40.

NOTE SUR L'INVOLUTION DE SIX LIGNES DANS L'ESPACE.

[*Comptes Rendus de l'Académie des Sciences*, LII. (1861), pp. 815—817.]

DÉSIGNONS six droites par les chiffres 1, 2, 3, 4, 5, 6. Prenons les équations de chacune de ces lignes sous la forme la plus générale (en nous servant de coordonnées tétraédrales). Ainsi, soit la ligne i définie par les équations

$$a_i x + b_i y + c_i z + d_i u = 0, \quad \alpha_i x + \beta_i y + \gamma_i z + \delta_i u = 0;$$

et de même la ligne j par les équations

$$a_j x + b_j y + c_j z + d_j u = 0, \quad \alpha_j x + \beta_j y + \gamma_j z + \delta_j u = 0,$$

et sous-entendons par i, j le déterminant

$$\begin{vmatrix} a_i, & b_i, & c_i, & d_i \\ \alpha_i, & \beta_i, & \gamma_i, & \delta_i \\ a_j, & b_j, & c_j, & d_j \\ \alpha_j, & \beta_j, & \gamma_j, & \delta_j \end{vmatrix}.$$

Cela étant fait, formons le déterminant (que je nommerai Δ_6):

$$\begin{vmatrix} & 1,2 & 1,3 & 1,4 & 1,5 & 1,6 \\ 2,1 & & 2,3 & 2,4 & 2,5 & 2,6 \\ 3,1 & 3,2 & & 3,4 & 3,5 & 3,6 \\ 4,1 & 4,2 & 4,3 & & 4,5 & 4,6 \\ 5,1 & 5,2 & 5,3 & 5,4 & & 5,6 \\ 6,1 & 6,2 & 6,3 & 6,4 & 6,5 & \end{vmatrix}.$$

Si les six droites 1, 2, 3, 4, 5, 6 sont en involution, on aura

$$\Delta_6 = 0;$$

et réciproquement si $\Delta_6 = 0$, les six lignes seront en involution.

En nous bornant aux cinq chiffres 1, 2, 3, 4, 5, on peut former un déterminant analogue à Δ_6 (disons Δ_5) qui ne contiendra que cinq lignes et cinq colonnes, et qui sera un déterminant mineur du premier ordre du grand déterminant Δ_6. De même pour Δ_4, etc.

Si $\Delta_6 = 0$ et $\Delta_5 = 0$ (sans que *tous* les déterminants mineurs du premier ordre de Δ_6 soient zéro), les cinq lignes 1, 2, 3, 4, 5 formeront un système en involution entre elles. Si tous les déterminants mineurs du premier ordre sont zéro (ce qui ne suppose qu'une seule condition de plus, c'est-à-dire trois conditions en tout), les six lignes de 1 à 6 seront toutes rencontrées par la même droite.

Si $\Delta_6 = 0$, $\Delta_5 = 0$, $\Delta_4 = 0$, alors en général les quatre lignes de 1 à 4 seront en involution entre elles; je n'insisterai pas ici sur les cas possibles d'exception; j'ajouterai seulement que si $\Delta_5 = 0$ sans autre condition, les cinq lignes de 1 à 5 seront toutes rencontrées par la même droite. Si $\Delta_4 = 0$, sans autre condition, les quatre lignes de 1 à 4 n'admettront qu'*une seule transversale* qui les rencontre toutes quatre, au lieu des deux qui existent ordinairement pour quatre droites dans l'espace. C'est M. Cayley qui le premier a fait cette dernière remarque. De plus il a trouvé indépendamment un déterminant qui est égal à la racine carrée de Δ_6 et qui conséquemment sert tout aussi bien que Δ_6 pour définir l'involution.

L'espace me manque pour produire ici cet autre déterminant, mais je dois ajouter que c'est d'une grande utilité dans l'étude analytique de la théorie d'involution.

Je prie qu'il me soit permis de profiter de cette occasion pour rectifier une erreur qui s'est glissée dans l'énoncé d'un théorème donné dans les *Comptes rendus* (26 janvier 1861). Dans le second paragraphe de la Note [p. 229 above], au lieu de "pour numérateurs le cycle toujours répété...par rapport à r," lisez "pour numérateurs le cycle toujours répété des nombres entiers congrus* à $\frac{1}{p}, \frac{2}{p}, \frac{3}{p}, \ldots, \frac{r}{p}$ et compris parmi les nombres $1, 2, 3, \ldots, r$." Et plus bas au lieu de "mais à cause... $10k+2$" lisez: "mais, à cause de $7 \times 3 \equiv 1 \pmod 5$,

$$\equiv \frac{3}{p-1} + \frac{1}{p-2} + \frac{4}{p-3} + \frac{2}{p-4} + \frac{5}{p-5} + \ldots$$

quand p est de la forme $10k+7$."

[* mod. r.]

41.

NOTE SUR LES 27 DROITES D'UNE SURFACE DU 3^e^ DEGRÉ.

[*Comptes Rendus de l'Académie des Sciences*, LII. (1861), pp. 977—980.]

MES recherches sur l'involution d'axes de rotation m'a forcément conduit à étudier les propriétés géométriques des 27 lignes droites qui sont situées sur chaque surface générale du 3^e degré, et j'ai trouvé un théorème pour représenter ces lignes d'une manière à ôter toute difficulté en approfondissant leurs rapports mutuels. En se servant de ce théorème que je lui ai communiqué, M. Cayley est parvenu avant moi à donner une construction géométrique de ces 27 lignes; mais sa construction exige la connaissance de 8 droites données, c'est-à-dire d'une ligne droite prise comme base, coupée par 3 paires de droites qui se croisent (et dont les traces sur la base forment un système de 6 points en *involution*), et coupée aussi par une 7^e droite. C'est une conséquence de la théorie connue de ces 27 lignes (comme l'a bien montré mon ami distingué), qu'une surface du 3^e degré peut être construite, qui contiendra ces 8 droites (la base et les 7 autres lignes qui la coupent).

En me prévalant d'une autre façon de mon théorème, je suis parvenu à donner une construction d'une nature semblable, mais plus symétrique et plus simple que celle de M. Cayley, au moins dans des données qui pour moi sont une ligne droite coupée par 5 autres droites sans autre condition.

C'est le système de droites qui s'offre tout naturellement dans la théorie de mécanique dont je m'occupais et dont je me fus proposé de prime abord de me servir pour résoudre la question au temps même que j'ai reçu de la part de M. Cayley la solution avec le nouveau système de données dont j'ai parlé plus haut. Voici une première observation qui sera utile dans la suite. En prenant 5 lignes droites tout à fait arbitraires, disons a, b, c, d, e, en les joignant quatre à quatre, on peut construire 5 systèmes de paires de transversales; mais si les 5 données rencontrent la même droite, disons x, il est évident que ces 5 paires se réduiront à cette droite et 5 autres transversales; or il est facile de démontrer que ces 5 dernières seront toutes rencontrées elles-mêmes par une autre droite, disons ξ; elles peuvent être convenablement nommées $\alpha, \beta, \gamma, \delta, \epsilon$; où α est la seconde transversale à b, c, d, e; β à a, c, d, e, etc.

Je fais une seconde observation très-importante, voir que 6 droites dont 5 sont coupées par la 6^e, sont situées sur la même surface du 3^e degré, et réciproquement tout système de 5 droites sur une surface du 3^e degré qui ne se coupent pas entre elles sont coupées par la même droit. Je dois ajouter que si 5 droites sont toutes coupées par les mêmes 2 lignes droites, on peut faire passer un nombre infini de surfaces du 3^e degré par ces 7 lignes, parmi lesquelles se trouveront comprises 2 surfaces réglées, et le théorème réciproque aura aussi lieu.

Écrivons les 12 lignes

$$\begin{array}{ccccc} & & x & & \\ a, & b, & c, & d, & e \\ \alpha, & \beta, & \gamma, & \delta, & \epsilon \\ & & \xi & & \end{array}$$

où on suppose que a, b, c, d, e sont rencontrées par x, mais non par aucune autre droite, et que $\alpha, \beta, \gamma, \delta, \epsilon$ sont les 5 transversales à a, b, c, d, e prises quatre à quatre, et ξ la transversale commune à $\alpha\beta\gamma\delta\epsilon$.

Formons encore le système $ABCDE$, où A est la transversale à $xa\alpha\xi$, B à $xb\beta\xi$, C à $xc\gamma\xi$, D à $xd\delta\xi$, E à $xe\epsilon\xi$; c'est-à-dire A est l'intersection des plans qui passent respectivement par xa, $\xi\alpha$ et de même pour B, C, D, E.

Finalement menons les 10 transversales désignées par la combinaison des symboles des 4 lignes qu'elles rencontrent respectivement, c'est-à-dire $a\alpha b\beta$, $a\alpha c\gamma$, $a\alpha d\delta$, $a\alpha e\epsilon$, $b\beta c\gamma$, $b\beta d\delta$, $b\beta e\epsilon$, $c\gamma d\delta$, $c\gamma e\epsilon$, $d\delta e\epsilon$. Il est bon de remarquer que les deux droites a, β se croisent, comme aussi b, α, et que $a\alpha b\beta$ signifie l'intersection des deux plans de $a\beta$, $b\alpha$. Une remarque semblable a lieu pour les autres droites de cette série de 10. On voit qu'on a obtenu

$$1 + 5 + 5 + 1 + 5 + 10 = 27 \text{ droites.}$$

Il est facile de démontrer géométriquement que toutes ces droites sont situées sur la même surface du 3^e degré, et que cette surface ne contiendra pas aucune autre ligne droite sur elle. Je dois ajouter, pour rendre plus complète l'image de ce système de 27 droites, que les 10 dernières couperont chacune 6 autres au-dessus des 4 exprimées par la notation quaternaire même, c'est-à-dire $a\alpha b\beta$ ne rencontrera pas seulement a, α, b, β, mais aussi C, D, E et $c\gamma d\delta$, $c\gamma e\epsilon$, $d\delta e\epsilon$ et ainsi pour les autres, de sorte qu'on trouvera facilement que chaque droite des 27 sera rencontrée par 10 autres, chaque combinaison de 3 qui ne se rencontrent pas par 3 autres qui ne se rencontrent pas, chaque combinaison de 4 qui ne se rencontrent pas par 2 autres sur la surface, etc.; conformément aux beaux résultats de MM. Salmon et Cayley, déjà, il y a longtemps, donnés dans le *Cambridge and Dublin Mathematical Journal.*

On peut résumer en peu de mots la construction précédente.

5 droites rencontrées par une 6^e étant données, on construit 5 autres rencontrées par une nouvelle 6^e, telles que chaque droite d'un des groupes de 5 rencontre 4 de l'autre groupe. Les 12 droites ainsi liées s'entrecoupent (par construction) en $2 \times 5 + 5 \times 4$, c'est-à-dire en 30 points, et conséquemment sont situées deux à deux en 30 plans dont chacun joint d'un rapport de réciprocité avec quelque autre. Les intersections de ces paires des plans réciproques donnent naissance à 15 nouvelles droites, lesquelles, combinées avec les 12 déjà nommées, constituent un système (le plus général qui peut exister) de 27 droites *réelles* appartenant à une surface du 3^e degré. Il va sans dire qu'il existe des surfaces de ce degré pour lesquelles les 27 droites ne sont pas toutes *réelles*.

Je me propose de faire construire en fil de fer ou d'archal un système de 27 droites par la méthode donnée en haut, et d'en faire des copies stéréographiques, de sorte qu'on pourra éprouver le plaisir inattendu de voir avec les yeux du corps toutes les droites (le squelette pour ainsi dire) d'une surface du 3^e degré avec leurs 135 points d'intersection, les 45 triangles les hexagones situés sur le même hyperboloïde et des autres non pas ainsi situés, et les autres merveilles de cette involution si compliquée, mais en même temps si symétrique.

Je prie qu'il me soit permis de profiter de cette occasion pour rectifier une erreur dans ma communication donnée dans les *Comptes rendus* (15 avril 1861): Dans le 4^e paragraphe* les mots "les deux droites perpendiculaires ....correspondants; en conséquence" doivent être rayés. Plus bas dans le même paragraphe les mots "perpendiculaire à la ligne des centres" doivent être rayés, et dans la ligne suivante pour "perpendiculaire" on doit lire "droite."

La belle observation de M. Chasles dans le même numéro des *Comptes rendus*, sur une méthode de trouver un système de 6 droites en involution au moyen des perpendiculaires aux trajectoires de 6 points dans le déplacement infiniment petit d'un corps rigide, se trouve confirmée par une application assez simple de la méthode des vélocités virtuelles.

Car en donnant à un corps rigide sollicité par 6 forces agissant suivant des lignes droites données 6 déplacements arbitraires, on obtiendra 6 équations indépendantes et homogènes auxquelles les valeurs des 6 forces doivent satisfaire pour qu'elles fassent équilibre entre elles; ce qui en général ne sera pas possible; mais en supposant qu'un des déplacements peut être effectué d'une telle manière, que toutes les vélocités virtuelles des 6 points d'application seront nulles, une des six équations disparaîtra, c'est-à-dire deviendra une identité, et le système de cinq équations linéaires qui restent admettra une solution."

[* p. 238 above.]

42.

GÉNÉRALISATION D'UN THÉORÈME DE M. CAUCHY*.

[*Comptes Rendus de l'Académie des Sciences*, LIII. (1861), pp. 644, 645.]

Dans son Mémoire sur les *arrangements*, 1844, M. Cauchy a établi le théorème suivant:

Soit n un nombre entier donné,

$$\alpha a + \beta b + \gamma c + \ldots + \lambda l = n;$$

en supposant $a, b, c, \ldots, l$ des nombres entiers et inégaux, $\alpha, \beta, \gamma, \ldots, \lambda$ des nombres entiers, et en faisant varier de toutes les manières possibles les valeurs du système $a, b, c, \ldots, l$, on aura

$$\Sigma \frac{1}{\pi\alpha . \pi\beta \ldots \pi\lambda a^\alpha b^\beta \ldots l^\lambda} = 1,$$

où πx signifie le produit $1 . 2 . 3 \ldots x$.

Je vais démontrer qu'on peut exprimer d'une manière très-simple la valeur générale de $\Sigma \frac{\omega^{\alpha+\beta+\ldots+\lambda}}{\pi\alpha . \pi\beta \ldots \pi\lambda a^\alpha b^\beta \ldots l^\lambda}$ pour une valeur quelconque d'une constante ω.

En effet, il est très-facile de voir qu'en posant l'équation en nombres positifs et entiers

$$x_1 + x_2 + x_3 + \ldots + x_r = n,$$

et en attribuant à $x_1, x_2, \ldots, x_r$ toutes les valeurs possibles qui satisfont à cette équation (en regardant comme distinctes les solutions qui diffèrent dans les valeurs de x, quoique contenant le même système de valeurs), on peut représenter la série (nommée fonction de n et ω) sous la forme

$$\sum_{r=\infty}^{r=1} \Sigma \frac{1}{x_1 x_2 \ldots x_r} \frac{\omega^r}{\pi(r)},$$

c'est-à-dire

$$\sum_{r=\infty}^{r=1} F(r, n) \frac{\omega^r}{\pi(r)}.$$

[* See below, p. 290.]

Or on voit immédiatement que $F(r, n)$ n'est autre chose que le coefficient de t^n dans le développement de la fonction génératrice $\left(t+\frac{t^2}{2}+\frac{t^3}{3}+\dots\right)^r$, c'est-à-dire dans le développement de $[\log(1-t)^{-1}]^r$. Donc évidemment la série totale sera le coefficient de t^n dans le développement de $e^{\omega \log[(1-t)^{-1}]}$, c'est-à-dire de t^n dans $\left(\frac{1}{1-t}\right)^\omega$.

En prenant $\omega=1$, on voit que la valeur est toujours l'unité pour toute valeur de n, ce qui est le théorème de Cauchy. En prenant $\omega=-i$, i étant un nombre entier quelconque plus petit que n, on trouve la valeur zéro. Pour le cas de $\omega=-1$, cette remarque avait déjà été faite par M. Cayley, dans le *Philosophical Magazine* (mars 1861). En prenant $\omega=\frac{1}{2}$, on trouve pour la valeur de la série $\frac{1.3.5\dots 2n-1}{2.4.6\dots 2n}$, ce qui peut se déduire aussi par la méthode des arrangements, en se servant du théorème que le nombre des *substitutions* de $2n$ lettres qui peuvent être représentées par des égales d'un rang exclusivement pair est $[1.3.5\dots(2n-1)]^2$, théorème que je crois être nouveau, mais qui est intimement lié au théorème célèbre de M. Cayley sur la valeur des déterminants dits *gauches*.

Voici une dernière observation que je fais sur le théorème général. On remarquera que l'exposant de $\omega^{\alpha+\beta+\dots+\lambda}$ est le nombre des parties dans la partition de n, représentée par α répétitions de a, β de b, ..., λ de l: je nommerai donc $\alpha+\beta+\gamma+\dots+\lambda$ l'*indice* de cette partition, et je dis qu'étant donné le *nombre* de ces indices, disons ν (nombre qu'on peut trouver pour une valeur quelconque de n par le théorème très-bien connu d'Euler sur les partitions indéfinies), on peut faire dépendre les valeurs de ces ν indices de la solution d'un système de 2μ équations algébriques à 2μ inconnues. Car pour une valeur quelconque de ω on connaîtra par le théorème du texte la valeur de $\frac{\omega^{i_1}}{q_1}+\frac{\omega^{i_2}}{q_2}+\dots+\frac{\omega^{i_\mu}}{q_\mu}$, où $i_1, i_2, \dots, i_\mu$ seront les indices cherchés, et $q_1, q_2, \dots, q_\mu$ des quantités inconnues, mais indépendantes de ω. En substituant pour ω successivement $\omega, \omega^2, \omega^3, \dots, \omega^{2\mu}$ et en écrivant $\omega^{i_r}=I_r$, on aura 2μ équations de la forme

$$\frac{I_1^k}{q_1}+\frac{I_2^k}{q_2}+\dots+\frac{I_\mu^k}{q_\mu}=C,$$

k prenant toutes les valeurs de 1 jusqu'à 2μ. On peut donc former une équation dont dépendra la valeur de chacune des quantités $I_1, I_2, \dots, I_\mu$, et conséquemment de leurs logarithmes $i_1, i_2, \dots, i_\mu$, les μ indices de la partition indéfinie de n.

43.

ADDITION À LA NOTE INTITULÉE: "GÉNÉRALISATION D'UN THÉORÈME DE M. CAUCHY," ET INSÉRÉE DANS LE "COMPTE RENDU" DE LA SÉANCE DU 7 OCTOBRE DERNIER.

[*Comptes Rendus de l'Académie des Sciences*, LIII. (1861), pp. 722—725.]

En suivant la même marche que dans la Note dont il s'agit [p. 245 above], on parvient très-facilement à résoudre une question un peu plus compliquée de la théorie des *arrangements*, savoir : *Trouver le nombre de substitutions de n lettres qu'on peut représenter par le moyen d'un nombre donné r de substitutions cycliques d'ordres impairs.*

Pour que ce nombre ne soit pas zéro, il faut que $n-r$ soit un nombre pair $2i$; alors le nombre demandé sera la somme suivante,

$$\Sigma\,[(\nu_1^2+\nu_1)(\nu_2^2+\nu_2)\dots(\nu_e^2+\nu_e)\dots(\nu_i^2+\nu_i)],$$

où le signe Σ se rapporte à tous les systèmes possibles de nombres entiers $\nu_1, \nu_2, \dots, \nu_e, \dots, \nu_i$ qui satisfont aux inégalités

$$\nu_e > \nu_{e-1}+1, \quad \nu_e < n-1.$$

Désignons par $[n, r]$ le nombre des substitutions exprimé par la somme précédente, et par (n, r) le nombre correspondant pour le cas où les r substitutions cycliques sont chacune indifféremment d'ordre pair ou d'ordre impair. On a déjà trouvé que (n, r) est la somme des produits de $n-r$ quelconques des nombres 1, 2, 3, ..., $(n-1)$, et l'on voit à présent que $[n, r]$ n'est autre chose que la somme des produits de $\frac{n-r}{2}$ facteurs dont chacun est le produit d'un terme de la même suite de nombres par le terme suivant. Et de même que (n, r) satisfait à l'équation fonctionnelle

$$\frac{(n+1, r+1)-(n, r)}{n}=(n-1, r),$$

la fonction $[n, r]$ satisfait à l'équation analogue

$$\frac{[n+2, r+2]-[n, r]}{n}=(n+1)[n-2, r]+(n-1)[n-3, r-1],$$

comme il est facile de s'en assurer.

On peut ajouter que (n,r) (pour $n-r$ positif) et $[n,r]$ $\left(\text{pour } \frac{n-r}{2} \text{ positif}\right)$ sont tous deux divisibles par n quand n est un nombre premier. Ce théorème est bien connu en ce qui concerne (n, r), mais il me paraît nouveau à l'égard de $[n, r]$. Au reste, on peut appliquer aux deux cas la même démonstration fondée sur ce que le nombre de produits de cycles correspondant à chaque partition de n est évidemment un multiple de n.

Voici un exemple du théorème énoncé au commencement de cette Note: Le nombre des substitutions de 6 lettres qu'on peut représenter par le produit de deux cycles d'ordres impairs sera, d'après notre formule générale,

$$2\times 12+2\times 20+6\times 20=184,$$

ce que l'on peut vérifier bien facilement en remarquant que ce nombre doit être

$$6\times 24+10\times 4=184.$$

On démontre encore très-facilement que le nombre total des substitutions de n lettres représentées par le produit de substitutions cycliques d'ordres impairs est

$$[1.3.5\ldots(n-1)]^2$$

quand n est pair (c'est le même nombre que nous avons déjà obtenu pour les substitutions cycliques d'ordre pair), et

$$n[1.3.5\ldots(n-2)]^2$$

quand n est impair.

On peut donner une extension* très-considérable aux théorèmes énoncés précédemment, en considérant le nombre des substitutions de n lettres formées avec les produits de r substitutions cycliques où l'ordre de chaque cycle est congru à un nombre ρ par rapport à un module donné μ.

La solution dépend toujours des combinaisons des nombres de la série $1, 2, 3, \ldots, (n-1)$. Je me bornerai ici au cas de $\rho=1$ qui est le plus simple, en exceptant celui de $\rho=0$, dont la solution est évidente. Dans le cas de $\rho=1$, le nombre cherché est exprimé par la somme

$$\Sigma\frac{\Pi(\nu_1+\mu-1)\,\Pi(\nu_2+\mu-1)\ldots\Pi(\nu_i+\mu-1)}{\Pi(\nu_1-1)\,\Pi(\nu_2-1)\ldots\Pi(\nu_i-1)},$$

[* Cf. below, p. 293.]

où l'on fait $i=\frac{n-r}{\mu}$ et où les nombres ν sont assujettis aux conditions

$$\nu_e>\nu_{e-1}+\mu-1,\quad \nu_e<n-1,$$

et, en conséquence, on peut énoncer le théorème suivant:

Si n est un nombre premier, r et μ deux nombres quelconques donnés, la somme des produits de r groupes de μ termes consécutifs de la série $1.2.3\ldots(n-1)$ (en supposant que chaque groupe contient des nombres distincts de ceux qui sont contenus dans chacun des autres groupes) sera divisible par n, pourvu que μr soit inférieur à n.

Dans le cas de $\mu=1$, on retombe sur le théorème si connu, associé au théorème de Wilson.

Comme exemple du nouveau théorème, prenons $n=11$, $\mu=3$, $r=3$. On doit trouver et l'on trouvera effectivement que la somme

$$\begin{aligned}&\ \ 1.2.3.4.5.6.7.8.\ 9\\ &+1.2.3.4.5.6.8.9.10\\ &+1.2.3.5.6.7.8.9.10\\ &+2.3.4.5.6.7.8.9.10\end{aligned}$$

est divisible par 11. En effet cette somme est le nombre de substitutions de 11 lettres formées par les produits de deux substitutions cycliques assujetties à ne contenir que 1, 4, 7 ou 10 lettres. Les quatre produits qui figurent dans cette somme font partie des cinquante-cinq produits qu'on devrait prendre dans le cas correspondant du théorème ordinaire associé à celui de Wilson.

44.

DÉMONSTRATION DIRECTE DU THÉORÈME DE LAGRANGE SUR LES VALEURS NUMÉRIQUES MINIMA D'UNE FONCTION LINÉAIRE À COEFFICIENTS ENTIERS D'UNE QUANTITÉ IRRATIONNELLE*.

[*Comptes Rendus de l'Académie des Sciences*, LIII. (1861), pp. 1267—1272.]

Après Euler, je me servirai du symbole $(a, b, c, \ldots, l)$ pour représenter le dénominateur de la fraction convergente dont $a, b, c, \ldots, l$ sont les quotients partiels, de sorte que $(b, c, \ldots, l)$ représentera le numérateur de la même fraction. Soit ν une quantité quelconque incommensurable à l'unité,

$$\frac{(b, \ldots, h, k)}{(a, b, \ldots, h, k)}, \quad \frac{(b, \ldots, h, k, l)}{(a, b, \ldots, h, k, l)},$$

deux réduites consécutives de ν. Comme à l'ordinaire, je nommerai ces convergentes $\frac{p}{q}, \frac{p'}{q'}$; on aura

$$\nu = \frac{[b, \ldots, h, (k+\theta)]}{[a, b, \ldots, h, (k+\theta)]} = \frac{N}{D}, \text{ où } \theta < \frac{1}{l};$$

on en conclut

$$p - \nu q = \frac{(b, \ldots, h, k)(a, b, \ldots, h, k+\theta) - (a, b, \ldots, h, k+\theta)(b, \ldots, h)}{D}$$

$$= \theta \frac{(b, \ldots, h, k)(a, b, \ldots, h) - (a, b, \ldots, h, k)(b, \ldots, h)}{D}$$

$$= (-1)^i \frac{\theta}{D},$$

i désignant le nombre des quantités $a, b, \ldots, h$.

Faisons $$p - \nu q = \Delta,$$

on aura $$D\Delta = (-1)^i \theta. \qquad (1)$$

[* Cf. p. 306, below.]

Prenons $$(p - \lambda) - \nu(q - \mu) = \Delta',$$

λ et μ étant des nombres entiers quelconques, tels que $\Delta'^2 < \Delta^2$, avec exclusion du cas où $p - \lambda = 0$, $q - \mu = 0$; alors

$$\Delta' = \Delta + \frac{\mu(b, \ldots, h, k+\theta) - \lambda(a, b, \ldots, h, k+\theta)}{D} = [(-1)^i \theta + A\theta + B] \div D$$

où

$$\left.\begin{array}{l} A = (b, \ldots, h)\mu - (a, b, \ldots, h)\lambda, \\ B = (b, \ldots, h, k)\mu - (a, b, \ldots, h, k)\lambda. \end{array}\right\} \qquad (2)$$

Donc, pour que Δ'^2 soit moindre que Δ^2, A et B doivent être de signes contraires, à moins que A ou B soit zéro.

Si $A = 0$,

$$\lambda = r(b, \ldots, h), \quad \mu = r(a, b, \ldots, h),$$

$$B = r[(a, b, \ldots, h)(b, \ldots, h, k) - (b, \ldots, h)(a, b, \ldots, h, k)] = (-1)^i r,$$

et $$D\Delta' = (-)^i (\theta + r),$$

ce qui serait contraire à l'hypothèse.

De même si $B = 0$,

$$\lambda = r(b, \ldots, h, k), \quad \mu = r(a, b, \ldots, h, k),$$

et $D\Delta'$ devient

$$(-)^i \theta (1 - r),$$

de sorte que Δ'^2 ne peut pas être au-dessous de Δ^2, à moins que $r = 1$, ce qui donnerait

$$p - \lambda = 0, \quad q - \mu = 0,$$

cas dont on a fait exclusion.

Donc, puisque A et B doivent avoir des signes contraires, $\frac{\lambda}{\mu}$ sera intermédiaire entre $\frac{(b, \ldots, h, k)}{(a, b, \ldots, h, k)}$ et $\frac{(b, \ldots, h)}{(a, b, \ldots, h)}$, c'est-à-dire $\frac{(b, \ldots, h, \infty)}{(a, b, \ldots, h, \infty)}$, et conséquemment, comme il est très-facile de le voir, $\frac{\lambda}{\mu}$ sera de la forme

$$\frac{\left(b, \ldots, h, \frac{\rho}{\sigma}\right)}{\left(a, b, \ldots, h, \frac{\rho}{\sigma}\right)}.$$

Or on peut supposer $\frac{\rho}{\sigma}$ ou un nombre entier ou une fraction irréductible plus grande que k; de plus, comme il est facile de démontrer que $\sigma . \left(b, \ldots, h, \frac{\rho}{\sigma}\right)$, $\sigma . \left(a, b, \ldots, h, \frac{\rho}{\sigma}\right)$ seront premiers entre eux, on aura nécessairement

$$\lambda = r(b, \ldots, g, h) + s(b, \ldots, g), \quad \mu = r(a, b, \ldots, g, h) + s(a, b, \ldots, g),$$

avec la condition $r > ks$.

Donc, en substituant ces valeurs en (2), $D\Delta'$ devient égal à

$$(-)^i \theta + rP + sQ,$$

$$P = (b, \ldots, h, k)(a, b, \ldots, h) - (a, b, \ldots, h, k)(b, \ldots, h) = (-1)^i,$$

$$Q = \theta\,[(b, \ldots, g, h)(a, b, \ldots, g) - (a, b, \ldots, h)(b, \ldots, g)]$$
$$+ (b, \ldots, g, h, k)(a, b, \ldots, g) - (a, b, \ldots, g, h, k)(b, \ldots, g)$$
$$= -\theta(-1)^i + (-)^\omega k,$$

ω étant le nombre des lettres $(a, b, \ldots, h, k)$, c'est-à-dire $i+1$.

Donc $$D\Delta' = (-1)^i(\theta - s\theta + r - sk). \qquad (3)$$

Maintenant, imposons à volonté sur λ la limite $\lambda < p + p'$, ou bien sur μ la limite $\mu < q + q'$; pour fixer les idées, disons $\lambda < p + p'$:

$$p' = (b, \ldots, h, k, l) = (kl+1)(b, \ldots, g, h) + l(b, \ldots, g),$$

$$p = (b, \ldots, h, k) = k(b, \ldots, g, h) + (b, \ldots, g);$$

donc $$p' + p = (kl + k + 1)(b, \ldots, g, h) + (l+1)(b, \ldots, g).$$

Mais $$\lambda = r(b, \ldots, g, h) + s(b, \ldots, g).$$

Donc je dis que s ne peut pas excéder l.

Car si $$s \geqslant l + 1,$$

r, qui est au moins $ks+1$, sera $\geqslant kl + k + 1$, et λ ne sera pas moindre que $p' + p$, ce qui est contraire à l'hypothèse. Donc

$$s\theta \leqslant l\theta < 1;$$

mais $$r - sk > 1,$$

donc $$(-)^i D\Delta' > \theta,$$

c'est-à-dire $$> (-)^i D\Delta,$$

et l'on peut, de la même manière, démontrer que, si $\mu < q + q'$,

$$(-)^i D\Delta' > (-)^i D\Delta.$$

Donc il est évident que $(p - q\nu)^2$ sera moindre que $(x - y\nu)^2$ si $x < p'$ ou si $y < q'$. Toujours excluant le cas, on a en même temps

$$x = 0, \quad y = 0.$$

Je nomme ce résultat la *conclusion A*.

J'ajoute une *observation* importante pour ce qui sort immédiatement de la forme de l'équation (2) : c'est que $(D\Delta')^2$ sera plus grand que $(D\Delta)^2$ si $\lambda = 0$ pour toute valeur de $\mu > 0$, et de même si $\mu = 0$ pour toute valeur de $\lambda > 0$. Je nomme cette observation *conclusion B*.

En vertu de ces deux conclusions, on peut démontrer très-facilement ce qui est le but du théorème Lagrange donné dans les Additions de l'Algèbre d'Euler, c'est-à-dire que la condition *nécessaire* et *suffisante* que $\frac{p}{q}$ soit une convergente de ν sera que la valeur $(p - q\nu)$ sera toujours augmentée en diminuant ou p ou q, ou tous les deux.

La nécessité de cette condition découle immédiatement et avec surabondance de la conclusion A, qui affirme qu'un changement quelconque de p qui ne le rend pas égal à p', ou de q qui ne le rend pas égal à q', aura l'effet d'augmenter $p - q\nu$.

Pour prouver que la condition est suffisante, il faut montrer que si a et b ne sont pas simultanément de la forme p, q, $a - b\nu$ peut être diminué en diminuant ou a ou b, ou tous les deux.

Si $\frac{p_e}{q_e}$ est une convergente de ν du rang e,

$\frac{p_i}{q_i}$ une autre convergente de ν du rang i,

1°. Si $a = p_e$, $b = q_i$, si $i > e$, il découle de la conclusion B, que $(p_e - q_e\nu)^2$ sera plus petit que $(p_e - q_i\nu)^2$, et de même si $e > i$, $(p_i - q_i\nu)^2$ sera plus petit que $(p_e - q_i\nu)^2$, et conséquemment $p_e - q_i\nu$ diminue en diminuant ou p_e ou q_i.

2°. Si l'une au moins des suppositions faites en 1° n'a pas lieu, par exemple si a tombe entre p_e et p_{e+1}, en vertu de la conclusion A, $(p_e - b\nu)$ sera plus petit que $a - b\nu$, et de même si b tombe entre q_i et q_{i+1}, $(a - q_i\nu)$ sera plus petit que $a - b\nu$.

Donc, à moins que $a = p_e$, $b = q_e$, $(a - b\nu)$ ne sera pas un minimum.

La conclusion A, quoiqu'elle n'ait pas été formellement énoncée par M. Hermite, était contenue implicitement, je dois le dire, dans une belle démonstration du théorème de Lagrange fondée sur d'autres principes et que M. Hermite a bien voulu me communiquer il y a un an ou deux.

45.

NOTE ON THE NUMBERS OF BERNOULLI AND EULER, AND A NEW THEOREM CONCERNING PRIME NUMBERS.

[*Philosophical Magazine*, XXI. (1861), pp. 127—136.]

FOLLOWING the accepted *Continental* notation, I denote by B_n* the positive value of the coefficient of t^{2n} in $\frac{t}{1-e^t}$, multiplied by the continual product $1 . 2 . 3 \dots 2n$.

The law which governs the fractional part of B_n was first given in Schumacher's *Nachrichten*, by Thomas Clausen in 1840; and almost immediately afterwards a demonstration was furnished by Professor Staudt in *Crelle's Journal*, with a reclamation of priority, supported by a statement of his having many years previously communicated the theorem to Gauss.

The law is this, that the positive or negative fractional residue of B_n (according as n is odd or even) is made up of the simple sum of the reciprocals of all the prime numbers which, respectively diminished by unity, are contained in $2n$. The proof, which is of an inductive kind, is virtually as follows: Suppose the law holds good up to $(n-1)$ inclusive; if we expand† Σx^{2n} under the form $\frac{1}{e^{\frac{d}{dx}}-1}x^{2n}$, we shall evidently obtain $\frac{\Sigma x^{2n}}{x} \pm B_n$ under the form of a finite series, of which the terms are numerical multiples of the products of powers of x by the Bernoullian numbers of an order inferior to the nth. If, now, we make x equal to the product of all the primes which, diminished by unity, are contained in $2n$, it will at once be

* Were it not for the general usage being as stated in the text, I certainly think it would be far more convenient to use a notation agreeing with the Continental method as to sign, and nearly, but not quite, with Mr De Morgan's as to quantity, namely, to understand by B_n the coefficient of t^n in $\frac{1}{2}t\frac{e^t+1}{e^t-1}$ taken positively, so that B_n should be equal to zero for all the odd values of n, not excepting $n=1$.

[† Σx^{2n} denotes $1^{2n}+2^{2n}+\dots+(x-1)^{2n}$. Cf. p. 227.]

seen (on inspection of the series) that all its terms become integer numbers, and consequently $\frac{\Sigma x^{2n}}{x} \pm B_n$ becomes an integer; and therefore the law will hold good up to n, since it may easily be shown, by an application of Fermat's theorem and elementary arithmetical considerations, that if N be the product of any prime numbers whatever, and if p is the general name of such of them as diminished by unity are factors of μ, then $\frac{\Sigma N^\mu}{N} + \Sigma\frac{1}{p}$ is an integer.

Hence, since the law holds good for $n=1$, it is universally true. This theorem, then, of Staudt and Clausen, *inter alia*, gives a rule for determining what primes alone enter into the denominators of the Bernoullian numbers when expressed as fractions in their lowest terms; it enables us to affirm that only simple powers of primes enter into those denominators, and to know *à priori* what those prime factors are. This note is intended to supply a law concerning the *numerators* of the Bernoullian numbers, which I have not seen stated anywhere, and which admits of an instantaneous demonstration, *to wit*, that the whole of n will appear in the numerator of B_n, save and except such primes, or the powers of such primes, as we know by the Staudt-Clausen law must appear in the denominator.

I am inclined to believe that this law of mine was not known, at all events, in 1840, from the circumstance that in Rothe's Table, published by Ohm in *Crelle's Journal* in that year, which gives the values of B_n up to $n=31$, the numerators are, with one exception (about to be named), all exhibited in such a form as to show such low factors as readily offer themselves, but for B_{23} the fact of the divisibility of the numerator by 23 is not indicated. This numerator is 596451111593912163277961, which in fact $= 23 \times 25932657025822267968607$. It is obvious, indeed, under my law, that whenever p is a prime number other than 2 and 3, the numerator of B_p must contain p, because in such case $p-1$ cannot be a factor of $2p$. When $p=3$ or $p=2$, $2p$ always contains $(p-1)$, so that 2 and 3 are necessarily constant factors of the Bernoullian denominators, and can therefore never appear in the numerators. In Schumacher the law of the denominator is given as "a passing" (or *chance*?) "specimen" of a promised memoir by Clausen on the Bernoullian numbers, as to which I shall feel obliged if any of the readers of this *Magazine* will inform me whether it has appeared anywhere, and if so, where. Now for my demonstration of the law of the numerators.

By definition, $B_n = \Pi(2n) \times$ coefficient of t^{2n-1} in $\frac{1}{e^t-1}$. Let μ be any integer number; then $\pm(\mu^{2n}-1)B_n = \Pi(2n) \times$ coefficient of t^{2n-1} in

$$\frac{\mu}{e^{\mu t}-1} - \frac{1}{e^t-1},$$

or in
$$\frac{(\mu-1)-(e^{(\mu-1)t}+e^{(\mu-2)t}+\ldots+e^{t})}{e^{\mu t}-1},$$

or in
$$-\frac{e^{(\mu-2)t}+2e^{(\mu-3)t}+\ldots+(\mu-2)e^{t}+(\mu-1)}{e^{(\mu-1)t}+e^{(\mu-2)t}+\ldots+e^{t}+1}.$$

But obviously, by Maclaurin's theorem, the coefficient of t^{2n-1} in the expansion of this last generating function will be of the form $\pm\frac{1}{\Pi(2n-1)}\cdot\frac{I}{\mu^{2n-1}}$, where I is an integer, and therefore B_n will be of the form $\frac{2nI}{\mu^{2n-1}(\mu^{2n}-1)}$.

Suppose now, when $\frac{2nI}{\mu^{2n-1}(\mu^{2n}-1)}$ is reduced to its lowest terms, that p (a prime contained in $2n$) does not appear in the numerator, this can only happen by virtue of p being contained in $\mu^{2n-1}(\mu^{2n}-1)$; let now μ be taken successively $2, 3, 4, \ldots (p-1)$, then $\mu^{2n}-1$ in all these cases is divisible by p; and therefore, by an obvious inverse of Fermat's theorem, $(p-1)$ must be contained in $2n$, that is, p must be a factor of the denominator of B_n under the Staudt-Clausen law, which proves my theorem.

As a corollary to the foregoing, using Herschel's transformation, we see that if μ be taken any integer whatever,

$$\pm B_n=\frac{2n}{\mu^{2n}-1}\cdot\frac{(1+\Delta)^{\mu-2}+2(1+\Delta)^{\mu-3}+\ldots+(\mu-1)}{\Delta^{\mu-1}+\mu\Delta^{\mu-2}+\mu\frac{\mu-1}{2}\Delta^{\mu-3}+\ldots+\mu}0^{2n}$$

$$=\frac{2n}{\mu^{2n}-1}\frac{\Delta^{\mu-2}+\mu\Delta^{\mu-3}+\mu\frac{\mu-1}{2}\Delta^{\mu-4}+\ldots+\mu\frac{\mu-1}{2}}{\Delta^{\mu-1}+\mu\Delta^{\mu-2}+\mu\frac{\mu-1}{2}\Delta^{\mu-3}+\ldots+\mu\frac{\mu-1}{2}\Delta+\mu}0^{2n};$$

and if we write 0^{2n+1} instead of 0^{2n}, the result vanishes. For the case of $\mu=2$, this theorem accords with one well known. As this subject is so intimately related to that of the Herschelian differences of zero, I may take this occasion of stating a proposition concerning the latter, which (simple as it is) appears to have escaped observation, namely, that $\frac{\Delta^r 0^{n+r}}{\Pi(r)}$ is in fact the expression for the sum of the homogeneous products of the natural numbers from 1 to r, taken n together. For

$$\frac{1}{(x-r)(x-r+1)\ldots(x-1)x}$$
$$=\frac{1}{\Pi(r)}\left\{\frac{1}{x-r}-\frac{r}{x-r+1}+\frac{r\cdot\frac{r-1}{2}}{x-r+2}\ldots\pm\frac{1}{x}\right\}.$$

Hence obviously

$$\frac{1}{\Pi(r)}\left\{r^n-r(r-1)^n+r\cdot\frac{r-1}{2}(r-2)^n\mp\&c.\right\},$$

that is

$$\frac{\Delta^r 0^n}{\Pi(r)}=\text{coefficient of }\frac{1}{x^n}\text{ in }\frac{1}{(x-r)(x-r+1)\ldots(x-1)}$$

$=$ the sum of the $(n-r)$ary homogeneous products of $1, 2, 3, \ldots r$.

Thus, then, we are able to affirm, from what is known concerning $\frac{\Delta^r 0^{r+n}}{\Pi r}$ (see Prof. De Morgan's *Calculus*), that the r-ary homogeneous product-sum of $1, 2, 3, \ldots n$ (which is of the degree $2r$ in n) always contains the algebraic factor $n(n+1)\ldots(n+r)$.

Addendum.—Since sending the above to press, I have given some further and successful thought to the Staudt-Clausen theorem. Staudt's demonstration labours under the twofold defect of indirectness and of presupposing a knowledge of the law to be established. In it the Bernoullian numbers are not made the subject of a direct contemplation, but are regarded through the medium of an alien function, one out of an infinite number, in which they are as it were latently embodied; and the proof, like all other inductive ones, whilst it convinces the judgment, leaves the philosophic faculty unsatisfied, inasmuch as it fails to disclose the reason (the title, so to say, to existence) of the truth which it establishes. I present below an immediate and a direct proof of this beautiful and important proposition, founded upon the same principle as gives the law of the necessary factor in the numerators (namely, the arbitrary decomposition of the generating function of Bernoulli's numbers into partial fractions), and resting upon a simple but important conception, that of *relative* as distinguished from absolute integers.

I generalize this notion, and define a quantity to be an integer relative to r (or, for brevity's sake, to be an rth integer) when it may be represented by a fraction of which the denominator does not contain r.

