible to cause any number of points, otherwise free, to remain always in a straight line with each other. The speaker believes that he is in possession of a *bonâ fide* valid proof of the proposition assumed on totally insufficient grounds by Peaucellier, namely, that every algebraical curve may be best described by link-work. The proof is founded on the union of the above statement (or still better, one founded on his own Kine-

The speaker illustrated these results by various models constructed in wood. By changing the length of the radial bar connecting one pole of the cell with a fixed point, the free pole was shown to describe arcs of circles convex or concave to the fulcrum, according as the ideal circle, in an arc of which the first-named pole moved, fell short of the fulcrum or contained that point within it; in the limiting case, when it passed through the fulcrum, the path was shown to be neither convex nor concave, but a straight line free from all curvature in either direction. This was further verified mechanically by connecting together at their free poles two perfectly equal and similarly mounted cells. If the tendency of either of these was to deviate from the straight path, the tendency of the other would be to deviate in the contrary direction; so that either the pair of mounted cells would become an

matical Paradox subsequently referred to) with Grassmann's method of describing algebraical curves by means of an apparatus of fixed points and lines; this proposition, as far as concerns curves of the first nine genera (that is, of a *cursality*, or, so to say, circuit-*complexity* not transcending the 9th degree), and also for curves of the first six orders, or for any order where the degree of one of the variables in the representing equation is 5 or less, he had already demonstrated by a direct method. In using this method he found it necessary to prove that a general algebraical equation of the fifth degree could always be reduced to a trinomial form by *real* transformations, which, by Tschirnhausen's (the only method hitherto applied), as often as not, is incapable of being done. By an extension of the principle of Tschirnhausen's method he succeeded in establishing this important algebraical proposition. A very much more important conclusion relating to the representation of every algebraical function (that is, the function that one quantity is of another connected with it by any algebraical equation), under a quasi-explicit form, he believes he can show may be deduced from the transformed Grassmannian construction above alluded to: by quasi-explicit, meaning a form capable of being obtained by the elementary processes of addition, multiplication, change of sign, and reciprocation with that of general form-inversion superadded. Thus Peaucellier's discovery seems likely to throw open a new chapter in the highest summits of Analysis, no less important in the theoretical direction than its numerous applications to the mechanical arts in the direction of practice.

In the lineo-circular or parallel-motion adjustment imagine the connectors to be detached from the angles of the diamond, and joined on to the two sides of the diamond, which meet at the "parallel point," at equal distances from it. Then the motion of that point will no longer be in a straight line, but in a circle.

This method of producing one circular motion from another (which was first given by the speaker in the *Educational Times*) may probably be found to possess important practical advantages over the circulo-circular adjustment of the Peaucellier cell described in the text above.

The speaker exhibited another modification of the Peaucellier cell; like it consisting of six links, but having the property that the sum of the squares of the two arms (instead of their product) remains constant. This he calls a quadratic-binomial extractor.

By means of this cell, mounted with a suitable radius, a perfect lemniscate may be described; and what is very interesting, and flows from this construction (but was first observed by Dr Henrici), the same curve may be described by means of a binomial-extractor, of a certain kind, reduced to a link-work by the *ablative* method of fixing one of the links: in other words, a perfect lemniscate may be described throughout its complete extent by means of 5-bar motion. Peaucellier refers to, without specifying, a combination, "assez compliquée," of cells (or, as he terms them, compound compasses) by means of which a lemniscate may be traced; whereas, in the method above described the number of links employed is less by a pair than in the single mounted Peaucellier cell.

absolute fixture, or the two would crush or tear each other to pieces; but in the experiment exhibited the pair of mounted cells were seen to move together (as if in happy wedlock), without let or hindrance to each other's motion. The circular motion of the free pole of a single mounted cell in the general case was also verified experimentally, and even more simply than in the rectilinear case, by the addition of a second radial bar, taken of a suitable length, determined by previous mathematical calculation. As a general rule, the total number of bars in a link-work machine must be odd, but here there were eight bars, and yet the combination admitted of being set in free motion,—any one of the eight being, in fact, what may be termed a lazy-bar, and capable of being removed without disturbing the motion, very much in the same way as any one of the four legs of a table may be removed without disturbing the equilibrium*.

The speaker pointed out the important applications of the two kinds of motion above referred to (which he proposed to call the circulo-linear and the circulo-circular respectively) to various constructions in machinery, such as the steam-engine, planing and grinding machines, the construction of maps on the stereographic projection, millwrights' work, laying out of railway curves, dioptric apparatus for lighthouses, ornamental tracery, pendulum suspension to effect motion in a practically exact cycloidal arc, &c., &c., and referred to the use which, as he was informed by the authorities at Woolwich, might have been made of the circulo-circular adjustment in saving several weeks' work, inconvenience, and expense in cutting out the fish-bellied torpedo casings recently constructed in the laboratory department at the Royal Arsenal there, and the use contemplated to be made of the circulo-linear, or perfect parallel motion, for guiding a piston-rod in certain machinery connected with some new apparatus for the ventilation and filtration of the air of the Houses of Parliament, now under course of construction.

He next referred to the unlimited command over the motion of a point furnished by a combination of cells. Returning to the simple Peaucellier

* Suppose four circles to be given, and that it is proposed to inscribe upon them a quadrilateral whose four sides are given in length.

This is a determinate problem which will in general admit of a definite number of solutions. (The method of correspondence and of bipartite equations founded thereon seeming to indicate thirty-two as the total number of such solutions, some or all of which may be imaginary.) But now the question may be put, "Under what circumstances can the number of such solutions become infinite and the problem undeterminate?" It follows from what is stated in the text above that this may happen (other conditions being satisfied) when two of the circles coincide and the four given lengths are all equal. It remains to be ascertained whether with any new set of conditions a like undeterminateness can be brought about for the case of four circles all distinct. If so, a solution would be obtained of the problem of converting by link-work circular into circular and conceivably (as an extreme case) into linear motion by an arrangement radically distinct from Peaucellier's, and involving the use of three instead of two fixed centres, but with the same number of links.

cell, its use may be modified in a very remarkable manner by setting free the point of junction of the two connectors (termed, in what precedes, the fulcrum), and fixing one of the poles as a centre of rotation in its place. If now the liberated fulcrum be made to describe any curve, the free pole will describe a curve corresponding to it, according to a certain easily-statable mathematical law. Imagine the first-named curve to be part of a circle passing through the fixed point—it may be shown that in that case the free pole will describe the inverse of a conic section in respect to a vertex of the conic as the origin of the inversion; consequently, by combining with this cell a second, used as a Reciprocator, we may, mounting with a suitable radius a pair of Peaucellier cells duly adjusted, cause a point to move in a parabola, ellipse, or hyperbola.

The speaker exhibited a combination of this kind, and caused a point to describe portions of an ellipse, a parabola, and of the two branches of a hyperbola in succession; the traversing pole of the first cell, which might be termed the first follower, being seen to describe beautiful nodal cubics (or the inverses of the conics), whilst the free pole of the second cell or second follower described the conics themselves*.

* The nodal cubics or conic-inverses above described are for the parabola, the common cissoid, and for the ellipse and hyperbola curves which may be termed trans-cissoid, and cis-cissoid, or less barbarously and more euphoniously the hyper-cissoid and hypo-cissoid respectively. The common cissoid, as is well known, has a cusp which here coincides with the fulcrum. In the hyper-cissoid this becomes a detached, or, as it is ordinarily termed, a conjugate point, and in the hypo-cissoid a node on the curve, which in this case possesses a loop in addition to an infinite branch. When the first follower moves in this infinite branch, the second follower describes a portion of that branch of the hyperbola in which the fulcrum lies—but of course can never reach the vertex, which coincides with the fulcrum; when the first follower moves in the loop the second follower describes the opposite branch of the hyperbola, and can be made to pass through the vertex of that branch.

The geometrical construction for the common cissoid, or cissoid proper, is well known to be as follows. Imagine a pencil of rays proceeding from one extremity of a diameter of a circle, and meeting a tangent to the circle drawn at the other extremity. Then if the portion of each ray intercepted between the circle and tangent be shifted along the ray until one point of it coincides with the centre of the pencil, the other point will mark out the cissoid. Now imagine everything to remain as above, with the exception that the tangent is moved parallel to itself and becomes fixed in a new position nearer to or further from the centre of the pencil than it was at first, then the curve marked out becomes the hypo-cissoid or hyper-cissoid respectively, a remark due to Mr Howard Elphinstone. The smoothness of the motion, and the facility with which the cissoidal curves and the corresponding curves were drawn was matter of general surprise and admiration to the audience. This circumstance, due in part to the skill of Dr Henrici in choosing the proportions of the parts, ably seconded by the mechanical experience and ingenuity of Mr Grant, modeller to University College, at the same time served to evince the extraordinary superiority of pure link-work motion, that is, motion due exclusively to the action of radiating bars about centres, over motion effected in whole or in part through the intervention of grooves and slides. It was the analogous superiority enjoyed by circular over linear construction for the purpose of graduating instruments of precision that actuated Mascheroni (the favourite geometer of the first Napoleon) in devising his admirable, most valuable, and most tedious exposition of the geometry of the compass. The superiority in question was still more

He next went on to state that by a combination of cells properly proportioned and suitably attached to each other in succession in a manner similar or analogous to that in which simple machines, as for example a number of levers, may be combined to produce a complex one, we are able to bring about any mathematical relation that may be desired between the

strongly evinced in the triple-cell combinations employed in the instrument for the extraction of cube-roots and the trisection of an angle.

Clairaut has given a method of constructing an instrument for extracting the roots of an equation by means of linear measurements described in Borgni's *Traité de Mécanique Appliquée* (volume on *Machines Imitatives*, p. 226); but the author's method, founded on Peaucellier's discovery, is beyond all comparison superior in the range of algebraical operations which come within its scope, in the simplicity, homogeneity, and smaller number of its parts, in the facility of its application, and the smoothness of the resulting motion. His instrument for solving cubic equations is far less complicated than that of Clairaut for quadratics (which he does not suppose has ever been realized) and infinitely easier of application. For instance, in working his cube-root machine, one point of the instrument is fixed as the zero-point; a second point, called the setter, is drawn out to a division on a scale corresponding to any proposed number; a third point, called the finder, will then automatically place itself over the division on the same scale, corresponding to the cube root of that number. The zero, setter and finder points in the calculating linkage are identical (or, as in the transformation scene of a pantomime, may be said to change characters) with the fulcrum, power and weight (or driver and follower), points in the corresponding link-work used as a machine. In Clairaut's and other similar machines the calculations are made by means of measurements made upon curves described by the machines. The author's method is direct and does not involve the use of any such intermediary process.

Returning to the subject, which has led to this digression, it will be noticed that by the method referred to in the text a mounted double reciprocating cell, that is, an apparatus of thirteen links, serves to describe a conic. Peaucellier's method, founded on the combination of what may be termed a collineator or radial protractor, with a mounted reciprocator, involves the use of fifteen links, besides a cross-piece rigidly attached to one of them, and, so far, is less simple, as well as less symmetrical than the author's method: but this must not be supposed to be said in derogation from the merit of the admirable invention of the collineator itself, by which Peaucellier has solved the beautiful and most important kinematical problem of devising a perfect linkage, enjoying the property, that however it is turned about, or drawn in and out, one point of it shall always remain upon or in the direction produced of a physical line rigidly attached to the linkage, but in different positions upon such line. It is believed that a conic-describing instrument (may one say conicograph?) on Peaucellier's plan has not been actually executed, and that a pure link-work for effecting conical motion was witnessed for the first time since the creation of the world in the lecture-room of the Royal Institution on the 23rd of January, 1874. Although it may be presumed that the Peaucellier conicograph would not work so simply as the one exhibited, it possesses a superiority in one respect, namely, that the fulcrum on this arrangement lying off the curve at the focus, the part of the curve described may be made to include the vertex of the parabola, which cannot be reached by the other method. It has been thought by competent judges, conversant with practical mechanism, that this (the writer's) method might be applied with advantage to constructing parabolic light-house reflectors; and as these, from the nature of the case, are made *without backs*, consisting of two paraboloids of revolution, situated *dos-à-dos*, having a common focus, at which the source of light is placed, from which the rays stream through the opening upon the surfaces of the two reflectors, the fact of the tracing or cutting or grinding instrument not being able to reach the vertex, would be no disadvantage in this case, since the portion of the surface in the neighbourhood of that point is not required, and, indeed, if formed would have to be subsequently cut away. But it should be added that by a generalized single-mounted cell an approximation to the parabolic form can be attained to a degree of precision far in excess of all practical needs.

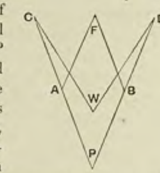
distances of two of the poles of a linkage from a third, and are thus potentially in possession of a universal calculating machine. He exhibited and worked a cube-root extracting machine constructed on this principle, and claimed to have given the first really practical solution of the famous problem proposed by the ancients of the duplication or multiplication of the cube. This machine consisted of a combination of three cells; by changing the modulus of one of the three, he explained that it was also quite easy to solve the cubic equation involved in the analytical solution of the problem of the trisection of the angle; and a working model of an instrument of this kind executed in zinc was exhibited by Professor Henrici after the lecture. He concluded by expressing his great obligations to this gentleman, without whose aid he would have been able to do little more than adumbrate in general terms the results which, thanks to his friend's practical knowledge and skill, he had had the pleasure of exhibiting in a tangible form, and submitting before his audience to the test of actual experiment; and expressed his conviction that Peaucellier's unhoped-for discovery (even if viewed merely on its practical side as a new vital element of mechanism) was destined to produce lasting and important results through innumerable applications to the useful and ornamental arts, and would hand down the name of its inventor to posterity as one of the benefactors of mankind.

Postscript.

In some possibly forthcoming number of *Nature* a detailed account, which was expected to appear two months ago, will be given, illustrated with the necessary diagrams, of the cube-root extractor and angle-trisector: the materials for this purpose are in the hands of the editor of that journal, and have been entrusted by him to the most competent person to draw them out into form—the writer not feeling within himself the necessary energy for accomplishing this task. He thinks it, however, desirable (indeed almost a moral duty on his part) to supplement those materials by the desultory remarks which follow, in order that some results, which he believes to be important to the progress of mechanical and algebraical science, may be rescued from the chances of total oblivion and virtual annihilation.

The first question which presents itself relates to the square-root extractor. It is a remarkable fact that a cellular system for extracting square roots is much more complicated than what is required for the cube root; and so in general all even-degreed extractors require a more extensive apparatus of link-work than is required for the odd degrees. Such extractions may be performed in all cases by a system consisting of Peaucellier cells exclusively; but the process may be abridged in the case of even degrees by the interpolation of another form of cell, alluded to in a previous foot-note under the name of the quadratic-binomial extractor, which deserves a somewhat more

detailed description. It is figured in the diagram below. $FAPB$ is a jointed rhomb or diamond; PC and PD are each doubles of the sides of the rhomb and CW, DW are two equal links. The difference between the squares of CP and CW is the modulus. FP, FW are the arms, and the difference between their squares is equal to the modulus. This is the instrument which, when F is fixed and P moves in a circle passing through W, describes a curve which may be called the Lemniscatoid, having the same general kind of relation to the Lemniscate that the Hypercissoid and Hypocissoid bear to the Cissoid proper. This Lemniscatoid becomes the Lemniscate when a certain simple arithmetical relation subsists between the modulus and the diameter of the circle described by P. If A as well as F be fixed, P will move in a circle passing through F, of which AP will be the radius, and consequently the five-bar link-work, consisting of the links CW, CP, DW, DP, FB (centred at F and A), will serve to describe the Lemniscate when the arithmetical relation above referred to subsists between CP and the modulus; that is, between CP and the difference of the squares of CP and CW; consequently, when the lengths CP, CW have a certain simple arithmetical proportion to each other, W will describe the Lemniscate: this proportion, it will be found, is such that when W comes to F the angle at P is a right angle. So much for the binomial-root extractor: obviously by aid of this kind of linkage when one arm is the tangent of any angle, the other arm may be made equal to the secant, and *vice versâ*. Again, it should be observed that, as in the Peaucellier cell (used as a reciprocator) the arms may be taken as x and $\frac{1}{x}$, by interchanging the fulcrum with one of the poles that is, reckoning the two arms as the distance between the fulcrum and one pole and from the other pole to the arm x, the new arm may be made to become $\frac{1}{x} - x$, which may be reciprocated into $\frac{2x}{1-x^2}$ by the use of a second Peaucellier cell. Hence by two Peaucellier cells an arm denoted by $\tan\theta$ may be, so to say, transformed into an arm $\tan 2\theta$. Thus we see that we may pass through the following series of transformations



$$\cos\theta,\ \sec\theta,\ \tan\theta,\ \tan 2\theta,\ \sec 2\theta,\ \tfrac{1}{2}\cos 2\theta$$

by means of a P.C., a Q.B.E., a pair of P.C.'s, a Q.B.E., and a P.C.,—that is, by an apparatus containing four Peaucellier cells and two cells of the new kind—making a linkage of six cells or 36 links in all. In other words, by means of such a linkage the arm x may be, so to say, converted into $x^2 - \frac{1}{2}$.

If, therefore, by a Q.B.E. we first convert x into the square root of $x^2 + \frac{1}{2}$ by superadding to this the linkage last named, that is, by a linkage of seven cells or 42 links, x becomes converted into x^2. Thus, then, seven cells are required for a squaring or square-root extractor instrument analogous to the cubing or cube-root instrument for which only three cells are required*.

The above investigation leads to a further construction of extraordinary interest, which the speaker is wont to describe as the Kinematical Paradox: every new flight in physics and mathematics, and the same seems equally true of politics, ethics, and philosophy†, is apt to commence with a paradox. Two perfect linkages have been described above, one of six, the other of seven cells. Let these linkages both be constructed simultaneously; they will have two detached points of the one (namely, the two extremities of the arm x) coincident with two of the other: their union will itself (according to a general principle) form a perfect linkage. In this linkage of 13 cells two points will lie in the same straight line with the original zero point from which the arms are measured, one at the distance x^2, the other at the distance $x^2 - \frac{1}{2}$ therefrom. Hence there will be two points in this linkage which are disconnected, but in whatever way the other links are drawn in and out, retain an invariable distance from each other! Any other two points of the apparatus may be made to vary their distances from each other, but no force that can be applied at these two points to force them nearer to or separate them further from each other can be of any effect. There is no immediate rigid connection between them, and yet they are as good as rigidly connected. Imagine now that they become connected by a material link: the linkage will not be a fixture, but a perfect linkage as before, consisting, however, of an odd number, namely, 79 links; any one of these may be regarded as a lazy-bar, and may be removed without affecting the motion of which the apparatus is susceptible. Returning to the original state of things, where there are 13 cells, if we fix the two points of invariable distance the instrument will not become a fixture (as would be the case if any two other disconnected points in it were fixed), but a free link-work with a superfluous or lazy-bar,

* The much simpler scheme for converting x into x^3, which explains the principle of the cube root machine, is as follows:

First conversion, $x - \frac{1}{x}$, that is, $\frac{x^2-1}{x}$.

Second conversion, $\frac{x}{x^2-1} - \frac{1}{x}$, that is, $\frac{1}{x^3-x}$.

Third conversion, $(x^3 - x) + x$, that is, x^3.

For the trisection of the angle it is necessary to solve kinematically the equation between $\cos 3\theta$ and $\cos\theta$, to effect which it is only necessary to replace the third conversion above by

$$4(x^3 - x) + x, \text{ that is, } 4x^3 - 3x.$$

† As for example Cramer's paradox (the foundation of the highest modern geometry) the ποῦ στῶ of Archimedes and the hydrostatic paradox, "The king can do no wrong," "It is better to suffer than to do wrong," "All proof is reducible to syllogisms, and the syllogism can prove nothing," "A heavy body begins to fall with no velocity." The Kantian antinomies. Helmholtz's vortices. A variable function which never varies, that is, an Invariant as distinguished from a Constant.

represented by any of the links at will; for by fixing these particular two points, not *four*, but only *three* degrees of liberty are abstracted. By fixing one of them two such degrees are taken away; but as the other is then not free, but compelled to move in a circle, fixing *it* takes away only one additional degree of liberty of motion.

By this link-work of 78 bars (one supererogatory) a remarkable Kinematical problem has been solved (and it is probably the simplest solution of which it admits), which may be stated as follows:—"Required to construct a link-work fixed or centred at two of its points, such that (when the machine is set in motion) some other point or points therein shall be compelled to move in the line of centres."

There are some similar questions to this, which ought, in a strict logical order, to have preceded it, which we may now take into consideration. By a single mounted Peaucellier cell fixed at two centres, one point is made to move perpendicular to the line of centres. Suppose now it were required to devise a link-work such that a point should move parallel to such line.

The motion perpendicular to the line of centres is due to the fact that by the Peaucellier cell the radius vector $C\cos\theta$ is transformed into $C\sec\theta$; in like manner to get the parallel direction a means must be found of passing from the cosine to the cosecant. Now although a single cell serves to change the tangent into the secant, *or vice versâ*, and consequently a single *imaginary* cell will serve to change the cosine into the sine (which of course could then be immediately Peaucellierized into the cosecant), he is not aware of any direct real process simpler than that about to be stated by which this can be effected. His actual law of deduction is as follows: Cosine; secant; tangent; cotangent; cosecant, involving the use of two Peaucellier cells and two quadratic-binomial extractors.

With one cell more, that is, with five in all, the cosine becomes converted into the sine, and consequently by introducing a pantigraphic cell $\cos\theta$ may be converted into $\cos(\theta + \alpha)$, and this reciprocated into $\sec(\theta + \alpha)$. Thus it seems (at all events after the present method) that four cells are required to obtain by link-work rectilinear motion parallel to the line of centres, and seven cells to convert it into motion oblique to the line of centres; or taking into account the mounting radius 7, 25, 43 links are required to obtain motions respectively perpendicular, parallel, and oblique to that line. In the Kinematical Paradox it will have been seen that there are 13 cells employed, that is, 78 links, of which any one is liable to removal at will, so that for motion in the very line of centres 77 links are requisite. Consider this system in its entirety. In a straight line with the two fixed points there will be 13 other medial points; and two parallel ranks on both sides, each also containing 13 points. The whole apparatus admits of being moved with a sort of see-saw motion backwards and forwards; and it may assist the imagination of the reader if he will conceive such an instrument armed with 13 picks in the line of centres, each at work to remove the asphalt of a pavement under repair; an idea suggested by a member or visitor at a soirée of the Amateur Mechanical Society of London, of which the ingenious and accomplished "Senior Member for Greenwich" acts as honorary secretary. Or we might describe the Kinematical Paradox as a kind of compound saw. If the "two points of invariable distance" be set free, and some other of the medial points be fixed as a fulcrum, the instrument may be used like Peaucellier's second invention referred to in a previous foot-note as a radial protractor to change the curve

$$\rho = \text{a given function of } \theta$$

into the curve $\qquad \rho + c = \text{the same function of } \theta;$

as, for instance, to pass from the circle to the limaçon of Pascal, or from a straight line to a conchoid. For while one of the two points of constant distance described any curve, the other would describe the same curve with all its radii vectores reckoned from the fixed point lengthened or shortened by a constant quantity. The Kinematical Paradox ought not to be regarded in the light of a mere luxury of speculation; it serves to represent a constant as a Kinematical function of the independent variable (corresponding to the use of the zero power of x to represent unity in algebra), without which the general analytical theory of linkages, and the very important theory of algebraical functions founded thereon, would fall to the ground, or rather be incapable of being constructed.

It would be difficult to quote any other discovery which opens out such vast and varied horizons as this of Peaucellier—in one direction, as has been shown, descending to the wants of the workshop, the simplification of the steam-engine, the revolutionizing of the millwright's trade, the amelioration of garden-pumps, and other domestic conveniences (the sun of science glorifies all it shines upon), and in the other soaring to the sublimest heights of the most advanced doctrines of modern analysis, lending aid to, and throwing light from a totally unsuspected quarter on the researches of such men as Abel, Riemann, Clebsch, Grassmann, and Cayley. Its head towers above the clouds, while its feet plunge into the bowels of the earth.

Prophetic and well-timed were the parting words to the speaker of the illustrious Tchebicheff: "Take to Kinematics, it will repay you; it is more fecund than geometry; it adds a fourth dimension to space." So also said Lagrange.

In the course of the foregoing exposition, incidental reference has been made to the addition of perfect linkages to each other*. This gives rise to

* Namely, by pivoting together two disconnected points of the one with two disconnected points of the other, each with each. The sum of two perfect linkages so connected will satisfy the same numerical linear equation between joints and links as its two constituents, and thus will itself constitute a perfect linkage.

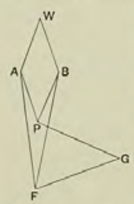
the important distinction of all perfect linkages into prime and composite—prime ones being such as can be resolved into the sum of two others, and composite those for which no two such components can be found. As an example of one kind, imagine an octagon with its four pairs of opposite angles (or, which will do as well, its four pairs of opposite sides) connected by links. There will then be 12 links and 16 joints; and since $3 \times 12 - 2 \times 16 = 4$, the linkage will be perfect. Such a linkage is prime, for it will be found impossible to resolve it into two others. Whereas, every cell previously described is capable of being formed by the successive accretions of single pairs of links, thereby justifying in a new and specialized sense the title of Compound Compass, used by Peaucellier to designate his cell. Moreover, cells belong to a very special class of compound linkages, those namely which by successive processes of decomposition can eventually be reduced to depend on sets of link-pairs, and which may accordingly be termed Dyadisms. Dyadisms, again, require to be classed according to their order. A dyadism of the first order is one that can be obtained by successive additions of single duads at a time. A dyadism of the second order is one that can be formed by successive additions of single dyadisms of the first order at a time, and so on; and it is very essential to notice that the addition together of two dyadisms of a given order will not in general be a dyadism of the same order. Thus we see that a pure tactical theory of colligation underlies the subject of linkages, a theory of the same nature as that which is known to underlie the doctrine of crystallography and polyhedra; and as that which, under the name of ramification (proposed by the speaker), gives the clearest notion of the modern chemical doctrine of the atom-groupings of the hydro-carbons, and in a manner supplies an *à priori* ground for the formula of the saturated hydro-carbons $C_n H_{2n+2}$, which, for the simpler case of the hydro-borons (if such series existed), would become $C_n B_{n+2}$.

It may be shown that every ramification may be subjected to a process of reduction (a sort of divulsion process, the number of steps of which fixes its genus, or order), which leads eventually to a single intrinsic centre or a pair of intrinsic centres, and consequently may be referred to one or the other of two great classes of forms which may be termed central and axial respectively; and it seems only reasonable to anticipate that the physical properties of such chemical compounds as the hydrocarbons will eventually be found to correspond to this distinction between their representative ramifications; and that they will accordingly range themselves under one or the other of two great families distinguished by properties at least as important and specific as those which serve to distinguish the crystalloidal and colloidal states of matter. The theory of ramification is one of pure colligation, for it takes no account of magnitude or position; geometrical lines are used, but have no more real bearing on the matter than those employed in genealogical tables have in explaining the laws of procreation.

The sphere within which any theory of colligation works is not spatial but logical—such theory is concerned exclusively with the necessary laws of antecedence and consequence, or in one word of *connection* in the abstract, or in other terms is a development of the doctrine of the compound parenthesis. M. Camille Jordan, independently of and anteriorly to the author, discovered and published in a memoir, the title of which would never suggest the notion of ramification, the existence of the intrinsic centre and centres here referred to—without having any suspicion of its bearing on modern chemical doctrine. He has moreover discovered the existence of another kind of intrinsic centre of ramification which was unknown to the author of these lines.

A ramification, it ought to be added, is a rootless tree, that is, one in which the root only ranks the same as the terminal of a branch, and saturated hydrocarbons are typified by ramifications in which every joint is trifurcated, meaning thereby that in tracing the wood outwards from any terminal assumed as the root, it splits and splits again, so that trifurcation takes place at each joint, or in other words, *four* lines radiate out from each joint*; the joints are supposed to adumbrate the carbon atoms and the terminal points the hydrogens.

To conclude, as he has begun, with the principal personage of his story, the author thinks it will be useful to several of his readers to have before their eyes the figure which contains the property of the admirable linkage which lies at the root of Peaucellier's conicograph.



In the given figure $APBW$ is a rhomb. PA is equal to PB, GP to GF, and G' is a point lying on FG, or FG produced such that $FG'W$ is a right angle. Then, however the links are moved about, the motion of W *relative* to FG will be always perpendicular to FG, from which it follows that $FG'W$ will always continue to be a right angle, and consequently an upright piece attached at G' perpendicular to FG will always continue to point to W. When W is fixed, the instrument serves as a radial protractor. One point of the upright can describe any curve, and any other point a radial protraction (or retraction) of that curve. When one point of the upright perpendicular is fixed, the combination becomes ideally equivalent to a revolving slot, in which W is free to traverse. The inverse of a conic in respect to a focus (that is, the Limaçon of Pascal) is a protraction or retraction of the circle. Hence the use of the instrument for describing conics.

* Observe that if there were *no* splitting, as in a bamboo cane, *two* lines would issue from each joint.

In the above linkage let a pair of equal links GP, GW be *substituted* for the pair GP, GF. It is easy to prove that if O be the intersection of the diagonals of the rhomb, GO and FO will then be at right angles to each other, and the sum of their squares will be a constant. If now any one link of the rhomb is transferred parallel to itself so as to pass through O, and is jointed on to the sides at the points where it meets them, and O is fixed, and F made to move in a circle containing O, the path of G will be the *inverse in respect to O of a conic* of which O is the centre, so that by the aid of a radius and a reciprocator in addition to the transformed linkage above described, a point may be made to move in any conic round its *centre* as a fixed point*. This is rather a simpler construction than Peaucellier's for motion in a conic round the *focus* as a fixed point, for the number of links is no greater, and the ungainly cross-piece disappears. Moreover, it possesses all the advantages of Peaucellier's method arising from the fulcrum lying off the curve to be described. Finally, as regards the most general motion that can be produced by a Peaucellier-mounted cell in its generalized form, if F be the junction of two links on which FA, FB are two equal segments, and FC, FD two other equal segments, and PA, PB and WC, WD be two pairs of equal links in the same plane with the first pair, such combination of three pairs is the generalized form of cell in question. In applying it to draw curves, F may be fixed, and a mounting radius of any length attached to P or W, or P or W may be fixed, and the mounting radius attached to W or P, or P or W be fixed, and the mounting radius attached to F. In a résumé of this general kind it would be out of place to enter into a discussion of the forms thus generated†.

* It follows as a particular case of the above, that an apparatus of nine links moving round two fixed centres will serve to generate motion in a circle whose centre is in a right line drawn through one of the given two, perpendicular to the line joining it to the other.

† It is too late to make any change in the many places where the term perfect linkage appears in the text, but the author regrets to have used the word *perfect* when *complete* would have expressed the meaning more clearly, and suggests this change of nomenclature to any writer who may hereafter have occasion to employ the term—besides being better in itself, it comes nearer to Peaucellier's "système de tiges à liaison complète"; two words (and those much more expressive) supplying the place of six. The existence of such words as surplusages, curtilage, equipage, assemblage, and many similar ones in the English language, appears quite sufficient to justify the innovation in the use of the final syllable in linkage. A question of great interest remains over, namely, "how to extend the above inquiry to linkages in space"; any two links being supposed free to move by means of universal joints in all directions round each other. As regards surfaces of revolution, the solution of the problem is virtually contained in the theory of plane linkages, and consequently as a plane may be regarded as a surface of revolution, the difficulty does not begin to be felt until the problem of producing motion in an ellipsoid or other surfaces of the second order, by means of solid link-work, comes under consideration. It seems to be a problem well worthy of being investigated and thought out, especially for the sake of its analytical consequences and the light it might be expected to throw upon the theory of algebraical functions of two variables.

3.

ON THE PLAGIOGRAPH *aliter* THE SKEW PANTIGRAPH.

[*Nature*, Vol. XII. (1875), pp. 168, 214—216;
also, *Archivo de Mat.* I. pp. 112—114.]

I HAVE been led by the study of linkages to the conception of a new instrument, or rather a simple modification of an old and familiar one, the Pantigraph, by means of which a figure in the act of being magnified or reduced may at the same time be slewed round the centre of similitude. Some of the readers of *Nature*, such possibly as my able and most ingenious friends, Messrs George Cayley and Francis Galton, may be able to pronounce with authority how far the invention is new and whether it is likely to be found in any way useful in practice as applied to the art of the designer or engine-turner. Already my invention of the Isagoniostat, or equal angle setter, which I shall take some other opportunity to communicate to this journal, has been deemed available in practice for working automatically the train of prisms of a spectroscope.

In Fig. 1, $AOBCQ$ represents an ordinary pantigraph. O is the fixed point, P is the tracer, and Q the corresponding follower; then, as everybody

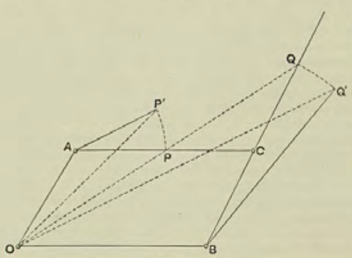


Fig. 1.

knows, any curve traced out by P will be imitated by Q, and the two curves

will be similarly situated in respect to O. The point of addition is the following:—

Let P be moved through any angle, $P'AP$ round A, and Q through *an equal angle* QBQ' in the *opposite* direction round B, and let P' and Q' be supposed to be in any manner rigidly connected with the bars AC, BC respectively. Then it admits of an easy proof that in whatever way the jointed parallelogram $AOBC$ is deformed, OQ' will bear to OP' the constant ratio of AC to AP, and moreover the angle $P'OQ'$ will always remain equal to the angles $P'AP$, $Q'BQ$.

It follows that whilst P' is made to move upon any curve the follower Q' will trace out a similar curve altered in magnitude, and at the same time turned round the first point O.

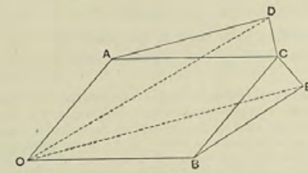


Fig. 2.

If, as in Fig. 2, we take AD equal to AC, BE equal to BC, and the angles CAD, CBE equal to each other, then the rays OD, OE will always remain equal and be inclined to each other at a constant angle. With this adjustment the instrument may be used to transfer a figure from one position in a sheet of drawing paper to any other position upon it, leaving its form and magnitude unaltered, but its position slewed round through any desired angle.

History of the Plagiograph.

I should like to add a few words to my description of the instrument called the Plagiograph* (the g to be pronounced soft, like j, as in Genesis

* It may be questioned whether a new-born child can have a history. Perhaps it might have been more correct to have used for my title, "History of the Birth of the Plagiograph," but this would have been long; moreover, the Plagiograph proves to be an unusually precocious child, having in its very cradle given birth to a greater than itself, the Quadruplane, a full-grown invention described in the sequel of the text.

Plagiarist, Oxygen) in *Nature*, Vol. XII. p. 168, for the purpose of explaining the order of ideas in which it took its rise, and also a very beautiful extension of another recent kinematical invention to which it naturally leads the way, and which, thus generalised, I propose to term the Quadruplane.

The true view of the theory of *linkages** is to consider every link as carrying with it an indefinitely extended plane, and to look upon the question as one of relative† motion which may be put under this form: When a *complete* linkage (meaning thereby a combination of jointed planes capable of only a definite series of relative movements) is set in motion, *what is the curve which any point in one of these planes will describe upon any other?*

In this mode of stating the question, the lines joining the pivots round which the planes can turn correspond to the jointed rods of the common theory. Fix any one of the planes, and the linkage becomes a link-work, or, to speak with more precision, a piece-work.

The curve described by a point in one plane upon any other plane has been termed by me with general acquiescence a Graph, and to keep the

* It is quite conceivable that the whole universe may constitute one great linkage, that is, a system of points bound to maintain invariable distances, certain of them from certain others, and that the law of gravitation and similar physical rules for reading off natural phenomena may be the consequences of this condition of things. If the Cosmic linkage is of the kind I have called complete, then determinism is the law of Nature; but, if there be more than one degree of liberty in the system, there will be room reserved for the play of free-will. We should thus revert to the Aristotelian view under a somewhat wider aspect of circular (the most perfect because the simplest form of motion) being the primary (however recondite) law of cosmical dynamics. Speaking of cosmical laws brings to my mind a reflection I have made upon the new chemical theory of atomicity. Suppose it should turn out that the doctrine of *Valence* should be confirmed by experience, and that the consequent logico-mathematical theory of colligation containing the necessary laws of consecution, or if one pleases so to say of cause and effect, should plant its foot and introduce a firm basis of predictive science into chemistry, how beautiful will be the analogy between this and the physical law of inertia! which really merely affirms the fact of each atom or point of matter carrying about with it a certain number, denoting its communicative and inverse receptive faculty of motion; for in such case Valency, also affirming a numerical capacity for colligation, will be the exact analogue in chemistry to Inertia in the theory of mass motion, and might properly assume the name of chemical inertia. Social individuals differ as egregiously as Isomers in their capacity for forming multifarious attachments.

† I believe it is to Mr Samuel Roberts that we are indebted for the idea of passing from mere copulated links to planes associated with the links, and for the observation that the order of the corresponding Graphs is not thereby augmented. The substitution of the more general idea of linkage for link-work, and of isolating completely the conception of relative in lieu of absolute motion, is due to the author of these lines. Take the case of a Quadruplane in which the four joints in their natural order of sequence form a contra-parallelogram. It is well known (and the fact has been applied to machinery under the name of "the parallelogram of Reulleux") that the relative motion of an opposite pair of planes may be represented by causing two curves to roll upon each other; but I add that this may be done simultaneously for both pairs of planes, giving rise to a beautiful and previously unthought-of double motion of rolling (without slip) between two ellipses for one pair and two hyperbolas for the other pair of planes. This is an immediate deduction from the conception of purely relative motion.

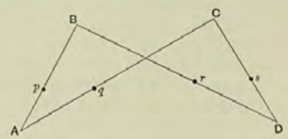
correlation closely in view, I have proposed to call the describing point the Gram*. We may further understand by canonigrams describing points taken in the lines connecting the joints and their corresponding curves, canonigraphs; Grams lying outside these lines and their appurtenant Graphs may be termed Epipedograms and Epipedographs; or, if these names are found too long for use, Planigrams and Planigraphs.

Now consider more particularly the generalised form of the linkage which corresponds to three-bar motion, of which Watt's parallel motion (so-called) offers a simple instance. If we were to revert to the old notion of link-work we should say that a three-bar motion is obtained by fixing one of the sides of a jointed quadrilateral of any form; but adhering to the more general conception of the matter here set forth, we may describe it as resulting from the fixation of any one of the planes of a quadriplane, that is, a system of four planes connected together by four joints. Mr A. B. Kempe, who has brought to light magnificent additions to Peaucellier's ever memorable discovery of an exact parallel motion in a paper which I have had the pleasure of presenting to the Royal Society of London, in the course of conversation with me made the very acute and interesting remark that in an ordinary three-bar motion, supposing the distance between the two fixed centres to be given, and the lengths of the two radial arms and the connecting rod to be also given, the order in which these three latter elements are arranged will not affect the nature of the canonigraphs described. Whichever of the three lengths is adopted as the length of the connector and the remaining two as the lengths of the radial arms, the very same system of curves will be described in all three cases so far as regards their form: every canonigram in the arrangement will have a canonigram corresponding to it in each of the other arrangements such that the corresponding curves described will be similar and similarly placed—a most remarkable, and, for the purposes of theory, an exceedingly important observation; but, as Prof. Cayley observed, when once stated, a self-evident deduction from the principle of the ordinary pantigraph†. It

* Gram is intended to suggest the notion of a *letter* discharging the duty of a point. In inventing new verbal tools of mathemathical thought, the following are the rules which I bear in mind:—1. The word must be transferable into the common currency of the mathematical centres of Europe, France, Germany, and Italy. 2. It must enter readily into combinations and be susceptible of inflexion fore and aft. 3. It should contain some suggestion of the function of the idea intended to be conveyed. 4. It should by similarity in quality or weight of sound conjure up association with the allied ideas. 5. When all these conditions are incapable of being simultaneously fulfilled, they should be observed as far as possible, and their relative importance estimated according to the order in which they are written above.

† Suppose AB, BC, CD to be three jointed rods fixed at A and D. Choose either of the fixed points, say A, and complete the parallelogram $ABCB'A$, regarding CB', $B'A$ as two additional jointed rods; through A draw any transversal, cutting the two indefinite straight lines CB, CB' in P and P' respectively; then whatever curve P describes when the system is set in motion, P' by the principle of the common Pantigraph will describe a curve similar and similarly situated

therefore occurred to me that a corresponding theorem ought to hold for all graphs whatever—for plagiographs just as well as for canonigraphs; and by a very simple application of the general double-algebra method of *Versors*, I found that this would be the case, the only difference being that now the corresponding graphs, instead of being similar and similarly situated, would be similar but not similarly situated; in other words, that the lines joining the centre of similitude with the corresponding points, instead of coinciding in direction, would make for each particular graph a constant angle with each other. Thus I passed from the conception of the common Pantigraph to that of the Quergraph, or Plagiograph, or Skew Pantigraph, as the new instrument described in the previous article may indifferently be called. I now come to



the second part of my story, and proceed to explain the remarkable extension a theorem analogous to and naturally suggested by the Plagiograph gives of Mr Hart's remarkable discovery of a *cell* consisting of only four jointed rods which possesses the same property of reciprocation as Peaucellier's six-sided cell.

This cell is exhibited in the figure above. The four jointed rods AB, AC, CD, BD are equal in pairs, AB and CD being equal, also AC and BD. In fact, the figure is nothing else but a jointed parallelogram twisted out of its position of combined parallelisms, and may be termed a contra-parallelogram. When the cell is in any position whatever, imagine a geometrical line to be drawn parallel to the lines joining A and D or B and C

thereto, A being the centre of similitude. Now, it will be noticed that $AB'CD$ is a system of four jointed rods in which the lengths AB', $B'C$ are the same as the lengths AB, BC in inverted order, namely, $AB'=BC$, and $B'C=AB$, and as we may proceed from the point D equally well as from A, it follows that all the six interchanges may be rung between the three lengths AB, BC, CD. This is the proof of Mr Kempe's admirable theorem; but does the simplicity of the principle involved take away in any degree from the beauty of the result, or rather, is not the interest of the conclusion enhanced by the simplicity of the means by which it is arrived at? In fact, as Kant has observed, the groundwork of all mathematical proof consists in putting things together by a free act of the imagination; and the essence of Euclid is to be sought in the constructions which antecede the formal proofs. The real proof is the construction, and no one has the right to call Mr Kempe's discovery "a truism."

(for these lines will obviously always remain parallel to each other), cutting the four links in the points p, q, r, s.

Now take up the cell and manipulate it in any manner you please so as to change its form, the same four points p, q, r, s will always remain in the same straight line, the distances pq and rs will always remain equal to one another, and the product of pq by pr, or, which is the same thing, of sr by sq, will never vary, so that pr remains (so to say) a constant inverse of pq, and sr of sq—the actual value of the constant product (called the modulus of the cell) being the difference between the squares of the unequal sides of the contra-parallelogram, multiplied by the product of the segments into which any one of the links is separated by the points p, q, r, or s, and divided by the square of such link. Now Mr Kempe and myself—he by the free play of his vivacious geometrical imagination, I by the sure and fatal march of algebraical analysis—have arrived at the following beautiful generalisation of Mr Hart's discovery. On AB, CA, BD, DC describe a chain of four similar triangles, the angles of which are arbitrary, but looking towards the same parts, and so placed that the equal angles in any two contiguous triangles are adjacent—call the vertices of these triangles P, Q, R, S (which will be in fact the analogues of the points p, q, r, s before mentioned): then it will be found that the figure $PQRS$ will be a parallelogram whose angles are invariable, and the product of whose unequal sides is constant; in a word, a parallelogram of constant area and constant obliquity*.

The modulus, or constant product of the sides, follows the same rule as in the special case, except that for the product of the segment of a link divided by the square of its entire length, must be substituted the product of the sines of the angles adjacent to any link divided by the square of the sine of the angle subtended by it.

* I early noticed the analogy between M. Peaucellier's six-linked reciprocator and the primitive form of the pantigraphic proportionator formed by two parallelograms having an angle and the directions of its two containing sides in common, also therefore consisting of six links; and indeed pointed out that, starting (to fix the ideas) from a negative Peaucellier-cell (such as is in successful use in the Houses of Parliament for ventilating the brains of our representative and hereditary legislators), we have only to unfix the two interior links from the angles to which they are attached, and attach them instead to two sides of the containing lozenge, so as to be parallel to the other two sides; and we pass from a Reciprocator to the comparatively barren Proportionator. Now as a Proportionator (the Pantigraph in common use) exists with only four sides, it ought to have been inferred as fairly probable that a Reciprocator also might be discovered having only four sides, that is, by analogy, the probable existence might have been inferred of a Hart cell—the contra-parallelogram first imagined by Mr Samuel Roberts, but rediscovered and hugged with the affection of a supposed original discoverer, and warmed into new and unsuspected uses by its foster-parent Mr Hart. I shall have no difficulty in finding a generalisation of the Peaucellier-cell standing to it in the same relation as the Quadruplane does to the Hart-cell, and similarly for the Proportionator, so that we shall have the fourfold proportion—Peaucellier-cell : Hart-cell : Quadruplane : New Peaucellier-cell :: Old Pantigraph : Common Pantigraph : Plagiograph : Oblique Old Pantigraph; but, except as completing a chain of analogies, the last terms in each quatrain will probably not prove of any practical importance.

Just as in the first case $pq \,.\, pr$ and $sr \,.\, sq$ are constant, so now $PQ \,.\, PR$ and $SR \,.\, SQ$ are constant; but whereas pq coincided in direction with pr and sr with sq, PQ and PR, like SR and SQ, remain inclined to each other at a constant angle; in a word, as the Plagiograph is to the Pantigraph, so is the Sylvester-Kempe Inverter or Reciprocator to Mr Hart's*. Do not let it be supposed that this new reciprocator is to be consigned to the limbo of barren mathematical generalities—very far from it; it is very likely indeed to find a most valuable application to mechanical practice, and to subserve the purposes of that immediate "Utilitarianism†" so dear to the Philistine mind; for, as by means of Mr Hart's Quadrilateral, when one of the four named points, say p, is absolutely fixed, and one of its non-conjugate points,

* In the case of a three-piece motion whose fundamental linkage (that is, the quadrilateral formed by the lines joining the pivots and the fixed points in their natural order of succession) is subject to the condition that either the two pairs of opposite sides or two pairs of contiguous sides are equal for each pair, the Planigraph (leaving out of account its circular portion) is the inverse of a conic. In the first case (that of the contra-parallelogram) the position of this node is seen immediately to be the opposite to the Planigram in respect to the centre of the figure in its untwisted (that is, parallelogrammatic) form. In the second case, that of the so-called kite-form, it was found to be far from easy to determine its position. Even our Cayley did not quite succeed in determining it from the analytical equations, and it was reserved for M. Manheim to deduce it geometrically by a most elegant but very elaborate construction given in a paper inserted in the *Proceedings of the Mathematical Society of London*. By the aid of the reciprocity established by me above we pass at once from the case of the contra-parallelogram to that of the kite-form, and the problem literally solves itself as easily as a musical passage can be transposed from one key to another. It is to that profound mathematician, Mr Samuel Roberts, that we are indebted for bringing to light these two cases of three-bar motion, in which the general three-bar sextic Graph breaks up into a circle, and the inside of a conic, and I have proved that no other such cases exist. Mr Roberts's papers are inserted in the *Proceedings of the London Mathematical Society*, which is indebted for its existence, at least in its present form (being originally little more than a juvenile mathematical debating society among the students of University College), to the organising talents of Mr Hirst, who has reason to be proud of his progeny. Similar societies on a precisely similar basis, and adopting the rules of its elder sister, have been subsequently founded in Paris, Warsaw, and, I believe, other capitals in Europe, and, it is safe to predict, are destined to play no unimportant part in the further evolution of our time-honoured yet ever young, ever fresh, and self-renovating science—Othello, Hamlet, and Romeo all in one. Meanwhile, in the University supposed to be peculiarly dedicated to the advance of mathematical science, a young and very promising mathematician (whose name shall not be divulged) *àpropos* of a movement kindly attempted, without my being previously consulted, to place me in a position where, in the vicinity of our central luminary, I might have been in my proper place, and helped to reflect some portion of his rays upon surrounding bodies, wrote to me lately: "You cannot imagine the bitter prejudice that prevails here against pure mathematics, &c." I freely forgive those, "the bigots of a narrow creed," who entertain such sentiments, knowing that "they know not what they do."

† What would our English statesmen say to the conduct of the proverbially parsimonious Prussian Government and the nineteenth century Richelieu, "der tolle Bismarck," in appropriating a million and a half of marks (75,000*l.* sterling) placed at the free disposal of the modern Aristotle, Helmholtz, for constructing the bare shell alone of the new Physical Laboratory at Berlin! If such an appropriation were proposed at home, would there not run through the land a frantic shriek or muttered growl of disapprobation at such a wanton waste of the public funds on mere speculative science?

say r, is attached to the end of a radius so centred and of such a length that the path of r is a circle which, *geometrically* completed, would pass through p, the remaining conjugate point q will be forced to describe a straight line perpendicular to the line joining the two fixed points—so by means of our Quadruplane, when P is fixed and R made to move in the arc of a circle passing through P, the point Q may be made to describe a straight line having any *desired obliquity* to the line of centres, the amount of such obliquity depending on the magnitude of the supplemental equal angles P, Q, R, S. Thus the Plagiograph (and in the first instance Mr Kempe's notice of the homœographic commutability of the lengths of the connecting rod and the radial bars in ordinary three-bar motion) has led by a devious path to the construction of a three-piece-work giving the most general and available solution of the problem of exact parallel motion that has been discovered or that can exist—I say the most available, for it is evident, in general, that piece-work must possess the advantage of greater firmness and steadiness from the more equal distribution of its strain over ordinary link-work.

The Peaucellier and Hart cells, duly mounted, afford the means by obvious methods of adjustment to cut straight lines at any distance from either of the fixed centres, but confined to lying perpendicular to the line of centres; whereas the Quadruplane puts it into our power with one and the same instrument affected with simple means of adjustment to make straight cuts (and, if desired, two parallel ones simultaneously) in all directions as well as at all distances in the plane of motion. So again the Plagiograph enables us to apply the principle of angular repetition (as, for instance, in making an ellipse with dimensions either fixed or varying at will, successively turn its axis to all points of the compass) to produce designs of complicated and captivating symmetry from any simple pattern or natural form, such as a flower or sprig; and as the head of a house at Oxford in the good old port-wine days was heard to complain about the electro-magnetic machine, that "he feared it would place a new weapon in the hands of the incendiary" (the power of Swing being then rampant in the land), so, but with better reason and upon the highest authority, it may be predicted that this simple invention will be found to place a new and powerful experimentative and executory implement in the hand of the engine-turner, the pattern-designer, and the architectural decorator.

P.S.—I rejoice to be able to state that the Institute of France has quite recently adjudged its great mechanical prize, the "Prix Montyon," to Col. Peaucellier for his discovery of an exact parallel motion when a lieutenant in 1864. The first practical application of this discovery, made by Mr Prim under the sanction of Dr Percy, may be seen daily at work in the Ventilating

Department of our Houses of Parliament. The workmen there, who never tire of admiring its graceful and silent action, have given it the pet name of the "Octopus," from some fancied resemblance between its backward and forward motion and that of the above-named distinguished Cephalopod. I feel a strong persuasion that when the inertia of our operative classes shall have been overcome, this application will prove to be but the signal, the first stroke of the tocsin, of an entire revolution to be wrought in every branch of construction; and that machinery is destined eventually to merge into a branch of the science of Linkage in the sense in which that word is used in the text above.

4.

ON A LADY'S FAN, ON PARALLEL MOTION, AND ON AN ORTHOGONAL WEB OF JOINTED RODS.

[*Proceedings of London Mathematical Society*, VI. (1875), pp. 196, 197.]

By means of Prof. Sylvester's Fan, it is possible to divide any angle into any assigned equal number of parts; and the trajectories of points taken in the several links connecting together the sticks of the fan have finite nodes, whose numbers are successively $1, 2^4, 3^4, 4^4, \ldots\ldots$

Prof. Sylvester stated, in his second communication, That parallel motions exist at all is a paradox more wonderful than ever, now that his method gives the means of determining the conditions to be satisfied, and comparing their number with that of the disposable constants. The orders for 3, 5, 7, ... bars are 6, 20, 72, ... Formerly the existence of *one* was doubted; now a finite number for *every* order of linkwork is rendered highly probable. In particular, Prof. Sylvester showed how to determine whether *Parallel Motions* exist, and, if so, how to find them for any given number of bars and mode of colligation. He showed how to form a determinant involving only the lengths of the bars and other quantities which fix their direction; this determinant, if a parallel motion exists, must vanish identically for all values of the latter set of quantities. This is called the *Determinant of Parallel Motion*. The determinant is formed as a Jacobian of Equations, involving only linear functions of the lengths, and of a determinant corresponding to a set of equations of the same form as the above. Its evanescence gives a system of conditions to be satisfied, all expressed as rational functions of the lengths; and, by known algebraical methods, these enable us to find *necessary* relations of the lengths, if a Parallel Motion exists. It must then be ascertained whether these solutions are sufficient, and the problem is solved.

Prof. Sylvester's remarks on "An Orthogonal Web of Jointed Rods" were to the following effect: If two sets of joints be taken respectively in two lines perpendicular to each other, either in a plane or in space, and a *linkage* be formed by connecting each point in one set with each point in the other by jointed rods, this constitutes an orthogonal web. It is *not* a fixture, and its motion is subject to this curious condition, that either each set of points must always continue to lie in the same right line, which may be called a neutral position, or else one set will lie in a right line, and the other in a plane at right angles to such line. Starting from the neutral position (a position of *double-lock*), the system may be said to be subject to an optional locking about one or the other of two perpendicular lines, and an unlocking about the others; but, when once put in motion, the system must again be brought into the same, or a new neutral position, before the one axis of lock can be got rid of, and another at right angles thereto substituted in its stead. If the whole motion be confined to a plane, the paradox consists in the link-combination forming one degree of liberty of deformation (ἀλλοίωσις, as distinguished by Plato from κίνησις), although a calculation of the amount of restraint by the general method applicable to such questions would seem to indicate that it ought to form an absolutely rigid system except in the case where there are only two joints in one at least of two sets. Taken in space, there is the further and more striking paradox, that the number of degrees of liberty of deformation, according to the choice made of one or the other of the two sets of points to be unlocked out of the rectilinear into the planar position, will be the *alternative of two numbers*, viz., the number of points in the one set or in the other set (which need not be the same), a kind of indeterminateness in the "Index of Freedom" without precedent in mathematical speculations. As lightning clears the air of impalpable noxious vapours, so an incisive paradox frees the human intelligence from the lethargic influence of latent and unsuspected assumptions. Paradox is the slayer of Prejudice.

5.

NOTE ON SPHERICAL HARMONICS.

[*Philosophical Magazine*, II. (1876), pp. 291—307, and p. 400.]

If for a moment we confine our attention to so-called "zonal" harmonics, and affect each element of a uniform spherical shell with a density varying as the product of two such harmonics of unequal degrees, we know that the mass of such shell is zero. A very slight consideration will serve to show that this is tantamount to affirming that if a given spherical surface be charged with a density inversely proportional to the product of the distances of each element from two fixed internal points lying in the same radius produced, then the mass of such shell will be a complete function of the product of the distances of the two points from the centre; and in fact, if we write dS for an element of a spherical surface, it is easy to find, by direct integration, that

$$\iint \frac{dS}{\sqrt{(c^2 - 2hx + h^2)}\,\sqrt{(c^2 - 2h'x + h'^2)}},$$

for the entire surface, is proportional to

$$\frac{1}{\sqrt{(hh')}} \log \frac{c^2 - \sqrt{(hh')}}{c^2 + \sqrt{(hh')}}.$$

In like manner, the truth of the more general theorem relating to the surface-integral of the product of any two harmonics of unequal degrees involves, and is involved in, the fact that the surface-integral $\iint \frac{dS}{R\,.\,R'}$, where

$$R^2 = (x-h)^2 + (y-k)^2 + (z-l)^2,$$
$$R'^2 = (x-h')^2 + (y-k')^2 + (z-l')^2$$

and $h^2 + k^2 + l^2$ and $h'^2 + k'^2 + l'^2$ are each less or each greater than the square of the radius of the sphere, is not merely a function (as we see *à priori* from the symmetry of the sphere must be the case) of the three quantities

$$h^2 + k^2 + l^2, \quad h'^2 + k'^2 + l'^2, \quad hh' + kk' + ll',$$

but, more definitely, is a complete function of the product of two of them, namely, $(h^2+k^2+l^2)(h'^2+k'^2+l'^2)$, and of the third. In other words, the fundamental law of spherical harmonics is exactly tantamount to the assertion that if each element of a sphere is charged with a density inversely proportional to the product of its distances from two internal or two external points, then the mass of the sphere will be a function only of the density at the centre and of the angle subtended at the centre by the line joining the given pair of points; or, venturing upon an irrepressible neologism, which explains its own meaning, the *Bipotential*, with respect to a given uniform sphere at any point-pair, is a function only of the Bipotential thereat with respect to a unit particle at the centre, and of the angle subtended at the centre by the line joining the two given points. Of course, if this is true for the volume of the sphere, it must be true for any shell of uniform thickness, or, in other words, for the surface, and *vice versâ*. In what immediately follows the volume of a spherical shell is to be understood. It is, I think, very noticeable that in that proof no process whatever of integration is employed; only the *idea* implied in integration is employed to acquire the fact that the integral in question cannot but be a function of three parts of the triangle, of which the centre of the sphere and the two given points are the apices. The rest of the proof follows as a matter of purely formal or algebraical necessity from the above fact, conjoined with that of each factor under the sign of integration being subject to Laplace's equation. In this feature of exemption from all use of integration as a process, this proof, I believe, stands alone.

It is further remarkable that its success depends on the proposition being stated as a whole; it would not be applicable, for example, to the simple case, taken *per se*, treated of at the beginning of this paper. It is by no means uncommon in mathematical investigation for this to happen, and (as regards the exigencies of reasoning) for the part to be in a sense greater than the whole—the groundwork of this wonder-striking intellectual phenomenon being that, for mathematical purposes, all quantities and relations ought to be considered (so experience teaches) as in a state of flux. In the particular case before us it is not difficult to see *à priori* why the general proposition should be more easily demonstrable than any special case of it, the reason being that more information as to the *form* of the function under consideration is made use of in dealing with the general than in dealing with any special case.

The integral under consideration is

$$\iiint \frac{dS}{RR'} \text{ (say } I\text{)},$$

where

$$R^2 = x^2+y^2+z^2-2hx-2ky-2lz+h^2+k^2+l^2,$$
$$R'^2 = x^2+y^2+z^2-2h'x-2k'y-2l'z+h'^2+k'^2+l'^2.$$

Call

$$h^2+k^2+l^2=r^2, \quad h'^2+k'^2+l'^2=t^2, \quad hh'+kk'+ll'=s.$$

Then $\frac{1}{RR'}$, expanded under the form of a converging series (x, y, z being for a moment regarded as constants), will be of the form $\frac{1}{rt}$ multiplied by a rational function of $\frac{1}{r}, \frac{h}{r^2}, \frac{k}{r^2}, \frac{l}{r^2}$ and of $\frac{1}{t}, \frac{h'}{t^2}, \frac{k'}{t^2}, \frac{l'}{t^2}$ when the two points are external, and (more simply) of h, k, l and of h', k', l' when they are both internal. I, we know, must turn out to be a complete function of r, s, t, and, when expressed in the form of a series derived from the above expansion, will be the sum of terms of the form $r^i . s^j . t^k$, where it is obvious that i and j must both be negative when the "pair-point" is exterior, both positive when it is interior to the shell, and one positive and one negative in the remaining case.

Now we have identically

$$\left(h\frac{d}{dk}-k\frac{d}{dh}\right) r=0,$$
$$\left(h'\frac{d}{dk'}-k'\frac{d}{dh'}\right) t=0,$$

and

$$\left\{\left(h\frac{d}{dk}-k\frac{d}{dh}\right)+\left(h'\frac{d}{dk'}-k'\frac{d}{dh'}\right)\right\} s=0.$$

Hence with respect to I as operand we have

$$\left(h\frac{d}{dk}-k\frac{d}{dh}\right)+\left(h'\frac{d}{dk'}-k'\frac{d}{dh'}\right)=0.$$

Operate on this identity with

$$\left(h\frac{d}{dk}-k\frac{d}{dh}\right)-\left(h'\frac{d}{dk'}-k'\frac{d}{dh'}\right),$$

and we obtain

$$\left(h\frac{d}{dk}-k\frac{d}{dh}\right)^2-\left(h\frac{d}{dh}+k\frac{d}{dk}\right)=\left(h'\frac{d}{dk'}-k'\frac{d}{dh'}\right)^2-\left(h'\frac{d}{dh'}+k'\frac{d}{dk'}\right);$$

and there will be two other equations of like form. Adding all these together, changing all the signs, and remembering that in regard to I as operand

$$\left(\frac{d}{dh}\right)^2+\left(\frac{d}{dk}\right)^2=-\left(\frac{d}{dl}\right)^2,$$
$$\left(\frac{d}{dh'}\right)^2+\left(\frac{d}{dk'}\right)^2=-\left(\frac{d}{dl'}\right)^2,$$

we obtain

$$\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right)^2+2\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right)$$
$$=\left(h'\frac{d}{dh'}+k'\frac{d}{dk'}+l'\frac{d}{dl'}\right)^2+2\left(h'\frac{d}{dh'}+k'\frac{d}{dk'}+l'\frac{d}{dl'}\right).$$

In this formula $\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right)^2$

stands for its algebraical value

$$h^2\left(\frac{d}{dh}\right)^2+2hk\frac{d}{dh}\frac{d}{dk}+\ldots;$$

but if we write $\left\{\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right)*\right\}^2$

to denote the operation twice repeated, then

$$\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right)^2$$

$$=\left\{\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right)*\right\}^2-\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right),$$

and so for the like expressions with the accented letters. The formula thus is

$$\left\{\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right)*\right\}^2+\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right)$$

$$=\left\{\left(h'\frac{d}{dh'}+k'\frac{d}{dk'}+l'\frac{d}{dl'}\right)*\right\}^2+\left(h'\frac{d}{dh'}+k'\frac{d}{dk'}+l'\frac{d}{dl'}\right);$$

or say $\{(E*)^2+E-(E'*)^2-E'\}\,I=0,$

or simply $(F-F')\,I=0.$

Let now $r^is^jt^k$ be any term in I; then since

$$Er=r,\quad Es=s,\quad Et=0,$$

$$E't=t,\quad E's=s,\quad E'r=0,$$

we have

$$Fr^is^jt^k=\{(i+j)^2+(i+j)\}\,r^is^jt^k,$$

$$F'r^is^jt^k=\{(k+j)^2+(k+j)\}\,r^is^jt^k,$$

and thence $(F-F')\,r^is^jt^k=(i^2+i+2ij-k^2-k-2kj)\,r^is^jt^k.$

Hence $\Sigma\,(i^2+i+2ij-k^2-k-2kj)\,r^is^jt^k$ must be identically zero; therefore $i-k=0$, or $i+k+2j+1=0$.

But when the two points to which the Bipotential is referred (and which I shall hereafter call the points of *prise*) are both external or both internal, i and k have the same sign; therefore $i=k$, and the integral is a function only of rs and t, or say of

$$(h^2+k^2+l^2)(h'^2+k'^2+l'^2),\quad (hh'+kk'+ll')\dagger.$$

† When the point corresponding to r is external and that corresponding to t is internal, the equation $i+k+2j+1=0$ applies, which shows that each term is of the form $\frac{1}{r}\left(\frac{t}{r}\right)^\lambda\cdot\left(\frac{s}{rt}\right)^\mu$; that is to say, the Bipotential multiplied by r is a complete function of $\frac{t}{r}$ and the cosine of the angle which the line joining the two fixed points subtends at the centre.

Thus the desired theorem has been established by virtue of an algebraical necessity of form alone; and the proof is of course applicable to space in any number of dimensions, substituting for the sphere or spherical surface its analogue in such space, and for the reciprocal of distance the proper power necessary for the satisfaction of Laplace's equation, that is, the $(q-2)$th power of the reciprocal, where q is the number of dimensions (supposed to be greater than 2).

For the case of two dimensions, substituting the logarithm for the reciprocal, so that, for example, we are able to affirm that if each element of a circular ring be affected with a density proportional to the product of the logarithms of its distances from two fixed internal points, the mass of such ring will depend only on the product of their distances from the centre of the ring and the angle between these distances—for this case, writing $E=h\frac{d}{dh}+k\frac{d}{dk}$ and $E'=h'\frac{d}{dh'}+k'\frac{d}{dk'}$ in the equation $(F-F')\,I=0$, $F=(E*)^2$ and $F'=(E'*)^2$; and if the two points are interior, every term in $\frac{1}{RR'}$ will be of the form $cr^i.s^j.t^k$, i and k being both positive, and we must have $i^2+2ij-k^2-2kj=0$, and consequently $i=k$—the other solution, $i+k+2j=0$, being applicable to the case of one point being external and the other internal. If the points are both external there will be four sets of terms. One set will consist of the single term $A\log r\log t$; a second, of terms of the form $c\log r\,.\,r^is^jt^k$; a third, of terms of the form $c\log t\,.\,r^is^jt^k$; and the last set, of terms of the form $cr^is^jt^k$: and it is easy to see that

$$F(\log r\log t)=0,\quad F'(\log r\log t)=0,$$

$$(F-F')\log r\,.\,r^is^jt^k=\{(i+j)^2\log r-(k+j)^2\log r+2(i-k)\}\,r^is^jt^k,$$

and consequently $i=k$ for the second and third sets; as regards the fourth set, $i=k$ for the same reason as in the case of three dimensions. Hence

$$I=A\log r\log t+\log r\phi(rt,s)+\log t\psi(rt,s)+\omega(rt,s);$$

and as r and t are interchangeable, we must have $\phi=\psi$, and consequently

$$I-A\log r\log t=F(rt,s);$$

so that not now the mass of the ring, but the difference between it and the mass due to the density at the centre is invariable when rt and s are given.