The lemma* upon which my demonstration rests is the following, which

* This lemma is the converse of a self-evident fact, and it virtually embodies a principle respecting an arithmetical fraction strikingly analogous to a familiar one respecting an algebraical one; namely, in the same way as a rational algebraical function of x can be expressed *in one, and only one, way* as an integral function augmented by a sum of negative powers of linear functions of x, so a rational arithmetical quantity can be expressed in one, and only one, way as an integer augmented by the sum of negative powers of simple prime numbers multiplied respectively by numbers less than such primes. In drawing this parallel, the arithmetical quantity $\frac{c}{p^i}$, where $c<p$, is regarded as the analogue of the algebraical one $\frac{1}{(ax+b)^i}$, as is quite

is itself an immediate corollary from the arithmetical theorem that if $a, b, c, \dots l$, with or without repetitions, are the distinct prime factors of the denominator of a fraction, the fraction itself may be resolved into the sum of simple fractions,

$$\frac{A}{a^\alpha} + \frac{B}{b^\beta} + \frac{C}{c^\gamma} + \&c. + \frac{L}{l^\lambda}$$

(itself a direct inference from the familiar theorem that if p, q be any two relative primes, the equation $px - qy = c$ is soluble in integers for all values of c). The lemma in question is as follows: If the quantity above described is representable under the several forms,

$$\frac{a'}{a^f} + \text{an } (a\text{th}) \text{ integer}, \quad \frac{b'}{b^g} + \text{a } (b\text{th}) \text{ integer}, \quad \dots \frac{l'}{l^k} + \text{a } (k\text{th}) \text{ integer},$$

then it is equal to

$$\frac{a'}{a^f} + \frac{b'}{b^g} + \dots + \frac{l'}{l^k} + \text{an \textit{absolute} integer}.$$

From what has been already shown, it is obvious that μ being any prime number, the highest power of μ which *can* enter into the denominator of $(\mu^{2n} - 1) B_n$ is μ^{2n}, and consequently $\mu^{2n} B_n$ is an integer relative to μ. Also it is clear that only those values of μ can appear in the denominator of B_n which, diminished by unity, are factors of $2n$. We have, moreover,

$$(-)^{n-1} (\mu^{2n} - 1) B_n = \Pi(2n) \times \text{coefficient of } t^{2n-1} \text{ in } \frac{\mu}{e^{\mu t} - 1} - \frac{1}{e^t - 1},$$

that is, coefficient of t^{2n-1} in $\dfrac{-N}{e^{\mu t} - 1}$, where

$$N = \Pi(2n) \{ e^{(\mu-1)t} + e^{(\mu-2)t} + \dots + e^t - (\mu - 1) \}$$

$$= \nu_1 t + \nu_2 t^2 + \dots\dots + \nu_{2n} t^{2n} + \&c.,$$

where obviously $\nu_1, \nu_2, \dots \nu_{2n}$ are all integers, and the last of them

$$= (\mu - 1)^{2n} + (\mu - 2)^{2n} + \dots + 2^{2n} + 1^{2n}.$$

proper, for both of them are fractions in their simplest forms, which would not be the case for the former were c equal to or greater than p, since in such case $\dfrac{c}{p^i}$ could be more simply expressed under the form $\dfrac{\gamma}{p^{i-1}} + \dfrac{\gamma'}{p^i}$.

This principle amounts to an affirmation that the equation in positive integers,

$$(b \dots kl) x + (ab \dots l) y + \dots + (ab \dots k) t - (ab \dots kl) u = N,$$

where $a, b, \dots k, l$ are relative primes, and $N < (ab \dots kl)$, always admits of a solution, which may be termed the primitive one, and which will be unique, that namely in which $x, y, \dots z, t$ are respectively less than $a, b, \dots k, l$.

Suppose now that $2n$ contains $(\mu - 1)$, then by Fermat's theorem

$$\nu_{2n} \equiv (\mu - 1) \quad [\text{mod } \mu].$$

Again, a very slight consideration* will serve to show that when μ is any prime other than 2, $e^{\mu t} - 1$ is of the form

$$\mu (t + \mu \delta_1 t^2 + \mu \delta_2 t^3 + \dots + \mu \delta_{2n-1} t^{2n} + \&c.),$$

where $\delta_1, \delta_2, \dots \delta_{2n-1} \dots$ are all integers *relative* to μ. Now suppose

$$\frac{\mu N}{e^{\mu t} - 1} = q_0 + q_1 t + q_2 t^2 + \dots + q_{2n-1} t^{2n-1} + \&c.;$$

then by multiplication and comparison of coefficients we obtain the identities following:

$$q_0 = \nu_1, \quad q_1 + \mu q_0 \delta_1 = \nu_2, \quad q_2 + \mu q_1 \delta_1 + \mu q_0 \delta_2 = \nu_3, \dots$$

$$q_{2n-1} + \mu q_{2n-1} \delta_1 + \dots + \mu q_0 \delta_{2n-1} = \nu_{2n};$$

obviously therefore $q_{2n-1} = \mu \times (\text{an integer relative to } \mu) + \nu_{2n}$. Hence

$$(-1)^n B_n = (\text{an integer relative to } \mu) - \frac{\nu_{2n}}{\mu}$$

$$= (\text{an integer relative to } \mu) + \frac{1}{\mu}.$$

And this relation obtains for any value of μ other than 2, which (or a power of which) *could* be contained in $2n$. When $\mu = 2$, the δ series will *not* all of them be the doubles of relative integers to 2; but the ν series, on account of the factor $\Pi(2n)$, will obviously, up to ν_{2n-1} inclusive, all contain 2 and $\nu_{2n} = 1$; consequently q_{2n} will be twice (an integer *quâ* 2) + 1, and B_n will

* For μ being a prime number greater than 2, if we put $\dfrac{\mu^r}{\Pi(r)}$ (the coefficient of t^r in $e^{\mu t} - 1$) under the form of (an integer *quâ* μ) $\times \mu^i$, we have

$$i = r - E\left(\frac{r}{\mu}\right) - E\left(\frac{r}{\mu^2}\right) - E\left(\frac{r}{\mu^3}\right) - \&c.$$

$$= \text{ or } > r - \frac{r}{\mu - 1} = \text{ or } > \frac{r}{2} > 1 \text{ when } r > 2; \text{ also when } r = 2, \ i = 2 - E\left(\frac{2}{\mu}\right) = 2.$$

When $\mu = 2$, this would be no longer true; and in fact it is easily seen that in this case, whenever r is a power of 2, i will be only equal to 1.

For the benefit of my younger readers, I may notice that the *direct* proof of the theorem that the product of any r consecutive numbers must contain the product of the natural numbers up to r, or, in other words, that the trinomial coefficient $\dfrac{\Pi n}{\Pi \nu \Pi \nu'}$, where $\nu + \nu' = n$, is an integer, is drawn from the fact that this fraction may be represented as an integer *quâ* μ (any prime) multiplied by μ^i, where

$$i = \left[E\left(\frac{n}{\mu}\right) - E\left(\frac{\nu}{\mu}\right) - E\left(\frac{\nu'}{\mu}\right)\right] + \left[E\left(\frac{n}{\mu^2}\right) - E\left(\frac{\nu}{\mu^2}\right) - E\left(\frac{\nu'}{\mu^2}\right)\right] + \&c.$$

($E(x)$ meaning the integer part of x), so that i is necessarily either zero or positive, because the value of each triad of terms within the same parenthesis is essentially zero or positive. This is the natural and only direct procedure for establishing the proposition in question.

still be (an integer relative to μ) $+\frac{1}{\mu}$ as before. Hence it follows from the lemma that $(-1)^n B_n =$ an absolute integer $+ \Sigma \frac{1}{\mu}$, or

$$B_n = \text{an integer} + (-)^n \Sigma \frac{1}{\mu},$$

which is the equation expressed by the Staudt-Clausen theorem*.

My researches in the theory of partitions have naturally invested with a new and special interest (at least for myself) everything relating to the Bernoullian numbers. I am not aware whether the following expression for a Bernoullian of any order as a quadratic function of those of an inferior order happens to have been noticed or not. It may be obtained by a simple process of multiplication, and gives a means (not very expeditious, it is true) for calculating these numbers from one another without having recourse to the calculus of differences or Maclaurin's theorem, namely

$$-\frac{B_n}{\Pi(2n)} = (2^2-1)\frac{B_1}{\Pi(2)}\cdot\frac{B_{n-1}}{\Pi(2n-2)} + (2^4-1)\frac{B_2}{\Pi(4)}\cdot\frac{B_{n-2}}{\Pi(2n-4)}$$

$$+ \&c. \ldots + (2^{2n-4}-1)\frac{B_{n-2}}{\Pi(2n-4)}\cdot\frac{B_2}{\Pi(4)}$$

$$+ (2^{2n-2}-1)\frac{B_{n-1}}{\Pi(2n-2)}\cdot\frac{B_1}{\Pi(2)},$$

in which formula the terms admit of being coupled together from end to end, excepting (when n is even) one term in the middle.

To illustrate my law respecting the numerators of the numbers of Bernoulli, and its connexion with the known law for the denominators, suppose twice the index of any one of these numbers to contain the factor $(p-1)p^i$, where p is any prime; then this number will contain the first power of p in its denominator; but if the factor p^i is contained in double the index in question, but $(p-1)$ *not*, then p^i will appear bodily as a factor of the numerator.

* I ought to observe that in all that has preceded I have used the word *integer* in the sense of positive or negative integer, and the demonstration I have given holds good without assuming B_n to be positive. That this is the case, or, in other words, that the signs of the successive powers in $\frac{e^t-1}{e^t+1}$ are alternately positive and negative, may be seen at a glance by putting $t = 2\sqrt{(-1)}\,\theta$, and remembering that all the coefficients in the series for $\tan\theta$ in terms of θ are necessarily positive, because $\left(\frac{d}{d\theta}\right)^i \tan\theta$ obviously only involves positive multiples of powers of $\tan\theta$ and $\sec\theta$.

It has occurred to me that it might be desirable to adhere to the common definition of "*Bernoulli's numbers*," but at the same time to use the term Bernoulli's *coefficients* to denote the actual coefficients in $\frac{e^t+1}{2(e^t-1)}$; so that if the former be denoted in general by B_n and the latter by β_n, we shall have

$$\beta_{2n} = (-)^{n-1} B_n,$$

$$\beta_{2n+1} = 0.$$

In the absence of some such term as I propose, many theorems which are really single when affirmed of the *coefficients*, become duplex or even multifarious when we are restrained to the use of the *numbers* only.

Postscript.—The results obtained concerning Bernoulli's numbers in what precedes, admit of being deduced still more succinctly; and this simplification is by no means of small importance, as it leads the way to the discovery of analogous and unsuspected properties of Euler's numbers (namely the coefficients of $\frac{\theta^{2n}}{\Pi(2n)}$ in the expansion of $\sec\theta$), and to some very remarkable theorems concerning prime numbers in general.

In fact, to obtain the laws which govern the denominators and numerators of Bernoulli's numbers, we need only to use the following principles:—(1) That μ being a prime*, $\Sigma\mu^n \equiv 0$, or $\equiv -1$ to the modulus μ, according as $\mu-1$ is not, or is, a factor of n,—the second part of this statement being a direct consequence of Fermat's theorem, the first part a simple inference from its inverse. (2) That $e^{\mu t}-1$ is of the form $\mu t + \mu^2 t^2 T$, where T is a series of powers of t, all of whose coefficients are integers relative to μ, except for the case of $\mu = 2$, when $e^{\mu t}-1$ is of the form $2t + 2t^2 T$. We have then $(\mu^{2n}-1)(-1)^n B_n = \Pi(2n)$. coefficient of t^{2n-1} in $\frac{e^{(\mu-1)t}+e^{(\mu-2)t}+\ldots+e^t-(\mu-1)}{e^{\mu t}-1}$; this by actual division (in virtue of principle (2)) $= I + \frac{R}{\mu}$, where I is an integer relative to μ, containing n, and $R = 1^{2n} + 2^{2n} + \ldots + (\mu-1)^{2n}$. Hence $(-)^n B_n =$ an integer relative to μ, or to such integer $+\frac{1}{\mu}$, according as $2n$ does not or does contain $(\mu-1)$, which proves the law for the numerators; and so if μ^i is a factor of n, but $(\mu-1)$ not a factor of $2n$, $\frac{R}{\mu}$ will vanish, and $\mu^{2n}-1$ will not contain μ; hence $(\mu^{2n}-1)B_n$, and consequently B_n will be the product of μ^i by an integer relative to μ, which proves my numerator law.

[* $\Sigma\mu^n$ denotes $1^n + 2^n + \ldots + (\mu-1)^n$.]

So by extending the same method to the generating function $\frac{1}{e^t + \sqrt{(-1)}}$, it may very easily be proved that if we write

$$\sec\theta = E_0 + E_1\frac{\theta^2}{1\,.\,2} + E_2\frac{\theta^4}{1\,.\,2\,.\,3\,.\,4} + \ldots + E_n\frac{\theta^{2n}}{1\,.\,2\,.\,3\ldots 2n} + \&c.,$$

every prime number μ of the form $4n+1$, such that $(\mu - 1)$ is a factor of $2n$, will be contained in E_n; and every such prime, when of the form $4n-1$, will be contained in $E_n + 2\,(-)^{n-1}$.

I call the numbers $E_1, E_2, \ldots E_n$ Euler's 1st, 2nd, ... nth numbers, as Euler was apparently the first to bring them into notice. In the *Institutiones Calculi Diff.* he has calculated their values up to E_9 inclusive: in this last there is an error, which is specified by Rothe in Ohm's paper above referred to; had Euler been possessed of my law this mistake could not have occurred, as we know that $E_9 + 2$ ought to contain the factors 19 and 7, neither of which will be found to be such factors if we adopt Euler's value of E_9, but both will be such if we accept Rothe's corrected value. But in still following out the same method, I have been led, through the study of Bernoulli's and the allied numbers, and with the express aid of the former, to a perfectly general theorem concerning prime numbers, in which Bernoulli's numbers no longer take any part. Fermat's theorem teaches us the residue of $r^{\mu-1}$ in respect to μ, namely, that it is unity; but I am not aware of any theorem being in existence which teaches anything concerning the relation of $\frac{r^{\mu-1}-1}{\mu}$ to μ (or, which is the same thing, of the relation of $r^{\mu-1}$ to the modulus μ^2). I have obtained remarkable results relative to the above quotient, which I will state for the simplest case only, namely, that where r as well as μ is a prime number. I find that when r is any odd prime,

$$\frac{r^{\mu-1}-1}{\mu} \equiv \frac{c_1}{\mu-1} + \frac{c_2}{\mu-2} + \frac{c_3}{\mu-3} + \ldots + \frac{c_{\mu-1}}{1}, \text{ (to mod. } \mu),$$

where $c_1, c_2, c_3, \ldots c_{\mu-1}$ are continually recurring cycles of the numbers $1, 2, 3, \ldots r$, the cycle beginning with that number r' which satisfies the congruence $\mu r' \equiv 1$ (mod. r). Since we know that

$$\frac{1}{\mu-1} + \frac{1}{\mu-2} + \frac{1}{\mu-3} + \ldots + \frac{1}{1} \equiv 0 \text{ (to mod. } \mu)$$

in place of the cycle $1, 2, 3, \ldots r$, we may obviously substitute the reduced cycle

$$-\frac{r-1}{2},\; -\frac{r-3}{2},\; \ldots -1, 0, 1, \ldots \frac{r-3}{2},\; \frac{r-1}{2}.$$

Thus*, for example, $\frac{3^{\mu-1}-1}{\mu}$, when μ is of the form $6n+1$,

$$\equiv \frac{1}{\mu-1} - \frac{1}{\mu-3} + \frac{1}{\mu-5} - \frac{1}{\mu-7} \ldots + 1, \text{ (to mod. } \mu),$$

and when μ is of the form $6n-1$,

$$\equiv \frac{-1}{\mu-2} + \frac{1}{\mu-3} - \frac{1}{\mu-4} + \frac{1}{\mu-5} \ldots - 1, \text{ (to mod. } \mu).$$

When r is 2, the theorem which replaces the preceding is as follows†:

$$\frac{2^{\mu-1}-1}{\mu},$$

when μ is of the form $4n+1$,

$$\equiv \frac{1}{\mu-1} + \frac{1}{\mu-2} - \frac{1}{\mu-3} - \frac{1}{\mu-4} + \frac{1}{\mu-5} + \frac{1}{\mu-6}$$

$$- \frac{1}{\mu-7} - \frac{1}{\mu-8} + \frac{1}{\mu-9} \pm \&c., \text{ (to mod. } \mu),$$

and when μ is of the form $4n-1$,

$$\equiv -\frac{1}{\mu-1} + \frac{1}{\mu-2} + \frac{1}{\mu-3} - \frac{1}{\mu-4} - \frac{1}{\mu-5}$$

$$+ \frac{1}{\mu-6} + \frac{1}{\mu-7} \mp \&c., \text{ (to mod. } \mu).$$

When r is not a prime, a similar theorem may be obtained by the very same method, but its expression will be less simple. The above theorems would, I think, be very noticeable were it only for the circumstance of their involving (as a condition) the primeness as well of the base as of the augmented index of the familiar Fermatian expression $r^{\mu-1}$,—a condition which here makes its appearance in the theory of numbers (as I believe) for the first time.

[* Cf. the formulae at the top of p. 230 above. The second of these had originally a wrong sign throughout, but has been corrected, after a sentence inserted by the author at the end of the paper 40 above (p. 241), not reproduced here.]

[† The sign of every term in the two following expressions should be changed.]

46.

NOTE ON THE HISTORICAL ORIGIN OF THE UNSYMMETRICAL SIX-VALUED FUNCTION OF SIX LETTERS.

[*Philosophical Magazine*, XXI. (1861), pp. 369—377.]

THE discovery and first announcement of the existence of the celebrated function of six letters having six values, and not symmetrical in respect to all the letters, is usually assigned to my illustrious friend M. Hermite, to whom M. Cauchy expressly ascribes it in a memoir inserted in the *Comptes Rendus* of the Institut for December 8, 1845, p. 1247, and again, January 5, 1846, p. 30.

M. Cauchy adds that the conversation he held with M. Hermite on this subject excited in himself a lively desire to sound to its depths the question of permutations, and to develope the consequences to be deduced from the application of the principles relative thereto, which he had himself long previously laid down.

I was not at that date in the habit of consulting the *Comptes Rendus*, or I should at once have made the reclamation of priority which I now do, not from any unworthy motive of self-love in so small a matter, but out of regard to historic truth. It is a year or two since I first learnt that the origin of this function was usually referred to M. Cauchy or M. Hermite; but although aware that its existence was known to myself long previous to the dates quoted, I did not recollect that I had ever communicated it to the world through the medium of the press, and I therefore kept silence on the subject.

Turning over, a few days ago, for another purpose, the pages of a back volume of this *Magazine*, my eye chanced to alight on a footnote to a paper of my own inserted therein*, under date of April 1844, "On the Principles of Combinatorial Aggregation," which I will take the liberty of quoting at length, as it proves incontestably the priority which I lay claim to.

[* p. 92 of Vol. I. of this Reprint.]

"When the modulus is four, there is only one synthematic arrangement possible, and there is no indeterminateness of any kind; from this we can infer, *à priori*, the reductibility of a biquadratic equation; for using ϕ, f, F to denote rational symmetrical forms of function, it follows that

$$F\begin{Bmatrix} f[\phi(a, b), \phi(c, d)] \\ f[\phi(a, c), \phi(b, d)] \\ f[\phi(a, d), \phi(b, c)] \end{Bmatrix} \text{ is itself a rational symmetric function of } a, b, c, d.$$

Whence it follows that if a, b, c, d be the roots of a biquadratic equation, $f[\phi(a, b), \phi(c, d)]$ can be found by the solution of a cubic: for instance, $(a+b)\times(c+d)$ can be thus determined, whence immediately the sum of any two of the roots comes out from a quadratic equation.

"To the modulus 6 there are fifteen different synthemes capable of being constructed. At first sight it might be supposed that these could be classed in natural families of three or of five each, on which supposition the equation of the sixth degree could be depressed; but on inquiry this hope will prove to be futile, not but what natural affinities do exist between the totals; but in order to separate them into families, each will have to be taken twice over; or in other words, the fifteen synthemes to modulus 6 being reduplicated, subdivide into six natural families of five each."

The six families above referred to (in which it is to be understood that $p.q$ and $q.p$ are identical in effect) are the following:—

$a.b$	$c.d$	$e.f$	$a.c$	$d.e$	$f.b$	$a.d$	$e.f$	$b.c$
$a.c$	$b.e$	$d.f$	$a.d$	$c.f$	$e.b$	$a.e$	$d.b$	$f.c$
$a.d$	$b.f$	$c.e$	$a.e$	$c.b$	$d.f$	$a.f$	$d.c$	$e.b$
$a.e$	$b.d$	$c.f$	$a.f$	$c.e$	$d.b$	$a.b$	$d.f$	$e.c$
$a.f$	$b.c$	$d.e$	$a.b$	$c.d$	$e.f$	$a.c$	$d.e$	$f.b$
$a.e$	$f.b$	$c.d$	$a.f$	$b.c$	$d.e$	$a.b$	$c.d$	$e.f$
$a.f$	$e.c$	$b.d$	$a.b$	$f.d$	$c.e$	$a.c$	$b.e$	$d.f$
$a.b$	$e.d$	$f.c$	$a.c$	$f.e$	$b.d$	$a.d$	$b.f$	$c.e$
$a.c$	$e.b$	$f.d$	$a.d$	$f.c$	$b.e$	$a.e$	$b.d$	$c.f$
$a.d$	$e.f$	$b.c$	$a.e$	$f.b$	$c.d$	$a.f$	$b.c$	$d.e$

And it will be observed that every two families have one, and only one, syntheme in common between them; and precisely in the same way as in the note above quoted it is especially shown that the one single natural family

$$\begin{vmatrix} a.b & c.d \\ a.c & b.d \\ a.d & b.c \end{vmatrix}$$

gives rise to a function of four letters with only one value, so the six functions analogously formed with these six families obviously give rise to six functions, which change into one another when any interchange is effected between the letters which enter into them; so that any one of these is a function of six letters having only six values. I conceive that, after this reference, no writer on the subject wishing to specify the function in question would hesitate to call it after my name.

I may also take occasion to observe that, in connexion with my researches in combinatorial aggregation, long before the publication of my unfinished paper in the *Magazine*, I had fallen upon the question of forming a heptatic aggregate of triadic synthemes comprising all the duads to the base 15, which has since become so well known, and fluttered so many a gentle bosom, under the title of the fifteen school-girls' problem; and it is not improbable that the question, under its existing form, may have originated through channels which can no longer be traced in the oral communications made by myself to my fellow-undergraduates at the University of Cambridge long years before its first appearance, which I believe was in the *Ladies' Diary* for some year which my memory is unable to furnish.

In order to relieve this notice from the mere personal character which it may thus far appear to bear, I will state another question concerning the combinatorial aggregation of fifteen things which may serve as a pendant to the famous school-girl problem.

The number of triads to the base 15 is $\frac{15 \times 14 \times 13}{3 \, . \, 2 \, . \, 1} = 5 \times 91$. Let it be required to arrange these into 91 synthemes, in other words, to set out the walks of 15 girls for 91 days (say a quarter of the year) in such a manner that the same three shall never *all* come together more than once in the quarter. Of the various ways in which it is probable this problem may be solved, the following deserves notice. Let 15 letters be arbitrarily divided into 5 sets, namely,

$$a_1b_1c_1; \quad a_2b_2c_2; \quad a_3b_3c_3; \quad a_4b_4c_4; \quad a_5b_5c_5.$$

The sets as they stand will represent one of the 91 arrangements sought for, which I call the basic syntheme. The remaining 90 may be obtained as follows in 10 batches of 9 each. Write down the 10 index distributions following:—

1 2 3; 4 5	1 4 5; 2 3
1 2 4; 3 5	2 3 4; 1 5
1 2 5; 3 4	2 3 5; 1 4
1 3 4; 2 5	2 4 5; 1 3
1 3 5; 2 4	3 4 5; 1 2.

Take any one of these distributions, as for instance 2 3 5; 1 4, and proceed

as follows:—In respect of 2, 3, 5, conjugate the three sets $\begin{matrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_5 & b_5 & c_5 \end{matrix}$; and in respect of 1, 4, conjugate the two remaining sets $\begin{matrix} a_1 & b_1 & c_1 \\ a_4 & b_4 & c_4 \end{matrix}$.

From the ternary conjugation form the nine arrangements,

	$a_2\ a_3\ a_5$	$b_2\ b_3\ b_5$	$c_2\ c_3\ c_5$
	$a_2\ a_3\ b_5$	$b_2\ b_3\ c_5$	$c_2\ c_3\ a_5$
	$a_2\ a_3\ c_5$	$b_2\ b_3\ a_5$	$c_2\ c_3\ b_5$
	$a_2\ b_3\ a_5$	$b_2\ c_3\ b_5$	$c_2\ a_3\ c_5$
	$a_2\ b_3\ b_5$	$b_2\ c_3\ c_5$	$c_2\ a_3\ a_5$
	$a_2\ b_3\ c_5$	$b_2\ c_3\ a_5$	$c_2\ a_3\ b_5$
	$a_2\ c_3\ a_5$	$b_2\ a_3\ b_5$	$c_2\ b_3\ c_5$
	$a_2\ c_3\ b_5$	$b_2\ a_3\ c_5$	$c_2\ b_3\ a_5$
	$a_2\ c_3\ c_5$	$b_2\ a_3\ a_5$	$c_2\ b_3\ b_5$,
which call	$L_1\ L_2\ L_3$	$L_4\ L_5\ L_6$	$L_7\ L_8\ L_9$.

Again, from the binary conjugation, form the nine arrangements,

	$a_1\ b_1\ c_4$	$a_4\ b_4\ c_1$
	$a_1\ b_1\ b_4$	$a_4\ c_1\ c_4$
	$a_1\ b_1\ a_4$	$c_1\ b_4\ c_4$
	$a_1\ c_1\ c_4$	$a_4\ b_4\ b_1$
	$a_1\ c_1\ b_4$	$a_4\ b_1\ c_4$
	$a_1\ c_1\ a_4$	$b_1\ b_4\ c_4$
	$b_1\ c_1\ c_4$	$a_4\ b_4\ a_1$
	$b_1\ c_1\ b_4$	$a_4\ a_1\ c_4$
	$b_1\ c_1\ a_4$	$a_1\ b_4\ c_4$,

which call $\quad M_1\ M_2\ M_3 \quad M_4\ M_5\ M_6 \quad M_7\ M_8\ M_9$.

Now combine the L with the M system, each L with some M in any order whatever; the 9 combinations or appositions thus obtained will give a batch of 9 synthemes; and proceeding in like manner with each of the 10 distributions of the indices 1, 2, 3, 4, 5, we shall obtain 90 synthemes, which together with the basic syntheme complete the system required. The M system corresponding to any distribution of the indices is the system which contains the synthematic arrangement of the bipartite* triads which can be constituted out of six things, separated in two sets or parts, and is unique. The L system is *one* of those which represents the synthematic arrangement

* See note at end of paper.

of the tripartite* triads of nine things separated into three sets or parts. I have set out above one in particular of these for the sake of greater clearness; but any other system having the same property will serve the same purpose, and a careful study will serve to show that the total number of *L*'s corresponding to a given distribution of indices will be ( )*. Consequently the total number of *LM*'s that we can form for a given distribution will be ( ) $\times 1.2.3.4.5.6.7.8.9$; and the number of *distinct* synthematic arrangements satisfying the given conditions corresponding to any assumed basic syntheme will be this number raised to the tenth power; and as this vastly exceeds the total number of permutations of fifteen things, we see, without even taking into consideration the diversity that may be produced by a change of the base, that this method must give rise to many distinct types of solution (arrangements being defined to belong to the same or different types, according as they admit or not of being deduced from each other by a permutation effected among their monadic elements). The common character of all these allotypical aggregations, and which serves to constitute them into a natural order or family, consists in their being derived from a base formed out of five sets, such that the monopartite triads corresponding to the base form one syntheme, and the other 90 synthemes each contain a conjugation of the tripartite triads belonging to three out of the five sets of the base with the bipartite triads belonging to the other two sets thereof. There is, moreover, no reason to suppose, or at all events no safe ground for affirming, that this family exhausts the whole possible number of types to which the arrangements satisfying the proposed condition admit of being reduced. A further question which I have somewhere raised, and which brings the two problems of the school-girls into *rapport*, is the following:—"To divide the system of 91 synthemes satisfying the conditions above stated into thirteen minor systems, each of which satisfies the conditions of the old problem, that is, of containing all the duads that can be made out of the fifteen elements once and once only"; or to put the question in a more exact form, to exhibit thirteen systems, each satisfying this last condition, which shall together include between them all the triads that can be made out of the fifteen elements.

The reader would have reason to be dissatisfied with the author's reticence, were he to leave altogether unmentioned the synthematic aggregation of the *binomial* triads appertaining to the same three triliteral sets or *nomes*; but space forbids my doing more at present than giving one of these aggregates, and indicating the number and mode of generation of all from this one. It will readily be seen that any such aggregate will be made up of two sub-aggregates, which I shall call A and B respectively, of which one bears

* Some day or another a new combinatorial calculus must come into being to furnish general solutions to the infinite variety of questions of *multifariousness* to which the theory of combinatorial aggregation, *alias* compound permutations, gives rise.

the same relation to the disposition of the nomes in the order 123 456 789, as the other to their disposition in the order 123 789 456. Thus we may take for our A and B the following, which will each contain 9 synthemes, the total number of synthemes in the two together being 18*:—

(A)			(B)		
124	567	893	127	894	763
125	468	739	128	795	436
126	459	783	129	786	453
134	568	279	137	895	246
135	469	278	138	796	245
136	457	289	139	784	256
234	569	187	237	896	154
235	467	189	238	794	156
236	458	179	239	785	146

The system of triads contained in A may be arranged in twelve different aggregates similar to the one given, and the same will be true for the triads in the B; so that the total number of the combined systems will be 144. All the permutations which leave A or B (separately considered) unaltered will form a natural group,—the theory of groups in this, as in every other case, standing in the closest relation to the doctrine of combinatorial aggregation, or what for shortness may be termed syntax. I have elsewhere given the general name of Tactic to the third pure mathematical science, of which order is the proper sphere, as is number and space of the other two. Syntax and Groups are each of them only special branches of Tactic. I shall on another occasion give reasons to show that the doctrine of groups may be treated as the arithmetic of ordinal numbers. With respect to the twelve varieties of the A or B aggregates, they may be obtained from the one given by combining the substitutions corresponding to the six permutations of the three constituents of one nome, as 7, 8, 9, with the permutation of any two constituents of another, as 5, 6. But I have said enough for my present purpose, which is to point out the boundless untrodden regions of thought in the sphere of order, and especially in the department of *syntax*, which remain to be expressed, mapped out, and brought under cultivation. The difficulty indeed is not to find material, of which there is a superabundance, but to discover the proper and principal centres of speculation that may serve to reduce the theory into a manageable compass.

* Thus, since there is evidently one mononomial syntheme, the total number of synthemes of all three kinds will be $1+18+9=28=\frac{8\times 7}{2}$, as it should be, the total number of triads being $\frac{9.8.7}{3.2}$, and $\frac{9}{3}$ of them going to a syntheme.

I put on record (as a Christmas offering on the altar of science) for the benefit of those studying the theory of groups, or compound permutations (to which the prize shortly to be adjudicated by the Institute of France for the most important addition to the subject may tend to give a new impulse), and with an eye to the geometrical and algebraical verities with which, as a constant of reason, we may confidently anticipate it is pregnant, an exhaustive table of the monosynthematic aggregates of the trinomial triads that are contained in a system of three triliteral nomes. Let these latter be called respectively 123; 456; 789; then we have the annexed:—

Table of Synthemes of Trinomial Triads to base 3 . 3.

(1)	(2)	(3)	(4)
147 258 369	147 258 369	147 258 369	147 258 369
148 259 367	148 259 367	148 259 367	148 259 367
149 257 368	149 257 368	149 257 368	149 267 358
157 268 349	157 268 349	157 269 348	157 268 349
158 269 347	158 269 347	158 267 349	158 269 347
159 267 348	159 267 348	159 268 347	159 247 368
167 248 359	167 249 358	167 248 359	167 248 359
168 249 357	168 247 359	168 249 357	168 249 357
169 247 358	169 248 357	169 247 358	169 257 348

(5)	(6)	(7)	(8)
147 258 369	147 258 369	147 258 369	147 258 369
148 267 359	148 267 359	148 269 357	148 269 357
149 268 357	149 257 368	149 257 368	149 267 358
157 249 368	157 268 349	157 268 349	157 268 349
158 269 347	158 269 347	158 249 367	158 249 367
159 248 367	159 248 367	159 267 348	159 247 368
167 259 348	167 259 348	167 248 359	167 248 359
168 257 349	168 249 357	168 259 347	168 259 347
169 247 358	169 247 358	169 247 358	169 257 348

The discussion of the properties of this Table, and the classification of the eight aggregates into natural families, must be reserved for a future occasion.

Note.—A triad is called tripartite if its three elements are culled out of three different parts or sets between which the total number of elements is supposed to be divided; bipartite if the elements are taken out of two distinct sets; unipartite if they all lie in the same set. The more ordinary method for the reduction of synthematic arrangements from a given base to a linear one which I employ, consists in the separate synthematization *inter se* of all the combinations of the *same* kind as regards the number of parts

from which they are respectively drawn. Thus, for example, if the distribution of the $\frac{30 \times 29 \times 28}{6}$ triads to the base 30 into $\frac{29 \times 28}{2}$ synthemes be required, this may be effected by dividing the 30 elements in an arbitrary manner into 15 parts, each part containing 2 elements. These 15 parts being now themselves treated as elements, are first to be conjugated as in the old 15-school-girl problem, and each of these 7 conjugations can be made to furnish 6 synthemes containing exclusively bipartite triads. The same 15 parts are then to be conjugated as in the new school-girl problem, and the 91 conjugations thus obtained will each furnish 4 synthemes, containing exclusively the tripartite triads. These bipartite and tripartite synthemes will exhaust the entire number of triads of both kinds, and accordingly we shall find

$$7 \times 6 + 91 \times 4 = 406$$

$$= \frac{29 \times 28}{2}.$$

A syntheme, I need scarcely add, is an aggregate of combinations containing between them all the monadic elements of a given system, each appearing once only. In the more general theory of aggregation, such an aggregate would be distinguished by the name of a monosyntheme. A disyntheme would then signify an aggregate of combinations containing between them the duadic elements, each appearing once only, and so forth. Thus the old 15-school-girl question in my nomenclature would be enunciated under the form of a problem "to construct a triadic disyntheme, separable into monosynthemes to the base 15"; the new school question, as a problem "to divide the whole of the triads to base 15 into monosynthemes"; the question which connects the two, as a problem "to exhibit the whole of the triads to base 15 under the form of 13 disynthemes, each separated into 7 monosynthemes."

A question of a more general kind, and embracing this last, would be the problem of dividing the whole of the same system of triads into 13 disynthemes, without annexing the further condition of monosynthematic divisibility. So there is the simpler question of constructing a single disyntheme to the base 15 without any condition annexed as to its decomposability into 7 synthemes.

47.

ON A PROBLEM IN TACTIC WHICH SERVES TO DISCLOSE THE EXISTENCE OF A FOUR-VALUED FUNCTION OF THREE SETS OF THREE LETTERS EACH.

[*Philosophical Magazine*, XXI. (1861), pp. 515—520.]

AT page 375 of the May Number of this *Magazine** (in that paragraph commencing at the middle of the page) I gave a Table of Synthemes, correct as far as it went, but left in a very imperfect state. It was intended to be supplemented with a material addition which escaped my recollection when, after a long delay, the proofs of the paper passed through my hands. The question to which this Table refers is the following:—

Three *nomes*, each containing three elements, are given; the number of *trinomial* triads (that is, ternary combinations, composed by taking one element out of each nome) will be 27, and these 27 may be *grouped* together into 9 synthemes (each syntheme consisting of 3 of the triads in question, which together include between them all the 9 elements). It is desirable to know:—1st. How many distinct *groupings* of this kind can be formed. 2nd. Whether there is more than one, and, if so, how many distinct types of groupings. The criterion of one grouping being cotypal or allotypal to another is its capability or incapability of being transformed into that other by means of an interchange of elements. Be it once for all stated that the question in hand is throughout one of combinations, and not of permutations; the order of the elements in a triad, of a triad in a syntheme, of a syntheme in a grouping is treated as immaterial. As we are only concerned with the elements as distributed into *nomes*, the number of interchanges of elements with which we are concerned is 6×6^3 or 1296; the factor 6^3 arises from the permutability of the elements of each nome *inter se*, the remaining factor 6 from the permutability of any nome with any other. I find, by a method which carries its own demonstration with it on its face, that the number of distinct *groupings* is 40, of which 4 belong to one *type* or *family*, and 36 to a second type or family.

[* p. 270 above.]

Let the nomes be 1.2.3, 4.5.6, 7.8.9, and let

c_1 denote 1.4, 2.5, 3.6 $\dot{c}_1$ denote 1.4, 2.6, 3.5
c_2 „ 1.5, 2.6, 3.4 $\dot{c}_2$ „ 1.5, 2.4, 3.6
c_3 „ 1.6, 2.4, 3.5 $\dot{c}_3$ „ 1.6, 2.5, 3.4

γ denote 7, 8, 9 / 8, 9, 7 / 9, 7, 8 γ' denote 7, 9, 8 / 9, 8, 7 / 8, 7, 9

b_1 denote 1.7, 2.8, 3.9 $\dot{b}_1$ denote 1.7, 2.9, 3.8
b_2 „ 1.8, 2.9, 3.7 $\dot{b}_2$ „ 1.8, 2.7, 3.9
b_3 „ 1.9, 2.7, 3.8 $\dot{b}_3$ „ 1.9, 2.8, 3.7

β denote 4, 5, 6 / 5, 6, 4 / 6, 4, 5 β' denote 4, 6, 5 / 6, 5, 4 / 5, 4, 6

a_1 denote 4.7, 5.8, 6.9 $\dot{a}_1$ denote 4.7, 5.9, 6.8
a_2 „ 4.8, 5.9, 6.7 $\dot{a}_2$ „ 4.8, 5.7, 6.9
a_3 „ 4.9, 5.7, 6.8 $\dot{a}_3$ „ 4.9, 5.8, 6.7

α denote 1, 2, 3 / 2, 3, 1 / 3, 1, 2 α' denote 1, 3, 2 / 2, 3, 1 / 3, 1, 2.

I take first the larger family of 36 groupings; these may be represented as follows:—

$a_1\alpha$	$a_1\alpha$	$a_1\alpha'$	$a_1\alpha$	$a_1\alpha'$	$a_1\alpha'$	$\dot{a}_1\alpha$	$\dot{a}_1\alpha$	$\dot{a}_1\alpha'$	$\dot{a}_1\alpha$	$\dot{a}_1\alpha'$	$\dot{a}_1\alpha'$
$a_2\alpha$	$a_2\alpha'$	$a_2\alpha$	$a_2\alpha'$	$a_2\alpha$	$a_2\alpha'$	$\dot{a}_2\alpha$	$\dot{a}_2\alpha'$	$\dot{a}_2\alpha$	$\dot{a}_2\alpha'$	$\dot{a}_2\alpha$	$\dot{a}_2\alpha'$
$a_3\alpha'$	$a_3\alpha$	$a_3\alpha$	$a_3\alpha'$	$a_3\alpha'$	$a_3\alpha$	$\dot{a}_3\alpha'$	$\dot{a}_3\alpha$	$\dot{a}_3\alpha$	$\dot{a}_3\alpha'$	$\dot{a}_3\alpha'$	$\dot{a}_3\alpha$
$b_1\beta$	$b_1\beta$	$b_1\beta'$	$b_1\beta$	$b_1\beta'$	$b_1\beta'$	$\dot{b}_1\beta$	$\dot{b}_1\beta$	$\dot{b}_1\beta'$	$\dot{b}_1\beta$	$\dot{b}_1\beta'$	$\dot{b}_1\beta'$
$b_2\beta$	$b_2\beta'$	$b_2\beta$	$b_2\beta'$	$b_2\beta$	$b_2\beta'$	$\dot{b}_2\beta$	$\dot{b}_2\beta'$	$\dot{b}_2\beta$	$\dot{b}_2\beta'$	$\dot{b}_2\beta$	$\dot{b}_2\beta'$
$b_3\beta'$	$b_3\beta$	$b_3\beta$	$b_3\beta'$	$b_3\beta'$	$b_3\beta$	$\dot{b}_3\beta'$	$\dot{b}_3\beta$	$\dot{b}_3\beta$	$\dot{b}_3\beta'$	$\dot{b}_3\beta'$	$\dot{b}_3\beta$
$c_1\gamma$	$c_1\gamma$	$c_1\gamma'$	$c_1\gamma$	$c_1\gamma'$	$c_1\gamma'$	$\dot{c}_1\gamma$	$\dot{c}_1\gamma$	$\dot{c}_1\gamma'$	$\dot{c}_1\gamma$	$\dot{c}_1\gamma'$	$\dot{c}_1\gamma'$
$c_2\gamma$	$c_2\gamma'$	$c_2\gamma$	$c_2\gamma'$	$c_2\gamma$	$c_2\gamma'$	$\dot{c}_2\gamma$	$\dot{c}_2\gamma'$	$\dot{c}_2\gamma$	$\dot{c}_2\gamma'$	$\dot{c}_2\gamma$	$\dot{c}_2\gamma'$
$c_3\gamma'$	$c_3\gamma$	$c_3\gamma$	$c_3\gamma'$	$c_3\gamma'$	$c_3\gamma$	$\dot{c}_3\gamma'$	$\dot{c}_3\gamma$	$\dot{c}_3\gamma$	$\dot{c}_3\gamma'$	$\dot{c}_3\gamma'$	$\dot{c}_3\gamma$

An example of the development of any one of the above symbolisms into its correspondent grouping will serve to render perfectly intelligible the whole Table.