For greater simplicity, and as bearing more immediately on the theory of spherical harmonics, I have hitherto regarded the points of the pair-point at which the "bipotential" is reckoned either both internal or both external. The results established in these two cases are not complementary, but mutually equivalent to each other, and to the theorem that the integral along a spherical surface of the product of two spherical harmonics of unequal degrees is zero. In the third case, where one point is internal and the other external, then for the case of space of three dimensions the equation between i and k will have to be satisfied, not by $i=k$ but by $i+k+2j+1=0$, as previously stated in a footnote; and for two dimensions the equation would have to be satisfied, not by $i=k$ but by $i+k+2j=0$.

The advantage of the method here indicated is that it is immediately applicable to space of any number of dimensions. I shall now proceed to show that it leads at once to the determination of the values of the surface-integral of the product of any two given types of spherical harmonics of equal degrees, and *mutatis mutandis* to the corresponding surface-integral in space of any order.

To prove that the degrees must be equal or else the integral will vanish, we have combined the two Laplacian operators applicable to R and R' respectively; to find the value of the integral in a series, I use either of these operators to act singly on the result acquired by their use in combination. For greater simplicity suppose the point-pair to be internal; then, calling

$$a+b+c=\mu=\alpha+\beta+\gamma,$$

the problem to be solved is in effect that of finding the value of the numerical coefficient of $h^a k^b l^c . h'^\alpha k'^\beta l'^\gamma$ in the integral I. Now we know by what precedes that the value, say I_μ, of that part of I which is of the μth order in the two sets h, k, l; h', k', l' respectively is a rational function of rt and s; and we may accordingly write

$$I_\mu = As^\mu + Bs^{\mu-2} . \theta + Cs^{\mu-4} . \theta^2 + \ldots,$$

where $$s = hh' + kk' + ll',$$

and $$\theta = (h^2+k^2+l^2)(h'^2+k'^2+l'^2) = \rho\rho'.$$

When $A, B, C, \ldots$ are determined, the problem is virtually solved, and we shall then know the coefficient of

$$h^a k^b l^c . h'^\alpha k'^\beta l'^\gamma$$

by mere binomial expansions.

Since $$\left(\frac{d}{dh}\right)^2 + \left(\frac{d}{dk}\right)^2 + \left(\frac{d}{dl}\right)^2,$$

say ∇, operating on the whole of I gives the result zero, the same must obviously be true for each part I_μ.

Now ∇s^p is obviously equal to

$$(p^2-p)\,\rho' s^{p-2},$$

and $$\nabla\theta^q = 2q\,(2q+1)\,\rho'\theta^{q-1};$$

for $$\frac{d^2}{dh^2}(h^2+k^2+l^2)^q = \Sigma \frac{d}{dh}\{2qh\,(h^2+k^2+l^2)^{q-1}\}$$
$$= \Sigma\,\{2q\rho^{q-1} + 4q\,(q-1)\,\rho . \rho^{q-2}\}$$
$$= \{6q + 4q\,(q-1)\}\,\rho^{q-1}.$$

Also $\nabla s^p \rho^q - \rho^q \nabla s^p - s^p \nabla \rho^q = 2pq\Sigma\left(\frac{d}{dh} s \frac{d}{dh}\rho\right) s^{p-1} . \rho^{q-1} = 4pqs^p . \rho^{q-1}.$

Therefore

$$\nabla s^p\theta^q = (p^2-p)\,s^{p-2}\rho^q\rho'^{q+1} + (4pq+4q^2+2q)\,s^p . \rho^{q-1} . \rho'^q,$$

or $$\nabla s^{\mu-2j}\theta^j = (\mu-2j)(\mu-2j-1)\,s^{p-2}(\rho\rho')^j\rho'$$
$$+ 2j\,(2\mu-2j+1)\,s^p(\rho\rho')^{j-1} . \rho'.$$

Hence, equating to zero the coefficients of the different combinations of ρ, ρ', s, we easily obtain by writing for j successively $0, 1, 2, 3, \ldots$,

$$\mu\,(\mu-1)\,A + 2\,(2\mu-1)\,B = 0,$$
$$(\mu-2)(\mu-3)\,B + 4\,(2\mu-3)\,C = 0,$$
$$(\mu-4)(\mu-5)\,C + 6\,(2\mu-5)\,D = 0,$$

....

$$B = -\frac{\mu\,(\mu-1)}{2\,(2\mu-1)}A,$$

$$C = \frac{\mu\,(\mu-1)(\mu-2)(\mu-3)}{2 . 4\,(2\mu-1)(2\mu-3)}A,$$

$$D = -\frac{\mu\,(\mu-1)(\mu-2)(\mu-3)(\mu-4)(\mu-5)}{2 . 4 . 6\,(2\mu-1)(2\mu-3)(2\mu-5)}A,$$

....

To find the value of A, I observe that when $k=0$, $l=0$, $k'=0$, $l'=0$, and $h'=h$, I_μ becomes

$$(A+B+C+\ldots)\,h^{2\mu}.$$

But in that case, taking the radius of the sphere equal to unity, I becomes the surface-integral of $\frac{1}{1-2hx+h^2}$, and is equal to

$$2\pi\int_{-1}^{1}\frac{dx}{1-2hx+h^2} = \frac{2\pi}{h}\log\left(\frac{1+h}{1-h}\right) = 4\pi\left(1+\frac{h^2}{3}+\cdots\frac{h^{2\mu}}{2\mu+1}+\ldots\right).$$

Therefore $$A+B+C+\ldots = \frac{4\pi}{2\mu+1},$$

or $$S_\mu A = \frac{4\pi}{2\mu+1},$$

where $$S_\mu = 1 - \frac{\mu\,(\mu-1)}{2\,(2\mu-1)} + \frac{\mu\,(\mu-1)(\mu-2)(\mu-3)}{2 . 4 . (2\mu-1)(2\mu-3)}$$
$$- \frac{\mu\,(\mu-1)(\mu-2)(\mu-3)(\mu-4)(\mu-5)}{2 . 4 . 6 . (2\mu-1)(2\mu-3)(2\mu-5)} \ldots.$$

This series admits of summation. And I find

$$S_1 = 1,\quad S_2 = \frac{2}{3},\quad S_3 = \frac{2}{5},\quad S_4 = \frac{8}{35},\quad S_5 = \frac{8}{63},\quad S_6 = \frac{16}{3 . 7 . 11},$$

$$S_7 = \frac{16}{3 . 11 . 13},\quad S_8 = \frac{128}{3 . 11 . 13 . 15},\quad S_9 = \frac{120}{5 . 11 . 13 . 17},$$

$$S_{10} = \frac{256}{11 . 13 . 17 . 19} \ldots,$$

and in general

$$S_{2m} = \frac{2 \,.\, 4 \,.\, 6 \ldots (2m)}{(2m+1)(2m+3)(2m+5)\ldots(4m-1)}$$

and $$S_{2m+1} = \frac{2m+1}{4m+1} S_{2m};$$

that is to say, S_μ is the reciprocal of the coefficient of h^μ in $(1-2h)^{-\frac{1}{2}}$.

Hence the values of $A, B, C \ldots$ in I_μ are completely determined, and I_μ, and consequently the value of the complete integral of

$$\iint dS \left\{\left(\frac{d}{dx}\right)^a \left(\frac{d}{dy}\right)^b \left(\frac{d}{dz}\right)^c \cdot \frac{1}{r}\right\} \left\{\left(\frac{d}{dx}\right)^\alpha \left(\frac{d}{dy}\right)^\beta \left(\frac{d}{dz}\right)^\gamma \cdot \frac{1}{r}\right\},$$

is known for all values of a, b, c; α, β, γ—and this by a method which is applicable step by step to any number of variables, provided in place of $\frac{1}{r}$ we write $\frac{1}{r^{n-2}}$ when n exceeds 2, and $\log r$ when $n = 2$, and consider dS to be the element of what in n dimensions corresponds to a spherical surface in three-dimensional space.

The method employed, of first using two Laplacian operators in combination to determine one property of the form under investigation and then a single one of them to act on the form thus partially determined, reminds one very much of the method for obtaining invariants of given orders from their two general partial differential equations. Combined, these two equations express the law of isobarism; then, assuming the isobarism, a single one of the two serves to determine the special values of the coefficients. The analogy between that process and the one here employed seems to me to be exact, although the subject-matter is so very unlike in the two problems—and is the more interesting on that very account.

The bipotential in the case where the two points of *prise* are both internal being known under the form $F\left(\frac{rr'}{a^2}, \cos\alpha\right)$, where a is the radius of the sphere, its value for the case where these points are both external, and for the case where they are one internal and the other external, may be assigned without any further calculation as follows:—

1. Suppose r greater than the radius of the sphere, but r' less. We know *à priori* from the result previously obtained (and stated in a footnote), that the bipotential for this case is of the form $\frac{1}{r} G\left(\frac{r'}{r}, \cos\alpha\right)$. Now in place of r, r' substitute $a, \frac{ar'}{r}$; then the bipotential becomes $\frac{1}{a} G\left(\frac{r'}{r}, \cos\alpha\right)$.

But we may by an easily justifiable application of the principle of continuity now regard a $\left(\text{as well as } \frac{ar'}{r}\right)$ as the distance of an internal point from the centre. Hence we have

$$\frac{1}{a} G\left(\frac{r'}{r}, \cos\alpha\right) = F\left(\frac{r'}{r}, \cos\alpha\right),$$

or $$\frac{1}{r} G\left(\frac{r'}{r}, \cos\alpha\right) = \frac{a}{r} F\left(\frac{r'}{r}, \cos\alpha\right),$$

which is the value of the bipotential of a spherical surface cut by the line of *prise*, r being the distance of the external point of *prise* from the centre.

2. Suppose r and r' to be each greater than the radius, and $r > r'$; we know the bipotential is of the form $H\left(\frac{rr'}{a^2}, \cos\alpha\right)$. For r, r' substitute respectively $a, \frac{rr'}{a}$. Then we may regard the case as that of an exterior and interior point of *prise*, and consequently from the last case we have

$$H\left(\frac{rr'}{a^2}, \cos\alpha\right) = \frac{a^2}{rr'} F\left(\frac{a^2}{rr'}, \cos\alpha\right).$$

If we compare the two expressions

$$F\left(\frac{rr'}{a^2}, \cos\alpha\right) \quad \text{and} \quad \frac{a^2}{rr'} F\left(\frac{a^2}{rr'}, \cos\alpha\right)$$

respectively applicable to two internal and two external points of *prise*, it will easily be seen that it leads to the following theorem. Let there be two concentric spheres, and let any two radii cut the first and second surfaces in the points P, Q and P', Q' respectively; then the bipotential of the first surface with respect to P', Q' as the points of *prise*, is to the bipotential of the second surface with respect to P, Q as the points of *prise* in the ratio of the squares of the radii of the two surfaces to each other.

This is a theorem of precisely the same kind as Ivory's for the comparison of the attractions (or, if we please, the potentials) of two confocal ellipsoids in the particular case when they become two concentric spheres, and may be verified by precisely the same geometrical method. For we have only to divide the two spherical surfaces into corresponding elements m, m' by radii drawn in all directions to meet the two surfaces, and it is evident that we shall have the distances mP' and $m'P$ equal, as also mQ' and $m'Q$. And, moreover, the ratio of any two corresponding elements m, m' will be as the square of the radii, which evidently establishes the theorem in question. It may further be noticed that the relations between the bipotentials in the three several cases considered may be deduced from the fact that each such radical as

$$\frac{1}{\sqrt{(1-2hx-2ky-2lz+h^2+k^2+l^2)}},$$

where $h^2+k^2+l^2$ is greater than unity, may be put under the form

$$\frac{1}{\sqrt{(h^2+k^2+l^2)}}\,\frac{1}{\sqrt{(1-2h_1x-2k_1y-2l_1z+h_1^2+k_1^2+l_1^2)}},$$

where h_1, k_1, l_1 and h, k, l are the coordinates of two points the inverses (or electrical images) of each other in regard to the origin, and consequently $h_1^2+k_1^2+l_1^2$ less than unity. This is going to the heart of the matter. So I may observe that if we would go to the root of the relation between positive- and negative-degreed solid spherical harmonics, the more logical mode of proceeding is not (as is usually done) to infer this by a lengthy *à posteriori* process, but immediately from the fact that since

$$\frac{1}{\sqrt{\{(x^2+y^2+z^2)-2(hx+ky+lz)+(h^2+k^2+l^2)\}}}$$

is nullified by the operator

$$\left(\frac{d}{dx}\right)^2+\left(\frac{d}{dy}\right)^2+\left(\frac{d}{dz}\right)^2,$$

so also must the same operator nullify the radical

$$\frac{1}{\sqrt{\{1-2(hx+ky+lz)+(h^2+k^2+l^2)(x^2+y^2+z^2)\}}}.$$

Before proceeding further, I ought to observe that I_μ in the above series for the bipotential may easily be shown to be $\frac{4\pi}{2\mu+1}$ multiplied into the coefficient of t^μ in the expansion of $\frac{1}{\sqrt{(1-2st+\theta t^2)}}$ *; or, in other words,

* s, it will be remembered, is $hh'+kk'+ll'$, and θ is the product $(h^2+k^2+l^2)(h'^2+k'^2+l'^2)$. The statement in the text follows as a consequence from the fact that $(1-2st+\theta t^2)^{-\frac{1}{2}}$ obeys Laplace's law, and, when expanded according to powers of t, is of the form found for I_μ, and must consequently be identical with it to a factor *près*, that factor being a function of μ, whose value is easily found by making $h=h'$ and k, l; k', l' each zero. In like manner it may be shown that in higher space of n dimensions the corresponding value of I_μ is a function of μ multiplied by the coefficient of t^μ in $\{1-2t\Sigma hh'+\Sigma(h^2)\Sigma(h'^2)t^2\}^{1-\frac{1}{2}n}$; and [writing m for μ] I find† that this function, say $\phi(m, n)$, (as will be shown in a sequel to this paper) is always a rational function in m, containing in the denominator, when n is odd, one factor of the form $2m+j$, all the others being of the form $m+i$—and when n is even, factors all of the form $m+i$. Whatever the form of these linear factors had been for even numbers, we could see *à priori* that the Bipotential for space of even dimensions could contain only algebraic and inverse circular or logarithmic functions. But as regards the case of space of odd dimensions, the fact of there being no factors except of the form $m+i$, $2m+j$, is prepotent in determining the form of the result. For space of two dimensions the Bipotential does not appear readily to yield to summation in finite

[† See below, p. 51.]

if the distances from the centre of a spherical surface of two points *in the interior* be r, t, and the angle which the line joining them subtends at the centre be ω, then [for a sphere of radius c] the value of the bipotential of the surface at this point-pair is the elliptic integral

$$\int_0^{\frac{\sqrt{(rt)}}{c}}\frac{4\pi\,dx}{\sqrt{(1-2x^2\cos\omega+x^4)}},$$

which I take leave to call the Cardinal Theorem of Spherical Harmonics; for it is the theorem from which spring all the properties relating to the "surface-integral" of the product of any two rational forms of Laplace's coefficients.

Since every spherical harmonic of integral degree is a linear function of the differential derivatives of $(x^2+y^2+z^2)^{-\frac{1}{2}}$, the whole theory of the diplo-spherical-harmonic-surface integral is contained in the annexed equation,

terms. Thus at one blow the theory of spherical harmonics has been extended to "globoidal" harmonics in general; and the chief cases of statical distribution of electricity heretofore solved may be regarded as virtually solved *mutatis mutandis* for space of any number of dimensions, of course with the proviso that the law of attraction (in consonance with the hypothetical principle of force-emanation to which the English school of physicists seem to be returning) is always to be supposed to vary as the $(i-1)$th power of the distance in space of i dimensions.

The actual expression for $\phi(m, n)$ when n is 3 we know is $\frac{4\pi}{2m+1}$. In general when n is any other odd number, I find that its value is

$$\frac{2(2\pi)^{\frac{n-1}{2}}}{(2m+n-2)(m+n-3)(m+n-4)\ldots\left(m+\frac{n-1}{2}\right)}.$$

As this expression may be split up into partial fractions, it is obvious that the value of the Bipotential may be expressed by means of the sum of integrals of the form

$$\int_\infty^1\frac{u^j\,du}{\{\sqrt{(u^2+Au+B)}\}^{n-2}},$$

and one of the form

$$\int_\infty^1\frac{du}{\{\sqrt{(u^4+Au^2+B)}\}^{n-2}};$$

so that it involves no transcendents of a higher order than an ordinary elliptic function. I think also that it follows from the limits to the value of j that the other integrals are mere algebraical functions. The less interesting case when n is an even number (being very much pressed for time and within twenty-four hours of steaming back to Baltimore) I have not taken the trouble to work out in detail.

The determination of the Bipotential constitutes in itself a vast accession to the theory of definite integrals, and promises to be fruitful in yielding whole new families of such when subjected to the usual processes performed under the sign of integration. But does the theory stop here? The success of my method for the Bipotential depends solely upon the discovery that, as regards internal points of *prise*, it may be regarded as a function of only two variables, rr' and $\cos\omega$. Now a Tripotential will obviously at first sight be a function of not more than six variables, viz. the three quantities r, r', r'' and the cosines of the angles between them; but it becomes a question whether this number also may not be reduced to be less than six, themselves simple functions of the six parts of a tetrahedron; and so for a multipotential of any order the question arises, Is it a function of $\frac{1}{2}m(m+1)$ quantities or of a smaller number? and if so, of what number of what variables?

which springs immediately from the expression found above for the bi-potential of a spherical surface at two internal points (slightly modified by taking $-h, -k, -l; -h', -k', -l'$ for the coordinates of the points) by means of the simple and familiar principle that any differential derivative with respect to x, y, z of a function of x, y, z is identical with what the corresponding derivative with respect to h, k, l of the like function of $x+h$, $y+k$, $z+l$ becomes when h, k, l are made to vanish.

Let U stand for $u^4 - 2u^2\Sigma hh' + \Sigma h^2 . \Sigma h'^2$, and let

$$V(h, k, l;\ h', k', l') = \int_\infty^1 \Phi\left(\frac{d}{dh}, \frac{d}{dk}, \frac{d}{dl}\right)\Psi\left(\frac{d}{dh'}, \frac{d}{dk'}, \frac{d}{dl'}\right)\frac{du}{\sqrt{U}},$$

where Φ and Ψ are forms of function which denote series, whether finite or infinite, containing only positive integer powers of the variables. Then, if $\rho = (x^2+y^2+z^2)^{-\frac{1}{2}}$ and dS is the element of a spherical surface of unit radius, the complete integral

$$\iint dS \left\{\Phi\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)\rho\Psi\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)\rho\right\} = 4\pi V(0, 0, 0;\ 0, 0, 0).$$

When Φ and Ψ are homogeneous forms of function, each of the degree i, if we write

$$T = 1 - 2\Sigma hh' + \Sigma h^2 . \Sigma h'^2,$$

and make

$$\Omega(h, k, l;\ h', k', l') = \Phi\left(\frac{d}{dh}, \frac{d}{dk}, \frac{d}{dl}\right)\Psi\left(\frac{d}{dh'}, \frac{d}{dk'}, \frac{d}{dl'}\right)\frac{1}{\sqrt{T}},$$

the value of the corresponding harmonic surface integral becomes

$$\frac{4\pi}{2m+1}\Omega(0, 0, 0;\ 0, 0, 0).$$

I am not aware that a rule for finding such integral so simple in form and of such absolute generality in operation as the one above has been given before; the interesting rule furnished by Professor Clerk Maxwell, *Electricity and Magnetism* (vol. I. p. 170), assumes that Φ and Ψ have been each reduced to the form of the product of linear functions of $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$—a reduction which cannot practically be effected, as it involves the solution of systems of equations of a high order—not, however, so high as might at first sight be inferred from Professor Maxwell's statement that, for the case of i factors, it depends on the solution of a system of $2i$ equations of the ith degree, as the equations referred to (evidently those obtained by the use of the method of indeterminate coefficients in its crude form) would be of a special character: thus, for example, when $i = 2$, the order of the system of the four quadratic equations sinks down from $4 . 2^3$ or 32 (its value in the general case) to be only 3, as will presently be seen.

The method of poles for representing spherico-harmonics, devised or developed by Professor Maxwell, really amounts to neither more nor less than the choice of an apt canonical form for a ternary quantic, subject to the condition that the sum of the squares of its variables (here differential operators) is zero; and I am quite at a loss to understand how it can at all assist "in making the conception of the general spherical harmonic of an integral degree perfectly definite," or what want of definiteness apart from the use of this canonical form can be said to exist in the subject.

Since $\left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 + \left(\frac{d}{dz}\right)^2$ retains its form when any orthogonal linear substitutions are impressed on x, y, z, we recognize *à priori* that a harmonic distribution on the surface of a sphere is invariantive in the sense that it bears no intrinsic relation to the particular set of axes which may happen to be used to express the value of the harmonic at each point of the surface; and the great merit, it seems to me, of Professor Maxwell's beautiful conception of harmonic poles is that it puts this fact in evidence: for it is easy to see at a glance, from the use of successive linear operators, that the harmonic at any variable point on the surface for any given degree (n) will depend in an absolutely determinate manner (save as to an arbitrary constant factor) on the cosines of the arcs joining it with n arbitrarily assumed fixed points on the sphere, and of the arcs joining those n points with one another (being in fact a symmetrical function of each of the two sets of cosines), so that intrinsic poles are substituted for extrinsic Cartesian axes. I am a little surprised that this distinguished writer should not have noticed that there is always one, and only one, *real* system of poles appertaining to any given harmonic, and that to find this system it is not necessary, as he has stated, to employ a system of n equations each of the order $2n$, but one single equation of that order. For calling $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ by the names ξ, η, ζ, then any given harmonic of the nth degree may be reduced by the use of mere linear equations to the form $(\xi, \eta, \zeta)^n \frac{1}{r}$, and the problem to be solved in order to find its poles is the purely algebraical one of converting the quantic

$$(\xi, \eta, \zeta)^n + \Lambda(\xi^2 + \eta^2 + \zeta^2),$$

where Λ is a quantic of the order $(n-2)$, into a product of linear factors. Now this again is merely the problem of finding a pencil of rays that shall pass through the intersections of the curve $(\xi, \eta, \zeta)^n$ with the curve $(\xi^2 + \eta^2 + \zeta^2)$; that is to say, any dispersal of the $2n$ intersections into n sets of two each will give a system of n polar factors in Professor Maxwell's problem. We have therefore only to find the values of $\xi : \eta : \zeta$ in the two simultaneous equations $(\xi, \eta, \zeta)^n = 0$, $\xi^2 + \eta^2 + \zeta^2 = 0$, and this leads to a resolving equation of the $2n$th order. From the form of the second equation we see that the values $x:y:z$ are *all imaginary*; consequently there will be one, and but one, system of real rays, that is, real polars corresponding to the distribution of the $2n$ roots of the resolving equation into n *conjugate pairs*. The remaining systems (there are in all $1\,.\,3\,.\,5\ldots(2n-1)$ of them) will each contain imaginary elements, so that all or some of the poles become imaginary.

In the case of $n=2$, the problem becomes the familiar one of finding the principal axes of a cone of the second order; and instead of employing a biquadratic resolvent we make the discriminant of $(\xi, \eta, \zeta)^2+(\xi^2+\eta^2+\zeta^2)$ vanish, which of course only requires the solution of a cubic equation; but as subsequently (when the pair is to be divided into its elements) a new quadratic surd is introduced, we are virtually solving a biquadratic, in accordance with the general rule that, to find the poles of a spherical harmonic of the degree n, it is necessary to solve an equation of the degree $2n$.

To put the coping-stone to Professor Clerk Maxwell's method of poles, I think it would be desirable to find an intrinsic definition of spherical harmonics to correspond with their representation referred to intrinsic axes: I mean we ought to be able to dispense with the Laplacian operator altogether, and to define a Harmonic with sole reference to some algebraical or geometrical (but certainly not physical) condition which it satisfies in regard to its poles. With all possible respect for Professor Maxwell's great ability, I must own that to deduce purely analytical properties of spherical harmonics, as he has done, from "Green's theorem" and the "principle of potential energy" (*Electricity and Magnetism*, vol. I. p. 168), seems to me a proceeding at variance with sound method, and of the same kind and as reasonable as if one should set about to deduce the binomial theorem from the law of virtual velocities, or make the rule for the extraction of the square root flow as a consequence from Archimedes' law of floating bodies.

POSTSCRIPT. NOTE ON SPHERICAL HARMONICS.

The value of $\phi(m, n)$ is stated inaccurately in the long footnote at pp. [46, 47]. If

$$\Omega_i=\frac{(2\pi)^i}{1\,.\,3\,.\,5\ldots(2i-1)}$$

and

$$R=\sqrt{(1-2\Sigma hh'\,.\,t^2+\Sigma h^2\,.\,\Sigma h'^2\,.\,t^4)}$$

then I find

$$\phi(m, 2i+1)=\frac{(2i-1)\,\Omega_i}{2m+2i-1};$$

and accordingly the Bipotential in space of $2i+1$ dimensions is

$$\int_1^0 \frac{\Omega_i dt^{2i-1}}{R^{2i-1}}.$$

Also I find that in space of $2i+2$ dimensions the prospherical Bipotential is

$$\frac{2\pi^i}{1\,.\,2\,.\,3\ldots i}\int_1^0 \frac{dt^i}{(1-2\Sigma hh'\,.\,t+\Sigma h^2\,.\,\Sigma h'^2\,.\,t^2)^i}.$$

The above results may be extended to general quadric surfaces and prosurfaces. Thus, for example, if an indefinitely thin ellipsoidal shell be contained between two concentric surfaces, the equation to one of which is $G(x, y, z)=1$, where G is a general quadric, and if the squared density at x, y, z is the reciprocal of

$$G(x-h,\ y-k,\ z-l)\,.\,G(x-h',\ y-k',\ z-l'),$$

then the mass of the shell divided by its volume is

$$\int_1^0 \frac{dt}{\sqrt{(1-At^2+Bt^4)}},$$

where

$$A=\Sigma\left(h\frac{d}{dx}\right).\,\Sigma\left(h'\frac{d}{dx}\right)G(x, y, z),$$

and

$$B=G(h, k, l)\,.\,G(h', k', l').$$

It is further noticeable that if F and G are contravariantive forms, each numerator of the fractions expressing the differential derivatives of $\frac{1}{\sqrt{\{G(x, y, z)\}}}$ is nullified by the operator

$$F\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right);$$

and conversely, every rational integer function of x, y, z so nullifiable is a linear function of such numerators. And so in general the Theory of Spherical and Prospherical Harmonics merges in a theory of Conicoidal and Proconicoidal Harmonics.

6.

SUR LES INVARIANTS FONDAMENTAUX DE LA FORME BINAIRE DU HUITIÈME DEGRÉ.

[*Comptes Rendus*, LXXXIV. (1877), pp. 240—244, 532—534.]

On sait que le nombre des invariants linéairement indépendants de l'ordre j, appartenant à une forme binaire du degré i, est égal à la différence de deux nombres dont l'un, le plus grand, est le nombre de manières de représenter $\frac{ij}{2}$ comme la somme de j nombres (avec des répétitions à volonté) choisis entre les nombres $0, 1, 2, \ldots, i$, et l'autre est le nombre de manières de former $\frac{ij}{2} - 1$ selon la même loi.

Ainsi le nombre d'invariants de l'ordre n, appartenant à la forme binaire du degré 8, est la différence entre deux *dénumérants*, l'un du système

$$y + 2z + 3t + 4u + 5v + 6w + 7\rho + 8\sigma = 4n,$$
$$x + y + z + t + u + v + w + \rho + \sigma = n;$$

l'autre du système

$$y + 2z + 3t + 4u + 5v + 6w + 7\rho + 8\sigma = 4n - 1,$$
$$x + y + z + t + u + v + w + \rho + \sigma = n.$$

On comprendra que le dénumérant d'une équation ou d'un système d'équations simultanées en nombres entiers veut dire le nombre de solutions que cette équation ou ce système admet en nombres entiers.

Or j'ai démontré ailleurs que le dénumérant d'un système quelconque d'équations simultanées peut toujours s'exprimer au moyen de dénumérants simples, c'est-à-dire appartenant chacun à une seule équation, et l'on trouvera sans difficulté que la différence entre les deux dénumérants dont il est ici question sera le coefficient de t^n dans la fonction génératrice

$$G = \frac{1 + t^8 + t^9 + t^{10} + t^{18}}{(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6)(1-t^7)}. \qquad (1)$$

Ce résultat est parfaitement d'accord avec l'expression donnée par M. Cayley dans son *Second Memoir on Quantics*, c'est-à-dire

$$\frac{(1-x)(1+x-x^3-x^4+x^6+x^7+x^8+x^9+x^{10}-x^{12}-x^{13}+x^{15}+x^{16})}{(1-x^2)^2(1-x^3)^2(1-x^4)(1-x^5)(1-x^7)};$$

car on trouvera, par un calcul algébrique des plus simples, que ces deux fonctions génératrices sont identiques en valeur.

En vertu de la forme donnée à G dans l'équation (1), on peut immédiatement déduire les conséquences suivantes, que je nommerai désormais, si j'ai occasion de les citer, *principes :*

(1) Il existe des invariants, appartenant à la forme binaire octavique des ordres 2, 3, 4, 5, 6, 7, que je nomme les *invariants primaires.*

(2) Il existe* quatre invariants des ordres 8, 9, 10, 18, disons x, y, z, θ, tels, que chaque autre invariant peut s'exprimer comme une fonction linéaire de x, y, z, θ, les coefficients et le terme constant de telle fonction étant des fonctions rationnelles et entières des invariants primaires.

(3) Il sera impossible de former aucune équation linéaire de la nature exprimée plus haut entre x, y, z, θ.

(4) x, y, z seront indépendants entre eux. Quant à θ, il y aura deux hypothèses à faire: ou il est indépendant de x, y, z, ou l'on peut prendre pour sa valeur une fonction linéaire quelconque de xz et y^2.

Je démontrerai que la dernière hypothèse doit être rejetée, c'est-à-dire qu'il existe en effet un invariant fondamental de l'ordre 18, de sorte que le système complet des invariants se composera de six, que je nomme *primaires*, dont les ordres sont 2, 3, 4, 5, 6, 7, et cinq dont les ordres sont 0, 8, 9, 10, 18, que je nommerai *secondaires*, car on ne doit jamais oublier que la constante 1 est un invariant du degré zéro.

Traitons désormais les invariants primaires comme des constantes, cela facilitera beaucoup la parole dans cette dissertation.

Supposons que y^2, zx ne puissent pas s'exprimer séparément comme fonctions linéaires de $x, y, z, 1$; puisque, pour une valeur quelconque de x, y, z, on peut substituer $ax + b$, $cy + d$, $ez + fx + g$, on verra facilement qu'on peut instituer les équations suivantes entre x, y, z:

$$xz - y^2 = T, \qquad (2)$$
$$x^2 = Ax + By + Cz + D, \qquad (3)$$
$$xy = A'x + B'y + C'z + D', \qquad (4)$$
$$yz = L'x + M'y + N'z + P', \qquad (5)$$
$$z^2 = Lx + My + Nz + P, \qquad (6)$$

[* Cf. p. 62 below.]

où l'on remarquera que l'on a fait disparaître le terme xz ou y^2 dans l'équation pour z^2 par le moyen du multiplicateur arbitraire λ, qu'on peut ajouter à z. Multiplions

(2) par x et z,
(3) y,
(4) x et z,
(5) x et z,
(6) y.

Il est facile de voir qu'en faisant les éliminations dialytiques convenables, on obtiendra cinq équations linéaires entre x, y, z, xz, y^2, 1. De plus, dans chacune de ces équations, le coefficient de xz doit être égal à celui de y^2; car, si c'est combiné avec (2), on trouvera xz et y^2 comme fonctions linéaires contraires à l'hypothèse de x, y, z. Ainsi chacune de ces cinq équations doit être une identité et fournira ainsi cinq liaisons entre les coefficients, de sorte qu'on pourrait attendre de trouver vingt-cinq de ces liaisons; mais, en faisant le calcul, on trouvera qu'il n'y a plus que onze indépendantes, que j'écris de la manière suivante:

(1) Le groupe $B'=A$, $C'=B$, $L'=M$, $M'=N$, $N'=A'$; de sorte que l'on peut substituer respectivement les lettres

$$\begin{matrix} A, & B, & C, \\ K, & A, & B, \\ M, & N, & K, \\ L, & M, & N, \end{matrix}$$

au lieu de

$$\begin{matrix} A, & B, & C, \\ A', & B', & C', \\ L', & M', & N', \\ L, & M, & N. \end{matrix}$$

Il y aura encore un groupe de cinq équations que voici:

$$D = BK - CN, \tag{12}$$
$$P = MK - AL, \tag{13}$$
$$D' = CM - AK, \tag{14}$$
$$P' = LB - NK, \tag{15}$$
$$T = CL - K^2, \tag{16}$$

et finalement

$$CL - AN = 0. \tag{17}$$

Mais on peut obtenir encore une nouvelle équation identique en multipliant (2) par xz, (3) par (6), et (4) par (5); car on a

$$xzT = x^2 . z^2 - xz . yz.$$

Les cinq liaisons qui en résultent seront indépendantes entre elles-mêmes, mais une d'elles ne sera qu'une répétition de (16). Les quatre qui restent sont toutes nouvelles et peuvent s'écrire

$$2(KBM - CMN - ALB + ANR) = 0, \tag{18}$$
$$LB^2 - 2KNB + CN^2 + AT = 0, \tag{19}$$
$$CM^2 - 2AKM + LA^2 + NT = 0, \tag{20}$$
$$2(BM - AN)T = 0. \tag{21}$$

Ainsi l'on a

$$\text{ou } T = 0, \quad \text{ou } BM = AN.$$

Si $BM = AN$, on obtient, en combinant avec (18) et (19), $AT = 0$, et, en combinant avec (18) et (20), $NT = 0$. Donc

$$\text{ou } T = 0, \quad \text{ou } A = 0, \quad N = 0, \quad \text{et } BM = 0.$$

Mais si

$$N = 0 \quad \text{et} \quad B = 0, \quad D = 0,$$

et si

$$A = 0 \quad \text{et} \quad M = 0, \quad P = 0.$$

Donc

$$\text{ou } T = 0, \quad \text{ou } A = 0, \quad B = 0, \quad D = 0, \quad \text{ou } N = 0, \quad M = 0, \quad P = 0.$$

Donc on a

$$xz = y, \quad \text{ou } x^2 = Cz, \quad \text{ou } z^2 = Lx;$$

mais chacune de ces équations est inadmissible. Donc l'hypothèse que θ n'est pas indépendant est fausse, et nous avons établi que les invariants *secondaires* de la forme binaire octavique sont respectivement de l'ordre

0, 8, 9, 10, 18.

J'ajoute que, pour obtenir les *principes* qui ont conduit à ce résultat, on n'a besoin de s'appuyer sur aucune autre chose que la forme même de la fonction génératrice prise en conjonction avec la vérité intuitive que chaque combinaison d'invariants est elle-même un invariant.

A cause d'une erreur qui s'est glissée dans le *Second Memoir on Quantics* de M. Cayley, dans son explication des conséquences qui découlent de la fonction génératrice pour les covariants appartenant aux formes au-dessus du quatrième et les invariants au-dessus du sixième degré, on a pensé (voir *Théorie des formes binaires*, de M. Faà de Bruno, p. 150) que la théorie elle-même est en défaut et que les équations linéaires qui conduisent à cette fonction, après qu'un certain point est passé, cessent d'être indépendantes. J'ai examiné cette question de près et j'arrive à la *certitude* du contraire.

En effet, l'indépendance de ces équations est une conséquence d'un théorème très-curieux que j'ai découvert, un théorème plutôt de position que d'arithmétique que voici. Prenons trois nombres quelconques i, j, w avec la seule condition que w ne soit pas plus grand que $\frac{ij}{2}$. Formons toutes les combinaisons possibles avec les chiffres 0, 1, 2, ..., i, qui donnent la somme w: que le nombre de ces partitions soit m et qu'elles soient nommées $P_1, P_2, \ldots, P_m$.

De même formons toutes les partitions semblables avec la somme $w-1$, que leur nombre soit μ et nommons-les $\Pi_1, \Pi_2, \ldots, \Pi_\mu$.

On doit observer que le nombre m ne peut jamais devenir plus petit que μ, à cause de la condition que w n'est pas plus grand que $\frac{ij}{2}$.

Quand un Π quelconque, disons Π_λ, peut être déduit d'un P quelconque, disons P_l, par moyen de diminuer un des chiffres qui y entrent par l'unité, je nomme Π_λ une dérivée de P_l et dans le cas contraire une non-dérivée.

Formons un rectangle de m sur μ et à la tête des colonnes écrivons les sommes P et à côté de chaque ligne les sommes Π. De cette manière on peut dire que chaque place dans le rectangle aura une certaine longitude désignée par un P et une latitude désignée par un Π. Dans chaque place dont la latitude est une *dérivée* de la longitude, écrivons un signe quelconque, par exemple une croix, et dans toutes les autres places insérons des zéros. Par une diagonale d'une *matrice* carrée, comprenons une combinaison quelconque des positions occupées par ces éléments qui entrent dans la valeur du déterminant qui y appartient. Ces diagonales se diviseront, selon la règle élémentaire pour le calcul des déterminants, en deux espèces positives et négatives. De plus on peut sous-entendre par une diagonale *effective* une diagonale dans laquelle il n'entre nul *zéro*.

Or, avec le rectangle dont j'ai parlé, formons toutes les matrices carrées *complètes* possibles, c'est-à-dire des carrés de μ^2 plans. Il peut arriver que, pour un certain nombre d'entre elles, il n'y aura nulle diagonale effective, mais on peut démontrer qu'il en existe toujours une au moins qui possède une ou plusieurs diagonales. S'il n'y a qu'une seule diagonale effective, évidemment le déterminant ne peut pas s'évanouir; mais s'il y en a plusieurs, alors je dis que toutes ces diagonales effectives pour un déterminant donné porteront le même signe, de sorte que, si l'on donne des valeurs positives quelconques aux éléments désignés par des croix, la somme des produits qui correspondent à ces diagonales ne peut pas devenir égale à zéro. Cette proposition, fort remarquable, suffit pour démontrer la suffisance de la règle mise en doute par M. de Bruno. Pour trouver le nombre total de covariants appartenant à une forme donnée du degré i, d'un ordre donné j dans les

coefficients, et d'un degré donné k dans les variables, on n'aura qu'à prendre la différence de deux dénumérants de deux systèmes de deux équations simultanées dans l'une desquelles les termes constants seront $\frac{ij-k}{2}$ et dans l'autre $\frac{ij-k}{2}-i$. Comme conséquence de ce théorème, il est facile de démontrer que le nombre total des covariants de l'ordre j n'est autre chose que le nombre de manières de former la somme $\frac{ij}{2}$ ou $\frac{ij-1}{2}$ avec j des chiffres 0, 1, 2, 3, ..., i.

Toutes ces conclusions se trouvent peut-être étendues à des *systèmes* de formes binaires.

Par exemple, si l'on considère le cas de deux formes binaires seulement, disons des degrés i et i', et si, pour plus de simplicité, on traite le problème du nombre *total* de covariants de l'ordre j dans ces coefficients par rapport à une des deux formes, et j' par rapport à l'autre, ce nombre sera le dénumérant d'un système ternaire d'équations simultanées en nombres entiers, que voici:

$$y+2z+3t+\ldots+it+\eta+2\zeta+\ldots+i'\tau=\frac{ij+i'j'+\epsilon}{2}, \tag{1}$$

$$x+y+z+\ldots+t=j, \tag{2}$$

$$\zeta+\eta+\quad\ldots+\tau=j', \tag{3}$$

ϵ étant égal à zéro si $ij+i'j'$ est pair, et à -1 dans le cas contraire, et ainsi, en général, pour un système contenant un nombre de formes quelconques.

Le théorème qui porte à la démonstration de l'indépendance des équations linéaires dont il a été fait mention plus haut peut être mis sous une forme plus générale, que voici:

Soit Q une quantité quelconque d'un ou plusieurs systèmes de variables $x, y, z, \ldots, x', y', \ldots, x'', y'', \ldots$.

Prenons l'émanant de cette quantité par rapport à $\xi, \eta, \ldots, \xi', \eta', \ldots, \xi'', \eta'', \ldots$.

Substituons, pour $\xi, \eta, \ldots$, des fonctions linéaires *omnipositives* quelconques de $x, y, z \ldots$, pour $\xi', \eta', \ldots$ des fonctions linéaires *omnipositives* de $x', y', \ldots$, de sorte qu'on obtiendra une nouvelle quantité tout à fait semblable, dans sa constitution, à Q, mais dont les coefficients seront fonctions linéaires des coefficients de Q. Alors je dis que ces fonctions linéaires seront nécessairement indépendantes entre elles. Par une fonction linéaire omnipositive on comprendra facilement que je désigne une fonction linéaire dont tous les coefficients sont des quantités qui ne sont ni négatives ni nulles.

7.

SUR UNE MÉTHODE ALGÉBRIQUE POUR OBTENIR L'ENSEMBLE DES INVARIANTS ET DES COVARIANTS FONDAMENTAUX D'UNE FORME BINAIRE ET D'UNE COMBINAISON QUELCONQUE DE FORMES BINAIRES.

[*Comptes Rendus*, LXXXIV. (1877), pp. 1113—1116, 1211—1213.]

J'AI complètement résolu ce grand problème de trouver le système complet des invariants et covariants fondamentaux, que j'appellerai désormais les radicaux (*grundformen*) d'une forme binaire ou d'une combinaison quelconque de formes binaires, par une méthode purement algébrique tirée de l'équation partielle différentielle, à laquelle chaque *différentiant* binaire est assujetti. Par le mot *différentiant*, je désigne une fonction rationnelle quelconque des différences des racines d'une forme binaire donnée ou de chacune de telles formes, s'il y en a plus d'une, données. A l'aide de cette équation, j'obtiens une fonction, dite *génératrice* pour le système, sous la forme d'une fraction rationnelle contenant une variable, en raison du nombre des formes dans le système donné, laquelle fraction étant développée d'une telle façon que, dans la série qui en résulte, toutes les puissances des variables portent des indices positifs; le coefficient de chacune de ces puissances répondra au nombre des covariants ou invariants, linéairement indépendants, dont le degré et les ordres sont égaux respectivement aux indices de la puissance. Pour obtenir les radicaux (*grundformen*) du système, cette fonction doit être présentée, non sous sa forme réduite, mais d'une telle façon, que les indices des facteurs dont le dénominateur sera composé répondront chacun au degré et aux ordres d'un invariant ou covariant actuellement existant, comme il est toujours possible de le faire. Alors les indices du dénominateur répondront aux indices, pour ainsi dire, d'un radical appartenant à ce que j'appelle la *classe des primaires*.

Les radicaux secondaires seront obtenus au moyen du numérateur de la génératrice, en soumettant à une règle très-simple de *tamisement* l'ensemble des termes portant des coefficients positifs qui s'y présentent. J'ajouterai, pour plus de clarté, la génératrice dans quatre cas où j'aurai

le moyen de comparer mes résultats avec ceux de M. le professeur Gordan. Pour les trois premiers cas, l'accord entre les deux méthodes est parfait; pour le quatrième cas, sur trente radicaux donnés par M. Gordan, vingt-huit se présentent dans mon résultat, les deux qui manquent ayant disparu dans le procédé dit de *tamisement*. J'ai démontré catégoriquement que M. Gordan s'est trompé sur ces deux formes en les supposant fondamentales; elles doivent être et sont, en effet, décomposables, c'est-à-dire peuvent être exprimées comme sommes de combinaisons des radicaux inférieurs, de sorte que, pour un système de deux formes du quatrième degré, le nombre des covariants fondamentaux biquadratiques est 7 et des covariants du sixième degré 5, et non pas 8 et 6, comme M. Gordan l'avait pensé. J'ai même déterminé les coefficients numériques qui entrent dans ces deux sommes, de sorte qu'il ne reste pas la moindre ombre de doute sur la justesse de cette rectification. C'est le grand avantage que possède cette nouvelle méthode sur l'ancienne. De l'aveu même de M. Gordan, on ne peut jamais, en se servant de cette méthode (la méthode des hyperdéterminants), s'assurer d'une manière absolue que les formes réputées fondamentales sont telles en effet. Dans ma méthode, qui distingue les radicaux en deux classes, les primaires se présentent immédiatement à première vue, et les secondaires s'obtiennent en *tamisant* (selon une règle numérique des plus simples) un ensemble de formes qui se présentent simultanément avec les primaires.

(1) Soit donnée une seule forme binaire du cinquième degré.

La génératrice, sous sa forme canonique, sera la fraction dont le dénominateur est

$$(1-t^4)(1-t^8)(1-t^{12})(1-tu^5)(1-t^2u^6)(1-t^2u^2)$$

et le numérateur

$$\begin{aligned} 1 + t^{18} &+ (t^5+t^7+t^{11}+t^{13})v + (t^6+t^8+t^{10}+t^{12}+t^{16}-t^{20})v^2 \\ &+ (t^3+t^5+t^9+t^{11})v^3 + (t^4+t^6+t^8+t^{10}+t^{14}-t^{18})v^4 \\ &+ (t^3+t^7+t^9-t^{19})v^5 + (t^4-t^{14}-t^{18}-t^{20})v^6 \\ &+ (t^5-t^9-t^{13}-t^{15}-t^{17}-t^{19})v^7 - (t^{12}+t^{14}+t^{18}+t^{20})v^8 \\ &+ (t^3-t^7-t^{11}-t^{13}-t^{15}-t^{17})v^9 - (t^{10}+t^{12}+t^{16}+t^{18})v^{10} \\ &- (t^5+t^{23})v^{11}. \end{aligned}$$

On déduit immédiatement du dénominateur 4, 0; 8, 0; 12, 0, trois invariants, 1, 5; 2, 6; 2, 2, trois covariants (dont le premier est la forme donnée elle-même); ces sept formes sont les radicaux primaires de la forme donnée.

Pour trouver les radicaux secondaires, on soumet au procédé de *tamisement* les formes ayant pour indices

18, 0; 5, 1; 7, 1; 11, 1; 13, 1; 6, 2; 8, 2; 10, 2; 12, 2; 16, 2; 3, 3; 5, 3; 9, 3; 11, 3; 4, 4; 6, 4; 8, 4; 10, 4; 14, 4; 3, 5; 7, 5; 9, 5; 1, 6; 1, 7.

La règle de tamisement enseigne à négliger les couples 10, 2; 12, 2; 16, 2; 11, 3; 10, 4; 14, 4; 9, 5, parce que ces couples se forment en additionnant des couples inférieurs (l'addition des couples $f.g$, $h.k$ signifie le couple $(f+k).(g+k)$). Il reste dix-sept couples qui répondent aux ordres et aux degrés des radicaux secondaires.

Voici la règle générale pour le tamisement:

Supposons que par le tamisement on ait déjà obtenu un certain nombre de couples irréductibles et qu'on trouve un nouveau couple $i.j$ avec le coefficient μ. On détermine le nombre M de manière à former ce dernier couple en additionnant les couples inférieurs avec eux-mêmes ou les uns avec les autres. Alors, si M est inférieur à μ, on aura $(\mu - M)$ radicaux secondaires avec les indices $i.j$, on comptera $\mu - M$ fois ce couple et l'on continuera le procédé de tamisement comme auparavant. Si $\mu - M$ est zéro ou négatif, il n'y aura aucun secondaire du type $i.j$. Dans le dernier cas, la valeur numérique de la différence $\mu - M$ indiquera l'existence de ce nombre de rapports syzygétiques entre les radicaux des deux espèces des degrés i pour les coefficients et j pour les variables.

Dans le cas traité ci-dessus, toutes les valeurs de μ sont l'unité. Il résulte de ce qui a été fait que l'ensemble du système radical contient vingt-trois formes que l'on trouvera identiques avec celles données par Clebsch dans son *Traité sur les formes binaires*, p. 277.

(2) Prenons la forme binaire du sixième degré. On trouve pour génératrice la fraction dont le dénominateur est

$$(1-t^2)(1-t^4)(1-t^6)(1-t^{10})(1-t^2v^4)(1-t^2v^8)(1-tv^6)$$

et le numérateur

$$\begin{aligned}(1+t^{15}) &+ (t^3+t^5+t^7+t^8+t^{10}+t^{12})v^2\\ &+ (t^4+t^5+t^6+t^7+t^8+t^9+t^{10}+t^{11}+t^{13}-t^{17})v^4\\ &+ (t^3+t^4+2t^6+t^8+t^9+t^{11}-t^{16})v^6\\ &+ (t^3+t^5+t^7-t^{13}-t^{15}-t^{17})v^8\\ &+ (t^4-t^9-t^{11}-t^{12}-2t^{14}-t^{16}-t^{17})v^{10}\\ &+ (t^3-t^7-t^9-t^{10}-t^{11}-t^{12}-t^{13}-t^{14}-t^{15}-t^{16})v^{12}\\ &- (t^9+t^{20})v^{16}-(t^8+t^{10}+t^{12}+t^{13}+t^{15}+t^{17})v^{14}.\end{aligned}$$

Le procédé de tamisement fera disparaître

6, 4; 8, 4; 10, 4; 11, 4; 13, 4; 8, 6; 9, 6; 11, 6; 7, 8.

Il y aura donc sept radicaux primaires et dix-neuf secondaires, en tout les vingt-six *bildungen* posés par Clebsch (*Formen binären*, p. 296).

(3) Prenons le système comprenant deux formes binaires, l'une biquadratique, l'autre quadratique. En faisant rapporter la variable T à la quadratique et t à la biquadratique, je trouve que la génératrice, sous sa forme canonique, aura pour dénominateur

$$(1-t^2)(1-t^3)(1-T^2)(1-T^2t)(1-T^2t^2)(1-Tv)(1-tv^2)(1-t^2v^2)$$

et pour numérateur

$$\begin{aligned}(1+T^3t^3) &+ [(T+T^2)t+(T+T^2)t^2+(T^2-T^4)t^3]v^2\\ &+ [Tt+Tt^2+(T-T^3)t^3-T^3t^4-T^3t^5]v^4\\ &+ [(1-T^2)t^3-(T^2+T^3)t^4-(T^2+T^3)t^5]v^6-(Tt^3+T^4t^6)v^8.\end{aligned}$$

Ici aucun des termes du numérateur ne disparaît par l'opération du tamisement, et il y aura 8 primaires, 10 secondaires, 18 *grundformen* en tout, ce qui est d'accord avec les résultats déjà obtenus. (Voir *Salmon's Lessons*, 3ᵉ édition, p. 200.)

Finalement, je considérerai le cas *crucial*, où M. Gordan et moi nous sommes en désaccord, de deux formes biquadratiques. Pour plus de brièveté, je ne donnerai que la première moitié des termes du numérateur; on peut obtenir le reste de ces termes (qui n'influe nullement sur le résultat, tous les coefficients positifs dans cette partie, 25 en nombre, s'évanouissant dans le procédé de *tamisement*) par la règle suivante: *A chaque terme, dans la première partie, correspondra un terme dans la seconde partie du numérateur, tel que le produit des deux termes sera* $T^7.t^7.v^{14}$.

Or je dis que le dénominateur de la génératrice sera

$$\begin{gathered}(1-T^2)(1-T^3)(1-t^2)(1-t^3)(1-Tt)(1-Tt^2)(1-tT^2)(1-Tv^4)\\ (1-T^2v^4)(1-tv^4)(1-t^2v^{14}),\end{gathered}$$

et la première partie du numérateur (la seule effective) sera

$$\begin{aligned}&(1+T^2t^2+T^4t^4)\\ &+[(Tt)+(T^2t+Tt^2)+(Tt^3+T^2t^2+T^3t)+(T^2t^3+T^3t^2)+T^3t^3]v^2\\ &+(Tt+Tt^2+T^2t+T^2t^3+T^3t^3+T^4t^3+T^3t^5-T^5t^4-T^4t^5-T^6t^4-T^4t^6)v^4\\ &+[(Tt+T^3+T^2t+Tt^2+t^3+T^2t^2-Tt^4-T^2t^3-T^3t^2-T^4t\\ &\quad -Tt^5-2T^2t^4-3T^3t^3-2T^4t^2-T^5t-T^5t^2-2T^4t^3\\ &\qquad -2T^3t^4-T^2t^5-T^3t^5-T^5t^3)]v^6.\end{aligned}$$

Par l'opération de *tamisement* opérée sur les termes du numérateur, il ne restera que les triplets

2.2.0, 1.1.2, 2.1.2, 1.2.2, 1.3.2, 2.2.2, 3.1.2, 2.3.2, 3.2.2,

1.1.4, 1.2.4, 2.1.4,

1.1.6, 3.0.6, 0.3.6, 2.1.6, 1.2.6.

Observez que les triplets 2.2.4, 2.2.6 disparaissent, comme étant respectivement les sommes de triplets inférieurs. Ainsi il y aura 17 *grundformen* secondaires et 11 primaires, faisant ensemble le nombre 28.

J'ai calculé aussi la génératrice pour la forme du huitième degré; mais elle est trop longue pour être reproduite ici. La partie de cette fonction appartenant aux invariants a été déjà donnée par moi, dans sa forme canonique, dans la première de mes deux Communications récentes à l'Académie.

Le dénominateur est

$$(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6)(1-t^7);$$

le numérateur est

$$1+t^8+t^9+t^{10}+t^{18}.$$

Je profite de cette occasion pour corriger une erreur dans la Communication que j'avais envoyée par dépêche télégraphique. Les radicaux primaires invariants seront, comme je l'avais remarqué, 6 en nombre et des degrés 2, 3, 4, 5, 6, 7 par rapport aux coefficients; mais les secondaires seront 3 et non pas 4 en nombre et des degrés 8, 9, 10 respectivement. Le tamisement fera disparaître l'indice 18 tout à fait, et, comme dans le cas de deux formes biquadratiques, cette opération de tamisement fait disparaître l'invariant double correspondant au terme T^4t^4 dans le numérateur.

J'ai obtenu la génératrice pour les invariants appartenant à la forme du septième et à la forme du dixième degré; dans cette dernière, c'est fort remarquable, un invariant de degré impair 9 figure parmi les radicaux secondaires. Je crois aussi être sur la voie pour faire l'extension de cette méthode aux formes et systèmes de formes de *dimensions* supérieures à la seconde, c'est-à-dire de formes ternaires, quaternaires, &c.; mais il faut réserver pour quelque autre occasion ce que j'ai à dire sur ce sujet et sur la méthode dont je me suis servi pour former la fonction génératrice des formes binaires. Je dois ajouter que l'erreur que j'ai commise dans ma démonstration prétendue de l'existence d'un radical du degré 18, pour les formes octaviques binaires, paraît consister dans l'hypothèse, mal fondée, de l'impossibilité de l'existence d'une équation syzygétique, dans laquelle les x, y, z figurent seulement au premier degré.

8.

SUR LE VRAI NOMBRE DES COVARIANTS ÉLÉMENTAIRES D'UN SYSTÈME DE DEUX FORMES BIQUADRATIQUES BINAIRES.

[*Comptes Rendus*, LXXXIV. (1877), pp. 1285—1289.]

DANS une récente Communication que j'ai eu l'honneur d'adresser à l'Académie, j'ai remarqué que ma méthode pour obtenir les *grundformen* d'un système de deux formes biquadratiques ne donne raison qu'à supposer l'existence de 28 invariants et covariants élémentaires, tandis que M. le professeur Gordan en a fourni une Table de 30. J'ai appris qu'outre M. Salmon, qui a adopté les conclusions de M. Gordan sans examen, M. le professeur Bertini pense aussi, de son côté, en avoir confirmé la justesse. Il importe donc dans le plus haut degré au progrès de l'Algèbre que ce point ne puisse rester douteux; c'est pourquoi j'ai pris la liberté d'exposer dans les *Comptes rendus* la preuve concluante que deux des formes données par M. Gordan sont superflues, c'est-à-dire qu'elles ne sont en effet que des combinaisons algébriques d'autres formes contenues dans sa Table.

On me présente deux corps qu'on affirme être des corps simples: sans me donner la peine de démontrer (comme il sera facile dans le cas actuel) l'impossibilité qu'il en existe de tels possédant les caractères qu'on leur attribue; je vais démontrer l'erreur de cette affirmation, en effectuant pour ainsi dire leur décomposition sous les yeux mêmes du lecteur.

Le travail de cette décomposition sera beaucoup abrégé par la considération suivante. Quand le premier terme d'un covariant quelconque est donné, le covariant lui-même est donné; car, en vertu de l'équation différentielle partielle à laquelle chaque covariant satisfait, de ce premier terme découlent tous les autres au moyen d'opérations explicites de différentiation et d'addition exclusivement. Ainsi, pour prouver qu'un covariant donné est la somme d'autres covariants, il suffit de démontrer que le coefficient du premier terme de l'un est la somme des coefficients des premiers termes des autres.

Or servons-nous en général du symbole *ijk* pour désigner le coefficient de x^k dans un covariant élémentaire dont l'ordre, par rapport aux coefficients d'une forme biquadratique binaire donnée, est i, par rapport aux coefficients d'une autre forme semblable j, et dont le degré, relatif aux variables x, y, est k.

Posons

$$a,\ 4b,\ 6c,\ 4d,\ e, \qquad \alpha,\ 4\beta,\ 6\gamma,\ 4\delta,\ \epsilon$$

pour les coefficients des deux formes biquadratiques données.

Alors, en suivant les prescriptions données par M. Gordan lui-même, on trouvera facilement les valeurs suivantes pour les invariants et covariants fondamentaux dont l'existence n'est pas douteuse, c'est-à-dire

$$1.1.2 = a\delta - 3b\gamma + 3c\beta - d\alpha, \qquad 1.1.4 = a\gamma - 2b\beta + c\alpha,$$

$$1.1.0 = a\epsilon - 4b\delta + bc\gamma - 4d\beta + e\alpha, \qquad 1.1.6 = a\beta - b\alpha,$$

$$1.0.4 = a,$$

$$1.2.2 = a(\gamma\delta - \beta\epsilon) + b(\alpha\epsilon + 2\beta\delta - 3\gamma^2) + 3c(\beta\gamma - \alpha\delta) + 2d(\alpha\gamma - \beta^2),$$

$$0.1.4 = \alpha,$$

$$2.1.2 = \alpha(be - cd) - \beta(ae + 2bd - 3c^2) + 3\gamma(ad - bc) - 2\delta(ac - b^2).$$

On comprend que dans ce qui précède j'aie réduit chaque expression à sa forme numérique la plus simple. Le signe algébrique est disponible à volonté, et j'ai attribué à chacune le signe le plus commode pour mettre en évidence le rapport numérique qui lie le produit de chaque couple à la forme 2.2.6 dite *élémentaire* par M. Gordan, dont la valeur (voir *Salmon's Lessons on higher Algebra*, 3[e] édition, p. 206) est obtenue de la manière suivante. Dans la *hessienne* d'une des formes données pour x, y, écrivez x_1, y_1, dans l'autre x_2, y_2. Multipliez ces deux hessiennes ainsi modifiées ensemble et opérez sur ce produit avec le symbole $\left(\frac{d}{dx_1}\frac{d}{dy_2} - \frac{d}{dx_2}\frac{d}{dy_1}\right)^2$, et, dans le résultat, remplacez x_1, x_2 par x; y_1, y_2 par y; c'est la méthode de M. Gordan pour obtenir sa forme 2.2.6, traduite dans le langage des hyperdéterminants. Moins 6 fois ce résultat pris dans sa forme arithmétique réduite (et affectée d'un signe algébrique convenable) sera la somme des quatre produits précédents, comme on le verra par la Table ci-jointe, où l'on remarquera que la somme des chiffres de chaque colonne sera égale à zéro.

La manière de comprendre cette Table s'explique d'elle-même. Par exemple, la seconde ligne enseigne que le produit 1.1.0 par 1.1.6 sera égal à

$$[a^2\beta e - ab\alpha\epsilon - 4ab\beta\delta + 6ac\beta\gamma \ldots],$$

et de même pour les autres lignes. La dernière ligne montre que la forme de l'ordre 2 dans chaque système de coefficients et du degré 6 en x et y, citée par M. Gordan comme un covariant fondamental calculé selon la règle donnée par lui, aura le coefficient de x^6 égal à

$$ac\alpha\delta - ac\beta\gamma - b^2\alpha\delta + b^2\beta\gamma - ad\alpha\gamma + ad\beta^2 + bc\alpha\gamma - bc\beta^2.$$

A.—Table pour effectuer la décomposition de la forme dite élémentaire du type 2.2.6 de M. Gordan

	aaβe	aaγδ	abαe	abβδ	abγγ	acαδ	acβγ	bbαδ	bbβγ	adαγ	adββ	bcαγ	bcββ	aeαβ	bdαβ	ccαβ	beαα	cdαα
1.1.2 × 1.1.4		−1		+2	+3	−1	−3		−6	+1		+3	+6		−2	−3		+1
1.1.0 × 1.1.6	1		−1	−4			6	4			−4	−6		1	4		−1	
1.0.4 × 1.2.0	−1	1	1	2	−3	−3	3			2	−2							
0.1.4 × 2.1.2						−2		2		3		−3		−1	−2	3	1	−1
−[2.2.6] de Gordan sextuplé						6	−6	−6.	6	−6	6	6	−6					

B.—Table pour effectuer la décomposition de la forme dite élémentaire du type 2.2.4 de M. Gordan

	aaγe	aaδδ	abβe	abγδ	acαe	acβδ	acγγ	bbαe	bbβδ	bbγγ	adαδ	adβγ	bcαδ	bcβγ	aeαγ	aeββ	bdαγ	bdββ	ccαγ	ccββ	beαβ	cdαβ	ceαα	ddαα
1.1.2 × 1.1.2		1		−6		6				9	−2			−18			6			9		−6		1
1.1.0 × 1.1.4	−1		2	4	−1		−6		−8			4	4	12	−1			−8	−6		2	4	−1	
0.2.0 × 2.0.4					1	−4	3	−1	4	−3														
2.0.0 × 0.2.4															1	−1	−4	4	3	−3				
1.0.4 × 1.2.0	1	−1	−2	2	1	2	−3				−2	2			1	−1								
0.1.4 × 2.1.0					1			−1			−2		2		1		2		−3		−2	2	1	−1
−[2.2.4] de Gordan doublé					−2	−4	6	2	4	−6	6	−6	−6	6	−2	2	−4	4	+6	−6				

Je passe à la considération de la forme 2.2.4, et, comme dans le cas précédent, je me sers du symbole ijk pour représenter le coefficient principal dans le covariant dont les ordres et le degré sont i, j, k.

On trouvera

$$1.1.2 = \quad a\delta - 3b\gamma + 3c\beta - d\alpha,$$

$$1.1.0 = -a\epsilon + 4b\delta - 6c\gamma + 4d\beta - e\alpha, \qquad 1.1.4 = a\gamma - 2b\beta + c\alpha,$$

$$0.2.0 = \quad \alpha\epsilon - 4b\delta + 3\gamma^2, \qquad 2.0.4 = ac - b^2,$$

$$2.0.0 = \quad ae - 4bd + 3c^2, \qquad 0.2.4 = \alpha\gamma - \beta^2,$$

$$1.0.4 = a,$$

$$1.2.0 = e(\alpha\gamma - \beta^2) - 2d(\alpha\delta - \beta\gamma)$$
$$+ c(\alpha\epsilon + 2\beta\delta - 3\gamma^2) - 2b(\beta\epsilon - \gamma\delta) + a(\gamma\epsilon - \delta^2),$$

$$0.1.4 = \alpha,$$

$$2.1.0 = \epsilon(ac - b^2) - 2\delta(ad - bc)$$
$$+ \gamma(ae + 2bd - 3c^2) - 2\beta(be - cd) + \alpha(ce - d^2).$$

Finalement la forme de M. Gordan, dont le coefficient principal est représenté par [2.2.4], s'obtient tout à fait comme la forme correspondant à [2.2.6] dans le cas précédent, avec la seule exception que l'opérateur différentiel sur le produit des hessiennes sera la puissance quatrième au lieu de la puissance deuxième du symbole $\frac{d}{dx_1}\frac{d}{dy_2} - \frac{d}{dx_2}\frac{d}{dy_1}$. Cette opération donnera

$$(b^2 - ac)(\alpha\epsilon + 2\beta\delta - 3\gamma^2) + (\beta^2 - \alpha\gamma)(ae + 2bd - 3c^2) + 3(ad - bc)(\alpha\delta - \beta\gamma)$$

pour valeur de [2.2.4], ou plutôt je préférerai considérer cette fonction comme la valeur de $-$[2.2.4]. Alors on trouvera que 2 fois [2.2.4] sera la somme des six produits précédents, comme on le voit par la Table ci-contre, où l'on remarquera que la somme de chaque colonne donne la somme zéro, comme dans le cas précédent.

J'ajouterai seulement que cette preuve éclatante de l'insuffisance de la méthode de M. Gordan et de son école, pour séparer les formes véritablement élémentaires des formes superflues qui s'y rattachent (insuffisance reconnue par M. Gordan lui-même de la manière la plus loyale dans son discours inaugural prononcé à Erlangen), n'ôte rien à la valeur immense du service qu'il a rendu à l'Algèbre, en ayant le premier démontré l'existence d'une limite au nombre de ces formes.

(9)

THÉORIE POUR TROUVER LE NOMBRE DES COVARIANTS ET DES CONTREVARIANTS D'ORDRE ET DE DEGRÉ DONNÉS LINÉAIREMENT INDÉPENDANTS D'UN SYSTÈME QUELCONQUE DE FORMES SIMULTANÉES CONTENANT UN NOMBRE QUELCONQUE DE VARIABLES.

[*Comptes Rendus*, LXXXIV. (1877), pp. 1359—1361, 1427—1430.]

Pour plus de clarté, je commencerai par le cas d'une seule forme du degré n à k variables. On se propose de trouver le nombre: (1) des covariants, (2) des contrevariants linéairement indépendants de degré i par rapport aux coefficients et d'ordre j par rapport aux variables.

(1) *Cas des covariants.*—Écrivons

$$\sigma = \frac{in + (k-1)i}{k}, \quad \sigma' = \sigma + 1,$$

et trouvons toutes les solutions en nombres positifs et entiers des équations

$$a_0 + a_1 + a_2 + \ldots + a_n = i, \qquad (1)$$

$$a_1 + 2a_2 + \ldots + na_n = \sigma. \qquad (2)$$

Pour une solution quelconque de ces équations, soit S le nombre des *invariants* indépendants appartenant à un système de formes des degrés n, $n-1$, $n-2$, ..., contenant chacun $k-1$ variables, les ordres de ces invariants quant aux coefficients de ces formes étant respectivement

$$a_0, a_1, a_2, \ldots, a_{n-1},$$

nous obtiendrons ainsi une somme de nombres que je nommerai ΣS.

Formons le même système d'équations en a', comme plus haut avec des a, avec la différence d'écrire σ' au lieu de σ, et soit S' le nombre des *contrevariants linéaires* appartenant au même système de formes qu'auparavant, les ordres de ces contrevariants par rapport aux coefficients étant respectivement

$$a'_0, a'_1, a'_2, \ldots, a'_{n-1};$$

nous obtiendrons ainsi une seconde somme $\Sigma S'$; la différence $\Sigma S - \Sigma S'$ sera le nombre de covariants du degré i et de l'ordre j pour la forme du $n^{\text{ième}}$ degré à k variables.

(2) *Cas des contrevariants.* Écrivons

$$\sigma = \frac{in - (k-1)j}{k}, \quad \sigma' = \sigma - 1,$$

et, avec la nouvelle valeur de σ, trouvons, comme dans le cas précédent, la valeur de ΣS. De même trouvons $\Sigma S'$, comme auparavant, en nous servant de la nouvelle valeur de σ', mais avec cette différence que, pour trouver un S' quelconque, il faut calculer le nombre non pas des contrevariants, mais des covariants linéaires des formes correspondantes. $\Sigma S - \Sigma S'$ sera le nombre des contrevariants du degré i et de l'ordre j, linéairement indépendants, appartenant à une forme du degré n à k variables.

Pour les invariants, on met $j = 0$, et l'on se sert indifféremment de l'une ou de l'autre méthode, c'est-à-dire on écrit

$$\sigma' = \sigma + 1 \quad \text{ou} \quad \sigma' = \sigma - 1$$

à volonté.

Quand $k = 3$, c'est-à-dire pour les formes ternaires, on comprend, en formant S', que la distinction entre les covariants et les contrevariants binaires devient superflue, puisque à chaque covariant d'une forme binaire correspond un contrevariant, et *vice versâ*.

Quand $k = 2$, en se rappelant que pour un système de formes *unitaires* simultanées

$$a_0 x^n, \; a_1 x^{n-1}, \; a_2 x^{n-2}, \ldots, \; a_{n-1} x$$

chaque combinaison des coefficients est un invariant, et multipliée par x un covariant ou contrevariant unitaire, la règle pour trouver le nombre des covariants et des contrevariants binaires revient à la règle connue.

Je passe à présent au cas plus général d'un système de formes $n_1, n_2, \ldots, n_q$ à k variables. On cherche le nombre des covariants et des contrevariants du degré j et des ordres $i_1, i_2, \ldots, i_q$ quant aux coefficients des formes données.

On écrit dans les deux cas respectivement

$$\sigma = \frac{\pm (k-1) j + \Sigma in}{k};$$

le rapport de σ' à σ reste le même, comme auparavant. Au lieu de l'équation (1), on écrit q équations de la forme

$$a_{0,q} + a_{1,q} + a_{2,q} + \ldots + a_{nq,q} = i_q \, [q = 1, \; 2, \; 3, \ldots, \; q],$$

et, au lieu de l'équation (2), on écrit la seule équation

$$\Sigma_{q=q}^{q=1} (a_{1,q} + 2a_{2,q} + \ldots, + n_q a_{nq,q}) = \sigma.$$

Alors, pour trouver S, on prend un système de formes à $k - 1$ variables, une de chaque degré de 1 jusqu'à n_1, encore une de chaque degré de 1 jusqu'à $n_2, \ldots$, et finalement une de chaque degré de 1 jusqu'à n_q, et l'on trouve pour S le nombre des invariants à $k - 1$ variables, dont les ordres respectifs, par rapport à ces formes, sont les valeurs des a données pour une solution quelconque des équations écrites plus haut : ainsi l'on obtient ΣS; de même, en substituant σ' pour σ et des contrevariants linéaires (si l'on s'occupe des covariants) ou des covariants linéaires (si l'on s'occupe des contrevariants), on trouve la valeur de $\Sigma S'$, et la différence $\Sigma S - \Sigma S'$ sera le nombre cherché.

Ainsi l'on voit que le problème pour des systèmes à k variables se réduit au même problème pour $k - 1$ variables, de sorte que, par déductions successives, le problème est complétement résolu par une méthode arithmétique pour un nombre quelconque de variables.

Avec l'aide de ce principe, on peut construire, simplifier et réduire à la forme canonique une fonction génératrice ayant par rapport aux formes ternaires, quaternaires, etc., le même genre de rapport que la fonction génératrice dont, sous la forme canonique, j'ai déjà donné des exemples pour les formes binaires : c'est de cela que je m'occupe en ce moment ; mais ce travail algébrique, quoique d'une nature très-élémentaire, devient, même pour les formes ternaires, extrêmement laborieux.

Je terminerai cette Note par un seul exemple numérique du calcul indiqué par mon théorème : qu'il soit demandé de trouver le nombre de contrevariants asyzygétiques du douzième ordre et du neuvième degré appartenant à la forme cubique ternaire.

Nous avons ici

$$i = 12, \quad \sigma = \frac{3 \,.\, 12 - 2 \,.\, 9}{3} = 6, \quad \sigma' = 5.$$

Je forme les deux Tables

0	1	2	3	S
10	0	0	2	0
9	1	1	1	1
9	0	3	0	1
8	2	2	0	3
7	4	1	0	5
6	6	0	0	2
3e	2e	1re		12

0	1	2	3	S'
10	0	1	1	0
9	2	0	1	1
9	1	2	0	3
8	3	1	0	7
7	5	0	0	0
3e	2e	1re		11

Dans la Table à gauche, en prenant une ligne horizontale quelconque, la somme du produit de chaque chiffre par le chiffre correspondant à la tête de la colonne où il se trouve est égale à 6; dans la Table à droite, cette somme de produits est 5; pour l'une et l'autre, la somme des chiffres de chaque ligne est 12. Les chiffres de ces dernières colonnes ne figurent pas dans les calculs; ces chiffres sont les valeurs des S et des S'; le nombre d'invariants ou de covariants linéaires appartenant à chaque partition, par exemple à un système composé d'une cubique quadratique et d'une forme linéaire à deux variables, le nombre des invariants des ordres 8, 2, 2 respectivement pour ces formes est 3, et, appartenant au même système de formes, le nombre des covariants linéaires des ordres 8, 3, 1 quant aux coefficients est 7. La somme des S étant 12 et des S' 11, la différence 1 sera le nombre des contrevariants à la forme cubique ternaire du type donné, et ainsi en général.

Comme second exemple, cherchons s'il y a des invariants cubiques pour les courbes du quatrième degré.

Ici $$i=3, \quad n=4, \quad j=0:$$

donc $$\sigma=\frac{4.3}{3}=4, \quad \sigma'=\sigma\pm 1.$$

Prenons $\sigma'=3$. On forme les deux Tables

0	1	2	3	4	S
2	0	0	0	1	1
1	1	0	1	0	1
1	0	2	0	0	1
0	2	1	0	0	0
4^e^	3^e^	2^e^	1^re^	r	3

0	1	2	3	4	S'
2	0	0	1	0	0
1	1	1	0	0	2
0	3	0	0	0	0
4^e^	3^e^	2^e^	1^re^	r	2

Tous les chiffres, dans les colonnes S et S', correspondent à des résultats ou évidents d'eux-mêmes ou donnés déjà par MM. Clebsch, Gordan et Gundelfinger, sauf la valeur 2 de S', qui représente le nombre des contrevariants linéaires, non pas seulement par rapport à leurs degrés, mais aussi par rapport à chaque système des coefficients appartenant à un système de trois formes des degrés 4, 3, 2 respectivement. Pour trouver ce nombre, on en forme un nouveau

$$\sigma=\frac{1.4+1.3+1.2-1}{2}=4$$

et un autre $$\sigma'=(\sigma-1)=3,$$

et l'on prend la différence de deux *dénumérants:* l'un le nombre de solutions en nombres positifs et entiers du système d'équations

$$\lambda'+2\lambda''+3\lambda'''+4\lambda^{\text{IV}}+\mu'+2\mu''+3\mu'''+\nu'+2\nu''=4,$$
$$\lambda+\lambda'+\lambda''+\lambda'''+\lambda^{\text{IV}}=1, \quad \mu+\mu'+\mu''=1, \quad \nu+\nu'+\nu''=1;$$

l'autre le dénumérant du même système d'équations quand on remplace

4 par 3. On voit facilement que le premier dénumérant est le nombre des combinaisons

4	0	0
3	1	0
3	0	1
2	2	0
2	0	2
2	1	1
1	3	0
1	2	1
1	1	2
0	2	1
0	2	2

et que le second est le nombre des combinaisons

3	0	0
2	1	0
2	0	1
1	2	0
1	0	2
1	1	1
0	3	0
0	2	1
0	1	2

c'est-à-dire le nombre des contrevariants linéaires qu'on cherche est $11-9$ ou 2. La somme des S moins la somme des S' est donc $3-2$, et conséquemment il n'y a qu'un et un seul invariant cubique appartenant aux courbes du quatrième degré.

10.

ADDRESS ON COMMEMORATION DAY AT JOHNS HOPKINS UNIVERSITY*
22 FEBRUARY, 1877.

Sir! Ladies and Gentlemen!

It is the custom of this country (which will take no denial) that, through the voice of our truly estimable President, calls upon me to appear before you and render an account of my experiences in connection with this great institution, which, so recently inaugurated, is steadily and solidly rising from its foundations, like the stately pile standing almost at its gates—the magnificent bequest of George Peabody to his fellow-citizens—where day by day, quietly but persistently, to the ring of the hammer and the merry click of the chisel, without haste as without pause, we may witness stone after stone lifted into its position, and each pillar set upright and securely on its base. Had I consulted only my own inclinations in the matter, I would have much preferred to remain silent, and let my work in the future tell its own tale.

It is with unaffected feelings of diffidence that I present myself before you, for, save on rare and exceptional occasions, it has not been my wont to make my voice heard in public assemblies. I know, indeed, and can conceive of no pursuit so antagonistic to the cultivation of the oratorical faculty—that faculty so prevalent in this country that the possession of it is not regarded as a gift, but the want of it as a defect—as the study of Mathematics. An eloquent mathematician must, from the nature of things, ever remain as rare a phenomenon as a talking fish, and it is certain that the more anyone gives himself up to the study of oratorical effect the less will he find himself in a fit state of mind to mathematicize. It is the constant

* The address was written on a rather sudden call, within a few hours, and many marks will be apparent to the practised eye of the haste with which it was composed. Two or three paragraphs have been inserted that were not contained in the address as delivered, and the writer is alone responsible for the opinions or sentiments which it expresses. Some copies of it will be forwarded to England, which he hopes to revisit in June next.

aim of the mathematician to reduce all his expressions to their lowest terms, to retrench every superfluous word and phrase, and to condense the Maximum of meaning into the Minimum of language. He has to turn his eye ever inwards, to see everything in its dryest light, to train and inure himself to a habit of internal and impersonal reflection and elaboration of abstract thought, which makes it most difficult for him to touch or enlarge upon any of those themes which appeal to the emotional nature of his fellow-men. When called upon to speak in public he feels as a man might do who has passed all his life in peering through a microscope, and is suddenly called upon to take charge of an astronomical observatory. He has to go out of himself, as it were, and change the habitual focus of his vision.

Yet it is not without a considerable admixture of feelings of a more agreeable nature that I have acquiesced in taking the part allotted to me in this day's proceedings.

It is always a satisfaction to meet those from whom we have received marks of regard, and whom we know to be favorably disposed towards us; and I should be heartless, indeed, and more callous than the oyster, who, twin-soul to the mathematician, working in silence and seclusion between the folding-doors of his mansion, elaborates the pearl that may, hereafter, deck an empress's brow, could I be insensible to the many proofs of kind and generous feeling which, both within and without the walls of this University, have been so widely and unequivocally accorded to me.

I scruple not to say (for it is strictly the truth) that I have experienced from the authorities of the University a degree of delicate consideration and forbearance from all claims that might be supposed to interfere, in any respect, with my comfort or ease of mind, that, as long as I live, will endear to me the name of the Johns Hopkins University.

But that pleasure of coming as a friend among friends is enhanced by the fact that I stand here a harbinger of glad tidings—that I can honestly congratulate all of you who are interested in the success of the University on the good seed that has been sown, and on the promise which it affords of a rich harvest in no distant future.

One of our great English judges observed on some occasion, when he was outvoted by his brethren on the bench (or, perchance, it may have been the twelfth outstanding juryman, who protested that never before in his life had he been shut up with eleven other such obstinate men) that "opinions ought to count by weight rather than by number," and so I would say that the good done by a university is to be estimated not so much by the mere number of its members as by the spirit which actuates and the work that is done by them. When I hear, as I have heard, of members of this University, only hoping to be enabled to keep body and soul together in order that they may continue to enjoy the advantages which it affords, it may be for a decade of years to come; when I find classes diligently attending lectures on the most abstruse branches of scholarship and science, remote from all the avenues which lead to fortune or public recognition; when I observe the earnestness with which our younger members address themselves to the studies of the place, and the absence of all manifestations of disorder or levity, without the necessity for the exercise of any external restraint, it seems to me that this establishment, even in its cradle, better responds to what its name should import, more fully embodies the true idea of a university, than if its halls and lecture-rooms swarmed with hundreds of idle and indifferent students, or with students, diligent, indeed, but working not from a pure love of knowledge, not even for the chaplet of olive, or the laurel crown, but for high places in examinations, for marks, as we say in England, the counters or vouchers to enable their fortunate possessor to draw large stakes out of the pool of sinecure fellowships or lucrative civil appointments.

But I look not only to our students, but to the means of instruction at our command, to our chemical and physical and biological laboratories, unsurpassed anywhere in the world for completeness in all essential particulars, furnished and replenished whenever called for without question and without stint, to our libraries and rooms for research and study, where any earnest student can work in comfort and seclusion, with all the materials and aids that he may require to assist him in his investigations, close at hand, and to the multifarious subjects in which (with a necessarily limited staff), even in our inchoate state, all who wish can receive instruction. In the course of time and as opportunities present themselves no doubt our staff of professors and lecturers will receive, as they need, considerable augmentation; but as I have heard it pithily and tellingly expressed, the object of our trustees is to found not chairs but professors.

I have had the pleasure myself of listening to a course of lectures on a very abstruse and important subject, allied to my own, which I am sure could not be surpassed for lucidity of arrangement, strictness of concatenation, aptness, fulness and variety of illustration and application, by lectures given in any university in the world, with which I am acquainted. These were the lectures of our colleague, unfortunately absent on this occasion, owing to ill health, Professor Rowland, on Thermodynamics, in which all the principal conclusions of this wonderful mathematical theory, perhaps the most wonderful since the discovery of universal gravitation, were deduced with geometrical rigor from the two great laws capable of being contained within a few words, the seven last words of the expiring Caloric theory, "*Heat is motion.*" "*Temperature seeks its level.*" *

* That is to say, temperature in regard to the categories of greater and less. Its *measure* Professor Rowland identifies with the integrating factor of a partial differential equation.

I have alluded, as a subject of congratulation, to the absence of all vexatious restraints upon the free action of our students which their conduct justifies. With equal reason may I congratulate myself and the professors and teaching staff with whom I have the happiness to be associated, on the confidence that is reposed in us, and on the free scope that is given to each to carry out in the manner that may seem to him most likely to be conducive to a useful result, the combined objects which this University has been founded to promote, under its two-fold aspect as a teaching body and as a corporation for the advancement and propagation of science and learning.

It has happened to myself, when in a state of despondency and embarrassment as to how I could best divide my energies between the contending claims of the teacher and the investigator, to be released from my difficulty by the cheering words graven lastingly on my memory, "The University desires from you your best and highest work."

And let me take this opportunity of making my profession of faith on a subject much mooted at the present day, as to whether the highest grade of university appointments should be conferred with or without the condition of teaching annexed.

I hesitate not to say that, in my opinion, the two functions of teaching and working in science should never be divorced. I believe that none are so well fitted to impart knowledge (if they will but recognize as existing, and take the necessary pains to acquire, the art of presentation) as those who are engaged in reviewing its methods and extending its boundaries—and I am sure that there is no stimulus so advantageous to the original investigator as that which springs from contact with other minds and the necessity for going afresh to the foundations of his knowledge, which the work of teaching imposes upon him. I look forward to the courses of lectures that I hope to deliver in succession within the walls of this University as marking the inauguration of a new era of productivity in my own scientific existence; nor need I consider any subject too low (as it is sometimes foolishly termed) for me to teach, when I remember to have seen the minutes of the conversation held between the delegates of the Convention, at the time of the French Revolution, and the illustrious Lagrange, the son of the pastry-cook of Turin, possibly the progenitor of the Marquis Lagrange, of turf celebrity (Citoyen Lagrange, as he is styled in the record), who, when asked what subject he would be willing to profess for the benefit of the community, answered meekly, "I will lecture on Arithmetic."

At this moment I happen to be engaged in a research of fascinating interest to myself, and which, if the day only responds to the promise of its dawn, will meet, I believe, a sympathetic response from the Professors of our divine Algebraical art wherever scattered through the world.

There are things called Algebraical Forms. Professor Cayley calls them Quantics. These are not, properly speaking, Geometrical Forms, although capable, to some extent, of being embodied in them, but rather schemes of processes, or of operations for forming, for calling into existence, as it were, Algebraic quantities.

To every such Quantic is associated an infinite variety of other forms that may be regarded as engendered from and floating, like an atmosphere, around it—but infinite in number as are these derived existences, these emanations from the parent form, it is found that they admit of being obtained by composition, by mixture, so to say, of a certain limited number of fundamental forms, standard rays, as they might be termed in the Algebraic Spectrum of the Quantic to which they belong. And, as it is a leading pursuit of the Physicists of the present day to ascertain the fixed lines in the spectrum of every chemical substance, so it is the aim and object of a great school of mathematicians to make out the fundamental derived forms, the Covariants and Invariants, as they are called, of these Quantics.

This is the kind of investigation in which I have for the last month or two been immersed, and which I entertain great hopes of bringing to a successful issue. Why do I mention it here? It is to illustrate my opinion as to the invaluable aid of teaching to the teacher, in throwing him back upon his own thoughts and leading him to evolve new results from ideas that would have otherwise remained passive or dormant in his mind.

But for the persistence of a student of this University in urging upon me his desire to study with me the modern Algebra I should never have been led into this investigation; and the new facts and principles which I have discovered in regard to it (important facts, I believe), would, so far as I am concerned, have remained still hidden in the womb of time*. In vain I represented to this inquisitive student that he would do better to take up some other subject lying less off the beaten track of study, such as the higher parts of the Calculus or Elliptic Functions, or the theory of Substitutions, or I wot not what besides. He stuck with perfect respectfulness, but with invincible pertinacity, to his point. He would have the New Algebra (Heaven knows where he had heard about it, for it is almost unknown in this continent), that or nothing. I was obliged to yield, and what was the consequence? In trying to throw light upon an obscure explanation in our text-book, my brain took fire, I plunged with re-quickened zeal into a subject which I had for years abandoned, and found food for thoughts which have engaged my attention for a considerable time past, and will probably occupy all my powers of contemplation advantageously for several months to come.

I remember, too, how, in like manner, when a very young professor, fresh from the University of Cambridge, in the act of teaching a private pupil the

* See Appendix [p. 85 below].

simpler parts of Algebra, I discovered the principle now generally adopted into the higher text books, which goes by the name of the "Dialytic Method of Elimination." So much for the reaction of the student on the teacher*. May the time never come when the two offices of teaching and researching shall be sundered in this University! So long as man remains a gregarious and sociable being, he cannot cut himself off from the gratification of the instinct of imparting what he is learning, of propagating through others the ideas and impressions seething in his own brain, without stunting and atrophying his moral nature and drying up the surest sources of his future intellectual replenishment.

I should be sorry to suppose that I was to be left for long in sole possession of so vast a field as is occupied by modern mathematics. Mathematics is not a book confined within a cover and bound between brazen clasps, whose contents it needs only patience to ransack; it is not a mine, whose treasures may take long to reduce into possession, but which fill only a limited number of veins and lodes; it is not a soil, whose fertility can be

* Not to speak of professor on professor. Thus it was in order to be able to meet the threatened interrogatories of my valued colleague, the irrepressible Mr Rowland, that I was led, on my return passage to England last summer, to look into Prof. Clerk Maxwell's extremely valuable, but ill-digested and somewhat unduly pretentious treatise on Electricity and Magnetism, which led to my theory of the Bipotential, and to my writing the paper published in the *Philosophical Magazine* for October last, which ought to have the effect of causing the author to rewrite one of his leading chapters on Statical Electricity.

I have at present a class of from eight to ten students attending my lectures on the Modern Higher Algebra. One of them, a young engineer, engaged from eight in the morning to six at night in the duties of his office, with an interval of an hour and a half for his dinner or lectures, has furnished me with the best proof, and the best expressed, I have ever seen of what I call the Law of Concomitant Interchange, applicable to permutation systems, *i.e.* the law which affirms that every complete set of permuted elements may be separated into two parts, or if we like to say so, be presented in the form of a diptych with two precisely similar Alæ, such that a single interchange between any two elements is accompanied with a total interchange between the two Alæ. This is the theorem which lies at the basis of the great theory of simple equations, which every school-boy is supposed to understand, but which was not really made out until a bevy of great Mathematicians, including Leibnitz, Laplace and Lagrange, had turned their attention to the subject. Jacobi, I have read somewhere, used to say that if he at all excelled other mathematicians, it was chiefly due to his greater facility in manipulating simple equations that he owed it. The same Jacobi, who, I remember, visited our English Cambridge, and so much relished the Trinity audit ale which he drank there, and who once being asked whether he was brother to the eminent physicist, Professor Jacobi, of St Petersburg, replied: "Quite the contrary—he is my brother." And *apropos* of the zeal of the student in question, let me mention for the benefit of my English friends, I have been agreeably surprised to find how widely diffused a spirit there exists in this country of disinterested love of learning. Out of Italy, especially Tuscany, where my friend Enrico Betti, as I had the opportunity of observing, and in his own country too, where no man is supposed to be a prophet, the neighbourhood of Pistoja, as a Professor is more influential, more honored and courted than he could be if he were a rich Marquis, I believe there is no nation in the world where ability with character counts for so much, and the mere possession of wealth (in spite of all that we hear about the Almighty dollar), for so little as in America, with exception it may be of certain of the Trans-Atlantic cities, which are really only colonies and emporiums for the trading classes of Europe.

exhausted by the yield of successive harvests; it is not a continent or an ocean, whose area can be mapped out and its contour defined: it is limitless as that space which it finds too narrow for its aspirations; its possibilities are as infinite as the worlds which are forever crowding in and multiplying upon the astronomer's gaze; it is as incapable of being restricted within assigned boundaries or being reduced to definitions of permanent validity, as the consciousness, the life, which seems to slumber in each monad, in every atom of matter, in each leaf and bud and cell, and is forever ready to burst forth into new forms of vegetable and animal existence.

I think that I am not claiming too much for my own special pursuit when I affirm that every science becomes more perfect, approaches more closely to its own ideal, in proportion as it imitates or imbibes the mathematical form and spirit. It is, therefore, I think, a just cause of congratulation to us that as shown by our official returns and the evidence of those best acquainted with the aims and pursuits of our students, their interest and their proficiency in mathematics is, to say the least, unsurpassed by that which is evinced by them in any other department of instruction carried on within our walls. Many gentlemen who have graduated years ago in other colleges have come up to us with the sole or principal object of continuing and extending their mathematical studies.

I have reason to think that the taste for mathematical science, even in its most abstract form, is much more widely diffused than is generally supposed over this great continent, and that there is really a demand for the higher instruction which we are, or hope to be, prepared to give.

I know that in response to a circular letter inviting opinions as to the expediency of founding a mathematical journal of a high character under the auspices of the authorities of this University, we have received some scores of replies expressing deep interest in the proposed undertaking, with hopes for its speedy realization, coupled with distinct pledges of support and co-operation.

Such a journal, I venture to vaticinate, would not fail to receive contributions from mathematicians of the highest eminence in Europe, and would form a new chain of connection, of which this University would hold the leading link in its hand, between America and the other nations of the world which lead the van of science.

In contributing my share to the matter and superintendence of this journal I should feel that I was discharging one important duty of my office. Another branch of my duty will consist, as now, in being open to communication, at a stated hour of the day, with all who wish to confer with me in relation to their studies.

A third branch of my duty will be to deliver a succession of lectures on subjects either of special interest in themselves, or in which I may happen to possess what may seem to me to be new views, or in which I may have succeeded in making discoveries of any general interest.

I ought not to omit to mention here the invaluable aid which I derive from the concurrence of the gentlemen associated with me in the work of mathematical instruction carried on under my general direction.

My associate, Dr Story, has had the advantage of studying for a long course of years in more than one German university, and I can speak, from personal attendance on one of his courses of lectures, from which I have derived both pleasure and instruction, of his thorough mastery over many of the most important and difficult branches of mathematical science.

Our students have thus the advantage of being put in direct communication with, and made participants of, all that has been done and is doing in that classical land of learning, in the way of mathematical research. In thoroughness of exposition, whatever may be the case as regards lucidity of presentation or spontaneity of initiative, I need hardly add my testimony to the general verdict of the world that our Teutonic brethren occupy the foremost rank. Many important *lacunæ*, which I should find otherwise a difficulty in filling up out of my own intellectual resources, are thus completely and efficiently supplied. Added to this, one of the most promising of our Fellows has lent his co-operation in bringing up to the standard of our University instruction such of our junior members as have come here insufficiently prepared, either from a too-short course of study or the lack of competent instruction in the schools or colleges in which they have received their preliminary education, and I am happy to be able to state that our trustees, with wise liberality, have recognized his services by raising him at once to the rank of a stipendiary lecturer.

Any one who will look through the syllabus of the lectures, not merely announced, but *bona fide* delivered and followed by attentive audiences within our walls, will see how respectable a range our courses of mathematical instruction comprehend: Analytical Geometry, Determinants, the Theory of Equations, the Differential and Integral Calculus, Definite Integrals, Rational Mechanics, Thermodynamics, the Theory of Elasticity and Modern Higher Algebra, are the subjects which have been actually taught here to smaller or larger classes of diligent students within the last few months.

Various other courses have been announced and will form part of our programme of instruction in this or future years.

The mention of Germany brings to my mind the importance of universities to the maintenance or development of a national spirit in the countries in which they are fostered and carried on with an animus free from local or sectarian prejudices.

I think that there can be little doubt that the greatest fact in modern history, the resuscitation of the German Empire, the resurrection of the

German people, is mainly to be attributed to the feeling of brotherhood and the spirit of nationality kept alive in those ganglions of thought, those centres of intellectual activity, the German universities.

It is the university professors who have made German unity a possibility, and I cannot but deplore the unpatriotic short-sightedness of those in my own country who, until so late a period, have struggled, and still covertly struggle, to make our universities in England not the representatives of the universal English mind, but the monopoly of a party and the appanage of a sect.

Their work it is that a separation deeper and a chasm more difficult to fill up has been created between the two most free and powerful nations in the world, England and America, than any due to political causes present or past.

Not the strained prerogative of a well-meaning but obstinate and narrow-minded monarch, nor the subservience of his ministers, nor the echoing voice of a misguided people it is, which has set up a permanent wall of separation between these two countries, a separation not founded on any opposition of material interests, but striking to the very groundwork of our mental constitution. Why is it that the flower of American youth resort not where the ties of a common language and of a common kindred would naturally have attracted them, to our English universities, to receive their mental impulse and their higher education, not to Oxford or Cambridge, but to Berlin, Leipzig, Göttingen, Jena or Heidelberg?

It is because there they were welcomed to whatever religious communion they were attached or unattached, without question and without distinction. It is because there they could rest on the bosom of a common mother, who shows kindness to all and favor to none.

If German professors have made Germany what it is, England may thank the narrow-minded class, or section of a class, of its university professors and chiefs (for there are numerous and nobleminded examples of English university leaders who combine the highest genius with the most liberal views; think of the Herschels, the Peacocks, the Sedgwicks, the De la Prymes, the Babbages, the Henslows and Lubbocks of the past, the Sidgwicks, the Stanleys, the Jowetts, the Liddells, the Brodies, the Mark Pattisons, the Prices, the Henry Smiths, His Grace of York, and many other illustrious men, leaders of thought, children of light, of the present generation,)—England, I say, may thank the obscurantist class of her university professors and heads, if the right arm of her spiritual power is shortened—if she is now, and it is to be feared will long remain, so much inferior in intellectual weight and influence in the world to what she ought to, and but for them would have been. They it is who, surrendering to party what was meant for

mankind, and laboring to cut out an English university upon the pattern of the University of Salamanca, have made a rent in the garment that should have been without seam, and alienated from us the intellectual sympathy of a mighty and kindred race. Driven to bay, like Rizzio at Holyrood, cowering behind *their* chairs and covered with *their* academic gowns, Intolerance found its last refuge and received, or is destined to receive, its last stab. Yet we shall probably live to see, as we have seen on former occasions, on the principle of "setting the cat to watch the cream," those very same men entrusted with the task of carrying out and shaping the promised university reforms who have passed their lives in endeavoring to frustrate or avert all substantial reforms up to this time.

I have been struck, almost from the first hour of my landing on these shores, by the manifestations I have everywhere witnessed of the close scholarly alliance which has sprung up between America and Germany. It is German books that are read, German professors who are quoted, German opinion on all matters of science and learning that is appealed to; and as regards community of work and intellectual ties, I do not think it at all extravagant to assert that Germany and America belong to one hemisphere, and we in England to another. If the English and American minds are ever again to be brought into contact, it will have to be on neutral and German soil.

I am old enough to remember when the great universities of England affixed their corporate seals to petitions to Parliament praying that the Crown would refuse to grant a charter to the University of London, then in the course of being founded, to enable it to give degrees, and that, too, at a time when, within their own walls, in many or most of the colleges, a religious test applied even to the admission of students*, and when no student, not a member of the Anglican communion, could be admitted to take a degree, so that not only would the universities not confer their own degrees, but they labored to prevent all Englishmen unwilling to sign the Thirty-nine Articles, from obtaining degrees elsewhere.

Then followed a struggle against Mr James Heywood's bill to open the degrees of the old universities to members of every faith; and in the third stage of this protracted contest, after the awakened intelligence and conscience of the magnanimous English people had overruled the monkish objections of the professorial and other chiefs of the retrograde party, the official head of

* The tutor on "one of the sides" at Trinity College, Cambridge, acting under the express directions of Dr Whewell, the then Master of the College, made strict inquisition of a gentleman, now occupying a Professor's chair in the University of Oxford, whether he professed the faith in which the founder of Christianity was educated, as in that case he must refuse to admit him as a student of the College. If I am not mistaken, the tutor in question was the present highly estimable and learned Master of the College. This incident was reported to me by the gentleman to whom and at the time when it occurred.

Physical Science in my own Alma Mater (for as such and not as an Injusta Noverca, or as a neglectful nurse who leaves her helpless charge whilst she perambulates with others more dear to her, will I ever continue to regard and cherish her) not merely signed, but was, (as I have been credibly informed, the projector and originator, and to my certain knowledge,) the active and leading canvasser for signatures to a petition* to the two Houses of Parliament to estop all others but members of the Church of England from holding any office of instruction in the university! This happened only a very few years ago.

Such is the blinding and blighting effect of early sectarian influences, one-sided culture, and narrow partisan connections, even on minds of a superior intellectual order, and on dispositions amiable by nature. There is a black drop of gall, a taint of congenital rancour and animosity, which infects all it comes in contact with, more indelible, more difficult to wring out or efface, than that dread smear on Lady Macbeth's hand, which could "the multitudinous sea incarnadine."† I doubt not that those who have taken this part, so prejudicial to their country's welfare, believe themselves to have been actuated by honest motives, just as I should not hesitate to admit that Torquemada was actuated by such and believed that he was doing a work acceptable to God when torturing heretics or presiding at the celebration of one of those *Auto da fé's* more horrid but scarcely more brutalizing than the bull-fights which I have seen supply their place.

It is difficult to estimate the lengths to which human self-delusion can be carried. No one questions that a great English statesman believes that he is prompted by the purest motives of philanthropy and patriotism when

* I ought to have said "to *two* petitions," one to shut out non-Anglicans from offices of emolument in the Colleges, the other to shut them out from Professors' Chairs in the University. I can understand upon what grounds (mistaken as they may appear to me) it may have been thought right to retain the management and endowments of the Colleges in the hands of a single denomination, but am really at a loss to conceive what reasonable plea can be offered for petitioning Parliament to exclude any one from teaching Anatomy, Latin and Greek or Mathematics in the University, who should happen not to say his prayers out of the same prayer-book as the signatories to the petition—a petition more worthy, it seems to me, to have proceeded from the members of some red-hot Irish Orange Lodge than from sober-minded Professors in a great National English University.

† A young gentleman, born in County Antrim, near the Giant's Causeway, was one night returning home from a dinner party with three-cornered hat and frilled shirt, "flushed," as we may suppose, not "with the juice of the Tuscan grape," but with run-claret or *crooked* potheen, when, in passing over a narrow plank bridge that spanned a mountain torrent, he espied coming towards him an aged priest with lantern in hand, probably on his way to perform some silent deed of charity and mercy. He did *not* throw him over the bridge, but for two years afterwards felt much troubled in mind at the thought of having let slip so favorable an opportunity of doing a good deed. He subsequently emigrated to America, where he often recounted the story, and lived to shudder at the temptation to which he had so nearly succumbed. I have taken this account from the lips of his grandson, one of the most respected and enterprising citizens of Baltimore.

agitating to paralyze a government and overthrow a rival odious to him, and to cast his country at the feet or into the arms of an insidious suitor and foe, or that one who treads in his footsteps believes that he is acting with a single eye to the interests of education when he cries up the University of London, and, like Ham, mocking the nakedness of his parent, with sublime self-abnegation or *matchless* cynicism, derides the venerable university where he was nurtured, where he taught, and which gave him his start in life.*

I think that you in this favored land are so far educated out of such pseudo-religious and antisocial views (survivals of a bygone age), that you will feel almost prompted to doubt the veracity of my statements, or the faithfulness of my recollections on the subject, and I am certain that not a score of signatures could be gathered to any document of such a nature in this country, were the continent canvassed from Maine to Florida, or from Chesapeake Bay to the shores of the Pacific. If I speak with some warmth on this subject, it is because it is one that comes home to me—because I feel what irreparable loss of facilities for domestic and foreign study, for full mental development and the growth of productive power, I have suffered, what opportunities for usefulness been cut off from, under the effect of this oppressive monopoly, this baneful system of protection of such old standing and inveterate tenacity of existence. I cannot easily express myself at any length in cold blood, but require to be warmed by a sense of personal interest in my subject, when I venture to address a public audience; with me *facit indignatio versus*, nor can I sit down to compose except in conformity with the dictates of the Muse to the impassioned Sidney, "Fool! look in thy heart," she said—"there learn to write."

Happy the young men gathered under our wing, who, unfettered and untrammeled by any other test than that of diligence and attainments, have here afforded to them an opportunity of filling up a complete scheme of education, such as a Milton or a Locke would have deemed adequate to their ideal.

How rejoiced should I be, were I of less ripe years and under less peremptory obligations as to the disposal of my time, branching out from mathematics as my natural mental centre of gravity, to diverge into the physical and chemical studies which lie so near to it, and which there are here such ample means accorded of studying under the most competent instructors, and with all the aids that modern ingenuity and the improvements in mechanical science can devise for putting direct questions to

* "Viewed her own feather on the fatal dart,
"And winged the shaft that quivered in her heart;
"Keen were her pangs, but keener far to feel
"She nursed the pinion that impelled the steel."
—*Byron's lines on the death of Henry Kirke White.*

Nature, and complementing and substantiating theory by visible and palpable Experience. For Experimental Physics, like the Practice of Gunnery, thanks in a great measure to the close alliance which the go-between Telegraphy has brought about between Science and Commerce, has in these days almost become a refined branch of Mechanical Engineering, very changed from those when a James Watt worked at his bench, when Priestley may have used a washhand basin for a Pneumatic Trough, or when a Woollaston could point to his cupboard as his Laboratory, and to a saucer holding a watch-glass, a lens and a blowpipe as his Philosophical Apparatus.

How delightful it were to be brought into contact with the treasures of antiquity and the music of the most perfect instrument of language, interpreted with Hellenic taste and wit and subtlest intellectual sympathy by my gifted colleague, whom you have just had the pleasure of listening to*, on whose lips all the bees of Hymettus seem to have settled and left their sweetest honey there; or, under the guidance of our enthusiastic and accomplished junior associate, penetrate to the foundations of our Indo-Germanic tongue; or, if that were not a dream too bright to be realized, to be led to the pure well of English undefiled, by one whose stay among us is, alas! only too short and transitory, who has effected among us what the rewards offered by an Eastern potentate were incompetent to bring forth—the Invention of a New pleasure—the eminent Chaucerian scholar, to whom I shall ever feel I owe a heavy debt of gratitude in supplying an unfailing source of delight, a pillow of repose for my declining years, in bringing to my knowledge and teaching me how to read and enter into the charm of another, a fresher and earlier Shakspeare!

'Αλλ' οὐ γὰρ αὐτὸς πάντ' ἐπίστασθαι βροτῶν
Πέφυκεν.

Even as the case stands, could our trustees but see their way to the institution of a certain number—I must not say of new mathematical chairs, but of additional mathematical professors, through whom I might supplement my deficiencies, and with them interchange ideas and carry on joint studies—I know not where in the wide world, out of my own country, I could feel more content to abide, or where I could find more conveniently within my reach all the materials for a complete mental equipment than within these walls, in this Temple of the Muses, in this free and law-abiding and

* Mr Gildersleeve, late Professor of Greek in the University of Virginia, who, like all of my other American colleagues, has drunk deep out of and been baptized in the perennial springs, in which whoever has been dipped comes out twice the man he was before he went in, which bubble up in those sacred precincts of science, the German Universities, and whose grammatical and other works are familiar to all scholars. I am informed that he is at present engaged in completing a Magnum Opus on Greek Syntax, which is likely to give him constant occupation for the next four or five years to come.

hospitable land—a land, to borrow the words of a fantastic but, to me, sympathetic rhymester,

Where tost bark a haven may find,
And new earth its roots to bind,
Drawing sap with instinct blind,
Willow stooping to each wind,
Oak, the monarch of its kind.

I thank you, ladies and gentlemen, for the kindness with which you have listened to my feeble utterances and, bird of night, given up to moping and brooding on my solitary perch, gladly make way for the lark, the herald of the morn, the star conspicuous amidst the effulgence of those Northern Lights*, which, "shooting madly from their spheres" for the last month past, have wandered into *our* latitudes and glowed in *our* sky, the poet whose name is honored wherever the English language is spoken or read, the author of the "Biglow Papers" and the "Ode to Washington."

APPENDIX.

There are three methods of treating the question of the Scale of Fundamental Invariants and Covariants—the *realistic*, the *symbolic* and the *fatalistic* or *peprotic*. In the first of these methods (the explicit or realistic) the derived Quantics, set out in full or abridged, through the intervention of canonical forms, are dealt with. It was thus that I established the scale for Ternary Cubics and for Binary Quartics and Quintics, in my early papers in the *Philosophical Magazine*, the *Cambridge and Dublin Mathematical Journal*, and in my Trilogy, published in the *Philosophical Transactions*. In the second, (the symbolic, schematic or embryonic method,) the derivations are not regarded as actually deduced, but are studied through the medium of the symbolic processes which gave the key to their existence (this is the method

* During the last month, Professors Childs and James Russell Lowell, both of Harvard University, have been giving lectures on Chaucer and Dante at Johns Hopkins University, and Mr Norton, Professor of the Fine Arts, and Mr Fiske, Assistant Librarian at Harvard, on English Cathedrals and the Aryan language and myths at the Peabody Institute. Professor Whitney is at present holding under the spell of his eloquence an audience of some hundreds of people, of both sexes, at the University, with lectures on the History of the Inflexional Structure of the Indo-European languages. There are many ladies in Baltimore who know Greek, and some who are about to enter upon a course of Sanskrit; others whose skill in singing and playing would command attention in any European concert room; and I have heard Professor Childs say that he never was in any city in the world where there was so pronounced a dramatic instinct as in Baltimore. Not to speak or read French and German is rather the exception than the rule. I mention these facts in order that my friends in Europe may well understand that my lot has not been cast among a barbarous or uncultivated race, and that the University has been planted in a congenial soil. Professor Lowell's recitation of his poem on Washington, at the Johns Hopkins Commemoration, "moved many of the audience, men as well as women, even to tears."

pursued with so large a measure of success by Prof. Gordan). The third, (the Deontological or Peprotic,) which precedes the one last named in the order of time, is the method indicated by Professor Cayley, in his memorable Second Memoir on Quantics, published in the *Philosophical Transactions*, which, owing to an error in its application, committed by its illustrious author, has fallen into neglect, and even the validity of whose substratum has been called into question. In this method the qualities of the derived forms, and the modes in which they can be brought into existence, are equally ignored: they are treated as mere Arithmetical existences, and, through the medium of that subtlest of all instruments for putting Nature and Reason to the question—a Partial Differential Equation—the numerical laws to which they are subject are made to depend on a problem in the Partition of Numbers.

This is the method followed in my researches, the validity of which I have established on an irrefragable basis, and which I have extended and modified so as to recover, by its aid alone, all the results obtained by Professors Clebsch and Gordan, and to go beyond them in showing how, with very great probability, (for at present I have not completed a strict apodictic proof of my Cardinal Principle,) an Algebraical Limit may be set to the degree and order of the Fundamental Derivations.

In order that I may not be supposed to be making a gratuitous assertion, I subjoin the Complete Generating Function, by means of which I can obtain the fundamental Invariants and Covariants given in Clebsch's Binären Formen for the Binary Sextic, demonstrate (with the aid of my Cardinal Principle) that there are none others, and establish all the fundamental Syzygants by which they are connected; for it ought to be noticed that alongside of the problem of determining the fundamental scale of Invariants or Covariants, there is the correlative problem, equally deserving of a solution, of determining the fundamental Syzygants, *i.e.* those rational integral functions of the Invariants or Covariants which, expressed in the terms of the coefficients of the Primary Quantic, are identically zero. I find that the total number of Covariants of the order m in the coefficients, and of n in the facients, (of course when n is zero the Covariants become Invariants,) is the coefficient of $t^m . v^n$ in the fraction whose Denominator is

$$(1-t^2)(1-t^4)(1-t^6)(1-t^{10})(1-t^2v^4)(1-t^2v^8)(1-tv^6)$$

and whose Numerator is

$$\begin{aligned}(1+t^{15}) &+ (t^3+t^5+t^7+t^8+t^{10}+t^{12})v^2\\ &+ (t^4+t^5+t^6+t^7+t^8+t^9+t^{10}+t^{11}+t^{13}-t^{17})v^4\\ &+ (t^3+t^4+2t^6+t^8+t^9+t^{11}-t^{16})v^6\\ &+ (t^3+t^5+t^7-t^{13}-t^{15}-t^{17})v^8\\ &+ (t^4-t^9-t^{11}-t^{12}-2t^{14}-t^{16}-t^{17})v^{10}\\ &+ (t^3-t^7-t^9-t^{10}-t^{11}-t^{12}-t^{13}-t^{14}-t^{15}-t^{16})v^{12}\\ -(t^5+t^{20})v^{16} &- (t^8+t^{10}+t^{12}+t^{13}+t^{15}+t^{17})v^{14}.\end{aligned}$$

I am thus enabled to show that the fundamental Covariants and Invariants are composed of two classes—Primaries, which are got from the Denominator, and Secondaries, from the Numerator—a distinction of the utmost importance, but which does not disclose itself in Professor Gordan's method. And it ought to be noticed that besides the problem of forming the fundamental scale, there is the not less important one in all cases of determining the *total* number of Invariants and Covariants of any given degree and order to which Gordan's method gives no general clue, but which is absolutely and completely resolved by my extension of the Peprotic method, above indicated. I repeat emphatically that no table of fundamental Invariants and Covariants will serve to calculate the *total number* of a given order and degree in the absence of a correlative table of the fundamental syzygies.

So completely had the Peprotic or partial-differential-equation method fallen into discredit, that I believe no allusion is made to it in Clebsch's work; only a slight reference is made to it in a note in Dr Salmon's new edition of his Modern Higher Algebra, and a condemnation is passed upon it by Professor Faà de Bruno, in his Treatise on Binary Forms, in so far as regards its application to Covariants of Quantics exceeding the fourth degree.

11.

ON A GENERALIZATION OF TAYLOR'S THEOREM.

[*Philosophical Magazine*, IV. (1877), pp. 136—140.]

CONNECTED with the study of the Theory of the symmetrical functions of the differences of the roots of an Algebraical Equation, a theorem presents itself in Dr Salmon's *Lessons on Higher Algebra*, 3rd edition, p. 59, art. 63, only partially indicated and insufficiently demonstrated there, which on a closer inspection will be found to be well deserving of notice as containing a true generalization of Taylor's theorem, leading to a development of the same form, subject to a like law of convergence, and easily demonstrable by the same method as that theorem.

Let f be *any function whatever* of $a, b, c, \ldots$, and f_1 the same function of $a_1, b_1, c_1, \ldots$, where

$$a_1 = a, \quad b_1 = b + ah, \quad c_1 = c + 2bh + ah^2,$$
$$d_1 = d + 3ch + 3bh^2 + ah^3, \ldots$$

and let Ω represent the operator

$$a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{d\,.\,d} + \ldots;$$

then the theorem in question affirms that

$$f_1 = f + \Omega\,.\,fh + (\Omega\,.)^2 f\frac{h^2}{1\,.\,2} + (\Omega\,.)^3 f\frac{h^3}{1\,.\,2\,.\,3} + \ldots.$$

On making $a = 1$, $b = x$, $c = 0$, $d = 0 \ldots$, the theorem becomes Taylor's. To prove it in its general form, let

$$\phi x = ax^n + nbx^{n-1} + n\frac{n-1}{2}cx^{n-2} + \ldots;$$

then, on substituting $x + h$ for x, ϕx becomes

$$= a_1x^n + nb_1x^{n-1} + n\frac{n-1}{2}c_1x^{n-2} + \ldots.$$

Let h become $h + \delta h$, then obviously

$$\delta f_1 = \frac{d}{dh}f_1\delta h.$$

But we may obtain the new values of $a_1, b_1, c_1, \ldots$ corresponding to the change of h into $h + \delta h$, by substituting in ϕx first $x + \delta h$ and then $x + h$ for x.

The effect of the first substitution is to change $a, b, c, \ldots$ into $a + \delta a$, $b + \delta b$, $c + \delta c, \ldots$, where

$$\delta a = 0, \quad \delta b = a\delta h, \quad \delta c = 2b\delta h, \quad \delta d = 3c\delta h, \ldots.$$

Hence the increment

$$\delta f_1 = \left(a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{d\,.\,d}\ldots\right)f_1\,.\,\delta h;$$

consequently

$$\frac{d}{dh}f_1 = \Omega\,.\,f_1.^*$$

Hence, if we write

$$f_1 = f + Bh + Ch^2 + Dh^3 + \ldots,$$

we shall have

$$B + 2Ch + 3Dh^2 + \ldots$$
$$= \Omega f + \Omega Bh + 2\Omega Ch^2 + \ldots.$$

Hence $\quad B = \Omega\,.\,f, \quad C = \tfrac{1}{2}(\Omega\,.)^2 f, \quad D = \dfrac{1}{2\,.\,3}(\Omega\,.)^3 f \ldots;$

and consequently

$$f_1 = f + \Omega\,.\,f + (\Omega\,.)^2 f\frac{h^2}{1\,.\,2} + (\Omega\,.)^3 f\frac{h^3}{1\,.\,2\,.\,3} + \ldots,^\dagger$$

and the first part of the theorem is demonstrated. It will of course be understood that $(\Omega\,.)^i$ means not $(\Omega^i)\,.$, but $\Omega\,.\,\Omega\,.\,\Omega\,.$ (to i factors).

* Or without introducing ϕx, the equations between $a_1, b_1, c_1 \ldots$ and $a, b, c, \ldots$ show by direct inspection that the effect upon the former is the same, whether we augment h by δh or $b, c, d \ldots$ respectively and simultaneously by $a\delta h$, $2b\delta h$, $3c\delta h, \ldots$ so that $\frac{d}{dh}f_1 = \Omega\,.\,f_1$, as in the text.

† Consequently, if Ωf vanishes, since also $(\Omega\,.)^i f$ will also vanish for all values of i, we shall have $f_1 = f$. It is this fact of $(\Omega f = 0)$ being the complete solution of $(f_1 = f)$ which constitutes the importance of the theorem in the Calculus of Invariants.

Lagrange's or any other rule for the Remainder in the old Taylor's theorem may be extended to this generalization of it; that is to say, if in the development of f_1 we stop at the nth term, the *remainder* will be equal to

$$\frac{h^n}{\Pi n}(\Omega\,.)^n f(\alpha, \beta, \gamma \ldots),$$

where $\alpha, \beta, \gamma \ldots$ are what $a_1, b_1, c_1 \ldots$ become when we write θh for h, θ being some proper positive fraction. The demonstration proceeds *pari passu* for the generalized form and for Taylor's case of it. Thus, consider Bertrand's proof as given in Williamson's *Calculus*, second edition, p. 64.

The *lemma* upon which the proof depends takes the form, that if f_1 (supposed continuous between two values of h) has the same value (zero, as it happens in the matter in hand) for two values of h, Ωf must vanish for some intermediate value of h; which is obviously true, since $\delta f = \Omega f \delta h$. The rest of the demonstration remains essentially the same, *mutatis mutandis*, at each point as for Taylor's theorem properly so called.

The theorem above established easily admits of extension to the case of $a_1, b_1, c_1 \ldots$ being the values assumed by $a, b, c \ldots$, when in the quantic $(a, b, c \ldots \between x, y, z)^n$ we substitute $x + hy + kz + \ldots$ for x. We may thus obtain a theorem which will bear to Taylor's theorem for any number of variables the same relation as the theorem given in the text to Taylor's theorem for a single variable.

Since the effect of changing x into $x + h + \delta h$ may be obtained either by first substituting $x + h$ for x and then $x + \delta h$ for x in ϕx, or by a reversal of the order of these two processes, we obtain the interesting consequence that the two operators

$$a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{d\,.\,d} + \ldots$$

and

$$a_1\frac{d}{db_1} + 2b_1\frac{d}{dc_1} + 3c_1\frac{d}{d\,.\,d_1} + \ldots$$

are absolutely identical,—a theorem which of course admits, but not without a somewhat complicated process, of an *à posteriori* direct proof; so that the operator Ω is to all intents and purposes what Professor Cayley calls a semi-invariant or pene-invariant, but to which I am accustomed to give the name of a *differentiant* to ϕx.

Finally, it may be observed that a development for f_1 may be obtained by the use of the ordinary Taylor's theorem for *several* variables. If we make use of this method, and write in addition to

$$\Omega = a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{d\,.\,d} + \ldots,$$

$$\Omega_1 = a\frac{d}{dc} + 3b\frac{d}{d\,.\,d} + 6c\frac{d}{d\,.\,e} + \ldots,$$

$$\Omega_2 = a\frac{d}{d\,.\,d} + 4b\frac{d}{d\,.\,e} + 10c\frac{d}{d\,.\,f} + \ldots,$$

&c. = &c.,

we shall obtain the noteworthy symbolical and absolute identity

$$e^{h\Omega\,.} = e^{h\Omega + h^2\Omega_1 + h^3\Omega_2 + \ldots}\ *,$$

which may be verified, but not without some little trouble, by direct expansion.

If we use $\Omega\,!$ to signify that Ω is to be used as a pure operator on the matter coming after it (operating that is to say solely on the symbols of quantity $a, b, c, \ldots$ and not on the operators $\frac{d}{da}, \frac{d}{db}, \frac{d}{dc} \ldots$), we shall have

$$\Omega_1 = \frac{(\Omega\,!\,\Omega)}{1\,.\,2}, \quad \Omega_2 = \frac{(\Omega\,!\,\Omega\,!\,\Omega)}{1\,.\,2\,.\,3},$$

and so on. Hence the "noteworthy" symbolical equation above written may be put under the hypersymbolical form

$$e^{h\Omega\,.} = e^{(e^{h\Omega\,!} - 1)\frac{\Omega}{\Omega\,!}}$$

a suggestive identity that may possibly call forth a sneer from the mathematical cynic, but not from the thoughtful mathematician, who, aware that algebra is in its essence a language which it is the proper business of his art to fathom and develop, is prompt to recognize every step in expression as a gain in power.

* If we write

$$\Delta = (-\Omega\,. + \Omega)\,h + \Omega_1 h^2 + \Omega_2 h^3 + \ldots,$$

we ought to have $e^{\Delta} - 1 = 0$, and the coefficients in the expansion of $e^{\Delta} - 1$ according to ascending powers of h ought all to vanish identically; and so they will be found to do, provided that in each such coefficient expressed as the sum of the product of powers of $\Omega_0, \Omega_1, \Omega_2, \ldots$ and of $\Omega\,.$ the power of the *dotted* Ω be taken last in order. As soon as that expansion is made (but of course not before) we may write $\Omega\,. - \Omega = 0$, and we may readily calculate *à priori* the value of each power of $(\Omega\,. - \Omega)$; thus we shall obtain

$$(\Omega\,. - \Omega)^2 = \Omega\,.^2 - 2\Omega\Omega\,. + \Omega^2 = \Omega\,.^2 - 2\Omega^2 + \Omega^2 = 2\Omega_1;$$

and so by a similar calculation, having first determined $\Omega\,.^2$, $\Omega\,.^3$, $\Omega\,.^4$, &c., we shall obtain

$$(\Omega\,. - \Omega)^3 = 6\Omega_2, \quad (\Omega\,. - \Omega)^4 = 24\Omega_3 + 12\Omega_1^2, \text{ \&c.};$$

on substituting these values in $\Delta + \frac{\Delta^2}{1\,.\,2} + \frac{\Delta^3}{1\,.\,2\,.\,3} + \ldots$ the coefficients of the several powers of h will be found to vanish.

The appearance in the above process of a zero whose powers are not zero is a phenomenon which will not shock those who are acquainted with Professor Peirce's discussions of possible algebras; but it is new to find it occur in working out a symbolical identity.

The theorem $f_1 = e^{h\Omega} \cdot f$ having, as far as I am aware, been first given by Dr Salmon in a form, if not quite complete, still sufficient for the immediate purpose to which it was to be applied, ought, I think, in justice to bear his name; and I see no reason why Salmon's Theorem in its totality should not be expected in the future to bear new fruit in algebraical expansions and other uses as important as have flowed from the one familiar and simplest case of it, known as Taylor's Theorem. Thus, *ex. gr.*, for the special case where f_1 becomes a function of one only of the quantities b_1, c_1, ... the Salmonian theorem reproduces Arbogast's celebrated one for expanding a rational integral function by the method of derivations, but under a greatly improved form of notation, and with the advantage of a test of convergency supplied by the *limit to the remainder* given in the text above. Who on a first casual reading could have imagined that Arbogast's problem in the differential calculus was virtually solved in an improved form in an article treating "on the symmetrical functions of the differences of the roots of an equation"? "*Que diable allait-il faire dans cette galère là?*" may rise to the lips of many a reader on being made acquainted with the fact*.

* Using Q to denote any rational integral function of x, Salmon's theorem is a theorem for expanding any function of Q, $\frac{dQ}{dx}$, $\frac{d^2Q}{dx^2}$, ... in terms of ascending powers of x.

12.

SUR LES INVARIANTS.

[*Comptes Rendus*, LXXXV. (1877), pp. 992—995, 1035—1039, 1091—1092.]

La théorie que j'ai exposée dans mes dernières Communications à l'Académie repose sur le théorème suivant. Commençons par le cas d'une seule quantique du degré i, fonction des variables x et y, soit $(a, b, c, \ldots, l)(x, y)^i$. Je nomme différentiant de cette quantique une fonction rationnelle et entière quelconque, qui retient sa valeur quand on substitue pour les coefficients de la quantique donnée les coefficients de la quantique qu'on obtient en substituant $x + hy$ pour x. Alors le nombre de ces différentiants de l'ordre j dans les coefficients et du poids w par rapport à x sera égal à la différence entre deux nombres dont l'un est le nombre de combinaisons de j quelconques des chiffres $0.1.2.\ldots i$ (répétées autant de fois qu'on veut) dont la somme est w, moins le nombre de combinaisons pareilles pour lesquelles la somme est $(w-1)$. Nommons l'opérateur $a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \ldots = \Omega$. La condition nécessaire et suffisante pour que D soit un différentiant est que ΩD soit identiquement zéro. De là on déduit facilement que le nombre des D linéairement indépendants, dont le poids est w et l'ordre s, soit $D\,(w : i, j)$, ne peut pas être moins que la différence dont j'ai parlé plus haut, soit la différence $(w : i, j) - \{(w-1);\ i, j\}$. Si les équations contenues dans l'identité $\Omega D = 0$ sont indépendantes, la valeur de $D(w : i, j)$ sera égale à

$$(w : i, j) - \{(w-1);\ i, j\};$$

si elles ne sont pas indépendantes, ce nombre sera *plus grand* que

$$(w : i, j) - \{(w-1);\ i, j\}.$$

Dans une Communication que je viens d'envoyer au *Journal de M. Borchardt*, j'ai réussi à donner une démonstration rigoureuse de l'égalité de $D(w : i, j)$ à la différence citée qu'on peut nommer $\Delta(w : i, j)$; car, si cette égalité n'était pas vraie pour toutes les valeurs de w, en commençant par la plus grande possible, c'est-à-dire $\frac{ij}{2}$ ou $\frac{ij-1}{2}$, alors on aurait pour cette valeur *maxima* de w

$$D(w : i, j) + D\{(w-1) : i, j\} + D\{(w-2) : i, j\} + \ldots + D(0 : i, j) > (wi, j),$$

laquelle inégalité ne peut pas avoir lieu, comme je le démontre par une méthode très-belle et très-facile. C'est à M. Cayley qu'on doit l'énoncé de la proposition $D(w:i,j)=\Delta(w:i,j)$; mais ce grand géomètre n'avait réussi qu'à démontrer rigoureusement l'inégalité $D(w:i,j) = \text{ou} > \Delta(w:i,j)$.

On avait même exprimé des doutes sur la vérité de la proposition, désormais mise à l'abri de toute objection, $D(w:i,j)=\Delta(w:i,j)$. Passons au cas de plusieurs quantiques $(a, b, c\ldots)(x, y)^i$, $(a, b, c\ldots)(x, y)^{i'}, \ldots$. J'ai étendu la méthode de M. Cayley à ce cas plus général. Par un procédé analogue au sien pour le cas d'une seule quantique, j'établis la proposition

$$D(w:i,j:i',j':\ldots) = \text{ou} > (w:i,j:i',j':\ldots) - \{(w-1):i,j:i',j':\ldots\},$$

où le premier membre de l'équation signifie le nombre de différentiants, linéairement indépendants, appartenant au système de quantiques donné de l'ordre $j, j', \ldots$, dans les quantiques successives et du poids w par rapport à x: $(n:i,j:i',j':\ldots)$, signifiant, pour une valeur quelconque de n, le nombre des combinaisons de j des chiffres $(0, 1, 2, 3, \ldots, i)$, de j' des chiffres $(0, 1, 2, \ldots, i'), \ldots$, dont la somme réunie est égale à n. Alors, par une méthode précisément identique avec celle que j'applique au cas d'une seule quantique, je démontre que l'inégalité

$$D(w:i,j:i',j':\ldots) + D\{(w-1):i,j:i',j':\ldots\} + \ldots \\ + D(0:i,j:i',j':\ldots) > (w:i,j:i',j':\ldots),$$

où w représente la valeur maxima du poids w, ne peut pas avoir lieu et que conséquemment, pour *toutes* les valeurs de w,

$$D(w:i,j:i',j':\ldots) = \Delta(w:i,j:i',j':\ldots).$$

Donc la théorie de la construction de la fonction génératrice dont je me suis servi reste aujourd'hui sur une base inattaquable. Mais, même en l'absence de cette démonstration nouvellement trouvée, l'évidence de sa vérité, fondée sur l'improbabilité *a priori* d'aucune dépendance sur les autres équations de condition données par la formule $\Omega D=0$, conjointe avec l'accord parfait des résultats obtenus, en les supposant indépendants, avec les résultats qu'on obtient par d'autres méthodes pour tous les cas où l'on pouvait faire la comparaison, suffisait provisoirement comme démonstration *morale* de la vérité supposée. Or, chose bien remarquable, une difficulté de même nature revient quand on se sert de la fonction génératrice non pas en l'appliquant au calcul du nombre des dérivées invariantes linéairement indépendantes d'un type donné, mais en déduisant par son moyen l'échelle des dérivées élémentaires (*grundformen*). En un mot, la difficulté qui, aujourd'hui, a disparu quant à la formation de la fraction génératrice subsiste encore quand on passe à l'interprétation de cette fraction qui conduit à l'échelle de *grundformen*, mais avec une certaine différence. Quant à la proposition qui vient d'être nouvellement démontrée, la difficulté autrefois

consistait à démontrer l'absence de rapports syzygétiques quelconques. Mais, dans l'application dont je parle, on admet par nécessité l'existence de certains de ces rapports, qui *se révèlent* comme conséquence de la loi élémentaire de toute combinaison algébrique d'invariants. L'hypothèse que l'on fait, c'est qu'il n'existe pas de tels rapports (pour ainsi dire *cachés*) en dehors de ceux dont l'existence est apparente.

Si l'on voulait nier l'exactitude de cette hypothèse, voici ce qui arriverait: les formes élémentaires (*grundformen*) obtenues en l'admettant ne cesseraient pas de subsister comme telles; seulement il y aurait la possibilité (pour ainsi dire métaphysique) de l'existence d'autres en plus. Prenons, par exemple, le cas de deux biquadratiques. M. Gordan en a donné 30, dont j'ai démontré que 2 sont superflues: il en reste donc 28. La méthode de M. Gordan ne suffit pas pour démontrer que ce nombre n'est pas encore assujetti à une réduction au-dessous de 28; mais ma méthode, au contraire, quoique laissant provisoirement peser un doute métaphysique sur l'existence de plus de 28, n'en laisse aucun sur la certitude qu'au moins ces 28 subsistent. Donc on est assuré que les 28 en question forment l'échelle fondamentale. La méthode de M. Gordan assure qu'il n'y a pas plus que 28, la méthode anglaise qu'il n'y a pas moins que 28 invariants et covariants élémentaires; donc le nombre est 28, ni plus ni moins. On comprend que l'incertitude dont je parle dans l'application de la méthode anglaise n'est que provisoire et, pour ainsi dire, métaphysique; l'évidence, à dire vrai, est accablante et ne peut laisser subsister aucun doute moral que les rapports syzygétiques cachés ou latents, dont j'ai parlé, n'ont aucun lieu dans la sphère de réalité. Cependant il semble bon de confirmer ce *postulatum*, en donnant encore des exemples, comme je vais le faire, de la conformité des résultats auxquels il conduit avec ceux qu'on obtient par d'autres méthodes. De plus, on doit se rappeler que chacune de mes fractions génératrices donne encore des résultats en dehors de la formation de l'échelle fondamentale, qu'on ne sait pas obtenir par la méthode de M. Gordan ni par aucune autre méthode connue. Elle donne absolument, et sans suggestion à aucun doute métaphysique, le nombre total des invariants, covariants, etc., les mouvements indépendants de degrés et d'ordres donnés, et, une fois la vérité absolue de la conclusion quant à l'échelle fondamentale pour un cas donné étant ou admise ou prouvée par l'évidence, elle donne en même temps et immédiatement tous les rapports syzygétiques qui peuvent lier ensemble les formes qui entrent dans l'échelle fondamentale. Bien plus, non-seulement les *grundformen* ne sont pas indépendantes, mais les équations qui les lient, en général, ne seront pas non plus indépendantes. Voici la vraie idée de ces rapports successifs:

On commence avec les *grundformen*. Alors il y aura des fonctions algébriques, qu'on peut nommer des *syzygants* du premier rang et qui auront la propriété de s'évanouir quand on substituera aux *grundformen* leurs valeurs comme fonctions des coefficients des quantités données. De même il y aura des fonctions algébriques de ces syzygants qu'on peut nommer des syzygants du second rang, qui auront la propriété de s'évanouir quand on substituera pour les syzygants du premier rang leurs valeurs comme fonctions des *grundformen*, et ainsi de suite, de sorte qu'il y aura une succession de syzygants de rangs de plus en plus élevés, et pour les syzygants de chaque rang il y aura une échelle fondamentale finie. Je crois que l'indice des rangs ascendants ne va jamais à l'infini. Sous ce point de vue, on voit que les formes fondamentales (*grundformen*) elles-mêmes peuvent être regardées comme des syzygants du rang zéro. Or ma fraction génératrice donne le moyen d'obtenir l'échelle fondamentale pour les syzygants d'un rang quelconque. Le procédé pour l'obtenir dans les cas du rang zéro et du rang unité est aussi simple pour l'un que pour l'autre. Quant aux syzygants de rang supérieur, le calcul peut être un peu plus compliqué, et je ne me suis pas permis jusqu'à présent d'entrer dans ce calcul. Il est singulier de remarquer l'inversion de rôles qui a lieu entre les deux problèmes, l'un de trouver les formes élémentaires et les syzygants successifs qui en découlent, l'autre de trouver le nombre total de formes dérivées d'un type donné. On aurait pensé *a priori* que la solution du premier problème serait nécessaire pour arriver à la solution du second. Mais, en réalité, la marche de l'investigation est toute contraire. Grâce à l'initiative admirable pour tout jamais de M. Cayley, dans son second Mémoire sur les *Quantics*, on sait comment résoudre d'un seul coup le second problème et de la forme même de cette solution on fait découler pas à pas la solution du premier.