Let it be required to develope

$$\dot{b}_1\beta$$
$$\dot{b}_2\beta'$$
$$\dot{b}_3\beta'.$$

Since

$$\begin{array}{lccc} \dot{b}_1 = 1.7 & 2.9 & 3.8 & \qquad 4,\ 5,\ 6 \qquad\qquad 4,\ 6,\ 5 \\ \dot{b}_2 = 1.8 & 2.7 & 3.9 & \qquad \beta = 5,\ 6,\ 4 \qquad\quad \beta' = 6,\ 5,\ 4 \\ \dot{b}_3 = 1.9 & 2.8 & 3.7 & \qquad 6,\ 4,\ 5 \qquad\qquad 5,\ 4,\ 6, \end{array}$$

the development required is the following:—

$$\left|\begin{array}{ccc} 1.7.4 & 2.9.5 & 3.8.6 \\ 1.7.5 & 2.9.6 & 3.8.4 \\ 1.7.6 & 2.9.4 & 3.8.5 \\ 1.8.4 & 2.7.6 & 3.9.5 \\ 1.8.6 & 2.7.5 & 3.9.4 \\ 1.8.5 & 2.7.4 & 3.9.6 \\ 1.9.4 & 2.8.6 & 3.7.5 \\ 1.9.6 & 2.8.5 & 3.7.4 \\ 1.9.5 & 2.8.4 & 3.7.6 \end{array}\right|.$$

The whole of this family of 36 may be represented under the following condensed form, according to the notation usual in the theory of substitutions.

$$\left(\left| \begin{array}{c} a_1\alpha \\ a_2\alpha \\ a_3\alpha \end{array} \right| \times \begin{pmatrix} 123 & 123 & 123 \\ 123 & 231 & 312 \end{pmatrix} \times \begin{pmatrix} a & \dot{a} \\ a & a \end{pmatrix} \times \begin{pmatrix} \alpha\alpha' & \alpha\alpha' \\ \alpha\alpha' & \alpha'\alpha \end{pmatrix} \times \begin{pmatrix} a\alpha & b\beta & c\gamma \\ a\alpha & a\alpha & a\alpha \end{pmatrix} \right).$$

It remains to describe the principal and most symmetrical family. This contains only 4 groupings, and may be represented indifferently under any of the three following forms:—

$$\begin{array}{lcccc} a_1\alpha\ a_1\alpha'\ \dot{a}_1\alpha\ \dot{a}_1\alpha' & & b_1\beta\ b_1\beta'\ \dot{b}_1\beta\ \dot{b}_1\beta' & & c_1\gamma\ c_1\gamma'\ \dot{c}_1\gamma\ \dot{c}_1\gamma' \\ a_2\alpha\ a_2\alpha'\ \dot{a}_2\alpha\ \dot{a}_2\alpha' & \text{or} & b_2\beta\ b_2\beta'\ \dot{b}_2\beta\ \dot{b}_2\beta' & \text{or} & c_2\gamma\ c_2\gamma'\ \dot{c}_2\gamma\ \dot{c}_2\gamma' \\ a_3\alpha\ a_3\alpha'\ \dot{a}_3\alpha\ \dot{a}_3\alpha' & & b_3\beta\ b_3\beta'\ \dot{b}_3\beta\ \dot{b}_3\beta' & & c_3\gamma\ c_3\gamma'\ \dot{c}_3\gamma\ \dot{c}_3\gamma'. \end{array}$$

In developing, it will be found that each of these three representations gives rise to the *same* family of groupings, which from its importance it is proper to set out in full as follows:—

1.4.7 2.5.8 3.6.9	1.4.7 2.5.9 3.6.8	1.4.7 2.6.8 3.5.9	1.4.7 2.6.9 3.5.8
1.4.8 2.5.9 3.6.7	1.4.9 2.5.8 3.6.7	1.4.8 2.6.9 3.5.7	1.4.9 2.6.8 3.5.7
1.4.9 2.5.7 3.6.8	1.4.8 2.5.7 3.6.9	1.4.9 2.6.7 3.5.8	1.4.8 2.6.7 3.5.9
1.5.7 2.6.8 3.4.9	1.5.7 2.6.9 3.4.8	1.5.7 2.4.8 3.6.9	1.5.7 2.4.9 3.6.8
1.5.8 2.6.9 3.4.7	1.5.9 2.6.8 3.4.7	1.5.8 2.4.9 3.6.7	1.5.9 2.4.8 3.6.7
1.5.9 2.6.7 3.4.8	1.5.8 2.6.7 3.4.9	1.5.9 2.4.7 3.6.8	1.5.8 2.4.7 3.6.9
1.6.7 2.4.8 3.5.9	1.6.7 2.4.9 3.5.8	1.6.7 2.5.8 3.4.9	1.6.7 2.5.9 3.4.8
1.6.8 2.4.9 3.5.7	1.6.9 2.4.8 3.5.7	1.6.8 2.5.9 3.4.7	1.6.9 2.5.8 3.4.7
1.6.9 2.4.7 3.5.8	1.6.8 2.4.7 3.5.9	1.6.9 2.5.7 3.4.8	1.6.8 2.5.7 3.4.9

It follows at once from the above Table, that if 3 cubic equations be given, we may form a function of the 9 roots, which, when any of the roots of any of the equations are interchanged *inter se*, or all the roots of one with all those of any other, will receive only *four distinct values*.

It also follows that we may form with 9 letters an intransitive group (of Cauchy) containing $\frac{216}{4}$, that is, 54, or a transitive group containing $\frac{1296}{4}$, or 324 substitutions. So the family of 36 groupings lead to the formation of an intransitive substitution group of $\frac{216}{12}$, that is, 18, and of a transitive group of $\frac{1296}{36}$, or 36 substitutions.

Since 9 letters may be thrown, in $\frac{8.7}{2} \times \frac{5.4}{2}$, that is, 280 different ways, into nomes of 3 letters each, it further follows that by repeating each of the above two families 280 times we shall obtain new families remaining unaltered by any substitution of any of the nine elements *inter se*, and consequently indicating the existence of substitution-groups containing

$$\frac{1.2.3.4.5.6.7.8.9}{280 \times 36} \text{ and } \frac{1.2.3.4.5.6.7.8.9}{280 \times 4},$$

that is, 36 and 324 substitutions respectively.

In the above solution a little consideration will show that the method is essentially based on the solution of a *previous* question, namely, of grouping together the synthemes of *binomial duads* of *two* nomes of three letters each, which can be done in two distinct modes, which (if, for example, we take 1.2.3, 4.5.6 as the two nomes in question) are represented in the notation used above by $\begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array}$ and $\begin{array}{c} \dot{c}_1 \\ \dot{c}_2 \\ \dot{c}_3 \end{array}$ respectively. So, more generally, the groupings of the q-nomial q-ads of r nomes of s elements may be made to depend on the groupings of the $(q-1)$-nomial $(q-1)$-ads of $(r-1)$ nomes of s elements each. The more general question is to discover the groupings and their families of the synthemes composed of p-nomial q-ads of r nomes of s elements, of which the simplest example next that which has been considered and solved is to discover the groupings of the synthemes composed of 54 *binomial* triads of 3 nomes of 3 elements each*.

The chief difficulty of calculating *à priori* the number of such groupings is of a similar nature to that which lies at the bottom of the ordinary theory of the partition of numbers, namely, the liability of the same groupings to make their appearance under distinct symbolical representations. Of this we have seen an example in the threefold representation of the principal family

* I have ascertained, by a direct analytical method, since the above was written, that the number of different groupings of the synthemes composed of these binomial triads is 144. The number of distinct types or families is *three*, one containing 12, another 24, and the third 108 groupings.

of 4 groupings just treated of. But for the existence of this multiform representation of the same grouping we could have affirmed *à priori* the number of groupings to be $2 \times 3 \times 2^3$ or 48, whereas the true number is only 40. I believe that the above is the first instance of the doctrine of types making its appearance explicitly, and illustrated by example in the theory of tactic. It were much to be desired that some one would endeavour to collect and collate the various solutions that have been given of the noted 15-school-girl problem by Messrs Kirkman (in the *Ladies' Diary*), Moses Ansted (in the *Cambridge and Dublin Mathematical Journal*), by Messrs Cayley and Spottiswoode (in the *Philosophical Magazine* and elsewhere), and Professor Pierce, the latest and probably the best (in the *American Astronomical Journal*), besides various others originating and still floating about in the fashionable world (one, if not two, of which I remember having been communicated to me many years ago by Mr Archibald Smith, F.R.S.), with a view to ascertaining whether they belong to the same or to distinct types of aggregation.

48.

CONCLUDING PAPER ON TACTIC.

[*Philosophical Magazine*, XXII. (1861), pp. 45—54.]

In my tactical paper in the May Number of this *Magazine* [p. 264 above], I considered the number of groupings and of types of groupings of *synthemes* formed out of triads of three *nomes* of three elements each. The first example of considering the *ensemble* of the groupings of a defined species of synthemes (each of such groupings being subjected to satisfy a certain exhaustive condition) was, as already stated, furnished by me in this *Magazine*, April, 1844. In that case the synthemes consisted of duads belonging to a single nome of 6 elements, and the total number of the groupings was observed to be 6, all contained in one type or family. The total number of synthemes in that instance being 15, and there being 6 groupings of 5 synthemes each, it followed that in the whole family every syntheme is met with twice over; once in one grouping, and once in another. In the case treated of in my last communication to this *Magazine*, the total number of the synthemes of the kind under consideration is 36 (for it may easily be shown that the number of synthemes of n-nomial n-ads of n nomes of q elements each is $(1.2.3...q)^{n-1}$); and as each grouping contains 9 synthemes, these 36 are distributed *without repetition* between the 4 groupings of the smaller of the two natural species,—a phenomenon of a kind here met with for the first time in the study of *syntax*. If now we go on (as a natural and irrepressible curiosity urges) to ascertain the groupings of the synthemes of *binomial* triads of the same 3 nomes of three elements each, we advance just one step further in the direction of type-complexity; that is to say, we meet with the existence of 3, and not more than 3, types or species in which all such groupings are comprised. The investigation by which this is made out appears to me well worthy to be given to the world, as affording an example of a new and beautiful kind of analysis proper to the study of *tactic*, and thus lighting the way to the further opening up of this fundamental doctrine of mathematic, the science of necessary relations, of which, combined with logic (if indeed the two be not identical), tactic appears to me to constitute the main stem from which all others, including even arithmetic itself, are derived and secondary branches. The key to success in dealing with the problems of this incipient science (as I suppose of most others) must be sought for in the construction of an apt and expressive notation, and in the discovery of language by force of which the mind may be enabled to lay hold of complex operations and mould them into simple and easily transmissible forms of thought. I must then entreat the indulgence of the reader if, in this early grappling with the difficulties of a new language and a new notation, I may occasionally appear wanting in absolute clearness and fulness of expression.

Let us, as before, represent the nine elements by the numbers from 1 to 9, and suppose the *nomes* to be 1, 2, 3 : 4, 5, 6 : 7, 8, 9.

If we take any syntheme formed out of the *binomial* triads belonging to the above nomes, and if out of such syntheme we omit the elements 1, 2, 3 (belonging to the 1st nome) wherever they occur, the slightest consideration will serve to show that the synthemes thus denuded will assume the *form* $l.m.r, p.q, n$, where l, m, r may be regarded as belonging to one of the remaining nomes, and p, q, n to the other. The total number of synthemes in a grouping which contains all the binomial triads is 18, because the total number of these triads is 54; and consequently it will be seen that every grouping will in fact consist of the same *framework*, so to say, of combinations of elements belonging to the second and third nomes variously compounded with the elements of the first nome.

This framework may be advantageously divided into two parts, each containing nine terms, and which I shall call respectively U and $\dot{U}$. Thus by U I shall understand the nine arrangements following:—

4.5.7, 8.9, 6; 4.5.8, 7.9, 6; 4.5.9, 7.8, 6
5.6.7, 8.9, 4; 5.6.8, 7.9, 4; 5.6.9, 7.8, 4
6.4.7, 8.9, 5; 6.4.8, 7.9, 5; 6.4.9, 7.8, 5

each *imperfect* or defective syntheme being separated from the next by a semicolon, or else by a change of line. So by $\dot{U}$ I shall understand the *complementary* part of the framework, namely:—

8.9.6, 4.5, 7; 7.9.6, 4.5, 8; 7.8.6, 4.5, 9
8.9.4, 5.6, 7; 7.9.4, 5.6, 8; 7.8.4, 5.6, 9
8.9.5, 6.4, 7; 7.9.5, 6.4, 8; 7.8.5, 6.4, 9.

It is of cardinal importance to notice that the order in which the *imperfect synthemes* are arranged in U and $\dot{U}$ is one of absolute reciprocity. It is in this reciprocity, and in the fact of U or $\dot{U}$ being each in *strict regimen* (so to say) with the other, that the cause of the success of the method about to be applied essentially resides.

The slightest reflection will serve to show that every *complete* syntheme of the kind required will be of the form

$$\left| \begin{matrix} U \times P \\ \dot{U} \times \dot{P} \end{matrix} \right|,$$

where the symbolical multipliers P and $\dot{P}$ are each of them some one of the forms (by no means necessarily the *same*) represented generally by the framework of defective synthemes hereunder written (defective in the sense that all the elements of the second and third nomes are supposed to be omitted),

$$\begin{matrix} , a, bc; & , b, ca; & , c, ab \\ , b, ca; & , c, ab; & , a, bc \\ , c, ab; & , a, bc; & , b, ca, \end{matrix}$$

or else by the cognate framework

$$\begin{matrix} , a, bc; & , c, ab; & , b, ca \\ , b, ca; & , a, bc; & , c, ab \\ , c, ab; & , b, ca; & , a, bc, \end{matrix}$$

where a, b, c are identical in some order or another with the elements of the first nome, namely, 1, 2, 3; so that there are six different systems of a, b, c in each of these two frameworks.

No other combination of the elements in U or $\dot{U}$ (all of which belong to the second and third nomes) with the elements in the first nome is possible; for any such combination would involve the fact of a *repetition* of the same *triad* or triads in the same grouping, contrary to the nature of a grouping. Hence, then, the number of forms of P and of $\dot{P}$ being twice six, or 12, we at once perceive that the total number of groupings is 12×12, or 144.

But now comes the more difficult question of ascertaining between how many distinct species or types these groupings are distributed. If we study the form of P or $\dot{P}$, it is obvious that it will be completely and distinctively *denoted in brief* by the twelve forms arising from the development of

$$\begin{matrix} a & b & c \\ b & c & a \\ c & a & b \end{matrix} \quad \text{and} \quad \begin{matrix} a & c & b \\ b & a & c \\ c & b & a \end{matrix}; \quad \text{videlicet}$$

$$\overset{(1)}{\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{matrix}} \quad \overset{(2)}{\begin{matrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{matrix}} \quad \overset{(3)}{\begin{matrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{matrix}} \quad \overset{(4)}{\begin{matrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{matrix}} \quad \overset{(5)}{\begin{matrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{matrix}} \quad \overset{(6)}{\begin{matrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{matrix}}$$

$$\overset{(7)}{\begin{matrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{matrix}} \quad \overset{(8)}{\begin{matrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{matrix}} \quad \overset{(9)}{\begin{matrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{matrix}} \quad \overset{(10)}{\begin{matrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{matrix}} \quad \overset{(11)}{\begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{matrix}} \quad \overset{(12)}{\begin{matrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{matrix}}$$

which we may for facility of future reference denote by

$$\pi_1, \pi_2, \pi_3, \pi_4, \pi_5 \ldots \pi_{12}.$$

Now as regards the types: since the order of the elements in one nome is entirely independent of the order of the elements in any other, it is obvious that it is not the particular form of P or of $\dot{P}$ which can have any influence on the form of the type, but solely the relation of P and $\dot{P}$ to one another. In order then to fix the ideas, I shall for the moment consider P equal to

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2. \end{matrix}$$

This at once enables us to fix a *limit* to the number of distinct types. In the first place, the essentially distinct FORMS of the first column in $\dot{P}$, with respect to that of P, may be sufficiently represented by taking the two columns identical, or differing by a single interchange, or else having no two elements in the same place. Hence $\dot{P}$, so far as the ascertainment of types is concerned, may be limited to the six forms following:—

(α)	(γ)	(ϵ)
1 2 3	2 1 3	2 3 1
2 3 1	1 3 2	3 1 2
3 1 2	3 2 1	1 2 3

(β)	(δ)	(η)
1 3 2	2 3 1	2 1 3
2 1 3	1 2 3	3 2 1
3 2 1	3 1 2	1 3 2.

But again, since (β) and (η) are each derivable from (α) (the assumed form of P) by an interchange of two columns *inter se*, it is clear that, as regards distinction of type, $\eta = \beta$, and consequently there are only *at utmost* five types remaining, which may be respectively described by the symbols

$$\left|\begin{matrix} U\alpha \\ \dot{U}\alpha \end{matrix}\right| \quad \left|\begin{matrix} U\alpha \\ \dot{U}\beta \end{matrix}\right| \quad \left|\begin{matrix} U\alpha \\ \dot{U}\gamma \end{matrix}\right| \quad \left|\begin{matrix} U\alpha \\ \dot{U}\delta \end{matrix}\right| \quad \left|\begin{matrix} U\alpha \\ \dot{U}\epsilon \end{matrix}\right|.$$

It must be noticed that α comprehends or typifies the squares numbered 1; β those numbered 7, 8, 9; γ those numbered 4, 5, 6; δ those numbered 10, 11, 12; ϵ those numbered 2, 3.

I say designedly that the number of types is *at utmost* limited to these five. But it by no means follows that the number is so great as five; for it will not fail to be borne in mind that these differences have reference to the peculiar mode in which we have chosen to decompose in idea each syntheme, by viewing it as a symbolical product of an arrangement containing only the elements of the second and third nomes by an arrangement containing only those of the first nome. But the nomes are interchangeable, and therefore it may very well be the case that two types which appear to be distinct are in reality identical, their elements in the groupings appertaining to such types having absolutely analogous relations to different orderings of the

nomes, so that the groupings will be convertible into each other by permutations among the given elements. We must therefore ascertain how the above types, or any specific forms of them, come to be represented when we interchange the first nome with either of the other two, or, to fix the ideas, let us say with the second.

To effect this, let $U\alpha$, $\dot{U}\alpha$, $\dot{U}\beta$, $\dot{U}\gamma$, $\dot{U}\delta$, $\dot{U}\epsilon$ be actually expanded; by the performance of the symbolical multiplications we obtain—

$$U\alpha = \left|\begin{array}{l} 4.5.7\ 8.9.1\ 6.2.3;\ 4.5.8\ 7.9.2\ 6.1.3;\ 4.5.9\ 7.8.3\ 6.2.1 \\ 5.6.7\ 8.9.2\ 4.1.3;\ 5.6.8\ 7.9.3\ 4.1.2;\ 5.6.9\ 7.8.1\ 4.2.3 \\ 6.4.7\ 8.9.3\ 5.1.2;\ 6.4.8\ 7.9.1\ 5.2.3;\ 6.4.9\ 7.8.2\ 5.1.3 \end{array}\right|$$

$$\dot{U}\alpha = \left|\begin{array}{l|l|l} 8.9.6\ 4.5.1\ 7.2.3 & 7.9.6\ 4.5.2\ 8.1.3 & 7.8.6\ 4.5.3\ 9.2.1 \\ 8.9.4\ 5.6.2\ 7.1.3 & 7.9.4\ 5.6.3\ 8.1.2 & 7.8.4\ 5.6.1\ 9.2.3 \\ 8.9.5\ 6.4.3\ 7.2.1 & 7.9.5\ 6.4.1\ 8.2.3 & 7.8.5\ 6.4.2\ 9.1.3 \end{array}\right|$$

$$\dot{U}\beta = \left|\begin{array}{l|l|l} 8.9.6\ 4.5.1\ 7.2.3 & 7.9.6\ 4.5.3\ 8.1.2 & 7.8.6\ 4.5.2\ 9.1.3 \\ 8.9.4\ 5.6.2\ 7.1.3 & 7.9.4\ 5.6.1\ 8.2.3 & 7.8.4\ 5.6.3\ 9.2.1 \\ 8.9.5\ 6.4.3\ 7.2.1 & 7.9.5\ 6.4.2\ 8.1.3 & 7.8.5\ 6.4.1\ 9.2.3 \end{array}\right|$$

$$\dot{U}\gamma = \left|\begin{array}{l|l|l} 8.9.6\ 4.5.2\ 7.1.3 & 7.9.6\ 4.5.1\ 8.2.3 & 7.8.6\ 4.5.3\ 9.1.2 \\ 8.9.4\ 5.6.1\ 7.2.3 & 7.9.4\ 5.6.3\ 8.1.2 & 7.8.4\ 5.6.2\ 9.1.3 \\ 8.9.5\ 6.4.3\ 7.1.2 & 7.9.5\ 6.4.2\ 8.1.3 & 7.8.5\ 6.4.1\ 9.2.3 \end{array}\right|$$

$$\dot{U}\delta = \left|\begin{array}{l|l|l} 8.9.6\ 4.5.2\ 7.1.3 & 7.9.6\ 4.5.3\ 8.1.2 & 7.8.6\ 4.5.1\ 9.2.3 \\ 8.9.4\ 5.6.1\ 7.2.3 & 7.9.4\ 5.6.2\ 8.1.3 & 7.8.4\ 5.6.3\ 9.1.2 \\ 8.9.5\ 6.4.3\ 7.1.2 & 7.9.5\ 6.4.1\ 8.2.3 & 7.8.5\ 6.4.2\ 9.1.3 \end{array}\right|$$

$$\dot{U}\epsilon = \left|\begin{array}{l|l|l} 8.9.6\ 4.5.2\ 7.1.3 & 7.9.6\ 4.5.3\ 8.1.2 & 7.8.6\ 4.5.1\ 9.2.3 \\ 8.9.4\ 5.6.3\ 7.1.2 & 7.9.4\ 5.6.1\ 8.2.3 & 7.8.4\ 5.6.2\ 9.1.3 \\ 8.9.5\ 6.4.1\ 7.2.3 & 7.9.5\ 6.4.2\ 8.1.3 & 7.8.5\ 6.4.3\ 9.1.2 \end{array}\right|$$

Let us form a *framework* with the nomes 1.2.3, 7.8.9 exactly similar to that which we formed before with 4.5.6, 7.8.9, and let V, $\dot{V}$ be its two parts respectively analogous to U, $\dot{U}$, we thus obtain for $\dot{V}$,

$$\begin{array}{l} 1.2.7,\ 8.9,\ 3;\quad 1.2.8,\ 7.9,\ 3;\quad 1.2.9,\ 7.8,\ 3 \\ 2.3.7,\ 8.9,\ 1;\quad 2.3.8,\ 7.9,\ 1;\quad 2.3.9,\ 7.8,\ 1 \\ 3.1.7,\ 8.9,\ 2;\quad 3.1.8,\ 7.9,\ 2;\quad 3.1.9,\ 7.8,\ 2, \end{array}$$

and for V,

$$\begin{array}{l} 8.9.3,\ 1.2,\ 7;\quad 7.9.3,\ 1.2,\ 8;\quad 7.8.3,\ 1.2,\ 9 \\ 8.9.1,\ 2.3,\ 7;\quad 7.9.1,\ 2.3,\ 8;\quad 7.8.1,\ 2.3,\ 9 \\ 8.9.2,\ 3.1,\ 7;\quad 7.9.2,\ 3.1,\ 8;\quad 7.8.2,\ 3.1,\ 9. \end{array}$$

We must now perform the unwonted process of symbolical division, and obtain the quotients of $U\alpha$ by V, and of $\dot{U}\alpha$, $\dot{U}\beta$, $\dot{U}\gamma$, $\dot{U}\delta$, $\dot{U}\epsilon$ by $\dot{V}$ (it will of course be perceived that it is known *à priori* that the dividend forms of arrangement are actual multipliers of the divisors V and $\dot{V}$). In writing down the results of these divisions, which will consist exclusively of elements belonging to the nome 4.5.6, and of which each term will be of the form $d, e.f$, we may, analogously to what we have done before for greater brevity, write down only the single element (d), and omit the residue (ef), which is determined when (*d*) is determined. We shall thus obtain the quotients following :—

$$\frac{U\alpha}{V} = \begin{matrix} 5 & 4 & 6 \\ 6 & 5 & 4 \\ 4 & 6 & 5 \end{matrix}$$

$$\frac{\dot{U}\alpha}{\dot{V}} = \begin{matrix} 5 & 4 & 6 \\ 6 & 5 & 4 \\ 4 & 6 & 5 \end{matrix} \qquad \frac{\dot{U}\beta}{\dot{V}} = \begin{matrix} 5 & 6 & 4 \\ 6 & 4 & 5 \\ 4 & 5 & 6 \end{matrix} \qquad \frac{\dot{U}\gamma}{\dot{V}} = \begin{matrix} 5 & 4 & 6 \\ 4 & 6 & 5 \\ 6 & 5 & 4 \end{matrix}$$

$$\frac{\dot{U}\delta}{\dot{V}} = \begin{matrix} 5 & 6 & 4 \\ 4 & 5 & 6 \\ 6 & 4 & 5 \end{matrix} \qquad \frac{\dot{U}\epsilon}{\dot{V}} = \begin{matrix} 4 & 6 & 5 \\ 5 & 4 & 6 \\ 6 & 5 & 4. \end{matrix}$$

It may be observed that these divisions may be effected with great rapidity ; because when three out of the nine figures (in any quotient) not in the same line or column are known, all the rest are known. Thus, for example, to find $\frac{\dot{U}\epsilon}{\dot{V}}$ it is only necessary to seek in $\dot{U}\epsilon$ the syntheme which contains 1 . 2 . 7, and then to take out the figure in that syntheme associated with 8 . 9 in that line, namely, 4; then again to seek the syntheme which contains 1 . 2 . 8, and to take out the figure in that syntheme associated with 7 . 9, which is 6 ; and finally to seek the syntheme which contains 2 . 3 . 7, and then to take out the figure associated with 8 . 9, namely 5 ; we thus obtain the three corner figures of the square which represents $\frac{\dot{U}\epsilon}{\dot{V}}$ as thus:

$$\begin{matrix} 4 & 6 & . \\ 5 & . & . \\ . & . & . \end{matrix}$$

of which the six remaining figures are given by the condition that in no line and in no column must the same two figures be found. In order to compare these quotients, or rather the relations of the first of them to the remaining five with those of α to α, β, γ, δ, ϵ, it will be convenient to subtract the constant number 3 from each figure, and to transpose the first and second columns; we thus obtain

$$\frac{U\alpha}{V} \equiv \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} \equiv \pi_1 \equiv \alpha,$$

$$\frac{\dot{U}\alpha}{\dot{V}} \equiv \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} \equiv \pi_1 \equiv \alpha, \qquad \frac{\dot{U}\beta}{\dot{V}} \equiv \begin{vmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{vmatrix} \equiv \pi_9 \equiv \beta,$$

$$\frac{\dot{U}\gamma}{\dot{V}} \equiv \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} \equiv \pi_{11} \equiv \delta, \qquad \frac{\dot{U}\delta}{\dot{V}} \equiv \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{vmatrix} \equiv \pi_6 \equiv \gamma,$$

$$\frac{\dot{U}\epsilon}{\dot{V}} \equiv \begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix} \equiv \pi_3 \equiv \epsilon.$$

Thus, for greater brevity, considering the five types to be represented by

$$\begin{matrix} \alpha & \alpha & \alpha & \alpha & \alpha \\ \alpha & \beta & \gamma & \delta & \epsilon, \end{matrix}$$

or still more briefly by

$$\alpha \quad \beta \quad \gamma \quad \delta \quad \epsilon;$$

and calling the nomes N_1, N_2, N_3, we find that the effect of interchanging N_1 and N_2 with each other is to change

$$\alpha \quad \beta \quad \gamma \quad \delta \quad \epsilon$$

into

$$\alpha \quad \beta \quad \delta \quad \gamma \quad \epsilon.$$

In like manner it may be ascertained (and the student is advised to satisfy himself by actual trial of the fact) that the effect of interchanging N_1 and N_3 with each other is to convert

$$\alpha \quad \beta \quad \gamma \quad \delta \quad \epsilon$$

into

$$\alpha \quad \delta \quad \gamma \quad \beta \quad \epsilon.$$

From these two calculations it follows that the effect of any permutation between N_1, N_2, N_3 is to produce a permutation in β, γ, δ *inter se*, but will leave α and ϵ unaltered*. Hence then we have arrived at the goal of our inquiry, having demonstrated that

$$\begin{vmatrix} V\alpha \\ \dot{V}\alpha \end{vmatrix}$$

indicates one type,

$$\begin{vmatrix} V\alpha \\ \dot{V}\beta \end{vmatrix}, \quad \begin{vmatrix} V\alpha \\ \dot{V}\gamma \end{vmatrix}, \quad \begin{vmatrix} V\alpha \\ \dot{V}\delta \end{vmatrix}$$

each of them another *the same* type, and

$$\begin{vmatrix} V\alpha \\ \dot{V}\epsilon \end{vmatrix}$$

a third type,—and bearing in mind that

(α)	belongs to	π_1	exclusively,
(ϵ)	„	π_2, π_3	„
(β)	„	π_7, π_8, π_9	„
(γ)	„	π_4, π_5, π_6	„
(δ)	„	π_{10}, π_{11}, π_{12}	„

* This result, by the aid of a fine observation, may be more rapidly established *uno ictu* (I mean by *one* calculation instead of two) as follows. Let $N_1N_2N_3$ be made to undergo a *cyclical* interchange, then it will be found that β, γ, δ also undergo a cyclical interchange, whilst α and ϵ remain unchanged. This proves that β, γ, δ are only different phases of the same type, which is *sufficient*; for as regards α and ϵ, the fact of the number of individuals which they represent being unequal *inter se*, and also unequal to the number contained in β, γ, δ, renders it *à priori* impossible to allow that they can either pass into each other or into the forms β, γ, δ, by virtue of any interchange among the elements.

and that each form of π comprehends 12 groupings due to the 12 forms of $V\alpha$, we are enabled to affirm that the total number of groupings of the binomial triads of 3 nomes of 3 elements each is 144, and that the number of types or species between which these 144 are distributed is 3, comprising 12, 24, and 108 respectively,—a conclusion which it would almost have exceeded the practical limits of human labour and perspicuity to have established by the direct comparison of the 144 groupings of 18 synthemes each with each other, with a view to ascertain which admit of being permutable into each other, and which not.

The largest species of 108 groupings, it may be observed, is subdivisible into 3 *varieties*, not really allotypical, of 36 each,—the characteristic of those groupings which belong to the same variety being that they permute *exclusively* into each other when the permutations of the elements are confined to perturbations of the order of the elements in the same nome or nomes, and the different nomes are subject to no interchange of elements between themselves.

Just so the species of 36 groupings of trinomial triads, treated of in my preceding paper, subdivides into 3 varieties of sub-families characterized by a similar property.

The total number of modes of subdivision of 9 elements between 3 nomes being 280, it follows, from considerations of the same kind as stated in the May Number of this *Magazine* [p. 264 above], that there exist transitive substitution-groups belonging to 9 elements of

$$\frac{\pi(9)}{280 \times 12}, \quad \frac{\pi(9)}{280 \times 24}, \quad \frac{\pi(9)}{280 \times 108},$$

that is, 108, 24 and 12 substitutions respectively.

Again, let us consider the question of forming the synthemes of the triads of a *single nome* of 9 elements into groupings where *every* triad shall be found without repetition. We may obtain such groupings by choosing arbitrarily any one of the 280 sets of 3 nomes into which the 9 elements may be segregated*, and then forming one syntheme with the three monomial triads (corresponding to such set so chosen), 18 synthemes (in any one of the 144 possible ways) of exclusively binomial triads, and 9 synthemes (in any one of the 40 possible ways) of exclusively trinomial triads; we shall thus obtain in all 280 × 144 × 40, or 1,612,800 solutions of the question proposed; I mean

* 280 is also evidently the number of synthemes of triads belonging to one nome of 9 elements. In general the number of r-ads belonging to one nome of mn elements is

$$\frac{\pi(mn-1)\,\pi\{(m-1)n-1\}\,\pi\{(m-2)n-1\}\dots\pi(n-1)}{\{\pi(n-1)\}^m\,\pi\{(m-1)n\}\,\pi\{(m-2)n\}\dots\pi(n)}.$$

1,612,800 groupings, all satisfying the imposed condition, and reducible to 6 *genera**, comprising respectively

$$4 \times 12 \times 280, \quad 4 \times 24 \times 280, \quad 4 \times 108 \times 280, \quad 36 \times 12 \times 280,$$
$$36 \times 24 \times 280, \quad 36 \times 108 \times 280,$$

that is, 13,440, 26,880, 120,960, 120,960, 241,920, 1,088,640 individual groupings. I conclude with putting a grand question, more easy to propose than to answer, namely, are these one million six hundred thousand (and upwards) groupings (classifiable under six distinct genera) all the possible modes and types of grouping which will satisfy the conditions of the question? and if not, what other mode or type of grouping can be found? Were I compelled to give an answer to this question, I would say that the balance of my mind leans to the opinion that the six types in question are the sole possible types of solution; but I do not pretend to rest this judgment upon any solid grounds of demonstration, nor to entertain it with any strong degree of assurance. It is a question which the effort to resolve cannot but react powerfully on our knowledge of the principles of tactic in general, and of the theory of substitution-groups in particular; and as such I submit it to the consideration of the rising chivalry of analysis, seeking myself meanwhile fresh fields and pastures new of meditation.

* The above *genera* must not be confounded with types or species. (In my preceding communications I may inadvertently have used the word *family* as coincident with type: *species* is the proper term.) The type of a total grouping in the problem referred to in the text will depend not only on the particular combination of the types of the binomial and trinomial partial groupings which give rise to these 6 (= 2 × 3) genera, but also on the relative *phases* of the types so combined. The number of groupings in one type or species is always a submultiple of the number of permutations of the elements; whereas it will be seen that the number of groupings in one of the above genera greatly exceeds that number, which in the present case is only

$$1.2.3.4.5.6.7.8.9, \text{ or } 362,880.$$

Whatever may be the case in natural history, the nature of a type or species, as distinguished from a genus, family, or any other higher kind of aggregation of individuals, in *pure syntax* is perfectly clear and unambiguous; those groupings form a species which are commutable into one another by an interchange of elements: thus the different *phases* of the same type or species are in analogy with the different values of the same function arising out of a change in a constant parameter. If it should turn out that the above sixteen hundred thousand and odd groupings are not the sole solutions of which the question admits, then it will follow that even in this early instance we shall have an example not only of species and genera, but of distinct families of genera, for it is certain that the above six genera constitute within themselves a complete natural family. It will form an interesting subject of inquiry to ascertain how many types are included within each of the six genera belonging to this family; and be it never forgotten that to each species corresponds, and from it is, so to say, capable of being extracted or sublimated, a Cauchian substitution-group.

49.

REMARK ON THE TACTIC OF NINE ELEMENTS.

[*Philosophical Magazine*, XXII. (1861), pp. 144—147.]

AT the end of my preceding paper in this *Magazine* for July [p. 284 above], I hazarded an opinion that any grouping of 28 synthemes comprising the 84 triads belonging to a system of 9 elements, might be regarded as made up of 1 syntheme of monomial triads, 18 synthemes of binomial triads, and 9 of trinomial triads, the denominations (monomial, binomial, and trinomial) having reference to a duly chosen distribution of the 9 elements into 3 nomes of 3 elements each. This conjecture is capable of being brought to a very significant, although not decisive test, by examining a peculiar and important distribution of the 28 synthemes into 7 sets of 4 synthemes each, the property of each set being that its 12 triads contain amongst them all the 36 *duads* appertaining to the 9 elements. I discovered this mode of distribution very many years ago; but it was first published independently by a mathematician whose name I forget, either in the *Philosophical Magazine* or in the *Cambridge and Dublin Mathematical Journal*, I think at some time between the years 1847–53. A similar mode of distribution exists for any system of elements of which the number is a power of 3. Without pausing to give the law of formation, I shall simply observe that for 9 elements we may take as a basic arrangement the square

1	2	3
4	5	6
7	8	9

and form from this, by a symmetrical method, the annexed six derived arrangements :—

7	1	2	7	2	3	9	2	3
3	5	6	1	4	5	1	5	6
4	8	9	6	8	9	4	7	8
4	3	1	5	2	3	4	2	3
7	5	6	7	6	4	8	5	6
2	8	9	1	8	9	1	9	7

and reading off each of these squares in lines, in columns, and in right and left diagonal fashion, we obtain the 7 sets of 4 synthemes each referred to, namely

	123 456 789	
	147 258 369	
	159 267 348	
	168 249 357	
712 356 489	723 145 689	923 156 478
734 158 269	716 248 359	914 257 368
759 164 238	749 256 318	958 264 317
768 139 254	758 219 346	967 218 354
431 756 289	523 764 189	423 856 197
472 358 169	571 268 349	481 259 367
459 362 178	569 241 378	457 261 389
468 379 152	548 279 361	469 287 351

If, now, we take any distribution of the 9 elements into nomes other than 123, 456, 789, we shall find that some of the synthemes will contain trinomials, some binomials only, but others (in number either 9 or 18, according to the distribution chosen) will contain binomials and trinomials mixed; but if we adopt 123, 456, 789 as the nomes, then it will be found that the remaining 27 synthemes (after excluding the monomial syntheme 123, 456, 789) will consist of 18 purely binomial triads, and 9 purely trinomial triads. The former will consist of the first, second, and fourth synthemes of the 6 derived groups; the latter of the second, third and fourth of the basic group, and of the second synthemes of each of the 6 derived groups.

It may be remembered that there are two types or species of groupings of trinomial triad synthemes appertaining to 3 nomes of 3 elements; one of these species contains 4, the other 36 individual groupings. It may easily be ascertained that the grouping above indicated belongs to the first (the less numerous) of these species. Again, there are 3 types or species of groupings of binomial triad synthemes appertaining to the same system of nomes; one containing 12, one 24, and the third 108 groupings. The grouping with which we are here concerned will be found to belong to the *second* of these species,—that denoted by the symbols $\left|\begin{matrix}\alpha\\ \epsilon\end{matrix}\right|$ in my paper of last month. Hence, then, we derive a very considerable presumption in favour of the opinion which I advanced at the close of my preceding paper on Tactic, and derived, too, from a case apparently unfavourable to the verisimilitude of the conjecture; for a natural subdivision of 28 things into 7 sets of 4 each seems at first sight hardly compatible with another natural division into 3 sets of 1, 18, and 9 respectively. Notwithstanding this seeming incompatibility, we have found that the two methods of decomposition do coexist, owing essentially to the fact that the 7 sets (of 4 synthemes each) stand not in a relation of indifference set to set, but are to be considered as composed of a base and 6 derivatives indifferently related to the base and to each other. The theory of these 7 sets is extremely curious, and well worthy of being fully investigated by the student of tactic, but cannot be gone into within the limits suitable to the pages of a philosophical miscellany.

Before taking final leave of the subject (at all events for the present, and in the pages of this *Magazine*), as I have been questioned as to the meaning of the important word "syntheme," derived from *συν θημα*, I repeat that a "syntheme" is the general name for any consociation of the single or combined elements of a given system of elements in which each element is once and once only contained. A *nome*, from *νεμω* (*to divide*), means a consociation of a certain number out of a given system of elements; and a binomial, trinomial, or r-nomial combination of any specified sort, means a combination whose elements are dispersed between 2, 3, or r of the nomes between which the entire system of elements is supposed to have been divided.

P.S. I have found the date and place of the resolution into 7 sets referred to in the text; it is given in a paper by Mr Kirkman, Vol. v. p. 261 of the *Cambridge and Dublin Mathematical Journal* for 1850. His 7 squares, whose horizontal, vertical, and two diagonal readings (like mine) constitute the 7 sets in question, are substantially as follows:—

	1 2 3	
	4 5 6	
	7 8 9	
1 2 4	4 5 7	7 8 1
5 6 7	8 9 1	2 3 4
8 9 3	2 3 6	5 6 9
7 1 2	1 4 5	4 7 8
3 4 5	6 7 8	9 1 2
6 8 9	9 2 3	3 5 6*.

* By changing the positions of the lines and columns of the six derivative squares, which may be done without affecting the value of their *readings*, they may be represented under the form following, which will be seen to render much clearer their relation to the primitive square:—

412	623	423	239	129	127
756	745	956	451	563	453
389	189	178	786	784	896.

On assuming 123, 456, 789 as the three nomes, the 28 synthemes contained in the sets will be found to consist of purely monomial, binomial, and trinomial synthemes.

Thus there would be an additional presumption in favour of the supposed law of *homonomial resolubility*, provided that Mr Kirkman's solution were essentially distinct in type from my own; his binomial and trinomial systems, taken separately, coincide in type with those afforded by my solution, notwithstanding which it would not be lawful to assume (indeed I had at first some reasons for doubting) the identity of type of the total groupings of which these systems form part; all we could have positively inferred from that fact would have been, that these two groupings both belong to the same class or genus containing 26,880 individuals, the second of the six referred to at the close of my last paper; a comparison of the two solutions has, however, satisfied me that they are absolutely identical in form.

50.

ON A GENERALIZATION OF A THEOREM OF CAUCHY ON ARRANGEMENTS*.

[*Philosophical Magazine*, XXII. (1861), pp. 378—382.]

IN a paper "On the Theory of Determinants" in the *Philosophical Magazine* for March in this year, Mr Cayley has referred and added to a theorem of Cauchy deduced from the latter's method of *arrangements*, namely, that if we resolve an integer n in every possible way into parts, to wit α parts of a, β parts of b, ... λ of l, ($a, b, c \ldots l$ being all distinct integers), then

$$\Sigma \frac{1}{\Pi\alpha \,.\, a^\alpha \Pi\beta \,.\, b^\beta \ldots \Pi\lambda \,.\, l^\lambda} = 1.$$

Now both Cauchy's theorem and Mr Cayley's addition to it (which essentially consists in the observation, that if before the numerator 1 in the above quantity under the sign of summation we write $(-)^{\alpha+\beta+\ldots+\lambda}$, the sum becomes zero) are no more than particular cases of the following theorem: namely, that if instead of 1 we write $\rho^{\alpha+\beta+\ldots+\lambda}$ in the numerator of the quantity under the sign of summation (ρ being any quantity whatever), the sum becomes expressible as a known function of ρ. Nothing can be easier than the proof.

Let the $\alpha, \beta, \gamma, \ldots \lambda$ in the preceding statement be supposed subject to the further condition that their sum is r; then for any assigned value of r (a positive integer) it is easy to see that the sum of the terms within the sign of summation in Cauchy's theorem is

$$S\left(\frac{1}{x_1 x_2 \ldots x_r} \cdot \frac{\rho^r}{\Pi(r)}\right),$$

where $x_1, x_2, \ldots x_r$ mean *every* system of values of $x_1, x_2, \ldots x_r$ (permutations *admitted*) which satisfy the equation

$$x_1 + x_2 + \ldots + x_r = n.$$

(It should here be observed that $\alpha, \beta, \gamma, \ldots \lambda$; $a, b, c, \ldots l$ are the systems

[* See above p. 245.]

which satisfy $\alpha a + \beta b + \gamma c + \ldots + \lambda l = n$, permutations being *excluded*; that is to say, if, for example, α, β, γ should happen to be equal for any partition of n, the values α, a; α, b; α, c would figure only *once*, and not *six* times, among the systems included under the sign of Σ.) Hence then we see that

$$\Sigma \frac{\rho^{\alpha+\beta+\gamma+\ldots+\lambda}}{\Pi\alpha \,.\, a^\alpha \Pi\beta \,.\, b^\beta \ldots \Pi\lambda \,.\, l^\lambda} = \sum_{r=\infty}^{r=1} S_r \frac{\rho^r}{\Pi(r)} \text{*},$$

where S_r is the coefficient of t^n in $\left(\frac{t}{1} + \frac{t^2}{2} + \frac{t^3}{3} + \text{\&c. } ad\ infin.\right)^r$, that is, in $\left\{\log\left(\frac{1}{1-t}\right)\right\}^r$; and the total sum designated by Σ will be consequently the coefficient of t^n in

$$\log\left(\frac{1}{1-t}\right)\rho + \left(\log\frac{1}{1-t}\right)^2 \frac{\rho^2}{1\,.\,2} + \text{\&c.},$$

that is, in $e^{\rho \log\left(\frac{1}{1-t}\right)}$, that is, in $\left(\frac{1}{1-t}\right)^\rho$.

Thus if $\rho = 1$, we have Cauchy's theorem, namely $\Sigma = 1$;

Thus if $\rho = -1$, we have Cayley's theorem, namely $\Sigma = 0$†;

and in general for any value of ρ,

$$\Sigma = \frac{\rho(\rho+1)\ldots(\rho+n-1)}{1\,.\,2 \ldots n} \text{‡}.$$

* For if we take a system of values satisfying the above equation, consisting of α equal values a, β equal values b, ... λ equal values l, such a system will give rise in $\Sigma \frac{1}{x_1 x_2 \ldots x_r}$ to $\frac{\pi(r)}{(\pi\alpha)(\pi\beta)\ldots(\pi\lambda)}$ repetitions of the term $\frac{1}{a^\alpha \,.\, b^\beta \ldots l^\lambda}$, and consequently in $\Sigma \frac{1}{x_1 x_2 \ldots x_r} \cdot \frac{1}{\pi r}$ to a total value $\frac{1}{(\pi\alpha)\, a^\alpha\, (\pi\beta)\, b^\beta \ldots (\pi\lambda)\, l^\lambda}$, condensed into a single term in Cauchy's theorem.

† Provided, however, that n exceeds 1, a limitation accidentally omitted in Mr Cayley's paper; and so in general

$$\Sigma \frac{(-\rho)^{\alpha+\beta+\ldots+\lambda}}{\Pi\alpha \,.\, a^\alpha \ldots \Pi\lambda \,.\, l^\lambda} = 0,$$

ρ being *any* positive integer provided n is greater than ρ.

‡ If $\rho = \frac{1}{2}$, we obtain

$$\Sigma = \frac{1\,.\,3\,.\,5 \ldots (2n-1)}{2\,.\,4\,.\,6 \ldots 2n};$$

from which it is easy to infer that the number of substitutions of $2n$ things representable by the product of cyclical substitutions, all of an even order, is $\{1\,.\,3\,.\,5 \ldots (2n-1)\}^2$. If $\rho = -\frac{1}{2}$, we obtain

$$\Sigma = \frac{1\,.\,1\,.\,3 \ldots (2n-1)}{2\,.\,4\,.\,6 \ldots (2n)},$$

combining which with the preceding result, it is easy to infer that the number of substitutions of $2n$ things representable by the product of an odd number of cyclical substitutions, all of an even order, is to the number of such representable by the product of an even number of cyclical substitutions, all of an even order, in the ratio of n to $(n-1)$. The former of these two theorems

In this theorem is in fact included another, namely, that if

$$aa + \beta b + \ldots + \lambda l = n \text{ and } \alpha + \beta + \ldots + \lambda = r$$

(permutations *not* admissible), then

$$\Sigma \frac{\Pi n}{\Pi \alpha . a^{\alpha} . \Pi \beta . b^{\beta} \ldots \Pi \lambda . l^{\lambda}}$$

is equal to the coefficient of ρ^{r-1} in

$$(\rho + 1)(\rho + 2) \ldots (\rho + n - 1).$$

This coefficient is accordingly (to return to Cauchy's theory of arrangements) the number of substitutions of n elements capable of being expressed by the product of r cyclical substitutions. As, for instance, the number of substitutions of four elements a, b, c, d capable of expression by the product of two cyclical substitutions ought to be the coefficient of λ in $(\lambda + 1)(\lambda + 2)(\lambda + 3)$, that is, 11, which is right; for the number of substitutions of the form $(a, b)(c, d)$ will be 3, and of the form $(a, b, c)(d)$ will be 8. In conclusion, I may notice that by an obvious deduction from this last theorem, we are led to the well-known one in the theory of numbers, that every coefficient in the development of

$$\Sigma (\rho + 1)(\rho + 2) \ldots (\rho + n - 1),$$

except the first and last, and the sum of these two, is divisible by n when n is a prime number; and indeed we can actually express by aid of it the quotient of every intermediate coefficient divided by n as the sum of separate integer terms free from the sign of addition.

Postscript. By an extension of the method of generating functions contained in the text above, it may easily be seen that the number* of substitutions of n letters represented by the products of r cyclical substitutions, where the number of letters of each cycle leaves a given residue e in respect

is intimately allied with Mr Cayley's celebrated theorem on "skew," or what, for good reasons hereafter to be alleged, I should prefer to call *polar* determinants, namely, that every such of the $2n$th order is the square of a *Pfaffian*. A Pfaffian is in fact a sum of quantities typifiable completely, both as to sign and magnitude, by a duadic *syntheme* of $2n$ elements, the number of which is readily seen to be $1 . 3 . 5 \ldots (2n - 1)$. I believe I shall soon be in a condition to announce a remarkable extension of this theory to embrace the case of Polar *Commutants* and *Hyperpfaffians*.

* For this number, divided by $\Pi(n)$, is the coefficient of x^n in

$$\frac{1}{\Pi r}\left(\int_0^x \frac{dx\, x^{e-1}}{1-x}\right)^r, \text{ say } \frac{1}{\Pi r}(\phi x)^r,$$

and therefore of $x^n \rho^r$ in $e^{\rho\phi x}$, say $\psi(x, \rho)$, and therefore $\left(\text{since } \frac{d\psi}{dx} = \frac{x^{e-1}}{1-x^m} \text{ and } \psi \text{ may be put under the form } \Sigma \frac{u_n}{n} x^n\right)$ of ρ^r in $\frac{u_n}{n}$, where u_n is defined as in the text.

to a given *modulus* μ, may be made to depend on the solution of the equation in differences

$$u_n - u_{n+\mu} = \frac{\rho}{n-e} u_{n-e}.$$

The case where $e = 1$ is deserving of particular notice.

It may be shown by means of the above equation in differences, that the number of substitutions of n letters formed by r cycles each of the form $\mu K + 1$ (μ being constant), say $\phi(n, r, \mu, 1)$, where $\frac{n-r}{\mu}$ is necessarily an integer, may be found by taking in every possible way $\frac{n-r}{\mu}$ *distinct* groups of μ consecutive terms of the series $1, 2, 3, \ldots (n-1)$; the sum of the products of every such combination of groups is the value required. For example, if

$$n = 8, \quad r = 3, \quad \mu = 2,$$

$$\phi(8, 3, 2, 1) = 1.2.3.4.5.6 + 1.2.3.5.6.7 + 1.2.3.5.6.8$$
$$+ 1.2.3.6.7.8 + 2.3.4.5.6.7 + 2.3.4.6.7.8$$
$$+ 3.4.5.6.7.8.$$

And as a corollary, since it may easily be seen that $\phi(n, r, \mu, e)$ is always divisible by n when n is a prime and $\mu r + e < n$, it follows that the sum of all the possible products of (any given number) i distinct groups of a given number r of consecutive terms of the series $1, 2, 3, \ldots (n-1)$ will be divisible by n when n is a prime and $ir < n - 1$*. When $r = 1$, this theorem becomes identical with Wilson's, already referred to.

Finally, it may be noticed that the number of substitutions of n letters formed by *any* number of cycles, all of an *odd* order, will be the coefficient of x^n in $\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$, that is, $[1 . 3 . 5 \ldots (n-1)]^2$ (the same as the number that can be formed with cycles all of an *even* order) when n is even, and

$$[1 . 3 . 5 \ldots (n-2)]^2 \, n$$

when n is odd†.

* For instance, making $n = 7$, $r = 2$, $i = 2$,

$$1.2.3.4 + 1.2.4.5 + 1.2.5.6 + 2.3.4.5 + 2.3.5.6 + 3.4.5.6 = 784$$

and is *divisible by* 7.

† By taking $\mu = 2$ in the general theorem, it is an easy inference that if we write

$$(\tan^{-1} x)^r = x^r - \frac{A_2 x^{r+2}}{(r+1)(r+2)} + \frac{A_4 x^{r+4}}{(r+1)(r+2)(r+3)(r+4)} \mp \&c.,$$

A_{2i} will be the sum of all the products of $2i$ integers comprised between 1 and $r + 2i - 1$ that can be formed with combinations of i distinct pairs of consecutive integers; thus, for example, the coefficient of x^{2m} in $(\tan^{-1} x)^2$ ought to be

$$\frac{1}{m}\left(1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2m-1}\right),$$

which may be easily verified.

51.

NOTE ON A DIRECT METHOD OF OBTAINING THE EXPANSION OF THE SINE OR COSINE OF MULTIPLE ARCS IN TERMS OF POWERS OF THE SINES OR COSINES OF THE SIMPLE ARC BY MEANS OF DE MOIVRE'S THEOREM.

[*Quarterly Journal of Mathematics*, IV. (1861), pp. 159—163.]

THE annexed appears to be the most direct and natural method for obtaining the known formulæ for the expansion of the sines and cosines of multiple arcs.

We know by De Moivre's theorem, that

$$\cos 2nx = (\cos x)^{2n} - 2n \cdot \frac{2n-1}{2} (\sin x)^2 (\cos x)^{2n-2} + \&c.$$

Let $(\sin x)^2 = \gamma$, then

$$\cos 2nx = (1-\gamma)^n - 2n \frac{2n-1}{2} \gamma (1-\gamma)^{n-1}$$

$$+ 2n \frac{2n-1}{2} \cdot \frac{2n-2}{3} \cdot \frac{2n-3}{4} \gamma^2 (1-\gamma)^{n-2}, \&c.$$

$$= A_0 - A_1\gamma + A_2\gamma^2 - A_3\gamma^3, \&c.$$

I use $\omega_r \phi x$ to indicate the coefficient of x^r in ϕx expanded in a series of powers of x. We have then

$$A_r = P_0 Q_0 + P_1 Q_1 + P_2 Q_2 + \&c.,$$

where

$$P_0 = \omega_r (1+t)^n = \omega_r (1-t)^{-(n-r+1)} = \omega_{2r} (1-t^2)^{-(n-r+1)},$$

$$P_1 = \omega_{r-1} (1+t)^{n-1} = \omega_{r-1} (1-t)^{-(n-r+1)} = \omega_{2r-2} (1-t^2)^{-(n-r+1)},$$

$$P_2 = \omega_{r-2} (1+t)^{n-2} = \omega_{r-2} (1-t)^{-(n-r+1)} = \omega_{2r-4} (1-t^2)^{-(n-r+1)},$$

&c. &c.

$$Q_0 = 1 = \omega_0 (1+t)^{2n},$$

$$Q_1 = 2n \frac{2n-1}{2} = \omega_2 (1+t)^{2n},$$

$$Q_2 = 2n \frac{2n-1}{2} \cdot \frac{2n-2}{3} \cdot \frac{2n-3}{4} = \omega_4 (1+t)^{2n},$$

&c. &c.

Hence evidently,

$$A_r = \omega_{2r} \{(1-t^2)^{-(n-r+1)} \times (1+t)^{2n}\} = \omega_{2r} \{(1-t)^{-(n-r+1)} \times (1+t)^{n+r-1}\}^*.$$

To fix the ideas, suppose $r = 2$, then

$$A_2 = \omega_4 \left\{ \begin{array}{l} \left\{1 + (n-1)t + \frac{(n-1)n}{2} t^2 \right. \\ \quad \left. + \frac{(n-1)n(n+1)}{2.3} t^3 + \frac{(n-1)n(n+1)(n+2)}{2.3.4} t^4 \right\} \\ \times \left\{1 + (n+1)t + \frac{(n+1)n}{2} t^2 \right. \\ \quad \left. + \frac{(n+1)n(n-1)}{2.3} t^3 + \frac{(n+1)n(n-1)(n-2)}{2.3.4} t^4 \right\} \end{array} \right\}$$

$$= \frac{(n+1)n(n-1)(n-2) + (n+2)(n+1)n(n-1)}{1.2.3.4}$$

$$+ \frac{(n+1)n(n-1)(n-1) + (n+1)(n+1)n(n-1)}{1.2.3 \times 1}$$

$$+ \frac{n(n-1)n(n+1)}{1.2 \times 1.2}$$

$$= \frac{2n(n-1)n(n+1)}{1.2.3.4} + \frac{2n(n-1)n(n+1)}{1.2.3 \times 1} + \frac{n(n-1)n(n+1)}{1.2 \times 1.2}$$

$$= \omega_4 \left(1 + t + \frac{t^2}{1.2} + \frac{t^3}{1.2.3} + \frac{t^4}{1.2.3.4}\right)^2 (n^2-1) n^2;$$

and so in general, we shall have

$$A_r = n(n-r+1)(n-r+2)\ldots(n+r-1)$$

$$\times \omega_{2r} \left(1 + t + \frac{t^2}{1.2} + \ldots + \frac{t^{2r}}{1.2\ldots 2r}\right)^2$$

$$= \omega_{2r} e^{2t} \times n \{(n-r+1)\ldots(n+r-1)\}$$

$$= \frac{2^{2r}}{1.2.3.4\ldots 2r} n^2 (n^2-1)(n^2-4)\ldots\{n^2-(r-1)^2\},$$

and thus

$$\cos 2nx = 1 - \frac{n^2}{1.2} (2\sin x)^2 + \frac{n^2(n^2-1)}{1.2.3.4} (2\sin x)^4 \mp \&c.$$

In like manner we have

$$\cos(2n+1)x = \cos x \{(1-\gamma)^n - \tfrac{1}{2}(2n+1) 2n\gamma (1-\gamma)^{n-1} + \&c.\}$$

$$= \cos x \{B_0 - B_1\gamma + B_2\gamma^2 + \text{etc.}\},$$

where

$$B_r = \omega_{2r} \{(1-t^2)^{-(n-r+1)} (1+t)^{2n+1}\}$$

$$= \omega_{2r} \{(1-t)^{-(n-r+1)} \times (1+t)^{n+r}\};$$

* Note well this simple change in the form of the generating function; in it the point and pith of the method resides.

and making, as before, $r = 2$, we see that

$$B_2 = \omega_4 \left\{ \begin{aligned} &\left\{1 + (n-1)t + \frac{(n-1)n}{2} t^2 \right. \\ &\quad \left. + \frac{(n-1)n(n+1)}{1.2.3} t^3 + \frac{(n-1)n(n+1)(n+2)}{1.2.3.4} t^4 \right\} \\ &\times \left\{1 + (n+2)t + \frac{(n+2)(n+1)}{2} t^2 \right. \\ &\quad \left. + \frac{(n+2)(n+1)n}{1.2.3} t^3 + \frac{(n+2)(n+1)n(n-1)}{1.2.3.4} t^4 \right\} \end{aligned} \right\}$$

$$= \omega_4 (e^{2t}) \times (n-1) n (n+1)(n+2),$$

and so in general,

$$B_r = \omega_{2r} e^{2t} \{(n-r+1)(n-r+2) \dots n \dots (n+r-1)(n+r)\}$$
$$= \frac{(n-r+1)(n-r+2) \dots (n+r)}{1.2.3 \dots 2r} 2^{2r},$$

and thus

$$\cos(2n+1)x = \cos x \left\{1 - \frac{n(n+1)}{2} (2 \sin x)^2 \right.$$
$$\left. + \frac{(n-1)n(n+1)(n+2)}{1.2.3.4} (2 \sin x)^4 \dots \&c. \right\}.$$

We might in like manner, and by precisely the same process, obtain the expressions for $\cos 2mx$, $\cos(2m+1)x$ in terms of $\cos x$, and of $\sin 2mx$, $\sin(2m+1)x$ in terms of $\sin x$ or of $\cos x$, but these results may, of course, be most readily found by means of obvious processes of differentiation in respect to the arc and by substitution of the complement for the arc itself in the results already obtained.

It may be worth while to show here how the same elementary theorem as we have employed above, furnishes, *uno ictu*, another important formula connected with multiple arcs:

$$\left(\frac{d}{dx}\right)^{n-1} (1-x^2)^{\frac{2n-1}{2}} = \left(\frac{d}{dx}\right)^{n-1} \left\{(1+x)^{\frac{2n-1}{2}} (1-x)^{\frac{2n-1}{2}}\right\},$$

by Leibnitz's Theorem,

$$= \frac{2n-1}{2} \cdot \frac{2n-3}{2} \dots \frac{3}{2} \sqrt{(1-x^2)} (1-x)^{n-1}$$
$$- (n-1) \times \frac{2n-1}{2} \cdot \frac{2n-3}{2} \dots \frac{5}{2} \times \frac{2n-1}{2} \sqrt{(1-x^2)} (1-x)^{n-2} (1+x)$$
$$+ (n-1) \frac{n-2}{2} \times \frac{2n-1}{2} \cdot \frac{2n-3}{2} \dots \frac{7}{2}$$
$$\times \frac{2n-1}{2} \cdot \frac{2n-3}{2} \sqrt{(1-x^2)} (1-x)^{n-3} (1+x)^2 \mp \&c.$$

$$= \frac{1.3.5 \dots (2n-1)}{2^{n-1}} \sqrt{(1-x^2)}$$
$$\times \{(1-x)^{n-1} - A_1 (1-x)^{n-2} (1+x) + A_2 (1-x)^{n-3} (1+x)^2 \mp \&c.\},$$

where

$$A_r = \frac{2n-1}{2} \cdot \frac{2n-3}{2} \dots \frac{2n-(2r-1)}{2}$$
$$\times \frac{(n-1)(n-2) \dots (n-r)}{1.2 \dots r}$$
$$\times \frac{2}{3} \cdot \frac{2}{5} \dots \frac{2}{2r+1}$$
$$= \frac{(2n-1)(2n-2)(2n-3)(2n-4) \dots \{2n-(2r-1)\}}{2.3.4.5 \dots (2r+1)}$$
$$= \frac{1}{2n} \left[\frac{2n(2n-1) \dots (2n-2r)}{1.2 \dots (2r+1)}\right].$$

Hence, making $x = \cos 2\phi$,

$$\left(\frac{d}{dx}\right)^{n-1} (1-x^2)^{\frac{2n-1}{2}} = \frac{1.3.5 \dots (2n-1)}{n}$$
$$\times \left\{2n (\sin\phi)^{2n-1} \cos\phi - \frac{2n(2n-1)(2n-2)}{1.2.3} (\sin\phi)^{2n-3} (\cos\phi)^3 \pm \&c.\right\}$$
$$= \frac{1.3.5 \dots (2n-1)}{n} \cdot \frac{\{\sin\phi + \sqrt{(-1)}\cos\phi\}^{2n} - \{\sin\phi - \sqrt{(-1)}\cos\phi\}^{2n}}{2\sqrt{(-1)}}$$
$$= (-)^{n-1} \frac{1.3.5 \dots (2n-1)}{n} \sin 2n\phi$$
$$= (-)^{n-1} \frac{1.3.5 \dots (2n-1)}{n} \sin\{n \sin^{-1} \sqrt{(1-x^2)}\},$$

or if we please to pass to the more general form by a linear transformation,

$$\left(\frac{d}{dx}\right)^{n-1} (A + 2Bx - Cx^2)^{\frac{2n-1}{2}}$$
$$= (-)^{n-1} \frac{1.3.5 \dots (2n-1)}{n} C^{\frac{2n-3}{2}} \sqrt{(AC+B^2)} \sin n \sin^{-1} \sqrt{\left(\frac{A + 2Bx - Cx^2}{A + \frac{B^2}{C}}\right)}.$$

52.

NOTE ON CERTAIN DEFINITE INTEGRALS.

[*Quarterly Journal of Mathematics*, IV. (1861), pp. 319—324.]

IN the *Institutiones Calculi Integralis*, Euler has investigated the value of the definite integral $\int_1^0 \frac{\log x\,dx}{\sqrt{(1-x^2)}}$, and his mode of statement seems to imply that the result, as well as the demonstration, was his own. How this may be, in fact, I do not pretend to know: in the *Philosophical Magazine** of December, 1860, I have (under another notation) investigated the values of

$$\int_1^0 \frac{\log x\,dx}{\sqrt{(1-x^2)(1-c^2x^2)}} \text{ and } \int_1^0 \frac{\log\{1+\sqrt{(1-c^2x^2)}\}}{\sqrt{\{(1-x^2)(1-c^2x^2)\}}}\,dx,$$

and shown them to be equal, but of course with contrary signs, and the former to be expressed† by $\frac{1}{2}\log cF(c)+\frac{1}{4}\pi F(b)$. The relation of which (in regard to the form of the functions of which it is composed) to the integral of its differential without $\log x$ in the numerator is so strikingly analogous to the relation of Euler's more simple integral (namely, $\frac{1}{2}\pi\log 2$) to $\int_1^0 dx\,\frac{1}{\sqrt{(1-x^2)}}$ as to suggest the existence of some general theorem in which both these results are comprised.

In proving the equality of the two definite integrals in question, a third integral of different form from either came to light. In fact, it is shown in the paper referred to, that

$$\frac{\log\{1+\sqrt{(1-t)}\}}{\sqrt{(1-t)}}$$
$$=\frac{2}{\pi}\int_{\frac{1}{2}\pi}^0 d\phi\,\{\log(\cos\phi)+(\cos\phi)^2\log(\cos\phi)\,t+(\cos\phi)^4\log(\cos\phi)\,t^2+\text{etc.}\},$$

with a tacit supposition that t is contained within the limits (both *inclusive*) $+1$ and -1, within which limits the series which expresses $\frac{\log\{1+\sqrt{(1-t)}\}}{\sqrt{(1-t)}}$ in powers of t remains convergent.

[* p. 213 above.] [† where $b^2=1-c^2$.]

If now we make $1-t=c^2$, and write θ in place of ϕ, we have

$$\int_{\frac{1}{2}\pi}^0 \frac{\log\cos\theta\,d\theta}{1-t(\cos\theta)^2}=\int_{\frac{1}{2}\pi}^0 \frac{\log\cos\theta\,d\theta}{(\sin\theta)^2+c^2(\cos\theta)^2}=\frac{\pi}{2}\frac{\log(1+c)}{c},$$

with the restriction that c must be positive.

The limits of convergency imply furthermore (as far as the demonstration given is concerned) that c^2 should not be greater than 2, but since neither side of the equation passes through *a critical* phase in any sense (that is, as regards either themselves or their successive differentials) for this, or indeed for any positive value of c, it seems to follow that the equation must continue good from $c=0$ to $c=\infty$, and that the limitation which the cessation of the convergency of the intermediary series might have required to be placed upon the subsistence of the equation may in effect be disregarded. Perhaps also it would be desirable to inquire whether the equality may not continue to subsist for imaginary values of c with a positive real part.

Knowing the value of $\frac{2}{\pi}\int_{\frac{1}{2}\pi}^0 \frac{d\theta\log\cos\theta}{\{(\sin\theta)^2+c^2(\cos\theta)^2\}^i}$, namely, $\frac{\log(1+c)}{c}$ when $i=1$, we may obviously obtain an expression involving only logarithms and algebraical quantities for all integer values of i; indeed, calling the above integral u_i, we easily obtain the formula of reduction,

$$u_{i+1}-u_i=-\frac{1-c^2}{2ic}\frac{d}{dc}u_i,$$

from which it may readily be shown that u_i will be of the form

$$\left(\frac{A_1}{c}+\frac{A_2}{c^3}+\ldots+\frac{A_i}{c^{2i-1}}\right)\log(1+c)+\frac{B_1}{c}+\frac{B_2}{c^2}+\frac{B_3}{c^3}+\ldots+\frac{B_{2i-2}}{c^{2i-2}},$$

where the two sets of numerators are *constant*, the law of which it may be desirable at some future time to investigate. It should be noticed that although $(1+c)$ appears in the denominator of $\frac{du_i}{dc}$, it does not make its appearance in u_i by reason of the numerator $1-c^2$ in the expression for Δu_i.

The series expressing $\frac{\log\{1+\sqrt{(1-t)}\}}{\sqrt{(1-t)}}$ from which the value of u_1 has been derived, is the following:

$$\frac{\log\{1+\sqrt{(1-t)}\}}{\sqrt{(1-t)}}=\log 2\left(1+\frac{1}{2}t+\frac{1.3}{2.4}t^2+\frac{1.3.5}{2.4.6}t^3+\text{\&c.}\right)$$
$$-\left\{\frac{1}{1.2}\frac{1}{2}t+\left(\frac{1}{1.2}+\frac{1}{3.4}\right)\frac{1.3}{2.4}t^2\right.$$
$$\left.+\left(\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}\right)\frac{1.3.5}{2.4.6}t^3+\text{\&c.}\right\},$$

(see *Philosophical Magazine**, December, 1860, p. 530, where t^2 is used in place of t) but since

$$\frac{d}{dt}\log\{1+\sqrt{(1-t)}\} = -\frac{1}{2\sqrt{(1-t)}}\,\frac{1-\sqrt{(1-t)}}{t} = \frac{1}{2t} - \frac{1}{2t\sqrt{(1-t)}},$$

and consequently

$$\log\{1+\sqrt{(1-t)}\} = \log 2 - \frac{1}{2^2}t - \frac{1.3}{2.4^2}t^2 - \frac{1.3.5}{2.4.6^2}t^3 - \&c.,$$

the more natural (as the more obvious) mode of deducing the coefficients of the powers of t in the expansion of $\dfrac{\log\{1+\sqrt{(1-t)}\}}{\sqrt{(1-t)}}$, would *seem* to be to multiply the series written above by the series

$$1+\frac{1}{2}t+\frac{1.3}{2.4}t^2+\frac{1.3.5}{2.4.6}t^3+\&c.$$

This method of proceeding would however in fact have left obscure the true nature of those coefficients. Let us perform the multiplication indicated; we shall then obtain, by comparison with the expansion already obtained, the following very far from obvious, indeed very unlikely to be suspected identity, which it is desirable to put on record: namely,

$$\frac{1}{2}\cdot\frac{1}{2}\,\frac{1.3.5\ldots(2i-3)}{2.4.6\ldots(2i-2)} + \frac{1}{4}\cdot\frac{1.3}{2.4}\,\frac{1.3.5\ldots(2i-5)}{2.4.6\ldots(2i-4)} + \ldots$$
$$+\frac{1}{2i-2}\,\frac{1.3.5\ldots(2i-3)}{2.4.6\ldots(2i-2)} + \frac{1}{2i}\,\frac{1.3.5\ldots(2i-1)}{2.4.6\ldots 2i}$$
$$= \frac{1.3.5\ldots(2i-1)}{2.4.6\ldots 2i}\left\{\frac{1}{1.2}+\frac{1}{3.4}+\ldots+\frac{1}{(2i-1)\,2i}\right\}.$$

Thus, for example, if $i=4$,

$$\frac{1}{2^2}\,\frac{1.3.5}{2.4.6}+\frac{1.3}{2.4^2}\,\frac{1.3}{2.4}+\frac{1.3.5}{2.4.6^2}\,\frac{1}{2}+\frac{1.3.5.7}{2.4.6.8^2}$$
$$=\frac{1.3.5.7}{2.4.6.8}\left(\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\frac{1}{7.8}\right).$$

The expansion above referred to leads to the value of u_1 through the intervention of the equality [see p. 212 above]

$$\frac{2}{\pi}\int_{\frac{1}{2}\pi}^{0}\log(\cos\theta)(\cos\theta)^{2n}\,d\theta = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\sum_{k=\infty}^{k=1}\frac{1}{(2n+2k-1)(2n+2k)},$$

established in the paper referred to: and it is not uninteresting to notice that the above equality enables us to determine the value of the integral on the left-hand side of the equation in a series of descending powers of n, from

[* p. 213 above.]

which doubtless many new conclusions may be deduced. The first term in this descending series being $\frac{1}{4n}$, we are enabled to fix the degree of the integral, when n becomes infinite, for in that case

$$\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n} = \sqrt{\left(\frac{2}{\pi}\cdot\frac{1}{2n}\right)} = \frac{1}{\sqrt{(\pi n)}},$$

so that for $n=\infty$,

$$\int_{\frac{1}{2}\pi}^{0}(\cos\theta)^{2n}\log\cos\theta\, d\theta = \frac{\pi}{2}\,\frac{1}{\sqrt{(\pi n)}}\cdot\frac{1}{4n} = \frac{\sqrt{(\pi)}}{(2n^{\frac{1}{2}})^3}.$$

The consideration of the expansion above referred to, namely, of

$$\frac{1}{m+1}-\frac{1}{m+2}+\frac{1}{m+3}+\&c.$$

in a series of descending powers of $\frac{1}{m}$, or which is the same thing, of

$$\frac{\mu}{\mu+1}-\frac{\mu}{2\mu+1}+\frac{\mu}{3\mu+1},$$

in a series of ascending powers of μ, suggests an observation which may appear to amount to a mere futile distinction, but which, closely examined, will be found to have a real signification and importance.

The above series being obtained by means of the equivalence $\Sigma = e^{\frac{d}{dx}}-1$ will readily be seen to import Bernoulli's numbers in such a manner into the development that the latter would commonly be said (like all the series of the same class) to be absolutely divergent, incapable, that is to say, of constituting an arithmetical equivalent to its generatrix for any value whatever of the variable μ. The distinction I would draw would be to say not that the circle of convergence of μ ceases to exist, but that it becomes indefinitely small, or which is the same thing, the *corona* of convergence for the series treated as a function of m, has its inner radius indefinitely large: so that for $\mu=0$ or $m=\infty$, we may reason, and reason with perfect safety, upon the equality between the generating function and the series as subsisting in an arithmetical sense as regards not only μ or m, but all successive powers of the same. [I mean that supposing

$$\phi\mu = a_0 + a_1\mu + a_2\mu^2 + \ldots,$$

we may affirm not only the equality

$$\phi\mu - a_0 = 0,$$

but also

$$\phi\mu - a_0 = 0,\quad (\phi\mu - a_0)\div\mu = a_1,\quad (\phi\mu - a_0 - a_1\mu)\div\mu^2 = a_2,$$

and so on when $\mu=0$.]

This fact of the character of arithmetical equivalence within certain limits not absolutely departing even in the case of series considered irreclaimably divergent, may, I think, serve to account, *à priori*, for the phenomenon of many conclusions being capable of being truthfully drawn from reasonings upon them in which they are treated as though they were in an ordinary sense convergent, because, in fact, part of the attributes of ordinary convergency (all such indeed as are not nullified by the radius of convergency becoming infinitely small) must continue to adhere to such series.

The expansion for

$$\frac{1}{2n+1}-\frac{1}{2n+2}+\frac{1}{2n+3}\ \&c.,$$

which occurs in the expression for $\int_{\frac{1}{2}\pi}^{0}\log\cos\theta(\cos\theta)^{2n}d\theta$ in a series proceeding according to powers of $\frac{1}{n}$, may be most readily obtained by means of the differences of zero, as follows: calling $2n=x$, we have

$$\frac{1}{x+1}-\frac{1}{x+2}+\frac{1}{x+3}-\frac{1}{x+4}+\&c.$$

$$=\{(1+\Delta)-(1+\Delta)^2+(1+\Delta)^3\ldots\}\frac{1}{x+0}$$

$$=\frac{1+\Delta}{2+\Delta}\left(\frac{1}{x}-\frac{1}{x^2}0+\frac{1}{x^3}0^2\pm\&c.\right)$$

$$=\frac{1}{x}-\frac{1}{2+\Delta}\left(\frac{1}{x}-\frac{1}{x^2}0+\frac{1}{x^3}0^2\pm\&c.\right),$$

so that the first term will be $\frac{1}{2x}$ or $\frac{1}{4n}$. This might also be shown in a strictly arithmetical method as follows: let

$$s=\frac{1}{(x+1)(x+2)}+\frac{1}{(x+3)(x+4)}+\ldots\ ad\ inf.=s_1+s_2+s_3+\ldots,$$

where

$$s_1=\frac{1}{(x+1)(x+2)}+\frac{1}{(x+3)(x+4)}+\ldots+\frac{1}{(\epsilon x-1)\,\epsilon x},$$

$$s_2=\frac{1}{(\epsilon x+1)(\epsilon x+2)}+\frac{1}{(\epsilon x+3)(\epsilon x+4)}+\ldots+\frac{1}{(\epsilon^2 x-1)\,\epsilon^2 x},$$

$$s_3=\frac{1}{(\epsilon^2 x+1)(\epsilon^2 x+2)}+\frac{1}{(\epsilon^2 x+3)(\epsilon^2 x+4)}+\ldots+\frac{1}{(\epsilon^3 x-1)\,\epsilon^3 x},$$

&c. = &c.,

ϵ being any real positive value superior to unity, and x being infinite. Then, observing that each partial series is a mean between the products of the number of terms in it, by the first and last respectively, we have obviously s always intermediate to

$$\frac{\epsilon-1}{2x}+\frac{(\epsilon-1)}{2\epsilon x}+\frac{(\epsilon-1)}{2\epsilon^2 x}+\&c.,$$

and

$$\frac{(\epsilon-1)}{2\epsilon x}+\frac{(\epsilon-1)}{2\epsilon^2 x}+\frac{(\epsilon-1)}{2\epsilon^3 x}+\&c.;$$

that is, between $\frac{\epsilon}{2x}$ and $\frac{1}{2x}$, and consequently, is equal to $\frac{1}{2x}$, as before.

53.

ON THE INVOLUTION OF AXES OF ROTATION.

[*Manchester British Association Report* (1861), p. 12.]

AFTER a brief statement as to the most general mode of representing the displacement of a rigid body in space by means of angular rotations about six distinct axes fixed in position, it was shown that under peculiar conditions the six axes would become insufficient, being, in fact, equivalent to a smaller number, in which case they would be said to form a system in involution. Various constructions for representing such and similar systems were stated, and the remarkable conclusion presented, that the necessary and sufficient condition for three, four, five, or six lines being thus mutually, as it were, implicated and involved, consists in their lying in ruled surfaces of the first, second, third, and fourth orders respectively. The theory of involution originated with Prof. Möbius, by whom, however, it had been left in an imperfect condition. The author referred for further information on the subject to some recent notes by himself in the *Comptes Rendus** of the Academy of Sciences of Paris, and to certain masterly geometrical investigations of M. Chasles and Mr Cayley, to which these had given rise.

[* p. 236 above.]

54.

ADDITION À LA DÉMONSTRATION DU THÉORÈME DE LAGRANGE SUR LES MINIMA D'UNE FONCTION LINÉAIRE À COEFFICIENTS ENTIERS D'UNE QUANTITÉ IRRATIONNELLE, DONNÉE DANS LA SÉANCE PRÉCÉDENTE*.

[*Comptes Rendus de l'Académie des Sciences*, LIV. (1862), pp. 53—55.]

ON peut à juste titre élever quelque objection contre la forme donnée au théorème cité en tant que j'ai posé comme *criterium* des réduites $\frac{p}{q}$ de l'irrationnelle ν, la condition que la valeur de $p - q\nu$ restera plus petite que toute valeur qui résulte de la diminution ou de p, ou de q, ou de p et q simultanément dans cette fonction, tandis que le *criterium* de Lagrange ne considère que l'effet de la substitution simultanée des nombres inférieurs à p et à q. On remédie à cet inconvénient et en même temps on simplifie la démonstration du théorème dont il est question en donnant un peu plus d'extension à la conclusion nommée A dans la Note précédente.

Dans l'équation (3), c'est-à-dire,

$$D\Delta' = (-1)^i (\theta - s\theta + r - ks),$$

si l'on pose $\quad s = l + 1, \quad r = ks + 1 = kl + k + 1$

(de sorte que $p - \lambda$, $q - \mu$ deviennent simultanément $-p'$, $-q'$), on aura

$$s\theta = (1 + l)\,\theta \begin{matrix} > 1 \\ < 1 + \theta \end{matrix} \quad \text{et} \quad \theta - s\theta + (r - ks) \begin{matrix} < \theta \\ > \theta' \end{matrix},$$

donc $\Delta'^2 < \Delta^2$, c'est-à-dire que les minima $p - q\nu$, $p' - q'\nu$, etc., vont toujours en diminuant; mais si, s restant égale à $l + 1$, r n'est pas prise égale à $ks + 1$, λ et μ tous les deux excéderont $p + p'$, $q + q'$ respectivement. Tel est donc l'effet des conditions caractéristiques du système p, q; pour qu'il soit possible que Δ'^2 soit moindre que Δ^2, $(p - \lambda)^2$ ne peut pas devenir p'^2 sans qu'en même temps $(q - p)^2$ devienne q'^2 et réciproquement.

[* p. 250 above.]

Conséquemment à la place de ladite conclusion A, on peut substituer l'énoncé suivant, c'est-à-dire $\frac{p}{q}$ *étant une réduite quelconque de* ν, $p - q\nu$ *s'augmentera en substituant pour* p *un nombre quelconque moindre que* p' *ou pour* q *un nombre moindre que* q', *pourvu qu'on ne substitue pas en même temps* p' *pour* p *et* q' *pour* q.

Avec cet énoncé, on peut se passer tout à fait de la conclusion B. La preuve que la condition de Lagrange est nécessaire découle et avec surabondance de cet énoncé: cela saute aux yeux; et quant à la suffisance ou *criterium*, on n'a qu'à remarquer que si $\frac{a}{b}$ n'est pas une réduite de ν, on peut prendre

$$a > p_e,\ a \gtreqless p_{e+1};\quad b > q_i,\ b \gtreqless q_{i+1}.$$

et alors

$$p_e - q_e\nu \text{ et } p_i - q_i\nu,$$

seront tous les deux $< a - b\nu$. De plus, on aura

$$p_e < a \text{ et } q_e < b,$$

ou bien

$$p_i < a \text{ et } q_i < b\,^*;$$

donc, dans tous les cas, $a - b\nu$ diminuera quand on diminuera dans une manière convenable a et b simultanément: ce qui démontre la différence du *criterium* dont il a été question.

* On n'a pas besoin de dire que rien n'empêche que e ne soit égal à i; mais dans ce cas, comme on ne peut pas avoir simultanément $a = p_{e+1}$, $b = q_{e+1}$, la conclusion du texte reste bonne.

55.

ON THE SOLUTION OF THE LINEAR EQUATION OF FINITE DIFFERENCES IN ITS MOST GENERAL FORM.

[*Cambridge British Association Report* (1862), p. 188.]

THE author exhibited (and illustrated with examples) a simple and readily applied method of obtaining the general term of, and consequently the complete, solution of an equation of finite differences with any number of independent variables, a question which, although touched upon by Libri and laboriously investigated by Binet, had hitherto, to the best of his knowledge, remained unsolved even in the case of an equation with but one independent variable with non-constant coefficients; when the coefficients are supposed constant, the well-known solution flows as an immediate corollary from the author's general form. Essentially the method depends upon the adoption of a natural principle of notation for the given coefficients, according to which each coefficient is to be denoted by a *twofold* group of indices, the number of the double indices in a group being equal to the number of independent variables in the given equation. Thus, supposing $u_{m,n,p\ldots}$ to be expressible, by means of the given general equation, as a sum of u's with inferior indices, the coefficient of $u_{\mu,\nu,\pi\ldots}$ in that sum must be denoted by the double index group

$$\begin{bmatrix} m, & n, & p \ldots \\ \mu, & \nu, & \pi \ldots \end{bmatrix}.$$

The process for obtaining the general term in $u_{x,y,z\ldots}$ is then shown to be reducible virtually to the problem of effecting the simultaneous *decomposition* of the integer variables $x, y, z\ldots$ into *parts* in every possible manner and order of relative arrangement, the magnitudes of such parts being limited by the degree or degrees of the given equation in respect of these variables. The collective value of the terms thus obtained constituting the complete solution may be termed, in the author's nomenclature, a hyper-cumulant, whose properties and their applications remain to be studied, as those of the elementary kinds of common cumulants have been, to a considerable extent, in the ordinary theory of continued fractions. The first stage in the process of constructing the terms of a general cumulant or general hyper-cumulant is almost identical with that of finding the coefficients in the expansion of a power of a polynomial function of one or several variables, differing from it indeed only in the circumstance that permutations which lead to repetitions in the latter case, represent distinct values in the former.

56.

SUR UNE CLASSE NOUVELLE D'ÉQUATIONS DIFFÉRENTIELLES ET D'ÉQUATIONS AUX DIFFÉRENCES FINIES D'UNE FORME INTÉGRABLE.

[*Comptes Rendus de l'Académie des Sciences*, LIV. (1862), pp. 129—132.]

COMMENÇONS par le cas des différences finies. Représentons par Δ_x le déterminant

$$\begin{vmatrix} u_x, & u_{x+1}, & u_{x+2} & \dots u_{x+i-1} \\ u_{x+1}, & u_{x+2}, & u_{x+3} & \dots u_{x+i} \\ u_{x+2}, & u_{x+3}, & u_{x+4} & \dots u_{x+i+1} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ u_{x+i-1}, & u_{x+i}, & u_{x+i+1} & \dots u_{x+2i-2} \end{vmatrix},$$

et considérons l'équation

$$\Delta_x = C, \dots, \qquad (1)$$

ce qui au fond est aussi général que si nous écrivions $\Delta_x = C\gamma^x$.

Je dis que l'équation (1) pourra être satisfaite par la même intégrale que celle qui satisfait à l'équation

$$u_x - p_1 u_{x+1} + p_2 u_{x+2} \dots (-1)^{i-1} p_{i-1} u_{x+i-1} + (-1)^i u_{x+i} = 0, \qquad (2)$$

$p_1, p_2, \dots, p_{i-1}$, étant des constantes. Car si cette dernière équation a lieu, on peut dans la première ligne du déterminant substituer à

$$u_x,\ u_{x+1} \dots u_{x+i-1}$$

les quantités

$$(-1)^{i-1} u_{x+i},\ (-1)^{i-1} u_{x+i+1} \dots (-1)^{i-1} u_{x+2i-1},$$

sans changer la *valeur* de ce déterminant.

Donc on voit immédiatement que Δ_x devient égal à Δ_{x+1}, c'est-à-dire Δ_x sera constant ; donc l'intégrale de $\Delta_x = C$ sera

$$u_x = a_1\alpha_1^x + a_2\alpha_2^x + \dots + a_i\alpha_i^x, \qquad (3)$$

avec la condition $\alpha_1\alpha_2 \dots \alpha_i = 1$. Cette condition est une conséquence de la forme du dernier coefficient, $(-1)^i$, dans l'équation (2); de plus une autre condition se présente à cause de la valeur spéciale qu'il faut attribuer à la constante C dans l'équation donnée.

Pour obtenir cette dernière condition nous pouvons considérer les a et les α comme étant données et C comme une fonction de ces quantités. Or en faisant un quelconque des a égal à zéro, le degré de l'équation (2) s'abaisse d'une unité, c'est-à-dire les i fonctions $u_x, u_{x+1}, \dots, u_{x+i-1}$ seront liées entre elles par une équation linéaire et conséquemment le déterminant Δ_x s'évanouira. Donc C contient le produit $a_1a_2 \dots a_i$ comme facteur. Mais on trouve aussi, en prenant $x=0$, C égal au déterminant à i lignes

$$\begin{vmatrix} \Sigma a, & \Sigma a\alpha & \dots \Sigma a\alpha^{i-1} \\ \Sigma a\alpha, & \Sigma a\alpha^2 & \dots \Sigma a\alpha^i \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \Sigma a\alpha^{i-1}, & \Sigma a\alpha^i & \dots \Sigma a\alpha^{2i-2} \end{vmatrix}$$

qui est du degré i par rapport aux quantités a.

Donc $$C = a_1a_2 \dots a_i F(\alpha_1, \alpha_2 \dots \alpha_i).$$

Pour déterminer F, on n'a qu'à supposer

$$a_1 = a_2 = \dots = a_i = 1,$$

et on obtient immédiatement par un théorème bien connu

$$F = (\alpha_1 - \alpha_2)^2(\alpha_1 - \alpha_3)^2(\alpha_2 - \alpha_3)^2 \dots (\alpha_{i-1} - \alpha_i)^2.$$

Donc finalement on aura pour l'intégrale complète de l'équation (1), qui est de l'ordre $(2i-2)$, le système d'équations

$$\left.\begin{aligned} u_x = a_1\alpha_1^x + a_2\alpha_2^x + \dots + a_i\alpha_i^x, \\ \text{avec les conditions} \quad \alpha_1\alpha_2 \dots \alpha_i = 1, \\ a_1a_2 \dots a_i[(\alpha_1-\alpha_2)^2 \dots (\alpha_{i-1}-\alpha_i)^2] = C, \end{aligned}\right\} \qquad (4)$$

système qui contient $(2i-2)$ constantes, le nombre qu'on doit avoir.