Je vais donner les fractions génératrices pour trois nouveaux cas pour lesquels on peut comparer les résultats quant à l'échelle fondamentale avec des résultats déjà connus. Ces trois cas seront: (1) celui d'un système contenant une forme linéaire et une forme cubique; (2) d'un système contenant une forme quadratique et une forme cubique; (3) d'un système de deux cubiques. Dans une Communication prochaine, je donnerai la théorie qui s'applique aux cas d'un nombre indéfini de formes linéaires et d'un nombre indéfini de formes quadratiques. Entre ces cas il existe un lien vraiment surprenant. Je n'ai pas besoin de dire que, par rapport aux considérations qui limitent l'horizon des recherches de l'école allemande en matière de formes algébriques, ces deux cas n'offrent à peine aucune prise pour construire une théorie, ou pour mieux dire la théorie qu'on construit s'épuise en quelques mots; au contraire, selon les idées constituantes de la méthode anglaise, ces deux cas mènent à une théorie très-étendue et à des recherches du plus haut intérêt. En effet, le premier cas est celui de la théorie des rapports syzygétiques de fonctions des différences d'un nombre

quelconque donné de quantités, théorie qui doit réagir puissamment sur celles de formes de degrés quelconques; de plus, dans le traitement de l'un et l'autre cas, j'aurai occasion de donner une solution de certains problèmes de l'Algèbre ordinaire de la plus grande beauté, en faisant appel à des principes algébriques que je crois être d'un genre tout à fait nouveau.

Commençons par le cas d'un système composé d'une forme linéaire et d'une cubique. Le dénominateur de la fraction génératrice sous la forme canonique sera

$$(1-b^4)(1-b^2a^2)(1-ba^3)(1-ax)(1-b^2x^2)(1-bx^3),$$

où a est le symbole pour la fonction linéaire, et b pour la cubique. Ainsi il y aura six formes fondamentales primaires: L'invariant et la hessienne de la cubique, les deux formes données, leur résultant (typifié par ba^3) et le résultant de la hessienne et la forme linéaire typifiée par b^2a^2.

Le numérateur est

$$1 + a^3b^3 \qquad + (-ab^3 - a^4b^6)x^4$$
$$+ (a^2b + ab^2 + a^2b^3 - a^4b^3)x \qquad + (b^3 - a^2b^3 - a^3b^4 - a^4b^5)x^3$$
$$+ (ab + ab^3 - a^3b^3 - a^3b^5)x^2.$$

Les termes positifs ne perdent rien en étant assujettis au tamisage. Il reste donc sept formes fondamentales secondaires:

1 invariant typifié par	3 . 3 . 0
3 covariants linéaires	2 . 1 . 1 1 . 2 . 1 2 . 3 . 1
2 covariants quadratiques	1 . 1 . 2 1 . 3 . 2
1 covariant cubique	0 . 3 . 3

ce dernier appartenant à la cubique prise séparément.

Prenons, en deuxième lieu, le système composé d'une quadratique et d'une cubique. Le symbole a appartiendra à la première, b à la seconde.

La fraction génératrice, sous sa forme canonique, aura pour dénominateur

$$(1-a^2)(1-b^4)(1-ab^2)(1-a^3b^2)(1-ax^2)(1-bx^3)(1-b^2x^2)$$

et pour numérateur

$$(1 + a^3b^4)$$
$$+ (ab + a^2b + ab^3 + a^2b^3)x$$
$$+ (ab^2 + a^2b^2 + a^3b^2 + a^2b^4 - a^4b^4 - a^3b^6)x^2$$
$$+ (ab + b^3 - a^3b^3 - ab^4 - a^2b^5 - a^3b^5)x^3$$
$$+ (-a^2b^4 - a^3b^4 - a^2b^6 - a^3b^6)x^4$$
$$+ (-ab^3 - a^4b^7)x^5.$$

Le produit constant de chaque couple conjugué est, comme on voit, $-a^4b^7x^5$, et le rapport, qui est toujours constant entre les termes conjugués qui figurent dans la partie sans x, et la partie qui multiplie la plus haute puissance de x de ces fractions génératrices, est $-ab^3$.

Ainsi on a sept formes fondamentales primaires : les deux invariants des formes données, prises séparément ; deux autres invariants dont l'ordre, dans les coefficients de la quadratique et de la cubique, respectivement, est pour l'un (1, 2) et pour l'autre (3, 2), les deux formes données elles-mêmes et la hessienne de la cubique.

Quant au numérateur, on voit que les seuls coefficients positifs qui disparaissent sous le tamisage sont : $a^2b^2x^2$, $a^3b^2x^2$, $a^2b^4x^2$. Il reste les sept formes fondamentales secondaires, figurées par ces nombres :

1 invariant	3 . 4 . 0
4 covariants linéaires........	1 . 1 . 1 2 . 1 . 1 1 . 3 . 1 2 . 3 . 1
1 covariant cubique	1 . 2 . 2
2 covariants cubiques.........	1 . 1 . 3 0 . 3 . 3

ce dernier appartenant à la cubique donnée, prise séparément.

Comme dernier cas prenons le système composé de deux cubiques binaires ayant a et b pour leurs symboles.

Le dénominateur de la fraction génératrice canonique sera

$$(1-a^4)(1-b^4)(1-ab)(1-ab^3)$$
$$\times(1-a^3b)(1-ax^3)(1-a^2x^2)(1-bx^3)(1-b^3x^2)$$

donnant neuf formes fondamentales primaires dont les invariants et les hessiennes des cubiques données constituent 6 et en outre les trois invariants ayant pour symboles $ab : ab^3 : a^3b$. Son numérateur sera

$$1+a^2b^2+a^3b^3+a^5b^5$$
$$+(ab^2+a^2b+ab^4+a^4b+a^3b^2+a^2b^3+a^3b^4+a^4b^3)x$$
$$+(ab+ab^3+a^2b^2+a^3b+a^4b^4-a^3b^7-a^7b^3)x^2$$
$$+(a^3+a^2b+ab^2+b^3-a^4b-ab^4-ab^6-2a^3b^4-2a^4b^3-a^6b-2a^5b^6-2a^6b^5)x^3$$
$$+(ab-ab^5-a^2b^4-a^3b^3-a^4b^2-a^5b-a^2b^6-2a^3b^5-2a^4b^4$$
$$-2a^5b^3-a^6b^2-a^3b^7-a^4b^6-a^5b^5-a^6b^4-a^7b^3+a^7b^7)x^4$$
$$+(-2a^2b^3-2a^3b^2-a^2b^7-2a^4b^5-2a^5b^4$$
$$-a^7b^2-a^4b^7-a^7b^4+a^5b^8+a^6b^7+a^7b^6+a^8b^5)x^5$$
$$+(-ab^3-a^3b+a^4b^4+a^3b^7+a^6b^6+a^7b^3+a^7b^7)x^6$$
$$+(a^4b^5+a^5b^4+a^5b^6+a^6b^5+a^4b^7+a^7b^4+a^6b^7+a^7b^6)x^7$$
$$+(a^3b^3+a^5b^5+a^6b^6+a^8b^8)x^8.$$

On remarquera que le produit constant général pour les termes conjugués est ici $+a^8b^8x^8$; bien entendu que chaque terme précédé par un coefficient, disons k, doit être compté comme k termes avec le coefficient unité dont chacun aura été conjugué. On remarquera aussi le rapport constant de $1 : a^3b^3$ entre les quatre termes au commencement et les coefficients des quatre à la fin, et de plus le produit constant partiel pour ces deux groupes, c'est-à-dire a^5b^5 pour l'un, et conséquemment $a^{11}b^{11}$ pour l'autre. Ces trois théorèmes, le produit constant général, le produit constant pour la partie qui symbolise ces invariants et le rapport constant entre les termes de cette partie et les coefficients en nombre égal à la fin, sont des caractères permanents pour toutes les fractions génératrices dont on se sert dans le calcul des invariants, et qu'on peut démontrer *a priori*.

En soumettant les termes positifs au tamisage, on trouvera sans peine que les seuls qui restent seront les suivants :

a^2b^2, a^3b^3.
ab^2x, a^2bx, ab^4x, a^4bx, a^3b^2x, a^2b^3x, a^3b^4x, a^4b^3x,
abx^2, ab^3x^2, $a^2b^2x^2$, a^3bx^2.
a^3x^3, a^2bx^3, ab^2x^3, b^3x^3.
abx^4.

Donc il y a 19 formes fondamentales secondaires, *savoir* :

2 invariants	typifiés par	2.2.0 3.3.0
8 covariants linéaires...	„	1.2.1 2.1.1 1.4.1 4.1.1 3.2.1 2.3.1 3.4.1 4.3.1
4 covariants quadratiques	„	1.1.2 1.3.2 2.2.2 3.1.2
4 covariants cubiques...	„	3.0.3 2.1.3 1.2.3 0.3.3
1 covariant biquadratique	„	1.1.4.

Les résultats sont en parfait accord avec le résumé de M. Salmon, fondé sur les travaux de MM. Clebsch et Gordan : *Lessons on Higher Algebra*, 3e édition, p. 186, qui se trouvent ainsi pleinement confirmés, de sorte qu'on sait *apodictiquement* que rien de superflu ne peut être contenu dans leur Table des *Grundformen* pour ce cas-ci.

Ici, il est nécessaire de faire une remarque très-importante sur une omission d'un certain procédé, qui, dans ma méthode, doit précéder celui de tamisage. Cette omission n'a aucune importance pour les cas que nous avons considérés, car les circonstances qui rendent nécessaire l'application de ce procédé additionnel n'existent pas pour ces cas-là, et il semble souvent ne pas arriver que dans le cas où il y a un très-grand nombre de formes comprises dans le système donné, lequel nombre, apparemment, croît avec les degrés de ces formes. C'est dans l'étude des systèmes de formes linéaires ou quadratiques que ce phénomène, dont je vais parler, s'était présenté pour la première fois, et seulement quand ce système ne comprend pas moins de quatre formes. Dans toutes les huit fractions génératrices que j'ai données dans cette Note et dans mes Communications précédentes, on trouvera facilement que, si l'on développe ces fractions en séries, les coefficients positifs ne subiront pas une diminution quelconque. Mais, quand cela arrive, c'est-à-dire quand un tel coefficient ou disparaît, ou subit une diminution, alors il faut substituer, au lieu du coefficient dans le numérateur, le chiffre diminué (qui peut être zéro, mais, comme je l'ai démontré dans l'article* cité, destiné au *Journal de Crelle*, jamais négatif). Donc, comme règle générale (quoique presque jamais nécessaire dans la pratique), il faut soumettre chaque coefficient à cet examen, auquel je donne le nom de *triage*. Voici donc le tableau complet de mes procédés pour arriver à l'échelle des formes invariantes des dérivées fondamentales:

(1) Formation de la fraction génératrice dans sa forme crue dont le développement donnerait une série allant vers l'infini dans deux directions qu'on pourrait nommer série *bivergente*;

(2) Retranchement de la partie contenant des indices négatifs et substitution d'une fraction génératrice réduite, dont le développement en série sera *univergent*;

(3) Multiplication du numérateur et du dénominateur de la fraction réduite par un facteur commun propre à mettre le dénominateur sous une forme telle, que chaque facteur, comme $1 - a^{\alpha} b^{\beta} c^{\gamma} \ldots x^{\lambda}$, qu'il contient, correspondra à un covariant ou invariant, dont le type est $\alpha, \beta, \gamma, \ldots, \lambda$, laquelle condition sera satisfaite si, en faisant le développement en série, le terme $a^{\alpha} b^{\beta} c^{\gamma} \ldots x^{\lambda}$ ne se trouve pas aboli. La fraction est alors canonique;

(4) Triage appliqué à la diminution ou suppression des coefficients positifs du numérateur, quand cela est nécessaire;

(5) Tamisage appliqué aux coefficients ainsi triés.

Il est bon aussi de remarquer que, sans former la fonction génératrice, on peut appliquer ma méthode à la solution complète par des méthodes purement arithmétiques du problème suivant, qui, en effet, est la partie laissée incomplète dans la théorie de M. Gordan:

Étant donnés les types d'une assemblée de formes entre lesquelles sont composées toutes les GRUNDFORMEN *d'un système de formes données, on désire éliminer toutes celles qui sont superflues.*

C'est ainsi que j'ai mis à l'épreuve les résultats donnés par M. Gundelfinger, pour le cas d'un système composé d'une forme cubique et une forme biquadratique, car j'ai reculé, pour le moment, devant le travail énorme qui serait nécessaire pour former la fraction génératrice applicable à ce cas, et, comme résultat de cet examen (sauf la possibilité d'erreurs d'Arithmétique), je crois pouvoir affirmer que, sur les soixante-quatre *grundformen* prétendues, deux sont superflues, mais que les autres soixante-deux restent bonnes. Je compte revenir sur ce cas spécial dans une autre Communication que j'espère avoir l'honneur de faire à l'Académie sur ce sujet.

[* p. 218 below.]

13.

ON THE LIMITS TO THE ORDER AND DEGREE OF THE FUNDAMENTAL INVARIANTS OF BINARY QUANTICS.

[*Proceedings of the Royal Society of London*, XXVII. (1878), pp. 11, 12.]

THE developments which I have recently given to Professor Cayley's second method of dealing with invariants (the first method being that which has been exclusively used by Professor Gordan), has led me through the theory of the Canonical Generating Fraction to the following results, showing that the degree and order of the fundamental invariants and covariants to a quantic or system of quantics are subject to algebraical limits of a very simple kind, and I think it right that these results should not be withheld from the knowledge of those who are pursuing another and, as it seems to me, much more arduous and less promising direction of inquiry into the same subject.

By order I mean the dimensions of a derived form in the coefficients of its primitive (Clebsch and Gordan's *grad*), and by degree the dimensions in the variables (Clebsch and Gordan's *ordnung*).

First as to degree.

If there be a system of $n, n', n'' \ldots$ odd degreed quantics and $\nu, \nu', \ldots$ &c., even ones, then (with the exception of the case when the system reduces to a single linear function or a single quadratic) the degree of any irreducible covariant to the system has for a superior limit $\Sigma\left(\frac{n^2+1}{2}\right) + \Sigma\left(\frac{\nu^2}{2}\right) - 2$.

Thus, for example, where there is but one quantic, the limit is $\frac{n^2-3}{2}$ or $\frac{\nu^2-4}{2}$, according as the degree is n odd or ν even.

Secondly, as to order.

As the expressions become somewhat complicated when there are several quantics, I shall confine myself to a statement applicable to a single quantic, distinguishing between the three cases when n (its degree) is evenly even, oddly even, and odd.

A. When n contains 4, the superior limits for the order of the invariants and covariants respectively are for the former $\frac{(n+1)(n-4)}{2}$, and for the latter $\frac{(n+2)(n-3)}{2}$.

B. When n is even, but not divisible by 4, and is greater than 2, the limits for the two species are $\frac{3n^2-6n-12}{4}$ and $\frac{(n+2)(3n-8)}{4}$ respectively.

C. When n is any odd number greater than 3, the order of the invariants has for its limit $\frac{3}{2}(n+1)(n-3)$, and when it is any odd number greater than unity, the order of the covariants has for its limit $\frac{3n^2-4n-9}{2}$.

Further investigations will, I have good reason to believe, lead to considerably lower limits than those given for cases B and C.

Although morally certain, the three formulæ A, B, C cannot be considered at present apodictically established; the formula respecting the limit to *degree* may, I believe, be regarded as admitting of a complete demonstration. There exists, however, a superior limit to the orders of the fundamental invariants or covariants, which may be regarded as subject to direct demonstration even in our present state of knowledge; this when n is even is n^2-2n-3 for invariants, and n^2-n-4 for covariants; and when n is odd, the corresponding limits are $2n^2-3n-5$ for invariants, and $2n^2-2n-5$ for covariants. But I have no moral doubt whatever of the validity of the formulæ B and C as they stand, and next to none of the validity of formula A.

14.

CHEMISTRY AND ALGEBRA.

[*Nature*, XVII. (1877—1878), pp. 284, 309.]

It may not be wholly without interest to some of the readers of *Nature* to be made acquainted with an analogy that has recently forcibly impressed me between branches of human knowledge apparently so dissimilar as modern chemistry and modern algebra. I have found it of great utility in explaining to non-mathematicians the nature of the investigations which algebraists are at present busily at work upon to make out the so-called *Grundformen* or irreducible forms appurtenant to binary quantics taken singly or in systems, and I have also found that it may be used as an instrument of investigation in purely algebraical inquiries. So much is this the case that I hardly ever take up Dr Frankland's exceedingly valuable *Notes for Chemical Students*, which are drawn up exclusively on the basis of Kekulé's exquisite conception of *valence*, without deriving suggestions for new researches in the theory of algebraical forms. I will confine myself to a statement of the grounds of the analogy, referring those who may feel an interest in the subject and are desirous for further information about it to a memoir which I have written upon it for the new *American Journal of Pure and Applied Mathematics*, the first number of which will appear early in February.

The analogy is between atoms and *binary* quantics exclusively.

I compare every binary quantic with a chemical atom. The number of factors (or rays, as they may be regarded by an obvious geometrical interpretation) in a binary quantic is the analogue of the number of *bonds*, or the *valence*, as it is termed, of a chemical atom.

Thus a linear form may be regarded as a monad atom, a quadratic form as a duad, a cubic form as a triad, and so on.

An invariant of a system of binary quantics of various degrees is the analogue of a chemical substance composed of atoms of corresponding *valences*.

The order of such invariant in each set of coefficients is the same as the number of atoms of the corresponding *valence* in the chemical compound.

A co-variant is the analogue of an (organic or inorganic) compound radical. The orders in the several sets of coefficients corresponding, as for invariants, to the respective valences of the atoms, the free valence of the compound radical then becomes identical with the degree of the co-variant in the variables.

The weight of an invariant is identical with the number of the bonds in the chemicograph of the analogous chemical substance, and the weight of the leading term (or basic differentiant) of a co-variant is the same as the number of bonds in the chemicograph of the analogous compound radical. Every invariant and covariant thus becomes expressible by a *graph* precisely identical with a Kekuléan diagram or chemicograph. But not every chemicograph is an algebraical one. I show that by an application of the algebraical law of reciprocity every algebraical graph of a given invariant will represent the constitution in terms of the roots of a quantic of a type reciprocal to that of the given invariant of an invariant belonging to that reciprocal type. I give a rule for the geometrical multiplication of graphs, that is, for constructing a *graph* to the product of in- or co-variants whose separate graphs are given. I have also ventured upon a hypothesis which, whilst in nowise interfering with existing chemicographical constructions, accounts for the seeming anomaly of the isolated existence as "monad molecules" of mercury, zinc, and arsenic—and gives a rational explanation of the "mutual saturation of bonds."

I have thus been led to see more clearly than ever I did before the existence of a common ground to the new mechanism, the new chemistry, and the new algebra. Underlying all these is the theory of pure colligation, which applies undistinguishably to the three great theories, all initiated within the last third of a century or thereabouts by Eisenstein, Kekulé, and Peaucellier.

15.

SUR LA LOI DE RÉCIPROCITÉ POUR LES INVARIANTS ET COVARIANTS DES QUANTICS BINAIRES.

[*Comptes Rendus*, LXXXVI. (1878), 446—448.]

A UN invariant ou covariant donné d'un quantic binaire du degré i de l'ordre j dans les coefficients, M. Hermite a montré qu'il répond toujours un invariant ou covariant (du même degré) de l'ordre i dans les coefficients, mais appartenant à un quantic du degré j, et il a fourni un procédé pour passer de l'un à l'autre.

Je vais donner une généralisation de ce théorème en l'étendant à un système de quantics binaires, et une méthode plus facile pour faire la transformation pour le cas ou d'un seul quantic ou d'un système. Soit D un invariant ou covariant du degré δ appartenant à un système de quantics binaires des degrés respectifs $i, i', i'', \ldots$, dont l'ordre dans les coefficients des quantics est respectivement $j, j', j'', \ldots$. Je dis qu'il y répond un invariant Δ ou covariant du degré δ appartenant à un système de quantics binaires des degrés respectifs $j, i', i'', \ldots$, dont l'ordre dans ces coefficients des quantics est respectivement $i, j', j'', \ldots$, de sorte qu'à une forme comprise dans le type $i, j; i', j'; i'', j'', \ldots, : \delta$ il en répond une autre comprise dans le type $j, i; i', j'; i'', j'', \ldots, : \delta$.

Cela étant vrai pour le couple d'indices i, j sera nécessairement vrai pour tous les couples ou pour une combinaison quelconque des couples $i, j, \ldots; i', j', \ldots; i'', j'', \ldots$; mais il suffit évidemment de donner les règles de transformation pour l'échange entre eux d'un seul couple d'indices conjugués i et j.

Pour l'effectuer, voici tout ce qui est nécessaire :

Regardons le coefficient de x^i dans le quantic du degré i comme égal à un ; alors tous les coefficients de ce quantic deviennent fonctions symétriques des racines $e_1, e_2, \ldots, e_i$. Qu'ils soient exprimés ainsi, alors chaque terme de D sera de la forme $Me_1^{\alpha} e_2^{\beta} e_3^{\gamma} \ldots e_i^{\lambda}$; bien entendu qu'un ou plusieurs des chiffres $\alpha, \beta, \gamma, \ldots, \lambda$ peuvent devenir zéro.

Au lieu de ce terme, écrivons

$$M\eta_\alpha\eta_\beta\eta_\gamma \ldots \eta_\lambda, \text{ où } \eta_\alpha = (-)^\alpha \epsilon_\alpha,$$

$\epsilon_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_a$, étant les éléments du quantic général $(\epsilon_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_j)(x, y)^j$.

L'expression ainsi obtenue sera évidemment de l'ordre i, quant aux coefficients ϵ, et de plus elle sera un invariant ou un covariant (du même degré que le primitif).

La preuve en est facile, ne dépendant que de l'application de l'équation partielle différentielle, qui sert pour définir un invariant ou différentiant : elle est donnée avec des exemples de son application dans un Mémoire qui doit paraître prochainement dans l'*American Journal of Mathematics*, publié à Baltimore.

Je me borne ici à ajouter quelques mots sur l'usage du terme *réciprocité*, sans lesquels on pourrait aisément se tromper sur la véritable portée du théorème; et, pour plus de clarté, je ne sortirai pas du cas le plus simple, celui d'un seul quantic du type $i, j : \delta$, dont le type conjugué sera $j, i : \delta$.

Supposons que de D appartenant au premier type on ait passé à Δ appartenant au type conjugué $j, i : \delta$. Qu'on répète le procédé, on retournera au type donné $i, j : \delta$. Or il importe beaucoup de savoir si ou non on retournera à la forme donnée D en regardant si l'on veut comme identiques les formes qui ne diffèrent l'une de l'autre que par un multiplicateur numérique.

Pour répondre à cette question, il sera bon de se servir d'une nouvelle définition. J'appelle la *multiplicité* d'un type $j, i : \delta$ le nombre de formes linéairement indépendantes qui y sont attachées, ou, ce qui revient au même, le nombre de paramètres numériques arbitraires de la forme la plus générale qui est représentée par ce type. On peut nommer ces formes ou ces types monadelphiques, diadelphiques, etc., selon la valeur de la multiplicité.

Or, pour ces types monadelphiques en retournant au même type, on retourne nécessairement à la même forme, de sorte que la question que j'ai proposée se limite nécessairement aux types polyadelphiques. Or je suis en mesure d'affirmer qu'en général, en transformant deux fois un quantic appartenant à un type de la multiplicité k, il n'y a que k formes particulières qui se reproduisent identiquement. En donnant des valeurs arbitraires aux k paramètres, on retourne au même type, sans retourner à la même forme, de sorte que D ne peut pas se déduire de Δ comme Δ de D; et ainsi la *réciprocité*, tellement nommée, est essentiellement une réciprocité de types et non pas de formes. Quant aux formes spéciales (disons principales) qui se reproduisent et qui possèdent des réciproques dans un sens étroit, il est facile de voir qu'on peut les déterminer avec l'aide d'une équation algébrique du degré k, très-analogue à l'équation pour trouver les axes principaux d'une courbe ou surface, ou hypersurface, etc., du second degré; j'ai expérimenté,

comme on peut voir dans le Mémoire cité, sur des types diadelphiques, et je trouve, dans les cas que j'ai étudiés, que les exercices de l'équation quadratique à résoudre sont rationnels; mais je ne puis affirmer que cela aura toujours lieu. L'équation dont je parle exprime le rapport numérique entre chaque forme principale et, si je puis me servir de l'expression, seconde *image*, c'est-à-dire l'image de Δ comme Δ est l'image de D. Ses racines ou au moins leurs rapports sont indépendants de toute convention, et sont en effet des constantes absolues de la raison humaine; ainsi il me paraît que la constitution de ces équations mérite d'être étudiée à fond. Sans la règle simplifiée que j'ai donnée pour trouver les images, le travail nécessaire dans le cas des types polyadelphiques serait, à cause de sa longueur, presque inexécutable, et même avec cette simplification le travail est assez pénible. Quoique la nouvelle méthode de former l'image d'une dérivée invariantive possède (il me semble) un avantage considérable quant à la facilité du calcul, cependant la route frayée par M. Hermite a une très-grande utilité, car avec son aide on voit instinctivement que chaque invariant ou covariant binaire équivaut à un hyperdéterminant, et l'on peut même calculer par un procédé direct l'hyperdéterminant qui représente un invariant ou covariant binaire donné.

16.

SUR LA THÉORIE DES FORMES ASSOCIÉES DE MM. CLEBSCH ET GORDAN.

[*Comptes Rendus*, LXXXVI. (1878), pp. 448—450.]

DANS le Traité de Clebsch sur les formes binaires, on trouve un théorème très-remarquable sur ce qu'il appelle les *formes associées*, et sur le système le plus simple des formes associées.

Je me bornerai à l'exposition et à la généralisation de cette dernière. Voici le théorème comme on le trouve dans le travail de M. Clebsch : Soient Q un quantic binaire quelconque du degré i, f un invariant ou covariant quelconque de Q. En choisissant convenablement le chiffre μ, $Q^\mu f$ sera une fonction entière et rationnelle de i invariants et covariants, constants et connus de Q, dont le premier sera Q et les autres successivement de l'ordre 2 et 3 dans les coefficients de Q. Si l'on examine de près ce théorème avec l'aide de la conception et des propriétés des différentiants, voici à quoi il équivaut : Prenons la forme $x^i + px^{i-1} + qx^{i-2} + \ldots + l$.

On sait bien qu'une fonction symétrique quelconque de ses racines sera une fonction rationnelle et entière des i coefficients donnés. Mais, si l'on se borne à une fonction symétrique des *différences* des racines, on peut ajouter (et voilà en quoi consiste essentiellement ce théorème de M. Clebsch ou de M. Gordan) qu'elle sera une fonction rationnelle et entière de $i-1$ fonctions alternativement de l'ordre 2 et de l'ordre 3 des coefficients, dont chacune sera elle-même une fonction des différences des racines.

C'est par une analyse assez compliquée que MM. Clebsch et Gordan établissent leur théorème. Je le déduis par un calcul tout à fait élémentaire et presque instantané en me servant seulement de l'équation partielle différentielle qui sert à définir les invariants et les différentiants et avec ce grand avantage que, avec son aide, je passe immédiatement à l'extension du théorème au cas de système de quantics. Voici en effet le résultat auquel j'arrive avec cette méthode.

Soit $Q_1, Q_2, Q_3, \ldots, Q_\lambda$ un système de quantics binaires. Prenons $(\lambda-1)$ jacobiens indépendants quelconques des Q combinés en paires qu'on peut nommer $J_1, J_2, \ldots, J_{\lambda-1}$ et de plus prenons les a formes associées dans leur forme la plus simple qui appartiennent à $Q_1, Q_2, \ldots, Q_\lambda$ prises séparément. Alors, je dis que, f étant un invariant ou covariant quelconque du système des Q, on aura, en choisissant convenablement les chiffres $\mu_1, \mu_2, \ldots, \mu_\lambda$, $Q_1^{\mu_1} Q_2^{\mu_2} \ldots Q_\lambda^{\mu_\lambda} f$ une fonction rationnelle et entière des formes associées propres à $Q_1, Q_2, \ldots, Q_\lambda$ et des quantités $J_1, J_2, \ldots, J_{\lambda-1}$.

J'ajouterai encore un théorème que je crois être nouveau et qui se déduit immédiatement de ce dernier.

Soient $a_1, b_1, \ldots; a_2, b_2, \ldots; a_\lambda, b_\lambda$ les deux premiers coefficients de $Q_1, Q_2, \ldots, Q_\lambda$ et prenons la forme linéaire $a_k x + b_k y$ (k étant choisi arbitrairement), que je nommerai u. Soit un invariant ou un covariant quelconque du système exprimé comme fonction de u et de y, alors tous les coefficients de F seront des différentiants en x, ce que M. Cayley nomme des *semi-invariants*. Ainsi, par exemple, si l'on prend le covariant bien connu

$$(ac-b^2)x^2 + (ad-cb)xy + (bd-c^2)y^2$$

appartenant à un seul quantic $(a, b, c, d)(x, y)^3$, on peut le mettre sous la forme

$$\frac{1}{a^2}\{(ac-b^2)(ax+by)^2 + (a^2d - 3abc + 2b^3)(ax+by)y - (ac-b^2)^2y^2\},$$

où, en supposant $a=1$, tous les coefficients deviennent des fonctions des différences des racines de $(1, b, c, d)(x, y)^3 = 0$.

La preuve de ces théorèmes sera donnée dans l'*American Journal of Mathematics* publié à Baltimore (États-Unis de l'Amérique), qui doit paraître prochainement.

17.

DÉTERMINATION D'UNE LIMITE SUPÉRIEURE AU NOMBRE TOTAL DES INVARIANTS ET COVARIANTS IRRÉDUCTIBLES DES FORMES BINAIRES.

[*Comptes Rendus*, LXXXVI. (1878), pp. 1437—1441, 1491, 1492, 1519—1522.]

La méthode que je vais exposer s'applique aux cas de systèmes quelconques des formes binaires; mais, pour plus de concision, je me bornerai au cas d'un seul quantic de degré pair: cela suffira pour donner une idée nette de la méthode, ce qui est tout ce que je me propose de faire dans cette première Communication.

Je démontre facilement que le nombre total des invariants ou covariants appartenant au quantic binaire du degré $2t$, de l'ordre μ, dans les coefficients du quantic, sera le coefficient de t^{μ} dans le développement de

$$\frac{F(t)}{(1-t)(1-t^2)^2(1-t^3)\dots(1-t^{2i-1})}$$

en puissances ascendantes de t, où $F(t)$ est une fonction rationnelle et entière de t, qu'on sait comment obtenir.

Je donne le nom de *covariants primaires* aux $2i$ covariants, pour lesquels les coefficients de la plus haute puissance de x [en représentant le quantic par $(a, b, c, d, e, f, \dots)(x, y)^{2i}$] sont

$$a : ac-b^2 : ae-4bd+3c^2 : a^2d-3abc+2b^3$$
$$: a(a^2f-\dots) : a^3(ag-\dots) : a^3(a^2h-\dots) : a^5(ak-\dots),$$

et je nomme *covariants* (invariants compris) *adjoints* ceux qui, pris en conjonction avec les primaires, formeront un système tel, que tout autre covariant sera une fonction rationnelle et entière de ceux qui sont compris dans ce système.

Je regarde la fonction $F(t)$, qui ne contient en effet qu'un nombre fini de termes actuels, comme si elle contenait un nombre infini de puissances positives de t, dont les coefficients qui correspondent aux termes qui manquent sont des zéros.

Prenons un terme quelconque en $F(t)$, disons lt^{λ}. Le nombre des adjoints linéairement indépendants de l'ordre λ peut être, ou égal à l, ou plus grand, ou plus petit. Quand ce nombre est plus grand, je nomme la différence *l'excès* pour l'indice λ; quand il est plus petit, *le défaut* (en faisant exception du cas $\lambda=0$, que je regarde comme n'ayant ni manque ni excès).

Quand il y a excès, je distingue arbitrairement les adjoints en deux groupes: l'un contenant le nombre l et l'autre l'excès; et, en mettant de côté pour le moment ces derniers, je regarde tous les autres adjoints comme formant un seul système, que je nomme *système d'auxiliaires.*

Soit σ la somme des coefficients positifs en $F(t)$, Δ la somme de tous les défauts, et conséquemment $\sigma-1-\Delta$ le nombre des auxiliaires. Or, supposons qu'il existe au moins n adjoints surnuméraires, c'est-à-dire des adjoints pour lesquels la somme des excès est n; je démontre rigoureusement qu'en nommant τ le nombre des coefficients négatifs (s'il y en a), il existera au moins $n+\tau-\Delta$ équations entre les primaires et les auxiliaires, linéaires par rapport à ces derniers, et linéairement indépendantes les unes des autres. Donc, puisque les primaires évidemment n'admettent pas de liaison quelconque entre elles-mêmes, il s'ensuit que le nombre $n+\tau-\Delta$ ne peut pas excéder $\sigma-\tau-1$; donc le nombre total des adjoints ne peut pas excéder $2\sigma-\tau-\Delta-2$ et, à plus forte raison, ne peut pas excéder $2\sigma-\tau-2$.

Parmi ces adjoints, se trouvera nécessairement la partie indépendante des puissances du quantic de tous les primaires, à l'exception des quatre premiers, qui sont les seuls indécomposables. Donc la limite supérieure totale devient $2\sigma-\tau+2$, ou bien $S+\sigma+2$ si l'on prend S égal à la somme algébrique des coefficients, c'est-à-dire à $\sigma-\tau$.

Quant aux valeurs de S et σ, j'ai trouvé par induction, et je ne doute nullement, que $\tau=0$. Pour prouver cette proposition, on n'a besoin que de l'Algèbre ordinaire; mais, en attendant la preuve, que je n'ai pas encore trouvée, on peut se servir d'une limite supérieure à σ a lieu de sa valeur exacte. Quand on aura démontré que $\tau=0$, la limite deviendra tout simplement $2S$.

Or on trouve facilement que

$$S=\frac{1}{i}\left[i^{2i-1}-2i(i-1)^{2i-1}+2i\frac{2i-1}{2}(i-2)^{2i-1}\right]\dots\pm\frac{\Pi 2i}{\Pi(i-1)\,\Pi(i+1)}1^{2i-1}$$

et

$$\sigma<\frac{1}{i}\left[i^{2i-1}+2i\frac{2i-1}{2}(i-2)^{2i-1}+\dots\right],$$

la dernière série ne contenant que les termes positifs de s. $S+\sigma+2$ est donc la limite supérieure rigoureusement démontrée; mais il n'est pas douteux, sous le point de vue moral, que $2S+2$ peut être pris pour cette limite.

J'ajouterai que le point de départ, dans cette démonstration nouvelle du théorème de Gordan, est la règle numérique trouvée par M. Cayley, qui exprime le nombre total des covariants linéairement indépendants d'un ordre et de degré donné appartenant à un quantic de degré donné, règle dont la démonstration rigoureuse a été faite, pour la première fois, par moi-même dans le *Philosophical Magazine** (mars 1878) et dans le dernier tome du *Journal de Borchardt*†. C'est ainsi que, dans le cas considéré plus haut, on établit que ce nombre total sera le coefficient de $t^j u^\epsilon$ dans le développement de la fraction génératrice

$$\frac{1}{(1-tu^{2i})(1-tu^{2i-2})\dots(1-tu^{-2i+2})(1-tu^{-2i})},$$

j étant l'ordre et ϵ le degré du covariant donné : cela mène à la représentation de ce nombre, comme le coefficient de t^j, dans la fraction plus simple

$$\frac{Ft}{(1-t)(1-t^2)^2(1-t^3)\dots(1-t^{2i-1})}.$$

De même, pour le cas où le degré du quantic donné est $2i+1$, on établit que le nombre correspondant sera le coefficient de t^j dans le développement en série de puissances ascendantes de t de la fraction

$$\frac{\Phi t}{(1-t)(1-t^2)(1-t^4)\dots(1-t^{4i})}.$$

Dans ce cas, on se sert d'une série connue de covariants dont les ordres successifs seront $1, 2, 4, \dots, 4i$ comme primaires, et, en nommant S la somme algébrique des coefficients de Φt et Σ la somme des coefficients positifs exclusivement, on trouvera, comme auparavant, que $S+\Sigma-2$ sera une limite supérieure au nombre total des adjoints ; et, comme la série de primaires que j'adopte, pour ce cas, ne contient que deux covariants irréductibles, la limite totale des formes irréductibles sera $S+\Sigma$. En admettant, ce qui est certainement vrai, mais non encore prouvé, que Φt comme Ft est omnipositif, on aurait pour la limite $2S$, c'est-à-dire le double d'une certaine série de termes exponentiels connus, qui seront successivement positifs et négatifs : en attendant la preuve de cette loi d'omnipositivité, la limite privée sera cette même série avec seulement les termes positifs doublés.

On peut obtenir d'autres limites supérieures en se servant de la forme canonique pour les invariants, pris séparément, et de la forme canonique à *deux* variables pour les invariants et les covariants combinés ; mais on introduit ainsi une difficulté de plus, car on aurait besoin de démontrer *a priori* l'existence et le caractère exact du dénominateur de ces formes canoniques : ce qui n'a pas été encore fait. De même, en se servant de la fonction génératrice que j'ai employée ici, pour des valeurs *données* de $2i$ et $2iH$, on peut trouver des dénominateurs plus simples que le dénominateur

[* p. 117 below.] [† p. 232 below.]

général, auxquels répondront aussi des primaires connues : par exemple, pour le cas de $2i=8$, on trouvera que l'on peut prendre pour le dénominateur

$$(1-t)(1-t^2)^2(1-t^3)^2(1-t^4)(1-t^5)(1-t^7),$$

au lieu de

$$(1-t)(1-t^2)^2(1-t^3)(1-t^4)(1-t^5)(1-t^6)(1-t^7);$$

et le numérateur restera encore omnipositif : ainsi la limite au nombre des adjoints sera réduite de la moitié ; mais mon objet a été de trouver une limite supérieure *universelle*, c'est-à-dire algébrique, et en même temps de ne pas admettre un principe quelconque reposant en aucun degré sur l'induction ou sur la probabilité. M. Camille Jordan a trouvé et publié, dans le *Journal de Liouville*, une méthode pour déterminer une limite supérieure à l'ordre ou degré des *grundformen* en se servant des principes de M. Gordan, mais je ne sais pas si ce grand géomètre ou aucun autre a réussi à déterminer une limite supérieure à leur nombre. La méthode de MM. Gordan et Jordan est le développement de la première de M. Cayley (celles des hyperdéterminants), comme la mienne est le développement de sa seconde méthode, celle qui repose sur l'emploi de l'equation partielle différentielle, nécessaire et suffisante pour déterminer l'existence des invariants et covariants proposés.

Je donne en conclusion les formes actuelles de la fonction génératrice pour les covariants pris sans distinction quant à leur degré (ce qui revient à dire la fonction génératrice pour les *différentiants*) pour les quantics binaires de tous degrés de 2 jusqu'à 8. Soit μ le degré du quantic, G la fonction génératrice qui y répond.

Quand $\mu=2$,

$$G=\frac{1}{(1-t)(1-t^2)}.$$

Quand $\mu=3$,

$$G=\frac{1+t^3}{(1-t)(1-t^2)(1-t^4)}.$$

Quand $\mu=4$,

$$G=\frac{1+t^3}{(1-t)(1-t^2)^2(1-t^3)}.$$

Quand $\mu=5$,

$$G=\frac{1+t^2+3t^3+3t^4+4t^5+4t^6+6t^7+6t^8+4t^9+5t^{10}+3t^{11}+3t^{12}+t^{13}+t^{15}}{(1-t)(1-t^2)(1-t^4)(1-t^6)(1-t^8)}.$$

Quand $\mu=6$,

$$G=\frac{1+t^2+3t^3+4t^4+4t^5+4t^6+3t^7+3t^8+t^{10}}{(1-t)(1-t^2)(1-t^2)(1-t^3)(1-t^4)(1-t^5)}.$$

Quand $\mu=7$,

$$G=\frac{\left[\begin{array}{l}1+2t^2+6t^3+10t^4+19t^5+28t^6+44t^7+61t^8+79t^9+102t^{10}+129t^{11}+156t^{12}\\ +173t^{13}+196t^{14}+215t^{15}+230t^{16}+231t^{17}+231t^{18}+230t^{19}+\dots+2t^{34}+t^{35}\end{array}\right]}{(1-t)(1-t^2)(1-t^4)(1-t^6)(1-t^8)(1-t^{10})(1-t^{12})}.$$

Quand $\mu = 8$,

$$G = \frac{1 + 2t^2 + 6t^3 + 12t^4 + 19t^5 + 25t^6 + 31t^7 + 36t^8 + 38t^9 + 36t^{10} + \ldots + t^{18}}{(1-t)(1-t^2)(1-t^2)(1-t^3)(1-t^3)(1-t^4)(1-t^5)(1-t^7)},$$

où l'on remarquera que, pour $\mu = 8$, G peut être changé dans la forme normale en multipliant son numérateur et son dénominateur par $1 + t^3$.

En commençant par la forme brute

$$\frac{1 - u^{-2}}{(1-tu^8)(1-tu^6)(1-tu^4)(1-tu^2)(1-t)(1-tu^{-2})(1-tu^{-4})(1-tu^{-6})(1-tu^{-8})},$$

on connaît que le nombre des covariants qui appartiennent à la forme binaire du huitième degré, et qui sont de l'ordre j dans les coefficients et du degré ϵ dans les variables, est le coefficient $t^j . u^\epsilon$ dans le développement de cette fraction selon les puissances ascendantes de t. De là on conclut que, j et ϵ étant positifs, on peut substituer à cette fraction la fraction dont

$$(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6) \times (1-t^7)(1-tu^8)(1-t^2u^{12})(1-t^2u^8)(1-t^2u^4)$$

est le dénominateur, et dont le numérateur est

$$\begin{aligned}
&1 + t^8 + t^9 + t^{10} + t^{18} \\
&+ u^2(t^5 + t^6 + 2t^7 + 2t^8 + 3t^9 + 2t^{10} + 2t^{11} + t^{12} + t^{13}) \\
&+ u^4(t^3 + 2t^4 + 2t^5 + 2t^6 + 2t^7 + 2t^8 + t^9 + t^{10} + t^{11} + t^{12} + 2t^{13} + 2t^{14} + t^{15} + t^{16} - t^{21}) \\
&+ u^6(t^3 + t^4 + 2t^5 + 3t^6 + 3t^7 + 3t^8 + 3t^9 + 2t^{10} + t^{11} + t^{12}) \\
&+ u^8(t^3 + t^4 + t^5 + 2t^6 + 2t^7 + 3t^8 + 2t^9 + t^{10} + t^{11} + t^{12} - t^{15} - t^{16} - t^{18} - t^{19} - t^{20}) \\
&+ u^{10}(t^3 + 2t^4 + 3t^5 + 2t^6 + 2t^7 + t^8 - t^9 - 2t^{10} \\
&\qquad - 4t^{11} - 4t^{12} - 3t^{13} - 3t^{14} - 2t^{15} - t^{16} - t^{17}) \\
&+ u^{12}(t^3 + t^4 - t^8 - t^9 - 2t^{10} - 2t^{11} - 2t^{12} - 2t^{13} - 4t^{14} \\
&\qquad - 4t^{15} - 4t^{16} - 3t^{17} - 2t^{18} - t^{19} - t^{20} + t^{21} + t^{22}) \\
&+ u^{14}(t^3 + t^4 + t^5 + t^6 - t^7 - t^8 - 3t^9 - 5t^{10} - 6t^{11} \\
&\qquad - 6t^{12} - 6t^{13} - 4t^{14} - 4t^{15} - 2t^{16} - t^{17}) \\
&+ u^{16}(-t^8 - 2t^9 - 4t^{10} - 4t^{11} - 6t^{12} - 6t^{13} - 6t^{14} \\
&\qquad - 5t^{15} - 3t^{16} - t^{17} - t^{18} + t^{19} + t^{20} + t^{21} + t^{22}) \\
&+ u^{18}(t^3 + t^4 - t^5 - t^6 - 2t^7 - 3t^8 - 4t^9 - 4t^{10} - 4t^{11} \\
&\qquad - 2t^{12} - 2t^{13} - 2t^{14} - 2t^{15} - t^{16} - t^{17} + t^{21} + t^{22}) \\
&+ u^{20}(-t^8 - t^9 - 2t^{10} - 3t^{11} - 3t^{12} - 4t^{13} - 4t^{14} \\
&\qquad - 2t^{15} - t^{16} + t^{17} + 2t^{18} + 2t^{19} + 3t^{20} + 2t^{21} + t^{22}) \\
&+ u^{22}(-t^5 - t^6 - t^7 - t^9 - t^{10} + t^{13} + t^{14} + t^{15} \\
&\qquad + 2t^{16} + 3t^{17} + 2t^{18} + 2t^{19} + t^{20} + t^{21} + t^{22}) \\
&+ u^{24}(t^{13} + t^{14} + 2t^{15} + 3t^{16} + 3t^{17} + 3t^{18} + 3t^{19} + 2t^{20} + t^{21} + t^{22})
\end{aligned}$$

$$\begin{aligned}
&+ u^{26}(-t^5 + t^9 + t^{10} + t^{11} + 2t^{12} + t^{13} + t^{14} + t^{15} \\
&\qquad + t^{16} + 2t^{17} + 2t^{18} + 2t^{19} + 2t^{20} + 2t^{21} + t^{22}) \\
&+ u^{28}(t^{12} + t^{13} + 2t^{14} + 2t^{15} + 3t^{16} + 2t^{17} + 2t^{18} + t^{19} + t^{20}) \\
&+ u^{30}(t^7 + t^{15} + t^{16} + t^{17} + t^{25}).
\end{aligned}$$

Cette fraction a été prise sous sa forme canonique au moyen de l'introduction, dans le numérateur et le dénominateur, du facteur commun

$$(1 + tu^6)(1 + tu^4)(1 + tu^2).$$

En opérant sur les termes positifs de ce numérateur par la méthode générale du tamisage et en combinant les résultats avec les *primaires* donnés par le dénominateur, on obtient la table suivante pour le système complet des *grundformen* du quantic du huitième degré :

Ordre dans les coefficients.	Degré dans les variables.									
	0	2	4	6	8	10	12	14	16	18
1.........					1					
2.........	1		1		1		1			
3.........	1		1	1	1	1	1	1		1
4.........	1		2	1	1	2	1	1		1
5.........	1	1	2	2	1	3		1		
6.........	1	1	2	3	1	1				
7.........	1	2	2	3						
8.........	1	2	2	2						
9.........	1	3	1							
10.........	1	2								
11.........		2								
12.........		1								

Dans cette table un chiffre quelconque dans l'intérieur du cadre exprime le nombre des formes dérivées irréductibles de l'ordre qui se trouve au commencement de la ligne et du degré que se trouve à la tête de la colonne dans laquelle le chiffre est situé. Ainsi, par exemple, il y aura trois covariants irréductibles de l'ordre 6 et du degré 6, 2 de l'ordre 8 et du degré 6, et ainsi en général. Le nombre total de ces formes irréductibles est 69, le degré le plus élevé 18, l'ordre le plus élevé 12. La limite supérieure donnée par la méthode expliquée dans ma dernière Communication (qui sort de la considération de la génératrice à une seule variable) est $2(302) + 2 = 606$, qui est beaucoup trop grand. Mais, en se servant de la même méthode appliquée à la fonction génératrice *à deux variables* dans sa forme canonique donnée ci-dessus, au lieu de la fonction génératrice à une seule variable, on obtiendra comme limite supérieure

$$(2\sigma - \tau - 2) + \epsilon + \nu,$$

σ étant la somme des coefficients positifs, τ la somme des coefficients négatifs dans le numérateur, ϵ le nombre des liaisons algébriques entre les *primaires* qui répondent aux indices des facteurs du dénominateur, et ν le nombre de ces facteurs.

On aura donc

$$\sigma = 70, \quad \tau = 70, \quad \nu = 10, \quad \epsilon = \nu - 8 = 2,$$

et la limite supérieure devient 80, qui n'est pas beaucoup plus grand que le nombre 69 qu'on a trouvé.

De même, pour le cas d'une fonction du sixième degré, la limite supérieure tirée de la fonction génératrice (dans sa forme canonique) à deux variables sera $(2\sigma - \tau - 2) + \epsilon + \nu$, où l'on trouvera

$$\sigma = 29, \quad \tau = 29, \quad \nu = 7, \quad \epsilon = \nu - 6 = 1,$$

et conséquemment la limite devient 35, le vrai nombre étant 27.

La limite inférieure est évidemment dans tous les cas le nombre donné par la règle du tamisage: par conséquent, dans tous les exemples qu'on a précédemment traités, cette limite coïncide avec le nombre actuel des *grundformen*. On peut à peine douter que cette identité, qui est conforme à la *loi de parcimonie*, et soutenue par une induction à peu près irrésistible, ne soit d'application universelle, et il serait fort à désirer que M. Gordan ou quelqu'un de ses élèves fît connaître, s'il ne l'a pas déjà fait, le système des *grundformen* pour le quantic du huitième degré obtenu par sa méthode, afin qu'on pût le comparer avec celui qui se déduit de la mienne.

Pour éviter toute ambiguïté, je dois ajouter que le fonction génératrice à une variable est celle qui sert à donner le nombre total des covariants d'un ordre donné dans les coefficients sans que le degré dans les variables soit spécifié, tandis que la fonction génératrice à deux variables est celle qui sert pour l'énumération des covariants dont l'ordre et le degré sont tous les deux donnés. Les deux fonctions deviennent algébriquement égales quand, dans la dernière, on aura fait $u = 1$; mais le facteur commun au numérateur et au dénominateur ne sera pas en général le même dans les deux expressions.

18.

PROOF OF THE HITHERTO UNDEMONSTRATED FUNDAMENTAL THEOREM OF INVARIANTS.

[*Philosophical Magazine*, v. (1878), pp. 178—188.]

I AM about to demonstrate a theorem which has been waiting proof for the last quarter of a century and upwards. It is the more necessary that this should be done, because the theorem has been supposed to lead to false conclusions, and its correctness has consequently been impugned*. But, of the two suppositions that might be made to account for the observed discrepancy between the supposed consequences of the theorem and ascertained facts—one that the theorem is false and the reasoning applied to it correct, the other that the theorem is true but that an error was committed in drawing certain deductions from it (to which one might add a third, of the theorem and the reasoning upon it being both erroneous)—the wrong alternative was chosen.

* Thus in Professor Faà de Bruno's valuable *Théorie des Formes Binaires*, Turin, 1876, at the foot of page 150 occurs the following passage:—"Cela suppose essentiellement que les équations de condition *soient toutes indépendantes entr'elles, ce qui n'est pas toujours le cas*, ainsi qu'il résulte des recherches du Prof. Gordan sur les nombres des covariants des formes quintique et sextique."

The reader is cautioned against supposing that the consequence alleged above does result from Gordan's researches, which are indubitably correct. This supposed consequence must have arisen from a misapprehension on the part of M. de Bruno of the nature of Professor Cayley's rectification of the error of reasoning contained in his second memoir on Quantics, which had led to results discordant with Gordan's. Thus error breeds error, unless and until the pernicious brood is stamped out for good and all under the iron heel of rigid demonstration. In the early part of this year Mr Halsted, a Fellow of Johns Hopkins University, called my attention to this passage in M. de Bruno's book; and all I could say in reply was that "the extrinsic evidence in support of the independence of the equations which had been impugned rendered it to my mind as certain as any fact in nature could be, but that to reduce it to an exact demonstration transcended, I thought, the powers of the human understanding."

At the moment of completing a memoir, to appear in Borchardt's Journal, demonstrating my quarter-of-a-century-old theorem for enabling Invariants to procreate their species, as well by an act of self-fertilization as by conjugation of arbitrarily paired forms, the unhoped and unsought-for prize fell into my lap, and I accomplished with scarcely an effort a task which I had believed lay outside the range of human power.

An error was committed in reasoning out certain supposed consequences of the theorem; but the theorem itself is perfectly true, as I shall show by an argument so irrefragable that it must be considered for ever hereafter safe from all doubt or cavil. It lies at the basis of the investigations begun by Professor Cayley in his *Second Memoir on Quantics*, which it has fallen to my lot, with no small labour and contention of mind, to lead to a happy issue, and thereby to advance the standards of the Science of Algebraical Forms to the most advanced point that has hitherto been reached. The stone that was rejected by the builders has become the chief corner-stone of the building.

I shall for greater clearness begin with the case of a single binary quantic $(a, b, c, \ldots, l \between x, y)^i$. Any rational integral function of the *elements* $a, b, c, \ldots l$ which remains unchanged in value when for them are substituted the elements of the new quantic obtained by putting $x + hy$ instead of x in the original one, I call a Differentiant in x to the given quantic.

By a differentiant of a given weight w and order j, I mean one in every term of which the combination of the elements is of the jth order and the sum of their weights w, the weights of the successive elements $(a, b, c, \ldots l)$ themselves being reckoned as $0, 1, 2, \ldots i$ respectively.

The proposition to be proved is, that the number of arbitrary constants in the most general expression for such differentiant is the difference between the number of ways in which w can be made up with j of the integers $0, 1, 2, 3, \ldots i$ (repetitions allowable), less the number of ways in which $w-1$ can be made up with the same integers. We may denote these two numbers by $(w : i, j)$, $\{(w-1) : i, j\}$ respectively, and their difference by $\Delta(w : i, j)$. Then, if we call the number of arbitrary constants in the differentiant of weight w and order j belonging to a binary quantic of the ith order $D(w : i, j)$, the proposition to be established is that $D(w : i, j) = \Delta(w : i, j)$.

Let us use Ω to denote the operator

$$a\frac{d}{db} + 2b\frac{d}{dc} + \ldots + ik\frac{d}{dl},$$

and O to denote the operator

$$ib\frac{d}{da} + (i-1)c\frac{d}{db} + \ldots + l\frac{d}{dk}.$$

Then it is well known that the necessary and sufficient condition for D being a differentiant in x is that the *identity* $\Omega D = 0$ be satisfied.

Let us study the relations of Ω and O in respect to D.

In the first place, let U be any rational integral function of the elements of order j and weight w; then I say that

$$\Omega . O . U - O . \Omega . U = (ij - 2w)\, U.$$

For if we use $*$ to signify the act of pure differential operation, it is obvious that

$$\Omega . O . U = (\Omega \times O)\, U + (\Omega * O)\, U,$$
$$O . \Omega . U = (\Omega \times O)\, U + (O * \Omega)\, U;$$

Therefore

$$\Omega . O . U - O . \Omega . U = \{(\Omega * O) - (O * \Omega)\}\, U$$
$$= ia\frac{d}{da} + 2(i-1)b\frac{d}{db} + 3(i-2)c\frac{d}{dc} + \ldots + ik\frac{d}{dk}$$
$$- ib\frac{d}{db} - 2(i-1)c\frac{d}{dc} - \ldots - 2(i-1)k\frac{d}{dk} - il\frac{d}{dl}$$
$$= ia\frac{d}{da} + (i-2)b\frac{d}{db} + (i-4)c\frac{d}{dc} - \ldots - (i-2)k\frac{d}{dk} - 2l\frac{d}{dl}.$$

If now $\rho a^p . b^q . c^r \ldots l^t$, where ρ is a number, be any term in U, we have

$$\left.\begin{array}{r} p + q + r + \ldots + t = j \\ q + 2r + \ldots + it = w \end{array}\right\} \text{ by hypothesis;}$$

therefore

$$\Omega . O . U - O . \Omega . U,$$

that is

$$i\left(a\frac{d}{da} + b\frac{d}{db} + c\frac{d}{dc} \ldots + l\frac{d}{dl}\right) U$$
$$-2\left(\qquad b\frac{d}{db} + 2c\frac{d}{dc} \ldots + il\frac{d}{dl}\right) U$$
$$= \Sigma\rho\,(ij - 2w)\,(a^p . b^q . c^r \ldots l^t)$$
$$= (ij - 2w)\, U, \text{ as was to be proved.}$$

If now for U we write D a differentiant in x, we have $\Omega D = 0$, and therefore

$$\Omega . O . D = \delta D,$$

where $\delta = ij - 2w$.

Again,

$$\Omega . O (O . D) - O . \Omega (O . D) = \{ij - 2(w+1)\}\, O . D;$$

for $O . D$ is of the weight $w + 1$;

therefore

$$\Omega^2 . O^2 . D = \Omega . O\, \delta D + (\delta - 2)\, \Omega . O . D$$
$$= (2\delta - 2)\, \Omega . O . D$$
$$= \delta(2\delta - 2)\, D.$$

Similarly it will be seen that

$$\Omega^3 . O^3 . D = \delta(2\delta - 2)(3\delta - 6)\, D,$$

and in general

$$\Omega^q . O^q . D = \delta(2\delta - 2)(3\delta - 6) \ldots \{q\delta - (q^2 - q)\}\, D$$
$$= (1 . 2 . 3 \ldots q)\, \{\delta . (\delta - 1)(\delta - 2) \ldots (\delta - q + 1)\}\, D,$$

the successive numbers δ, $2\delta - 2$, $3\delta - 6$, &c. being the successive sums of the arithmetical series δ, $\delta - 2$, $\delta - 4$, $\delta - 6$, &c.

To find the most general differentiant in question, we must take every combination of the elements whose weight is w and order j, of which the number is obviously $(w:i,j)$, and prefix an indeterminate constant to each such combination; then operating upon this form with Ω, we shall reduce its weight by unity, and shall obtain as many combinations of this reduced weight (the order j remaining unchanged) as there are units in $\{(w-1):i,j\}$. Each of these combinations will have for its coefficient a linear function of the assumed indeterminate coefficients; and in order to satisfy the identity $\Omega D = 0$, each such linear function must be made equal to zero. There are therefore $(w:i,j)$ quantities connected by $\{(w-1):i,j\}$ homogeneous equations. *Supposing the equations to be independent*, the number of the indeterminate coefficients left arbitrary is obviously the difference between these quantities, namely, $\Delta(w:i,j)$. The difficulty consists in proving this independence—a difficulty so great that I think any one attempting to establish the theorem, as it were by direct assault, in this fashion, would find that he had another Plevna on his hands. But a position that cannot be taken by storm or by sap may be *turned* or starved into surrender; and this is how we shall take our Plevna. Be the equations of condition linearly independent or not, it is obvious that we must have $D(w:i,j)$ equal to or greater than $\Delta(w:i,j)$. I shall show by aid of a construction drawn from the resources of the Imaginative Reason, and founded on the reciprocal properties that have just been exhibited by the famous O and Ω, that this latter supposition, of the first member of the equation being greater than the second, is inadmissible and must be rejected. Observe that $(0:i,j)$, the number of ways of making up 0 with j combinations of $0, 1, 2, \ldots i$, is 1; also that $D(0:i,j)$, the number of arbitrary constants in the most general differentiant in x to the quantic $(a, b, c, \ldots \between x, y)^i$ of order j and weight 0, is also 1; for such differentiant is obviously λa^n.

Thus we have for all values of w,

$$D(w:i,j) = \text{ or } > (w:i,j) - \{(w-1):i,j\},$$

and also

$$D(0:i,j) = (0:i,j);$$

therefore

$$D(w:i,j) + D\{(w-1):i,j\} + D\{(w-2):i,j\} + \ldots + D(0:i,j)$$
$$= \text{ or } > (w:i,j).$$

If in the above condition, for any assumed value of w, $>$ is the sign to be employed, then the equation $D(w:i,j) = \Delta(w:i,j)$ cannot be satisfied for all values of w. If, on the other hand, $>$ is not the sign to be employed, then this equation, for *every value of* w, commencing with the assumed one down to 0, must be satisfied. The greatest value of w for given values of i, j, it is well known, is $\frac{ij}{2}$ for ij even, and $\frac{ij-1}{2}$ for ij odd. Let us give to w this

maximum value in the above "greater or equal" relation; for brevity, denote the differentiants whose types are $[w, i, j]$, $[(w-1), i, j] \ldots$ by $[w]$, $[w-1]$, $[w-2]$, &c. respectively, i and j being regarded as constants. It will be convenient to substitute for the number of arbitrary constants in any of these differentiants the same number of linearly independent specific values of them; so that we shall have $D(w:i,j)$ of linearly independent $[w]$'s, $D\{(w-1):i,j\}$ of linearly independent $[w-1]$'s, and so on. Now, instead of $D\{(w-q):i,j\}$ differentiants $[w-q]$, let us substitute the same number of the derived forms $O^q[w-q]$. I shall prove that the quantities (*all of the same weight* w) thus obtained are linearly independent of one another.

For suppose that those belonging to any one set $O^q.[w-q]$ are not independent, but are connected by a linear equation. Then, operating upon this equation with $\Omega^q.$, we shall obtain a linear equation between the quantities $[w-q]$, for each quantity $\Omega^q . O^q.[w-q]$ is a numerical multiple of $[w-q]$; which is contrary to the hypothesis. Again, let there be a linear equation between the quantities contained in any number of sets of the form $O^q.[w-q]$ for which m is the greatest value of q. Then, operating upon this with Ω^m, it is clear that all the quantities in the sets for which $q < m$ will introduce quantities of the form $\Omega^{m-q}[w-q]$ where $m-q>0$, and which consequently vanish. There will be left, therefore, only quantities of the form $[w-q]$, between which a linear equation would exist, contrary to hypothesis, as in the preceding case. Therefore all the quantities in all the sets are linearly independent. But these are all of the weight w, that is,

$$\left[\frac{ij}{2} \text{ or } \frac{ij-1}{2}\right],$$

and are therefore linear functions of the number of ways in which the integers $0, 1, 2, 3, \ldots i$ can be combined i and j together so as to give the weight w. Therefore being linearly independent, as just proved, their number cannot exceed this last-named number, that is, cannot exceed $(w:i,j)$. That is to say,

$$D(w:i,j) + D\{(w-1):i,j\} + \ldots + D(O:i,j)$$

cannot exceed $(w:i,j)$. Therefore every one of the equations

$$D(w:i,j) = \Delta(w:i,j)$$

must be satisfied from the maximum value of w down to the value 0, which proves *the great hitherto undemonstrated fundamental theorem* for a single quantic.

For any number of quantics the demonstration is precisely similar at all points: there will be as many systems of i, j as there are quantics. $(w:i,j:i',j':\&c.)$ will denote the number of ways of making up w with j

of the integers 0, 1, 2, ... i, with j' of the integers 0, 1, 2, ... i', and so on. The theorem to be demonstrated will be

$$D(w:i,j:i',j':\ldots)=\Delta(w:i,j:i',j':\ldots).$$

$$\Omega \text{ will become } \Sigma\left(a\frac{d}{db}+2b\frac{d}{dc}+\ldots\right),$$

$$O \quad ,, \quad ,, \quad \Sigma\left(ib\frac{d}{da}+(i-1)c\frac{d}{db}+\ldots\right).$$

It will still be true that $\Omega^q . O^q . D$—where D is a differentiant in x (that is, a function of the elements in all the given quantics which withstand change when these are transformed by writing $x+hy$ for x)—is a numerical multiple of D; and D will be subject to the identity $\Omega D=0$. We shall still have

$$D(w:i,j:i',j':\ldots)= \text{ or } > \Delta(w:i,j:i',j':\ldots),$$

and

$$D(0:i,j:i',j':\ldots)=(0:i,j:i',j':\ldots),$$

and shall be able in precisely the same way as before to demonstrate the impossibility of $\sum_{k=w}^{k=0} D(w-k:i,j:i',j':\ldots)$ being *greater* than $(w:i,j:i',j':\ldots)$, and so shall be able to infer by the same logical scheme

$$\Delta(w:i,j:i',j':\ldots)=D(w:i,j:i',j':\ldots).$$

This is my extension of Professor Cayley's theorem, which leads direct to the Generating Fractions given in my recent papers in the *Comptes Rendus*.