On peut appliquer cette même méthode à un système d'équations beaucoup plus général. Car si on désigne par $P_1, P_2, \dots, P_{i-1}$ les fonctions algébriques de $u_x, u_{x+1}, \dots, u_{x+2i-2}$ qui satisfont au système simultané des $(i-1)$ équations

$$\begin{aligned} &u_x - P_1u_{x+1} + P_2u_{x+2} \dots - (-1)^i P_{i-1}u_{x+i-1} + (-1)^i u_{x+i} = 0, \\ &u_{x+1} - P_1u_{x+2} + P_2u_{x+3} \dots - (-1)^i P_{i-1}u_{x+i} + (-1)^i u_{x+i+1} = 0, \\ &\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots \\ &u_{x+i-2} - P_1u_{x+i-1} + P_2u_{x+i} \dots - (-1)^i P_{i-1}u_{x+2i-3} + (-1)^i u_{x+2i-2} = 0, \end{aligned}$$

et si, en conservant à Δ_x la même valeur que dans l'équation (1), on écrit

$$\Delta_x + \phi(P_1, P_2, \ldots, P_{i-1}) = 0, \tag{5}$$

il est évident qu'en faisant

$$u_x - p_1 u_{x+1} + p_2 u_{x+2} \ldots - (-1)^i p_{i-1} u_{x+i-1} + (-1)^i u_{x+i} = 0,$$

Δ_x sera égal à Δ_{x+1} et ϕ sera toujours constant, car on aura

$$P_1 = p_1,\ P_2 = p_2,\ \ldots,\ P_{i-1} = p_{i-1}.$$

Donc l'équation (5) sera satisfaite par l'intégrale

$$\left.\begin{aligned} &u_x = a_1\alpha_1^x + a_2\alpha_2^x + \ldots + a_i\alpha_i^x, \\ \text{avec les conditions}\quad &\alpha_1\alpha_2\ldots\alpha_i = 1, \\ &(a_1a_2\ldots a_i)\{(\alpha_1-\alpha_2)^2(\alpha_1-\alpha_3)^2(\alpha_2-\alpha_3)^2\ldots(\alpha_{i-1}-\alpha_i)^2\} \\ &\quad + \phi(\Sigma\alpha_1, \Sigma\alpha_1\alpha_2, \ldots, \Sigma\alpha_i\ldots\alpha_{i-1}) = 0. \end{aligned}\right\} \tag{6}$$

Passons au cas de la forme analogue des équations différentielles. En supposant y une fonction de x, j'écrirai $\frac{d^iy}{dx^i} = y_i$, et je nommerai D_x^iy le déterminant

$$\begin{vmatrix} y, & y_1, & y_2 & \cdots y_{i-1} \\ y_1, & y_2, & y_3 & \cdots y_i \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ y_{i-1}, & y_i, & y_{i+1} & \cdots y_{2i-2} \end{vmatrix}.$$

Considérons d'abord l'équation

$$D_x^iy = C. \tag{7}$$

Sans prendre la peine de passer par les moyens connus du cas des différences finies à des différences infiniment petites, il suffit de faire le rapprochement de la valeur de $\frac{u_{x+1}}{u_x}$ quand $u_x = \alpha^x$ avec celle de $\frac{d_xy}{y}$ quand $y = e^{\alpha x}$ pour conclure immédiatement de la forme de l'intégrale (1) celle de l'équation (7) qui sera évidemment

$$\left.\begin{aligned} &y = a_1e^{\alpha_1x} + a_2e^{\alpha_2x} + \ldots + a_ie^{\alpha_ix} \\ \text{avec les conditions}\quad &\alpha_1 + \alpha_2 + \ldots + \alpha_i = 0, \\ &a_1a_2\ldots a_i(\alpha_1-\alpha_2)^2(\alpha_1-\alpha_3)^2(\alpha_2-\alpha_3)^2\ldots(\alpha_{i-1}-\alpha_i)^2 = C. \end{aligned}\right\} \tag{8}$$

Avant de considérer quelques modifications très-intéressantes de cette équation, il sera utile d'établir un théorème élémentaire sur les rapports des formes consécutives D_x^iy entre elles.

Pour fixer les idées, bornons-nous pour le moment à la considération du déterminant

$$\begin{vmatrix} y, & y_1, & y_2, & y_3 \\ y_1, & y_2, & y_3, & y_4 \\ y_2, & y_3, & y_4, & y_5 \\ y_3, & y_4, & y_5, & y_6 \end{vmatrix},$$

c'est-à-dire D_x^4y, et des déterminants *mineurs* qu'il renferme.

Posons

$$D_x^3y = \begin{vmatrix} y, & y_1, & y_2 \\ y_1, & y_2, & y_3 \\ y_2, & y_3, & y_4 \end{vmatrix}.$$

En différentiant les quantités qui entrent dans ce déterminant *ligne* sur *ligne*, on formera trois déterminants nouveaux dont tous s'évanouiront identiquement à cause de l'égalité de deux lignes (terme à terme) qui en résultera, sauf toutefois le dernier qui sera

$$\begin{vmatrix} y, & y_1, & y_2 \\ y_1, & y_2, & y_3 \\ y_3, & y_4, & y_5 \end{vmatrix}$$

et qui exprimera conséquemment la valeur de $\frac{d}{dx}(D_x^3y)$.

De même en différentiant ce dernier déterminant (*colonne à colonne*), on obtiendra

$$\begin{vmatrix} y, & y_1, & y_3 \\ y_1, & y_2, & y_4 \\ y_3, & y_4, & y_6 \end{vmatrix}$$

comme la valeur de $\frac{d^2}{dx^2}(D_x^3y)$.

On remarquera que tous les termes du nouveau déterminant

$$\begin{vmatrix} D_x^3y, & \frac{d}{dx}(D_x^3y) \\ \frac{d}{dx}(D_x^3y), & \frac{d^2}{dx^2}(D_x^3y) \end{vmatrix}$$

seront des déterminants mineurs de D_x^4y, et par un théorème tres-connu on conclut que ce déterminant composé sera égal au produit $D_x^2y \times D_x^4y$, c'est-à-dire

$$D_x^2y \times D_x^4y = D_x^2(D_x^3y),$$

et dans la même manière on peut établir l'équation générale qui lie ensemble trois termes consécutifs quelconques de la série

$$D^1,\ D^2,\ D^3,\ D^4,\ D^5\ \ldots,$$

c'est-à-dire

$$D_x^{i-1}y \times D_x^{i+1}y = D_x^2\,(D_x^i y). \qquad (9)$$

Avec l'aide de cette équation on parvient facilement à l'intégration d'une classe très-intéressante d'équations différentielles du quatrième ordre, parmi lesquelles on peut distinguer les équations

$$D_x^3 y = Cy^3, \quad (D_x^3 y)^2 = C\,(D_x^2 y)^3,$$

lesquelles ne sont que deux cas particuliers d'équations qu'on peut intégrer par le moyen des fonctions elliptiques inverses.

57.

ÉQUATIONS DIFFÉRENTIELLES. ADDITION À UNE NOTE INSÉRÉE DANS LE COMPTE RENDU DE LA SÉANCE PRÉCÉDENTE SUR UNE FORME NOUVELLE D'ÉQUATIONS DIFFÉRENTIELLES ET INTÉGRABLES.

[*Comptes Rendus de l'Académie des Sciences*, LIV. (1862), pp. 170—174.]

En se rappelant la forme du coefficient différentiel de $D_x^i y$ par rapport à x que j'ai donnée dans la Note indiquée ci-dessus*, il est aisé de voir qu'on peut arriver, par une méthode directe, à la solution de l'équation

$$D_x^i y = Le^{\lambda x} \qquad (1)$$

en se servant de l'équation auxiliaire

$$y_i - \lambda y_{i-1} + \mu y_{i-2} + \nu y_{i-3} \ldots + \omega y = 0, \qquad (2)$$

$\mu, \nu, \ldots, \omega$ étant des constantes arbitraires.

En prenant, par exemple, $i = 3$, on voit qu'en supposant l'équation $y_3 - \lambda y_2 + \mu y_1 + \nu y = 0$ satisfaite, le déterminant

$$\begin{vmatrix} y, & y_1, & y_2 \\ y_1, & y_2, & y_3 \\ y_2, & y_3, & y_4 \end{vmatrix} \text{ deviendra égal à } \frac{1}{\lambda}\begin{vmatrix} y, & y_1, & y_2 \\ y_1, & y_2, & y_3 \\ y_3, & y_4, & y_5 \end{vmatrix};$$

car, au lieu de y_2, y_3, y_4, dans la dernière ligne du premier de ces déterminants, on peut alors substituer respectivement $\frac{1}{\lambda}y_3, \frac{1}{\lambda}y_4, \frac{1}{\lambda}y_5$, tout simplement.

Donc $\frac{d}{dx}(D_x^3 y)$ devient égal à $\lambda D_x^3 y$, et l'équation $D_x^3 y = Le^{\lambda x}$ peut être satisfaite par l'intégrale de l'équation (2) en déterminant convenablement

[* p. 310 above.]

les rapports entre les constantes arbitraires qui y figurent. De même on voit, en général, que l'intégrale de $D_x{}^i y = Le^{\lambda x}$ sera

$$y = a_1 e^{\alpha_1 x} + a_2 e^{\alpha_2 x} + \ldots + a_i e^{\alpha_i x} \tag{3}$$

avec les conditions
$$\alpha_1 + \alpha_2 + \ldots + \alpha_i = \lambda, \tag{4}$$

$$a_1 a_2 \ldots a_i (\alpha_1 - \alpha_2)^2 (\alpha_1 - \alpha_3)^2 \ldots (\alpha_{i-1} - \alpha_i)^2 = L. \tag{5}$$

Rien n'empêche de prendre un nombre quelconque des α et d'en faire différer les valeurs infiniment peu les unes des autres ; on peut aussi, en général, former plusieurs groupes distincts de cette espèce. En agissant de cette façon, on arrive, par une analyse facile à retrouver et par le moyen d'un lemme que j'exposerai tout à l'heure, à des formes spéciales (pour ne pas dire singulières) de l'équation (3), dont voici le type le plus général :

$$y = Xe^{\alpha x} + X_1 e^{\alpha_1 x} + \ldots \tag{6}$$

où
$$X = ax^{n-1} + bx^{n-2} \ldots + l,$$
$$X_1 = a_1 x^{n_1 - 1} + b_1 x^{n_1 - 2} \ldots + l_1,$$
....

avec les conditions suivantes :

$$\Sigma n = i, \tag{7}$$

$$\Sigma \alpha n = \lambda, \tag{8}$$

$$\Gamma n \,.\, a^n \,.\, \Gamma n_1 \,.\, a_1^{n_1} \,.\, \Gamma n_2 \,.\, a_2^{n_2} \ldots (\alpha - \alpha_1)^{2nn_1} (\alpha - \alpha_2)^{2nn_2} (\alpha_1 - \alpha_2)^{2n_1 n_2} \ldots = L. \tag{9}$$

Le nombre total de ces formes (l'intégrale générale y comprise) sera le nombre des partitions indéfinies du nombre i ; le nombre de ces formes qui ne doivent contenir qu'un nombre donné $2i - 2 - \omega$ de constantes arbitraires sera le nombre de partitions de i en $i - \omega$ parties, lequel, quand ω n'excède pas $\frac{i}{2}$, est identique avec le nombre des partitions indéfinies de ω.

Le lemme dont il a été question est qui sert pour obtenir l'équation (9) est le suivant :

Si on a un système de n équations de la forme

$$\lambda_1{}^\omega x_1 + \lambda_2{}^\omega x_2 + \ldots + \lambda_n{}^\omega x_n = p_\omega \quad \text{où} \quad \omega = 0, 1, 2, \ldots, n-1, \tag{10}$$

alors, quand les λ deviennent tous infiniment petits, la fonction

$$(\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 \ldots (\lambda_{n-1} - \lambda_n)^2 x_1 x_2 \ldots x_n,$$

reste finie et aura pour limite une valeur indépendante de $p_0, p_1, \ldots, p_{n-2}$, savoir :

$$(-)^{n\frac{n-2}{2}} p^n{}_{n-1}.$$

J'ajoute que la même méthode suffit également pour trouver et l'intégrale générale et les intégrales spéciales d'une équation d'une forme plus générale, à savoir l'équation

$$D_x{}^i y = \phi(P_1, P_2, \ldots, P_{i-1}) e^{\lambda x},$$

ϕ exprimant une forme de fonction quelconque donnée, et les P étant les fonctions algébriques de $y, y_1, y_2, \ldots, y_{2i-2}$, qui satisfont identiquement aux $(i-1)$ équations

$$y_{i+\omega-2} P_1 + y_{i+\omega-3} P_2 + \ldots + y_\omega P_{i-1} = \lambda y_{i+\omega-1} - y_{i+\omega}, \tag{11}$$

où
$$\omega = 0,\ 1,\ 2,\ \ldots,\ i-2.$$

Sans insister là-dessus, je passe à la considération plus intéressante de certaines équations différentielles qu'on peut immédiatement réduire à une forme intégrable par le moyen de la formule établie à la fin de la Note précédente, c'est-à-dire

$$D_x{}^{i-1} y \,.\, D_x{}^{i+1} y = D_x{}^i y \,.\, \left(\frac{d}{dx}\right)^2 D_x{}^i y - \left(\frac{d}{dx} D_x{}^i y\right)^2. \tag{12}$$

Pour plus de brièveté, je me servirai du symbole λ pour exprimer $\left(\frac{d}{dx}\right)^2 \log$, de sorte que la loi d'opération de λ sur des produits sera identique avec celle de log, c'est-à-dire qu'on aura

$$\lambda (uvw \ldots) = \lambda u + \lambda v + \lambda w + \ldots .$$

Je me servirai aussi du symbole D_i pour exprimer ce que j'ai précédemment désigné par $D_x{}^i y$; on aura ainsi par la formule (12)

$$yD_3 = (D_2)^2 \lambda D_2, \quad D_2 = y^2 \lambda y ;$$

donc
$$D_3 = y^3 (\lambda y)^2 \lambda (y^2 \lambda y) = y^3 (\lambda y)^2 (2\lambda y + \lambda^2 y).$$

Ainsi on voit que la solution de l'équation $D_3 = y^3 \phi\left(\frac{D_2}{y^2}\right)$ dépendra de celle de l'équation

$$(\lambda y)^2 (\lambda^2 y + 2\lambda y) = \phi(\lambda y) ; \tag{13}$$

ou bien (en mettant $\lambda y = u$, ce qui donne $y = e^{\iint u\, dx^2}$) pourra être ramenée à celle de l'équation

$$u^2 \left[\frac{d}{dx}\left(\frac{u'}{u}\right) + 2u\right] = \phi u. \tag{14}$$

Cette dernière équation, on le voit immédiatement, aura pour intégrale

$$x = \int \frac{du}{\sqrt{\left\{2u^2 \int du \left(\frac{\phi u}{u^3}\right) - 4u^3\right\}}}. \tag{15}$$

Ainsi on voit que si

$$\phi u = A + Bu + Du^3 + Eu^4, \tag{16}$$

x deviendra une fonction elliptique de u, et de même que si

$$\phi u = \alpha u + \beta u^{\frac{3}{2}} + \delta u^{\frac{5}{2}} + \epsilon u^3, \tag{17}$$

x deviendra une fonction elliptique de $u^{\frac{1}{2}}$.

Ainsi l'équation

$$D_3 = y^3 \phi\left(\frac{D_2}{y^2}\right) \tag{18}$$

sera intégrable par le moyen de fonctions elliptiques, toutes les fois que l'une ou l'autre des suppositions (16), (17) aura lieu. En prenant successivement $\phi u = A$, $\phi u = A^{\frac{1}{2}} u^{\frac{3}{2}}$, on obtient deux équations que je signalerai (quoiqu'elles ne soient que des cas particuliers) à cause de leur grande simplicité; ce sont les équations

$$D_3 = Ay^3, \tag{19}$$

$$D_3^2 = AD_2^3, \tag{20}$$

où $\quad D_2 = yy'' - y'^2$ et $D_3 = yy''y^{\text{IV}} - yy'''^2 - y'^2y^{\text{IV}} - y''^3 + 2y'y''y'''$.

Considérons d'abord l'équation (19); en faisant $\xi = A^{\frac{1}{6}}x$, elle prend la forme

$$'D_3 = y^3 \tag{21}$$

($'D_3$ ne différant de D_3 qu'en ce que ξ y remplace x). Alors par la formule (15) nous aurons $y = e^{\int\int u dx^2}$;

$$\xi + \gamma = \int \frac{du}{\sqrt{(-1 + \lambda_1 u^2 - 4u^3)}} = -\frac{1}{2^{\frac{2}{3}}} \int \frac{dv}{\sqrt{(-1 + \lambda v^2 + v^3)}}$$

$$= -\frac{1}{2^{\frac{2}{3}}} \int \frac{dv}{\sqrt{\left\{(2c + 4c^2 v + v^2)\left(-\frac{1}{2c} + v\right)\right\}}},$$

c et γ étant des constantes arbitraires. En écrivant

$$w = 2c^2 + v = 2c^2 - 2^{\frac{2}{3}}u, \quad C = -\sqrt{(4c^4 - 2c)}, \quad C_1 = 2c^2 + \frac{1}{2c},$$

on obtient immédiatement (en se servant de la substitution $w = C \cos 2\theta$)

$$\frac{C - 2c^2 + 2^{\frac{2}{3}}u}{2C} = \sin^2 \text{am}\left(2^{-\frac{1}{3}}\sqrt{(C + C_1)}\,(\xi + \gamma), \ \sqrt{\frac{2C}{C + C_1}}\right); \tag{22}$$

ce qui donne

$$\log y = 2^{\frac{1}{3}} C \iint dx^2 \left[\sin^2 \text{am}\left\{2^{-\frac{1}{3}}\sqrt{\left(\frac{C + C_1}{A^{\frac{1}{3}}}\right)}(x + \gamma'), \ \sqrt{\frac{2C}{C + C_1}}\right\}\right]$$

$$+ (2c^2 - C)\frac{x^2}{2^{\frac{5}{3}}}.$$

Cette équation est l'intégrale complète de l'équation donnée $D_3 = Ay^3$.

Maintenant considérons l'équation (20), c'est-à-dire $D_3^2 = AD_2^3$. La formule (15) donne

$$y = e^{\int\int dx^2 u}; \quad x + \gamma = \int \frac{du}{\sqrt{(-4A^{\frac{1}{2}}u^{\frac{3}{2}} + \lambda_1 u^2 - 4u^3)}},$$

ce qui, en mettant $u = \frac{1}{v^2}$, devient

$$x + \gamma = \int \frac{-dv}{\sqrt{(-A^{\frac{1}{2}}v^3 + \lambda v^2 - 1)}}. \tag{23}$$

En supposant $A = 1$ la somme en (23) devient identique avec celle qui a été trouvée plus haut en déterminant la valeur de $\xi + \gamma$, d'où il sera facile de conclure la valeur de $\log y$ qui contiendra la reciproque d'une double somme du carré d'une fonction linéaire du $\sin^2$am d'une fonction linéaire de x, et quant au cas général où A a une valeur quelconque, il se réduit au cas précédent en écrivant $\xi = A^{\frac{1}{6}}x$.

Il existe encore une infinité d'équations d'une forme symétrique et analogue à celle des équations (19) et (20) auxquelles on peut appliquer une pareille méthode, non pas, il est vrai en général, pour les intégrer complétement, mais au moins pour en abaisser le degré de 4 unités. C'est toujours l'équation fondamentale (12) qui sert à effectuer cette réduction.

58.

ON THE INTEGRAL OF THE GENERAL EQUATION IN DIFFERENCES.

[*Philosophical Magazine*, XXIV. (1862), pp. 436—441.]

THE most general form which can be given to a linear equation in differences may easily be seen to be reducible to the following,

$$a_x u_x + b_x u_{x-1} + c_x u_{x-2} + \&c. \textit{ ad lib.} = 0,$$

with the initial conditions

$$u_0 = 1, \quad u_{-e} = 0.$$

Consequently to find u_n, or let us rather say to find

$$(-)^n a_1 a_2 \ldots a_n u_n,$$

is really the problem of finding the value of a determinant belonging to a matrix of n^2 terms, whereof all the places below the diagonal line, with the exception of those in the oblique line immediately under the diagonal, are occupied by zeros, but of which all the other places are or may be occupied by finite quantities. For instance, supposing n to be 4, such a determinant would be

$$\begin{vmatrix} b_4, & c_4, & d_4, & e_4 \\ a_3, & b_3, & c_3, & d_3 \\ 0, & a_2, & b_2, & c_2 \\ 0, & 0, & a_1, & b_1 \end{vmatrix}.$$

Let us for a moment consider more particularly this determinant. If, using double indices to denote each coefficient, we were to write the above according to the usual method of notation as below,

$$\begin{vmatrix} 4.4, & 4.3, & 4.2, & 4.1 \\ 3.4, & 3.3, & 3.2, & 3.1 \\ 0, & 2.3, & 2.2, & 2.1 \\ 0, & 0, & 1.2, & 1.1 \end{vmatrix}$$

the law of formation of the general term would be very far from becoming evident on a cursory inspection; but a slight change, suggested by the very

system of equations in which the determinant originates, makes the law at once obvious. Nothing is more natural than that we should use $r.s$ or $s.r$, where $r > s$, to denote the coefficient of u_s in the equation of which r is the highest subindex of u; with this modification, the above determinant changes into the following:—

$$\begin{vmatrix} 4.3, & 4.2, & 4.1, & 4.0 \\ 3.3, & 3.2, & 3.1, & 3.0 \\ . & 2.2, & 2.1, & 2.0 \\ . & . & 1.1, & 1.0 \end{vmatrix}$$

(the terms with equal indices appearing not now in the diagonal, but in the oblique line below it). With this notation it becomes apparent (and the *reason* of the rule may be deduced by the most simple reasoning from following the course of the successive substitutions in the system of equations giving rise to the determinant) that to find the general term we must write all the descending series of integers which can be formed, beginning with 4 and ending with zero, namely,

43210
4310
4210
4320
430
420
410
40

and read them off respectively into products as below:—

$$4.3 \times 3.2 \times 2.1 \times 1.0$$
$$(4.3 \times 3.1 \times 1.0) \times (-2.2)$$
$$(4.2 \times 2.1 \times 1.0) \times (-3.3)$$
$$(4.3 \times 3.2 \times 2.0) \times (-1.1)$$
$$(4.3 \times 3.0) \times (2.2 \times 1.1)$$
$$(4.2 \times 2.0) \times (3.3 \times 1.1)$$
$$(4.1 \times 1.0) \times (2.2 \times 3.3)$$
$$(4.0) \times (1.1 \times 2.2 \times 3.3).$$

The sum of the above terms is the value of the determinant in question. And so in general, if we define u_n by means of the equation

$$(n.n)\, u_n + (n.n-1)\, u_{n-1} + (n.n-2)\, u_{n-2} + \ldots = 0;$$

with the initial conditions as above stated, the value of u_n to a factor *près* will be represented by

$$\Sigma\, (n, n_1, n_2, \ldots n_\omega, 0),$$

where $n > n_1 > n_2 \dots > n_\omega \, [\omega = 0, 1, 2, \dots (n-1)]$ and $(n, n_1, n_2, \dots n_\omega, 0)$ is to be interpreted as meaning

$$M \times n \,.\, n_1 \times n_1 \,.\, n_2 \times \dots \times n_\omega \,.\, 0,$$

where to find M we write the complementary integers

$$m_1,\ m_2,\ m_3,\ \dots\ m_{n-\omega+1},$$

which together with $n_1, n_2, \dots n_\omega$ make up the complete tally of all the integers from 1 to $(n-1)$, and then write

$$M = (-)^{n-\omega+1} (m_1 \,.\, m_1) \,.\, (m_2 \,.\, m_2) \dots (m_{n-\omega+1} \,.\, m_{n-\omega+1}).$$

In order to form by an exhaustive process all the descending series above described, we may if we please consider the differences of the terms of any such series, and write

$$\delta = n - n_1,\ \delta_1 = n_1 - n_2 \dots \delta_\omega = n_\omega,$$

we have then

$$\delta + \delta_1 + \delta_2 + \dots + \delta_\omega = n.$$

So that the question is reducible to that of finding all the partitions of n, and of permuting in every possible manner the terms in each such system of partitions; for it is obvious that in general the value of $(n, n_1, n_2, \dots n_\omega, 0)$ depends not only on the magnitudes, but on the order of sequence of $\delta, \delta_1, \delta_2, \dots \delta_\omega$.

If we suppose that the order of the differences is limited, as, for example, that the equation is of the ith order, then any such coefficient as $r \,.\, s$ is to be considered as zero when $r \sim s > i$, and consequently the partitions of n are to be limited to parts none greater than i. Moreover, if in such case the coefficients become constant, so that $r \,.\, s = \phi(r - s)$, it is apparent that the order of the arrangement of $\delta_1, \delta_2, \dots \delta_\omega$ becomes indifferent, and consequently the value of u_n, defined by the equation

$$u_n = (1)\, u_{n-1} + (2)\, u_{n-2} + \dots + (i)\, u_{n-i},$$

becomes the coefficient of t^n in $\dfrac{1}{1 - (1)\, t - (2)\, t^2 \dots - (i)\, t^i}$, as is well known.

The above rule may easily be extended to a linear equation in differences with any number of variables. Thus suppose, for greater simplicity, that we write

$$u_{x,\,y} = \Sigma \begin{pmatrix} x, & x' \\ y, & y' \end{pmatrix} u_{x',\,y'} \qquad \begin{bmatrix} x' = x-1,\ x-2,\ \dots\ 0 \\ y' = y-1,\ y-2,\ \dots\ 0 \end{bmatrix},$$

with the initial conditions $u_{0,0} = 1$, $u_{e,f} = 0$ wherever one or both of e, f are negative units; then to find the value of $u_{m,n}$ we must form all the possible descending series $\begin{bmatrix} m, & m_1, & m_2, & \dots & m_\omega, & 0 \\ n, & n_1, & n_2, & \dots & n_\omega, & 0 \end{bmatrix}$, subject only to the law that there

is a descent either from m_i to m_{i+1}, or from n_i to n_{i+1}, or at one and the same time from m_i to m_{i+1} and from n_i to n_{i+1}. The value of $u_{m,n}$ then becomes

$$\Sigma \begin{pmatrix} m, & m_1, & m_2, & \dots & m_\omega, & 0 \\ n, & n_1, & n_2, & \dots & n_\omega, & 0 \end{pmatrix},$$

with the understanding that the term within the parenthesis is to be read as meaning

$$\begin{pmatrix} m, & m_1 \\ n, & n_1 \end{pmatrix} \times \begin{pmatrix} m_1, & m_2 \\ n_1, & n_2 \end{pmatrix} \times \begin{pmatrix} m_2, & m_3 \\ n_2, & n_3 \end{pmatrix} \dots \times \begin{pmatrix} m_\omega, & 0 \\ n_\omega, & 0 \end{pmatrix}.$$

And in like manner and under a similar form we obtain the value of $u_{n_1, n_2 \dots n_\epsilon}$ defined by the general equation

$$u_{n_1, n_2, \dots n_\epsilon} = \Sigma \begin{pmatrix} n_1, & \nu_1 \\ n_2, & \nu_2 \\ \vdots & \vdots \\ n_\epsilon, & \nu_\epsilon \end{pmatrix} u_{\nu_1, \nu_2, \dots \nu_\epsilon}.$$

In defining the relations which connect one u with another, we may suppose that (r, s) means the coefficient of u_s in the equation

$$u_r = \Sigma\, (r, s)\, u_s \qquad [r > s,\ u_0 = 1,\ u_{-\epsilon} = 0];$$

but we may also suppose that (r, s) means the coefficient of v_r in the equation

$$v_s = \Sigma\, (r, s)\, v_r \qquad [r > s,\ v_n = 1,\ v_{n+\epsilon} = 0];$$

the value of u_0, on the latter supposition, it is obvious, becomes equal to that of u_n on the former—a fact that is well known, and deducible from the circumstance that u_n and v_0 will be represented by the same determinant turned round into a new position. But by means of our general representation for the case of any number ϵ of variables, we see that there is an analogous theorem which connects together 2^ϵ different results, and which is not so immediate a consequence of the theory of determinants.

To make my meaning more clear, if we suppose the four following systems of equations, in each of which $m > \mu$, $n > \nu$,

$$u_{m,n} = \Sigma \begin{pmatrix} m, & \mu \\ n, & \nu \end{pmatrix} u_{\mu,\nu} \, [u_{0,0} = 1,\ u_{-e,f} = 0,\ u_{e,-f} = 0,\ u_{-e,-f} = 0]^*,$$

$$v_{\mu,n} = \Sigma \begin{pmatrix} m, & \mu \\ n, & \nu \end{pmatrix} v_{m,\nu} \, [v_{m,0} = 1,\ v_{m+e,0} = 0,\ v_{m-e,-f} = 0,\ v_{m+e,-f} = 0],$$

$$w_{m,\nu} = \Sigma \begin{pmatrix} m, & \mu \\ n, & \nu \end{pmatrix} w_{\mu,n} \, [w_{0,n} = 1,\ w_{0,n+f} = 0,\ w_{-e,n-f} = 0,\ w_{-e,n+f} = 0],$$

$$\omega_{\mu,\nu} = \Sigma \begin{pmatrix} m, & \mu \\ n, & \nu \end{pmatrix} \omega_{m,n} \, [\omega_{m,n} = 1,\ \omega_{m+e,n-f} = 0,\ \omega_{m-e,n+f} = 0,\ \omega_{m+e,n+f} = 0],$$

we shall have $u_{m,n} = v_{0,n} = w_{m,0} = \omega_{0,0}$.

* Or, more simply and rather more accurately, in place of the three equations within the bracket it is better to write $u_{p,q} = 0$ when p or q or each of them is negative, and so analogously for the cases following:—

$v_{p,q} = 0$ when $m - p$ or q or each of them is negative,
$w_{p,q} = 0$ when m or $n - q$ or each of them is negative,
$\omega_{p,q} = 0$ when $m - p$ or $n - q$ or each of them is negative.

The theorem $u_n = v_0$ above given, when the equation of differences is of the second order, expresses the well-known theorem that the cumulant $[a, b, c, \dots h, k, l]$

$$\left(\text{the denominator of the continued fraction } \frac{1}{a+}, \frac{1}{b+}, \frac{1}{c+}, \cdots \frac{1}{k+}, \frac{1}{l}\right)$$

is the same as the cumulant $[l, k, h, \dots c, b, a]$.

There is no known property either of cumulants of this kind or those of the higher orders, nor can there be any found, but what does and must flow as an immediate consequence from the representation of the linear-difference integral above given. For instance, the law of formation of the above cumulant by rejecting consecutive pairs of terms becomes intuitive; for to meet this case we must write descending series of integers $n, n_1, n_2, \dots n_\omega, 0$, such that each difference between consecutive terms n_i, n_{i+1} is always 1 or 2, and when the latter, $(n_i, n_{i+1}) = 1$.

So more generally if we write $u_n = a_n u_{n-1} + u_{n-r}$, we obtain an analogous law for throwing out in every possible way groups of r consecutive terms in order to express u_n in terms of $a_n, a_{n-1}, a_{n-2}, \dots a_0$. So, too, if we write $u_n = u_{n-1} + b_n u_{n-\mu}$, we obtain Binet's law of "*discontiguous*" products given in his long memoir on the subject published in the *Mémoires* of the Institute,—the law of descent upon this supposition being that the difference between n_i and n_{i+1} is 1 or r; and if the former, $(n_i, n_{i+1}) = 1$.

We have seen above the convenience of shifting the system of subindices so as, for instance, to be able to treat the question of finding u_0 when we suppose $u_n = 1$ and $u_{n+e} = 0$, as well as that of finding u_n when we suppose $u_0 = 1$, $u_{-e} = 0$. More generally there is an advantage in writing $u_m = 1$ and $u_{m-e} = 0$ when it is a question of expressing u_n, which may then be conveniently denoted indifferently by $m:n$ or $n:m$,—the law being that regularly descending or ascending series are to be formed beginning with n and ending with m in every possible manner, each of which expresses a known product consisting of two parts—one made up of factors denoted by the conjunction of the consecutive terms in every such series, the other by the duplication of the integers between n and m not appearing in the series.

It is, moreover, convenient in some cases to express the limit which the descents are not to exceed (corresponding to the order of the equation). Thus $\underset{i}{\overline{\overline{n:m}}}$ may be used to denote the limitation of the differences in $n:m$ not to exceed i. The well-known theorem in continued fractions ordinarily denoted by the equation $pq' - p'q = \pm 1$ may then be expressed in a somewhat more general form in the manner following.

(*To be continued.*)

59.

ON THE QUANTITY AND CENTRE OF GRAVITY OF FIGURES GIVEN IN PERSPECTIVE, OR HOMOGRAPHY.

[*Newcastle-on-Tyne British Association Report* (1863), p. 2.]

In the first instance, the author showed how to find the point in the perspective representation of a plane figure into which the centre of gravity of such figure is projected. For this purpose it is only necessary to be furnished with the direction of the vanishing-line corresponding to the plane of the object put into perspective. The rule for finding the point in question is the following: every element of the picture is to be charged with a density equal to the inverse fourth power of its distance from the vanishing-line; the centre of gravity of the figure so charged will be the point required, and may of course be found by the rules of the integral calculus.

Next, as to the area of the unknown object. To determine this another datum (but only one other) is required besides the direction of the vanishing-line, which may be termed the constant of perspective, being determined when the position of the eye and that of the object-plane in reference to the picture are given. This constant is the product of the eye's distance from the vanishing-line into the square of the distance of the intersections of the object- and picture-planes from the same line. If now every element of the picture be charged with a density equal to the constant of perspective divided by the *cube* of the element's distance from the vanishing-line, the mass of the figure so charged will be the area of the unknown object-figure.

The author then proceeded to show how the area and the perspective centre, by aid of the preceding principles, admit of being reduced to depend on one single integral, closely analogous to the *potential* used in the theory of attractions to which he gives the name of *polar potential*. The polar potential of a plane figure in respect to a given line is defined to be the sum of the quotients of the elements by their respective distances from the line, and consequently the polar potential of the picture in respect to a vanishing-line in its plane becomes a function of the two parameters by which its position may be determined. The parameters which the author finds most convenient to employ are the distance of the vanishing-line from an arbitrary fixed point in the picture and the angle which it makes with a fixed line therein.

The author then supplied the formulæ (which are of a very simple character) for calculating the area of the object and the coordinates of its perspective centre of gravity, by means of differentiation processes performed upon the polar potential of the picture treated as a function of these parameters. He afterwards proceeded to extend the same method to figures, plane or solid, connected by the more general relation known under the name of homography, of which the relation between figures generated through the medium of perspective is only a particular kind. In the case of a solid figure, its polar potential in respect to a variable plane becomes a function of three parameters; and by means of differentiations performed upon it in respect to these parameters, the content and the coordinates of the point corresponding homographically to the centre of gravity of a solid figure may be expressed when its homograph and the position of a plane corresponding to the points at infinity in the otherwise unknown figure are given in addition (as regards the content) to a certain constant termed the homographic determinant.

Professor Rankine threw out a suggestion as to the possibility of a practical application of the preceding theory to the stability of structures standing to each other in a certain simple relation of homography.

60.

ON A QUESTION OF COMPOUND ARRANGEMENT.

[*Proceedings of the Royal Society of London*, XII. (1862–3), pp. 561—563.]

My successful but as yet unpublished researches into the Theory of Double Determinants have involved the consideration of the following curious case of arrangements.

There are given $m+n-1$ counters of n distinct *colours* just capable of being packed into m *urns*. The question refers to the distribution of the counters among the urns, subject to the condition that it shall *not* be possible to form a closed circuit of double colours between any number of the urns chosen arbitrarily; for example, we must allow no distribution of counters in which one urn contains blue and yellow, a second yellow and red, a third red and green, and a fourth green and blue, because here *blue, yellow, red,* and *green* would form a closed circuit. This condition, it is evident, excludes the same combination of colours from existing in any two of the urns, and also the repetition of any one colour in the same urn. Any distribution of counters obeying this condition may be called an excyclic distribution.

I annex two propositions, one qualitative, the other quantitative, referring to such distributions.

Qualitative Theorem.

In any excyclic distribution between m urns of $m+n-1$ counters of n different colours, any set of counters selected at will must be fewer in number than the number of distinct colours which they contain added to the number of urns from which they are drawn.

Before going on to enunciate the second proposition I must premise one or two simple definitions.

The *capacity* of an urn means the number of counters it will contain, the *frequency* of a colour the number of counters of that colour, so that the sum of all the capacities and the sum of all the frequencies must be each equal to the number of the counters.

Again, by the *diminished* capacity of any urn or *diminished* frequency of any colour, I mean such capacity or frequency respectively diminished by *unity*.

Finally, by the *polynomial function* of any set of numbers $a, b, \ldots l$, I mean the coefficient of $x^a . y^b \ldots z^l$ in the expansion of

$$(x + y + \ldots + z)^{a+b+\ldots+l}.$$

I can now enunciate the following

Quantitative Theorem.

The number of modes of excyclic distribution between m urns of $m + n - 1$ counters of n different colours is equal to the product of the polynomial function of the diminished frequencies *of all the several colours multiplied by the polynomial function of the* diminished capacities *of all the several urns.*

Observation.

A double determinant means the resultant of a system of $(m + n - 1)$ homogeneous equations, each containing mn terms and linear in respect to each of two systems of m and n variables taken separately, but of the second order in respect to the variables of these two systems taken collectively. Any such resultant is of the degree $\dfrac{(m + n - 1)!}{(m - 1)!\,(n - 1)!}$ in respect of the given coefficients, and may be represented by an ordinary determinant of the $(m + n - 1)$th order, every one of whose terms corresponds to a particular system of capacities of the m urns and of repetitions of the n colours in the question above treated.

The total number of such systems or terms will be

$$\left\{\frac{(m + n - 2)!}{(m - 1)!\,(n - 1)!}\right\}^2.$$

Every term in this determinant will itself be a sum of simple determinants of the $(m + n - 1)$th order, corresponding (each to each) with the totality of the excyclic distributions of $(m + n - 1)$ counters in respect of the particular systems of m capacities and n frequencies appertaining to that term; so that the number of simple determinants whose sum constitutes a term in the grand total determinant is always the product of two polynomial coefficients. In the particular case, where one of the systems contains only *two* variables, one of these polynomial coefficients becomes unity, and the other sinks down to a binomial coefficient. The only instance of a double determinant which is believed to have been considered up to the present moment is that given by Mr Cayley in the *Cambridge and Dublin Mathematical Journal*, vol. IX. 1854, for the case of $m = 2$, $n = 2$.

61.

ON A THEOREM RELATING TO POLAR UMBRÆ.

[*Proceedings of the Royal Society of London*, XII. (1862–3), pp. 563—565.]

By polar umbræ I mean such as obey in the strictest manner the polar law of sign, so that not only any two appositions or products of such umbræ derivable from one another by an interchange of two of their elements are to be considered each as the negative of the other, but also any such apposition or product becomes zero if the same element is found in it more than once.

Thus Sir W. Hamilton's i, j, k are not polar umbræ, because although $ijk = -jik = kij$, &c., ii, jj, kk, instead of being *nulls*, are in the Calculus of Quaternions taken as *unities**.

Let us now define any set arranged either in line or column of such *umbral* quantities to be multiplied by a corresponding set of *actual* quantities when each term of the one set is multiplied by the corresponding one of the other, and the sum taken of the products so obtained as in the ordinary case of the multiplication of the lines or columns of two determinants *inter se*. Thus, for example, $(a, b, c \between x, y, z)$, as also $\begin{pmatrix}a\\b\\c\end{pmatrix}\!\!\begin{pmatrix}x\\y\\z\end{pmatrix}$ is to mean the same product, namely,

$$ax + by + cz.$$

Again, imagine a rectangular (square or oblong) matrix of polar umbræ, and that each line thereof is multiplied by the same line of *actual* quantities, the product of the products so obtained I call a Factorial of the Matrix. I also call the product similarly obtained when the columns of the matrix are substituted for the lines, a Factorial of the same, but distinguish between the two by giving to one the name of a Transverse, the second of a Longitudinal Factorial of the matrix. We are now in a position to enunciate the following remarkable theorem:—

* If we use Vandermonde's condensed notation for a determinant $\begin{bmatrix}1, 2, \ldots n\\1, 2, \ldots n\end{bmatrix}$ to represent a "déterminant gauche," then, since on this supposition $rs = -sr$ and $rr = 0$, the elements $1, 2, 3, \ldots n$ will be polar umbræ by definition.

The product of any longitudinal by any transverse factorial of the same polar umbral matrix is identically zero.

For example, let $\begin{vmatrix} a, b, c \\ d, e, f \end{vmatrix}$ be a matrix of polar umbræ, but x, y, z and also ξ, η actual quantities. Then

$$(ax + by + cz)(dx + ey + fz)$$

is a transverse factorial,

$$(a\xi + d\eta)(b\xi + e\eta)(c\xi + f\eta)$$

a longitudinal factorial of the above matrix, and by the theorem their product should be zero. This is easily verified.

The two *factorials* expanded are respectively

$$adx^2 + bey^2 + cfz^2 + (ae + bd)xy + (bf + ce)yz + (af + cd)zx,$$

$$abc\xi^3 + (abf + aec + dbc)\xi^2\eta + (dec + dbf + aef)\xi\eta^2 + def\eta^3;$$

in their product the coefficient of

$$\begin{aligned} x^2\xi^3 &= abcad = 0, \\ xy\xi^3 &= abcae + abcbd = 0, \\ x^2\xi^2\eta &= abfad + aecad + dbcad = 0, \\ xy\xi^2\eta &= abfae + abfbd + aecae + aecbd + dbcae + dbcdb \\ &= aecbd + dbcae = aecbd - aecbd = 0, \end{aligned}$$

and so for all the other terms.

This is the fundamental theorem by aid of which I obtain the resultant of a lineo-linear system of equations in its most perfect form. It is easy to obtain two different solutions, each of them unsymmetrical in respect of the data of the question; the conversion and fusion of each of these into one and the same determinant, symmetrical in all its relations to the data, is effected instantaneously by a process derived from the above theorem. In that particular application of it, the umbræ involved each represent columns of actual quantities in number equal to the number of places in the width and length of the umbral matrix to which they belong, so that each coefficient in the product of a lateral by a longitudinal factorial represents an ordinary determinant made up of these columns, from which it is evident that the polar law of sign and nullity necessary for the truth of the theorem is satisfied in the case supposed.

62.

ON THE DEGREE AND WEIGHT OF THE RESULTANT OF A MULTIPARTITE SYSTEM OF EQUATIONS.

[*Proceedings of the Royal Society of London*, XII. (1862–3), pp. 674—676.]

LET there be $(1 + n)$ equations each homogeneous in any number of sets of variables, and suppose that the degrees of the several equations in respect to these sets are respectively

$$\begin{array}{l} a,\ b,\ c,\ \ldots,\ l, \\ a_1,\ b_1,\ c_1,\ \ldots,\ l_1, \\ a_2,\ b_2,\ c_2,\ \ldots,\ l_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_n,\ b_n,\ c_n,\ \ldots,\ l_n, \end{array}$$

where the a, b, c, &c. are any positive integers, zero not excluded.

Let the number of variables in the several sets be respectively $1 + \alpha$, $1 + \beta$, $1 + \gamma$, ... $1 + \lambda$, then in order that the system may have a resultant, since the number of ratios to be eliminated is $\alpha + \beta + \gamma + \ldots + \lambda$, this sum must be equal to n.

Let $$a_i\rho + b_i\sigma + c_i\tau + \ldots + l_i\omega = L_i,$$

and let $$LL_1L_2 \ldots L_n = P,$$

then 1st, the degree of the resultant in question in regard to the coefficients of the rth equation will be the coefficient of $\rho^\alpha . \sigma^\beta . \tau^\gamma \ldots \omega^\lambda$ in $\dfrac{P}{L_r}$.

2nd. As regards weight. By the weight of any letter in respect to any given variable is to be understood the exponent of that variable in the term affected with the coefficient; and by the weight of any term of the resultant in respect to such variable, the sum of the weights of its several simple factors;

each term in the resultant in respect to any given variable has the same weight; and this weight may also be proved to be alike for each variable in the same set, and may be taken as the weight of the resultant in respect to such set. This being premised, we have the following theorem :—

The value of the weight of the resultant in respect to any particular set of the variables, for example, the $(1+\alpha)$ set, will be the coefficient of

$$\rho^{1+\alpha} . \sigma^{\beta} . \tau^{\gamma} \ldots \omega^{\lambda} \text{ in } P.$$

In the particular case where $\alpha = \beta = \gamma \ldots = \lambda$, the above expressions for the degree and weight evidently become polynomial coefficients. Thus, for example, if we suppose each equation *linear* in respect to the variables of each set, the degree of the resultant in respect to the coefficients of any equation will be

$$\frac{(\alpha+\beta+\gamma+\ldots+\lambda)!}{\alpha!\,\beta!\,\gamma!\ldots\lambda!},$$

and its weight in respect to the $(1+\alpha)$ set will be

$$\frac{(1+\alpha+\beta+\ldots+\lambda)!}{(1+\alpha)!\,\beta!\,\gamma!\ldots\lambda!}.$$

In particular if each set is binary, so that $\alpha = \beta = \gamma \ldots = \lambda = 1$, the degree becomes $n!$, and the weight $\frac{1}{2}(n+1)!$.

The above theorems are, I believe, altogether new.

It may just be noticed (as a passing remark) that the total degree in the general case is the coefficient of

$$\rho^{\alpha} . \sigma^{\beta} . \tau^{\gamma} \ldots \omega^{\lambda} \text{ in } P\left\{\frac{1}{L} + \frac{1}{L_1} + \ldots + \frac{1}{L_n}\right\},$$

and the *total* weight the coefficient of the same argument in

$$P\left\{\frac{1}{\rho} + \frac{1}{\sigma} + \ldots + \frac{1}{\omega}\right\}.$$

63.

SEQUEL TO THE THEOREMS RELATING TO "CANONIC ROOTS" GIVEN* IN THE MARCH NUMBER OF THIS MAGAZINE.

[*Philosophical Magazine*, XXV. (1863), pp. 453—460.]

THE theorems kindly communicated from me by Mr Cayley in the March Number of this *Magazine* were originally designed to appear as a note or *excursus* to a memoir in preparation on the extension of Gauss's method of approximation from single to multiple integrals by a method which invariably leads to the construction of a *canonizant* whose roots are all real. To establish this reality, recourse may advantageously be had to a theorem of Jacobi, given at the end of his well known memoir "De Eliminatione Variabilis e duabus æquationibus," *Crelle*, vol. XV. p. 101, a very slight inspection of which at once leads to the further and interesting inference that the resultant of the canonizant of an odd-degreed function of x and *unity*, and of the canonizant of the second differential coefficient of that function in respect to x, is an exact power of the *catalecticant* of the first differential coefficient of x in respect to the same. This is the essence of the matter communicated by Mr Cayley; but subsequent successive generalizations of the theorem have led me on, step by step, to the discovery of a vast general theory of double determinants, that is, resultants of bipartite lineo-linear equations, constituting, I venture to predict, the dawn of a new epoch in the history of modern algebra and the science of pure tactic.

I will begin this note upon a note, by reproducing in brief the first of my two demonstrations of the simple theorem in question†. Let us write

$$X_0 = 1, \quad X_1 = \begin{vmatrix} 1, & x \\ a, & b \end{vmatrix}, \quad X_2 = \begin{vmatrix} 1, & x, & x^2 \\ a, & b, & c \\ b, & c, & d \end{vmatrix}, \quad X_3 = \begin{vmatrix} 1, & x, & x^2, & x^3 \\ a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \end{vmatrix},$$

and so on. And again, let

$$\lambda_1 = a, \quad \lambda_2 = \begin{vmatrix} a, & b \\ b, & c \end{vmatrix}, \quad \lambda_3 = \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix},$$

* [Cayley, *Coll. Math. Papers*, vol. v. p. 104.]

† The second has been communicated by Mr Cayley in the March number of this *Magazine*.

and so on. The theorem in effect to be proved is simply this, that the resultant of X_i and X_{i-1} is an exact power of λ_i, which (as will at once be seen) is the coefficient of x^i in X_i. In what follows, I shall use $R(P, Q)$ or $R(Q, P)$ to denote indifferently the positive or negative resultant of any two functions P and Q, ignoring for greater simplicity all considerations as to the proper algebraical sign to be affixed to a resultant of two functions taken in assigned order.

Jacobi's theorem above referred to, stated so far as necessary for the purpose in hand, is as follows:

$$X_n = (Ax + B)\, X_{n-1} - \frac{\lambda_n^2}{\lambda^2_{n-1}}\, X_{n-2}.$$

Hence, by virtue of a general theorem of elimination*,

$$R(X_n, X_{n-1}) = \lambda^2_{n-1}\, R\left(-\frac{\lambda_n^2}{\lambda^2_{n-1}}\, X_{n-2}, X_{n-1}\right);$$

or, neglecting as premised all considerations of algebraical sign,

$$= (\lambda_{n-1})^2 \left(\frac{\lambda_n}{\lambda_{n-1}}\right)^{2(n-1)} . R(X_{n-1}, X_{n-2}),$$

that is
$$\frac{R(X_n, X_{n-1})}{\lambda_n^{2(n-1)}} = \frac{R(X_{n-1}, X_{n-2})}{\lambda_{n-1}^{2(n-2)}} = \frac{R(X_{n-2}, X_{n-3})}{\lambda_{n-2}^{2(n-3)}}$$

$$= \&\text{c.} = \frac{R(X_1, X_0)}{\lambda_1} = 1;$$

or if any one of my readers finds a difficulty in admitting that $R(ax - b, 1) = a$, he can stop short at $\frac{R(X_2, X_1)}{\lambda_2^2}$, which may easily be verified to be equal to unity. Hence

$$R(X_n, X_{n-1}) = \lambda_n^{2n-2}. \quad \text{Q.E.D.}$$

Thus we see that if X_n, X_{n-1} have one root in common, λ_n must vanish; but then, by the cited theorem of Jacobi, it follows that X_n completely contains X_{n-1}; from this it was easy to infer the necessity of the function† of which X_n is the canonizant, having infinity for one of its "canonic roots"—or, in other words, of its being reducible to the form

$$k_1 (x + h_1)^{2n-1} + k_2 (x + h_2)^{2n-1} + \ldots + k_{n-1} (x + h_{n-1})^{2n-1} + k_n.$$

* This theorem is best seen by dealing in the first instance with U, V, any two homogeneous functions of x, y of degrees n, $n-1$ respectively satisfying the identity $U = (Ax + By) V + y^2 W$; we have then

$$R(U, V) = R(V, y^2 W) = R(V, W) \times \{R(V, y)\}^2,$$

where evidently $R(V, y)$ is the coefficient of x^{n-1} in V; let y become unity, then on calling U, V, $R(V, y)$ respectively X_n, X_{n-1}, λ_{n-1}, and giving to W its corresponding value, we have the theorem as it is used in the text.

† For in general if X_n be the canonizant to F, X_{n-1} will be the canonizant to $\frac{d^2F}{dx^2}$.

And so it became natural to establish *à priori* the existence of this condition, and thus to obtain the proof virtually reproduced by Mr Cayley in the article referred to.

In what precedes, X_{n-1} was *a first principal minor* of X_n; and it occurred to me to institute an inquiry into the form of the resultant of two functions related to each other as X_{n-1} is to X_n, with the sole but important difference that the constants in X_n are not to be contained in a concatenated order from one line to another, but to be taken perfectly independent as in the example

$$X_3 = \begin{vmatrix} 1, & x, & x^2, & x^3 \\ a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{vmatrix}$$

and
$$X_4 = \begin{vmatrix} 1, & x, & x^2, & x^3, & x^4 \\ a, & b, & c, & d, & e \\ a', & b', & c', & d', & e' \\ a'', & b'', & c'', & d'', & e'' \\ a''', & b''', & c''', & d''', & e''' \end{vmatrix}$$

Or, according to a suggestion of Mr Cayley, putting the question into a more general and simple form, we may inquire into the resultant of any two complete determinants, functions of x of the $(n-1)$th degree, which belong to the rectangular matrix

$$\begin{array}{llll} 1, & x, & x^2 & \ldots x^{n-1}, \\ a_{1,1}, & a_{1,2}, & a_{1,3} & \ldots a_{1,n}, \\ a_{2,1}, & a_{2,2}, & a_{2,3} & \ldots a_{2,n}, \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n-1,1}, & a_{n-1,2}, & a_{n-1,3} & \ldots a_{n-1,n}, \\ a_{n,1}, & a_{n,2}, & a_{n,3} & \ldots a_{n,n}; \end{array}$$

as, for instance, the resultant of the two determinants which may be obtained by suppressing successively the last and last but one line in the matrix above written: and by aid of the most elementary principles of the calculus of determinants the instructed reader will find no difficulty in proving that this resultant will resolve itself into two distinct parts—one a power of the determinant obtained by suppressing the uppermost (or x) line in the above matrix, the other the Resultant of the matrix obtained by suppressing simultaneously the two lowermost lines*. This last suppression leaves a rectangular matrix which, written in a homogeneous form, becomes

$$\begin{array}{llll} y^{n-1}, & y^{n-2}x, & y^{n-3}x^2 & \dots x^{n-1}, \\ a_{1,1}, & a_{1,2}, & a_{1,3} & \dots a_{1,n}, \\ a_{2,1}, & a_{2,2}, & a_{2,3} & \dots a_{2,n}, \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n-2,1}, & a_{n-2,2}, & a_{n-2,3} & \dots a_{n-2,n}, \end{array}$$

consisting of n columns and $(n-1)$ lines†.

The Resultant of this matrix means the quantity R which, equated to zero, will indicate the possibility of the simultaneous nullity of *all* its first minors, so that R will be the factor common to the resultants of every couple of these minors. If we name the columns of the matrix taken in any arbitrary order $C_1, C_2 \dots C_n$, and call R' the resultant of

$$C_1 C_3 \dots C_{n-1} C_n,\ C_2 C_3 \dots C_{n-1} C_n,$$

it may readily be made out that $\frac{R'}{R}$ is equal to a power of the determinant obtained by suppressing the uppermost (or x) line of the rectangular matrix $C_3 \dots C_{n-1} C_n$.

To find R, we may proceed in the general case in the manner indicated in the example following, where $n-1$ is made 4. Taking the two extreme first minors and dividing them respectively by y and x, we have two equations of the following form for determining R, namely,

$$\begin{vmatrix} y^3, & y^2x, & yx^2, & x^3 \\ a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{vmatrix} = 0, \quad \begin{vmatrix} y^3, & y^2x, & yx^2, & x^3 \\ b, & c, & d, & e \\ b', & c', & d', & e' \\ b'', & c'', & d'', & e'' \end{vmatrix} = 0.$$

By rejecting, as we have done, the factors x and y from the above equations, certain factors, it is true, are lost to their resultant (R'); but it will easily be seen that these factors are each of them powers of one and the same determinant, namely, the determinant

$$\begin{vmatrix} b, & c, & d \\ b', & c', & d' \\ b'', & c'', & d'' \end{vmatrix}$$

* For, on making the last-named determinant referred to in the text zero, it may easily be shown, by aid of a familiar theorem in compound determinants, that the two determinants whose resultant is under investigation have all the coefficients of the one in the same ratio to each other as the corresponding coefficients of the other.

† The reader may notice that the real interest of the subject under consideration commences with the independent inquiry into the form of the Resultant of the above matrix—the original question, as to the quasi-canonizant, being important only as leading up to the appearance of this Resultant.

and that their product is contained in the irrelevant factor $\frac{R'}{R}$, itself a power of that determinant, as above explained. To find R, we may write down the oblong matrix

$$\begin{array}{lll} y^2, & yx, & x^2, \\ b, & c, & d, \\ b', & c', & d', \\ b'', & c'', & d'', \end{array}$$

and make its three first minors respectively equal to u, v, w, that is,

$$\begin{vmatrix} y^2, & yx, & x^2 \\ b', & c', & d' \\ b'', & c'', & d'' \end{vmatrix} = u, \quad \begin{vmatrix} y^2, & yx, & x^2 \\ b'', & c'', & d'' \\ b, & c, & d \end{vmatrix} = v, \quad \begin{vmatrix} y^2, & yx, & x^2 \\ b, & c, & d \\ b', & c', & d' \end{vmatrix} = w;$$

then we shall obtain the equations following, of which the intermediate ones result solely from the equations last assumed, but the first and last from those combined with the original two given ones, namely,

$$(bu + b'v + b''w)\,y - (au + a'v + a''w)\,x = 0,$$
$$(cu + c'v + c''w)\,y - (bu + b'v + b''w)\,x = 0,$$
$$(du + d'v + d''w)\,y - (cu + c'v + c''w)\,x = 0,$$
$$(eu + e'v + e''w)\,y - (du + d'v + d''w)\,x = 0.$$

These equations may be satisfied by making simultaneously

$$u = 0, \quad v = 0, \quad w = 0,$$

all of which (since u, v, w are minors of the same rectangular matrix) may exist simultaneously, provided

$$\begin{vmatrix} b, & c, & d \\ b', & c', & d' \\ b'', & c'', & d'' \end{vmatrix} = 0.$$

Rejecting (as before) this irrelevant factor, it remains to find the resultant of the system of equations in x, y; u, v, w, above written, defined as the characteristic of the possibility of their coexistence for some particular system of values of x, y; u, v, w, but with joint and *several exclusion* of the system $x = 0, y = 0$, and of the system $u = 0, v = 0, w = 0$.

So, in like manner, in the general case we shall obtain a similar system of $(m+1)$ homogeneous equations linear in x, y, and also in $u_1, u_2, \dots u_m$; and R will be the resultant of this system, subject to the same condition as to the exclusion of zero systems of x, y, and $u_1, u_2, \dots u_m$ as in the particular instance above treated. Such a resultant, as hinted at the outset, is entitled to the name of a double determinant. In general a double determinant will refer to two systems of variables, one p, the other q in number, and to $(p+q-1)$ equations between them.

In the particular instance before us, one of these quantities, say q, is the number 2. There is, moreover, a further particularity (but which as it happens does not at all influence the form of the solution), consisting in the fact that the equations are of the *recurring* form

$$\begin{aligned} &L_1 y \quad - L_0 x = 0, \\ &L_2 y \quad - L_1 x = 0, \\ &L_3 y \quad - L_2 x = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &L_{p+1} y - L_p x = 0, \end{aligned}$$

where $L_0, L_1, \ldots L_{p+1}$ are each of them linear homogeneous functions of $u_1, u_2, \ldots u_p$. This gives rise to an identification of the resultants of two matrices of very different appearance—one matrix, for example, being

$$\begin{matrix} y^4, & y^3x, & y^2x^2, & yx^3, & x^4, \\ a, & b, & c, & d, & e, \\ a', & b', & c', & d', & e', \\ a'', & b'', & c'', & d'', & e'', \end{matrix}$$

and the other being

$$\begin{matrix} au + a'v + a''w, & bu + b'v + b''w, & cu + c'v + c''w, & du + d'v + d''w, \\ bu + b'v + b''w, & cu + c'v + c''w, & du + d'v + d''w, & eu + e'v + e''w. \end{matrix}$$

I have ascertained, and hope shortly to publish, the method of obtaining the explicit value of double determinants in the most general case and under their most symmetrical form: for the particular case before our eyes, this resultant will be as follows:—

$$\left|\begin{matrix}
\begin{vmatrix} a, & b, & a', & a'' \\ b, & c, & b', & b'' \\ c, & d, & c', & c'' \\ d, & e, & d', & d'' \end{vmatrix}, \begin{vmatrix} a, & b, & b', & a'' \\ b, & c, & c', & b'' \\ c, & d, & d', & c'' \\ d, & e, & e', & d'' \end{vmatrix} + \begin{vmatrix} a, & b, & a', & b'' \\ b, & c, & b', & c'' \\ c, & d, & c', & d'' \\ d, & e, & d', & e'' \end{vmatrix}, \begin{vmatrix} a, & b, & b', & b'' \\ b, & c, & c', & c'' \\ c, & d, & d', & d'' \\ d, & e, & e', & e'' \end{vmatrix} \\
\begin{vmatrix} a', & b', & a'', & a \\ b', & c', & b'', & b \\ c', & d', & c'', & c \\ d', & e', & d'', & d \end{vmatrix}, \begin{vmatrix} a', & b', & b'', & a \\ b', & c', & c'', & b \\ c', & d', & d'', & c \\ d', & e', & e'', & d \end{vmatrix} + \begin{vmatrix} a', & b', & a'', & b \\ b', & c', & b'', & c \\ c', & d', & c'', & d \\ d', & e', & d'', & e \end{vmatrix}, \begin{vmatrix} a', & b', & b'', & b \\ b', & c', & c'', & c \\ c', & d', & d'', & d \\ d', & e', & e'', & e \end{vmatrix} \\
\begin{vmatrix} a'', & b'', & a, & a' \\ b'', & c'', & b, & b' \\ c'', & d'', & c, & c' \\ d'', & e'', & d, & d' \end{vmatrix}, \begin{vmatrix} a'', & b'', & b, & a' \\ b'', & c'', & c, & b' \\ c'', & d'', & d, & c' \\ d'', & e'', & e, & d' \end{vmatrix} + \begin{vmatrix} a'', & b'', & a, & b' \\ b'', & c'', & b, & c' \\ c'', & d'', & c, & d' \\ d'', & e'', & d, & e' \end{vmatrix}, \begin{vmatrix} a'', & b'', & b, & b' \\ b'', & c'', & c, & c' \\ c'', & d'', & d, & d' \\ d'', & e'', & e, & e' \end{vmatrix}
\end{matrix}\right|$$

And it may be noticed that if we return to the original question, in which the coefficients are no longer independent, but where the column $a'b'c'd'e'$ is

identical, term for term, with $bcdef$, and $a''b''c''d''e''$ with $cdefg$, the above determinant becomes

$$\left|\begin{matrix}
* & * & \begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix} \\
* & \begin{vmatrix} b, & c, & d, & a \\ c, & d, & e, & b \\ d, & e, & f, & c \\ e, & f, & g, & d \end{vmatrix} & * \\
\begin{vmatrix} c, & d, & a, & b \\ d, & e, & b, & c \\ e, & f, & c, & d \\ f, & g, & d, & e \end{vmatrix} & * & *
\end{matrix}\right|$$

that is to say, it becomes a power of the determinant

$$\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}$$

as we know *à priori* it ought to do, by virtue of the theorem originating out of Jacobi's theorem stated at the beginning of this paper: in fact the two factors of the resultant of X_3, X_4 each of them becomes equal to λ_4^3; and so in general we shall find, if we use n instead of 4, each factor of the corresponding resultant becomes λ_n^{n-1}, giving λ_n^{2n-2} as the complete resultant for that singular case, as previously determined.

The author is conscious that some apology may appear due for the cursory mode of elucidation pursued in the preceding extended note, and for the absence as regards certain points of the appropriate proofs; but to have gone into all the details of demonstration would have swollen the paper to a length out of proportion to its importance. Let him be permitted also in all humility to add (as can be vouched by more than one contributor to this *Magazine*), that in consequence of the large arrears of algebraical and arithmetical speculations waiting in his mind their turn to be called into outward existence, he is driven to the alternative of leaving the fruits of his meditations to perish (as has been the fate of too many foregone theories, the stillborn progeny of his brain, now for ever resolved back again into the primordial matter of thought), or venturing to produce from time to time such imperfect sketches as the present, calculated to evoke the mental cooperation of his readers, in whom the algebraical instinct has been to some extent developed, rather than to satisfy the strict demands of rigorously systematic exposition.

64.

OBSERVATIONS ON THE METHOD FOR FINDING THE CENTRE OF GRAVITY OF A QUADRILATERAL GIVEN IN THE PRESENT NUMBER OF THE JOURNAL.

[*Quarterly Journal of Mathematics*, VI. (1863), pp. 130—133.]

THE method given in *Mechanical Solutions of Geometrical Problems*, p. 127, for finding the centre of gravity of a quadrilateral, leaves nothing to be wished for in point of elegance and conciseness; it is new* to the Editors and stands in advantageous contrast with all other methods of effecting the same end. It involves only four lines of construction and two bisections; in some elementary works on Mechanics, in use at our Universities, a method is given involving no less than 9 or 11 auxiliary lines. It must henceforth take rank as the best method of effecting the end in view; the second best is that which has been treated analytically, by Mr Stephen Fenwick, of the Royal Military Academy, in the *Mathematician*, 1847, Vol. II., p. 292, but admits of a simple and pleasing geometrical proof.

Let us call the intersection of the two diagonals the *cross-centre*, and the intersection of the two bisectors of opposite pairs of sides the *mid-centre* of a quadrilateral.

If we take the centres of gravity of the four triangles into which a given quadrilateral is divided by its two diagonals, it is clear that the cross-centre of the new quadrilateral, of which these four points are the summits, will be the centre of gravity of the original quadrilateral. But it may easily be seen that this new quadrilateral is only a miniature image of the original one, and that each of the two quadrilaterals has the same mid-centre; in a word, the new quadrilateral may be obtained by reducing the linear dimensions of the original one in the ratio of 1 to 3, and then swinging it through half a revolution round the mid-centre. Hence the new cross-centre will be in opposition with the original one, in respect to the mid-centre, and at a distance from it equal to one-third of the distance of the former one from the same.

* I should say new in *form*; in substance it is identical with that given, Vol. II., p. 292, of the *Mathematician*.

This method involves only four auxiliary lines, but requires four bisections and one trisection, instead of merely two bisections, according to the method of the text above.

The substitution of heavy points for areas or volumes admits of an extension which the author of this note believes to be new, and which occurred to him incidentally in treating of the extension of Gauss' method of approximation from simple to multiple quadratures.

It will be convenient to call the sum of the masses of any system of bodies into the nth powers of their distances from a fixed plane their nth *moments* in respect to the plane. (Thus the second moments will mean the sum of the masses into their squared distances.) It may then be affirmed as a universal proposition that such nth moments of a line, triangle, and tetrahedron (and so on for the higher dimensions of space) may always be replaced by suitable weights at fixed points symmetrically situated about the centre of gravity of such figures. For example, the *second* moments of lines, triangles, and tetrahedra (say each of mass unity) in respect to any plane may be replaced by masses of $\frac{1}{6}$ at the two angular points for the line, of $\frac{1}{12}$ at the three angular points for the triangle, and of $\frac{1}{20}$ at the four angular points for the tetrahedron, the balance $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$ being of course placed at the centre of gravity in these cases respectively. Hence it follows obviously that the same law will be true for the *moments of inertia* of a line, triangle, or tetrahedron about any *axis*, and consequently the centres and times of oscillation of these figures about any axis will be the same as for equal weights placed at the angles and weights respectively 4, 9, and 16 times as great placed at their centres of gravity. The ingenious author of the matter which has called forth these observations may probably be able to draw interesting inferences from this equivalence, and also from combining his own unrivalled method for finding the centre of gravity of a quadrilateral with the miniature-image method hereinbefore explained.

One word more before I conclude; the rule given in the text may be expressed in general terms by aid of a simple verbal definition. Let two points situated in a limited line be said to be *opposite* when their respective distances from opposite ends of the line are equal.

The centre of gravity of a quadrilateral may then be stated to be identical with the centre of gravity of a triangle whose apices are the point of intersection of the two diagonals and the opposite points thereto on these two diagonals*.

* It is difficult to resist the impression that some similar construction must apply to the determination of the centre of gravity of the frustum of a pyramid. Two points in a line with the centre of gravity of a triangle and at equal distances on opposite sides may be defined as opposite points in respect to the triangle. As a mere conjecture to be subjected to ulterior verification I suggest the possibility of the following construction being applicable; if not true it may at least serve to set the mind athinking in the right direction for the discovery of the truth.

The Barycentric principle employed in the text leads me to make an observation which will be found somewhat prolific in consequences and may be made instrumental (as I have satisfied myself by actual trial) in the edification of a complete theory of the parabola by processes greatly exceeding in simplicity those depending on Cartesian coordinates.

The secret of the utility of the Barycentric principle consists essentially in the plasticity of the axes to which the moments may be referred. Equally advantageous will be found the introduction of the laws of motion into the theory of the parabola, aided by the *plastic* condition that the motion of a projectile acted on by a constant force, *reckoned in any direction*, depends only on the actual velocity and force respectively estimated in such direction.

Let Aa, Bb, Cc be the edges of the frustum.

Let the three diagonal triangles Abc, Bca, Cab intersect in the point P and let the opposites to P in these three triangles be respectively P', P'', P'''. Similarly, by means of the other system of diagonal planes aBC, bCA, cAB, let a second system of four points p, p', p'', p''' be obtained. I conjecture that the centre of gravity of equal weights at these eight points, or, which is the same thing, the centre of gravity of the pyramid, whose apices bisect respectively the lines Pp, $P'p'$, $P''p''$, $P'''p'''$, *may be* the centre of gravity of the frustum. This intermediate pyramid appears to be the natural measure of the distortion of the frustum from the prismatic form, as the triangle formed by the cross-centre and its two opposites is that of the distortion of the quadrilateral, which may be regarded as the frustum of a triangle.

I have verified the conjectural construction for the case where the frustum becomes a prism and also for the case where it becomes a tetrahedron by the vanishing of one of its triangular faces.

À propos of the relation between the trapezium and the pyramidal frustum, I am not aware whether it has been observed that as a trapezium may be divided in two ways into a pair of triangles, so may the frustum of a pyramid be divided in six ways into a triplet of tetrahedrons. Using the same letters as before, one such division will be represented by the table following:

a	b	c	A
b	c	A	B
c	A	B	C

and permuting simultaneously and conformably the two systems of letters a, b, c; A, B, C, we obtain all the six systems in question. This stereotomic division leads to a direct and almost instantaneous geometrical proof of the known expression

$$\delta = \frac{h}{4}\frac{a^2 + 2ab + 3b^2}{a^2 + ab + b^2},$$

for finding the position, of the centre of gravity of a frustum bounded by parallel faces, in the line joining the centres of the parallel faces.

Obviously for space of any number of dimensions, say n, an analogous dissection may be effected in $n!$ different ways; the scheme above given serving fully to disclose the tactical law of the symbols.

For example, for space of four dimensions one such dissection out of the twenty-four will be denoted by the scheme

a	b	c	d	A
b	c	d	A	B
c	d	A	B	C
d	A	B	C	D

By aid of this principle I have reconstructed all the essential properties of the curve in respect to its directrix, focus, and tangents, and obtained, as it were instantaneously, various theorems, some of which, if not new, could only be obtained by long processes through the ordinary methods, whether of Geometry or of Cartesian coordinates.

Since penning the above observations, the author has found without difficulty the two geometrical constructions for the centre of gravity of a pyramidal frustum, precisely analogous to those alluded to for the centre of gravity of a quadrilateral: which will probably appear* in the August Number (or, if not, in the September Number) of the *Philosophical Magazine*. The true *mid-centre* is the centre of gravity of six equal weights placed at the six angles of the frustum; the true *cross-centre* is the point of intersection of either of two ternary systems of planes, which have the property of intersecting in the same point; one of these planes, for example, will be the plane passing through the middle point of ab, the middle point of aC, and the middle point of BC.

This brings to mind an analogous generalization long ago made known by the writer of this note, namely, that as a quadratic surface is cut by any tangent plane in two straight lines, so is a cubic hyper-surface by a tangent hyper-plane in six, a quartic transhyper-surface by a tangent transhyper-plane in twenty-four right lines, and so on indefinitely. Passing by an abrupt flight from a transcendental analogy to what many may regard as a mere platitude, let me notice that it is not a truism but a proposition and no insignificant one to affirm that a convex figure of five [plane] faces capable of being formed by joining conformably the angles of one triangle with those of another can only be the *frustum of a pyramid*; it is in fact equivalent to the assertion that three right lines of which every two intersect must either lie in one plane or pass through one point.

I ought not to conclude without alluding to a second conjectural method for finding the centre of gravity of such frustum which will in all probability stand or fall with that already given, bearing to it the same relation as the second best bears to the best determination of the analogous problem for the quadrilateral. Taking as Q (the *cross-centre*) the point mid-way between P, p the respective intersections of the two systems of diagonal planes, and as O (the *mid-centre*) the point where the axis joining the centres of gravity of the two triangular faces meets the plane containing the centres of gravity of the three quadrilateral faces, QO being joined and produced through O to G, so that $GO = \frac{QO}{4}$, G will be the conjectural position of the centre of gravity of the frustum. This is easily verified for the case where the frustum becomes an entire pyramid.

P.S.—Since the above was in press I have ascertained that each of the above two conjectural methods is erroneous. Apparently the *euristic* problem to be solved is to discover in the pyramidal frustum, the analogue to the cross-centre in the quadrilateral; this, there is every reason to believe, is closely connected with the points P and p above described; it is, however, certainly not the point mid-way between them.

[* Below, p. 342.]

65.

ON THE CENTRE OF GRAVITY OF A TRUNCATED TRIANGULAR PYRAMID, AND ON THE PRINCIPLES OF BARYCENTRIC PERSPECTIVE.

[*Philosophical Magazine*, XXVI. (1863), pp. 167—183.]

THERE is a well-known geometrical construction for finding the centre of gravity of a plane quadrilateral, which may be described as follows.

Let the intersection of the two diagonals (say Q) be called the *cross-centre*; the intersection of the lines bisecting the middle points of pairs of opposite sides (say O) the *mid-centre* (which, it may be observed, is the centre of gravity of the four angles viewed as equal weights); then the centre of gravity is in the line joining these two centres produced past the latter (the mid-centre), and at a distance from it equal to one-third of the distance between the two centres; in a word, if G be the centre of gravity of the quadrilateral, QOG will be in a right line, and $OG = \frac{1}{3}QO$.

The frustum of a pyramid is the nearest analogue in space to a quadrilateral *in plano*, since the latter may be regarded as the frustum of a triangle. The analogy, however, is not perfect, inasmuch as a quadrilateral may be regarded as a frustum of either of two triangles, but the pyramid to which a given frustum belongs is determinate. Hence *à priori* reasonable doubts might have been entertained as to the possibility of extending to the pyramidal frustum the geometrical method of centering the plane quadrilateral. The investigation subjoined dispels this doubt, and will be found to lead to the perfect satisfaction, under a somewhat unexpected form, of the hoped-for analogy.

Let abc, $\alpha\beta\gamma$ be the two triangular faces, $a\alpha$, $b\beta$, $c\gamma$ the edges of the quadrilateral faces of a pyramidal frustum. Then this frustum may be

resolved in six different ways into the sum total of three pyramids, as shown in the annexed double triad of schemes,

$$\begin{array}{lll} a,\ b,\ c,\ \alpha, & b,\ c,\ a,\ \beta, & c,\ a,\ b,\ \gamma, \\ b,\ c,\ \alpha,\ \beta, & c,\ a,\ \beta,\ \gamma, & a,\ b,\ \gamma,\ \alpha, \\ c,\ \alpha,\ \beta,\ \gamma, & a,\ \beta,\ \gamma,\ \alpha, & b,\ \gamma,\ \alpha,\ \beta, \end{array}$$

$$\begin{array}{lll} b,\ a,\ c,\ \beta, & a,\ c,\ b,\ \alpha, & c,\ b,\ a,\ \gamma, \\ a,\ c,\ \beta,\ \alpha, & c,\ b,\ \alpha,\ \gamma, & b,\ a,\ \gamma,\ \beta, \\ c,\ \beta,\ \alpha,\ \gamma, & b,\ \alpha,\ \gamma,\ \beta, & a,\ \gamma,\ \beta,\ \alpha. \end{array}$$

If, then, taking any one of the above schemes we draw a plane through the centres* of the three pyramids of which it is composed, the six planes thus drawn will meet in a point, which will be the centre of the frustum†.

Let the point in which αa, βb, γc meet when produced be the origin of coordinates, and $bc\beta\gamma$, $ca\gamma\alpha$, $ab\alpha\beta$ be taken as the planes of x, y, z; and let $4a, 0, 0$; $0, 4b, 0$; $0, 0, 4c$ be the coordinates of a, b, c, and $4\alpha, 0, 0$; $0, 4\beta, 0$; $0, 0, 4\gamma$ those of α, β, γ. Consider the first of the schemes above written.

$$\begin{array}{llll} a+\alpha, & b, & c & \text{will be the coordinates of the centre of } abc\alpha, \\ \alpha, & b+\beta, & c & \quad ,, \qquad ,, \qquad ,, \qquad bc\alpha\beta, \\ \alpha, & \beta, & c+\gamma & \quad ,, \qquad ,, \qquad ,, \qquad c\alpha\beta\gamma; \end{array}$$

because, as everyone knows, the centre of a pyramid is the same as that of its angles regarded as of equal weight. But again, if we define as the mid-centre the centre of the six angles of the frustum regarded as of equal weight, its coordinates will be

$$\frac{2a+2\alpha}{3}, \quad \frac{2b+2\beta}{3}, \quad \frac{2c+2\gamma}{3};$$

and if we substitute for each of the three centres last named points lying respectively in a right line with them and the mid-centre on the opposite side of the mid-centre and at distances from it double those of these centres themselves, these quasi-images of the centres in question will have for their coordinates

$$\begin{array}{lll} 0\ , & 2\beta, & 2\gamma, \\ 2a, & 0\ , & 2\gamma, \\ 2a, & 2b, & 0\ . \end{array}$$

These points are accordingly the centres of the lines $\beta\gamma$, γa, ab respectively.

And a similar conclusion will apply to each of the six schemes. Hence using in general (p, q) to mean the middle of the line p, q, and by the

* I shall throughout in future for greater brevity hold myself at liberty to use the word *centre* to mean *centre of gravity*.

† I shall hereafter show that these six planes all touch the same cone, of which, as also of its polar reciprocal, I have succeeded in obtaining the equations.

collocation of the symbols for three points understanding the plane passing through them, it is clear

1. That the six planes,

$$(\beta, \gamma);\ (\gamma, a);\ (a, b);\quad (\gamma, \alpha);\ (\alpha, b);\ (b, c);\quad (\alpha, \beta);\ (\beta, c);\ (c, a);$$
$$(\gamma, \beta);\ (\beta, a);\ (a, c);\quad (\alpha, \gamma);\ (\gamma, b);\ (b, a);\quad (\beta, \alpha);\ (\alpha, c);\ (c, b),$$

will meet *in a single point* which may be called the *cross-centre*, being the true analogue of the intersection of the two diagonals of a quadrilateral figure in the plane.

2. That if we join this cross-centre (say Q) with O the mid-centre, and produce QO to G making $OG = \frac{1}{2}QO$, G will be the centre of the frustum $abc\alpha\beta\gamma$.

It may be satisfactory to some of my readers to have a direct verification of the above.

Let, then,

$$A = \frac{a^2bc - \alpha^2\beta\gamma}{abc - \alpha\beta\gamma}, \quad B = \frac{ab^2c - \alpha\beta^2\gamma}{abc - \alpha\beta\gamma}, \quad C = \frac{abc^2 - \alpha\beta\gamma^2}{abc - \alpha\beta\gamma}.$$

A moment's reflection will serve to show that A, B, C are the coordinates of the centre of the frustum.

Again, the first three of the six planes last referred to will be found to have for their equations respectively,

$$\beta\gamma x + \gamma a y + ab z = 2a\gamma(b + \beta),$$
$$bc x + \gamma\alpha y + \alpha b z = 2b\alpha(c + \gamma),$$
$$\beta c x + ca y + \alpha\beta z = 2c\beta(a + \alpha).$$

The determinant

$$\begin{vmatrix} \beta\gamma, & \gamma a, & ab \\ bc, & \gamma\alpha, & \alpha b \\ \beta c, & ca, & \alpha\beta \end{vmatrix} = (abc - \alpha\beta\gamma)^2.$$

The determinant

$$\begin{vmatrix} \gamma a, & ab, & 2a\gamma(b + \beta) \\ \gamma\alpha, & \alpha b, & 2b\alpha(c + \gamma) \\ ca, & \alpha\beta, & 2c\beta(a + \alpha) \end{vmatrix}$$
$$= 2\alpha a(bc - \beta\gamma)(abc - \alpha\beta\gamma),$$
$$= 2\{(\alpha^2\beta\gamma - a^2bc)(abc - \alpha\beta\gamma) + (a + \alpha)(abc - \alpha\beta\gamma)^2\}.$$

Hence if x, y, z be the coordinates of the intersection of the above-mentioned three planes,

$$x = -2A + 2(a + \alpha),$$
$$y = -2B + 2(b + \beta),$$
$$z = -2C + 2(c + \gamma);$$

and the same will evidently be true of the other ternary system of planes; so that all six planes intersect in a single point Q, of which x, y, z above written are the coordinates. And the coordinates of O being

$$\frac{2a + 2\alpha}{3}, \quad \frac{2b + 2\beta}{3}, \quad \frac{2c + 2\gamma}{3},$$

and those of G being

$$A, \quad B, \quad C,$$

it is obvious QOG is a right line, and $OG = \frac{1}{2}QO$, as was to be shown.

The analogy with the quadrilateral does not end here. There is a construction* for the centre of a quadrilateral still easier than that above cited, which may be expressed in general terms by aid of a simple definition. Agree to understand by the *opposite* to a point L on *a limited line* AB a point M, such that L and M are at equal distances from the *centre* of AB but on opposite sides of it; then we may affirm that the centre of a quadrilateral is the centre of the triangle whose apices are the intersection of its two diagonals (that is, the cross-centre), and the *opposites* of that intersection on those two diagonals respectively. So now if we agree to understand by opposite points on a limited triangle two points in a line with the centre of the triangle and at equal distances from it on opposite sides, and bear in mind that the cross-centre of a pyramidal frustum is the intersection of *either* of two distinct ternary systems of triangles which may be called the two systems of cross-triangles†, we may affirm that the centre of a pyramidal frustum is the centre of a pyramid whose apices are its cross-centre, and the opposites of that centre on the three components of either of its systems of cross-planes. This is easily seen; for if we take the first of the two systems, their respective centres will evidently be

$$\frac{4a}{3}, \quad \frac{2b + 2\beta}{3}, \quad \frac{4\gamma}{3},$$
$$\frac{4\alpha}{3}, \quad \frac{4b}{3}, \quad \frac{2c + 2\gamma}{3},$$
$$\frac{2a + 2\alpha}{3}, \quad \frac{4\beta}{3}, \quad \frac{4c}{3}.$$

* This is the mode of statement (except that the important notion of opposite points was not explicitly contained in it) which, accidentally meeting my eye in a proof sheet of some Geometrical Notes (by an anonymous author) intended for insertion in the forthcoming (if not forthcome) Number of the *Quarterly Journal of Mathematics*, led to the long train of reflections embodied in this paper, which but for that casual glance would never have seen the light. The same construction, under another and somewhat less eligible form, is given in the *Mathematician* (a periodical now extinct, edited by Dr Rutherford and Mr Fenwick, both of the Royal Military Academy), 1847, Vol. II. p. 292, and is therein stated by the latter gentleman to have, "as he believes, first appeared in the *Mechanics' Magazine*, and subsequently in the *Lady's Diary* for 1830."

† From the description given previously, it will be seen that *a cross-triangle* of the frustum is one which has its apices at the centres of either diagonal of any quadrilateral face and of the two edges coterminous but not in the same face with that diagonal.

Thus the three opposites to the cross-centre whose coordinates are

$$-2A+2(a+\alpha), \quad -2B+2(b+\beta), \quad -2C+2(c+\gamma),$$

will have for their x coordinates

$$\frac{2a}{3}-2\alpha+2A,$$

$$-2a+\frac{2\alpha}{3}+2A,$$

$$-\frac{2a}{3}-\frac{2\alpha}{3}+2A;$$

for their y coordinates

$$\frac{2b}{3}-2\beta+2B,$$

$$-2b+\frac{2\beta}{3}+2B,$$

$$-\frac{2b}{3}-\frac{2\beta}{3}+2B;$$

and for their z coordinates

$$\frac{2c}{3}-2\gamma+2C,$$

$$-2c+\frac{2\gamma}{3}+2C,$$

$$-\frac{2c}{3}-\frac{2\gamma}{3}+2C;$$

and consequently the centre of the pyramid whose apices are the cross-centre and its three opposites will be A, B, C, that is, will be the centre of gravity of the frustum, as was to be shown*.

* I at one time supposed that a, b, c; α, β, γ formed two systems of *diagonal* planes, and that there were thus two cross-centres; and dreamed a dream of the construction for the centre of gravity of the pyramidal frustum based upon this analogy, inserted (it is true as a conjecture only) in the *Quarterly Journal of Mathematics*; but the nature of things is ever more wonderful than the imagination of men's minds, and her secrets may be won, but cannot be snatched from her. Who could have imagined *à priori* that for the purposes of this theory a diagonal of a quadrilateral was to be viewed as a line drawn through two opposite angles of the figure regarded, not as themselves, but as their *own centres of gravity*! Some of my readers may remember a signal case of a similar autometamorphism which occurred to myself in an algebraical inquiry, in which I was enabled to construct the canonical form of a six-degreed binary quantic from an analogy based on the same for a four-degreed one, by considering the *square* of a certain function which occurs in the known form as consisting of two factors, one the function itself, the other a function morphologically derived from, but happening for that particular case to coincide with the function. This parallelism is rendered more striking from the fact of 4 and 6 being the *numbers* concerned in each system of analogies, those numbers referring to degrees in the one theory and to angular points in the other. It is far from improbable that they have their origin in some common principle, and that so in like manner the parallelism will be found

It is clear that these results may be extended to space of higher dimensions. Thus in the corresponding figure in space of four dimensions bounded by the hyperplanar quadrilaterals $abcd$, $\alpha\beta\gamma\delta$, which will admit of being divided into four hyperpyramids in twenty-four different ways, all corresponding to the type

$$\begin{matrix} a, & b, & c, & d, & \alpha, \\ b, & c, & d, & \alpha, & \beta, \\ c, & d, & \alpha, & \beta, & \gamma, \\ d, & \alpha, & \beta, & \gamma, & \delta, \end{matrix}$$

there will be a cross-centre given by the intersection of any four out of twenty-four hyperplanes resoluble into six sets of four each,—one such set of four being given in the scheme subjoined, where in general pqr means the point which is the centre of (p, q, r) and the collocation of four points means the hyperplane passing through them, namely,

$$\begin{matrix} \beta\gamma\delta, & \gamma\delta a, & \delta ab, & abc, \\ \gamma\delta\alpha, & \delta ab, & abc, & bca, \\ \delta\alpha\beta, & \alpha\beta c, & \beta cd, & cdb, \\ \alpha\beta\gamma, & \beta\gamma d, & \gamma da, & dac. \end{matrix}$$

The mid-centre will mean the centre of the eight angles $a, b, c, d, \alpha, \beta, \gamma, \delta$, regarded as of equal weight; and to find the centre of the hyperpyramidal frustum, we may either produce the line joining the cross-centre with the mid-centre through the latter and measure off three-fifths of the distance of the joining line on the part produced (as in the preceding cases we measured off two-fourths and one-third of the analogous distance), or we may take the four opposites of the cross-centre on the four components of any one of the six systems of hyperplanar tetrahedrons of which it is the intersection, and find the centre of the hyperpyramid so formed. The point determined by either construction will be the centre of gravity of the hyperpyramidal frustum in question. And so on for space of any number of dimensions. It will of course be seen that a general theorem of determinants* is contained

to extend in general to any quantic of the degree $2n$, and the corresponding barycentric theory of the figure with $2n$ apices (n of them in one hyperplane and n in another), which is the problem of a hyperpyramid in space of n dimensions. The probability of this being so is heightened by the fact of the barycentric theory admitting, as is hereafter shown, of a *descriptive* generalization, descriptive properties being (as is well known) in the closest connexion with the theory of invariants. Much remains to be done in fixing the canonic forms of the higher even-degreed quantics; and this part of their theory may hereafter be found to draw important suggestions from the hyper-geometry above referred to, if the supposed alliance have a foundation in fact.