In a series of articles which I hope to publish in the *American Journal of Pure and Applied Mathematics*, I propose to give a systematic development of the Calculus of Invariants, taking a differentiant as the primordial germ or unit. I have spoken of a differentiant in x, and of course might have done so equally of a differentiant in y. If we call the former D_x, it is capable of being shown, from the very natures of the forms O and Ω, that if the quantity $ij-2w$, which may be called the *degree* of D_x, be called δ, then $O^\delta D_x$ becomes a differentiant in y. These may be termed simple differentiants; but the principle of continuity forbids that we should omit to comprise in the same scheme the intermediate forms $O^p D_x$ or $\Omega^q D_y$, through which simple differentiants in x and y pass into each other. These may be termed mixed differentiants; $O^p D_x$ may be termed a differentiant p removed (as we speak of *cousins* once, twice, &c. removed) from x, which will be the same thing as $O^q D_y$ (a differentiant q removed from y) if $p+q$ is equal to the degree, namely, $ij-2w$. Now all these differentiants, whether simple or mixed, possess a wonderful property, which may be deduced by means of Salmon's Theorem, given in the *Philosophical Magazine* for August 1877. They are all, in an enlarged sense of the term, Invariants—in this sense to wit, that if the elements are made to undergo a substitution consequent upon or, as we may say, induced by a general linear substitution impressed on the variables, which for greater simplicity of enunciation may be

supposed to have unity for the determinant of its matrix, then every differentiant, whether single or double (the latter being equivalent to an invariant), and whether simple or mixed, will remain a Constant Function of the Coefficients of the impressed substitution. To wit, if the differentiant be p removes from x and q removes from y (so that its degree is $p+q$), and if the impressed substitution be $lx+\lambda y$ for x, and $mx+\mu y$ for y, where $l\mu-\lambda m=1$, then will the differentiant be a constant bipartite quantic in the two sets of coefficients l, m and λ, μ, of the degree q in the former and p in the latter—a theorem which amounts almost to a revolution in the whole sphere of thought about Invariants.

I have borrowed the term "Imaginative Reason" from a recent paper of Mr Pater on Giorgione, in which, as in many of those of Mr Symonds (I will instance one on Milton in particular), I find a continued echo of my own ideas, and in the latter many of the very formulæ contained in my *Laws of Verse*, where versification in sport has been made æsthetic in earnest. Surely the claim of Mathematics (its "*Andersstreben*") to take a place among the liberal arts must be now admitted as fully made good. Whether we look to the advances made in modern geometry, in modern integral calculus, or in modern algebra, in each of these a free handling of the material employed is now possible, and an almost unlimited scope left to the regulated play of the fancy. It seems to me that the whole of æsthetic (so far as at present revealed) may be regarded as a scheme having four centres, which may be treated as the four apices of a tetrahedron, namely Epic, Music, Plastic, and Mathematic. There will be found to be a *common* plane to every three of these, *outside* of which lies the fourth; and through every two may be drawn a common axis *opposite* to the axis passing through the remaining two. So far is certain and demonstrable. I think it also possible that there is a centre of gravity to each set of three, and that the lines joining each such centre with the outside apex will intersect in a common point the centre of gravity of the whole body of æsthetic; but what that centre is or must be I have not had time to think out.

Postscript.—In the first fervour of a new conception, I fear that in the manuscript which is now on its way to England I may have expressed myself with some want of clearness or precision on the subject of pure and mixed differentiants. I will therefore add a few more explanatory and vaticinatory words on this subject, through the medium of which I catch a glimpse of the possibility of obtaining a simple proof of Gordan's theorem, just as through the medium of pure differentiants taken *per se* I caught a glimpse (almost immediately afterwards to be converted into a certainty) of the proof of Cayley's theorem given in this memoir. I conceive that what the *ensemble* of pure differentiants have done for the one, the larger *ensemble* of all sorts of differentiants, pure and mixed, taken together, will enable me or some one else to accomplish for the other.

Any function of the coefficients of a quantic which is nullified by the operation upon it of Ω, which we may call the *revector* symbol, or in other words, whose first *revect* is zero, is a pure differentiant in x. So, of course, if nullified by the operation upon it of O, which may be called the *provector* symbol, it is a pure differential in y. We may call $ij-2w$, where i is the degree of the quantic, j the order of a pure differentiant, and w its weight in x, the grade of the differentiant, and denote this grade by δ.

The δth provect of a pure differentiant in x is of course a pure differentiant in y, which is δ *removes* from x, as the pure differentiant in x is δ removes from y. If q be less than δ, the qth provect of a pure differentiant in x is a mixed differentiant q removes from x, or, if we like to say so, $(\delta-q)$ removes from y. The *grade* of a mixed differentiant may be defined to be the same as that of the pure differentiant of which it is a provect or revect.

Then, in the first place, we have this proposition:—If any linear substitution whatever be impressed on the variables of a quantic, the transformed value of any of its differentiants will separate into two factors, of which one will be the determinant of substitution raised to the power w, where w is the weight corresponding to the order and grade of the differentiant and the degree of the quantic. The remaining factor will be a function of the coefficients of substitution, and may be called the outstanding factor. Of this I shall proceed to speak.

Let x be replaced by $hx+ly$,
y „ „ $kx+my$.

Then the outstanding factor for the transformed D (a pure differentiant in x of the grade δ) may be proved by repeated applications of Salmon's theorem to be equal to

$$\left(1+\frac{k}{m}O.+\left(\frac{k}{m}\right)^2\frac{O^2.}{1.2}+\dots\right)m^{\delta}D,$$

where of course the series of terms in the development will, after the $(\delta+1)$th term, vanish spontaneously. In other words, the outstanding factor of the transformed D is $m^{\delta}e^{\frac{kO.}{m}}.D$, where it will be noticed that only the coefficients of substitution due to the change in y make their appearance.

If now we take any mixed differentiant, say the qth provect of D, that is, $O^q.D$, its outstanding factor, I find, will be the qth emanent of the outstanding factor for D, that is, will be

$$\left(h\frac{d}{dk}+l\frac{d}{dm}\right)^q\left(m^{\delta}e^{\frac{kO.}{m}}\right)D.$$

And here for the present I end. The subject is, as it was, a vast one; and this conception of mixed differentiants opens out still vaster horizons. Every thing grown on American soil, or born under the influence of its skies, as its lakes, its rivers, its trees, and its political system, seems to have a tendency to rise to colossal proportions.

I will merely add one remark which has occurred to me relating to Sturm's theorem and the process of Algebraical common measure in general. If $f(x, y)$ be a rational integral function of x, y, and $f'(x, y)$ its derivative in respect to x, and we perform the process of common measure between them regarded as functions of x, we know that the irreducible part of the successive remainders taken in ascending order, say $U_0, U_1, U_2, \dots$, will have for their leading coefficients (say $D_0, D_1, D_2 \dots$) the discriminants of f and of its successive derivatives in respect to x respectively.

Here D_0 is an invariant of the given form;

D_1 (a differentiant in x) will be the leading coefficient of the covariant

$$D_1x^2+O.D_1xy+\frac{O^2.}{1.2}D_1y^2;$$

D_2 (another differentiant in x) will be the leading coefficient of the covariant

$$D_2x^4+O.D_2x^3y+\frac{O^2.}{1.2}D_2x^2y^2+\frac{O^3.}{1.2.3}D_2xy^3+\frac{O^4.}{1.2.3.4}D_2y^4,$$

and so on until we come back to the first Sturmian remainder of $(x, y)^i$, the irreducible part of which (or we may call it the Sturmian Auxiliary Proper) is the Hessian differentiated down from being of the degree $2i-4$ to the degree $i-2$, that is, to half of what it was at first; and so in like manner every Sturmian Auxiliary Proper is, so to say, a Covariant differentiated down to half its original dimensions.

The above invariant and the following covariants may be called $V_0, V_1, V_2, \dots$ respectively. The interesting point in question is that (to numerical factors *près*)

$$U_0=V_0,\quad U_1=\frac{d}{dx}V_1,\quad U_2=\left(\frac{d}{dx}\right)^2V_2,\quad U_3=\left(\frac{d}{dx}\right)^3V_3,$$

and so on.

So more generally for any two functions $f(x, y)$, $\phi(x, y)$, the irreducible part of the remainders obtained by common-measuring them with respect to x will all be derivatives in regard to x of covariants of the two given quantics. If we take for our quantics

$$(a, b, c, \dots h, k, l \between x, y)^i : (a', b', c', \dots h', k', l' \between x, y)^i,$$

the covariants in question will all be educts of (that is, functions having for their leading coefficients) the successive resultants of the forms

$$[(a, \dots h, k, l),\ (a', \dots h', k', l')],$$

of the forms $[(a, \ldots h, k), \ (a', \ldots h', k')]$,

of the forms $[(a, \ldots h), \ (a', \ldots h')]$,

and so on, the discriminants of which may be called *partial* resultants of the given forms; in a word, the simplified residues arising in the process of common-measuring in respect to one of their variables two given binary quantics are differential derivatives, in respect to that variable, of the educts of their partial resultants (of course with the understanding that the last simplified residue is the complete resultant itself).

This seems to point to the existence of some generalized statement of Sturm's theorem in which the same Educts as above referred to shall appear, but where, instead of their derivatives in respect to one of the variables being made use of, perfectly general Emanants of them shall be employed as the Criterion functions. For I need hardly add that all Educts (although not written so as to show it in what precedes) are in fact symmetrical in respect to the two sides of the quantic to which they belong.

On various *à priori* grounds I suspect the generalized theorem to be as follows. If $X_{2\mu}$ is the covariant (of degree 2μ) whose μth derivative in respect to x is a Sturmian Auxiliary Proper to $F(x, y)$, we may substitute throughout for all the values of μ, instead of each such derivative, the more general one $\left(f\frac{d}{dx} - g\frac{d}{dy}\right)^{\mu} X_{2\mu}$, where f and g are any assumed positive constants, of course with the understanding that the second criterion also is to be $\left(f\frac{d}{dx} - g\frac{d}{dy}\right)f$ in lieu of $\frac{dF}{dx}$. And the method of Sturm will still be applicable for finding the positions of the real roots of $\frac{x}{y}$ in $f(x, y) = 0$ when we use these more general derivatives as the criteria instead of Sturm's own. When $g = 0$ the theorem is that of Sturm; when $f = 0$ it is an immediate deduction from this theorem applied to finding the positions of the root values of $\frac{y}{x}$, when it is borne in mind that the motions of $\frac{x}{y}$ and of $\frac{y}{x}$, as regards ascent and descent (excluding the moment for which either of these ratios is indefinitely near to zero) are inverse to each other. It is this that accounts for the negative sign which precedes g.

It is difficult to conceive by what theorem other than the assumed one the chasm between those extreme cases can be bridged over; and all analogy and all belief in continuity veto the supposition that no such bridge exists. "Divide *et impera*" is as true in algebra as in statecraft; but no less true and even more fertile is the maxim "auge *et impera*." The more to do or to prove, the easier the doing or the proof.

19.

SUR LES COVARIANTS FONDAMENTAUX D'UN SYSTÈME CUBO-BIQUADRATIQUE BINAIRE.

[*Comptes Rendus*, LXXXVII. (1878), pp. 242—4, 287—9.]

Le seul cas du dénombrement des *grundformen* binaires qui restait à déterminer par ma méthode, hors de ceux qui ont été calculés par la méthode de Gordan, est celui de la combinaison d'une forme biquadratique avec une forme cubique binaire.

Grâce à la coopération intelligente et à la grande habileté, comme calculateur, de M. J. Franklin, un de mes élèves à Baltimore, je suis en état de présenter à l'Académie le tableau des invariants et covariants fondamentaux, donné par la méthode de tamisage.

En partant de la forme primitive

$$\frac{1 - u^{-2}}{(1 - tu^4)(1 - tu^2)(1 - t)(1 - tu^{-2})(1 - tu^{-4})(1 - \tau u^3)(1 - \tau u)(1 - \tau u^{-1})(1 - \tau u^{-3})},$$

on parvient à la fraction génératrice canonique, dont le dénominateur est

$$(1 - t^2)(1 - t^3)(1 - t^2u^4)(1 - tu^4)(1 - \tau^4)(1 - \tau^2u^2)(1 - \tau u^3)(1 - t^2\tau^4)$$
$$(1 - t\tau^4)(1 - t^3\tau^2)(1 - t^3\tau^4),$$

et dont le numérateur contient 338 termes, dont ceux qui portent des coefficients positifs sont égaux en nombre à ceux qui portent le signe négatif. En effet, à chaque terme $kt^{\alpha} . \tau^{\beta} . u^{\gamma}$ correspond un terme

$$- kt^{\alpha'} . \tau^{\beta'} . u^{\gamma'},$$

où $\alpha + \alpha'$, $\beta + \beta'$, $\gamma + \gamma'$ sont des nombres constants, lesquels (si je ne me trompe, car j'ai eu le malheur de perdre le manuscrit) sont respectivement 12, 17, 11.

En représentant un terme $kt^{\alpha} . \tau^{\beta} . u^{\gamma}$ par le symbole $(\alpha . \beta . \gamma)^k$, voici le tableau des termes positifs.

2. 4.0	1. 1.3	7. 8.4	$(10.11.5)^2$	7.13.7	$(6.7.9)^2$
2. 6.0	1. 3.3	7.10.4	10.13.5	8. 7.7	$(6.9.9)^2$
$(3.4.0)^2$	2. 1.3	$(7.12.4)^4$	3. 0.6	$(8.9.7)^5$	7.7.9
$(3.6.0)^3$	$(2.3.3)^3$	$(7.14.4)^3$	4.10.6	$(8.11.7)^4$	$(7.9.9)^3$
$(4.4.0)^2$	3. 1.3	$(8.8.4)^2$	4.12.6	$(9.9.7)^3$	$(8.9.9)^3$
$(4.6.0)^2$	$(3.3.3)^5$	8.10.4	$(5.10.6)^2$	$(9.11.7)^4$	$(9.9.9)^3$
5. 4.0	$(3.5.3)^2$	$(8.12.4)^2$	$(5.12.6)^2$	10. 9.7	10.9.9
5. 6.0	3.11.3	$(8.14.4)^2$	6. 8.6	$(10.11.7)^2$	4.4.10
1. 1.1	$(4.3.3)^3$	9. 8.4	$(6.10.6)^2$	11.11.7	$(4.6.10)^2$
$(1.3.1)^2$	$(4.5.3)^3$	9.14.4	$(6.12.6)^2$	11.13.7	4.8.10
1. 5.1	4.11.3	10.14.4	6.14.6	3. 4.8	5.4.10
2. 1.1	5. 3.3	11.14.4	7. 8.6	3. 6.8	$(5.6.10)^3$
$(2.3.1)^3$	$(5.5.3)^2$	1. 1.5	$(7.10.6)^4$	4. 6.8	$(5.8.10)^2$
$(2.5.1)^2$	5.11.3	2. 1.5	$(7.12.6)^3$	4. 8.8	$(6.6.10)^2$
$(3.3.1)^2$	5.13.3	4.11.5	(7.14.6)	5. 6.8	$(6.8.10)^3$
$(3.5.1)^3$	6.13.3	$(5.11.5)^2$	8. 8.6	$(5.8.8)^3$	7.6.10
(4. 3.1)	$(7.13.3)^2$	5.13.5	$(8.10.6)^5$	5.10.8	$(7.8.10)^2$
$(4.5.1)^2$	7.15.3	$(6.11.5)^2$	$(8.12.6)^4$	$(6.8.8)^3$	8.8.10
5. 5.1	8.13.3	$(6.13.5)^3$	(8.14.6)	$(6.10.8)^3$	9.8.10
6. 5.1	8.15.3	6.15.5	$(9.10.6)^4$	$(7.8.8)^4$	3.3.11
$(1.2.2)^2$	9.15.3	7. 9.5	$(9.12.6)^5$	$(7.10.8)^3$	5.7.11
$(1.4.2)^2$	1. 2.4	$(7.11.5)^4$	9.14.6	$(8.8.8)^2$	5.9.11
$(2.2.2)^3$	$(2.2.4)^2$	$(7.13.5)^4$	$(10.10.6)^2$	$(8.10.8)^3$	$(6.7.11)^2$
$(2.4.2)^4$	$(3.2.4)^3$	7.15.5	$(10.12.6)^3$	$(9.10.8)^4$	$(6.9.11)^3$
$(3.2.2)^2$	$(3.4.4)^2$	$(8.9.5)^2$	11.12.6	$(10.10.8)^2$	$(7.7.11)^2$
$(3.4.2)^5$	4. 2.4	$(8.11.5)^5$	5.11.7	11.10.8	$(7.9.11)^2$
4. 2.2	4. 4.4	$(8.13.5)^4$	5.13.7	11.12.8	8.7.11
$(4.4.2)^3$	4.12.4	8.15.5	$(6.9.7)^2$	3. 5.9	8.9.11
4.12.2	$(5.12.4)^3$	$(9.9.5)^2$	$(6.11.7)^3$	$(4.5.9)^2$	
5. 4.2	5.14.4	$(9.11.5)^4$	$(6.13.7)^2$	$(4.7.9)^2$	
5.12.2	6.10.4	$(9.13.5)^4$	7. 7.7	$(5.5.9)^2$	
9.16.2	$(6.12.4)^4$	(9.15.5)	$(7.9.7)^5$	$(5.7.9)^3$	
0. 3.3	$(6.14.4)^2$	10. 9.5	$(7.11.7)^4$	5. 9.9	

En effectuant le tamisage, ces combinaisons se réduisent aux 50 suivantes:

2.4.0	1.1.1	$(1.2.2)^2$	0.3.3	1.2.4	1.1.5	3.0.6
2.6.0	$(1.3.1)^2$	$(1.4.2)^2$	1.1.3	2.2.4	2.1.5	
$(3.4.0)^2$	1.5.1	$(2.2.2)^2$	1.3.3	3.2.4		
$(3.6.0)^3$	2.1.1	$(2.4.2)^2$	2.1.3			
$(4.4.0)^2$	$(2.3.1)^3$	3.2.2	2.3.3			
$(4.6.0)^2$	$(2.5.1)^2$		3.1.3			
5.4.0	$(3.3.1)^2$		3.3.3			
5.6.0	$(3.5.1)^2$					
	4.3.1					

En ajoutant à ces 50 *grundformen* secondaires les 11 primaires qui proviennent du dénominateur dont les types sont

2.0.0	0.2.2
3.0.0	0.1.3
0.4.0	1.0.4
1.4.0	2.0.4
2.4.0	
3.2.0	
3.4.0	

on retrouve les 64 types calculés par M. Gundelfinger, selon la méthode de M. Gordan, avec l'exception des 3 suivants: 3.4.2, 3.4.2, 4.5.1.

Il reste à considérer les 3 covariants qui y correspondent; pour cela, je n'ai pas besoin de savoir la construction des *grundformen* données par M. Gundelfinger, car on peut procéder par un calcul algébrique direct pour déterminer si, oui ou non, le nombre des covariants linéairement indépendants appartenant à un quelconque de ces types peut être comblé par la combinaison de certains des 61 covariants connus. Ce nombre, on peut toujours le déterminer *a priori* par le théorème fondamental de M. Cayley, et, de plus, étant donné le type d'un covariant, on peut toujours trouver le covariant lui-même.

C'est par cette méthode, abrégée avec l'aide de quelques considérations appartenant à la théorie générale de la fraction génératrice, que je me suis convaincu de l'exactitude des résultats donnés par le tamisage pour le cas de deux biquadratiques, et que les deux formes, dites *irréductibles*, qui se trouvaient dans le tableau de M. Gordan, mais qui ne figuraient pas dans le mien, étaient superflues.

C'est la méthode la plus courte. Cependant, afin d'ôter toute nécessité d'expliquer la base du raisonnement, au lieu de suivre cette méthode dans la Note insérée dans les *Comptes rendus*, je jugeai préférable de prendre les deux formes qu'on obtient par la construction donnée par M. Gordan et d'en effectuer la décomposition, pour ainsi dire, sous les yeux du lecteur. J'espère, dans une prochaine Communication à l'Académie, par l'une ou l'autre de ces méthodes, pouvoir démontrer que les 3 *grundformen* supposées dont il est question sont superflues aussi, et que le véritable nombre des invariants et covariants irréductibles pour le système cubo-biquadratique binaire est effectivement 61 et non pas 64, comme le pensait M. Gundelfinger. En tout cas, je ferai savoir le vrai nombre de ces *grundformen*.

Pour m'assurer de l'exactitude des résultats précédemment donnés, j'ai fait calculer la fraction génératrice (fonction seulement de t et τ) dont le développement ne contient que les puissances positives de ces lettres, et tel que le coefficient numérique de $t^n . \tau^\nu$ coïncide avec le nombre des covariants (d'un ordre *quelconque* dans les variables) des degrés n, ν dans les coefficients de la biquadratique et la cubique respectivement. Cette fraction se déduit de la génératrice primitive

$$\frac{1}{(1-tu^4)(1-tu^2)(1-t)(1-tu^{-2})(1-tu^{-4})(1-\tau u^3)(1-\tau u)(1-\tau u^{-1})(1-\tau u^{-3})}$$

(qui ne diffère de celle dont je me suis déjà servi que dans le numérateur où se trouve 1 au lieu de $1-u^{-2}$) de la manière suivante. En la traitant comme une fonction de u, et en la décomposant en fractions partielles, on prend la somme des coefficients (fonctions de t et τ) de celles de ces fractions qui ont pour dénominateurs les facteurs de

$$1-tu^4,\ 1-tu;\ 1-\tau u^3,\ 1-\tau u:$$

cette somme sera la fraction génératrice cherchée. Or il est facile de démontrer que, en mettant $u=1$ dans la fraction génératrice canonique déjà obtenue, les deux fractions doivent devenir égales: on a fait ce calcul et, en comparant les deux expressions, on a trouvé entre elles un accord parfait sans qu'il y ait eu occasion d'introduire, dans l'une ou l'autre, un changement numérique quelconque, preuve satisfaisante de l'exactitude des résultats et, en même temps, de l'habileté très-peu commune du calculateur (M. Franklin), qui, par son dévouement consciencieux et opiniâtre à ce long et pénible travail, a rendu un véritable service au progrès de la science algébrique.

Ce qui ajoute considérablement à la difficulté du travail est la circonstance suivante, qui est assez intéressante en elle-même pour que je la cite ici. En faisant la décomposition en fractions partielles de la génératrice primitive, on trouvera contenus, dans les coefficients de celles mêmes qu'on doit conserver, les facteurs

$$\frac{1}{t-\tau^2},\ \frac{1}{t-\tau^4},\ \frac{1}{t^3-\tau^2},\ \frac{1}{t^3-\tau^4},$$

lesquels ne doivent et ne peuvent pas paraître dans la fraction canonique, de sorte qu'on sait d'avance que $t-\tau^2$, $t-\tau^4$, $t^3-\tau^2$, $t^3-\tau^4$ seront diviseurs exacts du numérateur de la fraction qui conduit à la fraction canonique. C'est, en effet, un théorème général que (quel que soit le nombre des *quantics* donnés), le dénominateur de la fraction génératrice canonique ne peut jamais contenir des facteurs où les lettres prises avec des exposants positifs sont distribuées entre deux groupes.

Toujours des facteurs de cette forme se présenteront dans le cours du calcul; mais, à la fin, quand toutes les sommations auront été effectuées, ils doivent nécessairement disparaître par voie de division dans le numérateur. Sans cette propriété, qu'on peut démontrer *a priori*, un théorie de la fonction génératrice pour des systèmes de *quantics* binaires aurait été impossible ou tout à fait inutile.

En ajoutant aux fractions canoniques que j'ai déjà données dans les *Comptes rendus* celle qui appartient à deux quadratiques, c'est-à-dire

$$\frac{1-t\tau u^2}{(1-t^2)(1-\tau^2)(1-t\tau)(1-tu^2)(1-\tau u^2)},$$

on voit qu'on est à présent en possession des génératrices canoniques pour tous les systèmes binaires qui proviennent des combinaisons deux à deux des ordres 2, 3, 4, c'est-à-dire 2.2, 2.3, 2.4, 3.3, 3.4, 4.4; et en ajoutant les génératrices déjà connues pour les *quantics* linéaires, quadratiques, cubiques et biquadratiques, pris séparément, à celles que j'ai données dans les *Comptes rendus* pour les *quantics* des ordres 5, 6, 8, on aura de même les génératrices appartenant aux *quantics* pris séparément d'un ordre quelconque, compris entre les limites 1 et 8, avec l'exception de 7, lequel cas M. Cayley a entrepris de calculer. De plus, j'ai donné, dans le second numéro du *American Mathematical Journal*, la génératrice pour la partie invariantive du *quantic* de l'ordre 10, et je me propose de la compléter en faisant calculer, en outre, sa partie covariantive.

J'ai aussi obtenu la génératrice générale pour un nombre quelconque donné des formes linéaires, et la même pour les formes quadratiques, entre lesquelles deux génératrices il existe un rapport algébrique vraiment remarquable, de sorte que, par le moyen d'une substitution algébrique des plus simples, on peut passer immédiatement de l'une à l'autre; mais ce travail n'a pas encore été publié.

Si quelqu'un voulait bien entreprendre le calcul de la génératrice des formes fondamentales pour le *quantic* de l'ordre 9, on aurait une collection très-compacte et assez étendue de ces fonctions importantes.

Je saisis cette occasion pour renouveler mes instances auprès des disciples de M. Gordan, si nombreux et si largement disséminés dans l'Allemagne, l'Italie et ailleurs, de vouloir bien faire exécuter entre eux, par sa méthode, les travaux nécessaires pour confirmer ou réfuter le dénombrement, que j'ai récemment publié dans les *Comptes rendus*, des covariants irréductibles appartenant au *quantic* du huitième degré. Ce serait manquer aux devoirs imposés par la science et la grande renommée de M. Gordan que de ne pas répondre à cet appel. Quant aux résultats que j'ai donnés ici pour le cas de la combinaison des ordres 3 et 4, il est bon d'ajouter que l'ordre le plus élevé des covariants irréductibles 6 est d'accord avec la limite supérieure pour le cas d'un nombre quelconque de *quantics* dont l'ordre de chacun n'excède pas 4, selon la formule donnée par M. Camille Jordan dans une séance toute récente de l'Académie. On trouvera, en effet, que, pour le cas supposé, cette limite est le nombre 6 lui-même.

20.

SUR LE VRAI NOMBRE DES FORMES IRRÉDUCTIBLES DU SYSTÈME CUBO-BIQUADRATIQUE.

[*Comptes Rendus,* LXXXVII. (1878), pp. 445—448.]

En addition aux 61 formes irréductibles que j'ai trouvées dans une Communication précédente faite à l'Académie, M. Gundelfinger affirme l'existence de trois autres: deux du type 3.4.2 et une du type 4.5.1, où le premier, le deuxième et le troisième chiffre expriment respectivement le degré ou l'ordre de la forme dans les coefficients de la biquadratique, de la cubique et dans les variables.

Je me bornerai dans cette Note à démontrer qu'il n'existe nul covariant du type 3.4.2.

Ce que M. Gordan nomme une *Ueberschiebung,* je le nommerai une *alliance*: si f et ϕ représentent

$$(a_0, a_1, a_2, \ldots)(x, y)^m; \quad (b_0, b_1, \ldots)(x, y)^n,$$

l'alliance $(f, \phi)^i$, i n'étant pas plus grand ni que m ni que n, sera un covariant de l'ordre $m+n-2i$, dont le coefficient de x^{m+n-2i}, que je nommerai *son représentant,* est

$$(1, -1)^i (a_0 b_i, a_1 b_{i-1}, a_2 b_{i-2}, \ldots, a_i b_0).$$

Je considérai le système spécial composé de $(a, b, c, d, e)(x, y)^4$ et de $(1, \beta, 0, 1)(x, y)^3$, où β sera traité comme un infinitésimal.

On aura donc

$$0.1.3 = x^3 + 3\beta x^2 y + y^3,$$
$$0.2.2 = 0x^2 + xy + \beta y^2,$$
$$0.3.3 = x^3 + 3\beta x^2 y - y^3,$$
$$0.4.6 = x^6 - y^6 + 6\beta x^5 y,$$
$$1.0.4 = ax^4 + 4bx^3 y + 6cx^2 y^2 + 4dxy^3 + ey^4,$$
$$2.0.4 = Ax^4 + 4Bx^3 y + 6Cx^2 y^2 + 4Dxy^3 + Ey^4,$$

où
$$A = ac - b^2, \quad C = \frac{2ae - bd - c^2}{3}, \quad D = \frac{be - cd}{2};$$

$$0.4.4 = x^2 y^2 + \ldots,$$
$$2.0.8 = \ldots + e^2 y^8,$$
$$0.3.9 = x^9 + \ldots.$$

Faisons

$l = (1.0.4,\ 0.1.3)^3$, dont le type est 1.1.1,
$m = (2.0.4,\ 0.1.3)^3$, „ 2.1.1,
$n = \ldots$, „ 2.0.0,
$p = \ldots$, „ 3.2.0,
$r_1 = (1.0.4,\ 0.3.3)^3$, „ 1.3.1,
$r_2 = (1.0.4,\ 0.3.5)^4$, „ 1.3.1,
$s_1 = (2.0.4,\ 0.3.3)^3$, „ 2.3.1,
$s_2 = (2.0.4,\ 0.3.5)^4$, „ 2.3.1,
$s_3 = (2.0.8,\ 0.3.9)^8$, „ 2.3.1,
$t_1 = (1.0.4,\ 0.4.6)^4$, „ 1.4.2,
$t_2 = (1.0.4,\ 0.4.4)^3$, „ 1.4.2,
$u = \ldots$ „ 0.2.2.

Alors les huit produits $mr_1, mr_2, ls_1, ls_2, ls_3, nt_1, nt_2, pu$ seront tous du type 3.4.2.

En se servant de la notation $R\phi$ pour exprimer le coefficient de la plus haute puissance de x dans la forme la plus générale de ϕ, on obtient, pour le système spécial dont il s'agit,

$$Rl = a + 3c\beta - d, \quad Rm = A + 3C\beta - D,$$
$$Rr_1 = a + 3c\beta + d, \quad Rs_1 = A + 3C\beta + D,$$
$$Rr_2 = a + 12c\beta - 4d, \quad Rs_2 = A + 12C\beta - 4D;$$

donc

$$Rmr_1 = (a + d + 3c\beta)(A - D + 3C\beta),$$
$$Rmr_2 = (a - 4d + 12c\beta)(A - D + 3C\beta),$$
$$Rls_1 = (a - d + 3c\beta)(A + D + 3C\beta),$$
$$Rls_2 = (a - d + 3c\beta)(A - 4D + 12C\beta).$$

Rs_3 possédera évidemment le terme e^2.

Rt_1, en négligeant les termes contenant β, sera formé au moyen des deux séries de coefficients

$$\begin{matrix} a & b & c & d & e, \\ 1 & 0 & 0 & 0 & 0 \end{matrix}$$

et sera égal à e, et de même, sous la même supposition, Rt_2 sera formé au moyen des deux séries

$$\begin{matrix} a & b & c & d, \\ 0 & 0 & 1 & 0, \end{matrix}$$

et sera égal à b.

De plus, $R(u)$ est absolument zéro, et

$$n = ae - 4bd + 3c^2.$$

On voit donc que $R(ls_3)$, seul des huit produits, contiendra le terme de^2, et conséquemment ne peut pas entrer dans une équation numérique quelconque entre ces produits. En le mettant de côté, on voit que, des sept produits qui restent, $R(nt_1)$ et $R(nt_2)$ contiendront, le premier, à lui seul, le terme c^2e, le second, à lui seul, le terme c^2b; conséquemment, en se souvenant que $R(pu) = 0$, ce n'est qu'entre Rmr_1, Rmr_2, Rls_1, Rls_2 qu'une liaison numérique (s'il y en a aucune) peut exister. Quant à ces quatre quantités, si même on ne tenait nul compte de β, une seule combinaison linéaire existe entre elles, pour laquelle la valeur est zéro, c'est-à-dire

$$3R(ls_1) - 2R(ls_2) - 3R(mr_1) + 2R(mr_2),$$

laquelle, en ayant égard à β, devient

$$(a - d + 3c\beta)(5A - 5D - 15C\beta) - (A - D + 3C\beta)(5a - 5d - 15c\beta),$$

c'est-à-dire $$30[(A-D)c - (a-d)C]\beta,$$

qui, évidemment, n'est pas zéro. Donc les huit covariants réductibles du type 3.4.2, mr_1, mr_2, ls_1, ls_2, ls_3, nt_1, nt_2, pu pour le système spécial qu'on a considéré, et *à plus forte raison* pour le système cubico-biquadratique général, sont linéairement indépendants.

Trouvons le nombre total des covariants linéairement indépendants de ce type. En général, pour deux formes dont les ordres sont i, i', les covariants du type j, j', ϵ linéairement indépendants sont en nombre égal à $S - S'$, ou

$$S = \sum_{m=w}^{m=0} (m:i, j)(w - m:i', j') \quad \text{et} \quad S' = \sum_{m=w'}^{m=0} (m:i, j)(w' - m:i', j'),$$

$$w = \frac{ij + i'j' - \epsilon}{2}, \quad w' = w - 1,$$

$(m:i, j)$ représentant le nombre des compositions qu'on peut effectuer de m avec j chiffres (zéro y compris) dont nul ne surpasse i, ou bien avec i chiffres dont nul ne surpasse j.

Dans le cas actuel,

$$w = \frac{4.3 + 3.4 - 2}{2} = 11, \quad w' = 10,$$

$$i = j' = 4, \quad j = i' = 3.$$

En donnant à m les valeurs successives de 0 jusqu'à 11, on trouve pour $(m:3, 4)$ ou bien $(m:4, 3)$ les valeurs

1, 1, 2, 3, 4, 4, 5, 4, 4, 3, 2, 1,

et, en faisant la progression dans le sens inverse,

1, 2, 3, 4, 4, 5, 4, 4, 3, 2, 1, 1.

On a conséquemment

$$S = 1 + 2 + 6 + 12 + 16 + 20 + 20 + 16 + 12 + 6 + 2 + 1,$$

$$S' = \quad 2 + 3 + \ 8 + 12 + 20 + 16 + 20 + 12 + 8 + 3 + 2$$

et $$S - S' = 1 \quad + 3 + \ 4 + \ 4 \quad + \ 4 - \ 4 \quad - 2 - 1 - 1$$

$$= 8.$$

Conséquemment le nombre *total* des covariants linéairement indépendants du type 3.4.2 n'est pas plus grand que le nombre des covariants de ce même type linéairement indépendants et *réductibles*: il n'y a donc pas de place *in rerum natura* pour les deux covariants quadratiques *irréductibles* du type 3.4.2 imaginés par M. Gundelfinger.

Dans une prochaine Communication j'entreprendrai l'examen de la seule forme qui reste à discuter, c'est-à-dire le covariant linéaire des degrés 4, 5 dans les coefficients, qui se trouve dans la Table de M. Gundelfinger, mais en dehors de la mienne. On sait déjà que le nombre des formes irréductibles pour le système en question est ou 61 ou 62. Il me semble peu douteux que c'est le premier de ces nombres qui sortira victorieux de la discussion du type 4.5.1.

21.

DÉTERMINATION DU NOMBRE EXACT DES COVARIANTS IRRÉDUCTIBLES DU SYSTÈME CUBO-BIQUADRATIQUE BINAIRE.

[*Comptes Rendus*, LXXXVII. (1878), pp. 477—481.]

Le seul type donné par M. Gundelfinger qui reste à discuter est le covariant linéaire des degrés 4 et 5 dans les coefficients de la biquadratique et la cubique respectivement. Un type quelconque étant représenté par $\alpha.\beta.\gamma$ quand ce type est monadelphique, je me servirai de $\alpha.\beta.\gamma$ indifféremment pour signifier le type et la forme qui y appartient et de $[\alpha.\beta.\gamma]$ pour signifier le coefficient de la plus haute puissance de x dans cette forme. On trouvera que le type $4.5.1$ qui est à discuter peut être produit de douze manières diverses, par la combinaison entre eux des types inférieurs déjà reconnus comme appartenant à des formes irréductibles, et j'écrirai les douze produits correspondants sous la forme

$$Z_1 = (3.0.6,\ 0.2.6)^6(1.0.4,\ 0.3.3)^3, \quad X = (3.0.0)(1.0.4,\ 0.5.5)^4,$$
$$Z_2 = (3.0.6,\ 0.2.6)^6(1.0.4,\ 0.3.5)^4, \quad Y_2 = (2.0.0)(2.0.8,\ 0.5.9)^8,$$
$$U_1 = (1.1.1)(3.0.0)(0.4.0), \quad Y_1 = (2.0.0)(2.0.4,\ 0.5.5)^4,$$
$$U_2 = (1.1.1)(3.0.6,\ 0.4.6)^6, \quad J_1 = (2.1.1)(2.0.4,\ 0.4.4)^4,$$
$$U_3 = (1.1.1)(3.0.8,\ 0.4.8)^8, \quad J_2 = (2.1.1)(2.0.0)(0.4.0),$$
$$U_4 = (1.1.1)(3.0.12,\ 0.4.12)^{12}, \quad J_3 = (2.1.1)(2.0.8,\ 0.4.8)^8.$$

Ecrivons

$$0.1.3 = (1,0,0,1)(x,y)^3, \quad 1.0.4 = (a,b,c,d,e)(x,y)^4,$$

on aura

$$2.0.4 = (A,B,C,D,E)(x,y)^4,$$

où $A = ac - b^2$, $B = \dfrac{ad - be}{2}$, $C = \dfrac{ae + 2b - 3c^2}{6}$, $D = \dfrac{be - cd}{2}$, $E = ce - d^2$,

$$3.0.6 = (L,M,N,P,Q,R,S)(x,y)^6,$$

où $L = a^2d - 3abc + 2b^3$, $2P = b^2e - d^2a$, $S = -e^2b + 3edc - 2d^3$,

$[1.1.1] = [(1.0.4,\ 0.1.3)^3] = a - d$, $[2.1.1] = [(2.0.4,\ 0.1.3)^3] = A - D$,

$$0.2.2 = xy, \quad 0.3.3 = x^3 - y^3, \quad 0.5.5 = x^4y - xy^4, \quad 0.3.5 = x^4y + xy^4.$$

Donc $[(2.0.4,\ 0.5.5)^4] = A + 4D$, $[(1.0.4,\ 0.5.5)^4] = a + 4d$,

$$0.2.6 = x^6 + 2x^3y^3 + y^6,$$

donc

$$[(3.0.6,\ 0.2.6)^6] = L - 2P + S,$$
$$0.4.6 = x^6 - y^6,$$

donc

$$[(3.0.6,\ 0.4.6)^6] = L - S.$$

Faisons $a = 1,\quad c = b^2,\quad e = bd;$

alors $A = 0$, $D = 0$.

Donc $Y_1 = 0,\quad J_1 = 0,\quad J_2 = 0,\quad J_3 = 0.$

Je vais démontrer que nulle liaison linéaire ne subsistera entre les coefficients de la plus haute puissance de x dans les huit covariants X, Y_2, Z_1, Z_2, U_1, U_2, U_3, U_4. $3.0.12$ représente $(1.0.4)^3$, et $0.4.12$ représente $(0.1.3)^4$; donc U_4 contiendra a^4, c'est-à-dire 1, et, comme on va voir, sera la seule des huit formes nommées qui le contient; donc la liaison, si elle existe, ne peut pas contenir U_4.

$$2.0.0 = ae - 4bd + 3c^2 = 3(b^4 - bd),$$
$$0.5.9 = (0.1.3)^2(0.3.3) = (x^3 + y^3)^2(x^3 - y^3) = x^9 + x^6y^3 - x^3y^6 - y^9,$$
$$2.0.8 = (1.0.4)^2 = e^2y^8 + \dots.$$

Donc $[(2.0.8,\ 0.5.9)^8]$ contiendra le terme e^2, et Y_2, par conséquent, le terme b^4e^2 ou b^6d^2.

$$[(1.0.4,\ 0.3.3)^3] = a + d, \quad [(1.0.4,\ 0.3.5)^4] = a - 4d;$$

ainsi on peut remplacer (Z_1), (Z_2) par les combinaisons linéaires T_1, T_2, où

$$T_1 = L - 2P + S, \quad T_2 = d(L - 2P + S),$$

et

$$L = d - b^3, \quad 2P = b^3d - d^2, \quad S = 2b^3d^2 - 2d^3,$$
$$(X) = (1 + 4d)\Delta, \quad (U_1) = (1 - d)\Delta,$$

où

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix} = \begin{vmatrix} 1 & b & b^2 \\ b & b^2 & d \\ b^2 & d & bd \end{vmatrix},$$

de sorte qu'on peut substituer, au lieu de (X) et (U_1), Δ et $d\Delta$,

$$(U_2) = (1 - d)(L - S),$$
$$3.0.8 = (a,b,c,d,e)(x,y)^4.(A,B,C,D,E)(x,y)^4,$$
$$0.4.8 = xy(x^3 + y^3)^2 + x^7y + 2x^4y^4 + xy^7.$$

Donc $(U_3) = (1 - d)\Lambda$, où Λ est une fonction linéaire de aB, bA, cb, dB, bD, aE, eA, dE, eD, c'est-à-dire, puisque $A = 0$, $D = 0$, Λ est une fonction linéaire de

$$d - b^3;\ b^3d + 2b^3 - 3b^4;\ d^2 - b^3d;\ b^3d - d^2;\ b^3d^2 - d^3.$$

On voit que b^6d^2 n'entre comme terme dans aucune des quantités

$$T_1,\ dT_1,\ \Delta,\ d\Delta,\ (1-d)(L-S),\ (1-d)\Lambda;$$

donc la liaison dont on discute l'existence ne peut pas contenir.

Quant aux six quantités qui restent, Δ seul contient b^6, $d\Delta$ seul db^6, et Λ seul b^4; donc la liaison, si elle existe, doit avoir lieu entre

$$T_1,\ dT_1,\ (1-d)(L-S),$$

et conséquemment entre les trois quantités

$$L-2P+S,\ (1-d)(L-P),\ (1-d)(S-P),$$

dont la dernière seule contient d^4 et les deux premières, c'est-à-dire

$$(1-d)[d+2d^2-(1-d)b^3],\quad \tfrac{1}{2}(1-d)[2d+d^2-(2+d)b^3],$$

ne sont pas l'une un multiple de l'autre. Donc il n'y a nulle liaison linéaire entre des coefficients du même rang des douze covariants qu'on considère pour le cas où $1.0.4$ et $0.1.3$ sont de la forme

$$(1, b, b^2, d, bd)(x, y)^4,\quad (1, 0, 0, 1)(x, y)^3$$

respectivement, et conséquemment, dans le cas général, une telle liaison, si elle existe, ne peut avoir lieu qu'entre les quatre dont les coefficients en question s'évanouissent pour le cas spécial, c'est-à-dire entre Y_1, J_1, J_2, J_3, mais cela est inadmissible; car, sur cette supposition, on aurait

$$\lambda(2.0.0)(2.5.1)+\mu(2.1.1)(2.4.1)=0,$$

où les quatre facteurs sont irréductibles. Il y a donc douze covariants réductibles, mais linéairement indépendants, du type $4.5.1$.

Or le nombre total des covariants de ce type linéairement indépendants est $S-S'$, ou

$$S=\sum_{q=0}^{q=w}(q:4, 4)(w-q:3, 5)\quad \text{et}\quad w=\frac{4.4+3.5-1}{2}=15,$$

et S' est ce que S devient quand on substitue $w-1$ (c'est-à-dire 14) à w. Or, en donnant à q les valeurs successives de 0 jusqu'à 15, $(q:4, 4)$ prend les valeurs

1, 1, 2, 3, 5, 5, 7, 7, 8, 7, 7, 5, 5, 3, 2, 1

et $(q:3, 5)$

1, 1, 2, 3, 4, 5, 6, 6, 6, 6, 5, 4, 3, 2, 1, 1.

On a donc

$$S=1+1+4+9+20+25+42+42+48+42+35+20+15+6+2+1,$$

$$S'=1+2+6+12+25+30+42+42+48+35+28+15+10+3+2$$

et

$$S-S'=1+2+3+8+12+6-6-8-4-1-1=12,$$

c'est-à-dire le nombre total des covariants linéairement indépendants du type $4.5.1$ est entièrement épuisé par les covariants réductibles et linéairement

indépendants de ce type. Donc il n'y a nul covariant irréductible du type $4.5.1$, et conséquemment le montant des *grundformen* pour le système cubo-biquadratique binaire est 61, comme j'ai trouvé, et non pas 64 comme M. Gundelfinger avait pensé.

Je conclus par l'observation importante que ma méthode serait parfaitement démontrée *à priori* si l'on pouvait démontrer le théorème suivant:

Soit σ le nombre total de formes linéairement indépendantes d'un type donné appartenant à un système donné de quantics, c'est-à-dire $\sigma=S-S'$ pour les formes binaires obtenues par composition des formes irréductibles de types inférieurs, et σ' le nombre de formes du même type; alors, si σ n'est pas plus petit que σ', le nombre des formes irréductibles du type sera $\sigma-\sigma'$ et dans le cas contraire zéro: c'est-à-dire que, dans le premier cas, il n'existera nulle liaison linéaire entre les formes composées et, dans le cas contraire, seulement $\sigma'-\sigma$ telles liaisons. Ce principe, indubitablement vrai pour les quantics binaires, s'étend probablement à des quantics en général et, puisque j'ai donné la règle universelle pour trouver le nombre total des formes linéairement indépendantes d'un type donné, il s'ensuit que, si l'on possède la connaissance d'une assemblée de formes ou plus simplement la connaissance des types numériquement exprimes qui figurent dans une assemblée, parmi lesquels se trouvent toutes les formes irréductibles, on a le moyen de trouver par un calcul purement arithmétique quels sont les types qui correspondent à des formes irréductibles et combien il y en a pour chaque type.

On aurait donc la solution arithmétique et sans tâtonnement du problème qui vient à la fin de la méthode de M. Gordan, dont la difficulté a créé tant d'embarras dans l'application de cette méthode et produit des erreurs tellement graves dans les résultats obtenus et jusqu'à ce jour acceptés comme vrais.

22.

SUR LES COVARIANTS IRRÉDUCTIBLES DU QUANTIC DU SEPTIÈME ORDRE *.

[*Comptes Rendus*, LXXXVII. (1878), pp. 505—509.]

M. CAYLEY a eu la bonté de calculer pour moi, par une méthode propre à lui, la fraction génératrice pour le quantic $(x, y)^7$ dans sa forme réduite. Il trouve que son numérateur est

$$
\begin{aligned}
& a^0 . 1 \\
&+ a^1 (-x - x^3 - x^5) \\
&+ a^2 (+x^2 + x^4 + 2x^6 + x^8 + x^{10}) \\
&+ a^3 (-x^7 - x^9 - x^{11} - x^{13}) \\
&+ a^4 (2x^4 + x^8 + x^{14}) \\
&+ a^5 (x + 2x^3 - x^9 - x^{11}) \\
&+ a^6 (-1 + 2x^2 - x^4 - x^8 - x^{10} + x^{12}) \\
&+ a^7 (4x + 4x^5 - x^7 - x^9 + x^{11} - x^{13}) \\
&+ a^8 (2 - x^2 - 3x^6 - 3x^8 - x^{10} - x^{12}) \\
&+ a^9 (x + 3x^3 + x^5 - x^7 + 2x^9 + 2x^{13}) \\
&+ a^{10} (-1 + 4x^2 - x^6 - 2x^8 - 2x^{10} - x^{14}) \\
&+ a^{11} (5x + 3x^3 + 2x^5 - x^7 - 2x^9 - x^{11} + x^{13}) \\
&+ a^{12} (5 + x^2 - 4x^6 - 6x^8 - 4x^{10} - x^{12} - 2x^{14}) \\
&+ a^{13} (x - 4x^5 - 4x^7 - x^9 + x^{11} + 4x^{13}) \\
&+ a^{14} (2 + 5x^2 + x^4 + x^6 - 2x^8 + 3x^{12} - x^{14}) \\
&+ a^{15} (3x - x^3 - x^5 - 7x^7 - 5x^9 - x^{11} - x^{13}) \\
&+ a^{16} (6 + 3x^2 + 3x^4 - 4x^6 - 3x^8 - x^{12} + 5x^{14}) \\
&+ a^{17} (-x - 2x^3 - 9x^5 - 8x^7 - 4x^9 - 3x^{11} + 4x^{13}) \\
&+ a^{18} (2 + 6x^2 + x^4 + 2x^6 + 2x^8 + x^{10} + 6x^{12} + 2x^{14})
\end{aligned}
$$

$$
\begin{aligned}
&+ a^{36} x^{14} \\
&+ a^{35} (-x^9 - x^{11} - x^{13}) \\
&+ a^{34} (x^4 + x^6 + 2x^8 + x^{10} + x^{12}) \\
&+ a^{33} (-x - x^3 - x^5 - x^7) \\
&+ a^{32} (1 + x^6 + 2x^{10}) \\
&+ a^{31} (-x^3 - x^5 + 2x^{11} + x^{13}) \\
&+ a^{30} (x^2 - x^4 - x^6 - x^{10} - 2x^{12} - x^{14}) \\
&+ a^{29} (-x + x^3 - x^5 - x^7 + 4x^9 + 4x^{13}) \\
&+ a^{28} (-x^2 - x^4 - 3x^6 - 3x^8 - x^{12} + 2x^{14}) \\
&+ a^{27} (2x + 2x^5 - x^7 + x^9 + 3x^{11} + x^{13}) \\
&+ a^{26} (-1 - 2x^4 - 2x^6 - x^8 + 4x^{10} - x^{14}) \\
&+ a^{25} (x - x^3 - 2x^5 - x^7 + 2x^9 + 3x^{11} + 5x^{13}) \\
&+ a^{24} (2 - x^2 - 4x^4 - 6x^6 - 4x^8 + x^{10} + 5x^{14}) \\
&+ a^{23} (+4x + x^3 - x^5 - 4x^7 - 4x^9 + x^{13}) \\
&+ a^{22} (-1 + 3x^2 - 2x^4 + x^8 + x^{10} + 5x^{12} + 2x^{14}) \\
&+ a^{21} (-x - x^3 - 5x^5 - 7x^7 - x^9 - x^{11} + 3x^{13}) \\
&+ a^{20} (5 - x^2 - 3x^6 - 4x^8 + 3x^{10} + 3x^{12} + 6x^{14}) \\
&+ a^{19} (4x - 3x^3 - 4x^5 - 8x^7 - 9x^9 - 2x^{11} - x^{13})
\end{aligned}
$$

Quant au dénominateur, on sait d'avance qu'il est

$$(1 - ax)(1 - ax^3)(1 - ax^5)(1 - ax^7)(1 - a^4)(1 - a^6)(1 - a^8)(1 - a^{10})(1 - a^{12}).$$

Pour obtenir la fraction génératrice sous sa forme canonique, je multiplie le numérateur et le dénominateur de cette forme réduite chacun par

$$(1 + a^6)(1 + a^{10})(1 + ax)(1 + ax^3)(1 + ax^5).$$

[* See below, p. 144.]

Alors le dénominateur devient évidemment

$$(1 - a^4)(1 - a^8)(1 - a^{12})^2(1 - a^{20})(1 - a^2x^2)(1 - a^2x^6)(1 - a^2x^{10})(1 - ax^7)$$

et le numérateur devient $P + Q$ où, pour trouver Q, on n'a qu'à substituer, pour un terme quelconque Ka^jx^ϵ, le terme $Ka^{j'}x^{\epsilon'}$, avec la condition que

$$j + j' = 55 \quad \text{et} \quad \epsilon + \epsilon' = 23.$$

On voit que la fraction sera alors sous sa forme canonique, par la raison qu'on ne trouvera ni a^4, ni a^8, ni a^{12}, ni a^{20} dans le numérateur affecté du signe $-$. On comprend qu'en effectuant le développement de l'une ou l'autre expression, selon les puissances ascendantes de a et de x, le coefficient de a^jx^ϵ exprimera le nombre total des covariants du degré j dans les coefficients du quantic du septième ordre et de l'ordre ϵ dans les variables.

Je trouve alors, pour la valeur de P, l'expression suivante:

$$
\begin{aligned}
& a^0 . 1 \\
&+ a^3 (x^3 + x^5 + x^7 + x^9 + x^{11} + x^{15}) \\
&+ a^4 (2x^4 + x^6 + 2x^8 + x^{10} + x^{14}) \\
&+ a^5 (x + 2x^3 + 2x^5 + 2x^7 + 2x^9 - x^{17} - x^{21}) \\
&+ a^6 (x^2 + 2x^4 + 3x^6 + 2x^8 + 2x^{12} - x^{14} - x^{16}) \\
&+ a^7 (3x + x^3 + 5x^5 + x^7 + x^{11} - x^{13} - 2x^{15} - x^{19} + x^{23}) \\
&+ a^8 (2 + 3x^2 + 3x^4 + 6x^6 + 3x^{10} - 2x^{12} - 2x^{14} - x^{16} - 2x^{18}) \\
&+ a^9 (3x + 5x^3 + 7x^5 + 2x^7 + 4x^9 - x^{11} - 2x^{13} - 2x^{15} - 3x^{17} - x^{19}) \\
&+ a^{10} (5x^2 + 4x^4 + 6x^6 + 6x^8 - 3x^{10} - 3x^{12} + x^{14} - 4x^{16} - x^{18} - x^{22}) \\
&+ a^{11} (5x + 8x^3 + 11x^5 - 4x^7 - 2x^{11} + x^{13} - 3x^{15} - x^{17}) \\
&+ a^{12} (4 + 9x^2 + 9x^4 + 12x^6 + 2x^{10} - 7x^{12} - 5x^{14} - 4x^{16} - x^{20} + x^{22}) \\
&+ a^{13} (9x + 8x^3 + 13x^5 + 5x^7 - x^9 \\
&\qquad - 3x^{11} - 13x^{13} - 9x^{15} - 3x^{17} - x^{19} + x^{21}) \\
&+ a^{14} (4 + 9x^2 + 12x^4 + 15x^6 - 2x^8 - 3x^{10} \\
&\qquad - 10x^{12} - 11x^{14} - 8x^{16} - 3x^{18} + 3x^{22}) \\
&+ a^{15} (9x + 12x^3 + 16x^5 + 6x^7 + 6x^9 \\
&\qquad - 7x^{11} - 11x^{13} - 9x^{15} - 4x^{17} - x^{19} + 2x^{21} + 2x^{23}) \\
&+ a^{16} (5 + 14x^2 + 15x^4 + 12x^6 + x^8 - x^{10} \\
&\qquad - 13x^{12} - 4x^{14} - 10x^{16} - x^{18} + 3x^{20} + 2x^{22}) \\
&+ a^{17} (12x + 14x^3 + 17x^5 - 5x^7 - 3x^9 - 17x^{11} \\
&\qquad - 16x^{13} - 11x^{15} - 5x^{17} + 2x^{19} + 3x^{21}) \\
&+ a^{18} (9 + 14x^2 + 14x^4 + 14x^6 - 4x^8 - 13x^{10} \\
&\qquad - 21x^{12} - 18x^{14} - 18x^{16} - x^{18} + 2x^{20} + 5x^{22}) \\
&+ a^{19} (15x + 16x^3 + 18x^5 + 27x^7 - 8x^9 \\
&\qquad - 19x^{11} - 20x^{13} - 20x^{15} - 6x^{17} + 2x^{21} + 4x^{23})
\end{aligned}
$$

$$+ a^{20}(6 + 14x^2 + 18x^4 + 12x^6 - 8x^8 - 14x^{10}$$
$$+ 2x^{12} - 18x^{14} - 13x^{16} + 2x^{18} + 5x^{20} + 6x^{22})$$
$$+ a^{21}(14x + 17x^3 + 19x^5 - x^7 - 8x^9 - 25x^{11}$$
$$- 23x^{13} - 14x^{15} - 2x^{17} + 4x^{19} + 8x^{21} + 4x^{23})$$
$$+ a^{22}(9 + 17x^2 + 15x^4 + 11x^6 - 8x^8 - 18x^{10}$$
$$- 31x^{12} - 17x^{14} - 13x^{16} + 6x^{18} + 9x^{20} + 9x^{22})$$
$$+ a^{23}(17x + 17x^3 - 20x^5 - 43x^7 - 18x^9 - 32x^{11}$$
$$- 26x^{13} - 22x^{15} - 4x^{17} + 9x^{19} + 9x^{21} + 5x^{23})$$
$$+ a^{24}(8 + 17x^2 + 14x^4 + 9x^6 - 19x^8 - 66x^{10}$$
$$- 37x^{12} - 24x^{14} - 17x^{16} + 8x^{18} + 9x^{20} + 12x^{22})$$
$$+ a^{25}(15x + 15x^3 + 17x^5 - 7x^7 - 27x^9 - 30x^{11}$$
$$- 32x^{13} - 23x^{15} + 3x^{17} + 9x^{19} + 12x^{21} + 9x^{23})$$
$$+ a^{26}(9 + 13x^2 + 14x^4 + 6x^6 - 20x^8 - 23x^{10}$$
$$- 35x^{12} - 19x^{14} - 10x^{16} + 10x^{18} + 14x^{20} + 14x^{22})$$
$$+ a^{27}(14x + 15x^3 + 13x^5 - 15x^7 - 18x^9 - 37x^{11}$$
$$- 31x^{13} - 17x^{15} + 3x^{17} + 14x^{19} + 14x^{21} + 6x^{23}).$$

Pour effectuer le tamisage, en observant qu'en vertu des formules de M. C. Jordan on peut négliger toute puissance de x dont l'exposant excède 15, on obtient, pour les termes positifs de P et de Q qu'on doit obtenir, la table suivante:

$$+ x^0(1 + 2a^8 + 4a^{12} + 4a^{14} + 5a^{16} + 9a^{18} + 6a^{20} + 9a^{22}$$
$$+ 8a^{24} + 9a^{26} + 6a^{28} + 9a^{30} + 5a^{32} + 4a^{34} + 4a^{36} + 2a^{40} + a^{48})$$
$$+ x\,(a^5 + 3a^9 + 5a^{11} + 9a^{15} + 12a^{17} + 15a^{19} + 14a^{21} + 17a^{23} + 15a^{25}$$
$$+ 14a^{27} + 14a^{29} + 12a^{31} + 9a^{33} + 6a^{35} + 5a^{37} + 2a^{39} + 3a^{41} + a^{43})$$
$$+ x^2(a^6 + 5a^8 + 5a^{10} + 4a^{12} + 9a^{14} + 14a^{16}$$
$$+ 14a^{18} + 14a^{20} + 17a^{22} + 17a^{24} + 13a^{26} + 14a^{28}$$
$$+ 12a^{30} + 9a^{32} + 8a^{34} + 2a^{36} + 3a^{38} + 2a^{40} + a^{42})$$
$$+ x^3(a^3 + 2a^5 + a^7 + 5a^9 + 8a^{11} + 8a^{13} + 12a^{15}$$
$$+ 14a^{17} + 16a^{19} + 17a^{21} + 17a^{23} + 15a^{25}$$
$$+ 15a^{27} + 14a^{29} + 9a^{34} + 9a^{33} + 5a^{35} + 2a^{37} + 3a^{39})$$
$$+ x^4(2a^4 + 2a^6 + 3a^8 + 4a^{10} + 9a^{12} + 12a^{14} + 15a^{16} + 14a^{18} + 14a^{20}$$
$$+ 15a^{22} + 14a^{24} + 14a^{26} + 14a^{28} + 9a^{30} + 9a^{32} + 4a^{34} + 2a^{36})$$
$$+ x^5(a^3 + 2a^5 + 5a^7 + 7a^9 + 11a^{11} + 13a^{13} + 16a^{15} + 17a^{17}$$
$$+ 18a^{19} + 19a^{21} + 17a^{25} + 13a^{27} + 10a^{29} + 8a^{31} + 6a^{33} + 2a^{35})$$
$$+ x^6(a^4 + 3a^6 + 6a^8 + 6a^{10} + 12a^{12} + 15a^{14} + 12a^{16}$$
$$+ 14a^{18} + 12a^{20} + 11a^{22} + 9a^{24} + 6a^{26} + 3a^{28} + 3a^{30})$$
$$+ x^7(a^3 + 2a^5 + a^7 + 2a^9 + 5a^{13} + 6a^{15} + 27a^{19})$$

$$+ x^8(2a^4 + 2a^6 + 6a^{10} + a^{16} + a^{52})$$
$$+ x^9(a^3 + 2a^5 + 4a^9 + 6a^{15} + a^{45} + a^{51})$$
$$+ x^{10}(a^4 + 3a^8 + 2a^{12} + a^{44})$$
$$+ x^{11}(a^3 + a^7 + 2a^{35} + 2a^{49})$$
$$+ x^{12}(2a^6 + 2a^{20} + a^{48} + a^{52})$$
$$+ x^{13}(a^{11} + 2a^{43} + 3a^{47} + a^{51})$$
$$+ x^{14}(a^4 + a^{10} + 6a^{40} + 4a^{46} + 2a^{50} + a^{52})$$
$$+ x^{15}(a^3 + a^{39} + 6a^{45} + 2a^{49} + 2a^{51}).$$

Le tamisage étant effectué (ce qu'on peut aisément opérer par simple inspection), les termes et les coefficients numériques, qui seuls restent sains et saufs, toute soustraction faite, seront les suivants·

$1,\ 2a^8,\ 4a^{12},\ 4a^{14},\ 5a^{16},\ 9a^{18},\ a^{22},$
$a^5x,\ 3a^9x,\ 5a^{11}x,\ 9a^{15}x,\ 2a^{17}x,\ a^{19}x,$
$a^6x^2;\ 5a^8x^2,\ 5a^{10}x^2,\ 4a^{12}x^2,\ 4a^{14}x^2,$
$a^3x^3,\ 2a^5x^3,\ a^7x^3,\ 5a^9x^3,\ 5a^{11}x^3,$
$2a^4x^4,\ 2a^6x^4,\ 3a^8x^4,\ 4a^{10}x^4,$
$a^3x^5,\ 2a^5x^5,\ 5a^7x^5,\ 2a^9x^5,$
$a^4x^6,\ 2a^6x^6,\ 3a^8x^6,$
$a^3x^7,\ 2a^5x^7,$
$2a^4x^8,\ a^6x^8,$
$a^3x^9,\ 2a^5x^9,$
$a^4x^{10},$
$a^3x^{11},\ a^7x^{11},$
$a^4x^{14},$
$a^3x^{15}.$

En ajoutant à ces termes ceux qui sont fournis par le dénominateur, c'est-à-dire

$$a^4,\ a^8,\ 2a^{12},\ a^{20},\ a^2x^2,\ a^2x^6,\ a^2x^{10},\ ax^7,$$

on a le tableau complet des invariants et covariants irréductibles du quantic du septième ordre, sous la convention qu'on comprend, par Ka^jx^ϵ, K covariants du degré j et de l'ordre ϵ. De même $2a^8$, a^8 signifiera trois invariants du degré 8; $4a^{12}$, $2a^{12}$ six invariants du degré 12. Le covariant dénoté en haut par a^3x^{15} démontre que la limite inférieure pour l'ordre des covariants d'un système illimité de quantics, chacun d'ordre inférieur à n, est actuellement atteinte quand $n=7$, et même quand le système illimité se réduit à un seul quantic, ce qui aussi a lieu pour $n=8$ et pour tous les ordres inférieurs, sauf pour $n=3$, dans lequel cas la limite 4, il est vrai, est atteinte; mais le système doit contenir au moins deux quantics. L'apparence des invariants, dont les degrés sont 14, 18 et 22 (nombres nécessairement pairs), est aussi digne d'observation. On en conclut (et même un seul de ces covariants servirait à établir la même conclusion) que $1-a^7$ paraîtra comme facteur dans la partie invariantive du dénominateur de la fraction génératrice pour tout quantic dont l'ordre est pair et plus grand que 10.

23.

SUR LA FORME BINAIRE DU SEPTIÈME ORDRE.

[*Comptes Rendus*, LXXXVII. (1878), pp. 899—903.]

IL y a une erreur dans la Table pour la fraction réduite sur laquelle j'ai basé mon calcul des covariants irréductibles de la forme binaire du septième ordre. Le terme qui multiplie a^7, au lieu de

$$4x + 4x^3 - x^7 - x^9 + x^{11} - x^{13},$$

doit être écrit $4x + x^3 + 3x^5 - x^9 + x^{11}$, et, conséquemment, le terme complémentaire qui multiplie a^{38}, au lieu d'être

$$4x^{13} + 4x^9 - x^7 - x^5 + x^3 - x,$$

doit être écrit $4x^{13} + x^{11} + 3x^9 - x^5 + x^3$. Mais, de plus, pour ne pas parler d'erreurs de multiplication, le calcul a besoin d'être modifié, par suite d'une circonstance qui s'est présentée ici pour la première fois dans l'application de ma méthode: c'est que l'existence d'un invariant irréductible du degré 20 a été présumée, tandis qu'il y a toute raison de croire qu'il n'existe nul invariant dont le degré soit 20 ou même un multiple quelconque de 10, appartenant à la forme du septième ordre.