* We learn indirectly from this how to represent under the form of determinants of the ith order, and that in a certain number of different ways, the general expressions

$$(l_1 l_2 \ldots l_i - \lambda_1 \lambda_2 \ldots \lambda_i)^{i-1}$$

and

$$l_1 \lambda_1 (l_2 l_3 \ldots l_i - \lambda_2 \lambda_3 \ldots \lambda_i)(l_1 l_2 \ldots l_i - \lambda_1 \lambda_2 \ldots \lambda_i)^{i-2},$$

in the assertion that for space of n dimensions there will be $n!$ quasi-planes all intersecting in the same point, as also in the general relation connecting this point (the cross-centre) with the mid-centre and centre of gravity, of each of which it is easy to assign the value of the coordinates in the general case.

But returning to the case of the ordinary pyramidal frustum, the preceding results lead at once to an easy geometrical proof of the well-known analytical formula for finding the centre of gravity of a pyramidal frustum in the case where the base and its opposite plane are parallel.

As we know that the centre of gravity in this case is in the line joining the centres of the opposite faces, what is wanted here is merely the proportion of the segments into which this joining line is divided at the centre in question, or, in other words, the ratio to each other of the distances of the centre from the parallel faces.

Let $$ab : \alpha\beta = bc : \beta\gamma = ca : \gamma\alpha = l : \lambda.$$

Then obviously
$$\text{vol. } abc\alpha : \text{vol. } bc\alpha\beta = ab\alpha : b\alpha\beta = l : \lambda,$$
$$\text{vol. } bc\alpha\beta : \text{vol. } c\alpha\beta\gamma = bc\alpha : c\alpha\gamma = l : \lambda :$$

hence $$abc\alpha : bc\alpha\beta : c\alpha\beta\gamma = l^2 : l\lambda : \lambda^2 ;$$

also if h be the distance between abc, $\alpha\beta\gamma$, the distances of the centres of $abc\alpha$, $bc\alpha\beta$, $c\alpha\beta\gamma$ respectively from abc will be $\frac{h}{4}, \frac{h}{2}, \frac{3h}{4}$.

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a strange conclusion to be able to draw incidentally from a hyper-theory of centre of gravity! Thus, for example, on taking $i=4$, we shall find

$$\begin{vmatrix} bcd, & cda, & da\beta, & \alpha\beta\gamma \\ \beta\gamma\delta, & cda, & da\beta, & \alpha\beta\gamma \\ b\gamma\delta, & \gamma\delta a, & dab, & ab\gamma \\ bc\delta, & c\delta a, & \delta a\beta, & abc \end{vmatrix} = (abcd - \alpha\beta\gamma\delta)^3.$$

And again,

$$\begin{vmatrix} ad\,(bc + c\beta + \beta\gamma), & cda, & da\beta, & \alpha\beta\gamma \\ \beta a\,(cd + d\gamma + \gamma\delta), & cda, & da\beta, & \alpha\beta\gamma \\ \gamma b\,(da + a\delta + \delta a), & \gamma\delta a, & dab, & ab\gamma \\ \delta c\,(ab + ba + a\beta), & c\delta a, & \delta a\beta, & abc \end{vmatrix} = aa\,(bcd - \beta\gamma\delta)\,(abcd - \alpha\beta\gamma\delta)^2.$$

The number of these representations will not be twenty-four, that is, $4!$, but only twelve, the half of that number, because it will easily be seen that the cycles $abcd$, $\alpha\beta\gamma\delta$ will lead to the same determinants, only differently arranged, as the cycles $bcda$, $\beta\gamma\delta\alpha$. I believe the law is, that the number of varieties of such representations is $(i)!$, or $\frac{1}{2}(i)!$, according as i is odd or even. The expression $ab - \alpha\beta$ at once conjures up the idea of a determinant. We now see that there is an equally natural determinantive representation, or system of representations, of $(abc - \alpha\beta\gamma)^2$, $(abcd - \alpha\beta\gamma\delta)^3$, &c.

Hence the distance of the centre of the frustum from abc will be $\frac{h}{4}\left(\frac{l^2 + 2l\lambda + 3\lambda^2}{l^2 + l\lambda + \lambda^2}\right)$, and so from $\alpha\beta\gamma$ it will be $\frac{h}{4}\left(\frac{\lambda^2 + 2l\lambda + 3l^2}{l^2 + l\lambda + \lambda^2}\right)$, agreeing with the well-known formula applicable to this case*.

But I pass on to a subject of much deeper interest.

The geometrical constructions included in the preceding inquiry (such for instance as depend on the properties of *centres* and *opposites*), like those which occur in the more ordinary theory of the triangle and pyramid, at once suggest the existence of descriptive propositions in which harmonic centres and harmonic opposites, and in general harmonic multiplications and divisions, take the place of the corresponding arithmetical operations.

To make my meaning perfectly clear, let us conceive a fixed plane; and by a harmonic succession of points $A, B, C, D \ldots$ in a line meeting the fixed plane† (which we may term the plane of relation) in O, let us understand that $ABCO$, $BCDO$, &c. form so many harmonic systems of points; B may be then called a harmonic centre of AC, A and C opposites to B; also we may call AB, BC harmonic steps of the succession, so that by multiplying a line AB n times, or making AX equal to n times AB, we are constructing the point X to which A will be transferred by n harmonic steps, of which AB is the first; and by n-secting a line AX, we mean finding a point B in it such that a succession of n harmonic steps, commencing with AB, will carry A to X.

In all this there is of course nothing new: these principles are familiar to all geometers, and have received their fullest development at the hands of Professor Cayley. We know *à priori* that the descriptive properties included in the preceding (or similar) constructions, such, for example, as that the six cross-triangles of a frustum all meet in a point, will remain true when, adopting a fixed plane of relation, we substitute harmonic centres in respect to that plane in lieu of arithmetical centres‡. Or, again, we may affirm that

* If we agree to denote by a, b, c; α, β, γ, the planes $\alpha\beta\gamma$, $b\gamma a$, $c\alpha\beta$; abc, $\beta c\alpha$, γab respectively, it may easily be shown that each quaternary system of planes a, b, α, β; b, c, β, γ; c, a, γ, α passes through a single point; we have thus given three points which determine a plane; the intersection of this plane with the line a, b, c; α, β, γ is a sort of centre to the frustum, and must possess properties deserving closer investigation.

† It will of course be understood that in dealing with figures lying in the same plane, a line of relation (namely, the intersection of the plane of relation with the plane of the figures) may be substituted instead of the former plane, since the distances from the one and the other are in an invariable ratio; and so for different segments in a right line, we may substitute a point of relation on the line itself instead of the plane. I deal with a plane of relation as comprising implicitly all the subordinate cases; were it required to go out into space of four or a higher number of dimensions, it would of course become necessary to deal with hyper-planes of relation.

‡ Geometers have long been familiar with the idea of the *pole* or *harmonic centre* of a triangle in respect to a line in its plane; the principles now about to be developed will enable us to attach a precise signification to the *pole* or *harmonic centre* of every geometrical figure of any form whatever.

the lines joining the harmonic centres of the opposite edges of a tetrahedron will all *intersect* and *harmonically* bisect each other, and so on. But what is further wanted, and what I will proceed to supply, is a firm quantitative basis to this enlarged theory, so formed as that we shall be able in the general case to follow step by step the reasoning used in the common theory where the plane of relation goes off to infinity, and to assign to every point determined in the general constructions as distinctive a character as it possesses in the special ones. This may be done by the aid of very elementary considerations, which I proceed to unfold, and which will be seen at once to bring the general or perspective theory under the dominion of the so-called integral calculus or calculus of continuity.

The arithmetical centre of two *points* A, B is the centre of gravity of two equal atoms at A and B; let us then so assign the weights of the atoms A, B in the general case as to make their centre of gravity fall on the harmonic centre: this may evidently be done by considering their weights as proportional to their inverse distances from the plane of relation, and accordingly we shall understand by the weight of an atom at any point a quantity proportional to its inverse distance from the plane of relation. But, moreover, the centre of gravity of the homogeneous line AB ought to fall at this same point, which we may if we please consider as an inference at the limit from the same thing being true for equal atoms at distances dividing the line into any even number of equal parts. Hence in the general analogical theory we must take the infinitesimal intervals of our atoms at points in *harmonic* succession.

Let P, Q, R be any three such points, and let x, $x+dx$, $x+2dx+d^2x$ be their respective distances from the plane of relation; and let q be the *frequency* at P, that is a quantity proportional to the number of atoms which occur in a given infinitesimal space about P; then evidently qdx is constant, and $qd^2x+dxdq=0$; but by virtue of the harmonic relation between P, Q, R, we have

$$(x+2dx+d^2x)(dx)=x(dx+d^2x),$$

or

$$xd^2x=2(dx)^2, \text{ or } -\frac{dq}{q}=2\frac{dx}{x},$$

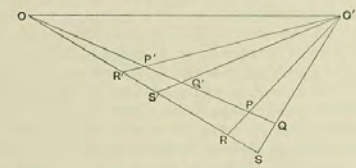
that is q varies as

$$\frac{1}{x^2}.$$

Moreover the weight of each atom varies as $\frac{1}{x}$, hence the *density* of any element in a line must be taken to vary as the inverse cube of its distance from the plane of relation.

Let us now endeavour to obtain the law of density for any element of a *plane*. Let O, O' be any two points in the line in which the plane in

question meets the plane of relation, and let the plane be divided into infinitesimal elements similar to $PQSR$ in the figure by pencils whose rays are in harmonic succession proceeding from O and O'; then one atom belongs to



every such element, which will be the analogue of a rectangular element in the common theory; but the area of this element, as compared with any similar element, say $P'Q'S'R'$ in the infinite sector QOS, varies as

$$OP \, . \, RS + OR \, . \, PQ,$$

where PQ, RS, by what has been last shown, vary as the square of the distance of the element from the plane of relation, and OP, OR vary directly as the distance; hence the *frequency* of the atoms at any element in *either sector* will vary as the inverse cube of its distance from the plane of relation, and hence this will be the law of frequency for elements all over the plane, and is irrespective of the particular positions of O, O'; and consequently, the density being proportional to the product of the frequency of the atoms by their atomic weights, the law of density is that it varies about any point as the inverse *fourth* power of its distance from the plane of relation. In like manner, by taking three points O, O', O'' in the plane of relation and dividing space into solid elements by plane bundles passing through OO', OO'', $O'O''$ respectively, it may be proved that the law of density for a solid figure will be that it varies as the inverse *fifth* power of the distance from the plane of relation*.

Atoms whose weights vary inversely as their distances from the plane of relation may be termed *like* atoms; lines, areas, and solids whose elements vary in *density* inversely as the cubes, fourth powers and fifth powers respectively, may be termed *qualiform* figures, or figures of *qualiform* density, the terms *like* and *qualiform* being adopted as the closest analogues to *equal* and *uniform*. It now becomes true, and may easily be verified, that

* The law of density for a solid is the inverse fifth power, for an area the inverse fourth power, and for a line the inverse third power. Here we must stop, for a *point* is that which has no parts: we can speak of the law of atomic *weights* at a point, but not of density, for the latter implies the existence of *elements* which are wanting to the point. In a hyper-ontological sense there would be no objection to saying that for an element of a point the law of density in this theory is as the inverse square, always remembering that no such element exists.

the centres of gravity of a *qualiform* finite line, triangle, and tetrahedron are respectively identical with the centres of gravity of *like* atoms placed at their apices*; and so every known or discoverable theorem whatever relating to the centre of gravity of uniform figures bounded by right lines or planes becomes immediately transferable to that of *qualiform* figures of the same kind. Thus, to take a most simple example, since the centre of gravity of a parallelogram is at the intersection of its diagonals, it must be and is true that the centre of gravity of a quadrilateral whose density at any point varies as the inverse fourth power at that point from the line joining the intersections of its two pairs of opposite sides, will also be at the intersection of the diagonals of that figure. I am informed by Professor Cayley that a somewhat analogous consideration of altered density has been employed by our eminent friend Professor William Thomson in his theory of images, in reference to the distribution of electricity, given in *Liouville's Journal*.

* As regards the finite line, these results may be very easily verified by the integral calculus. For the triangle, it may be made to depend on the preceding case by drawing from the point where the direction of any side intersects the plane of relation, rays dividing the triangle into infinitesimal portions; the centre of gravity of every one such portion will easily be seen to be in the right line joining the harmonic centre of the intersecting side with the opposite angle; and an analogous method applies to the tetrahedron.

The same results may also be obtained analytically. Thus, for example, for a qualiform triangle whose apices are distant h, k, l from the opposite sides, and $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ from the plane of relation, the distances of the centre of gravity from the respective sides will be

$$\frac{h\alpha}{\alpha+\beta+\gamma}, \quad \frac{k\beta}{\alpha+\beta+\gamma}, \quad \frac{l\gamma}{\alpha+\beta+\gamma}.$$

The masses, say M, of a qualiform line, triangle, or tetrahedron, using α, β; α, β, γ; $\alpha, \beta, \gamma, \delta$ for the inverse distances of the apices from the plane of relation, and V for the length, area, or volume, in the three cases respectively become expressible under the very noticeable forms

$$\frac{\alpha+\beta}{2}\alpha\beta V, \quad \frac{\alpha+\beta+\gamma}{3}\alpha\beta\gamma V, \quad \frac{\alpha+\beta+\gamma+\delta}{4}\alpha\beta\gamma\delta V,$$

their moments in respect to the plane of relation being respectively

$$\alpha\beta V, \quad \alpha\beta\gamma V, \quad \alpha\beta\gamma\delta V;$$

so that the *mean* density $\frac{M}{V}$ is in each case a simple symmetric function of the atomic weights of the apices (it being of course understood that the *absolute* atomic weight and frequency are each taken as unity). As the same figure may be variously partitioned, and the sum of the component areas and of their moments is unaffected by the mode of partition, the preceding formulæ obviously give rise to, or imply the existence of, a class of purely geometrical theorems relating to systems of points. It may be here observed that the moment of a qualiform figure in respect to its plane of relation represents the size, so to say, of (that is, the number of atoms contained in) the single molecule which, placed at the centre of gravity, will be the statical equivalent of such figure; for if n be this number, and d the distance of the centre from the plane of relation, and w the weight of the figure, since the atomic weight is $\frac{1}{d}$, we must have $\frac{n}{d}=w$, or

$$n = dw = \text{moment of } w \text{ in respect to the plane of relation.}$$

So in like manner, wherever the plane of relation is situated, two molecules A and B, placed at two *points*, will be equivalent to the molecule $A+B$ placed at their centre of gravity.

It is an easy inference* from what has been established concerning the law of *frequency*, that if in the perspective of any plane figure, by tinting or relief, we express the degree of crowding of any element, and proportion the tint or elevation to the inverse *cube* of its distance from the vanishing line, then any portion of the picture will accurately represent (and indeed if we use relief, the *volume* or weight of such portion will be strictly proportional to) the area (or its weight) of the corresponding part in the object plane. Supposing different object planes to be represented in perspective on the same picture plane, with liberty for the position of the eye to vary, it may be shown without difficulty† that if the *absolute* intensity of tint or relief for any object plane varies as the square of the distance of its trace upon the picture plane from its vanishing line, and as the first power of the distance of the eye from the same line, the ratio between corresponding portions of object and picture will be alike for every plane.

In the corresponding problem for right lines, the relief or tint of any element in the perspective of a given right line must vary as the inverse square of the distance from the vanishing point, and the absolute intensity for different lines must vary as the product of the distance between the trace and the vanishing point into the distance of the eye from that point. In *barycentric* perspective we have seen the further consideration of atomic weight enters, so that the density follows the law of the inverse fourth and third powers for planes and lines respectively, instead of third and second powers as in geometrical perspective; in fact in the geometrical theory the quantities visibly represented correspond to the *moments*‡ in respect to

* It may here also incidentally be noticed that the area of the primitive of any perspective projection of a figure in a given plane is proportional to the *attraction* exercised upon it by the object plane indefinitely extended, the force of attraction between any two elements being supposed to vary inversely as the fifth power of the distance.

† For if we take T the trace of an object line, V its vanishing point, and through O (the eye) draw OPp meeting TV in P and the object line in p, Tp the quantity of $TP = \frac{\mu TP}{TV.PV}$, so that $\mu = TV\frac{Tp}{TP}PV = TV.OV$; and again, if tTt' be the trace of an object plane, V the foot of the perpendicular from O on the vanishing line VT perpendicular to tTt', P a point in VT, and p the point where OP meets the object plane, we have tpt' (the quantity of tPt') $= \mu\frac{tPt'}{TV.TV.PV}$, or

$$\mu = TV^2.\frac{tpt'}{tPt'}.PV = TV^2.\frac{Tp}{TP}.PV = TV^2.OV.$$

The preceding calculations assume the expressions $\mu\alpha\beta$, $\mu\alpha\beta\gamma$ applicable to a linear and triangular space, given in a preceding footnote.

‡ And consequently if, in the pictorial representation of any plane surface, there is taken a triangular patch of given area, the quantity in the object corresponding thereto will vary inversely as the product of the distances of the three angles of the patch from the vanishing line,—a proposition in perspective which I imagine to be new, and at all events is certainly little known. This may be applied to determine instantaneously the area of an ellipse of which the perspective projection is a circle of radius r, and whose centre is at the distance h from the vanishing line. Writing μ equal to the distance of the vanishing line from the eye, multiplied by the square of its

the vanishing line of the quantities visibly represented in the barycentric theory *.

I have termed this a theory of barycentric perspective, because it includes a method whereby the centre of gravity of a plane figure is retained in perspective with the centre of gravity of its projection; by what has pre-

distance from the trace of the ellipse upon the plane of the circle, the area of the ellipse (regarded as made up of infinitesimal sectors with the centre of the projection for their common vertex) becomes

$$\int_0^{2\pi} d\theta \frac{\frac{1}{2}\mu r^2}{h(h - r\sin\theta)^2} = \frac{\mu\pi r^2}{h^3\left[1 - \left(\frac{r}{h}\right)^2\right]^{\frac{3}{2}}} = \frac{\mu\pi r^2}{(h^2 - r^2)^{\frac{3}{2}}};$$

so that the area of any ellipse in a *given* plane, the perspective representation of which ellipse is a circle, will vary directly as the area of the circle, and inversely as the cube of the tangent drawn to meet it from the orthogonal projection of its centre on the vanishing line. More generally, if the figure in the plane of projection be an ellipse with semiaxes a, b, eccentricity e, inclination of minor axis to vanishing line α, and distance of one of its foci from that line h, then calling V the area of the primitive and μ the absolute ratio between a primitive element and its projection, we shall have

$$V = \frac{\mu}{2h}\int_0^{2\pi} d\theta \frac{r^2}{(h - r\sin\theta)^2}, \text{ where } r = \frac{a(1-e^2)}{1 + e\sin\theta}.$$

This integration may be performed with extreme facility, and gives

$$V = \mu\pi ab\,[h^2 + 2hea\cos\alpha - a^2(1-e^2)]^{-\frac{3}{2}},$$

say

$$\frac{\mu}{D^3}\pi ab,$$

where to find D we may use the following construction:—*Draw a circle in the plane of, and concentric with, the projection, and such that a common tangent to the two shall be parallel to the vanishing line, and from the foot of the perpendicular upon that line from the centre draw a tangent to the circle, the length of the tangent so drawn will be* D; so that the area of any ellipse will be to the area of its perspective projection as the product of the square of the distance of the trace into that of the eye from the vanishing line is to the cube of the tangent just described,—a very remarkable proposition in perspective, if new. By varying the origin of our polar coordinates, as by taking it, for instance, at the centre of the projection or any other point, we may obtain a new class of definite integrals of known values, and which it might be exceedingly difficult to determine by any direct method. It may be added that all ellipses in the same plane will bear a constant ratio to their projections if these latter have a common tangent parallel to the vanishing line, and their centres be in another line also parallel to the same.

* The above statements, combined with the varying law of *frequency*, amount to the following propositions in perspective:—

1. If O be a *linear* element, P its perspective representation, H, h the distances of the eye and P from the line of O, and d of the eye from the line of P, then

$$O : P :: dH : (H-h)^2.$$

2. If O be a *plane* element, P its perspective, H, h the distances of the eye and P from the plane of O, and d the distance of the eye from the plane of P, then

$$O : P :: dH^2 : (H-h)^3.$$

These formulæ would become necessary in applying (as might be done perhaps advantageously) in some cases the integral calculus to the quantification of curved lines and surfaces by a perspective method more general than the one in ordinary use, which is essentially a method of orthogonal projection.

ceded, it appears that this may be effected by regarding its projection, not as of uniform density, but of a density following the law of the inverse cube of the distance. From this it follows that the distance of the perspective position in the picture of the centre of gravity of the primitive from the vanishing line becomes immediately known by a process of differentiation when the area of the primitive is expressed as a function of the distance of any arbitrarily fixed point in the plane of projection from the vanishing line. For if this area, which is the moment of the qualiform projection in respect to the vanishing line, be called M, and the mass of the same be termed Q, and if h, d be the distances of the origin and of the centre of gravity from the vanishing line, we have $d = \frac{M}{Q}$, where

$$M = \mu\int_0^{2\pi} \frac{r^2 d\theta}{(h - r\sin\theta)^2 h},$$

$$Q = \frac{1}{3}\mu\int_0^{2\pi} \frac{r^2 d\theta}{(h - r\sin\theta)^2 h}\left(\frac{1}{h - r\sin\theta} + \frac{1}{h - r\sin\theta} + \frac{1}{h}\right);$$

hence $$Q = -\frac{1}{3}\frac{dM}{dh},$$

and $$d = \frac{\frac{1}{3}M}{\frac{dM}{dh}}.$$

Thus, for example, if we wish to find the perspective position of the centre of gravity of the primitive of a given elliptic projection, we have found in a preceding footnote,

$$M = \mu(h^2 + 2hae\cos\alpha + a^2e^2 - a^2)^{-\frac{3}{2}};$$

hence $$d = \frac{h^2 + 2hae\cos\theta + a^2e^2 - a^2}{h + ae\cos\alpha};$$

or, calling R the radius of the circle concentric with the given projection, and having with it a common tangent parallel to the vanishing line, and H the distance of the centre of this circle from that line, $d = \frac{H^2 - R^2}{H}$, an equation the geometrical interpretation whereof is readily obtained.

More generally, if we take $x\cos\alpha + y\sin\alpha - h = 0$ as the equation to the vanishing line, using, as before, M to denote the moment of the qualiform projection in respect to that line (well worthy in this theory of being termed the principal moment), or, which is the same thing, the area of the primitive, and take M_x for the moment of the same in respect to the axis of y, we shall have

$$M = \iint \frac{dx\,dy}{(x\cos\alpha + y\sin\alpha - h)^3},$$

$$M_x = \iint \frac{dx\,dy\,x}{(x\cos\alpha + y\sin\alpha - h)^4},$$

from which it is easy to deduce

$$M_x = \cos\alpha\left(M + \frac{1}{3}h\frac{d}{dh}M\right) + \frac{1}{3}\sin\alpha\frac{d}{d\alpha}M;$$

and consequently $\frac{M_x}{Q} - h\cos\alpha$, which is the distance of the perspective of the centre of gravity of the primitive in the direction of x from the foot of the perpendicular from the assumed origin upon the vanishing line, will be

$$\frac{3\cos\alpha \, . \, M + \sin\alpha\frac{dM}{d\alpha}}{\frac{dM}{dh}}.$$

And thus we are led to the remarkable proposition, that when we know the area of the primitive in terms of the parameters of its vanishing line, we can completely determine the perspective position of its centre of gravity by means of processes of differentiation only; so that a method closely akin to (if not identical with) that of potentials in the theory of attraction has a necessary place also in the theory of perspective.

If, as is most convenient, we fix the perspective of the centre of gravity of the object figure by its distance from the vanishing line and its distance from the line through the origin perpendicular to the vanishing line, we see, by making α successively zero and $\frac{1}{2}\pi$ in the above formula, that these distances are $\frac{3M}{\frac{d}{dh}M}$ and $\frac{\frac{d}{d\alpha}M}{\frac{d}{dh}M}$ respectively*. Analogous results may be obtained for

* In the case of the ellipse, we have found in a preceding footnote,

$$M = \mu\left(h^2 + 2aeh\cos\alpha + a^2e^2 - a^2\right)^{\frac{3}{2}},$$

so that

$$\frac{3M}{\frac{d}{dh}M} = \frac{h^2 + 2eah\cos\alpha + a^2e^2 - a^2}{h + ea\cos\alpha} = y,$$

$$\frac{\frac{dM}{d\alpha}}{\frac{dM}{dh}} = \frac{ea\sin\alpha h}{h + ea\cos\alpha} = x,$$

where y and x are the coordinates of the point referred to in the text, if we take the vanishing line and a line perpendicular thereto from the focus for the axes of x and y. Consequently, if we remove the origin of coordinates to the centre of the ellipse, preserving the directions of the axes, and call x', y' the new coordinates, we shall have

$$x' = ae\sin\alpha - x = \frac{a^2e^2\sin\alpha\cos\alpha}{h + ae\cos\alpha},$$

$$y' = h + ae\cos\alpha - y = \frac{a^2\left[1 - e^2(\sin\alpha)^2\right]}{h + ae\cos\alpha},$$

$$\frac{y'}{x'} = \frac{1 - e^2(\sin\alpha)^2}{e^2\sin\alpha\cos\alpha},$$

solid figures, substituting the more general notion of homography for that of perspective, as will more fully appear in the sequel.

Remembering that M is the area of the primitive plane object, it seems to result as an indirect inference from the preceding theory, that whenever we can determine the area of an oval section (whether the bounding curve be the whole or a part of the curve of section) of an algebraical cone, then we can determine the position of the centre of gravity of that oval in its own plane by processes of differentiation only; and, *mutatis mutandis*, the same conclusion will admit of extension to solids bounded by algebraical surfaces; so that $\iint dxdy$ or $\iiint dxdydz$ being given, subject to certain conditions of limit, $\iint (ax + by)\,dxdy$, $\iiint (ax + by + cz)\,dxdydz$, subject to the same conditions, become known by algebraical and differentiation processes only, and so obviously for any number of variables*.

which may easily be shown to be the equation to the diameter drawn to the point of the ellipse where the tangent is parallel to the vanishing line; and consequently the perspective of the centre of gravity of the original lies in this diameter, as evidently it ought to do, since every infinitesimal slice of the *qualiform* area contained between parallels to the vanishing line is of uniform density throughout, and is bisected by the diameter conjugate to the direction of that line.

* The inference made hesitatingly in the text, upon further reflection appears to me perfectly clear, and will become so, I think, to the reader with the aid of a few words of explanation.

Let Q be a closed curve of the kind supposed lying in a plane which will be treated as a constant plane of projection; and for greater simplicity, and in order to steady the ideas, imagine that the vanishing plane (meaning thereby the plane passing through the eye and the vanishing line), and the plane of the object to be put in perspective, are retained at a constant distance from each other and always perpendicular to the picture plane, and also that the height of the eye above the vanishing line is invariable. Take any fixed line and point in the picture Q, and determine the equation to the curve boundary of its primitive O corresponding to a given distance h between the fixed point and the variable vanishing line and to a given angle of inclination α between the fixed line and this variable line. Then by hypothesis the area of O, say M, is known in terms of its coefficients, which will be known functions of α and h; hence $\frac{dM}{d\alpha}$ and $\frac{dM}{dh}$ are known, and consequently the position of the perspective of the centre of gravity of O on the picture is known; and from this the position of that centre in its own plane can be constructed, and therefore will have been found by aid of algebraical and differentiation processes only, as was to be shown.

The above explanation may be made still more distinct if we suppose that we *begin* with an object Ω (the curve for which is expressed by an equation in its most general form), wherein we have, say, $\alpha = 0$ and $h = 1$; that from this we deduce the equation of P in the preceding investigation, and from P pass to O as before; then, having found the coordinates of the perspective of the centre of gravity of O as functions of h and α, make $\alpha = 0$, $h = 1$, and pass back to the coordinates of the centre of gravity in Ω, of which the centre of gravity last named then becomes the perspective.

66.

NOTE ON A THEOREM OF THE INTEGRAL CALCULUS.

[*Philosophical Magazine*, XXVI. (1863), pp. 293, 294.]

I PROPOSE briefly to lay before the mathematical readers of the Magazine a wide generalization, and at the same time a more precise statement, of the theorem contained at the close of my paper in the last Number. The theorem, as therein enunciated, was drawn from geometrical considerations, it having first manifested itself dimly to the author by a sort of indirect reflection from a metrical theory of perspective. I have since obtained a very easy proof of it in its extended form, which in spirit amounts to a free algebraical paraphrase of the method indicated in the final footnote of the paper in question. The ultimate form of the perfected theorem is particularly interesting from its simplicity of application, and from its connexion with the grand and growing theory of invariants. The proof of it will appear in its proper place in the continuation of the paper in which, in its incipient state, it first came to light*.

Theorem.—Let a figure, whether plane, solid, or hyperspatial, be supposed to be limited by a locus or loci defined by one or more algebraical equations, not necessarily the most general of their respective degrees, but each at least the most general of its degree and kind†, and let the density at any point of the figure be any *homogeneous* function of the coordinates, and let the mass of such figure be supposed to be known in terms of the constants which enter into the defining equations; next let the density at each point of the mass be multiplied by a new factor, which may be any rational integral homogeneous function of the coordinates. Then the theorem affirms that the expression

* Strange cradle this for the inception of a quasi-invariantive theory of integration, "A geometrical construction of the centre of gravity of a truncated pyramid"! Où la vérité va-t-elle se nicher?

† By *kind* I mean descriptive character, that is such character as is not affected by perspective or homographical deformation. Thus, for example, the case of a cone may be treated apart from the more general case of a surface of the second degree. So, again, a curve of the third degree with a multiple point, or having one or both of its fundamental invariants zero, may be treated apart from the case of a general cubic curve.

for the new mass may be obtained by operating upon the expression for the original one with differential operators precisely identical with combinations of certain of those which serve to define an invariant of the given system of equations, and which will be found set forth in my paper "On the Calculus of Forms," in* the *Cambridge and Dublin Mathematical Journal*†. Thus, for example, by means of the known expressions for the area or content of a triangle, ellipse, pyramid, ellipsoid, or cone, this theorem enables us by differentiation and algebraical processes alone to obtain the parameters which define the centres of gravity, moments of inertia, principal axes, &c., of such figures.

I must add an important observation, namely, that the theorem remains true when one of the defining equations (supposing there to be more than one), instead of being the most general of a certain degree and kind, is affected with arbitrary numerical coefficients (zeros or others), provided only that it be *homogeneous* in the variables. Again, the theorem continues to hold when the original density, instead of being a homogeneous function of the variables, is such function multiplied by any Covariant of the defining equations taken separately or in groups—using the word covariant in its most extended sense, so as to comprehend fractional and irrational as well as integral forms,—the only effect of the introduction of such new factor into the density being to modify the form of the differential operators. There are certain very special cases, to which it is not necessary to allude here in detail, in which the theorem becomes illusory: such will be the case, for example, for a plane area when the given density is a homogeneous function in the variables of the negative degree 3, and for a solid content when that density is of the negative degree 4‡.

* [Volume I. of this Reprint, p. 356.]

† The partial differential equations for invariants, covariants, and contravariants will be found therein stated with absolute generality for any number of functions and any number of variables. Dr Aronhold, in the last Number of *Crelle's Journal*, states erroneously that these equations were given by me for binary functions only, and subsequently generalized by Cayley and Clebsch.

‡ A similar method applied to *extents* (as curves, surfaces, &c.) gives rise to curious theorems. Thus I find that the mass of a plane curve affected with a density varying at each point as the square of the cosine of the inclination of the curve to a fixed line, is a differential derivative of the length of the curve. So, again, the moment of inertia of a curve in respect to any axis perpendicular to its plane, is a differential derivative of its moment in respect to an arbitrary line in its plane.

67.

THÉORÈME SUR LA LIMITE DU NOMBRE DES RACINES RÉELLES D'UNE CLASSE D'ÉQUATIONS ALGÉBRIQUES.

[*Comptes Rendus de l'Académie des Sciences*, LVIII. (1864), pp. 494, 495.]

SOIENT $u_1, u_2, \ldots, u_n$ des fonctions linéaires d'une seule variable, à coefficients réels, et supposons qu'on ait l'équation

$$\lambda_1 u_1^{2i} + \lambda_2 u_2^{2i} + \ldots + \lambda_n u_n^{2i} = 0\,;$$

il est évident que si tous les coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ portent les mêmes signes, le nombre des racines réelles est nul.

En général, supposons que le nombre des signes de même nom soit r, et de nom opposé soit s. Si r est égal ou moindre de s, on peut parler de r comme étant le nombre inférieur des signes semblables de la série $\lambda_1, \lambda_2, \ldots, \lambda_n$; et alors on peut affirmer que le nombre des racines réelles dans l'équation donnée ne peut jamais excéder le double du nombre inférieur de signes semblables dans ses coefficients λ.

Je crois que cette proposition est nouvelle, mais elle n'est qu'une conséquence très-particulière du théorème plus spécifique que voici:

Soient $c_1, c_2, \ldots, c_n$ une série croissante ou décroissante composée avec des quantités réelles, et soit donnée l'équation

$$\lambda_1 (x + c_1)^m + \lambda_2 (x + c_2)^m + \ldots + \lambda_n (x + c_n)^m = 0.$$

Formons la suite $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n, (-1)^m \lambda_1$: je dis que le nombre des racines réelles dans l'équation donnée ne peut pas excéder *le nombre de variations de signe* dans cette suite, et comme corollaire on déduit aisément que ce nombre dans tous les cas ne peut pas excéder le double du nombre inférieur de signes semblables quand m est pair, ni ce double augmenté de l'unité quand m est impair.

Il est bon de remarquer que le maximum spécifique du nombre des racines réelles donné par la suite *déterminée* $\lambda_1, \lambda_2, \ldots, \lambda_n, (-1)^m \lambda_1$ *ne change pas* quand on transforme l'équation donnée en effectuant une substitution homographique réelle quelconque sur la variable x, de sorte qu'on peut dire que chaque *maximum spécifique* est un nombre jouant le rôle d'*invariant*, ce qui n'a pas lieu quand on se sert de la méthode ordinaire pour limiter le nombre des racines réelles de $fx = 0$, en considérant le nombre des racines imaginaires de $f'x = 0$.

68.

SUR UNE EXTENSION DE LA THÉORIE DES ÉQUATIONS ALGÉBRIQUES.

[*Comptes Rendus de l'Académie des Sciences*, LVIII. (1864), pp. 689—691.]

QUELQUES recherches que j'ai faites tout récemment sur la règle donnée sans démonstration par Newton dans l'*Arithmetica universalis* (voir le chapitre *De resolutione æquationum*), pour trouver une limite inférieure au nombre de racines imaginaires d'une équation, m'ont conduit forcément à reconnaître l'existence d'un nouveau et très-intéressant genre d'équations algébriques qui ont exactement le même degré de généralité que les équations ordinaires et jouissent de propriétés parfaitement analogues à celles de ces dernières.

Ce sont les équations pour lesquelles, en partant des deux extrémités de la fonction égalée à zéro, les coefficients se composent, deux à deux, de quantités conjuguées de la forme

$$\lambda + i\mu, \quad \lambda - i\mu$$

respectivement, sauf (pour les équations de degré pair) le coefficient central qui reste seul et nécessairement réel.

Une telle équation peut se mettre sous la forme

$$U + iV = 0,$$

et, en supposant que tout facteur algébrique commun à U et V a été préalablement chassé, elle peut être nommée équation conjuguée. Les équations conjuguées ainsi définies ne peuvent contenir ni racines réelles ni paires de racines imaginaires de la forme

$$\rho e^{i\theta}, \quad \rho e^{-i\theta};$$

mais néanmoins leurs racines, comme celles des équations ordinaires, se diviseront en deux classes, c'est-à-dire classe de racines solitaires et classe de racines associées. Ces deux classes seront chacune du même ordre de généralité. Les racines solitaires seront quantités complexes avec l'unité pour module, c'est-à-dire de la forme $e^{i\theta}$; les racines associées seront quantités complexes dont le rapport est réel et les modules réciproques, c'est-à-dire de la forme

$$\rho e^{i\theta}, \quad \frac{1}{\rho} e^{i\theta}.$$

Il va sans dire que les racines solitaires sont les analogues aux racines réelles, et que les racines associées sont les analogues aux racines imaginaires des équations ordinaires. Dans une forme conjuguée du degré n, comme dans une forme ordinaire du même degré, le nombre de paramètres sera évidemment $n+1$. Tous leurs invariants (sauf le facteur i pour quelques-uns) seront réels, et toutes leurs formes, invariants des dérivées, covariants, contre-variants, etc., seront, elles aussi, des formes conjuguées.

Les théorèmes et les propriétés fondamentales des équations ordinaires se reproduisent (sans exception) sous une forme convenablement modifiée dans la théorie des équations conjuguées; je cite comme exemples la règle pour connaître si le nombre des racines réelles renfermées entre deux quantités réelles est pair ou impair, la liaison de position entre les racines réelles des équations et celles de leurs dérivées différentielles, les théorèmes pour reconnaître le nombre ou une limite au nombre des racines réelles, et en particulier la règle de Sturm et la règle merveilleuse et jusqu' aujourd'hui non démontrée de Newton. Je dois ajouter comme auxiliaire à ce genre de recherches un théorème qui donne une loi d'inertie pour les formes quadratiques (à un nombre quelconque de variables) assujetties à subir des substitutions qui peuvent être qualifiées comme étant substitutions conjuguées au lieu de réelles.

Il n'est pas sans intérêt de faire remarquer que, de même que les racines des équations ordinaires peuvent être représentées géométriquement au moyen de points solitaires situés sur une ligne droite, et par des points associés en couples qui se trouvent deux à deux et à distances égales sur les deux côtés de cette ligne, de sorte que ces derniers points constituent, pour ainsi dire, des images optiques les uns aux autres par rapport à la ligne, de même les racines géométriquement représentées des équations conjuguées se divisent en des points simples situés sur la circonférence d'un cercle dont le rayon est l'unité, et des points qui se trouvent deux à deux à des distances réciproques du centre sur les mêmes rayons, et qui constituent ainsi, pour me servir du langage de M. William Thomson, des images électriques les uns des autres. Ces principes auront prochainement leur développement dans un supplément au Mémoire sur le théorème de Newton déjà cité, que j'ai lu récemment devant la Société Royale de Londres.

69.

SUR UNE EXTENSION DE LA THÉORIE DES RÉSULTANTS ALGÉBRIQUES.

[*Comptes Rendus de l'Académie des Sciences*, LVIII. (1864), pp. 1074—1079.]

Je me propose de dire quelques mots sur une nouvelle classe très-bien définie d'invariants appartenant à l'ordre des combinants et admettant des applications importantes pour la Géométrie. Pour fixer les idées, imaginons un système de surfaces de degré quelconque chacune. Commençons avec le cas de quatre surfaces. En général, elles ne se rencontreront pas: pour que cela ait lieu, une condition doit être satisfaite entre les coefficients, ou, si l'on veut bien, une certaine fonction des coefficients des équations qui représentent ces surfaces doit s'évanouir.

Passons au cas de trois surfaces: ces surfaces s'entrecouperont dans un système de points qui en général seront tous-distincts. Mais il peut arriver que deux de ces points se confondent, c'est-à-dire que les trois surfaces se rencontreront en deux points consécutifs, ou, si l'on veut bien, seront toutes trois touchées par la même ligne droite; pour que cela ait lieu, une certaine fonction des coefficients doit s'évanouir, laquelle, pour le moment, manque de nom. Continuons en supprimant encore une surface. Les deux surfaces qui restent se couperont dans une courbe qui, en général, ne possédera aucune singularité. Mais il peut arriver que cette courbe possède un point double, dans lequel cas les deux surfaces seront touchées par le même plan. Pour que cela arrive, une certaine fonction des coefficients doit s'évanouir, à laquelle, comme exprimant la condition de tangence, notre grand géomètre M. Cayley a proposé de donner le nom de *tact-invariant*.

On peut exprimer sous une forme générale la nature des conditions analytiques qui doivent être satisfaites dans tous ces cas, et dans le cas le plus général où il y aura i fonctions $U_1, U_2, \ldots, U_i$ de n variables $x_1, x_2, \ldots, x_n$. Ecrivons

$$U_1 = 0, \quad U_2 = 0, \quad \ldots, \quad U_i = 0,$$
$$\lambda_1 \delta U_1 + \lambda_2 \delta U_2 + \ldots + \lambda_i \delta U_i = 0,$$

$\lambda_1, \lambda_2, \ldots, \lambda_i$ étant des quantités indéterminées. Puisque

$$\delta U = \frac{dU}{dx_1}\delta x_1 + \frac{dU}{dx_2}\delta x_2 + \ldots + \frac{dU}{dx_n}\delta x_n,$$

cette dernière équation donne lieu à $(n-i+1)$ équations indépendantes: donc le nombre total des équations homogènes à satisfaire avec les n variables sera

$$(n-i+1)+i=n+1;$$

pour que cela soit possible dans le cas général d'un tel nombre d'équations avec un tel nombre de variables, deux conditions entre les coefficients devraient être satisfaites; mais dans le cas actuel une seule sera suffisante, car il existera toujours un rapport syzygétique entre les équations. Dans le cas où il n'y a qu'une seule fonction U, l'équation $U=0$ devient tout à fait superflue, et dans le cas où $i=n$, l'équation $\Sigma\lambda\delta U=0$, qui exprime que la jacobienne des n fonctions est égale à zéro, devient également superflue. Mais dans tout autre cas, quoique en vertu de l'identité

$$U = \Sigma\left(x_1\frac{dU}{dx_1}\right)$$

il existe un rapport syzygétique entre les équations, il n'est pas permis de se passer d'une quelconque d'entre elles, sous peine d'introduire des facteurs étrangers dans l'expression finale. J'espère ne pas trop encourir l'indignation de mon très-honoré confrère M. Poncelet, en donnant un nom spécifique à la fonction dont l'évanouissement exprime la condition suffisante et nécessaire pour que ce système d'équations soit simultanément satisfait, et je propose de lui donner le nom, qui n'est pas tout à fait étranger à la Géométrie, d'*osculant*; ainsi on peut partir de l'osculant d'un système de i fonctions homogènes quelconques de n variables, et on voit que les discriminants, les *tact-invariants* de M. Cayley et les résultants ne sont que des espèces particulières des osculants: pour les discriminants $i=1$, pour les *tact-invariants* $i=2$, pour les résultants $i=n$.

Il importe beaucoup au développement de cette théorie de bien fixer le degré des osculants par rapport à chaque système de coefficients contenu dans les fonctions auxquelles ils appartiennent.

Pour les deux extrémités de l'échelle d'osculants, c'est-à-dire les discriminants et les résultants, les expressions pour ce degré sont très-simples et bien connues. Pour les *tact-invariants* le degré n'a été trouvé (je crois par M. Cayley) que pour le seul cas où $n=3$, c'est-à-dire pour les contacts des courbes. Le théorème suivant donne l'expression absolument générale pour les osculants de chaque ordre n et de chaque classe i.