Voici la marche à suivre, à cause de cette circonstance. La fraction réduite a pour dénominateur

$$(1-a^4)(1-a^6)(1-a^8)(1-a^{10})(1-a^{12})(1-ax)(1-ax^3)(1-ax^5)(1-ax^7).$$

Je multiplie le numérateur et le dénominateur par

$$(1+a^6)(1+ax)(1+ax^3)(1+ax^5).$$

Cela me donne une Table dont celle qui suit est la moitié :

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}	x^{18}	x^{19}	x^{20}	x^{21}	x^{22}	x^{23}
a^0	1		0		0		0		0		0		0		0		0		0		0		0	
a^1		0		0		0		0		0		0		0		0		0		0		0		0
a^2	0		0		0		0		0		0		0		0		0		0		0		0	
a^3		0		1		1		1		1		1		0		1		0		0		0		0
a^4	0		0		2		1		2		1		0		1		0		0		0		0	
a^5		1		2		2		2		2		0		0		0		−1		0		−1		0
a^6	0		3		2		3		3		0		2		−1		−1		0		0		0	
a^7		3		2		4		4		0		1		0		−2		0		−1		0		1
a^8	2		3		4		6		1		3		−1		−2		0		−1		0		0	
a^9		3		5		7		1		4		0		−2		−1		−2		0		1		0
a^{10}	−1		5		8		6		4		1		−4		0		−3		−1		0		0	
a^{11}		5		8		8		8		4		−4		−1		−5		−1		0		0		0
a^{12}	4		9		9		12		4		−1		−3		−5		−6		0		−1		1	
a^{13}		9		8		11		5		−2		−4		−8		−10		−3		−1		1		0
a^{14}	4		9		11		10		−3		−4		−9		−11		−7		−2		0		3	
a^{15}		8		10		14		1		0		−10		−11		−8		−2		0		4		2
a^{16}	5		11		13		9		−2		−5		−18		−8		−8		−1		3		3	
a^{17}		9		13		12		2		−3		−18		−13		−13		−5		3		3		−1
a^{18}	7		11		11		8		−4		−16		−19		−13		−15		3		2		5	
a^{19}		12		11		11		−1		−12		−18		−18		−18		−1		3		4		4
a^{20}	7		9		10		6		−14		−17		−21		−19		−9		3		5		9	
a^{21}		9		9		11		−9		−12		−23		−24		−11		−1		4		8		4
a^{22}	5		8		10		−1		−12		−17		−28		−12		−9		6		10		8	

Pour la compléter, on n'a qu'à se rappeler que, pour chaque terme $ka^\alpha x^\lambda$ dans la moitié donnée, il faut suppléer un terme $ka^\beta x^\mu$ dans la partie supprimée, où $\alpha+\beta=45$, $\lambda+\mu=23$; ainsi, toutes les colonnes de chiffres dans la partie donnée se répéteront en sens inverse, par rapport en même temps à la direction verticale et à la direction horizontale, dans la partie supprimée. Je suppose ce numérateur multiplié par

$$1 + a^{10} + a^{20} + \dots$$

à l'infini, et le facteur $1 - a^{10}$ chassé du dénominateur, qui ne contiendra alors que les facteurs

$$1-a^4,\ 1-a^{12},\ 1-a^8,\ 1-a^{12},\ 1-a^2x^2,\ 1-a^2x^6,\ 1-a^2x^{10},\ 1-ax^7,$$

dont chacun représente par ses indices le degré et l'ordre d'un covariant irréductible; c'est-à-dire, au lieu de multiplier le numérateur et le dénominateur par $1 + a^{10}$, je divise chacun par $1 - a^{10}$.

Alors j'opère par tamisage successivement sur les séries qui multiplient les puissances successives de x dans le numérateur, ce qui, nonobstant le nombre infini des termes dans ces séries, est très-facile à faire, à cause de la récurrence constante des mêmes chiffres. En combinant avec les restes du tamisage ainsi opéré les invariants et les covariants représentés par les facteurs du dénominateur, j'obtiens la Table suivante, où l'on remarquera que nul invariant du degré 20 ne figure:

Table des 124 *covariants irréductibles de la forme binaire du septième ordre.*

Degré dans les coefficients	Ordre dans les variables													
	0	1	2	3	4	5	6	7	8	9	10	11	14	15
1............								1						
2............			1				1				1			
3............				1		1		1		1		1		1
4............	1				2		1		2		1		1	
5............		1		2		2		2		2				
6............			3		2		2		2					
7............		3		2		4		2						
8............	3		3		3		3							
9............		3		5		2								
10............			4		3									
11............		5		3										
12............	6		6											
13............		7												
14............	4													
15............		3												
16............	2													
17............		2												
18............	9													
22............	1													

Ce qui est absolument démontré, c'est qu'il existe les 124 covariants irréductibles indiqués par cette table. Ce qui est assujetti au doute métaphysique dont j'ai fréquemment parlé, c'est la possibilité de l'existence d'autres irréductibles en dehors de la Table. Si le cas est ainsi, il sera en contradiction avec le *postulatum* qu'il ne faut jamais supposer l'existence de plus de rapports syzygétiques entre les irréductibles qu'il n'est nécessaire pour satisfaire aux valeurs connues du nombre total des covariants linéairement indépendants pour chaque degré et ordre, ou, ce qui revient à la même chose, que des covariants irréductibles et des syzygies indécomposables ne peuvent pas coexister pour le même ordre et degré. En faisant l'énumération des invariants de tous les degrés jusqu'à 20, on trouvera facilement que, selon ce principe, on n'avait pas le droit d'admettre préalablement l'existence d'un invariant irréductible du degré 20. C'est pour la première fois, dans tous les cas si nombreux que j'ai discutés, que cette difficulté

s'est présentée, c'est-à-dire l'impossibilité de trouver une fraction canonique avec un numérateur fini, équivalente à la fraction réduite. Mais les résultats que j'obtiens ne sont nullement moins certains, à cause de cette difficulté que j'ai trouvé le moyen sûr et commode de vaincre. Les détails du calcul seront donnés dans une prochaine partie de l'*American Journal of Mathematics*.

Je terminerai ici par une observation qui me paraît très-significative: c'est qu'il résulte du calcul qui a été fait que l'effet du tamisage est précisément le même que si l'on avait multiplié le numérateur de la forme réduite par $1+a^{10}$ au lieu de le diviser par $1-a^{10}$, de sorte qu'on aurait pu agir précisément comme si l'invariant irréductible du degré 20 existait; seulement, au bout du compte, on aurait exclu cet invariant de la Table des formes irréductibles.

Quant à ce qui se rapporte au tamisage que j'ai appliqué aux séries simplement infinies, il est bon de se rappeler que l'usage qu'on fait de la fraction génératrice (pour un quantic binaire) mise sous une forme canonique n'est qu'une méthode abrégée, et pour ainsi dire artificielle, pour obtenir le même résultat qu'on pourrait obtenir, mais avec beaucoup plus de difficulté, en opérant directement le tamisage sur la série de nombres, doublement infinie, qu'on obtient en développant cette fraction en série de puissances de a et x, de laquelle série les coefficients représenteront le nombre des covariants linéairement indépendants pour chaque degré et chaque ordre, de zéro jusqu'à l'infini. Cette remarque fait voir aussi que la distinction entre les irréductibles primaires et secondaires ne tient à aucune différence essentielle de nature entre les deux, mais seulement à la méthode qu'on emploie pour les obtenir, et, en variant cette méthode, les irréductibles peuvent changer leur nom de *primaires* en *secondaires*, et *vice versâ*.

24.

ON AN APPLICATION OF THE NEW ATOMIC THEORY TO THE GRAPHICAL REPRESENTATION OF THE INVARIANTS AND COVARIANTS OF BINARY QUANTICS,—WITH THREE APPENDICES.

[*American Journal of Mathematics* I. (1878), pp. 64—125.]

[The figures are given on p. 163.]

By the *new* Atomic Theory I mean that sublime invention of Kekulé which stands to the *old* in a somewhat similar relation as the Astronomy of Kepler to Ptolemy's, or the System of Nature of Darwin to that of Linnæus;—like the latter it lies outside of the immediate sphere of energetics, basing its laws on pure relations of form, and like the former as perfected by Newton, these laws admit of exact arithmetical definitions.

Casting about, as I lay awake in bed one night, to discover some means of conveying an intelligible conception of the objects of modern algebra to a mixed society, mainly composed of physicists, chemists and biologists, interspersed only with a few mathematicians, to which I stood engaged to give some account of my recent researches in this subject of my predilection, and impressed as I had long been with a feeling of affinity if not identity of object between the inquiry into compound radicals and the search for "Grundformen" or irreducible invariants, I was agreeably surprised to find, of a sudden, distinctly pictured on my mental retina a chemico-graphical image serving to embody and illustrate the relations of these derived algebraical forms to their primitives and to each other which would perfectly accomplish the object I had in view, as I will now proceed to explain.

To those unacquainted with the laws of atomicity I recommend Dr Frankland's *Lecture Notes for Chemical Students*, vols. 1 and 2, London (Van Voorst), a perfect storehouse of information on the subject arranged in the most handy order and put together and explained with true scientific accuracy and precision. On the algebraical side of the subject my readers may consult Salmon's *Lessons on Higher Algebra*, Clebsch's *Binären Formen*

or Faà de Bruno's treatise more elementary than the former, *Sur les formes binaires* (Turin, 1876). I propose also to run a course of articles on the Invariantive Theory, beginning from the beginning, through the pages of this Journal, from my own particular point of view, which will be found, I hope, considerably to simplify the subject.

Any binary quantic may be denoted by a single letter with a number attached corresponding to its degree, and may therefore be adumbrated by a chemical symbol with corresponding *valence*. Thus hydrogen, chlorine, bromine, or potassium will serve to denote so many distinct binary linear forms; oxygen, zinc, magnesium, &c., binary quadrics; boron, gold, thallium, cubics; carbon, lead, silicon, tin, quartics; nitrogen, phosphorus, arsenic, antimony, &c., quintics; sulphur, iron, cobalt, nickel, &c., sextics. The sixth appears to be the highest degree of valency at present recognizable in natural substances.

The factors of any algebraical form may be regarded as in some sense the analogues of the rays of atomicity in the equivalent chemical atom—these rays being what Dr Frankland, according to his nomenclature, would have to designate as free bonds; such rays between two consecutive atoms in a molecule are conceived as blending in some manner so as to represent some unknown kind of special relation existing between them; they may then with propriety be called bonds or lines of connexion.

An invariant of a form or system of algebraical forms must thus represent a saturated system of atoms in which the rays of all the atoms are connected into bonds. Thus, for example, O_2 (oxygen combined with itself) will represent a quadratic invariant of a quadric. Its graph is seen in Fig. 1 (a). Potash, a combination of potassium, oxygen and hydrogen, having for its graph that of Fig. 2, will represent the invariant to a system of one quadratic and two linear forms which is linear in each set of coefficients. This is in fact the *Connective* between the given quadratic and another obtained by taking the product of the two linear forms. Phosphorus and arsenic are quinquivalent, but form "tetratomic molecules." An isolated element of phosphorus may possibly, therefore, be represented by the graph of Fig. 3, which will correspond, if the figure is indecomposable (which requires examination to determine), to the quart-invariant of a quintic, and the same for arsenic. So too the graph to nitric anhydride (Fig. 4) may possibly serve to express the resultant of a binary quadric and quintic, or this blended with any other invariant of the system included under the same type [10: 5, 2; 2, 5]*. And in general, the Jacobian to any two quantics will be completely expressed by their two corresponding atoms connected by a pair of bonds. Nitric acid has for its graph that of Fig. 5. This will

* 10 is the weight; 5, 2 the degree and order in the coefficients of the quintic; 2, 5 the degree and order in the coefficients of the quadric. See p. [151].

correspond to an invariant of a quintic, quadric and linear form of the first order in the coefficients of each extreme and of the third order in those of the middle form. Such an invariant as is well known (by virtue of a general principle about to be stated), is, in substance, the same thing as a lineo-cubic linear covariant of a quintic and quadric. The general arithmetical rule (also hereafter to be set forth) for determining the number of asyzygetic derivatives of a given type, enables us to see that such a covariant exists and is monadelphic. It may readily be obtained by making the given quintic (after substituting $\frac{d}{dy}$ and $-\frac{d}{dx}$ for x and y respectively) operate on the cube of the given quadratic.

The general principle above referred to, which is extremely easily proved from the partial differential equation (but which I believe I was the first to enunciate), is that every covariant of one quantic or several simultaneous quantics may be transformed into an invariant of the same quantic or set of quantics enlarged by the addition thereto of one additional linear form; the degree in the variables becoming replaced by the order in the new set of coefficients, and the orders in the original sets of coefficients remaining unchanged.

Thus, covariants might altogether be dispensed with and invariants alone made the object of study. But algebraists have found and will continue to find it more convenient to dispense with the additional linear form and to retain in use covariants as well as invariants. With me, covariants are to be regarded as simple emanations, so to say, from differentiants which are functions of the coefficients alone, and of which invariants are merely a particular species satisfying a certain condition of maximum; this is why the properties of invariants can with difficulty be made out so long as they are studied alone; it was only by contemplating the whole group of differentiants simultaneously, that I was enabled, after a suspense of more than a quarter of a century, to set on an irrefragable basis Professor Cayley's fundamental arithmetical theorem for calculating the number of asyzygetic invariants and covariants to a given quantic, and also the more general theorem which I have shown applies to a system of quantics*.

I will here give this rule, as it may be useful to us in the sequel. First, for a single quantic.—Let i be its degree, j the order of any covariant, w its weight (that is, the weight of its root-differentiant). Then we may call its type $[w: i, j]$. Now let us, in general, employ $(m: i, j)$ to signify *the number of ways* in which m can be made up with j parts of which each is either 0, 1, 2, 3, &c. up to i, and let us use the symbol $\Delta\,(m: i, j)$ to denote $(m: i, j) - \{(m-1): i, j\}$; then $\Delta\,(w: i, j)$ is the number of arbitrary

* The demonstration is given in a paper inserted in the *Philosophical Magazine* for March of this year [p. 117, above].

numerical parameters in the most general covariant or invariant answering to the type $[w: i, j]$. It is a known theorem in partitions of numbers that $(m: i, j) = (m: j, i)$, from which it follows that the number of arbitrary parameters remains unaltered when the degree of the primitive and the order of the derivative are interchanged. It is sometimes more convenient to use the degree of the derivative in lieu of the weight to express its type; let then ϵ be the degree, so that $\epsilon = ij - 2w$; then I shall employ, when desirable, $[i, j: \epsilon]$ to signify the same thing as $[w: i, j]$. If there be several quantics, the type may be expressed in like manner by $[w: i, j;\ i', j';$ &c.$]$, or by $[i, j;\ i', j';$ &c.$: \epsilon]$. The rule for finding the number of independent parameters, or the most general covariant or invariant corresponding to either of these types, then becomes as follows. Let $(m: i, j;\ i', j';$ &c.$)$ denote the number of ways in which m can be made up of j elements each comprised between 0 and i, combined with j' elements each comprised between 0 and i', and so on, and let $\Delta\,(m: i, j;\ i', j';$ &c.$)$ denote $(m: i, j;\ i', j';$ &c.$) - (m-1: i, j;\ i', j';$ &c.$)$. The number of parameters in question is $\Delta\,(w: i, j;\ i', j';$ &c.$)$ and I may observe that the value of Δ remains unaltered when *any one* i is interchanged with the corresponding j, and consequently when any number of i's are interchanged, each respectively with its corresponding j. This theorem of reciprocity for a single quantic is due to M. Hermite. The above statement, applicable to a quantic system, constitutes a notable and important generalization of it. In Note D to Appendix 2, it will be shown that this theorem still further generalized by employing the method of Emanation (virtually the same thing as Regnault's law of substitution) admits of the following simple chemico-algebraical statement. *In an algebraical compound (in an algebraical sense) m n-valent atoms may be replaced by n m-valent ones.* But it should be observed that this replacement involves an entire reconstruction of the representative graph and conveys the notion of respondence or contraposition rather than similarity of type. (See Appendix 2.)

It may be well here (as it will be useful in the sequel) to say a few words more on these differentiants in their relation to covariants. Every covariant may be regarded as arising from either of two differentiants, as from a root. One, the coefficient of the highest power of x, is called a differentiant in x; the other, the coefficient of the highest power of y, a differentiant in y. It is not, for ordinary purposes such as present themselves in this study, requisite to consider more than one of these at a time, and for greater brevity it will be understood that, unless I give notice to the contrary, a *differentiant* will always be understood to mean one in x. I shall also suppose, when dealing with a single binary quantic, that the successive coefficients beginning with the highest power of x, are $a, b, c, \dots h, k, l$ multiplied successively by the binomial coefficients proper to the degree of the form.

A differentiant, D, may then be defined as a rational integer function of the coefficients of equal weight in all its terms in respect to either variable subject to satisfy the equation

$$\left(a\frac{d}{db}+2b\frac{d}{dc}+3c\frac{d}{dd}+\dots\right)D=0.$$

An invariant again may be regarded as a rational integer isobaric function of the coefficients which is a differentiant both in regard to x and y, but it may be best defined as a differentiant (meaning in one of the variables as x) to a given form or form-system whose weight (in respect of the selected variable) is the greatest possible that its order in the coefficients admits of. [The doubleness of the character and the symmetry, direct or skew, of a differentiant satisfying this condition of maximum then become matter of deduction from the definition.] To each covariant corresponds but one differentiant (in a given variable), and *vice versa*, to each differentiant will correspond only one covariant. In fact, D being the differentiant in x, the covariant taking its rise in D is

$$Dx^{\epsilon}+\Omega.Dx^{\epsilon-1}y+\frac{1}{1.2}(\Omega.)^2Dx^{\epsilon-2}y^2+\dots,$$

where $\Omega.$ represents the operator,

$$\left(l\frac{d}{dk}+2k\frac{d}{dh}+3h\frac{d}{dg}+\dots\right)$$

if D belongs to a simple quantic, and

$$\Sigma\left(l\frac{d}{dk}+2k\frac{d}{dh}+\dots\right)$$

if it belongs to a quantic system, and where ϵ is $ij-2w$ for a single quantic, and $\Sigma ij-2w$ for a quantic system, i representing the degree of any one form in the variables, j the order of the differentiant in the corresponding set of coefficients, and w the weight of the differentiant. As ϵ can never become negative, we see that the maximum value of w, when each i and its corresponding j is given, will be $\frac{1}{2}ij$ for one form, and $\frac{1}{2}\Sigma ij$ for a form system. By the weight of any covariant I shall understand the weight of the differentiant in which it may be regarded as originating. Precisely as algebraists find their advantage in using covariants when invariants alone might be made to suffice, chemists find theirs in the use of organic or inorganic compound radicals, as unsaturated forms capable of becoming saturated by the addition of the right number of monad elements to the unsatisfied atoms, that is, those through which a sufficient number of *bonds* do not pass to exhaust their valency. Thus, for example, Hydroxyl H – O – is the linear covariant of the quadratic form oxygen, and the linear form hydrogen; this, combined with the linear form potassium, expresses the invariant potash denoted by H – O – K.

As the free valence of a single atom corresponds to the degree of a single quantic, so the free valence of a molecule formed by an aggregate of atoms will express the degree of the corresponding covariant. Let us understand by the *toti-valence* of a molecule the sum of the absolute valences of the separate atoms of which it is composed. This toti-valence will obviously correspond to the sum, Σij, above mentioned. Since every bond or connecting line in the graph passes through two atoms, this toti-valence must be equal to the free valence of the molecules increased by twice the number of bonds; but Σij is the toti-valence, and ϵ (the degree of the covariant) is the number of unsatisfied bonds, and we have already stated in effect that ϵ increased by twice the weight of the root differentiant (which for brevity we call the weight of the covariant) is equal to Σij; hence the weight of a covariant (meaning that of its root differentiant), represented by any chemicograph, is the number of bonds or connecting lines between the atoms.

Let us consider an invariant or a covariant belonging to a type containing only one numerical parameter, which I shall call a monadelphic form*. Then this is either decomposable into factors or not; in the former case it may be termed composite, in the latter case prime. When prime its graph will also be prime, when composite its graph will be composite in a sense which will be made more clear by one or two examples. Let us take as a first example a graph composed of four triadic atoms of the same name, as in Fig. 6, where each atom, for instance, represents boron and in ordinary chemical symbolism would be denoted by the same letter B, but where for facility of reference I use four different letters to mark the positions of the several atoms. This corresponds to a covariant of a cubic for which the complete type, if we use the weight or number of bonds, is [4 : 3, 4], or, if we use the free valency, is [3, 4 : 4]. Now for a cubic the fundamental types, expressed in terms of the order and degree alone, omitting the constant number 3, which refers to the given degree, are

1 . 3
4 . 0
2 . 2
3 . 3.

Consequently, there is but one covariant corresponding to the given graph, and that is the product of the primitive by the covariant whose order and degree are each 3, the well-known skew covariant of $(a, b, c, d\between x, y)^3$ whose root or base is the differentiant $ad^2-3abc+2b^3$.

* The type itself may also be termed a monadelphic type: so I shall speak when necessary of diadelphic, triadelphic, &c. types and designate any forms contained under such types as diadelphic, triadelphic, &c. forms. A family comprising many brothers, or any member of such a family, may each without doing violence to the laws or usage of language be termed *polyadelphic*.

It must be well understood that the bonds are not rigid, but capable of being curved or bent into any desired form. In this case the mode of decomposition is self-evident; for the skew covariant is represented by the triangle of Fig. 7, and we have only to draw out the elastic bond AC into the position ADC and place the atom D anywhere upon it to obtain the given graph. On the contrary the skew covariant itself is indecomposable and its graph ABC is obviously so too. Now let us consider the graph of Fig. 8. If the atoms at the angles are all triadic, there is no free valency, and the figure represents the invariant to a cubic form corresponding to 4 . 0 in the above table. It will be found, on trial, impossible to decompose it. But now suppose the atoms to be tetradic, the graph will represent a covariant of the fourth order and of the fourth degree to a quartic, each atom having one degree of valency unsatisfied. The fundamental derivatives of a quartic, of which all others are algebraical combinations, are represented in the following table of order and degree

1 . 4
2 . 0
3 . 0
2 . 4
3 . 3.

The complete covariant answering to the graph will therefore be $\lambda U + \mu V$, where, λ, μ being arbitrary numbers, U is the product of the primitive (1 . 4) by the cubinvariant 3 . 0, and V the product of the Hessian 2 . 4 by the quadrinvariant 2 . 0. Since, on making either $\lambda = 0$ or $\mu = 0$, the covariant breaks up and in two different ways into factors, we ought to expect that the graph should be capable of two corresponding modes of decomposition, and such we shall easily see is the case. For 1°, the invariant 3 . 0 may be represented by the graph of Fig. 9. Now imagine the three points E, F, G to come together and blend at D, and at D place a fourth atom. The given graph is thus recovered. Observe that this could not be done for the case of triads (corresponding to a cubic form) because, in the figure last referred to, the valence at each atom A, B, C is quadrivalent. Next, for the decomposition corresponding to the case of $\lambda = 0$ where the covariant breaks up into 2 . 0 multiplied by 2 . 4, the decomposition will be more easily followed by considering the graph to be pulled out into the form seen in Fig. 10. We may conceive this as the superposition of two carbon graphs, one in which the carbon atoms are at A and B connected by the *four* bonds AB, ACB, BDA, $ACDB$ denoting the quadrinvariant, and another in which the carbon atoms C, D are connected by the *two* bonds CAD, CBD, leaving two degrees of valence free at each atom and thus representing the quadro-quart-invariant or Hessian of the primitive.

I will now pass to the very interesting case which corresponds to one of the proposed graphs for benzole (or rather for the compound radical obtained by striking off its hydrogen atoms), a sextivalent hexad molecule of carbon—not the one proposed by Kekulé and which I believe still commands the general assent of chemists, but that suggested by Ladenburg* and put by him under the form of a wedge or prism. As, however, the question is one purely of colligation or linkage in the abstract, it is sufficiently described as a hexagon in which the three pairs of opposite angles are joined, or, if we please, as two triangles in which each angle of one is connected with a corresponding angle of the other. In regard of the atomicity theory, all these modes of colligation are identical, and the supposition that there is any real difference between them, or that figures in space are distinguishable from figures in a plane (as I heard suggested might be the case by a high authority at a meeting of the British Association for the Advancement of Science, where I happened to be present), is a departure from the cautious philosophical views embodied in the theory as it came from the hands of its illustrious authors and continued to be maintained by their sober-minded successors and coadjutors, and affords an instructive instance of the tendency of the human mind to the worship, as if of self-subsistent realities, of the symbols of its own creation.

The order (or number of atoms) being 6 and the unexhausted valences (one at each atom) also 6, we must turn to our table of fundamental derivatives to the quartic and shall find that the combination 6 . 6 is not amongst them, but that it can be obtained, and in only one way, by composition of the combinations therein contained. It is, in fact, the product of the cubic invariant 3 . 0 by the skew covariant 3 . 6, which has the very same *root* $a^2d - 3abc + 2b^3$ as the skew covariant to the cubic and accordingly has the same graph, namely a simple triangle. (It may be well to remark here incidentally, that it follows as an immediate consequence from the conditioning partial differential equation, that a root-differentiant to any quantic or system of quantics of given degree or degrees remains such to every other system in which one or more of those degrees is augmented.) On the other hand the cubic invariant has for its graph a triangle in which each line is doubled or looped. I shall show that Ladenburg's graph for the radical to benzole may be obtained by the superposition of these two forms. Let $ABC\gamma\beta\alpha$ represent a sextivalent tetradic hexad (Fig. 11); ABC, with the three loops $A\alpha\gamma C$, $C\gamma\beta B$, $B\beta\alpha A$, will represent a saturated triple atom of carbon, or the cubinvariant of a binary quartic. Again, $\alpha\gamma\beta$ taken alone will represent a sextivalent compound atom, or the fundamental skew covariant of the quartic, and the superposition of the two figures obviously gives the graph as it stands.

Another form of the product of the same two graphs would be a triangle inscribed in another, as in Fig. 12. Here $\alpha\beta\gamma$, as before, is the sextivalent

* *Berichte der deutschen chemischen Gesellschaft*, 1869, 141. I am indebted for this reference to my able colleague, Professor Ira Remsen.

molecule and ABC with the additional bonds $A\beta C$, $B\gamma A$, $C\alpha B$, the saturated one.

A simple hexagon of triadic atoms (Fig. 13) being sextivalent will serve to represent a derivative from a cubic of the sixth order and sixth degree. Such a covariant, in its most general form, will contain two parameters and be represented by $\lambda U^3 + \mu V^2$ where U is the Hessian 2.2 and V the skew cube covariant 3.3, and it is easy to see that this figure may be decomposed either into 3 bivalent, or 2 trivalent graphs. Thus AB, CD, EF, with the additional bonds $BCDEFA$, $DEFABC$, $FABCDE$, will represent the former; two atom groups such as A, C, E (with the bonds ABC, $AFEDC$, CDE, $CBAFE$, EFA, $EDCBA$) and B, D, F (with the bonds BCD, $BAFED$, DEF, $DCBAF$, FAB, $FEDCB$) the other. The first method of regarding the hexagon as a combination of three dyads may perhaps be admitted to throw some light on what Dr Frankland styles the two distinct molecular weights of sulphur. When two atoms of sulphur, regarded as bivalent, are combined by two loops, we have a representation of an isolated element of it as "a diatomic molecule." When three of these letters, regarded now as submolecules, are combined, or multiplied together into the hexagon, we have a representation of the isolated element as "a hexatomic molecule." More generally, let μ be the number of solutions of the equation in positive integers $2x + 3y = m$, then μ arbitrary parameters will enter into the most general representation of a covariant to a cubic of the order m in the coefficients and the degree m in the variables. Its graph will be a simple polygon of m sides and this will be capable of being decomposed, in μ essentially distinct ways, into elementary graphs consisting either, of binary groups or, ternary groups exclusively or, the two sorts of groups intermixed.

It may be easily shown (see Appendix 3) that every covariant of a binary form multiplied by a suitable power of its primitive, is capable of being represented by a rational integer function of covariants consisting, in addition to the primitive, of covariants exclusively of the second and third orders in the coefficients. I have already given an example of the mode in which a graph may be augmented by an additional atom corresponding to the multiplication of a covariant by the primitive.

The important proposition above referred to (given in Clebsch's *Binären Formen*) amounts then to affirming that any homogeneous graph augmented by a suitable number of atoms of the same, may be decomposed, in one or more ways, into bilooped dyads and single-sided triangles. Such a proposition ought to admit of graphical proof. The theorem has considerable graphical importance because it enables us, in some cases at least, to discriminate the true from the spurious graphs, or as we might say, pseudographs, representing a given type. Thus, it serves to show that Fig. 14 and not Fig. 15 is the

graph to the discriminant of a cubic; for, in accordance with Clebsch's theorem, this discriminant, namely

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd,$$

multiplied by a^2 becomes equal to the square of $a^2d - 3abc + 2b^3$, together with four times the cube of $ac - b^2$, and consequently its graph, after combination with two additional points, should be decomposable, at will, into 3 double-looped lines, or into 2 single-lined triangles, which is the case with Fig. 14, inasmuch as its combination with two points gives rise to a simple hexagon, but not with Fig. 15.

If we call the apices of the two figures, 14, 15, a, b, c, d, the true graph (on substituting negative signs for bonds and prefixing a sign of summation) reads as

$$\Sigma (a-b)^2 (c-d)^2 (a-c)(b-d),$$

which is the cubinvariant of the quartic whose roots are a, b, c, d, so that a graph to an invariant of the type [3, 4 : 0] gives the algebraical expression in terms of the roots of an invariant of the reciprocal type [4, 3 : 0]. On the other hand, the pseudograph treated in the same way reads as

$$\Sigma (a-b)(b-c)(c-d)(d-a)(a-c)(b-d),$$

the value of which is zero; a similar remark may probably be found to be true of reciprocal graphs of invariants in general. This is abundantly confirmed by subsequent investigation; see remarks at end of Appendix 1.

So again, if we take the graph of Fig. 42, which represents an invariant to the type [3, 2; 1, 2 : 0], it reads off into

$$\Sigma (B_1 - B_2)^2 (B_1 - H_1)(B_2 - H_2),$$

belonging to the reciprocal type [2, 3; 2, 1 : 0], and the Σ is in fact the discriminant of one binary quadratic multiplied by the connective between it and another.

So if we take the graph represented in (a), Fig. 45,

$$\Sigma (O_1 - O_2)(O_1 - H)(O_2 - K)$$

will represent an invariant to the type [2, 2; 1, 1; 1, 1 : 0]. If, however, we were to substitute H_1, H_2 in lieu of H and K, so as to form the hydroxyl graph of Fig. 45 (b), it would not be true that $\Sigma (O_1 - O_2)(O_1 - H_1)(O_2 - H_2)$ would represent an invariant to the type [2, 2; 2, 1 : 0]; on the contrary it would be zero. But hydroxyl is *not an invariant*, for to the combination of a quadratic and a linear form there appertains no invariant of the second degree in the coefficients of each of them. This may be easily proved by the rule I have given at the commencement of this paper. I have gone through this calculation for the benefit of those new to the subject and to show how the arithmetical "rule of multiplicity" is to be applied. Had I been writing solely for algebraists it would have been unnecessary to prove so familiar a fact. We have here

$$i=2,\quad j=2,\quad i'=1,\quad j'=2,\quad w=\frac{ij+i'j'}{2}=3.$$

To find $(w: i, j; i', j')$ we have to count the combinations

2 . 1	0 . 0
2 . 0	0 . 1
1 . 1	0 . 1
1 . 0	1 . 1 ;

the number of these is 4. Again to find $(w-1: i, j; i'; j')$ we have to count the combinations

2 . 0	0 . 0
1 . 1	0 . 0
1 . 0	0 . 1
0 . 0	1 . 1 ,

of which the number is also 4. Hence

$$\Delta(3: 2, 2; 1, 2) = 4 - 4 = 0.$$

So that hydroxyl, being of the type [3 : 2, 2 ; 1, 2], cannot be an invariant.

So far then the supposed law is safe; but I think I see other difficulties in the way of its application to heteronymous types, so that if it shall be capable of being made universally applicable, other parts of the graphical theory, as it has been laid down, will possibly require reconsideration. What I advance is to be regarded not as dogmatic but as tentative and open to correction.

It is obvious that not every chemico-graph, potential or even actual, corresponds to an invariantive derivative. Of this I have already given examples. Were the case otherwise we should have surprised the secret of nature, for, as we know how to obtain all possible fundamental forms to binary quantics, we should know *à priori* all possible compound radicals. As a matter of fact the cases of algebraical invariance in nature seem to be rare and rather the exception than the rule. Thus while muriatic acid (H – Cl), is an invariant, self-saturating hydrogen (H – H), is a non-invariant, there being a linear invariant to two linear forms but not to a single one. In like manner ozone (Fig. 16) is also non-invariantive, there being no cubic invariant to a quadratic form. But there is an essential difference to be observed between the two cases. A graph consisting of a single or an odd number of bonds between two atoms of the same kind can *never*, for any species of such atoms, be invariantive, because no covariant of the second order in the coefficients can have an *odd* weight. If that were possible, then, by the theorem of reciprocity, a quadratic function could have an invariant or covariant of an odd weight, which is, of course, not true. Whereas a triangle of n-ads, although it does not picture an invariant when $n=2$, does do so when $n=3$ or any higher number. When an homonymous graph is given in weight (the number of bonds) and in order (the number of atoms) two of the elements of its type $(w: i, j)$ say w, j are known and the third i is left indeterminate. For all values of i which make $\Delta(w: i, j)$ greater than zero, there will be one or a plurality of such graphs according to the value of Δ. If no value of i makes Δ greater than zero, there will be no such graph possible, but it is not necessary, to ascertain this, to make an indefinite number of trials, for it is obvious that for all values of i equal to or greater than w, Δ has the same value, namely $\Delta(w: \infty, j)$, since the condition that a number w shall not be made up of numbers greater than i, when i is equal to w, becomes nugatory.

It will be instructive to consider the case of $w=5, j=3$, and consequently the free valence $\epsilon = 3i - 10$; this implies that i must be at least equal to 4. But if we take $i=4$, $\epsilon=2$, as there is no covariant to a binary quartic whose order is 3 and degree 2, we may be sure that $\Delta(5: 4, 2) = 0$. Hence we have only to consider the case of $i = w = 5$, $\epsilon = 5$. $\Delta(5: 5, 3)$ is the number of covariants of the fifth order and fifth degree to a cubic of which there is but *one*, formed by the multiplication together of the Hessian and skew-covariant. If now we proceed to form the graph corresponding to the type [5: 5, 3], we have the choice of two figures, 17, 18. In the former figure there are three degrees of vacancy from saturation at A and one at each of the points B, C. In the latter, one at A and two at each of the points B and C. The graph, we must recollect, is to correspond to a cubic covariant of the fifth degree to a fifthic which is unique and indecomposable. This enables us to fix upon the true representation. It cannot be the graph of Fig. 17, for that may be considered as generated by the combination of one isolated nitrogen atom with two atoms of nitrogen, B, C, connected by five bonds; two of these being subsequently welded together and bent out into the angle having A at its vertex. [The hypothetical nitrogen pair exists in chemistry but not as an algebraical invariant.] Hence the true figure can but be that given in Fig. 18, where the free valence is separated into the parcels 2, 1, 2, and not as in Fig. 17 into the parcels 1, 3, 1. And it should be observed that, for all higher values of i beyond 5, this will continue to be the one and only true graph to the corresponding covariant. It thus appears that every given homogeneous graph has an intrinsic character of capability or incapability of respondence to algebraical in- or co-variance, irrespective of the particular valence assigned to its atoms, and it is natural to suppose that there must be some immediate intrinsic criterion for determining this character, so as to dispense with the necessity of any algebraical considerations to establish it; but if such criterion exists, I have not yet been able to make out what it is*. In common with this view we may consider the theory of reciprocity of algebraical derived forms. It has already been stated that to every m-ad of n-ad atoms having a given number of bonds corresponds an n-ad of m-ad atoms with the same number of bonds. As for example, to a quasi carbon-ad (so to say) of sulphur will correspond a quasi sulphur-ad of carbon, the number of bonds and consequently the amount of free atomicity remaining the same in the two molecules. This suggests the possibility of there being some mode of passing from a graph to its reciprocal (this reciprocity being seemingly of quite a different kind from that which connects correlated girders or frameworks in graphical statics). I offer the subjoined instance of such transformation tentatively and with a view to stimulate inquiry, rather than as possessing any assurance of the validity of the process employed.

Suppose the case of $i=4$, $j=2$, $w=4$; the one and only corresponding graph will be a system of 4 bonds connecting two atoms A, B. If now we take a pair of these bonds, stretch them out, weld them together and form a knot between them at C, and in like manner convert the other pair of bonds into a pair knotted at D, we shall have a graph consisting of a simple quadrilateral which will correspond to the case of $i=2$, $j=4$.

Again, suppose $i=6$, $j=4$, $w=12$. We may consider either of the graphs quasi in Figures 19, 20. In the first of these figures we may take four bonds connecting respectively AC, CB, AD, DB, stretch and weld them together and form a knot between them at a new point E which will then be attached by four bonds to the atom $ABCD$. I mean that we may stretch out AC, CB, to meet in E (Fig. 21) and have EC common, and in like manner stretch out AD, DB to E and have ED common and then knot together the four bonds of the strings at E. In like manner we may form another knot F with bonds through AB, BC, AD, DC, and shall thus obtain the reciprocal graph of Fig. 21, where now $i=4$, $j=6$, $w=12$. So again it will be found that we may distort Fig. 20 (if I can trust to my recollection of the result of previous work) in two different ways into a reciprocal graph.

At the risk of provoking the ire or ridicule of my chemical friends and the chemical public, I will venture to throw out a few remarks on the substructure, so to say, of the accepted theory of atomicity and to offer a suggestion as to a possible mode of getting rid of some imperfections under which it appears at present to labour. First there is the inconsistency of admitting the isolated existence of single atoms of mercury, cadmium and zinc, as monads with their bonds or tails absorbed or suppressed or else swinging loose and unsatisfied in direct opposition (as it seems to me) to the fundamental postulate of the theory. Next, one cannot get over a somewhat uncomfortable feeling at the representation of isolated oxygen in the state

* The law of reciprocity, however, exemplified above can obviously be made to supply the criterion in question.

of ozone by a triangular graph, which, although conceivable, is supported by no analogous case unless that of baric peroxide, or any similar graph, be regarded as such. Thirdly, there is the vague and unsatisfactory (not to say unthinkable) explanation of the variability of the valence of a given atom by what Dr Frankland calls "the very simple and obvious assumption that one or more pairs of bonds belonging to the atom of an element can unite and having saturated each other become, as it were, latent."

Now these stumbling-blocks to the acceptance of the theory may be removed by one simple, clear and unifying hypothesis, which will in no wise interfere with any actually existing chemical constructions. It is this: leaving undisturbed the univalent atoms, let every other n-valent atom be regarded as constituted of an n-ad of *trivalent* atomicules arranged along the apices of a polygon of n sides. Thus, sextivalent, quinquivalent and quadrivalent atoms in their state of maximum valence will be represented by Figures 22, 23, 24, where the letters denote *trivalent atomicules*. When the valence is reduced by two we need only conceive any one of the side loops doubled or a new loop as formed by the coalescence of a pair of free bonds or tails, and when in the Figures 22 and 23 the valence is reduced by 4, we may in like manner either suppose existing loops doubled, or fresh ones inserted, or both changes to go on simultaneously, by the coalescence of two pairs of tails. We have thus a conceivable and conformable-to-analogy method of accounting for the variability in question. So likewise, a trivalent atom with maximum state of valence will be represented by Fig. 25, and when univalent by Fig. 26. Again, an isolated zinc element will have for its graph Fig. 1 (b), the two letters Z signifying the zinc atomicules, and so in like manner isolated cadmium and mercury may be represented. On the other hand O_2, isolated oxygen in its ordinary state, will be represented by the graph of Fig. 27, whilst ozone will have for its representative graph the well known Kekuléan hexad (which, in its importance to chemistry, would seem to vie with Pascal's mystic hexagons to geometry) represented in Fig. 28, where as in Fig. 27, each letter O represents an atomicule of oxygen. So an isolated element of carbon would be represented by the graph of Fig. 29.

This hypothesis of atomicules, if unobjectionable on other grounds, would not be open to the charge of having any tendency to disturb or complicate the existing graphology; for we should still be at perfect liberty to substitute for the graphs (a) of Figures 30, 31, 32 the abridged notation (b), and should naturally do so when considering the relations of atoms to each other. The beautiful theory of atomicity has its home in the attractive but somewhat misty border land lying between fancy and reality and cannot, I think, suffer from any not absolutely irrational guess which may assist the chemical enquirer to rise to a higher level of contemplation of the possibilities of his subject. I have therefore ventured to make the above suggestion.

Chemical graphs, at all events, for the present are to be regarded as mere translations into geometrical forms of trains of priorities and sequences having their proper *habitat* in the sphere of order and existing quite outside the world of space. Were it otherwise, we might indulge in some speculations as to the directions of the lines of emission or influence or radiation or whatever else the bonds might then be supposed to represent as dependent on the manner of the atoms entering into combination to form chemical substances. Such not being the case, what follows is to be considered as having relation to mere *algebraical* atoms, or atomicules (quantics) and their bonds which may be regarded as represented by the linear factors of such quantics.

Let us consider a symmetrical trivalent atomicule whose three bonds or rays make angles of 120° with each other. Calling τ, τ', τ'', the tangents of the angles which the axis of y makes with its rays, we have

$$\tau' = \frac{\tau + \sqrt{(3)}}{1 - \sqrt{(3)}\,\tau}, \qquad \tau'' = \frac{\tau - \sqrt{(3)}}{1 + \sqrt{(3)}\,\tau},$$

so that its equation will be easily found to be

$$(1 - 3\tau^2)\,x^3 + (9\tau - 3\tau^3)\,x^2y + (9\tau^2 - 3)\,xy^2 + (\tau^3 - 3\tau)\,y^3 = 0,$$

which may be identified with the standard form

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0$$

by writing $\qquad a = 1 - 3\tau^2 = -c, \qquad b = 3\tau - \tau^3 = -d.$

Suppose the three atomicules to become condensed into a single atom after the manner of the graph of Fig. 25. The combination will be represented by the cubic covariant (see Tables des Invariants et Covariants, Table V, annexed to Faà de Bruno's *Théorie des Formes Binaires*)

$$(a^2d - 3abc + 2b^3)\,x^3 + (3abd - bac^2 - 3b^2c)\,x^2y$$
$$+ (3bc^3 + 6b^2d - 3acd)\,xy^2 + (3bcd - ad^2 + 2c^3)\,y^3,$$

which, for the present case, becomes

$$2\,(1 + \tau^2)^3\,[(3\tau - \tau^3)\,x^3 + (9\tau^2 - 3)\,x^2y + (3\tau^3 - 9\tau)\,xy^2 + (1 - 3\tau^2)\,y^3].$$

Hence the new ray-directions will have for their equation

$$-dx^3 + 3cx^2y - 3bxy^2 + ay^3 = 0,$$

or the pencil of the atom will be identical with that of each of the separate atomicules, but accompanied with a rotation (whatever that may mean) of the whole pencil of rays through a right angle in its own plane. Again, suppose that only two atomicules are brought into connexion as in (a) of Fig. 30. The quadricovariant which expresses the atom (Faà de Bruno *ante*) is

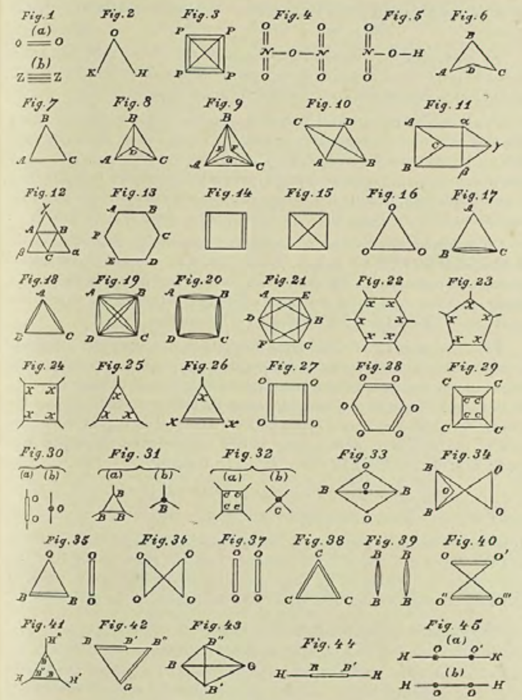
$$(ac - b^2)\,x^2 + (ad - bc)\,xy + (bd - c^2)$$

which here becomes $\qquad -(1 + \tau^2)^3\,(x^2 + y^2).$

Hence the ray-directions will be given by the equation

$$y^2 + x^2 = 0, \qquad y = \pm x\sqrt{(-1)},$$

which we may, if we please, according to the usual convention concerning the



square root of minus unity, explain by supposing that the original rays are situated in planes perpendicular to the joining line XX, and that these are replaced by two rays lying in opposite directions along the line XX, where the atomicules are condensed into one atom. But it would be idle to pursue this speculation further.

The most remarkable point in the theory which I have endeavoured to unfold in the preceding pages is the relation between it and that of reciprocal types.

We have seen that the graph to an invariant of one type read off as it stands (each bond being construed as the sign *minus*) with the sign Σ prefixed expresses an invariant of the reciprocal type.

This rule may be extended from homogeneous to heterogeneous graphs, provided only that the reciprocity be *total*, by which I mean that every i and every j in the type $[i, j; i', j'; i'', j'' \ldots : 0]$ are interchanged. It may be observed, in passing, that in the case of types to which resultants belong, the type is identical in form with its total reciprocal. As, for example, boric anhydride (consisting of two of boron and three of oxygen) is of the type [3, 2; 2, 3: 0].

On referring to "System of Cubic and Quadratic," Salmon's Lessons, third edition, p. 179, it will be seen that besides the resultant there is another invariant represented in Dr Salmon's notation by "$\Delta(0, 2) \times I(2, 1)$"; a linear combination of these two with arbitrary multipliers will express the most general form belonging to the type in question.

From the property of these types being their own complete reciprocals, it follows that a complete set of independent graphs of any such type will represent the constitution of a complete set of independent forms belonging to the type. Thus, in the case suggested by boric anhydride we have the two independent graphs of Figures 33, 34. Hence the complete representation of the invariants appertaining to the self-reciprocal diadelphic type [3, 2; 2, 3: 0] is $\lambda U + \mu V$, where U is the resultant

$$(a-\alpha)(a-\beta)(a-\gamma)(b-\alpha)(b-\beta)(b-\gamma)$$

and V is $$\Sigma(a-\gamma)(a-\beta)(b-\alpha)(b-\gamma)(b-a)(\beta-\alpha).$$

U is derived from the graph of Fig. 33 by replacing the several O's by α, β, γ, and the B's by a, b, and V in like manner from the graph of Fig. 34. This latter graph is replaceable by the disjoined graph of Fig. 35, to which, by the rule for combination of graphs, it is easily seen to be equivalent.

Hence, instead of $\lambda U + \mu V$ we may write $\lambda V + \mu V'$ where

$$V' = \Sigma(\alpha-\beta)^2(a-b)^2(a-\gamma)(b-\gamma);$$

a, b of course will be understood to be the roots of a general quadric and α, β, γ of a general cubic. A very good similar instance of this kind of equivalence is afforded by the quadrinvariant of a quartic whose type is [4, 2: 0]. The reciprocal of this, namely [2, 4: 0], may be represented, either by the connected graph of Fig. 36, or by the disjoined one of Fig. 37,

and accordingly the noted quadrinvariant $ae - 4bd + 3c^2$ may be expressed (to a numerical factor près) either by the symmetrical function

$$\Sigma(a-c)(a-d)(b-c)(b-d)$$

corresponding to the first, or by $\Sigma(a-b)^2(c-d)^2$ corresponding to the second graph. Again, let us consider the contrary types [4, 3: 0], [3, 4: 0]. The former has for its graph Fig. 38, and admits of no other representation. This gives $\Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\delta)^2$ for the discriminant of the cubic which belongs to the contrary type. The latter may be figured chemically by the graph (consisting of two molecules of boron) of Fig. 39, or by the equivalent Fig. 27 (capable of being derived from it by the mechanical rule for conversion of graphs). These two latter, algebraically speaking, will be pseudographs, because $\Sigma(\alpha-\beta)^3(\gamma-\delta)^3$ and $\Sigma(\alpha-\beta)(\beta-\gamma)(\gamma-\delta)(\delta-\alpha)(\alpha-\gamma)(\beta-\delta)$ are each zero. The graph of Fig. 27 may be mechanically converted, in the manner shown in the preceding case, into the graph of Fig. 40; but the type of the colligation remains unaltered by this conversion and whichever of the two we employ, we obtain $\Sigma(\alpha-\beta)^2(\gamma-\delta)^2(\alpha-\gamma)(\beta-\delta)$ as the representation in terms of the roots, of the cubic invariant to the quartic, namely to a numerical factor près $ace - b^2e - ad^2 + 2bcd - c^3$.

Thus we see that the graphical method suggested by the theory of atomicity is a real instrument not merely for the representation but also for the calculation and comparison of algebraical results. The important bearing upon it of the principle of contrary or reciprocal graphs, renders it desirable that I should put the algebraical theory or law of reciprocity, in its most complete form, before my readers; it will form the subject of Appendix 2.

I might have noticed explicitly at the commencement of this paper, instead of tacitly assuming it as I have done, that the chemical fact of a compound molecule playing the part of an atom with a valence equal to the free valence of the radical, is the precise homologue to the algebraical fact that every invariant or covariant of a covariant, or set of covariants, to a quantic, or system of quantics, is itself an invariant or covariant to such quantic, or system of quantics; and again that Regnault's chemical principle of substitution and the algebraical one of emanation* are identical; and again, the modern notion of two semi-molecules, simple or compound, combining or uniting to form a chemical substance is tantamount to the construction of an invariant, the connective (or in Professor Gordan's language, the final "Ueberschiebung") of a quantic, or of the derivee of a quantic or a set of quantics,

* By which I mean in this place the operation upon an invariant or covariant of the symbol $(a'\delta_a + b'\delta_b + \ldots)$ performed any number of times in succession; a, b, for instance, may refer to Hydrogen $(ax + by)$ and a', b' to Chlorine $(a'x + b'y)$, and then the emanantive operator, according to a notation used, if I mistake not, by Professor Clerk Maxwell in his theory of poles, might be denoted by $Cl\delta_H$.

with itself. So again, it will hereafter be seen* that Hermite's law of reciprocity applied to quantic systems and stated in its widest terms, amounts to affirming in chemical language that in any compound an arbitrarily selected group of m n-adic atoms may be replaced by a group of n m-adic atoms, but how far this law of replacement has objective validity in the chemical sphere, I am not able to say.

Attention might also have been called to the fact that every chemico-graph may, for anything that has been shown to the contrary, and probably in all cases does admit of algebraical interpretation, provided that each given atom however often repeated in a graph counts as a distinct quantic with its own distinct set of coefficients. I do not know whether chemists are of opinion that every chemico-graph exists or is capable of existence in nature; if this is not the case, the condition of the possibility of such existence (should it be discovered) must admit of being stated in mathematical terms. The condition for its existence in algebra may be gathered from what precedes, to be certainly for monadelphic types and probably in all cases, as follows, namely: *if the difference between every two letters of an algebraically existent graph be raised to the power whose index is the number of bonds connecting them, the permutation sum of the product of those powers must not vanish.* Finally, an irreducible covariant is the homologue of a compound radical. Thus we see that chemistry is the counterpart of a province of algebra as probably the whole universe of fact is, or must be, of the universe of thought.

APPENDIX 1.

Remarks on Differentiants Expressed in Terms of the Differences of the Roots of their Parent Quantics.

Since the preceding matter was written, in dwelling upon the law of reciprocal graphs, I came to what appeared to be a formidable difficulty in the way of its reception, a very lion in my path, so formidable that, for a time, I thought that it would be necessary, either to abandon this law, or else to admit the unwelcome conclusion that not every type of invariant was susceptible of graphical representation.

But further consideration has shown me that this apprehension was

* In Note D to Appendix 2. The proposition stated in the text results from the joint effect of the law of substitution or emanation combined with Hermite's law extended to quantic systems.

entirely groundless owing to an algebraical fact on which I had not previously reflected, but which this difficulty forced upon my notice. The difficulty in question arose out of the expressions given by M. Hermite and le père Joubert respectively for the skew invariants of the binary quintic and sextic. I shall first address myself to the consideration of the former. Following Dr Salmon's notation (Lessons, Third Edition, p. 230), let α, β, γ, δ, ϵ be the roots of a quintic, and let

$$F = (\alpha-\beta)(\alpha-\epsilon)(\delta-\gamma) + (\alpha-\gamma)(\alpha-\delta)(\beta-\epsilon)$$
$$G = (\alpha-\beta)(\alpha-\gamma)(\epsilon-\delta) + (\alpha-\delta)(\alpha-\epsilon)(\beta-\gamma)$$
$$H = (\alpha-\beta)(\alpha-\delta)(\epsilon-\gamma) + (\alpha-\gamma)(\alpha-\epsilon)(\delta-\beta).$$

Then it will be found as will presently be shown that the product $F.G.H$ is a symmetrical function of the four roots $\beta, \gamma, \delta, \epsilon$, consequently, on forming four other similar products symmetrical in respect to $\alpha, \gamma, \delta, \epsilon$: $\alpha, \beta, \delta, \epsilon$: $\alpha, \beta, \gamma, \epsilon$: $\alpha, \beta, \gamma, \delta$ respectively, the product of these five products will be symmetrical in respect to $\alpha, \beta, \gamma, \delta, \epsilon$ and being a function of the differences of the roots of order 18 and of weight 45, that is of the type [45 : 5, 18], must be (paying no attention to a mere numerical factor) I, the skew invariant to the quintic.

Now consider the type reciprocal to this [45 : 18, 5] (monadelphic like the preceding), and expressing the invariant of the fifth order to an octodecadic. Suppose this has a graph. It will follow from the law of reciprocal graphs that I may be expressed under the form

$$\Sigma(\alpha-\beta)^a(\alpha-\gamma)^b(\alpha-\delta)^c(\alpha-\epsilon)^d(\beta-\gamma)^e(\beta-\delta)^f(\beta-\epsilon)^g(\gamma-\delta)^h(\gamma-\epsilon)^k(\delta-\epsilon)^l,$$

where $a+b+c+\ldots=45$ and each letter $\alpha, \beta, \gamma, \delta, \epsilon$ is conditioned to appear the same number of times, which at first might seem contradictory to what has just been established, but in reality is in perfect accordance with it. For imagine the product of the 15 quantities

$$FGHF'G'H'F''G''H''F'''G'''H'''F^{IV}G^{IV}H^{IV}$$

to be actually written out giving rise to 2^{15}, or 32768 terms, and to each of these terms prefix the sign Σ indicating that the sum is to be taken of the 120 values which it assumes on permuting the five letters $\alpha, \beta, \gamma, \delta, \epsilon$. The sum of all these partial sums is $120I$; hence some, at least, of them cannot vanish. Let ΣT be any one that does not vanish. Then ΣT is a function of the differences of the roots of the same weight and order as the entire expression; it is therefore to a numerical factor près identical with I, just as every fragment of a mirror is itself a mirror, or as every particle of diamond dust, a diamond.

Thus, as many distinct non-vanishing forms as there may be of ΣT, so many different graphs to the quint-invariant of a binary octodecadic shall we be able to construct agreeing respectively with the different representations of I of the form

$$\Sigma(\alpha-\beta)^a(\alpha-\gamma)^b(\alpha-\delta)^c\ldots$$

and it is probable that the virtual equivalence of all these several graphs may admit of being made out by inspection, as we saw was the case with the two graphs (one dissociated, the other connected) corresponding to the two algebraical representatives of the quadrinvariant of a quartic. Thus, what seemed, at first sight, to be fatal to the admissibility of the algebraico-graphical theory only serves to set in a clearer light its value as an instrument of research.

If we analyse M. Hermite's form of the skew invariant* to the quintic we shall see that it depends upon this simple but not obvious fact, that writing

$$F=(c,\ d)(a-b)+(a,\ b)(c-d)$$

$$G=(b,\ d)(a-c)+(a,\ c)(d-b)$$

$$H=(b,\ c)(a-d)+(a,\ d)(b-c)$$

and interpreting any such quantity as (a, b) to mean either 1 or $(a+b)$ or ab the product FGH is a symmetrical function of a, b, c, d, because on interchanging any two letters (say for example c, d) that one of the three quantities F, G, H (in this example H) in which those two letters are affected with the same sign, will remain unaltered in value whilst the other two (here G and F) change, each into the negative of the other.

Consequently we may interpret (a, b) to mean $(e-a)(e-b)$ and then the product of the five products corresponding to FGH is a function of the coefficients which expressed in terms of the differences of the roots will be of the weight 15 and of the order $1.6+4.3$ or 18 because in one of the five products each letter will enter in six dimensions and in each of the other four products in three dimensions; thus in FGH, e^6 will appear, but in each of the other four products e^3 will be the highest power of e. Hence the quindenary product is the invariant in question. No further step is necessary, the proof is complete as stated.

This remark will enable us to illustrate the process of transformation, which I have compared with grinding a diamond into dust, by an example

* I am wont to compare in my mind this symmetrical and translucent form to the Pitt Diamond and Père Joubert's to the Koh-i-Noor. In Note D to Appendix 2 a method is given whereby these forms may be transmuted into one another subject, however, to the bare possibility that the one, put into the algebraical alembic at a certain stage of the process, instead of passing into the other may, so to say, evaporate and be reduced to nothing. In the theory of forms, all-embracing Zero is the source and reconciler of contradictions, because, algebraically speaking, *everything is contained in nothing*, and so in a morphological sense "nought is everything" though not "everything is nought."

that can be completely pursued to the end. For let us now regard a, b, c, d as the roots of a binary quartic; then

$$\{(a-b)+(c-d)\}\{(a-c)+(d-b)\}\{(a-d)+(b-c)\}$$

will be a differentiant thereto of weight three and order three; it will, in fact, represent the root-differentiant of the skew sextic covariant.

Imagine this multiplied out without disturbing the marks of coupling so as to give eight terms or fragments analogous to the 32768 fragments spoken of in the preceding case. These terms will be of only four different patterns, one of the pattern $(a-b)(a-c)(a-d)$, three of the pattern $(a-b)(a-c)(b-c)$, three of the pattern $(a-b)(b-c)(d-b)$ and one of the pattern $(c-d)(d-b)(b-c)$. Prefixing Σ to each of these pattern terms to signify the sum resulting from the 24 permutations of a, b, c, d, we know *à priori* that not all of these can be zero since a linear function of them will be 24 times the differentiant in question, and on examination we find that the second and fourth Σ will vanish, but that the first and third will not. Accordingly, we shall have two new expressions

$$\Sigma(a-b)(a-c)(b-c), \qquad \Sigma(a-b)(b-c)(b-d),$$

each of which represents a differentiant of the same type as the original one, and this type being monadelphic or henparametric, the original product and these two sums will only be different representations of the same differentiant. Thus we see that each independent form belonging to a given type is susceptible (when expressed as a function of the differences of the roots) of a number of distinct phases, or, as we may express it, an algebraical form, in this theory, is in general polyphasic and accordingly its Icon or linkage exponent will be in general polygraphic, and each phase will have its own appropriate graph. It is a work of some difficulty, in general, to recognize the substantial identity of the different phases of the same algebraical form, and in like manner it may not, in all cases, be easy to recognize the substantial identity of the different graphs of its Icon, but sufficient has been shown to indicate the possibility and method of establishing such identity. The more I study Dr Frankland's wonderfully beautiful little treatise the more deeply I become impressed with the harmony or homology (I might call it, rather than analogy) which exists between the chemical and algebraical theories. In travelling my eye up and down the illustrated pages of "the Notes," I feel as Aladdin might have done in walking in the garden where every tree was laden with precious stones, or as Caspar Hauser when first brought out of his dark cellar to contemplate the glittering heavens on a starry night. There is an untold treasure of hoarded algebraical wealth potentially contained in the results achieved by the patient and long continued labour of our unconscious and unsuspected chemical fellow-workers.

We have seen that M. Hermite's beautiful expression for the skew invariant of the quintic proves its own character. A similar analysis may be applied to père Joubert's equally beautiful and even more remarkable expression for that of the sextic. M. de Bruno's statement of this, Table IV^{11}, contains two very perplexing typographical errors, namely, 4th line from foot of page, in V_0, $x_1x_2(x_\infty + x_0 - x_3 - x_2)$ should read $x_1x_2(x_\infty + x_0 - x_3 - x_4)$, and 3rd line from foot of page, in W_0, $x_2x_4(x_2 + x_3 - x_\infty - x_0)$ should be $x_2x_4(x_1 + x_3 - x_\infty - x_0)$. Moreover, the form in which the expression is presented in M. de Bruno's pages tends to mask its true nature and to suggest an analogy, which has no existence in fact, between it and M. Hermite's form; the latter is intrinsically a quinary group of triadic products, but such representation in the case of M. Joubert's form is purely conventional and confusing, it really being a single indecomposable quindenary product. Call a, b, c, d, e, f the six roots of a sextic, and let ab; cd; ef be any one of the 15 *duadic synthemes** which can be formed with them, and

$$F = \pm \begin{Bmatrix} ab.(c+d-e-f) \\ +cd.(e+f-a-b) \\ +ef.(a+b-c-d) \end{Bmatrix}$$

The external sign is arbitrary, but must be considered as *determined* once for all for each of the 15 values of F. The product of these 15 values is a symmetrical function of the roots. For suppose any two letters, as a, b, to be interchanged; then three of the factors F in which a and b are coupled will undergo no change, but the remaining twelve will evidently be resoluble into six pairs reciprocally related, so that each F of a pair is transformed either into the other or into its negative and on either supposition the product of the pair remains unaltered in value. Also this product is a differentiant, for $\Sigma\delta_a$ operating on any one factor evidently reduces it to zero. It is also of the weight 45 and of the order 15. Hence the product of the fifteen values of F is the skew invariant to the sextic.

It seems desirable to make the *differentiantive* character of the form self-apparent. This may be done by virtue of the remark that $\pm F$ may be replaced by the form

$$\begin{Bmatrix} (a-d)(b-f)(c-e) + (a-f)(b-d)(c-e) \\ +(a-c)(b-e)(d-f) + (a-e)(b-c)(d-f) \\ +(a-c)(b-f)(d-e) + (a-f)(b-c)(d-e) \\ +(a-d)(b-e)(c-f) + (a-e)(d-b)(c-f) \end{Bmatrix}$$

* A duadic syntheme of $2n$ letters is a combination of n duads containing between them all the letters. In it the order of the duads and of the letters in each duad is disregarded. Hence the number of such is $\frac{\Pi 2n}{2^n \Pi n}$ or $1.3.5...(2n-1)$. For an odd number of letters simple synthemes do not exist but in lieu of them we may construct diplo-synthemes containing every letter taken twice over.

This sum contains 64 terms, of which 48 are the terms in F taken 4 times over, and the other 16 are the 8 quantities *ace, bdf, acf, bde, bce, adf, bcf, ade*, each appearing twice with opposite signs. If we expand the product of the 15 values of F, we shall obtain 35,184392,568832, or upwards of 35 billions of terms distributable among a certain number of patterns; on prefixing Σ to one of each pattern a certain number of such sums will be zero, but the remaining ones of which there must be some (and there will probably be a very large number) will all be (except as to a numerical multiplier) identical with each other and with père Joubert's formula. We see by these examples that there is a sort of polymorphism or pheno-polymorphism, as it may be termed, which is of a much more superficial character than and ought to be carefully distinguished from true polymorphism, eteo-polymorphism as we may call it, and this distinction as it has a marked bearing upon the theory of algebraical linkages, it is reasonable to expect may not be without importance in the study and construction of chemical graphs. Although I have been dealing, in what precedes, with particular cases, the reasoning is general in its nature and leads to conclusions which I will proceed to express in exact terms.

Let us understand by a permutation-sum of a function of letters belonging to one or more sets ($n, n', n'', \ldots$ being the number of letters in the respective sets) the sum of the $\Pi n \Pi n' \Pi n'' \ldots$ values which the function assumes when the letters in each several set are permuted *inter se*; and let us understand by a monomial differentiant one which (with the usual convention as to $a = 1$) may be expressed as a permutation-sum of a single product of differences of roots of the parent quantic, or quantic system; then in the first place it has virtually been proved, in what precedes, and is undoubtedly true that every monadelphic differentiant is monomial, and it may easily be proved in like manner that a differentiant of multiplicity k may be represented by the sum of k monomial differentiants.

For greater simplicity let us confine ourselves to the case of monadelphic invariants and let us consider any two such belonging to reciprocal types; then the algebraical value of either one, in terms of the roots of its parent quantic or quantic system, will be represented by the permutation-sum of the product of the differences of every two letters in the other taken as many times as there are connecting bonds between them, such letters being for this purpose regarded as the roots in question. Hence also we may derive the rule previously given for determining whether or not any given graph, in which the number of bonds is equal to half the toti-valence, represents or not an algebraical invariant—the condition of its doing so being that the permutation-sum of the product of the differences between the connected letters (each bond giving one such difference) shall be other than zero. This rule will stand good whether the type of the graph be monadelphic or not.

A very simple instance occurs to me of the monomial law for monadelphic types. Let α, β, γ be the roots of a cubic. It will easily be found that the type (4 : 3, 4) to which

$$\{(\alpha-\beta)^2+(\alpha-\gamma)^2+(\beta-\gamma)^2\}^2$$

belongs is monadelphic; prefix to it the sign of summation, which is merely equivalent to multiplying it by 6. It will not be a monomial permutation-sum as it stands, but it may be replaced by $2\Sigma(\alpha-\beta)^2(\alpha-\gamma)^2$ or $\Sigma(\alpha-\beta)^4$ each of which monomial sums is a half of

$$\{(\alpha-\beta)^2+(\alpha-\gamma)^2+(\beta-\gamma)^2\}^2.$$

POSTSCRIPT. Subsequently to the printing of the foregoing sheets I have seen in an editorial notice in the English Journal *Nature* (Feb. 14, 1878) a statement of the claims of Dr Frankland to be the discoverer and first promulgator of the law of atomicity, and I appear unconsciously to have done injustice to this great English chemist by attributing the discovery to Kekulé. I derived my impression on the subject from the popular belief and from the account of it given by Wurz in his *Histoire des doctrines chémiques*. If the facts of the case are as set forth in *Nature* and admit of no qualifying statements, I am unable to understand how such a discovery as that of valence or atomicity, which furnishes the master-key to our knowledge of the transformations of matter and raises chemistry to the rank of a mathematical and predictive science (it was previously only arithmetical), can have escaped receiving the award of a Copley Medal from the society in whose Transactions it appeared. I can hardly imagine that, if the first announcement and proof of universal gravitation or the circulation of the blood had been communicated to the world in a paper inserted in the *Philosophical Transactions* in these days, its author would have failed to receive for it the highest mark of recognition in the power of the Royal Society of London to bestow, and in my humble judgment the law of atomicity in its far-reaching importance and the labour, and mental acumen required for its discovery, stands fully on a level with either of these great landmarks in the history of natural science. It seems also from the same article in *Nature* that my distinguished friend, Professor Crum Brown, to whose personal teaching at Edinburgh I owe the very slight acquaintance with the subject I can lay claim to, was the first to use the admirable method of chemico-graphs.

The conception of hydro-carbon graphs as "trees with nodes, branches and terminals" and the indispensable notion of constructing them by starting from "an intrinsic central node or pair of nodes, so as to get rid of the otherwise unsurmountable difficulty of having to recognize equivalent forms appearing several times over in the same construction," are exclusively my own and were used by me in my communications with Professor Crum Brown on the subject and stated by me in a letter to Professor Cayley, who has

adopted them as the basis of his own isomerical researches. In the account of this method given in German chemical journals I am informed that all reference (or at least all adequate reference) to my name as the author of it "fine by degrees and beautifully less," has at length entirely evaporated. M. Camille Jordan was led by quite a different order of considerations and with quite a different object in view to a discovery of the same centres before me, but I was not acquainted with this fact when I rediscovered them and made the application above mentioned. The idea of this application stands in the same relation to Professor Cayley's perfected use of it, as his idea of the use to be made of the equation $\Delta(w: i, j) =$ the number of linearly independent covariants of the type $[i, j: ij-2w]$ stands to my completed method founded thereon, for obtaining the scale and connecting syzygies of the irreducible covariants to a quantic, laying me thereby under an obligation which I should take it in very ill part if any translator of my papers on the subject failed to acknowledge in unmistakable terms.

The hydro-carbon graphs, it may be noticed, belong to the limiting case of chemico-graphs; where no cyclical system of bonds connects any groups of atoms in a graph, it becomes an arborescence.

I have found it a profitable exercise of the imagination, from a philosophical point of view, to build up the conception of an *infinite* arborescence and to dwell on the relations of time and causality which such a concept embodies. An example of the good to be gained by these limitless mental constructions (new tracts and highways, so to say, opened out in the all-embracing "grand continuum" which we call space) is afforded by the valuable applications to the theory of local probability and the integral calculus in general made by Professor Crofton (my successor at Woolwich) of his new idea of an infinite reticulation (warp and woof), every finite portion of which contains an infinite number of meshes, being formed by the crossings of two sets of parallel lines all infinitely extended in both directions and those of the same set equidistant and infinitely near to each other. So the largest idea of an arborescence is that of an infinite number of nodes with an infinite number of branches proceeding from each of them.

APPENDIX 2.

NOTE ON M. HERMITE'S LAW OF RECIPROCITY.

I take for granted that the treatise of M. Faà de Bruno represents this theory as it at present stands, in which case it seems to have made no advance since it was first promulgated by M. Hermite in his well known paper in the *Cambridge and Dublin Mathematical Journal*, 1854. It will be seen, however, I think from what follows, that it admits of being presented in a somewhat simpler and more general form. It rests essentially on the proposition of reciprocity in the theory of partitions that $(w: i, j) = (w: j, i)$, from which it follows as an immediate consequence that the number of arbitrary constants in the general covariant (or invariant) whose type is $[w: i, j]$, is the same as that whose type is $[w: j, i]$ since that number will be $\Delta(w: i, j) = \Delta(w: j, i)$ for each. Let now $\phi(a, b, c, \ldots l)$ be any differentiant of the order j in the coefficients, and of the weight w to a binary quantic $F(x, y)$ of the degree i in the variables; then ϕ is the root of a single covariant whose order is j and degree in the variables $ij - 2w$. Let ϕ be expressed (as from the definition of a differentiant must necessarily be possible) as a function of the differences of the roots $\alpha_1, \alpha_2, \ldots \alpha_i$ of F when y is made unity. For any difference $\alpha_p - \alpha_q$ substitute $\frac{d}{dx_p}\cdot\frac{d}{dy_q} - \frac{d}{dx_q}\cdot\frac{d}{dy_p}$, and let ϕ be converted into $\dot{\phi}$ by this substitution. Now operate with $\dot{\phi}$ upon the product of the i forms $G(x_1, y_1)$, $G(x_2, y_2), \ldots G(x_i, y_i)$, $G(x, y)$ signifying the general form of the degree j in the variables, and after the operation has been performed turning each subscript x into x and each subscript y into y, after the manner of Professor Cayley's original method of generating invariants or covariants as "Hyper-determinants;" we shall thus obtain an in- or co-variant to a form of the degree j which will be of the order i in the coefficients and of the degree $ij - 2w$ in the variables, for there are w factors in $\dot{\phi}$ and each factor is of the second dimension in two of the x's and the corresponding two y's. Thus we shall have passed from a form of the type $[i, j: ij - 2w]$ to another of the type $[j, i: ij - 2w]$, or which is the same thing, from one of the type $[w: i, j]$ to another of the type $[w: j, i]$.

This latter may be called the *image* of the first. For facility of reference, let the number of arbitrary parameters in the one and the other type be called the multiplicity. If we repeat upon this image the process by which it was deduced from its primitive, we shall obviously get back the original type, but it by no means follows that if the multiplicity exceed unity, we shall get back the primitive form itself. It may be possible to revert to the same type without reverting to the same individual specimen of it *; and such, we shall presently see, is what in general happens.

Before proceeding further I shall give a very simple methodical rule for finding the image to any given invariantive form. Since, for any given value of i, the form and its image are each given when their root-differentiants are respectively given, it will be sufficient to assign the law for passing from the differentiant of the primitive to that of its image.

* Just as, if I rightly understand the explanation given of fluorescence, a ray of light may give birth to some other form of motion and that again to another ray of light but of a different colour from the first. The theory of reciprocity treated of in the text is, in fact, a theory of alternate generation.

For this purpose, let the given in- or co-variant be expressed in terms of symmetrical functions of the roots of the quantic when the leading coefficient (a), is made equal to unity. Then it will consist of terms, any one of which, apart from its numerical coefficient, will be of the form

$$\Sigma(\alpha_1\alpha_2\ldots\alpha_\lambda)^0(\beta_1\beta_2\ldots\beta_\mu)^1(\gamma_1\gamma_2\ldots\gamma_\nu)^2(\delta_1\delta_2\ldots\delta_\pi)^3\ldots$$

$\alpha_1\alpha_2\ldots\alpha_\lambda$, $\beta_1\beta_2\ldots\beta_\mu$, $\gamma_1\gamma_2\ldots\gamma_\nu$, &c. being all distinct and comprising between them *all* the i roots and of course $\mu + 2\nu + 3\pi + \&c.$ will be equal to the weight; to pass from a differentiant expressed in terms of roots of a given quantic to the expression in terms of coefficients of the allied quantic of its image it will be found that the only thing necessary is to change any such factor as α^λ (where α is any root of the given quantic) into C_λ, the coefficient of the term containing y^λ in the allied one. This rule is a consequence (obtainable by ordinary algebraical processes) from the method above explained, where it is to be borne in mind that in order to obtain the image from the given form we have only to substitute for each root α_κ which occurs in ϕ, the fraction $\frac{dx_\kappa}{dy_\kappa}$ and to multiply the result by such a power of $\frac{d}{dy_1}\cdot\frac{d}{dy_2}\cdots\frac{d}{dy_\kappa}$, as will just serve to make it integral. A much simpler demonstration of this rule will be given in the sequel, and it will be shown that it not only holds good for deriving the leading term of the reciprocal (in the case of a covariant) from that of the primitive (that is, the root-differentiant of the one from the root-differentiant of the other) but that it is applicable to deriving the whole of one expression from the whole of the other.

As an example, take the differentiant whose type is [3: 3, 3], the root or base of the skew covariant to a cubic $(a, b, c, d \between x, y)^3$. Its value is $a^2d - 3abc + 2b^3$; expressed in terms of the roots α, β, γ, making $a = 1$, this becomes

$$\alpha\beta\gamma - \frac{3(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma)}{9} + 2\frac{(\alpha+\beta+\gamma)^3}{27},$$

$$\text{or } \frac{1}{27}\left\{27\alpha\beta\gamma - 9(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma) + 2(\alpha+\beta+\gamma)^3\right\},$$

$$\text{or } \frac{1}{27}\left\{2\Sigma\alpha^3 - 3\Sigma\alpha^1\beta^2 + 12\alpha\beta\gamma\right\}, \text{ that is, } \frac{1}{27}\left\{2\Sigma\alpha^0\beta^0\gamma^3 - 3\Sigma\alpha^0\beta^1\gamma^2 + 12\alpha^1\beta^1\gamma^1\right\}.$$

Applying the rule, this becomes converted into

$$\frac{1}{27}\left\{6C_0^2C_3 - 18C_0C_1C_2 + 12C_1^3\right\},$$

or, reverting to the letters a, b, c, d, the image becomes the primitive affected with the factor $\frac{6}{27}$ and may be seen to be its own conjugate. Or again, let the primitive be the discriminant of a cubic, that is,

$$\frac{1}{27}(\alpha-\beta)^2(\alpha-\gamma)^2(\beta-\gamma)^2 \text{ or } (\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha-\alpha\beta^2-\beta\gamma^2-\gamma\alpha^2)^2;$$

this is equal to

$$\Sigma\,(\alpha^2\beta^4 + 2\Sigma\alpha\beta^2\gamma^3 - 2\Sigma\alpha^3\beta^3 - 6\alpha^2\beta^2\gamma^2 - 2\Sigma\alpha\beta\gamma^4).$$

Hence, by our rule, the image will be

$$\frac{1}{27}\,(6c_0c_2c_4 + 12c_1c_2c_3 - 6c_0c_3^2 - 6c_2^3 - 6c_1^2c_4),$$

or, using a, b, c, d, e in lieu of c_0, c_1, c_2, c_3, c_4, we obtain the form

$$-\frac{6}{27}\,(ace + 2bcd - ad^2 - c^3 - b^2e),$$

that is, $-\frac{2D}{9}$, where D is the well known quadrinvariant to a quartic

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}.$$

Treating this quadrinvariant as a function of the roots of a biquadratic form and proceeding as before to form its image, we shall obtain a second image which will be a numerical multiple of the original invariant.

But now let us consider the case of polyadelphic forms belonging to reciprocal types and for greater brevity, as the calculations are necessarily long, take a quantic of the self-contrary type $[w : i, i]$, as, for example $[6 : 4, 4]$ which belongs to the covariant of the fourth order and fourth degree to a quartic. This will be diadelphic; its general form is a linear combination of two products, one of the quartic itself by its cubinvariant, the other of the Hessian by the quadrinvariant. It will therefore have for its leading coefficient the differentiant

$$\lambda a\,(ace + 2bcd - ad^2 - c^3 - b^2e) + \mu\,(ac - b^2)\,(ae - 4bd + 3c^2),$$

say $\lambda U + \mu V$. Let us first find the image of U. Expressed in terms of the roots $\alpha, \beta, \gamma, \delta$, it is

$$\frac{1}{6}\,(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)\,(\alpha\beta\gamma\delta)$$

$$+\frac{1}{48}\,(\alpha + \beta + \gamma + \delta)\,(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)\,(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)$$

$$-\frac{1}{16}\,(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)^2 - \frac{1}{216}\,(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)^3$$

$$-\frac{1}{16}\,(\alpha + \beta + \gamma + \delta)^2\,\alpha\beta\gamma\delta,$$

which is

$$\frac{[6]\,(\alpha\beta\gamma^2\delta^2)}{6} + \frac{[24]\,(\alpha\beta^2\gamma^3) + [48]\,(\alpha\beta\gamma^2\delta^2) + [12]\,(\alpha^2\beta^2\gamma^2) + [12]\,(\alpha\beta\gamma\delta^3)}{48}$$

$$-\frac{[4]\,(\alpha^2\beta^2\gamma^2) + [12]\,(\alpha\beta\gamma^2\delta^2)}{16}$$

$$-\frac{[6]\,(\alpha^3\beta^3) + [90]\,(\alpha\beta\gamma^2\delta^2) + [72]\,(\alpha\beta^2\gamma^3) + [24]\,(\alpha^2\beta^2\gamma^2) + [24]\,(\alpha\beta\gamma\delta^3)}{216}$$

$$-\frac{[4]\,(\alpha\beta\gamma\delta^3) + [12]\,(\alpha\beta\gamma^2\delta^2)}{16},$$

where any term, as for example $[48]\,(\alpha\beta\gamma^2\delta^2)$, means the sum of the quantities of the type $\alpha\beta\gamma^2\delta^2$ each taken a sufficient number of times to make up 48 combinations, so that it is identical in meaning with $8\Sigma\,(\alpha\beta\gamma^2\delta^2)$ in the common notation. This convention is useful in saving the unnecessary labour of performing divisions in this first part of the process which have to be exactly reversed by multiplications in the transformation process which follows. The value of the above sum is, for purposes of transformation, equivalent to

$$\frac{1}{36}\left\{3\alpha\beta\gamma^2\delta^2 + 6\alpha\beta^2\gamma^3 - 4\alpha^2\beta^2\gamma^2 - 4\alpha\beta\gamma\delta^3 - \alpha^3\beta^3\right\},$$

which gives for the image of U

$$\frac{1}{36}\,(3b^2c^2 + 6abcd - 4ac^3 - 4db^3 - a^2d^2)$$

or $\frac{1}{36}\,(U - V)$, where it will be observed that $(V - U)$ is identical with the discriminant to $(a, b, c, d \between x, y)^3$. Let us now proceed to find the image of $(U - V)$. Using σ to denote the sum of the combinations of $\alpha, \beta, \gamma, \delta$ taken i and i together, where $\alpha, \beta, \gamma, \delta$ are the roots of the general quartic, we have

$$U - V = \frac{\sigma_1^2\sigma_2^2}{192} + \frac{\sigma_1\sigma_2\sigma_3}{16} - \frac{\sigma_2^3}{54} - \frac{\sigma_3\sigma_1^3}{64} - \frac{\sigma_3^2}{16}$$

$$= \frac{1}{1728}\,(9\sigma_1^2\sigma_2^2 + 108\sigma_1\sigma_2\sigma_3 - 32\sigma_2^3 - 27\sigma_3\sigma_1^3 - 108\sigma_3^2).$$

Expanding and transforming, it will be found that the image of $(U - V)$ is $\left(\frac{21}{432}\,U - \frac{1}{432}\,V\right)$ and the second image of U which is $\frac{I\,(U - V)}{36}$ does not revert to the form U.

As a simpler example we may take the covariant to a quartic, still of the fourth order in the coefficients as before, but of the eighth degree in the variables. This will have for its root-differentiant

$$\lambda a^2\,(ae - 4bd + 3c^2) + \mu\,(ac - b^2)^2, \text{ say } \lambda U + \mu V.$$

Here

$$U = \sigma_4 - \frac{\sigma_1\sigma_3}{4} + \frac{\sigma_2^2}{12} = \frac{1}{12}\,(12\sigma_4 - 3\sigma_1\sigma_3 + \sigma_2^2),$$

and for the purpose of transformation is equivalent to

$$\frac{1}{12}\left\{12\alpha\beta\gamma\delta - 3\,(4\alpha\beta\gamma\delta + 12\alpha^2\beta\gamma) + 6\alpha\beta\gamma\delta + 6\alpha^2\beta^2 - 12\alpha^2\beta\gamma\right\}$$

$$= \frac{1}{12}\left\{6\alpha\beta\gamma\delta + 6\alpha^2\beta^2 - 12\alpha^2\beta\gamma\right\}.$$

Hence, using I to denote "image of,"

$$IU = \frac{1}{2}\left\{b^4 + a^2c^2 - 2acb^2\right\} = \frac{1}{2}V.$$

Again $$V = \left(\frac{\sigma_2}{6} - \frac{\sigma_1^2}{16}\right)^2$$

$$= \frac{1}{48^2}\left\{3\alpha^2 + 3\beta^2 + 3\gamma^2 + 3\delta^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)\right\}^2,$$

which, for purposes of transformation, will be found equivalent to

$$\frac{1}{48^2}\left\{36\alpha^4 + 132\alpha^2\beta^2 - 48\alpha^2\beta\gamma - 144\alpha^3\beta + 24\alpha\beta\gamma\delta\right\}.$$

Consequently

$$IV = \frac{1}{192}\left\{3a^3e + 11a^2c^2 - 4ab^2c - 12a^2bd + 2b^4\right\}$$

$$= \frac{1}{192}\left\{3a^2(ae - 4bd + 3c^2) + 2(b^2 - ac)^2\right\}$$

$$= \frac{1}{192}(3U + 2V).$$

Let now $\lambda : \mu$ be so chosen that

$$I(\lambda U + \mu V) = \rho(\lambda U + \mu V).$$

This gives $$\frac{\mu U}{64} + \left(\frac{\lambda}{2} + \frac{\mu}{96}\right)V = \rho(\lambda U + \mu V),$$

or $$\frac{\mu^2}{64} - \frac{\lambda\mu}{96} - \frac{\lambda^2}{2} = 0,$$

that is, $$3\mu^2 - 2\lambda\mu - 96\lambda^2 = 0.$$

The two values of $\frac{\mu}{\lambda}$ derived from this equation are 6 and $-\frac{16}{3}$. The corresponding values of ρ will be 6 and $-\frac{1}{12}$. There are thus two definite systems of $\lambda : \mu$, and no more, which will make $\lambda U + \mu V$ self-conjugate and it is obvious that there will be no other values of $\lambda : \mu$ which will make

$$I^2(\lambda U + \mu V) = \rho(\lambda U + \mu V),$$

for, I^2U and I^2V being determinate linear functions of U, V, we shall have a quadratic equation for determining $\lambda : \mu$, but the two values of $\lambda : \mu$ which make $\lambda U + \mu V$ self-conjugate must satisfy this equation, and hence there can be no others. Reverting to the preceding example of the type [6 : 4, 4], we have found

$$IU = \frac{1}{36}U - \frac{1}{36}V$$

$$I(U - V) = \frac{21}{432}U - \frac{1}{432}V.$$

Hence $$IV = -\frac{9}{432}U - \frac{11}{432}V,$$

and making $$I(\lambda U + \mu V) = \rho(\lambda U + \mu V),$$

the equation for finding ρ will be

$$\begin{vmatrix} \frac{12}{432} - \rho, & -\frac{12}{432} \\ -\frac{9}{432}, & -\frac{11}{432} - \rho \end{vmatrix} = 0,$$

whence $$\rho_1 = -\frac{1}{27}, \quad \rho_2 = \frac{5}{144};$$

also, since $$\left(\frac{12}{432}\lambda - \frac{9}{432}\mu\right) = \rho\lambda,$$

we shall have $$\frac{\lambda_1}{\mu_1} = -\frac{9}{28}, \quad \frac{\lambda_2}{\mu_2} = 3.$$

What intrinsic peculiar properties are possessed by the principal forms* is a question as to which we are at present quite in the dark, as are we also with regard to the general character of the equation in ρ. It were much to be wished that some one would work out the case of a triadelphic type, as for example the type of covariants of the 6th order in the coefficients and the 6th degree in the variables, to a sextic. It might be supposed from the two preceding examples that the values of ρ are necessarily rational, but it will be shown hereafter that such is not the case.

It is easy to see that the relation between any form belonging to a given type of multiplicity 2 or 3 and its second image may be geometrically represented by means of a quadric curve or surface. Thus suppose the multiplicity is three, and that the three values of ρ are A, B, C. Construct an ellipsoid or hyperboloid whose semiaxes are $\frac{1}{\sqrt{A}}, \frac{1}{\sqrt{B}}, \frac{1}{\sqrt{C}}$. Draw r any radius vector making angles α, β, γ with the principal axes, p a perpendicular from the centre upon the tangent plane at the point where r meets the quadric, making angles λ, μ, ν with these axes. Then if

$$K(\cos\alpha U + \cos\beta V + \cos\gamma W)$$

be any given form of the system for which U, V, W are the principal forms,

$$\frac{K}{pr}(\cos\lambda U + \cos\mu V + \cos\nu W)$$

will be its second image. And we may say that, if a form lies in the

* By a principal form (in general), as hereafter stated in the text, I mean one which is the reciprocal of its first image in the sense that it bears a numerical ratio to its second image. The numerical quantity by which it must be multiplied to give the second image, I call a principal multiplier.

direction of the axis of instantaneous rotation, its second image will lie in the perpendicular upon the invariable plane: or more simply if by the direction of a form $\lambda U + \mu V + \nu W$ we understand that of a straight line whose direction cosines are as $\lambda : \mu : \nu$ and by its modulus $\sqrt{(\lambda^2 + \mu^2 + \nu^2)}$, we may say that if a radius vector of the ellipsoid (or other quadric) represent the direction and modulus of an in- or co-variant the corresponding radius vector of the polar reciprocal to the quadric will represent the direction and modulus of its second image.

The true nature of the reciprocity theorem, in the general case where i, j have any values whatever, is now obvious. Let $U_1, U_2, \ldots U_q$ be independent forms belonging to the type $[w: i, j]$, whose multiplicity is q, and $V_1, V_2, \ldots V_q$ as many forms belonging to the reciprocal type $[w: j, i]$. We may, by virtue of the transformation process, express each IU in terms of linear functions of the forms V and *vice versa*, so that each I^2U will be a known linear function of all the U's. For clearness sake suppose $q = 3$ and let

$$I^2U_1 = a\,U_1 + b\,U_2 + c\,U_3$$
$$I^2U_2 = a'\,U_1 + b'\,U_2 + c'\,U_3$$
$$I^2U_3 = a''\,U_1 + b''\,U_2 + c''\,U_3.$$

Now make

$$I^2(\lambda U_1 + \mu U_2 + \nu U_3) = \rho\,(\lambda U_1 + \mu U_2 + \nu U_3).$$

We shall have for finding ρ the equation

$$\begin{vmatrix} (a-\rho), & b & , & c \\ a' & , & (b'-\rho), & c' \\ a'' & , & b'' & , & (c''-\rho) \end{vmatrix} = 0,$$

and then the three systems of values of $\lambda : \mu : \nu$, which make the second image of $\lambda U_1 + \mu U_2 + \nu U_3$ coincide to a numerical factor près, with itself, will be rational functions of the respective roots. So, in general, when the multiplicity of the type $[w: i, j]$ is q, there will be in general q special forms, and no more, which have reciprocal forms belonging to the type $[w: j, i]$, and if the interchangeable elements, i, j are equal, then these q forms will all be self-conjugate. It is conceivable that in certain cases the equation in ρ may have equal roots; in that event each such equality would introduce a corresponding indeterminateness in the forms admitting of conjugates. For example, if the multiplicity were 2 and the two roots of ρ equal, that would signify that *every* form belonging to the type would have a conjugate—a fact analogous to an ellipse becoming a circle, or an ellipsoid a spheroid—and so in general.

A form having a conjugate, that is, whose second image is a numerical multiplier of itself, may be called a principal form. If the multiplicity of the type is q, there will be q such. All but these will give rise to an endless succession of images such that any $q + 1$ of an even order (the form itself included among these) will be connected by a linear equation. That the succession is endless is clear from the consideration that if an image, say of the $(2p)$th rank, is identical (to a numerical factor près) with the form, we have an equation of the qth degree for finding the values of the systems of multipliers λ, μ, ν of U, V, W; therefore there are only q such systems, but the systems which satisfy $I^2F = \rho F$ must also satisfy $I^{2p}F = \rho' F$, and consequently there are no others.

To illustrate this, suppose

$$I^2U = aU + bV$$
$$I^2V = cU + dV;$$

then

$$I^4U = (a^2 + bc)\,U + (ab + bd)\,V$$
$$I^4V = (ca + ad)\,U + (cb + d^2)\,V.$$

If now we put

$$\begin{vmatrix} a-\rho, & b \\ c \quad , & d-\rho \end{vmatrix} = 0,$$

to find the values of $\lambda : \mu$ which make $I^2(\lambda U + \mu V) = \rho\,(\lambda U + \mu V)$ we have

$$(a-\rho)\,\lambda + c\mu = 0.$$

In like manner, if we make

$$\begin{vmatrix} a^2 + bc - \rho, & ab + bd \\ ca + ad \quad , & cb + d^2 - \rho \end{vmatrix} = 0,$$

to find the values of Λ and M which make $I^4(\Lambda U + \mathrm{M}V) = R\,(\Lambda U + \mathrm{M}V)$, we have

$$(a^2 + bc - R)\,\Lambda + (ca + ad)\,\mathrm{M} = 0,$$

and it will be found that

$$a - \rho = \frac{a-d}{2} \pm \frac{1}{2}\sqrt{\{(a-d)^2 + 4bc\}}$$

$$a^2 + bc - R = \frac{a^2 - d^2}{2} \pm \frac{a+d}{2}\sqrt{\{(a-d)^2 + 4bc\}},$$

so that the values of $\lambda : \mu$ and $\Lambda : \mathrm{M}$ are the same, and such we know *à priori* must be the case.

It ought to be noticed that the method explained in the preceding pages furnishes a complete solution of the problem following. Given any in- or co-variant, say of the jth order in the coefficients to a form Q of the ith degree, to find the process of differentiation which performed upon the product

$$Q(x_1, y_1) \,.\, Q(x_2, y_2) \ldots .\, Q(x_j, y_j)$$

shall produce the j-partite-emanant of the in- or co-variant so given, and it proves incidentally that every binary in- or co-variant may be represented as a hyperdeterminant. To make this clear, let us call the above product, or rather that product divided by $(\Pi i)^j$, the j-ary norm of Q and denote it by NQ. Again, let G be any given differentiant to the type $[w: j, i]$, say $G(\rho_1, \rho_2, \dots \rho_j)$ which is necessarily identical with

$$G\{0; (\rho_2-\rho_1); (\rho_3-\rho_1); \dots (\rho_j-\rho_1)\}.$$

For $\rho_\kappa - \rho_1$ write $\frac{d}{dx_\kappa}\cdot\frac{d}{dy_1} - \frac{d}{dy_\kappa}\cdot\frac{d}{dx_1}$ and let the quantity so formed be called the hyperdeterminant to G and be denoted by HG. Then if E be any principal form to the type $[w: i, j]$, of the multiplicity q and belonging to a quantic Q, and G be its first image, we shall have

$$(HG)(N_jQ) = \rho F,$$

where ρ is one of the roots of a known equation of the qth degree in ρ. Consequently, since any form belonging to the given type is a linear function of its q principal forms, every such form may be expressed by means of the hyperdeterminant

$$\Sigma_{\lambda=q}^{\lambda-1} \frac{c_\lambda}{\rho_\lambda}(HG_\lambda)\, NQ,$$

the given form being supposed to be expressible by $\Sigma_{\lambda=q}^{\lambda=1} c_\lambda F_\lambda$, where F is any one of the q principal forms.

It follows from what has been shown above that in general from any one particular given form belonging to a type of multiplicity q may be deduced the $(q-1)$ others (by taking the successive second images) and thus the general form obtained; the exception is when the given form happens to be a linear function of less than q of the principal forms. A further consequence is that any in- or co-variant given in terms of the roots of its quantic may be converted by explicit processes into a function of the coefficients. Thus, for example, suppose that the multiplicity of the type is 3; call the given form R_0 and the successive second images R_1, R_2, R_3, R_4. These latter will be all known by the rule of transformation and we shall have R_4 a known linear function of the three preceding forms, say equal to

$$\alpha R_1 + \beta R_2 + \gamma R_3.$$

Hence if we put $\qquad R_0 = \lambda R_1 + \mu R_2 + \nu R_3,$

we must have $\qquad R_1 = \lambda R_2 + \mu R_3 + \nu(\alpha R_1 + \beta R_2 + \gamma R_3);$

hence $\qquad \nu = \frac{1}{\alpha}, \quad \mu = -\frac{\gamma}{\alpha}, \quad \lambda = -\frac{\beta}{\alpha}$

and thus R_0, given in terms of the roots, becomes known in terms of the coefficients of its quantic. And so in general, q being the multiplicity, $(q+1)$ forms deduced from the given function of the roots will serve to determine its value as a function of the coefficients. In fact by regarding R_0 as a linear function of the principal forms, it is easy to see it and all its

successive secondaries (that is, second images) form a recurring series, the scale of relations being

$$R_0 - \Sigma\frac{1}{\rho}R_1 + \Sigma\frac{1}{\rho^2}R_2 - \Sigma\frac{1}{\rho^3}R_3 + \dots = 0,$$

where $1:\rho$ is the ratio of any principal form to its immediate secondary. Thus E_0 being given in terms of the roots and consequently $E_1, E_2, \dots E_q$, in terms of the coefficients, E_0 becomes known in terms of the coefficients and of the quantities $\Sigma\frac{1}{\rho}, \Sigma\frac{1}{\rho^2}, \dots$; these latter are identical with the quantities previously mentioned and furnish the simplest means of forming the equation in ρ, which (if we agree to call $\rho_1, \rho_2, \dots \rho_q$ the moduli of the several principal forms $F_1, F_2, \dots F_q$, that is, the ratios of their respective second images to themselves) may be termed the modular equation for any given type*.

It might have been useful, had I thought of it in time, and may be useful when the subject comes again under consideration, to treat a form and its second image, in which the type is restored as *antecedent* and *consequent*, and to describe the first image as the *alternate* form to the primitive, inasmuch as we pass, by what biologists term alternate generation, from one type to the other. It has been shown, in what precedes, that the transformation by images at each second step leads back to the original type, but, contrary to what might have been supposed, does not in general imply the resuscitation of the individual form.

The theorem of reciprocity has been seen to be, in its essence, a theorem of differentiants, and ought therefore to admit of being proved by means of the necessary and sufficient partial differential equation to which differentiants are subject. This may be done as follows. If we call $\epsilon_0, \epsilon_1, \epsilon_2, \dots \epsilon_r$ the successive elements to a binary quantic expressed in its customary form, so that ϵ_r is the coefficient of the term containing y^r divested of its numerical binomial coefficient, and if we write

$$U = \frac{d}{d\alpha} + \frac{d}{d\beta} + \frac{d}{d\gamma} + \dots,$$

where $\alpha, \beta, \gamma, \dots$ are the roots of the quantic, it is very easily proved that

$$U\epsilon_r = -r\epsilon_{r-1}\dagger.$$

Let $C\Sigma\alpha^r\beta^s\gamma^t\dots$ be any term in a given differentiant F, the indices $r, s, t, \dots$ being any whatever with no condition as to their being distinct from each

* But it will be better to adhere to the previous convention and to designate the ρ's as the principal multipliers and the equation in ρ as the principal equation.

† In fact it may easily be proved by the ordinary rule for the change of one system of independent variables into another that, if $a_1, a_2, \dots a_i$ be the roots of $(\epsilon_0, \epsilon_1, \epsilon_2, \dots \epsilon_i \between x, y)^i$,

$$\Sigma\frac{d}{da} = -\sum_{q=0}^{q=i} q\epsilon_{q-1}\frac{d}{d\epsilon_q}.$$

other, and let $N(r, s, t, \ldots)$ signify the number of combinations comprised in Σ; also let $CN(r, s, t, \ldots) . \epsilon_r \epsilon_s \epsilon_t \ldots$ be called the image of the term above written and G the image of F, that is, the sum of the images of the several terms in F; where it must be observed that the ϵ quantities do not necessarily refer to roots the same in number or name as the roots $\alpha, \beta, \gamma, \ldots$. Now suppose that we have any term, such as $Q\Sigma\alpha^l\beta^m\gamma^n \ldots$ in UF, where U refers to the given roots $\alpha, \beta, \gamma, \ldots$ and means $\frac{d}{d\alpha} + \frac{d}{d\beta} + \frac{d}{d\gamma} + \ldots$. This term must arise from terms of the several forms

$$\left.\begin{array}{l} A\,\Sigma\alpha^{l+1}\,\beta^m\,\gamma^n \ldots \\ B\,\Sigma\alpha^l\,\beta^{m+1}\,\gamma^n \ldots \\ C\,\Sigma\alpha^l\,\beta^m\,\gamma^{n+1} \ldots \end{array}\right\} \text{in } F;$$

&c. &c. ...

corresponding to these there will be the images

$$\left.\begin{array}{l} AN(l+1, m, n, \ldots)\, \epsilon_{l+1} . \epsilon_m . \epsilon_n \ldots \\ BN(l, m+1, n; \ldots)\, \epsilon_l . \epsilon_{m+1} . \epsilon_n \ldots \\ CN(l, m, n+1, \ldots)\, \epsilon_l . \epsilon_m . \epsilon_{n+1} \ldots \end{array}\right\} \text{in } G,$$

&c. &c. ...

where G belongs to a quantic whose type is reciprocal to that of F, and it is clear that the effect of operating upon F with U will be to give

$$Q = A\rho N(l+1, m, n, \ldots)(l+1) + B\rho N(l, m+1, n, \ldots)(m+1) + C\rho N(l, m, n+1, \ldots)(n+1) + \&c. \ldots$$

ρ being a number easily determinable, but which there is no occasion to express. Again if $R\epsilon_l . \epsilon_m . \epsilon_n \ldots$ be the correlative term in G, we have by virtue of the formula $U\epsilon_r = -r\epsilon_{r-1}$, where the operator U refers to the roots of the quantic of reciprocal type,

$$(-)^w R = AN(l+1, m, n, \ldots)(l+1) + BN(l, m+1, n, \ldots)(m+1) + CN(l, m, n+1, \ldots)(n+1) + \&c. \ldots$$

Consequently, since on account of the identity $F=0$, we must have $Q=0$ for every term $Q\Sigma\alpha^l . \beta^m . \gamma^n \ldots$, we must also have $R = \rho^{-1} Q = 0$ and therefore, this being true for all the arguments $\epsilon_l . \epsilon_m . \epsilon_n \ldots$, we must have $UG = 0$. Hence, when any quantity F is a differentiant of a given quantic, its image (as defined in the text) is also a differentiant to a quantic of reciprocal type to the given one. This is the simplest method of establishing the theorem, but still the method originally employed in the note is valuable as serving to establish the important proposition that every in- or co-variant of a binary quantic is a hyperdeterminant.

I will proceed to show that for a system of two or more quantics of degrees $i, i', i'', \ldots$, we may pass from a covariant of the type $[w: i, j; i', j'; i'', j''; \ldots]$

to one of the type $[w: j, i; i', j'; i'', j''; \ldots]$ by taking its image in respect to the quantic whose indices, i, j, are to be interchanged precisely according to the same rule as if there were no other quantic present. As regards the law of reciprocity, a combination of quantics is analogous to a mixture of gases, according to Dalton's view, each playing the part, as it were, of a vacuum in respect to the other.

Let $[w: i, j; i', j'; \ldots]$ be the type, $[w: j, i; i', j'; \ldots]$ one of the antitypes, $(\epsilon_0, \epsilon_1, \epsilon_2, \ldots \epsilon_j \between x, y)^j$ the general form of the jth degree, $\alpha, \beta, \gamma, \ldots$ its roots when $\epsilon_0 = 1$. Let $\eta_r = (-)^r \epsilon_r$; then, since

$$\Sigma \frac{d}{d\alpha} \epsilon_r = -r\epsilon_{r-1}$$

$$\Sigma \frac{d}{d\alpha} \eta_r = r\eta_{r-1}.$$

Let D be any differentiant of the given type, $a, b, c, \ldots$ the roots of the quantic of degree i, $a', b', c', \ldots$ the roots of the quantic of degree i', with the usual convention as to the leading coefficients becoming unities. Let $\Sigma a^l b^m \ldots, \Sigma a'^{l'} b'^{m'} \ldots \Sigma \ldots$ be the arguments of any term in

$$\left(\Sigma \frac{d}{d\alpha} + \Sigma \frac{d}{d\alpha'} + \ldots\right) D,$$

say UD, then the coefficient of the term last written will arise from operating with U upon

$$\left.\begin{array}{l} A . \Sigma a^{l+1} . b^m \ldots \Sigma a'^{l'}\, b'^{m'} . \&c. \ldots \\ + B . \Sigma a^l . b^{m+1} . \ldots \Sigma a'^{l'}\, b'^{m'} . \&c. \ldots \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + A' \Sigma a^l . b^m \ldots \Sigma a'^{l'+1} b'^{m'} . \&c. \ldots \\ + B' \Sigma a^l . b^m \ldots \Sigma a'^{l'}\, b'^{m'+1} . \&c. \ldots \\ \quad \&c. \ldots \qquad \&c. \ldots \end{array}\right\}$$

and the value of the coefficient will be

$$\left.\begin{array}{l} A\,(l+1)\,N(l+1, m, \ldots)\,N(l', m', \ldots\ldots\ldots) \ldots \\ + B\,(m+1)\,N(l, m+1, \ldots)\,N(l', m', \ldots\ldots\ldots) \ldots \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + A'\,(l'+1)\,N(l, m, \ldots\ldots)\,N(l'+1, m', \ldots) \ldots \\ + B'\,(m'+1)\,N(l, m, \ldots\ldots)\,N(l', m'+1, \ldots) \ldots \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}$$

$$\div N(l, m, \ldots)\,N(l', m', \ldots) \ldots$$

To these feeders or contributory terms will correspond, in the image,

$$\begin{array}{l} AN\,(l+1, m, \ldots)\,\eta_{l+1} . \eta_m \ldots \Sigma a'^{l'} . b'^{m'} \ldots \\ + BN\,(l, m+1, \ldots)\,\eta_l . \eta_{m+1} \ldots \Sigma a'^{l'} . b'^{m'} \ldots \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + A'N\,(l, m, \ldots\ldots)\,\eta_l . \eta_m \ldots \Sigma a'^{l'+1} . b'^{m'} \ldots \\ + B'N\,(l, m, \ldots\ldots)\,\eta_l . \eta_m \ldots \Sigma a'^{l'} . b'^{m'+1} \ldots \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

and it is obvious that by operating upon this with the U corresponding to its roots we shall obtain the argument $\eta_l . \eta_m \ldots \Sigma a'^{l'} . b'^{m'}$. &c.... affected with the very same coefficient as that above written, except that in its denominator the factor, $N(l, m, \ldots)$, will not appear. Hence, when D is a differentiant of the given type, its image (obtained by expressing the i set of coefficients in terms of roots and then replacing every power, ρ^q, of any such root, ρ, by η_q, leaving all the other coefficients unchanged) will also be a differentiant of the type transformed by interchanging i with its conjugate j*.

When there is but one quantic the effect of substituting ϵ_q instead of η_q will evidently only be to introduce a common factor $(-)^w$ into each term, which is immaterial and we may accordingly in that case reflect ρ^q into ϵ_q. Of course, in the general case, if all the letters i are simultaneously interchanged with the letters j, a similar conclusion follows.

As an example, let us take the two quadratics,

$$ax^2 + 2bxy + cy^2,$$

$$\alpha x^2 + 2\beta xy + \gamma y^2,$$

their resultant $(a\gamma - c\alpha)^2 + 4(a\beta - b\alpha)(c\beta - b\alpha)$, belongs to the type [4: 2, 2; 2, 2] which is its own reciprocal whichever of the interchangeable elements we permute. This resultant, treating a as unity, will be equal to

$$(\alpha\rho_1^2 + 2\beta\rho_1 + \gamma)(\alpha\rho_2^2 + 2\beta\rho_2 + \gamma)$$

$$= \alpha^2\rho_1^2\rho_2^2 + 2\beta\alpha(\rho_1^2\rho_2 + \rho_1\rho_2^2) + 4\beta^2\rho_1\rho_2 + \alpha\gamma(\rho_1^2 + \rho_2^2) + 2\beta\gamma(\rho_1 + \rho_2) + \gamma^2$$

the image of which will be

$$\alpha^2\epsilon_2^2 - 4\alpha\beta\epsilon_1\epsilon_2 + 4\beta^2\epsilon_1^2 + 2\alpha\gamma\epsilon_0\epsilon_2 - 4\beta\gamma\epsilon_0\epsilon_1 + \gamma^2\epsilon_0^2,$$

or as we may write it,

$$\alpha^2c^2 - 4\alpha\beta bc + 4\beta^2b^2 + 2\alpha\gamma ac - 4\beta\gamma ab + a^2\gamma^2,$$

which is $(c\alpha - 2b\beta + a\gamma)^2$, the square of the well known connective. Again, if we combine $ax^3 + 3bx^2y + 3cxy^2 + dy^3$ with $\alpha x + \beta y$, we have the invariant

$$a\beta^3 - 3b\alpha\beta^2 + 3c\alpha^2\beta - d\alpha^3, \text{ say } I,$$

* Thus the rule of images for passing from a differentiant of a given type belonging to a single quantic to one of the opposite type is extended to the case of passing from a differentiant of a given type belonging to a system of quantics to any associated type, that is, to any type in which one or more of the numbers i chosen at discretion is or are interchanged with the corresponding numbers j, and it will presently be seen that this implies the extension of the rule without any alteration from differentiants or invariants to covariants of a quantic or system of quantics. In Note A it will further be shown that for any inversions whatever (or, to speak more accurately, for any cycle of inversions leading back to the original type), although the principal multipliers change their values as the cycle of inversion changes, the *principal forms themselves remain the same*,—a surprising conclusion but very easily proved. In other words, however many quantics there may be in the parent system, there is never more than one single set of principal forms of derivatives to it of a given type. A cycle of arbitrarily intercalated pairs of reversals (here of successive i's and j's), by which a type returns to itself, comes under the category, "Verschlingung," or "Knotting" of Gauss, Listing and Tait.

belonging to the type [3: 3, 1; 1, 3]. Write $\alpha = 1, \beta = -\rho$; this becomes

$$-a\rho^3 - 3b\rho^2 - 3c\rho - d,$$

of which an image, say J, belonging to the type [3: 3, 1; 3, 1],

$$a\epsilon_3 - 3b\epsilon_2 + 3c\epsilon_1 - d\epsilon_0$$

is the connective of

$$\begin{Bmatrix} ax^3 + 3bx^2y + 3cxy^2 + dy^3 \\ \epsilon_0x^3 + 3\epsilon_1x^2y + 3\epsilon_2xy^2 + \epsilon_3y^3 \end{Bmatrix}.$$

Similarly $\quad (a^2d - 3abc + 2b^3)\beta^3 \ldots + \ldots + (d^2a - 3dbc + 2c^3)\alpha^3,$

say I, belonging to the type [6: 3, 3; 1, 3], will have for a reciprocal

$$(a^2d - 3abc + 2b^3)\epsilon_3 + \ldots (d^2a - 3dbc + 2c^3)\epsilon_0,$$

say J, belonging to the type [6: 3, 3; 3, 1]. The graph of I will be that of Fig. 41 and the graph of J, that of Fig. 42, where I use B and G (the initials of boron and gold, instead of Au for the latter) and H (the initial of hydrogen) to represent the algebraical atoms (that is quantics) of valencies (that is degrees) 3, 3, and 1 respectively. Prefixing Σ to the I graph and substituting G_1, G_2, G_3, the three roots of G, for H, H', H'' and B_1, B_2, B_3 for B, B', B'' we obtain

$$\Sigma(B_1 - B_2)(B_1 - B_3)(B_2 - B_3)(B_1 - G_1)(B_2 - G_2)(B_3 - G_3),$$

which by inspection is the root representative of J, and prefixing Σ to the J graph and substituting H for G, we obtain in like manner

$$\Sigma(B_1 - B_2)^2(B_2 - B_3)(H - B_1)(H - B_3)^2,$$

as the root representative of I.

It may be observed that Fig. 43 is, algebraically speaking, a pseudograph of J, for its reading would give zero for the value of I.

It follows as an immediate consequence from the preceding extension of the law of images to quantic-systems, that the rule for deducing the first term of the reciprocal to a covariant from that of the covariant itself by writing η_r for α^r holds good as a rule for deducing each term of the one from the corresponding term of the other. To see this we have only to recall that every covariant to a quantic or quantic system may be regarded as an invariant of a new system containing the given quantic or system augmented by a linear quantic whose coefficients are y and $-x$.

Note A to Appendix 2.

Completion of the Theory of Principal Forms.

In the case of a derivative from a system of k parent quantics, it at first sight would seem that since reversion (the act of forming the second image, or process, as we may term it, of double reflexion) may be effected in regard to each system of coefficients separately, the method in the text ought to furnish in general k distinct systems of principal forms, but this is a mere mirage of the understanding which disappears as soon as the question is submitted to close examination. There is always an unique set of μ forms (μ being the multiplicity of the type) which revert unchanged (barring a numerical multiplier) whichever system of coefficients undergoes double reflexion. But a caution is necessary for the right interpretation of this statement. $U, V, W\ldots$ may be the principal forms in regard to one set of coefficients, $\lambda U + \mu V, W\ldots$, or $\lambda U + \mu V + \nu W\ldots$, where λ, μ, ν are indeterminate, in regard to some other. In any such case we may still say that $U, V, W\ldots$ is the principal system in regard to both sets and so in general. We have an example of this if we take any covariant to a single quantic Q and translate it into an invariant of Q and a linear form L. If $U, V, W\ldots$ are principal forms in respect to Q, $\lambda U + \mu V + \nu W + \ldots$ (that is the absolutely general form of the type) may be easily shown to undergo reversion in respect to L unaltered. $U, V, W\ldots$ may consequently still be seen to be a principal form system in respect to Q and L, as each of these quantities is unaltered by reversion in respect either to Q or to L.

Suppose now a diadelphic system of which U, V are the principal forms quâ one set of coefficients. Let R denote a reversion quâ this set, R' quâ some other set. Let $RU = aU$, $RV = bV$ and suppose $R'U = \alpha U + \beta V$. Then

$$R'RU = a\alpha U + b\beta V \text{ and } RR'U = a\alpha U + b\beta V.$$

But by the nature of the process of reversion $RR' = R'R$; hence $a\beta = b\beta$. If $a = b$, every linear combination of U, V is a principal form quâ R. Hence the principal form quâ the R' set, is such for both sets. But if a is not equal to b, we must have $\beta = 0$. Hence U will be a principal form quâ R' as well as R, and the same will be true of V. For if

$$R'V = \gamma U + \delta V$$
$$RR'V = a\gamma U + b\delta V$$
$$R'RV = R'bV = b\gamma U + b\delta V.$$

Therefore $a\gamma = b\gamma$ and $\gamma = 0$. Thus U, V will each of them be common as principal forms to each set. I have gone through the same somewhat tedious process of proof for triadelphic forms and with the same result. The very beautiful conclusion follows that whatever the multiplicity of a type may be and whatever number of sets of coefficients it involves, there is always *a single system of principal forms* common to all the sets*.

* Suppose there are k quantics in the parent system and that a derivative type μ is given. Each simple inversion of a pair of permutable indices (i, j) will give rise to a distinct principal equation; there will therefore be k such equations. Let ρ be a root of one of these, σ a root of any other. Then a principal form may be expressed as a linear function of any μ independent special forms connected by coefficients which are rational integer functions of ρ. Hence σ may be found as a rational function of ρ; but in like manner ρ may be found as a rational function of σ. Hence ρ, σ must be related by an equation of the form

$$A\rho\sigma + B\rho + C\sigma + D = 0,$$

Note B to Appendix 2.

Additional Illustrations of the Law of Reciprocity.

Acetic aldehyde contains two atoms of carbon, one of oxygen and four of hydrogen*. It thus corresponds to the quartic covariant of a quadratic and quartic, linear and quadratic in respect to the coefficients of the first and second respectively; such a form exists algebraically (*Higher Algebra*, third edition, p. 200) and may easily be proved to be monadelphic. Let us treat it as an invariant: if we were to take for its graph a triangle of which C, C, O were the apices and attach two atoms of hydrogen to each C, the permutation-sum of the product of the differences of the connected letters is zero; this then is a pseudograph. A true graph of it is given by the figure

$$\begin{array}{c} H \cdot C \cdot O \cdot H \\ : \\ H \cdot C \cdot H \end{array},$$

where each single dot between two letters means a single bond and the two dots between the upper and lower C's stand for a pair of bonds between them. This belongs to the invariantive type [4, 2; 2, 1; 1, 4: 0], the complete reciprocal to which is [2, 4; 1, 2; 4, 1: 0]. The constitution of the latter in terms of the roots is found from the above graph by writing O for C, C for H and H for O and is accordingly

$$\Sigma (O - O')^2 (O - C)(O - O')(O' - C'')(O' - H)(H - C),$$

where the factor $(O - O')^2$ may be put outside the sign of summation. We may therefore take for its graph a detached molecule of oxygen + a molecule of formic acid, which latter contains two of oxygen, one of carbon and two of hydrogen

$$\begin{array}{c} H \cdot C \cdot O \cdot H \\ : \\ O \end{array}$$

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and thus we see that all the k principal equations are homographically related, that is, that each may be obtained from any other by a substitution of the form

$$\rho = \frac{C\sigma + D}{A\sigma + B}.$$

In a word, the multiplicity μ (whatever the *diversity* k) determines the number of principal forms; and the k sets of principal multipliers are given by k algebraical equations of the μth degree, homographically transformable into one another.

* I originally took chloral as the subject of this investigation, being interested in examining its algebraical constitution in consequence of having had personal experience of its use as an escharotic. But for greater simplicity I have substituted acetic-aldehyde of which chloral is a third emanant, three hydrogen atoms of the former being replaced by three of chlorine in the latter.

will be a graph of it, from which, turning O into C, H into O and C into H we obtain

$$\Sigma (C-O')^2 (C''-H)(C'''-O')(C''''-H)(H-O)$$

as the value, in terms of its roots, of the algebraical equivalent to acetic aldehyde. The graph for formic acid, it may be noticed, exists algebraically (*Higher Algebra*, p. 300).

Instead of the dissociated molecules of oxygen and formic acid, we may exhibit them combined in the graph

$$\begin{array}{c} C \cdot O \cdot O \cdot O \cdot H \\ \vdots \\ O \end{array}$$

which will give another form to the value of the reciprocal in question, namely

$$\Sigma (C-H)^2 (H-O)(H-C')(C'-C'')(C''-C''')(C'''-O)$$

which, not being zero and the type being monadelphic*, must be in a pure numerical ratio to the sum above written.

Chemistry has the same quickening and suggestive influence upon the algebraist as a visit to the Royal Academy, or the old masters may be supposed to have on a Browning or a Tennyson. Indeed it seems to me that an exact homology exists between painting and poetry on the one hand and modern chemistry and modern algebra on the other. In poetry and algebra we have the pure idea elaborated and expressed through the vehicle of language, in painting and chemistry the idea enveloped in matter, depending in part on manual processes and the resources of art for its due manifestation.

A peculiar case might possibly arise in applying the theory of principal forms to a self-reciprocal type $[w: i, i]$ which it is proper to mention. For greater simplicity suppose the type to be diadelphic and let M, N be forms of the type which satisfy the equations

$$IM = \rho M, \qquad IN = \rho' N;$$

* As an exercise the reader may satisfy himself that this type is monadelphic by the direct application of the rule for finding the multiplicity. It corresponds to a quadratic covariant of the type [2, 4 ; 4, 1 : 2], which is the same (introducing the weight $\frac{2 \cdot 4 + 4 \cdot 1 - 2}{2}$ in lieu of the degree) as the type [5 : 2, 4 ; 4, 1] and has the same multiplicity μ by the law of reciprocity as the type [5 : 4, 2 ; 4, 1], namely, the difference between the number of modes of composing 5 and of composing 4 with two of the numbers 0, 1, 2, 3, 4 and with one of a *distinct* set of the same numbers. The arrangements for the weight 5 will be

4. 1: 0, 4. 0: 1, 3. 2: 0, 3. 1: 1, 3. 0: 2, 2. 2: 1, 2. 1: 2, 2. 0: 3, 1. 1: 3, 1. 0: 4,

and for the weight 4,

4. 0: 0, 3. 1: 0, 3. 0: 1, 2. 2: 0, 2. 1: 1, 2. 0: 2, 1. 1: 2, 1. 0: 3, 0. 0: 4.

The numbers of the combinations in the two sets of arrangements are respectively 10 and 9. Hence $\mu = 10 - 9 = 1$, or the type is monadelphic. The same result of course follows from the known fundamental scale for a quadro-biquadratic system.

the M and N have tacitly been defined to be the principal forms for such a type. Now in general this definition merges into and is coincident with the definition of principal forms for the general case, namely, that I^2M and I^2N must be multiples of M and N and the latter condition might be substituted for the former. But this is not always true, for if $\rho + \rho' = 0$, we shall have

$$I^2M = \rho^2 M, \qquad I^2N = \rho^2 N,$$

and consequently,

$$I^2(M + \lambda N) = \rho^2 (M + \lambda N),$$

so that if we were to follow the general definition the principal forms might become indeterminate, whereas by following the definition special to the self-reciprocal case they are determinate. Thus for example, suppose that P, Q, two particular forms of the type, satisfy the equations

$$IP = \rho Q, \qquad IQ = \sigma P;$$

the principal forms will then be

$$\sqrt{(\sigma)}\, P + \sqrt{(\rho)}\, Q \text{ and } \sqrt{(\sigma)}\, P - \sqrt{(\rho)}\, Q,$$

and the two principal multipliers become $\sqrt{(\rho\sigma)}$ and $-\sqrt{(\rho\sigma)}$, so that the principal forms according to the general definition would be indeterminate, but according to the definition proper to self-reciprocal forms strictly determinate.

Let us, as a final example of self-reciprocal type, consider the type [10: 5, 5] which is the same as [5, 5: 5] and corresponds to the covariant of the fifth order in the coefficients and of the fifth degree in the variables to a quintic. This is diadelphic, as may be found by consulting the table of irreducible forms for the quintic, which will show that it can arise only from the multiplication of the parent quantic itself by its quartinvariant or from that of the quadratic quadricovariant by the cubic cubo-covariant or from a linear combination of the two products. But without this, the same conclusion may be arrived at by direct calculation of the value of (10: 5, 5) − (9: 5, 5) and the multiplicity will be found to be 18 − 16, or 2 as premised. Let us take as our special forms,

$$P = (ae - 4bd + 3c^2)(ace + 2bcd - ad^2 - c^3 - b^2e),$$

$$Q = a(a^2f^2 - 10abef + 4acdf + 16ace^2 - 12ad^2e + 16b^2df + 9b^2e^2 - 12bc^2f$$
$$- 76bcde + 48bd^3 + 48c^3e - 32c^2d^2),$$

where $\frac{Q}{a}$ is the quartinvariant J given by Salmon, p. 207 (third edition), being in fact the discriminant of the quadricovariant whose root-differentiant is $ae - 4bd + 3c^2$. Call $\alpha, \beta, \gamma, \delta, \epsilon$ the five roots of the quintic and make $a = 1$. Q contains the term f^2 which is the image of $\alpha^5\beta^5$ which can only arise from combinations of the coefficients into which d, e, f none of them enter. But all the terms of Q contain d, e, or f, moreover P has no term containing f^2, therefore IQ does not contain Q but is simply a multiple of P. Again ce^2, which enters into P, is the image of combinations of the form

$\alpha^2\beta^4\gamma^4$, and the only term in Q which can give rise to such combinations is $-32c^2d^2$, or

$$-\frac{32}{10^4}(\Sigma\alpha\beta)^2(\Sigma\alpha\beta\gamma)^2,$$

and each such combination will have unity for its coefficient and their number is 30. Hence

$$IQ = -\frac{30.32}{10000}P = -\frac{12}{125}P.$$

Again, Q contains $-10bef$, and bef is the image of such root-combinations as $\alpha^5\beta^4\gamma$ (60 in number) the only terms in P capable of producing which are $10bc^3d$ and $-3c^5$ or $\frac{1}{5000}\Sigma\alpha(\Sigma\alpha\beta)^3\Sigma\alpha\beta\gamma - \frac{3}{100000}(\Sigma\alpha\beta)^5$. And bef does not appear in P, hence one part of IP will be

$$\left(\frac{60}{-50000} + \frac{3.5.60}{1000000}\right)Q, \quad \text{or } -\frac{3}{10000}Q.$$

Again, ce^2 is the image of such combinations as $\alpha^4\beta^4\gamma^2$ (30 in number) and the only terms in P giving rise to such are $-3c^5 - 8b^2cd^2 + 10bc^3d - 3c^2d^2$; $-3c^5$ is $-\frac{3}{100000}(\Sigma\alpha\beta)^5$ and will give rise to $-\frac{3.20.30}{100000}ce^2$ in IP; $-8b^2cd^2$ is $-\frac{8}{25000}(\Sigma\alpha)^2(\Sigma\alpha\beta)(\Sigma\alpha\beta\gamma)^2$ and will give rise to $-\frac{2.8.30}{25000}ce^2$ in IP; $10bc^3d$ is $\frac{10}{50000}\Sigma\alpha(\Sigma\alpha\beta)^3\Sigma\alpha\beta\gamma$ and will give rise to $\frac{7.10.30}{50000}ce^2$ in IP; $-3c^2d^2$ is $-\frac{3}{10000}(\Sigma\alpha\beta)^2(\Sigma\alpha\beta\gamma)^2$ and will give rise to $-\frac{3.30}{10000}ce^2$ in IP. Hence the total coefficient of ce^2 in IP is

$$-\frac{9}{500} - \frac{12}{625} + \frac{21}{500} - \frac{9}{1000} = \frac{-90-96+210-45}{5000} = -\frac{21}{5000},$$

and consequently, since P contains the term ce^2 and Q the term $16ce^2$, if $IP = \theta P - \frac{3}{10000}Q$,

$$\theta - \frac{3.16}{10000} = -\frac{21}{5000}, \text{ so that } \theta = \frac{3}{5000},$$

and therefore $$IP = \frac{3}{5000}P - \frac{3}{10000}Q,$$

and thus the equation for finding the principal multipliers ρ is

$$\begin{vmatrix} \frac{3}{5000} - \rho, & -\frac{3}{10000} \\ -\frac{12}{125}, & -\rho, \end{vmatrix} = 0,$$

or, if $$\rho = \frac{3\sigma}{10000}, \quad \begin{vmatrix} 2-\sigma, & -1 \\ -320, & -\sigma \end{vmatrix} = 0.$$

Thus $\sigma^2 - 2\sigma - 320 = 0$, the roots of which are irrational. I have thought it advisable to set out the work in this example with some explicitness in order to remove an impression that might otherwise arise from the examples which precede, that the principal multipliers and consequently the principal forms, for self-reciprocal types, necessarily contain only rational numbers.

The work is very much longer for the case of non-self-reciprocal types. The simplest example of such that presents itself to my mind is that of the sextinvariant of a quartic and the quartinvariant of a sextic, for either of which the type is diadelphic. The discussion of this case forms the subject of the annexed Note, for all the calculations of which I am indebted to the labour and skill of Mr F. Franklin, Fellow of Johns Hopkins University. For the sake of brevity the steps of the work have been suppressed and only the final results set out.

Note C to Appendix 2.

On the Principal Forms of the General Sextinvariant to a Quartic and Quartinvariant to a Sextic.

Let

$$L = (ae - 4bd + 3c^2)^3 = \left[\frac{1}{2^3 \cdot 3}\Sigma(\alpha-\beta)^2(\gamma-\delta)^2\right]^3,$$

$$M = \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}^2 = (ace + 2bcd - ad^2 - b^2e - c^3)^2 = \left[\frac{1}{2^4 \cdot 3^3}\Sigma(\alpha-\beta)^2(\gamma-\delta)^2(\alpha-\gamma)(\beta-\delta)\right]^2,$$

$$P = (ag - 6bf + 15ce - 10d^2)^2 = \left[-\frac{1}{2^4 \cdot 3 \cdot 5}\Sigma(\alpha-\beta)^2(\gamma-\delta)^2(\epsilon-\phi)^2\right]^2,$$

$$Q^* = \begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix} = \begin{cases} aceg - acf^2 - ad^2g + 2adef \\ -ae^3 - b^2eg + b^2f^2 + 2bcdg \\ -2bcef - 2bd^2f + 2bde^2 - c^3g \\ +2c^2df + c^2e^2 - 3cd^2e + d^4 \end{cases}$$

$$= \frac{1}{2^5 \cdot 3^3 \cdot 5^3}\Sigma(\alpha-\beta)^4(\gamma-\delta)^4(\epsilon-\phi)^4 - \frac{71}{2^{10} \cdot 3^4 \cdot 5^4}\left[\Sigma(\alpha-\beta)^2(\gamma-\delta)^2(\epsilon-\phi)^2\right]^2.$$

* M. Faà de Bruno, in the tables at the end of his *Théorie des Formes Binaires*, designates Q and $\Sigma(\alpha-\beta)^4(\gamma-\delta)^4(\epsilon-\phi)^4$ by the same symbol I_4; a misleading circumstance which gave rise in this instance, and might in others to a large amount of useless labour. As can easily be seen from the above, the true value of $\Sigma(\alpha-\beta)^4(\gamma-\delta)^4(\epsilon-\phi)^4$ is

$$120(71P + 900Q) = 120(71a^2g^2 - 852abfg + 3030aceg - 900b^2eg - 2320ad^2g + 1800bcdg - 900c^3g - 900acf^2 + 3456b^2f^2 + 1800adef - 14580bcef + 6720bd^2f + 1800c^2df - 900ae^3 + 1800bde^2 + 16875c^2e^2 - 24000cd^2e + 8000d^4).$$

It should also be observed that in the expression for Q (the catalecticant) given in the same table, the signs of the terms $-2bd^2f + 2bde^2$ have been interchanged.

Then $IL = \frac{P - 6Q}{2^5 \cdot 3^2}$, $IM = \frac{P - 33Q}{6^5}$,

$IP = \frac{L + 2M}{2^4 \cdot 5}$, $IQ = \frac{9L - 142M}{2^6 \cdot 3^2 \cdot 5^3}$,

$I^2L = \frac{7614L + 23868M}{2^{11} \cdot 3^6 \cdot 5^3}$, $I^2M = \frac{201L + 2162M}{2^{11} \cdot 3^6 \cdot 5^3}$.

In order that $\lambda L + \mu M$ shall be a principal form we must have

$$(7614 - 2^{11} \cdot 3^6 \cdot 5^3\rho)\lambda + 201\mu = 0,$$

$$23868\lambda + (2162 - 2^{11} \cdot 3^6 \cdot 5^3\rho)\mu = 0,$$

$$\begin{vmatrix} 7614 - 2^{11} \cdot 3^6 \cdot 5^3\rho, & 201 \\ 23868 \quad , & 2162 - 2^{11} \cdot 3^6 \cdot 5^3\rho \end{vmatrix} = 0,$$

or, putting $\sigma = 2^8 \cdot 3^6 \cdot 5^3\rho$,

$$\sigma^2 - 1222\sigma + 182250 = 0,$$

where it may perhaps be worth noticing that the last term is $2 \cdot 3^6 \cdot 5^3$ and the coefficient of the second term $2 \cdot 13 \cdot 47$. We obtain from this equation

$$\rho = \frac{611 \pm \sqrt{(191071)}}{2^8 \cdot 3^6 \cdot 5^3}*.$$

The principal forms in L and M will then be found to be

$$201L + \{-2726 + 8\sqrt{(191071)}\}M, \qquad 201L + \{-2726 - 8\sqrt{(191071)}\}M;$$

and those in P and Q

$$101P + \{-11436 + 24\sqrt{(191071)}\}Q, \qquad 101P + \{-11436 - 24\sqrt{(191071)}\}Q.$$

Or, if we please, the principal forms in the two cases may be taken as the factors of

$$201L^2 - 5452LM - 23868M^2 \text{ and } 101P^2 - 22872PQ + 205200Q^2$$

respectively†. The question, what reduced quadratic forms can appear in the theory of diadelphic types, may one day or another become the subject of *à priori* investigation and form a new connecting link between the Calculus of Invariants and the Theory of Numbers. The linear functions of L and M and of P and Q, corresponding to the reduced forms of the above expressions might perhaps be termed the principal *rational* forms of the two types respectively.

* The number under the radical sign is, I believe, a prime number, but I have not within reach the tables necessary for verifying this. Professor Newcomb, by an exceedingly ingenious combination of a table of squares with Crelle's table of multipliers (a real stroke of genius), was able to ascertain by an inspection (the work of a few minutes) that 191071, if not a prime number, must contain a factor not greater than a certain moderate sized integer (137 if my memory serves me right) which reduces the trials necessary to be made to a very small compass.

† These are reducible to

$(201, 68, -60800\between L', M)^2$, $(101, -23, -1089667\between P', Q)^2$, where $L' = L - 14M$, $P' = P - 113Q$.

It may be well to notice that if $I^2U = \rho U$, then $I^2 \cdot IU = I \cdot I^2U = \rho IU$, and consequently the principal forms for two reciprocal types are images respectively of one another, and the principal multipliers are the same for the two systems.

Note D to Appendix 2.

On the Probable Relation of the Skew Invariants of Binary Quintics and Sextics to one another and to the Skew Invariant of the same Weight of the Binary Nonic.

The law of reciprocity extended, as it has already been in these pages, to systems of quantics, admits of an additional important generalization.

We know that Regnault's law of substitution holds good for algebraical forms, and in fact when transferred to the algebraical sphere becomes identical with the method which I believe I was the first to employ (now familiar to algebraists through the use made of it by Professors Clebsch and Gordan) to which I gave the name of emanation (Faà de Bruno, p. 198).

The principle, stated in chemico-algebraical language, is that in algebraical compounds any number of atoms of a given valence may be replaced by the same number of *new* equi-valent atoms. [In algebra it is essential to lay a peculiar stress on the word *new*; for if the substituted atoms should be homonymous with the remaining atoms, there is a possibility of the transformed compound reducing to zero. As for instance in the algebraical compound $ab' - a'b$ (the representative, say, of potassic iodide), if the atom of potassium should be changed into another of iodine (or *vice versa*), the compound, viewed algebraically, would disappear.]

The law of reciprocity as I have previously given it, translated into chemico-algebraical language amounts to saying that the total number of atoms of one kind (say m n-valent of one kind) may be replaced by n m-valent atoms of another kind; but by applying the rule of substitution first and then that of reciprocity we may see that the condition of *totality* may be done away with and the proposition reduced to the simplified form that in any algebraical compound *m n-valent atoms may be replaced by n m-valent ones.* Whether this law has any application in the chemical sphere, I must leave to chemists to determine.

In addition to the well known fact that a quintic possesses an invariant of the 18th order, and a sextic one of the 15th order, having obtained a complete scheme of the irreducible invariants for the binary quantic of the 10th degree, I was put in possession of the new fact that this last form possesses an invariant of the 9th order and consequently that the nonic possesses an invariant of the 10th order*.

Now the weight of each of these skew invariants is the same number 45, and I was thus led to suspect that they coexisted in virtue of some secret connexion. What that connexion is I think that I am now (very unexpectedly) in a position to explain and to show (with a high degree of probability) how the values of these three invariants may be actually deduced and calculated from one another. This follows as a consequence of the combined laws of reciprocity and substitution otherwise called emanation. For suppose we have an invariant of a quantic of the mth degree, of the order np in the coefficients. By the principle of emanation we may transform this into an invariant to a system of n quantics, each of the degree m and of the order p in each set of coefficients, and by the generalized law of reciprocity this may be again transformed into an invariant to a system of n quantics, each of degree p and of the order m in each set of coefficients. If now finally these n quantics be all made identical with one another, then the transformed invariant, *provided it does not vanish*, becomes an invariant of the order mn to a single quantic of the degree p, and accordingly we may pass in certain

* I have calculated, with the kind assistance of Mr Halsted, the expression in its canonical form of the generating function to a binary quantic of the 10th degree. The coefficient of t^m in this fraction developed, represents the number of parameters in the general invariant of the mth order of the given decadic. Its denominator is

$$(1-t^2)(1-t^4)(1-t^6)^2(1-t^8)(1-t^9)(1-t^{10})(1-t^{14})$$

and its numerator is the rational integer function

$$1+2t^6+\ldots+2t^{42}+t^{48},$$

the successive coefficients being

1, 0, 0, 0, 0, 0, 2, 0, 4, 2, 7, 6, 15, 13, 16, 25, 22, 31, 34, 40, 41, 47, 46, 49, 48, 49, 46, 47, 41, 40, 34, 31, 22, 25, 16, 13, 15, 6, 7, 2, 4, 0, 2, 0, 0, 0, 0, 0, 1,

showing that the primary fundamental invariants are of the orders 2, 4, 6, 6, 8, 9, 10, 14, and that (by the law of "Tamisage" *anglice* siftage) the secondary (or as they might be better termed the auxiliary) ones are of the orders 6, 8, 9, 10, 11, 12, 13, 14, 15, 17 taken 2, 4, 2, 7, 6, 12, 13, 18, 21, 11 times respectively. Any other invariant of the decadic can be represented as a linear function of a limited number of combinations of the secondaries, having for its coefficients some combination of powers of the primaries.

Suppose that the same numerical order occurs among the primaries and secondaries, as for example 6, which occurs twice among the former and twice among the latter. This will indicate in the first place that, calling A and B the quadric and quartic invariants, the general sextic one will be of the form

$$\lambda A^3+\mu AB+\nu_1Q_1+\nu_2Q_2+\nu_3Q_3+\nu_4Q_4$$

and that any two independent special values of $\nu_1Q_1+\nu_2Q_2+\nu_3Q_3+\nu_4Q_4$ may be taken as primaries and any other independent two as secondaries, and so in general; I mention this to prevent the false suggestion, which might otherwise arise, that the secondaries and primaries are different in internal constitution. This remark receives a beautiful illustration in an algebraical theory (recently developed by me) of chemical isomerism, which gives rise to a generating function precisely similar in character to that applicable to in- and co-variants and is subject to a similar law of interpretation, graphs taking the place of algebraical forms, and atomicules and the numbers of grouped atoms, of degrees and orders.

cases from the type $[m, np: 0]$ to the type $[p, mn: 0]$. So in all probability we may pass from the type $[5, 18: 0]$ to the type $[6, 15: 0]$ and to the type $[9, 10: 0]$. As there is only one invariant of the type $[6, 15: 0]$, or of the type $[9, 10: 0]$, it follows that, if the passage from type to type is real and not nugatory, the three invariants of these second types may be deduced, any one from any other, by the explicit processes above described. There is nothing at all doubtful in the course of the transformation except what arises from the possibility that in the last step of it the effect of rendering identical the different sets of coefficients—that is of finding the counter-emanant, so to say, of the invariant containing n sets of variables—may be to render the whole expression null. This of course would happen if we attempted to pass from the type $[5, 18: 0]$ to the type $[3, 30: 0]$, or to the type $[2, 45: 0]$, which we know are void of forms. But there is no reason why we should expect this to happen when we pass from the given type to other types known to contain one or more forms. It would require no impracticable amount of labour to actually verify the fact of the transformation being effectual between the skew invariants of the sextic and quintic forms. The survival of a *single* known term in either of them, in the process of attempting to deduce it from the other, would be sufficient to establish the effectualness of the method, and that it will be found to be effectual, for reasons too long to dwell upon here, I scarcely entertain a doubt. The process to be employed may be summarily comprehended under the three rubrics of diversification, reciprocation and unification. The first is one of differentiation alone; the second involves the expansion of functions of the coefficients of an equation in terms of roots and the substitution of η_i for α^i; the third consists merely in replacing distinct sets of letters (η) by a single set. In practice the two latter processes would be of course combined into one. It will be instructive to consider some simple example of this method of transformation of types.

Let us take $(ac-b^2)^2$ regarded as belonging to the type $[2, 6: 0]$. I shall show how to pass from this to a form of the type $[3, 4: 0]$. Taking a third emanant of the given form, that is the result of the operation upon it of $\frac{1}{1\cdot2\cdot3}(a'\delta_a+b'\delta_b)^3$, we obtain

$$(ac'+a'c-2bb')^3+2(ac-b^2)(a'c'-b'^2)(ac'+a'c-2bb').$$

Let us call $\alpha, \beta, \alpha', \beta'$ the roots of the two forms $[1, b, c], [1, b', c']$ respectively; then the emanant last found (multiplied by 8) becomes

$$(2\alpha\beta+2\alpha'\beta'-\alpha\alpha'-\alpha\beta'-\beta\alpha'-\beta\beta')$$
$$\{(2\alpha\beta+2\alpha'\beta'-\alpha\alpha'-\alpha\beta'-\beta\alpha'-\beta\beta')^2+(\alpha-\beta)^2\cdot(\alpha'-\beta')^2\}.$$

After performing all the multiplications and introducing the zero powers of $\alpha, \alpha', \beta, \beta'$ in such terms as do not contain one or more of these letters, all that remains is to substitute

$$\begin{aligned} \alpha^0 = \alpha'^0 = \beta^0 = \beta'^0 &= a, \\ \alpha = \alpha' = \beta = \beta' &= -b, \\ \alpha^2 = \alpha'^2 = \beta^2 = \beta'^2 &= c, \\ \alpha^3 = \alpha'^3 = \beta^3 = \beta'^3 &= -d, \end{aligned}$$

the letters a, b, c, d for greater simplicity being used instead of $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3$, that is $\eta_0, -\eta_1, \eta_2, -\eta_3$. The result will not vanish. To show this consider the group of terms which change into a^2d^2. These are the binary combinations of $\alpha^3, \alpha'^3, \beta^3, \beta'^3$. $2\alpha\beta$ and $2\alpha'\beta'$ in the first factor give rise to $8\alpha^3\beta^3$, $8\alpha'^3\beta'^3$ and the remaining four terms to $-2\alpha^3\alpha'^3, -2\alpha^3\beta'^3, -2\beta^3\alpha'^3, -2\beta^3\beta'^3$ respectively. Hence the term a^2d^2 will survive with the multiplier $8+8-2-2-2-2$, that is, 8. So again the only terms introducing ac^3 will be the ternary combinations of $\alpha^2, \alpha'^2, \beta^2, \beta'^2$. $2\alpha\beta$ and $2\alpha'\beta'$ will be found to produce as many positive as negative terms of this kind, but $-\alpha\alpha'$ will produce $4\alpha^2\alpha'^2\beta^2 + 4\alpha^2\beta^2\beta'^2$, giving rise to $8ac^3$, and as the same will be true for $-\alpha\beta', -\beta\alpha', -\beta\beta'$, we see that $32ac^3$ will emerge in the result. Hence the given invariant becomes converted into

$$(a^2d^2 + 4ac^3 + \ldots),$$

that is the discriminant of the cubic whose type is [3, 4: 0] as was to be shown.

I think it is little doubtful that wherever there exist forms contained under each of two types, the product of whose rank and order is identical, we may pass from the one to the other by means of the combined processes of emanation and reciprocation, as in the foregoing example*. [The case is much the same as with transvection. That process may produce a null form, but any actually existent form may be produced by it and exhibited as a transvect.] To pass from Hermite's to Cayley's skew form, we must first by emanation change [5, 18: 0] into [5, 6; 5, 6; 5, 6: 0] and then this latter into [6, 15: 0]; by means of the process last exemplified.

* Call

$$(b^2-ac)^3 = A,\ a^2d^2+4ac^3+\ldots = B,\ a'\delta_a + b'\delta_b + c'\delta_c = E,\ a\delta_{a'} + b\delta_{b'} + c\delta_{c'} + d\delta_{d'} = H^{-1}.$$

Then it follows from the text that

$$B = \tfrac{1}{12} H^{-2} I E^3 A,$$

where it may be observed that E^3A is diadelphic, for it will be proved that $(6: 3, 2; 3, 2) = 16$, and $(5: 3, 2; 3, 2) = 14$, so that any form whatever coming under the same type as E^3A is a linear function of $(ac'+a'c-2bb')^3$ and $(ac'+a'c-2bb')(ac-b^2)(a'c'-b'^2)$, say L and M (whose difference, $L-M$, is $\frac{1}{4}E^3A$), and operated on by $H^{-2}I$ would produce a multiple of B (whose type is monadelphic) with the sole exception of $\lambda L - 2\mu M$, the result of operating upon which would be zero. Similarly we may see that in any given case the chances are infinitely in favour of the expectation that the process will *not* be nugatory by which it has been shown we may pass from one known type $[m, np: 0]$ to another known one $[p, nm: 0]$.

APPENDIX 3.

On Clebsch's Theory of the "Einfachstes System associirter Formen" (*vide Binären Formen*, p. 330) and its Generalization.

Let $(a, b, c, \ldots k, l \between x, y)^n$ be any binary quantic. Let the provector symbol $(l\delta_k + 2k\delta_h + 3h\delta_g + \ldots)$ be denoted by Ω, and the revector symbol $(a\delta_b + 2b\delta_c + 3c\delta_d + \ldots)$ by $\mho$. Let Q_{2i} represent the quadrinvariant of the above form when $n = 2i$. Now let Ω and $\mho$ be made to comprise the $2i+1$ letters $a, b, c, \ldots l, m$; then $a\Omega Q_{2i} - 2bQ_{2i}$* will be nullified by the operation of $\mho$ and will therefore be a cubinvariant for the case of $n = 2i+1$, which we may call Q_{2i+1}. Also let $Q_0 = a$; then $Q_0, Q_1, Q_2, \ldots Q_\mu$ will be differentiants to all binary quantics of degree equal to or greater than μ. The above I call basic differentiants. Their distinguishing characteristic is that the highest letter in each of them enters into it only in the first degree multiplied by a or by a^2 and by no other letter. Now let D be any given differentiant of degree μ and for the moment make $a = 1$. Then it is obvious that D may be expressed—by means of successive substitutions of its ultimate, its penultimate, its antepenultimate, etc. letters up to c inclusive, in terms of the corresponding basic differentiants and the anterior letters,—as a rational integer function of $Q_1, Q_2, \ldots Q_\mu, b$; or, restoring to a its general value, will be a rational integer function of $Q_0, Q_1, Q_2, \ldots Q_\mu, b$, say F, divided by a power of a. But I say that b will have disappeared in the process. For $\mho D = 0$; and $\mho Q_0 = 0$, $\mho Q_1 = 0 \ldots \mho Q_\mu = 0$. Hence, regarding each Q as a constant, $\left(a\dfrac{d}{db}\right)F = 0$, or F does not contain b.

Again, suppose we take a system of two quantics and let $Q_0, Q_1, \ldots Q_\mu$ be the basic differentiants of the one, $Q_0', Q_1', \ldots Q_\nu'$ of the other, and let D be any differentiant of the system. Then by the same method as before we shall find

$$D = \frac{F(Q_0, Q_2 \ldots Q_\mu : Q_0', Q_2' \ldots Q_\nu' : b, b')}{a^m \cdot a'^n}.$$

Also each Q will be nullified by $\mho$, and each Q' by $\mho'$, and therefore each Q and Q' as well as D will be nullified by the operator $\mho + \mho'$. Hence we shall have

$$\left(a\frac{d}{db} + a'\frac{d}{db'}\right)F = 0,$$

or

$$F = \phi(ab' - a'b),$$

* For by a well-known formula if D is a differentiant in x of the type $[w: i, j]$,

$$\mho\Omega D = (ij - 2w) D.$$

Consequently when Q_{2i} is regarded as a differentiant in x of the type $[2i: 2i+1, 2]$

$$\mho\Omega Q_{2i} = Q_{2i} \text{ also } \mho Q_{2i} = 0 \text{ and } \mho b = a.$$

Hence

$$\mho(a\Omega Q_{2i} - 2bQ_{2i}) = 0.$$

ϕ being a rational integral form of function. In like manner for a system of three quantics, regarding the several sets of its basic differentiants as constant, we shall have

$$F = \phi\,(ab' - a'b : ac' - a'c : bc' - b'c),$$

where ϕ is a rational integral form of function, or

$$F = \psi\,(ab' - a'b : ac' - a'c : a, a'),$$

and so in general. Hence, remembering that any relation between differentiants must continue to subsist between the covariants of which they are the roots, and now, understanding by base forms the complete covariants of which the basic coefficients are the roots, we may pass from differentiants to in- or co-variants and obtain the following theorems.

(1) For a single quantic of degree i, any in- or co-variant is expressible by a fraction whose numerator is a rational integer function of its i base forms and whose denominator is a power of the quantic. This is Clebsch's theorem.

(2) For a system of quantics, any in- or co-variant is expressible by a fraction whose numerator is a rational integer function of the separate base forms of its several quantics and of any complete system of $(\mu - 1)$ independent Jacobians of the quantics taken in pairs, and whose denominator is a product of powers of the quantics of the system.

Also it will be observed that these theorems will continue to subsist when the base forms have for their roots in lieu of the basic differentiants, as above defined, any ascending scale of differentiants in which the letters enter successively one at a time and each letter on its first appearance figures only in the first degree and combined exclusively with powers of a.

On the theory of basic forms may be grounded a method for obtaining, *in propriâ personâ*, the fundamental in- and co-variants to a quantic or system of quantics in regular succession, by a process which continues so long as there are many more to be elicited and comes to a self-manifesting end as soon as the last irreducible form has been obtained, like an air pump that refuses to act as soon as the exhaustion has become complete. In a word, the cataloguing of the irreducible in- or co-variants is transferred to the province of, and becomes a problem in, ordinary algebra.

I have previously observed that any expression which represents a differentiant in regard to a quantic of a given degree necessarily does the same for quantics of all higher degrees. And I may take this occasion to remark, or to repeat, that a differentiant may be irreducible in respect to the quantic of minimum degree to which it can be referred, and yet not so for quantics of higher degrees. Thus, if we take the expression

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd,$$

this referred to a cubic is irreducible (as is well known), but regarded as a differentiant of a quartic or higher degreed quantic, is reducible, being in fact identical with

$$(ac - b^2)(ae - 4bd + 3c^2) - a\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}.$$

Let us suppose a linear function $yu - xv$ combined with a quantic into a system. Then it follows as a corollary from (2) at [p. 200], that if the quantic belongs to the form $(a, b, c, \ldots l \between u, v)^i$, or say more simply to the form $[a, b, c, \ldots l]$ any covariant of such quantic multiplied by a suitable power of a will be a function of y, $ax + by$ and of the differentiants, or in a word, every covariant of the quantic expressed as a function of x and $ax + by$ will have no coefficients but what are differentiants, or to use Professor Cayley's term, semi-invariants. Thus, for example, the Hessian of the cubic $(a, b, c, d \between x, y)^3$ may be put under the form

$$\frac{1}{a^2}\left\{(ac - b^2)(ax + by)^2 + (a^2d - 3abc + 2b^3)(ax + by)\,y + (ac - b^2)^2 y^2\right\}.$$

So it will be found that the Hessian of the quintic, namely

$$(ae - 4bc + 3c^2)\,x^2 + (af - 3be + 2cd)\,xy + (bf - 4cd + 3d^2)\,y^2$$

on writing $ax + by = X$, becomes

$$\frac{1}{a^2}\Big\{(ae - 4bc + 3c^2)\,X^2 + (a^2f - 5abe + 2acd + 8b^2d - 6bc^2)\,Xy$$
$$- \Big[(ac - b^2)(ae - 4bd + 3c^2) + 3a\,(ace + 2bcd - ad^2 - b^2e - c^3)\Big]\,y^2\Big\},$$

where all the coefficients are semi-invariants-in-x, the second coefficient being one of the basic differentiants and the latter part of the third coefficient, the catalecticant

$$\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix},$$

and so more generally, it may be shown to follow from (2), that if there be any number of binary quantics

$$[a, b, c \ldots],\ [a', b', c', \ldots],\ [a'', b'', c'', \ldots],$$

every covariant of such system, expressed as a function of y and of *any one* of the quantics

$$ax + by,\ a'x + b'y,\ \ldots$$

chosen at will, has differentiants-in-x exclusively for its coefficients.

It is easy to express the base-covariants in terms of the roots. Those of weight $2n$ and order 2 will be of the form

$$\Sigma F\,(a_1, a_2, a_3, \ldots a_{2n})(x - a_{2n+1})^2 (x - a_{2n+2})^2 \ldots$$

where F may be expressed as

$$(a_1 - a_2)^2 (a_3 - a_4)^2 \dots (a_{2n-1} - a_{2n})^2,$$

or,

$$(a_1 - a_2)(a_2 - a_3)(a_3 - a_4) \dots (a_{2n-1} - a_{2n})(a_{2n} - a_1),$$

or under a variety of other forms all equal to a numerical factor près; for the type $[2n: 2n, 2]$ and the more general one $[2n: 2n + \nu, 2]$ are monadelphic. And again those of the weight $2n + 1$ and order 3 may take, or at all events be replaced by, the form

$$\Sigma \{(a_1 - a_2)(a_2 - a_3) \dots (a_{2n-1} - a_{2n})(a_{2n} - a_1)(a_1 - a_{2n+1})(x - a_1)(x - a_2) \dots (x - a_{2n+1})(x - a_{2n+2})^3 (x - a_{2n+3})^3 \dots\}.$$

It is proper to notice that the type $[2n + 1 : 2n + 1 + \nu;\ 3]$ is only monadelphic so long as $2n + 1$ is less than 9, so that we cannot, without an investigation which might be tedious, determine whether the above representation coincides with the basic forms of the third order in the coefficients adopted in [p. 199]; but such investigation would be a work of supererogation, for the only *material* character for any of the base-covariants in question to possess is, that its root differentiant-in-x shall be not higher than of the third order in the coefficients and shall contain the element ϵ_{2n+1}. Any formula having this property (which is enjoyed by the root function above given) is just as good as any other for the purposes of this theory*.

It will be seen to follow from the theorem I have given for differentiants from which Clebsch's follows as an immediate consequence, that all the permutation-sums of any rational integer function of the differences of the roots of an algebraical equation of the nth degree are rational integer functions of $(n-1)$ of them of the second and third order alternately; so, for example, all the coefficients in Lagrange's equations to the squares of the differences of the roots of an algebraical equation in its ordinary form are rational integer

* Writing the type under the form $[2n+1: 2n+1+\nu, 3]$, the degree of the corresponding covariant in the variables is $2n+1+3\nu$, which is the degree in x of the symmetrical function assumed in the text; also each letter in this function occurs 3 times agreeing with the order 3 of the type, and the number of factors in the coefficient of the highest power of x is $2n+1$, which is right for the weight. It is obvious also by inspection that the product $a_1 . a_2 \dots a_{2n+1}$ will arise from each term of the assumed symbolical function affected always with the same sign, so that ϵ_{2n+1} will occur (as required) in its expression in terms of the coefficients. Of course all the same conclusions will apply if in the formula

$$(a_1 - a_2)^2 (a_3 - a_4)^2 \dots (a_{2n-1} - a_{2n})^2$$

is substituted in lieu of

$$(a_1 - a_2)(a_2 - a_3) \dots (a_{2n-1} - a_{2n})(a_{2n} - a_1).$$

That the type to which Q_{2n+1} belongs is non-monadelphic from and after $2n+1=9$ is obvious from the fact that that type, when the degree of the parent quantic is made a minimum, is of the form $[2n+1: 2n+1, 3]$, the multiplicity of which is the same as that of $[2n+1: 3, 2n+1]$, or set out in full $[2n+1: 3, 2n+1: 2n+1]$; but cubics include covariants of orders and degrees 2 : 2 and 3 : 3 among their fundamental forms, and 9 : 9 can be formed either by taking a triplication of 3 : 3, or by combining 3 : 3 with a triplication of 2 : 2, so that when $2n+1=9$ the type is diadelphic, and *a fortiori*, it is non-monadelphic for values of $2n+1$ superior to 9.

functions of $(n-1)$ known quantities. Thus, for instance, the equation to the squares of the differences of a cubic equation will be

$$\rho^3 + 18(b^2 - ac)\rho^2 + 81(b^2 - ac)^2 + 27\Delta = 0,$$

where the coefficients are given in terms of two differentiants $(b^2 - ac)$ and Δ.

Throughout this paper the perspicuity of expression has been considerably marred by want of a complete nomenclature which the theory of graphs and types necessarily calls for and which I shall hereafter employ whenever I may have occasion to revert to the subject. It is as follows:

In the first place, w, the weight in respect to the selected variable, and j, the order in the coefficients, are terms well understood and need no change or further illustration; i, the degree of the parent quantic, I shall hereafter call the *rank* of the type, $ij - 2w$ which becomes the degree of a covariant got by expanding the differentiant of type $[w: i, j]$ may be called the *grade*. The order and rank may be termed collectively the *permutable indices*.

When a differentiant is given algebraically its weight and order are given but *not* its rank; in addition to the weight and order a third number which may be called the *range* (and which I shall denote by a Greek ϵ) is given, being the number less 1 of the letters which enter into it. The relation between *rank* and *range* is one of inequality. The former may be equal to, or greater than, but not less than the latter.

The multiplicity of the type to which a given differentiant belongs is a function of the *weight, order* and *rank* and is consequently not known until the *rank* is assigned. Thus, for example $(ac - b^2)^2$, considered as having the lowest possible rank, namely 2 (the *range*) is monadelphic; its type is then $[2: 2, 4]$, but if the rank 4 be assigned to it so that its type is $[2: 4, 4]$, it becomes diadelphic. We have then, in general, 6 characters (not all independent) appertaining to a differentiant, namely, *weight, rank, order, grade, range* and *multiplicity*. The theory of types has never hitherto formed the subject of distinct contemplation, and that is why the necessity for the use of some of the above terms has not been previously felt. But it will have been observed that throughout the preceding memoir it has forced itself upon our notice, and in particular, that it is impossible to go to the bottom of the so-called law of reciprocity or that of the radical representation of forms without keeping in view the question of type and multiplicity.

I have also to remark that since the preceding matter was completed I have been surprised to learn that recent chemical research favours the notion of simple elements (hydrogen atoms in special) being distinguishable from each other in chemical composition. If this view is confirmed, the discrepancy, which I have pointed to, between the known conditions for the existence of algebraical graphs and the unknown natural laws which govern the production of chemical substances may become partially or wholly obliterated, so that, for example, the hydrogen molecule and the extended derivatives from marsh gas may exist in accordance with, and not in contradiction to, algebraical law, and thus it is possible to conceive that all the phenomena of chemistry and algebra may ultimately be shown to be identical.

Since the above matter was sent to press I have been led to study algebraically what may be termed the direct problem of isomerism, that is to say the determination of the number of combinations subject to given conditions that can be formed between the constituents of groups each containing a given number of equivalent chemical atoms, the valences of the several groups being either independent or given linear functions of a certain number of independent parameters. In this problem the numbers of atoms are given and the valences left indeterminate. In the inverse problem the valences are given and the numbers left indeterminate.

The problem of the enumeration of the saturated hydro-carbons, investigated by Professor Cayley, is a simple example of the inverse problem. The direct problem admits of a uniform and unfailing method of solution by generating functions, the exposition of which may probably form the subject of an additional Appendix in the following number*. This method is

* The principle employed in this method leads to the following theorem only a particular case of which comes into play in the general partition problem which covers the ground occupied by the allied invariantive and isomeric theories. Let there be given a product of a limited number of rational functions of

$$u_1^{a_1}.u_2^{a_2}\ldots u_i^{a_i};\quad u_1^{a_1'}.u_2^{a_2'}\ldots u_i^{a_i'};\ \text{etc., etc.,}$$

where all the indices are *positive or negative* integers, and let $\mu_1, \mu_2, \ldots \mu_i$ be given linear functions of $\nu_1, \nu_2, \ldots \nu_j$ (j being not greater than i), then it is always possible to find a limited product of rational functions of

$$v_1^{\beta_1}.v_2^{\beta_2}\ldots v_j^{\beta_j};\quad v_1^{\beta_1'}.v_2^{\beta_2'}\ldots v_j^{\beta_j'};\ \text{etc., etc.,}$$

where the indices are exclusively *positive*, such that the coefficient of $v_1^{\nu_1}.v_2^{\nu_2}\ldots v_j^{\nu_j}$, in their product developed according to ascending powers of $v_1, v_2, \ldots v_j$, shall be the same as the coefficient of $u_1^{\mu_1}u_2^{\mu_2}\ldots u_i^{\mu_i}$ in the original product developed according to ascending powers of $u_1, u_2, \ldots u_i$. Previous to the discovery of this principle the problem of isomerism, now completely solved potentially for the direct case, must have remained unattackable by any existing methods, such for example as were known to Euler, the inventor of the application of the method of generating functions to the theory of partitions. It renders supererogatory a large part of the methods devised by myself for the treatment of the problem of compound partitions contained in the printed notes of my lectures on Partitions, delivered at King's College, London, in the year 1859†. As an example of the direct problem of isomerism, suppose that three atoms of the same valence j are to combine with ϵ atoms of hydrogen which do not combine *inter se*; then the number of combinations which can be so formed is the coefficient of a^jx^ϵ in the development of the generating function

$$\frac{1+ax+a^2x^2}{(1-a^2)(1-ax)^2(1-ax^3)}$$

if the three atoms are all unlike, and of the generating function

$$\frac{1}{(1-a^2)(1-ax)(1-a^2x^2)(1-ax^3)}$$

if they are all alike.

[† Volume II of this Reprint, p. 119.]

substantially the same as that which I have described* in general terms in the *Comptes Rendus* as applicable to the theory of ternary and other higher varieties of quantics but less difficult of application to the Isomeric Problem on account of the greater simplicity of the crude forms subject to reduction, which appear in it. Appendix 4 will contain the application of the theory of "Associirter Formen" to the algebraical deduction of the irreducible forms of the quintic and certain other cases which but for the press of matter awaiting publication in the *Journal* would have formed part (as announced) of the present Appendix.

As already stated in a previous footnote, the theory of irreducible forms reappears in the isomeric investigation, the general character of the reduced generating function to be interpreted in it being precisely the same as in the invariantive theory, which constitutes an additional and a closer and more real bond of connexion between the chemical and algebraical theories than any which I had in view when I commenced the subject of this memoir.

Note on the Ladenburg Carbon-Graph.

The reasoning by which I have† established, in the preceding number of the *Journal*, the validity of the Ladenburg graph (and the invalidity of the Kekulean one) as a representative of the root differentiant to a covariant of the 6th degree in the variables and of the 6th order in the coefficients to a quartic, is so peculiar and it may seem to some of my readers so far-fetched, that it appears highly desirable to confirm it by a direct demonstration founded on the principle, that the permutation-sum of the product of the bonds in a valid graph interpreted as differences between the letters which they connect, shall not vanish. Previous to applying this principle to Ladenburg's graph we must convert it into an invariant by attaching hydrogen atoms to the six apices. Let these apices be called a, b, c, d, e, f, and the hydrogen atoms $\alpha, \beta, \gamma, \delta, \epsilon, \phi$: then the permutation-sum under consideration is

$$\Sigma(a-b)(a-c)(b-c)(d-e)(d-f)(e-f)(a-d)(b-e)(c-f)(a-\alpha)(b-\beta)(c-\gamma)$$
$$(d-\delta)(e-\epsilon)(f-\phi)$$

where the 6 letters a, b, c, d, e, f are interpermutable, as are also the 6 letters $\alpha, \beta, \gamma, \delta, \epsilon, \phi$.

It may be well to observe at this point that if we struck off the hydrogen atoms and treated the graph as representing an invariant to a cubic form, the permutation-sum

$$\Sigma(a-b)(a-c)(b-c)(d-e)(d-f)(e-f)(a-d)(d-c)(c-f)$$

would be found to vanish, as may easily be shown and as it ought to do, because there exists no invariant of the 6th order in the coefficients to a cubic form. Let a and d be interchanged in the term given under the sign of summation in the permutation-sum formed from the Ladenburg graph; then the sum of this together with the original term becomes

$$(a-d)(b-e)(c-f)(b-c)(e-f)(b-\beta)(c-\gamma)(e-\epsilon)(f-\phi)$$

[* p. 100 above.] [† p. 155 above.]

multiplied by

$$(a\delta - da)\{a^2-(b+c)a+bc\}\{d^2-(e+f)d+ef\}-(d\delta-aa)\{d^2-(b+c)d+bc\}\{a^2-(e+f)a+ef\},$$

which last named multiplier will be found to contain the quantity $(a^3d^2-a^2d^3)(a+\delta)$. Again, in the multiplicand, let b and c be interchanged; then, since

$$(b-e)(c-f)-(c-e)(b-f)=(b-c)(e-f),$$

the sum of the original and permuted multiplicand will contain a term

$$(a-d)(b-c)^2(e-f)^2 bc(e-\epsilon)(f-\phi),$$

and accordingly the entire permutation-sum will contain the terms

$$(a+\delta)(a-d)(a^3d^2-a^2d^3)(b-c)^2(e-f)^2 bc\Sigma(e-\epsilon)(f-\phi).$$

The partial sum last written is

$$4ef+4\epsilon\phi-2(e+f)(\epsilon+\phi).$$

Hence we may readily see that the total permutation-sum will contain *inter alia* a positive multiple of the combination $a^4b^3e^3d^2cfa$ and will not vanish, and consequently the graph is valid and not illusory; I presume that the same method applied to Kekulé's graph regarded as a representation of the covariant to the type [9 : 4, 6 : 6], which is the same thing (except that the hydrogen atoms are suppressed) as the graph to the invariant [15 : 4, 6; 1, 6 : 0], would serve to show it to be illusory as previously inferred from other considerations.

25.

NOTE ON THE THEOREM CONTAINED IN PROFESSOR LIPSCHITZ'S PAPER.

[*American Journal of Mathematics*, I. (1878), pp. 341—343.]

I THINK it may be useful to state the principle to which the theorem demonstrated in the preceding paper leads, in the shape in which it has always presented itself to my mind, but which I found difficult to express when writing under the constraint of a foreign language.

It amounts simply to the statement that in a *prepared* form just as the variables $(x, y, z, \ldots)$ are contragredient to their symbolic inverses $\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \ldots\right)$ so the coefficients $(a, b, c, \ldots)$ are contragredient to theirs $\left(\frac{d}{da}, \frac{d}{db}, \frac{d}{dc}, \ldots\right)$; the latter statement in fact includes the former inasmuch as the so-called variables may be regarded as the coefficients of an auxiliary linear form.

In applying this principle it is expedient to enlarge our conception of invariants, covariants, etc., and to predicate invariance of functions not only of quantities ordinarily so termed, but of their symbolic inverses, or of functions in which quantities and operators enter conjointly*. To draw the

* It was through this idea that I was originally led to an intuitive perception of the theorems concerning the prepared form. For suppose $(a, b, c, \ldots l)(x, y, z)^n$ to be any prepared form; then if x', y', z'; x'', y'', z'' are cogredient with x, y, z, and we operate upon the given form with

$$(\dot{a}, \dot{b}, \dot{c}, \ldots \dot{l})(y'z''-y''z',\ z'x''-z''x',\ x'y''-x''y')^n,$$

the result is the nth power of the determinant $\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}$ which is a covariant. Hence we may conclude that the operator is a covariant; just as, if a covariant multiplied by any form is a covariant, we may conclude that the multiplier must be so too. Consequently

$$(\dot{a}, \dot{b}, \dot{c}, \ldots \dot{l})(x, y, z)^n$$

is a contravariant, and the same reasoning will apply whatever may be the number of variables. I originally used two forms, one the ordinary form for $(a, b, c, \ldots l)(x, y, z, \ldots)$ and the other the conclusions which flow from this conception, we have only to add the rule that all *combinations* of invariants, or of covariants, or of contravariants, etc., are themselves invariants, or covariants, or contravariants, etc., respectively. We may then state that the effect of substituting in a covariant or contravariant (to a prepared form) in place of the variables, or in place of the coefficients, the symbolic inverses of the one or the other, is to reverse their character and convert covariants into contravariants and *vice versa*, leaving of course the character of invariants unaltered; and I may remark incidentally that we are thus provided with a means of making any two *invariants* operate on each other so as to produce a third, a mode of operation which was not possible previous to the introduction of the prepared form*.

Moreover, the word combination must be taken in its widest sense, as there is more than one *mode* of combination possible. For example, if F, G, H are covariantive and Φ a contravariantive function of $(a, b, c, \ldots; x, y, z, \ldots)$, where $a, b, c, \ldots$ are the coefficients and $x, y, z, \ldots$ the variables of a prepared form, we have of course $\Phi\left(\frac{d}{da}, \frac{d}{db}, \ldots; x, y, \ldots\right)F$ and

$$\Phi\left(a, b, \ldots; \frac{d}{dx}, \frac{d}{dy}, \ldots\right)F \text{ and } G\left(\frac{d}{da}, \frac{d}{db}, \ldots; \frac{d}{dx}, \frac{d}{dy}, \ldots\right)F$$

all of them covariants. This may be termed an external mode of combination, but we shall equally have covariants derived by an internal mode of combination, for example

$$\Phi\left(\frac{dF}{da}, \frac{dF}{db}, \ldots; x, y, \ldots\right), \quad \Phi\left(a, b, \ldots; \frac{dF}{dx}, \frac{dF}{dy}, \ldots\right),$$

$$H\left(\frac{dF}{da}, \frac{dF}{db}, \ldots; \frac{dG}{dx}, \frac{dG}{dy}, \ldots\right)$$

will also be covariants.

ordinary form *divested* of its numerical coefficients for $(\dot{a}, \ldots \between y'z'' - y''z', \ldots)^n$, and of course with the same result.

The reasoning is perhaps not absolutely rigorous, but sufficiently so to bring conviction of the fact to be established. Of course when we have proved that $(\dot{a}, \dot{b}, \dot{c}, \ldots \dot{l} \between x, y, z)^n$ is a contravariant, it follows more generally that if $(A, B, C, \ldots \between x, y, z, \ldots)^N$ is a covariant

$$(\dot{A}, \dot{B}, \dot{C}, \ldots \between x, y, z, \ldots)^N,$$

where $\dot{A}, \dot{B}, \dot{C}, \ldots$ are the same functions of $\dot{a}, \dot{b}, \dot{c}, \ldots$ as $A, B, C, \ldots$ are of $a, b, c, \ldots$, will be a contravariant and *vice versa*.

* The case may be stated thus: previous to the introduction of the *prepared form*, invariants of systems could be made to operate upon invariants solely through the instrumentality of the coefficients of the linear forms of the system; since its introduction the same operation may be made to take effect through the instrumentality of the coefficients of all the forms, linear or non-linear, indiscriminately. The first named mode of operation is equivalent to the hyperdeterminantive method, which includes that of *Ueberschiebung*; the latter transcends the sphere of hyperdeterminants.

So again it may be observed that these modes of combination admit of being applied in more than one way: thus, to confine ourselves for a moment to the case of two forms, their external operation on each other may be simple, or concurrent, or reciprocal: simple when in *one* of them one set of quantities are converted into operators, concurrent when both sets are so converted, but reciprocal when in one of the two forms the variables and in the other the coefficients undergo such conversion. As an example, suppose we take the prepared form $ax^3 + \ldots + dy^3$, and its skew covariant

$$(a^2d + \ldots)\,x^3 + \ldots - (ad^2 + \ldots)\,y^3.$$

We may combine the contravariant

$$\left(\frac{d}{da}x^3 + \ldots + \frac{d}{dd}y^3\right)^2$$

with the contravariant

$$\left(a^2d + \ldots\right)\left(\frac{d}{dx}\right)^3 + \ldots - \left(ad^2 + \ldots\right)\left(\frac{d}{dy}\right)^3,$$

and the result will be a numerical multiple of the contravariant $dx^3 + \ldots - ay^3$. If, in the above instance, we denote the square of the primitive and its skew covariant according to their degree and order by $2\cdot 6$, $3\cdot 3$ respectively, we may explain their mutual action stenographically by saying that $\dot{2}\cdot 6$ and $3\cdot\dot{3}$ have acted reciprocally on each other, the *dot* signifying that the quantities typified by the number so marked have been replaced by their symbolic inverses; we cannot well represent this mutual action by writing $\dot{2}\cdot 6 * 3\cdot\dot{3}$ or $3\cdot\dot{3} * \dot{2}\cdot 6$, but may employ for the purpose $\overset{\dot{2}\cdot 6}{\underset{3\cdot\dot{3}}{*}}$. So from the square of a quintic $2\cdot 10$ and its linear covariant $5\cdot 1$ we may derive by reciprocal action $\overset{\dot{2}\cdot 10}{\underset{5\cdot\dot{1}}{*}}$, or the contravariant $3\cdot 9$: or, again, we may take any even number of covariants and cause them to operate in various manners, the variables on the variables and the coefficients on the coefficients, so as to form a closed circuit, as, for example, with four, we may make the coefficients of the first operate on those of the second, the variables of the second on those of the third, the coefficients of the third on those of the fourth, and the variables of the fourth on those of the first. Thus we have passed from reciprocal to the more general notion of simultaneous or circulatory action between any even number of covariants. And it is not unlikely that further applications may be made of this fertile conception: when dealing with a principle (an intellectual force) as distinguished from a theorem (a mere law), we never can feel sure that its uses are exhausted, or its plastic power spent.

26.

A SYNOPTICAL TABLE OF THE IRREDUCIBLE INVARIANTS AND COVARIANTS TO A BINARY QUINTIC, WITH A SCHOLIUM ON A THEOREM IN CONDITIONAL HYPERDETERMINANTS.

[*American Journal of Mathematics*, I. (1878), pp. 370—378.]

It is well known that every binary quintic can be expressed, and in only one way, as the sum of three fifth powers of linear functions of its variables, or which is the same thing, as the sum of the fifth powers of three variables connected by a linear equation, or finally, under the form

$$ax^5 + by^5 + cz^5,$$

subject to the equation

$$x + y + z = 0.$$

If ϕ, ψ be any two covariants of a binary quintic in x, y, the most general expression of the covariant produced by their operation on each other through the variables is

$$\left(\dot{x}\frac{d}{dy} - \dot{y}\frac{d}{dx}\right)^i \phi\psi,$$

where i is any positive integer and $\dot{x}$, $\dot{y}$ $\left(\text{abbreviations for } \frac{\delta}{\delta x}, \frac{\delta}{\delta y}\right)$ operate on ϕ only whilst $\frac{d}{dx}, \frac{d}{dy}$ operate on ψ.

Suppose now that ϕ, ψ are expressed as functions, say Φ, Ψ, of x, y, z, between which there exists the linear relation $lx + my + nz = 0$; it may be shown that the preceding expression becomes identical with

$$\begin{vmatrix} l, & m, & n \\ \dot{x}, & \dot{y}, & \dot{z} \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz} \end{vmatrix}^i \Phi\Psi,$$

where x, y, z are to be treated as independent variables. In the present case, therefore, writing

$$(\dot{y} - \dot{z})\frac{d}{dx} + (\dot{z} - \dot{x})\frac{d}{dy} + (\dot{x} - \dot{y})\frac{d}{dz} = \Lambda,$$

$\Lambda^i\Phi\Psi$, or (which will be more convenient for writing) $\Psi\Lambda^i\Phi$ will represent the covariant derived from the alliance of Φ and Ψ.

The twenty-three irreducibles of the quintic may be arranged in the following partially symmetrical order, which is that which I shall adopt as the order of their successive deduction: the first figure denotes the degree in the coefficients, the second the order in the variables*.

			2·2		4·0					
	1·5		3·3		5·1					
	2·6	3·5	4·4	5·3	6·2	7·1	8·0			
		4·6		6·4		8·2				
3·9		5·7		7·5		9·3		11·1	12·0	
									13·1	
										18·0

* Comparing this arrangement to the distribution of stars in a firmament, it will be observed that there is a tendency to concentration, or the formation of a sort of milky-way, in the zone situated towards the centre, consisting of three bands which comprise between them 15 out of 23, the total number of forms. This phenomenon becomes very much more distinctly marked in the distribution of the 124 irreducible forms appertaining to the septic, the *corrected* table of which I anticipate will have appeared, about simultaneously with the publication of this, in the *Comptes Rendus*†. The table previously given in that journal for the seventhic is affected with some inaccuracies chiefly arising from arithmetical errors of calculation, as I made the computation hurriedly and on the point of leaving England for this continent, and also, in part, from the existence of some errors in the table of the reduced generating function, which I accepted, without sufficient examination, as the basis of my work. It may perhaps be worthy of notice that, if we add a unit to the ordinarily received number of irreducible forms in each case (which it is proper to do, since an absolute number is an invariant of the order zero), the numbers of the irreducibles for the 1st, 3rd, 5th and 7th orders become 2, 5, 24, 125 respectively. As I am about to compute the irreducibles for the 9th order, we shall soon be in a position to ascertain whether the law indicated in this progression has any foundation in nature: if so, the number for that case should be 626, or thereabouts, but it is not unlikely that the fact of 9 being a composite number may have a tendency to affect the result, probably in the direction of decrease. For binary quantics of the even orders 0, 2, 4, 6, 8 the number of irreducible covariants is 1, 3, 6, 27, 70 respectively (for the last see *Comptes Rendus*‡, June 24, 1878), which appear to indicate a geometrical progression with the common ratio 3, subject to diminution for higher powers of 2 entering into the order of the quantic.

[† p. 146 above.] [‡ p. 114 above.]

There will be two sources of indeterminateness in the expressions obtained for these forms, one universal, arising from the arbitrary addition

$$(x + y + z)\, M,$$

the other special to those forms (such as 13·1) which can be obtained by the multiplication of lower forms (as 8·0, 5·1). Our object must be to seek in all cases the simplest expressions that can be obtained.

$2{\cdot}2 = 1{\cdot}5\Lambda^4 1{\cdot}5 \equiv \Sigma\,(abxy + acxz) \equiv \Sigma abxy.$

I use the sign of equivalence to signify that numerical common multipliers are to be rejected.

$$4{\cdot}0 = 2{\cdot}2\Lambda^2 2{\cdot}2$$
$$= \Sigma\,(\dot y - \dot z)(\dot z - \dot x)(abxy + acxz + bcyz)\frac{d}{dx}\cdot\frac{d}{dy}(abxy + acxz + bcyz)$$
$$= \Sigma\,(-ab + ac + bc)\,ab \equiv a^2b^2 + b^2c^2 + c^2a^2 - 2abc\,(a + b + c)$$
$$1{\cdot}5 = ax^5 + by^5 + cz^5$$
$$3{\cdot}3 = 2{\cdot}2\Lambda^2 1{\cdot}5 \equiv \Sigma ax^3 (\dot y - \dot z)^2 (abxy + bcyz + cazx) \equiv abc\Sigma x^3.$$

Since $x^3 + y^3 + z^3 = 3xyz + (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$ we have

(*bis*) $3{\cdot}3 = abcxyz$

$$5{\cdot}1 = 3{\cdot}3\Lambda^2 2{\cdot}2 \equiv abc\Sigma x\,(\dot y - \dot z)^2 (abxy + bcyz + cazx) \equiv abc\Sigma bcx$$
$$2{\cdot}6 = 1{\cdot}5\Lambda^2 1{\cdot}5 \equiv \Sigma ax^3 (\dot y - \dot z)^2 (ax^5 + by^5 + cz^5) \equiv \Sigma ax^3 (by^3 + cz^3) \equiv \Sigma abx^3y^3$$
$$3{\cdot}5 = 2{\cdot}2\Lambda\ 1{\cdot}5 = \Sigma\,(aby + acz)(\dot y - \dot z)(ax^5 + by^5 + cz^5)$$
$$\equiv \Sigma\,(aby + acz)(by^4 - cz^4) = \Sigma a\,(b^2y^5 - c^2z^5) + abc\Sigma\,(zy^4 - yz^4)$$
$$4{\cdot}4 = 3{\cdot}3\Lambda^2 1{\cdot}5 \equiv abc\Sigma x\,(\dot y - \dot z)^2 (ax^5 + by^5 + cz^5) \equiv abc\Sigma\,(bxy^3 + cxz^3)$$
$$= abc\Sigma\,[(ax^3 + by^3 + cz^3)\,x - ax^4] \equiv abc\Sigma ax^4$$
$$5{\cdot}3 = 2{\cdot}2\Lambda\ 3{\cdot}3 = \Sigma\,(aby + acz)(\dot y - \dot z)\,abc\,(x^3 + y^3 + z^3)$$
$$\equiv abc\Sigma\,(y^2 - z^2)(aby + acz)^* \equiv abc\Sigma ax\,(by^2 - cz^2)$$
$$6{\cdot}2 = 3{\cdot}3\Lambda^2 3{\cdot}3 \equiv a^2b^2c^2\Sigma x\,(\dot y - \dot z)^2 (x^3 + y^3 + z^3) \equiv a^2b^2c^2\Sigma\,(xy + xz)$$
$$\equiv a^2b^2c^2 (xy + yz + zx) \equiv a^2b^2c^2 (x^2 + y^2 + z^2)$$
$$7{\cdot}1 = 4{\cdot}4\Lambda^4 3{\cdot}5 \equiv abc\Sigma\,a\,(\dot y - \dot z)^4\Sigma\,\{(ab^2y^5 - ac^2z^5) + abc\,(zy^4 - yz^4)\}$$
$$\equiv a^2b^2c^2\Sigma\,a\,(\dot y - \dot z)^4\Sigma\,(zy^4 - yz^4) = a^2b^2c^2\Sigma a\,(y - z)$$
$$8{\cdot}0 = 4{\cdot}4\Lambda^4 4{\cdot}4 \equiv a^2b^2c^2\Sigma a\,(\dot y - \dot z)^4 (ax^4 + by^4 + cz^4) \equiv a^2b^2c^2 (ab + ac + bc)$$
$$4{\cdot}6 = 3{\cdot}3\Lambda\ 1{\cdot}5 \equiv abc\Sigma x^2 (\dot y - \dot z)(ax^5 + by^5 + cz^5) \equiv abc\Sigma a\,(y^2 - z^2)\,x^4$$
$$6{\cdot}4 = 2{\cdot}2\Lambda\ 4{\cdot}4 = \Sigma\,(aby + acz)(\dot y - \dot z)\,abc\,(ax^4 + by^4 + cz^4)$$
$$= abc\,(aby + acz)(by^3 - cz^3)$$
$$= abc\Sigma\,(ab^2y^4 + ac^2z^4) + a^2b^2c^2\Sigma\,(zy^3 - yz^3),$$

which, since $\Sigma\,(zy^3 - yz^3)$ contains $x + y + z$,

$$= abc\Sigma\,(c - b)\,a^2x^4$$
$$8{\cdot}2 = 4{\cdot}4\Lambda^4 4{\cdot}6 = a^2b^2c^2\Sigma a\,(\dot y - \dot z)^4\,[\Sigma a\,(y^2 - z^2)\,x^4] = a^2b^2c^2\Sigma ab\,(x^2 - y^2)$$
$$3{\cdot}9 = 2{\cdot}6\Lambda\ 1{\cdot}5 \equiv \Sigma\,(abx^2y^3 + acx^2z^3)(\dot y - \dot z)(ax^5 + by^5 + cz^5)$$
$$\equiv \Sigma\,(abx^2y^3 + acx^2z^3)(by^4 - cz^4)$$
$$= \Sigma ax^2 (b^2y^7 - c^2z^7) + abcx^2y^2z^2\Sigma\,(zy^2 - yz^2)\dagger$$

* For $y^2 - z^2$ I substitute $xz - xy$.

† Possibly this expression may be simplifiable by the addition of a suitable multiple of $x + y + z$.

$$5{\cdot}7 = 4{\cdot}4\Lambda\ 1{\cdot}5 \equiv abc\Sigma ax^3 (\dot y - \dot z)(ax^5 + by^5 + cz^5) \equiv abc\Sigma ab\,(x - y)\,x^3y^3$$
$$7{\cdot}5 = 4{\cdot}4\Lambda\ 3{\cdot}3 \equiv a^2b^2c^2\Sigma ax^3 (\dot y - \dot z)(x^3 + y^3 + z^3) \equiv a^2b^2c^2\Sigma ax^2 (y^3 - z^3)$$
$$11{\cdot}1 = 5{\cdot}1\Lambda\ 6{\cdot}2 \equiv a^3b^3c^3\Sigma bc\,(\dot y - \dot z)(x^2 + y^2 + z^2) \equiv a^3b^3c^3\Sigma bc\,(y - z)$$
$$9{\cdot}3 = 6{\cdot}2\Lambda\ 3{\cdot}3 \equiv a^3b^3c^3\Sigma x\,(\dot y - \dot z)(x^3 + y^3 + z^3) \equiv a^3b^3c^3 (x - y)(y - z)(z - x)$$
$$12{\cdot}0 = 6{\cdot}2\Lambda^2 6{\cdot}2 \equiv a^4b^4c^4\Sigma\,(\dot y - \dot z)^2 (x^2 + y^2 + z^2) \equiv a^4b^4c^4$$
$$13{\cdot}1 = 7{\cdot}1\Lambda\ 6{\cdot}2 = a^4b^4c^4\Sigma\,(b - c)(\dot y - \dot z)(x^2 + y^2 + z^2) \equiv a^4b^4c^4\Sigma\,(b - c)(y - z)$$
$$= a^4b^4c^4\,\{\Sigma\,(by + cz) - \Sigma\,(bz + cy)\}$$
$$= a^4b^4c^4\,\{2\,(ax + by + cz) + (bx + cz + ay)\} \equiv a^4b^4c^4\Sigma ax$$
$$18{\cdot}0 = 13{\cdot}1\Lambda\,5{\cdot}1 = a^4b^4c^4\Sigma a\,(\dot y - \dot z)(bcx + cay + abz) = a^4b^4c^4\Sigma a\,(c - b)$$
$$= a^5b^5c^5 (a - b)(b - c)(c - a).$$

18·0 may also be obtained by the operation of 11·1 on 7·1, or instantaneously as the resultant of 1·5, $abcxyz$ and $x + y + z$. In the following table the preceding results are collected; for greater brevity instead of the sign of summation I employ the sign + or − to signify respectively the symmetrical or semi-symmetrical completion of the terms to which it is affixed; m is used to signify abc.

1—2	$abxy + : (a^2b^2 - 2abc^2) +$
3—5	$ax^5 + : mx^3 +$, or $mxyz : mbcx +$
6—12	$\begin{cases} abx^3y^3 + : a^2bx^5 + myz^4 - : max^4 + : \\ mabxy^2 - : m^2x^2 + : m^2bx - : m^2ab + \end{cases}$
13—15	$max^4y^3 - : ma^2cx^4 - : m^2abx^2 -$
16—21	$\begin{cases} ab^2x^2y^7 + mx^2y^4z^3 - : mabx^4y^3 - : m^2ax^2y^3 - : \\ m^3bcy - : m^3x^2y - : m^4 \end{cases}$
22	$m^4ax +$
23	$m^5a^2b -$

I propose, at some future time, to apply a similar method to obtain an explicit representation of the irreducible forms appertinent to the binary seventhic, an arduous undertaking, but one that seems likely to lead to the apperception of new forms of complex symmetry. The primitive may, for that case, be represented by $x^7 + y^7 + z^7 + t^7$, connected by the linear equations $(l, m, n, p \between x, y, z, t) = 0$, $(\lambda, \mu, \nu, \pi \between x, y, z, t) = 0$, and Λ, the symbol of alliance, will be represented by

$$\begin{vmatrix} \dfrac{d}{dx}, & \dfrac{d}{dy}, & \dfrac{d}{dz}, & \dfrac{d}{dt} \\ \dot x, & \dot y, & \dot z, & \dot t \\ l, & m, & n, & p \\ \lambda, & \mu, & \nu, & \pi \end{vmatrix}.$$

Every in- and co-variant will then be a rational integer function of x, y, z, t and the six minor determinants, which are the parameters of the line represented by the above two linear equations.

It may be worth while to notice the representations of the irreducible derivatives of the quartic when put under the indeterminate form

$$ax^4 + by^4 + cz^4,$$

subject to the relation $x + y + z = 0$. We get

$$\begin{aligned}
2{\cdot}0 &= 1{\cdot}4\Lambda^4 1{\cdot}4 = \Sigma a\,(\dot{y} - \dot{z})^4 (ax^4 + by^4 + cz^4) \equiv ab + bc + ca \\
2{\cdot}4 &= 1{\cdot}4\Lambda^2 1{\cdot}4 = \Sigma ax^2 (\dot{y} - \dot{z})^2 (ax^4 + by^4 + cz^4) = abx^2y^2 + acx^2z^2 + bcy^2z^2 \\
3{\cdot}0 &= 1{\cdot}4\Lambda^4 2{\cdot}4 = \Sigma a\,(\dot{y} - \dot{z})^4 (abx^2y^2 + acx^2z^2 + bcy^2z^2) \equiv abc \\
3{\cdot}6 &= 1{\cdot}4\Lambda\ 2{\cdot}4 = \Sigma ax^3 (\dot{y} - \dot{z})(2{\cdot}4) \\
&= \Sigma\,(a^2bx^5y - a^2cx^5z) + abcxyz\Sigma\,(yz^2 - y^2z).
\end{aligned}$$

As regards the sextic form, the first idea would be to regard it as the resultant, in respect to one of the variables (say z), of the canonical system discovered by me so long ago,

$$\left.\begin{matrix} ax^6 + by^6 + cz^6 + mxyz\,(x - y)(y - z)(z - x) \\ x + y + z \end{matrix}\right\},$$

but this will be found to give rise to expressions for the invariants and covariants of extreme complexity. The representations will, I think, be simplified by adopting the new canonical system

$$\left.\begin{matrix} x^3 + y^3 + z^3 + 3mxyz \\ ayz + bxz + cyx \end{matrix}\right\} \begin{matrix} (1) \\ (2) \end{matrix}$$

and considering the sextic as the resultant of (1) and (2). It will then be found that every covariant proper (calling its order, which is always an even number, 2ϵ) will still be a resultant of (2) and of some new form in x, y, z of order ϵ*. The fact of the lowering, by one-half, the order of the form in x, y, z, corresponding to a covariant of any given order in x, y, gives a great (though it may be not an unbalanced) advantage to the new canonical system over the old. On setting out the equation connecting the four completely symmetrical invariants with the square of the skew one of the sextic, and then making this latter equal to zero, we obtain an equation between three absolute invariants of the sextic which may be regarded as the equation to a surface, the analogue of my Bicorn, the *Nomen Triviale* for the bicuspidal unicursal quartic curve. This surface will divide space into two parts, one corresponding to equations of the sixth order with real, the other with conjugate coefficients, or by real linear substitutions transformable into such, the surface itself being the locus of equations of the recurrent form. The facultative part of space, that is, the part corresponding to the case of real coefficients will then separate into two pairs of regions, one pair belonging to the case of 0 and 4, the other to that of 2 and 6 imaginary roots. By this method, however laborious, the solution of the problem of determining the invariantive criteria of the quality of the roots of the sextic (to borrow a term

* For every quantic of an even order in x, y is a ternary quantic in x^2+xy, y^2+yx, $-xy$, which quantities are proportional to x, y, z connected by the equation $xy+xz+yz=0$.

from the chess table) becomes *forced*, and no other mode of attacking the question appears to me to be practicable; nor can it fail to bring into view a surface possessed of remarkable properties*.

Scholium.

The mode of representing the covariants to a sextic above employed made it imperative, or at least expedient, to discover a method by aid of which the process of alliance, or hyperdetermination, could be performed upon the representative forms themselves, without eliminating one of the variables by means of the equation of condition, and I have obtained the following very general theorem, which, it will presently be seen, contains a solution of the problem in question, and which, as the first example of conditional alliance, or hyperdetermination, it seems to me desirable to put on record.

Let $\phi, \psi, \ldots, \theta$ be i homogeneous functions of the orders $\alpha, \beta, \ldots, \lambda$ in $i+j$ variables, $x, y, \ldots, t$ being i of them and $u, v, \ldots, z$ the j others, and let the variables be connected by the j homogeneous equations

$$L=0,\ M=0,\ \ldots,\ N=0.$$

Call the Jacobian $$\frac{d\,(L, M, \ldots, N)}{d\,(u, v, \ldots, z)} = \Omega.$$

Let $\Phi, \Psi, \ldots, \Theta$ be the values of $\phi, \psi, \ldots, \theta$ expressed in terms of $x, y, \ldots, t$ alone, and let

$$\left|\begin{matrix} \delta_{x_1}, & \delta_{y_1}, & \ldots, & \delta_{t_1} \\ \delta_{x_2}, & \delta_{y_2}, & \ldots, & \delta_{t_2} \\ \cdots & \cdots & \cdots & \cdots \\ \delta_{x_i}, & \delta_{y_i}, & \ldots, & \delta_{t_i} \end{matrix}\right|^q (\Omega_1^{\alpha}\Omega_2^{\beta}\ldots\Omega_i^{\lambda}\Phi_1\Psi_2\ldots\Theta_i)$$

be called D, it being understood that the meaning of any subscript, say μ, is to cause the letters $x, y, \ldots, t$ to be changed into $x_\mu, y_\mu, \ldots, t_\mu$. Again let the operative determinant of the $(i+j)$th order written below

$$\left|\begin{matrix} \delta_{x_1}, & \delta_{y_1}, & \ldots\ldots, & \delta_{z_1} \\ \delta_{x_2}, & \delta_{y_2}, & \ldots\ldots, & \delta_{z_2} \\ \cdots & \cdots & \cdots & \cdots \\ \delta_{x_i}, & \delta_{y_i}, & \ldots\ldots, & \delta_{z_i} \\ (\delta_x L)_\epsilon, & (\delta_y L)_\epsilon, & \ldots\ldots, & (\delta_z L)_\epsilon \\ (\delta_x M)_\epsilon, & (\delta_y M)_\epsilon, & \ldots\ldots, & (\delta_z M)_\epsilon \\ \cdots & \cdots & \cdots & \cdots \\ (\delta_x N)_\epsilon, & (\delta_y N)_\epsilon, & \ldots\ldots, & (\delta_z N)_\epsilon \end{matrix}\right|$$

* One may see at a glance that this surface cannot be of a higher order than 7, the integer part of 30 : 4. Possibly however, it may not be so high; there will be no difficulty in finding the actual order by means of the known expression for R^2 (Clebsch, *Binäre Formen*, p. 299), in terms of the invariants of even degrees.

be called J_ϵ (ϵ being any of the suffixes 1, 2, 3, ..., i) then it will be found that to a numerical factor près

$$(\Omega_1^{\alpha-q}\Omega_2^{\beta-q}\ldots\Omega_i^{\lambda-q})(J_1+J_2+\ldots+J_i)^q(\phi_1\psi_2\ldots,\theta_i)=D.$$

As a corollary, if the functions $L, M, \ldots, N$ are all linear in respect to $u, v, \ldots, z$, and if in respect to $1, u, v, \ldots, z$ the resultants of $\phi, L, M, \ldots, N$; $\psi, L, M, \ldots, N$; ... are $[\Phi], [\Psi], \ldots, [\Theta]$ (which is what we mean by saying that $\phi, \psi, \ldots, \theta$ represent $[\Phi], [\Psi], \ldots, [\Theta]$), it will be easily seen to follow from the above theorem that the qth alliance of these quantics will be itself *represented by*

$$(J_1+J_2+\ldots+J_i)^q(\phi_1\psi_2\ldots,\theta_i)^*.$$

Thus in the particular case where $x, y, \ldots, t$ becomes x, y and $u, \ldots, z$ becomes z and $L, M, \ldots, N$ becomes the single function $xy+yz+zx$, we see that the qth alliance of the quantics represented by ϕ, ψ will be itself represented by

$$\left\{\begin{vmatrix}\delta_{x_1}, & \delta_{y_1}, & \delta_{z_1}\\ \delta_{x_2}, & \delta_{y_2}, & \delta_{z_2}\\ y_1+z_1, & z_1+x_1, & x_1+y_1\end{vmatrix}+\begin{vmatrix}\delta_{x_1}, & \delta_{y_1}, & \delta_{z_1}\\ \delta_{x_2}, & \delta_{y_2}, & \delta_{z_2}\\ y_2+z_2, & z_2+x_2, & x_2+y_2\end{vmatrix}\right\}^q(\phi_1\psi_2)$$

on replacing x_1, y_1, z_1; x_2, y_2, z_2 by x, y, z after the differentiations have been executed. It will, of course, be understood that the factors in each cross product of the determinants above are to be taken in *their natural order*, that is,

$$\begin{vmatrix}\delta_{x_1}, & \delta_{y_1}, & \delta_{z_1}\\ \delta_{x_2}, & \delta_{y_2}, & \delta_{z_2}\\ y_1+z_1, & z_1+x_1, & x_1+y_1\end{vmatrix}^\mu$$

is to be understood to mean, not

$$[\Sigma(x_1+y_1)(\delta_{x_1}\delta_{y_2}-\delta_{y_1}\delta_{x_2})]^\mu,$$

but

$$[\Sigma(\delta_{x_1}\delta_{y_2}-\delta_{y_1}\delta_{x_2})(x_1+y_1)]^\mu,$$

and so in general.

* This expression may be put under the more compact form J^q, J being a matrix in which the first i lines are the same as those common to $J_1, J_2, \ldots, J_i$, and the last j lines are the sums of the corresponding ones in $J_1, J_2, \ldots, J_i$. Although I had submitted it to a mental process of demonstration (or what seemed such) before sending it to the press, I am not without some little misgiving as to the exactitude of the theorem so far as it regards the higher alliances; for those of the first order it is easily verifiable, and, in that case, it should be noticed that each of the i terms in the expression given by it will reproduce separately (but under quite a distinct form) the value of the Jacobian of $\phi, \psi, \ldots, \theta$; $L, \ldots, N$. Some corresponding simplification in practice, it is not improbable, will apply in the general case, supposing my doubts as to the validity of the theorem to prove unfounded. It is important, and greatly enlarges the horizon of the subject, to remark that, inasmuch as any ternary quadric is linearly transformable into the form $xy+yz+zx$, it will follow that any binary quantic of an even order, with its train of covariants, may be represented by corresponding ternary forms of half their respective orders, combined with a perfectly general final conic, so that, for example, instead of the form $xy+yz+zx$, useful though it be as an intermediate step in the evolution of the theory, we may substitute the handier and more advantageous one $x^2+y^2+z^2$ as the auxiliary quadric.

The result of this investigation has been to open my eyes to the unquestionable fact that, as we know that the first "Ueberschiebung," or "transvectant," or "alliance," of two or more quantics (names significant and useful enough to indicate the particular modes under which they are considered to be generated) is the ordinary Jacobian, so the right general name for the Ueberschiebung or alliance of any order viewed *per se* (as a *Ding an sich*) and without reference to its mode of origination, which ought to supersede all others, is the *Jacobian of the corresponding order*; or, in other words, the theory of invariants falls into the theory of compound differentiation, and just as $\left(\frac{du}{dx}\frac{dv}{dy}-\frac{du}{dy}\frac{dv}{dx}\right)$ is called a Jacobian and designated by $\frac{d(u,v)}{d(x,y)}$, so $\frac{d^2u}{dx^2}\frac{d^2v}{dy^2}-2\frac{d^2u}{dxdy}\frac{d^2v}{dxdy}+\frac{d^2u}{dy^2}\frac{d^2v}{dx^2}$ is entitled to be called the second Jacobian and to be designated by $\frac{d^2(u,v)}{d(x,y)^2}$, and more generally every hyperdeterminant may be designated as a compound differential coefficient (or derivative) of the type $\frac{d^\alpha d^\beta\ldots}{d(\quad)^\alpha d(\quad)^\beta\ldots}$, where the vacant spaces are to be filled up by the insertion of a certain number of letters, with liberty for any number of them in each parenthesis to be identical with the like number in any other. Since we are now in possession of a definite analogue to ordinary differential coefficients of all orders, I do not know whether I shall be considered too bold or fanciful in suggesting that there ought to exist, in the nature of things, some theorem of development for several sets of variables analogous to Taylor's for a single set: what such theorem is or could be I have at present no conception, but as little, be it remembered, could anyone, even Jacobi himself, before the creation of hyperdeterminants, have had the remotest conception in regard to a function of several variables bearing to $\left(\frac{d}{dx}\right)^i\phi$ the same relation of analogy as the ordinary functional determinant to $\frac{d\phi}{dx}$, whether such function could exist, and, if so, what it would be. I have always thought and felt that beyond all others the algebraist, in his researches, needs to be guided by the principle of faith, so well and philosophically defined as "the substance of things hoped for, the evidence of things not seen."