Soient $m_1, m_2, \ldots, m_i$ les degrés des variables des i fonctions, et pour plus de simplicité écrivons

$$m_1 = 1+\mu_1, \quad m_2 = 1+\mu_2, \quad \ldots, \quad m_i = 1+\mu_i.$$

En général, soit $H_w(\mu_2, \mu_3, \ldots, \mu_i)$ la somme des puissances et des produits homogènes de $\mu_2, \mu_3, \ldots, \mu_i$, et soit G_k le degré de l'osculant du système par rapport aux coefficients de la fonction U_k. Alors je dis que

$$\frac{1}{m_2 m_3 \ldots m_i} G_1 = H_{n-i}(\mu_2, \mu_3, \ldots, \mu_i) + 2H_{n-i-1}(\mu_2, \mu_3, \ldots, \mu_i)\mu_1$$
$$+ 3H_{n-i-2}(\mu_2, \mu_3, \ldots, \mu_i)\mu_1^2 + \ldots$$
$$+ (n-i)H_1(\mu_2, \mu_3, \ldots, \mu_i)\mu_1^{n-i-1} + (n-i+1)\mu_1^{n-i},$$

et on trouve de même les valeurs de $G_2, G_3, \ldots, G_i$.

Pour les *tact-invariants* $i=2$, et le théorème devient

$$G_1 = m_2[\mu_2^{n-2} + 2\mu_2^{n-3}\mu_1 + 3\mu_2^{n-4}\mu_1^2 + \ldots + (n-1)\mu_1^{n-2}],$$
$$G_2 = m_1[\mu_1^{n-2} + 2\mu_1^{n-3}\mu_2 + 3\mu_1^{n-4}\mu_2^2 + \ldots + (n-1)\mu_2^{n-2}],$$

ou, si l'on veut,

$$G_1 = m_2\frac{\mu_2^n - n\mu_2\mu_1^{n-1} + (n-1)\mu_1^n}{(\mu_1-\mu_2)^2},$$

$$G_2 = m_1\frac{\mu_1^n - n\mu_1\mu_2^{n-1} + (n-1)\mu_2^n}{(\mu_1-\mu_2)^2}.$$

Si $n=3$,

$$G_1 = m_2[(m_2-1)+2(m_1-1)] = m_2(m_2+2m_1-3), \qquad G_2 = m_1(m_1+2m_2-3):$$

c'est le cas du contact de deux courbes. Quand $n=4$, c'est-à-dire qu'on veut trouver le degré de la condition pour le contact de deux surfaces, on trouve

$$G_1 = m_2(m_2^2 + 2m_1m_2 + 3m_1^2 - 4m_2 - 8m_1 + 4).$$

Pour trouver les degrés de la condition de rencontre en deux points consécutifs de trois surfaces, il faut prendre $i=3$, $n=4$; alors on trouve

$$G_1 = m_2m_3(m_2+m_3+2m_1-4).$$

Pour le cas des polaires réciproques, on a

$$i=2, \quad m_1=m, \quad m_2=1,$$

et on retombe sur les résultats connus pour ce cas. Si on suppose dans le cas général $m_1=m_2=\ldots=m_i$, on obtient pour le degré de l'osculant, dans un système quelconque de coefficients,

$$\frac{n(n-1)\ldots(n-i+1)}{1 \,.\, 2 \ldots i} m^{i-1}(m-1)^{n-i}.$$

Pour mettre en plein jour la véritable identité de nature de ce genre compréhensif des osculants, je ferai l'extension à une classe de ces fonctions d'un théorème bien connu pour le discriminant de deux fonctions.

On sait bien que le discriminant du produit de deux fonctions homogènes en x et y est égal au produit de leurs discriminants multiplié par le carré de leur résultant. Ainsi, en se servant de Ω comme le symbole universel de l'osculation et supposant F et F' ces deux fonctions, on peut écrire

$$\Omega(FF') = \Omega F \times \Omega F' \times [\Omega(F, F')]^2.$$

Remarquons bien qu'on ne peut pas étendre ce théorème dans sa forme actuelle à des fonctions de plus de deux variables, car quand F, F' sont des fonctions de 3 ou un plus grand nombre de variables, on a identiquement

$$\Omega(FF') = 0.$$

Or, considérons $F_1, F_2, \ldots, F_i, F_i'$, $(i+1)$ fonctions de $(i+1)$ variables; j'énonce le théorème suivant:

$$\begin{aligned}&\Omega(F_1 F_2 \ldots F_{i-1} F_i F_i')\\ &= \Omega(F_1, F_2, \ldots, F_{i-1}, F_i) \times \Omega(F_1, F_2, \ldots, F_{i-1}, F_i')\\ &\times [\Omega(F_1, F_2, \ldots, F_i, F_i')]^2,\end{aligned}$$

où on peut remarquer que le dernier des trois facteurs est le carré d'un résultant. De plus, j'affirme que si les F deviennent fonctions de plus de $(i+1)$ variables, la quantité

$$\Omega(F_1 F_2 \ldots F_{i-1} F_i F_i')$$

s'évanouit identiquement. Mais je passe outre à un autre théorème sur les discriminants d'une fonction vue comme un *quantic* de quantics dont j'ai eu occasion de me servir dans quelques recherches récentes sur le théorème de Newton pour la découverte des racines imaginaires.

Soit F une fonction rationnelle homogène et entière du degré m en ϕ et ψ, ϕ et ψ étant elles-mêmes fonctions rationnelles homogènes et entières du degré μ en x et y. Servons-nous du symbole D pour désigner *discrimination* par rapport à x, y, et de D' pour désigner la même chose par rapport à ϕ, ψ; R sera le symbole du résultant par rapport à x, y, et J représentera la fonction *jacobienne*

$$\frac{d\phi}{dx}\frac{d\psi}{dy} - \frac{d\phi}{dy}\frac{d\psi}{dx}.$$

Alors je trouve que

$$D(F) = [R(\phi, \psi)]^{m^2-2m} [D'(F)]^{\mu} R(F, J).$$

Dans le cas où ϕ, ψ sont des fonctions linéaires de x, y, $R(F, J)$ devient égale à $[R(\phi, \psi)]^m$, et on retombe sur la formule connue pour les transformations linéaires

$$D_{x,y} F = [R(\phi, \psi)]^{m^2-m} D_{\phi,\psi} F.$$

Quand F est une fonction symétrique par rapport à x, y, F sera une fonction homogène et entière de (x^2+y^2) et de xy, dont la jacobienne a pour racines $\frac{x}{y} = \pm 1$, et conséquemment on voit que son discriminant prend la forme

$$I^2 F(1, 1) \,.\, F(1, -1).$$

Or, pour généraliser le théorème, soient $F_1, F_2, \ldots, F_{i-1}$, des fonctions homogènes et entières des degrés $m_1, m_2, \ldots, m_{i-1}$ des i quantités $\phi_1, \phi_2, \ldots, \phi_i$, dont chacune est une fonction homogène et entière de degré μ en $x_1, x_2, \ldots, x_i$. Servons-nous de Ω pour exprimer osculation par rapport à $x_1, x_2, \ldots, x_i$, Ω' pour exprimer la même chose par rapport à $\phi_1, \phi_2, \ldots, \phi_i$, de J pour exprimer la jacobienne de $\phi_1, \phi_2, \ldots, \phi_i$ par rapport à $x_1, x_2, \ldots, x_i$, et soit

$$M = (m_1 + m_2 + \ldots + m_{i-1} - i)(m_1 m_2 \ldots m_{i-1}).$$

Alors on aura l'identité suivante:

$$\begin{aligned}&\Omega(F_1, F_2, \ldots, F_{i-1})\\ &= [\Omega(\phi_1, \phi_2, \ldots, \phi_i)]^M \,.\, [\Omega'(F_1, F_2, \ldots, F_{i-1})]^{\mu^{i-1}} \,.\, \Omega(F_1, F_2, \ldots, F_{i-1}, J).\end{aligned}$$

Il me semble qu'on peut reconnaître ici l'approche de la véritable aurore de cette science des formes dont on ne voit qu'une phase bornée et passagère dans la théorie des transformations linéaires. Les actions mutuelles des formes, les unes sur les autres, constituant une espèce de chimie algébrique, me paraît le vrai but de cette science naissante.

P.S. Il n'est pas inutile de remarquer qu'on peut donner une définition des osculants qui montre d'une manière immédiate leur identité avec les discriminants. Soient $U_1, U_2, \ldots, U_i$, i fonctions homogènes rationnelles et entières de n variables, et soit R le résultat de l'élimination de $(i-1)$ quelconques des variables entre les équations

$$U_1 = 0, \ U_2 = 0, \ \ldots, \ U_i = 0.$$

Alors l'osculant du système donné de fonctions U sera contenu comme facteur dans le discriminant de R. De même on peut démontrer que si on combine ensemble $i-k$ des équations $U=0$ et si on prend $(k+1)$ de telles combinaisons, et si pour chaque combinaison on forme un résultant en éliminant les mêmes $i-k-1$ variables, l'osculant du système donné sera contenu comme facteur dans l'osculant de ces $(k+1)$ résultants.

70.

ADDITION À UNE NOTE INSÉRÉE DANS LE *COMPTE RENDU* DE LA SÉANCE PRÉCÉDENTE.

[*Comptes Rendus de l'Académie des Sciences*, LVIII. (1864), p. 1130.]

DANS la Note que j'ai eu l'honneur de soumettre à l'Académie sur une extension de la théorie des résultants algébriques, on trouve la formule générale pour le degré de l'*osculant* d'un système d'un nombre quelconque i de fonctions d'un nombre n quelconque de variables, et j'ai cité comme déjà connu le degré pour le cas de $n=3$, $i=2$, qui correspond à la condition de contact de deux courbes [p. 364].

Je dois citer en même temps comme également connus les degrés de l'osculant pour les cas de $n=4$, $i=2$, et de $n=4$, $i=3$, c'est-à-dire les cas qui correspondent à deux surfaces qui se touchent et à trois surfaces qui se rencontrent en deux points consécutifs.

Les degrés des conditions pour ces cas ont été donnés dans un excellent article par M. Th. Moutard, dans les *Nouvelles Annales de Mathématiques*, t. XIX. ce que j'ignorais au moment où j'ai écrit la Note en question.

71.

ADDITION À LA NOTE SUR UNE EXTENSION DE LA THÉORIE DES RÉSULTANTS ALGÉBRIQUES.

[*Comptes Rendus de l'Académie des Sciences*, LVIII. (1864), pp. 1178—1180.]

ON peut mettre la formule pour exprimer le degré d'un osculant de r fonctions homogènes de n variables sous une forme très-simple qu'il importe de signaler.

En sous-entendant toujours par $H_k(a, b, c, \ldots, l)$ la somme des puissances et des produits homogènes du degré k de $a, b, c, \ldots, l$, c'est-à-dire le coefficient de τ^k dans le développement en série de

$$\frac{1}{(1-a\tau)(1-b\tau)\ldots(1-l\tau)},$$

on verra sans aucune difficulté que la série donnée [p. 365 above] dans les *Comptes rendus* du 13 juin pour $\frac{1}{m_2 m_3 \ldots m_i} G_1$ n'est autre chose que la quantité

$$H_{n-i}(\mu_1, \mu_1, \mu_2, \ldots, \mu_i),$$

de sorte qu'on aura en général

$$m_\omega G_\omega = \Pi(m) \,.\, H_{n-i}[(m_1-1), (m_2-1), \ldots, (m_i-1), (m_\omega-1)],$$

où, dans la série écrite entre les crochets, $m_\omega - 1$ sera deux fois rencontré.

Pour les résultants, $(n-i)$ étant zéro, on trouve

$$m_\omega G_\omega = \Pi(m).$$

Pour les discriminants, $n-i=n-1$ et G devient égal à

$$H_{n-1}[(m-1), (m-1)] = n(m-1)^{n-1}.$$

J'apprends de la part de M. Salmon qu'il y a grand nombre d'années qu'il a trouvé le degré des osculants pour le cas de deux courbes; il paraît donc que je me trompais en attribuant cette détermination (qui de plus n'offre aucune difficulté) à M. Cayley.

72.

SUR LA THÉORIE DES RACINES RÉELLES ET IMAGINAIRES DES ÉQUATIONS DU CINQUIÈME DEGRÉ.

[*Comptes Rendus de l'Académie des Sciences*, LIX. (1864), pp. 749—753.]

On sait la découverte faite par M. Hermite et insérée dans le tome IX. du *Journal de Mathématiques de Cambridge et Dublin*. C'est là que M. Hermite a fait la belle observation, qu'aux conditions fournies par le théorème de Sturm on peut substituer des fonctions des invariants d'une forme binaire de degré impair quelconque, pour déterminer le nombre de ses racines réelles et imaginaires. De plus, M. Hermite, en suivant une marche toute particulière, a donné les *criteria* actuels, qui servent à peu près pour distinguer entre les trois cas qui se présentent dans la considération des formes du cinquième degré, c'est-à-dire le cas où toutes les racines sont réelles, celui où trois seulement sont réelles et le cas où il n'y a qu'une seule réelle. Cependant ce grand travail avait laissé quelque chose à désirer; car pour remplir cet objet, M. Hermite a été conduit à se servir de cinq invariants, un du degré 4, un (le discriminant) du degré 8 et trois chacun du degré 12, tandis que la méthode de M. Sturm n'exige que l'emploi de quatre *criteria*. De plus, le système de conditions donné par M. Hermite n'est pas absolument complet, mais laisse une certaine lacune à combler: je veux dire qu'il y a de certaines combinaisons de ses *criteria* pour lesquelles il reste douteux si la forme possède cinq ou bien une seule racine réelle; c'était une omission dont M. Hermite avait conscience et qu'il aurait sans doute trouvé le moyen de remplir. En me pénétrant de l'esprit de la méthode de M. Hermite, mais en suivant úne tout autre voie d'application, je suis parvenu à trouver la solution la plus générale de ce problème important sous une forme d'une simplicité qui ne laisse rien à désirer, et à laquelle aucun cas n'échappe. Dans cette solution, au lieu d'excéder le nombre des *criteria* donnés par la méthode générale de M. Sturm, on se sert d'un de moins; en effet, en outre du discriminant, on n'a besoin que d'un invariant (le seul qui existe) du quatrième ordre et un du douzième ordre. Nommons D le discriminant de la forme proposée, J le discriminant de son covariant quadratique le plus simple multiplié par -4, L le discriminant de son covariant cubique le plus simple multiplié par -27, et de plus écrivons

$$\Lambda = J^3 - 2^{11}L;$$

J, D, Λ suffisent pour déterminer le caractère des racines selon la règle suivante:

Quand D est négatif, trois racines sont réelles, deux imaginaires.

Quand D est positif, si J et $\Lambda + \mu JD$ sont tous les deux négatifs, les racines seront toutes réelles; dans le cas contraire, une seule sera réelle.

μ est un paramètre numérique variable à volonté entre certaines limites que j'ai trouvées, mais que je n'ose rapporter, n'ayant pas les calculs sous mes yeux. Je crois cependant pouvoir affirmer en toute sûreté que ces limites sont ou $1, -2$, ou bien $-1, 2$. Avec ces mêmes *criteria* on peut aussi déterminer le caractère des racines dans le cas où D devient zéro, mais je n'entrerai pas ici dans ce détail.

La valeur $\mu = -\frac{4}{5}$ ne sort pas des limites permises, et on trouvera que $\Lambda - \frac{4}{5}JD$ s'exprime facilement en fonction des racines. Nommons-les a, b, c, d, e en désignant par K un certain coefficient numérique et positif, on aura

$$\tfrac{4}{5}JD - \Lambda$$
$$= K\Sigma\,[(a-b)^2(a-c)^2(b-c)^2(a-d)^4(a-e)^4(b-d)^4(b-e)^4(c-d)^4(c-e)^4].$$

De plus, en nommant q un autre multiplicateur numérique et positif, on aura

$$-J = q\Sigma\,[(a-b)^2(a-c)^2(b-c)^2(d-e)^4].$$

Posons $$Q(d, e) = (a-b)^2(a-c)^2(b-c)^2(d-e)^4.$$

Alors, pour distinguer entre le cas où il n'y a pas de racines imaginaires et le cas où il y en a quatre (les seuls qui se présentent quand D est positif), la règle donnée ci-dessus conduit à l'observation que si les racines ne sont pas toutes réelles et si D est positif, $\Sigma Q(d, e)$ et $\Sigma \frac{1}{Q(d, e)}$ ne peuvent pas rester tous les deux positifs. Dans le cas contraire il est évident que ΣQ et $\Sigma \frac{1}{Q}$ sont tous les deux positifs. J'ajouterai quelques mots sur la marche que j'ai suivie pour obtenir ces résultats. Je démontre qu'en général la forme $(x, y)^5$ peut être réduite par des substitutions linéaires et réelles à l'expression $au^5 + bv^5 + cw^5$, où w est une fonction linéaire et réelle de x, y; u, v des fonctions linéaires, mais pas nécessairement réelles, et où de plus $u + v + w = 0$. Le cas d'exception, c'est celui où le covariant cubique du troisième ordre par rapport aux coefficients (dit le *canonisant*)

contient des racines égales ou bien s'évanouit. Dans ce cas, sauf la supposition de trois racines égales et quand, conséquemment, tous les invariants s'évanouissent, la proposée se réduit par des substitutions linéaires à la forme de Jerrard $ax^5 + exy^4 + fy^5$. De là on conclut facilement que, étant donnés J, D, L (pourvu qu'on n'ait pas en même temps $J = 0$, $D = 0$, $L = 0$), le caractère des racines, quant à la distinction entre le réel et l'imaginaire, est absolument déterminé, et de plus que J, D, L, non-seulement doivent être réels, mais encore (comme l'a remarqué le premier mon devancier M. Hermite) doivent satisfaire à une certaine condition d'inégalité, c'est-à-dire qu'une certaine fonction (nommons-la G) de J, D, L doit rester toujours positive. Je prends J, D, L pour coordonnées d'un point dans l'espace. Alors la surface $G = 0$ divisera l'espace en deux portions pour l'une desquelles (qu'on peut nommer *la portion facultative*) tous les points correspondront à des familles d'équations avec des coefficients réels et dans l'autre (qu'on peut nommer *la portion non facultative*) tous les points correspondront à des familles d'équations avec des coefficients conjugués. Ces deux portions d'espace sont exactement égales et contraires, étant disposées symétriquement par rapport à l'axe de D. Cela étant, je trouve que la première (en faisant pour le moment abstraction du plan de D) se divise en trois régions. Toute la portion facultative au-dessous du plan de D constitue une seule région, tandis que la portion facultative au-dessus de ce plan se divise en deux régions qui se rencontrent dans la ligne où la surface G touche le plan de D, c'est-à-dire la ligne parabolique

$$\Lambda = 0,\ D = 0.$$

La condition qui fixe les limites de ces trois régions ou, si l'on veut, de ces trois circonscriptions limitrophes, c'est qu'on doit pouvoir passer dans une région donnée d'un point à un autre sans percer ni toucher le plan de D. Cela étant ainsi, on démontre facilement que pour chaque région les familles des formes représentées par un point qui y est renfermé appartiennent à la même catégorie, quant au nombre de leurs racines réelles et imaginaires, et on assigne sans aucune difficulté son propre caractère radical à chaque région. En exprimant dans la langue de l'analyse les conditions qui servent pour déterminer à quelle région répond un système donné de valeurs de J, D, L, on établit la règle donnée ci-dessus pour fixer le caractère des racines de la forme à laquelle ces trois invariants appartiennent. On devinera facilement comment le paramètre μ vient s'offrir dans ces conditions: en effet,

$$\Lambda + \mu JD = 0$$

représente une surface qui, passant par la ligne limitrophe aux deux régions supérieures, ne passe par aucun point facultatif au-dessus du plan de D, c'est-à-dire ne rencontre nulle part la surface $G = 0$ au-dessus de ce plan.

Le perfectionnement que j'ai eu le bonheur d'ajouter ainsi à la découverte de mon confrère s'est offert à moi comme une conséquence (dans l'ordre subjectif des idées) du théorème que j'ai eu l'honneur déjà de publier dans les *Comptes rendus* de cette année*, et qui se rapporte à la limite du nombre des racines réelles de l'équation

$$\lambda_1 (x + c_1)^m + \lambda_2 (x + c_2)^m + \ldots \dot{+} \lambda_i (x + c_i)^m = 0.$$

Dans cette équation, en supposant $c_1, c_2, \ldots, c_m$ arrangés en ordre de leurs grandeurs et en écrivant la suite $\lambda_1, \lambda_2, \ldots, \lambda_i, (-1)^m \lambda_1$, le théorème consiste en ce que le nombre des racines réelles ne peut pas dépasser le nombre de changements de signe dans la suite; mais j'avais imposé la condition que m doit être un nombre entier et positif; cette dernière restriction au moins est superflue; le théorème reste vrai quand m est un nombre négatif, tout aussi bien comme quand il est positif. Cette extension suit comme conséquence immédiate d'un théorème algébrique qu'on peut établir sans aucune difficulté, mais que je ne me rappelle pas d'avoir jamais rencontré.

Soient† $f(x, y)$, $\phi(x, y)$ deux fonctions homogènes quelconques en x, y; J la jacobienne de f, ϕ, c'est-à-dire $\frac{df}{dx}\frac{d\phi}{dy} - \frac{df}{dy}\frac{d\phi}{dx}$. Alors je dis qu'un nombre impair des racines de J sera compris entre chaque paire de racines réelles et consécutives de f, comme évidemment aussi entre chaque paire de racines réelles et consécutives de ϕ, de sorte que le nombre des racines réelles de f ni de ϕ ne peut excéder de plus d'une unité le nombre des racines réelles de J. Si on prend $\phi(x, y) = y$, on retombe sur le théorème d'algèbre élémentaire qui donne la disposition des racines réelles de $f'x$ par rapport aux racines réelles de fx.

* [p. 360 above.]

† [See p. 375 below.]

73.

CORRECTION DE LA NOTE INSÉRÉE DANS LES *COMPTES RENDUS* POUR LA SÉANCE DU 7 NOVEMBRE.

[*Comptes Rendus de l'Académie des Sciences*, LIX. (1864), pp. 944, 945.]

UNE erreur assez grave, mais n'ayant nul rapport à l'objet principal de la communication mentionnée ci-dessus, s'est glissée dans le théorème donné vers sa fin [p. 374]. En supposant ϕ et ψ deux fonctions homogènes et entières en x, y et J leur jacobienne, j'ai affirmé qu'entre deux racines consécutives quelconques de ϕ (comme aussi de ψ) se trouvera une racine ou un nombre impair de racines de J. J'aurais dû dire qu'entre deux telles racines de ψ se rencontrera une racine ou un nombre impair de racines de ϕJ, et pareillement pour ϕ, ψJ. En conséquence, l'extension que je m'imaginais avoir faite du théorème pour les équations de la forme $\Sigma\lambda_i (x + c_i)^m$ au cas de m négatif n'aura pas lieu.

74.

ALGEBRAICAL RESEARCHES, CONTAINING A DISQUISITION ON NEWTON'S RULE FOR THE DISCOVERY OF IMAGINARY ROOTS, AND AN ALLIED RULE APPLICABLE TO A PARTICULAR CLASS OF EQUATIONS, TOGETHER WITH A COMPLETE INVARIANTIVE DETERMINATION OF THE CHARACTER OF THE ROOTS OF THE GENERAL EQUATION OF THE FIFTH DEGREE, &c.

[*Philosophical Transactions of the Royal Society of London*, CLIV. (1864), pp. 579—666.]

AN INQUIRY INTO NEWTON'S RULE FOR THE DISCOVERY OF IMAGINARY ROOTS.

[*Proceedings of the Royal Society of London*, XIII. (1863-4), pp. 179—183.]

(*Abstract.*)

IN the *Arithmetica Universalis*, in the chapter *De Resolutione Equationum*, Newton has laid down a rule, admirable for its simplicity and generality, for the discovery of imaginary roots in algebraical equations, and for assigning an inferior limit to their number. He has given no clue towards the ascertainment of the grounds upon which this rule is based, and has stated it in such terms as to leave it quite an open question whether or not he had obtained a demonstration of it. Maclaurin, Campbell, and others have made attempts at supplying a demonstration, but their efforts, so far as regards the more important part of the rule, that namely by which the limit to the number of imaginary roots is fixed, have completely failed in their object. Thus hitherto any opinion as to the truth of the rule rests on the purely empirical ground of its being found to lead to correct results in particular arithmetical instances. Persuaded of the insufficiency of such a mode of verification, the author has applied himself to obtaining a rigorous demonstration of the rule for equations of specified degrees. For the second degree no demonstration is necessary. or cubic equations a proof is found without difficulty. For biquadratic

equations the author proceeds as follows. He supposes the equation to be expressed homogeneously in x, y, and then, instituting a series of infinitesimal linear transformations obtained by writing $x + hy$ for x, or $y + hx$ for y, where h is an infinitesimal quantity, shows that the truth of Newton's rule for this case depends on its being capable of being shown that the discriminant of the function $(1, \pm e, e^2, \pm e, 1 \between x, y)^4$ is necessarily positive for all values of e greater than unity, which is easily proved. He then proceeds to consider the case of equations of the 5th degree, and, following a similar process, arrives at the conclusion that the truth of the rule depends on its being capable of being shown that the discriminant, say D, of the function $(1, \epsilon, \epsilon^2, \eta^2, \eta, 1 \between x, y)^5$, which for facility of reference may be termed "the (ϵ, η) function," is necessarily positive when $\epsilon^4 - \epsilon\eta^2$ and $\eta^4 - \eta\epsilon^2$ are both positive. This discriminant is of the 12th degree in ϵ, η. But on writing $x = \epsilon\eta$, $y = \epsilon^5 + \eta^5$, it becomes a rational integral function of the 6th degree in x, and of the second degree in y, and such that, on making $D = 0$, the equation represents a sextic curve, of which x, y are the abscissa and ordinate, which will consist of a single close. It is then easily demonstrated that all values of ϵ, η which cause the variable point x, y to lie inside this curve, will cause D to be negative (in which case the function ϵ, η has only two imaginary factors), and that such values as cause the variable point to lie outside the curve, will make D positive, in which case the ϵ, η function has four imaginary factors. When the conditions concerning ϵ, η above stated are verified, it is proved that the variable point must be exterior to the curve, and thus the theorem is demonstrated for equations of the 5th degree.

The question here naturally arises as to the significance of the sign of D when such a position is assigned to the variable point as gives rise to *imaginary* values of ϵ, η, which in such case will be conjugate quantities of the form $\lambda + i\mu$, $\lambda - i\mu$ respectively.

The curve D will be divided by another sextic curve into two portions, for one of which the couple ϵ, η corresponding to any point in its interior is *real*, and for the other *conjugate*. This brings to view the necessity of there being in general a theory for equations with conjugate coefficients, which for greater brevity may be termed conjugate equations, analogous to that for real equations in respect of the distinction between real and imaginary roots in the latter. A conjugate equation is one in which the coefficients, reckoning from the two ends of the equation, go in pairs of the form $p \pm iq$, with the obvious condition that when there is a middle coefficient this must be real. Such an equation may be supposed to be so prepared that, when thrown into the form $P + iQ$, P and Q shall have no common algebraical factor; and when this is effected, it may easily be shown that the conjugate equation can neither have real roots nor roots paired together of the form $\lambda + i\mu$, $\lambda - i\mu$ respectively. How, then, it may be asked, is the

analogy previously referred to possible? On investigation it will be found that the roots divide themselves into two categories, each of exactly the same order of generality,—namely *solitary* roots of the form $\epsilon^{i\theta}$, and *associated* roots which go in pairs, the two roots of each pair being of the form $\rho\epsilon^{i\theta}$, $\frac{1}{\rho}\epsilon^{i\theta}$ respectively; so that, following the ordinary mode of geometrical representation of imaginary quantities, the roots of a conjugate equation may be denoted by points lying on the circumference of a circle of radius unity (corresponding to solitary roots), and points (corresponding to the associated roots) lying in couples on different radii of the circle at reciprocal distances from the centre, each couple in fact constituting, according to Prof. W. Thomson's definition, electrical images of each other in respect to the circle. If the circle be taken with radius infinity instead of unity (so as to become a straight line), then we have the geometrical *eidolon* of the roots of an ordinary equation, the solitary roots lying on a straight line, and the associated or paired (imaginary) roots on each side of, and at equal distances from, the line.

In the inquiry before us, whether the variable point belong to the real or conjugate part of the plane of the D curve, it is shown to remain true that the number of *associated* roots will be two, if it lie inside the curve, and four if it lie outside. The author then suggests a probable extension of Newton's rule to conjugate equations of any degree. In conclusion, he deals with a question in close connexion with, and arising out of the investigation of this rule, relating to equations of the form $\Sigma \pm (ax+b)^m = 0$, to which, for convenience, he gives the provisional name of "superlinear equations" (denoting the function equated to zero as a superlinear form), and establishes a rule for limiting the number of real roots which they can contain, which is, that if such equation be thrown under the form

$$\lambda_1 (x+c_1)^m + \lambda_2 (x+c_2)^m + \ldots + \lambda_n (x+c_n)^m = 0,$$

and $c_1, c_2, \ldots c_n$ be an ascending or descending order of magnitudes, the equation cannot have more real roots than there were variations of sign in the sequence $\lambda_1, \lambda_2, \ldots, \lambda_n, (-)^m \lambda_1$.

This theorem was published by the author, but without proof, in the *Comptes Rendus* for the month of March in this year*.

But the method of demonstration now supplied is deserving of particular attention in itself; for it brings to light a new order of purely tactical considerations, and establishes a previously unsuspected kind of, so to say, algebraical polarity. The proof essentially depends upon the character of every superlinear form being associated with, and capable of definition by means of a pencil of rays, which may be called the type pencil, subject to

* [Above, p. 360.]

a species of circulation of a different nature according as the degree of the form is even or odd, which he describes by the terms "per-rotatory" in the one case, and "trans-rotatory" in the other; so that the types themselves may be conveniently distinguished by the names "per-rotatory" and "trans-rotatory." By per-rotatory circulation is to be understood that species in which, commencing with any element of the type, passage is made from it to the next, from that to the one following, from the last but one to the last, from the last to the first, and so on, until the final passage is to the element commenced with, from the one immediately preceding. By trans-rotatory circulation, on the other hand, is understood that species in which, commencing with any element and proceeding in the same manner as before to the end element, passage is made from that, not to the first element itself, but to its polar opposite, from that to the polar opposite of the next, and so on, until the final passage is made to the polar opposite of the element commenced with, from the polar opposite of its immediate antecedent. The number of changes of sign in effecting such passages, whether in a per-rotatory or a trans-rotatory type, is independent of the place of the element with which the circulation is made to commence, and may be termed the variation-index of the type, which is always an even number for per-rotatory, and an odd number for trans-rotatory types. A theorem is given whereby a relation is established between the variation-index of a per-rotatory or trans-rotatory and that of a certain trans-rotatory or per-rotatory type capable of being derived from them respectively; and this purely tactical theorem, combined with the algebraical one, that the form $f(x, y)$ cannot have fewer imaginary factors than any linear combination of $\frac{df}{dx}, \frac{df}{dy}$, leads by successive steps of induction to the theorem in question, but under a more general form, which serves to show intuitively that the limit to the number of real roots of a superlinear equation which the theorem furnishes must be independent of any homographic transformation operated upon the form. The author believes that, whilst it is highly desirable that a simple and general method should be discovered for the proof of Newton's rule as applicable to equations of any degree, and that the strenuous efforts of the cultivators of the New Algebra should be directed to the attainment of this object, his labours in establishing a proof applicable as far as equations of the 5th degree inclusive will not have been unproductive of good, as well on account of the confirmation they afford of the truth of the rule, towards the establishment of which on scientific grounds they constitute the first serious step yet made, as also, and still more, by reason of the accessions to the existing field of algebraical speculation to which they have incidentally led.

"Turns them to shapes and gives to airy nothing
A local habitation and a name."

(1) THIS memoir* in its present form is of the nature of a trilogy; it is divided into three parts, of which each has its action complete within itself, but the same general cycle of ideas pervades all three, and weaves them into a sort of complex unity. In the first is established the validity of Newton's rule for finding an inferior limit to the number of imaginary roots of algebraical equations as far as the fifth degree inclusive. In the second is obtained a rule for assigning a like limit applicable to equations of the form $\Sigma(ax+b)^m=0$, m being any positive integer, and the coefficients a, b real. In the third are determined the absolute invariantive criteria for fixing unequivocally the character of the roots of an equation of the fifth degree, that is to say, for ascertaining the exact number of real and imaginary roots which it contains. This last part has been added since the original paper was presented to the Society. It has grown out of a foot-note appended to the second, itself an independent offshoot from the first part, but may be studied in a great measure independently of what precedes, and constitutes, in the author's opinion, by far the most valuable portion of the memoir, containing as it does a complete solution of one of the most interesting and fruitful algebraical questions which has ever yet engaged the attention of mathematicians[1]. I propose in a subsequent addition to the memoir to resume and extend some of the investigations which incidentally arise in this part. The foot-notes are numbered and lettered for facility of reference, and will be found in many instances of equal value with the matter in the text, to which they serve as a kind of free running accompaniment and commentary.

PART I.—ON NEWTON'S RULE FOR THE DISCOVERY OF IMAGINARY ROOTS.

(2) In the *Arithmetica Universalis*, in the first chapter on equations, Newton has given a rule for discovering an inferior limit to the number of imaginary roots in an equation of any degree, without proof or indication of the method by which he arrived at it, or the evidence upon which it rests[2]. Maclaurin, in vol. XXXIV. [1726–7], p. 104, and vol. XXXVI. p. 59 of the

* [The Author's Table of Contents is given on p. 477.]

(1) I owe my thanks to my eminent friend Professor De Morgan for bringing under my notice, in a marked manner, the original question from which all the rest has proceeded. As all roads are said to lead to Rome, so I find, in my own case at least, that all algebraical inquiries sooner or later end at that Capitol of Modern Algebra over whose shining portal is inscribed "Theory of Invariants."

(2) It appears to be the prevalent belief among mathematicians who have considered the question, that Newton was not in possession of other than empirical evidence in support of his rule.

Philosophical Transactions, Campbell[3] in vol. XXXV. p. 515 of the same, and other authors of reputation have sought in vain for a demonstration of this marvellous and mysterious rule[4]. Unwilling to rest my belief in it on mere empirical evidence, I have investigated and obtained a demonstration of its truth as far as the fifth degree inclusive, which, although presenting only a small instalment of the desired result, I am induced to offer for insertion in the *Transactions* in the hope of exciting renewed attention

(3) Campbell's memoir is rather on an analogous rule to Newton's than on the rule itself, to which he refers only by way of comparison with his own. In it the same singular error of reasoning is committed as in the notes of the French edition of the *Arithmetica*, namely, of assuming, without a shadow of proof, that if each of a set of criteria indicates the existence of some imaginary roots, a succession of sets of such criteria must indicate the existence of at least as many distinct imaginary pairs of roots as there are such sets (see paragraph at foot of p. 528, *Phil. Trans.*, vol. XXXV.)—much as if, supposing a number of dogs to be making a point in the same field, the existence could be assumed of as many birds as pointers.

(4) Mr Archibald Smith has obligingly called my attention to Waring's treatment of the question of Newton's rule in the *Meditationes Analyticæ*. On superficial examination the reader might be induced to suppose that in part 9, p. 68, ed. 1782, Waring had deduced a proof of the rule from the preceding propositions; but on looking into the case will find that there is not the slightest vestige of proof, the rule being stated, but without any demonstration whatever being either adduced or alleged. In fact, on turning to the preface of this (the last) edition of the *Meditationes*, the reader will find at p. 11 an explicit avowal of the demonstration being wanting. After referring in order to Campbell's, Maclaurin's, and Newton's rules, as well as his own, for discovering the existence of impossible roots, he adds these words:

"At omnes hæ regulæ prædictæ perraro invenerunt verum numerum impossibilium radicum in æquationibus multarum dimensionum *et adhuc demonstratione egent*; vulgares enim demonstrationes solummodo probant impossibiles radices in data æquatione contineri, non vero quod *saltem tot sunt quot invenit regula.*"

"Vera resolutio problematis est perdifficilis et valde laboriosa; cognitum est radices ex possibilitate per æqualitatem transire ad impossibilitatem; ergo in generali resolutione hujusce problematis necesse est invenire casum in quo radices datæ æquationis evadunt æquales; resolutio autem hujus casus valde laboriosa est; et consequenter resolutio generalis prædicti problematis magis *erit* laboriosa."

Written in Latin, and when the proper language of algebra was yet unformed, it is frequently a work of much labour to follow Waring's demonstrations and deductions, and to distinguish his assertions from his proofs. I find he agrees with the opinion expressed by myself, that Newton's rule will *not* "pene," as stated by Newton, but only "perraro," give the true number of imaginary roots. Like myself, too, in the body of the memoir Waring has given theorems of probability in connexion with rules of this kind, but without any clue to his method of arriving at them. Their correctness may legitimately be doubted.

[Since the above was sent to press, I have been enabled to ascertain that the great name of Euler is to be added to the long list of those who have fallen into error in their treatment of this question: see *Institutiones Calculi Differentialis*, vol. II. cap. XIII. He says (p. 555, edition of Prony), "videndum est utrum hæc duo criteria (meaning Newton's criteria of imaginariness) sint contigua necne; priori casu numerus radicum imaginarium non augebitur; posteriori vero *quia criteria litteras prorsus diversas involvunt*, unumquodque binas radices imaginarias monstrabit."

The force of the supposed argument is contained in the words in italics. It is sufficiently met by the question, why or how the conclusion follows from them? Moreover the letters of two non-contiguous criteria are *not* necessarily *prorsus diversæ*; for two criteria with but a single other intervening between them will contain one letter in common.]

to a subject so intimately bound up with the fundamental principles of algebra.

Before commencing the inquiry I ought to state that, in addition to the rule for detecting the existence of a certain number of imaginary roots, Newton has given a remarkable subsidiary method for dividing this number into two parts, representing respectively how many of the positive and how many of the negative roots indicated by Descartes's rule are, so to say, absorbed, and thereby obtains two distinct limits to the number of positive and the number of negative roots separately: of the grounds of this method, as far as I am aware, no one has even attempted an explanation, nor do I propose here to enter upon it; the rule, as I treat it, may be stated, not in Newton's own words, but most simply as follows:—

If the literal parts of the coefficients of an equation affected with the usual binomial coefficients be $a, b, c, d, e \ldots h, k, l$, and if we form the successive criteria $b^2 - ac$; $c^2 - bd$; $d^2 - ce$; $\ldots$; $k^2 - hl$, or, which is the same thing differently expressed, if we write down the determinants[5] *of all the successive quadratic derivatives of the given equation, then as many sequences as there are of negative signs in the arithmetical values of these criteria, so many pairs of imaginary roots at least there will be in the given equation.* If we choose to consider a^2 and l^2 also as criteria, appearing at the beginning and end of the series, then we may vary the expression of the rule by saying that there will be at least as many imaginary roots as there are variations of sign in the complete series so formed.

It will, however, be found more convenient for our present purpose to confine the designation of criteria to the determinants above alluded to.

(3) I shall deal with the homogeneous equation $f(x, y) = 0$ so that the question of the reality of the roots is that of the reality of the ratios $\frac{x}{y}$ or $\frac{y}{x}$. It is obvious, from known principles, that f cannot have fewer imaginary roots than exist in $\frac{df}{dx}$ or $\frac{df}{dy}$[6], or, more generally, than in

(5) To avoid the possibility of misapprehension, I state here once for all, that in the *discriminant* of a form of any degree I suppose the sign to be so taken as to render *positive* the term which is a power of the product of the first and last coefficients; and it may be well to remember that with this definition the number of real roots in any equation $\equiv 0$ or 1 to modulus 4 when the discriminant is positive, and $\equiv 2$ or 3 when the discriminant is negative; whereas the Determinant of a Quadratic form is to be taken in the same sense as that in which it is used by Gauss, and is the same for such form as the Discriminant with the sign changed.

(6) This rule I find merges in the following more general and symmetrical one. Let f, ϕ be any two quantics in x, y; call J the Jacobian of f, ϕ; then the difference between the number of real roots in f and the like number in ϕ, taken positively and augmented by unity, cannot exceed the number of real roots in J. When ϕ is made equal to y, this theorem recurs to the familiar one alluded to in the text.

$\left(\frac{d}{dx} + \lambda \frac{d}{dy}\right) f$; from which it immediately follows[7] that if f have all its roots real, and the quadratic derivatives of f be called $Q_1, Q_2, \ldots Q_{n-1}$, and the coefficients of any function F of two degrees lower than f, whose roots are also *all* real, be $p_1, p_2, \ldots p_{n-1}$, the quadratic function

$$p_1Q_1 + p_2Q_2 + \ldots + p_{n-1}Q_{n-1}$$

must have its roots real, that is its discriminant must be positive: a particular consequence of this is, that by causing F to consist successively of the single terms $x^{n-2}, x^{n-3}y, \ldots xy^{n-3}, y^{n-2}$, we see that the determinants of $Q_1, Q_2, \ldots Q_{n-1}$ must each of them be positive; or, in other words, if any of the Newtonian criteria of an equation are negative, it must have *some* imaginary roots, which is all that Maclaurin, Campbell, and others have succeeded in proving.

(4) The labour of proof of the cases hereinafter considered will be much lightened by the following rule of induction, namely, granting Newton's rule to be true for the degree $n - 1$, it must be true for all those cases appertaining to the degree n in which the series of the signs of the criteria does not commence with $-+$ and end with $+-$: to prove this, we have only to remember that f must have at least as many imaginary roots as $\frac{df}{dx}$ or $\frac{df}{dy}$, and that the criterion-series corresponding to $\frac{df}{dx}$ and to $\frac{df}{dy}$ will be found by cutting off from the series of f one term to the right and left respectively[8]. If, now, the series for f begins with $++$ or $--$ or $+-$, the number of negative *sequences* is the same as when the left-hand sign is removed; so that it is only necessary to prove that the number of imaginary roots in f is not less than the number of negative sequences in $\frac{df}{dx}$; but this, by hypothesis, is not greater than the number of pairs of imaginary roots in $\frac{df}{dx}$, and, *à fortiori*, not greater than the number of such in f. In like manner, if the two *last* criteria of f are not $+-$, it may be shown that the truth of the rule for such form of f is implied in what is supposed to be known to be true for $\frac{df}{dy}$.

(7) By operating upon f successively with *any* $(n-2)$ distinct factors each of the form

$$\left(\frac{d}{dx} + \lambda_i \frac{d}{dy}\right).$$

(8) For $\frac{d}{dx}(a, b, \ldots k, l)(x, y)^n = n(a, b, \ldots k\ \ )(x, y)^{n-1}$,

and $\frac{d}{dy}(a, b, \ldots k, l)(x, y)^n = n(\ \ b, \ldots k, l)(x, y)^{n-1}$.

We may therefore limit our attention, as we ascend in the scale of proof, to those forms of f in which the criterion-series begins with $-+$ and ends with $+-$. Accordingly, since the rule is a truism for $n=2$, it is at once proved, by virtue of the above considerations, for $n=3$ [9].

If all the criteria are zero, it is evident that, whatever n may be, all the roots are real. In every other case we shall find that *zero* may be made positive or negative at will. Thus in the case before us, if the two criteria are $0+$ or $0-$, there will be a pair of imaginary roots, as the first may be read as $-+$ and the second as $+-$.

To prove this, we have only to observe that in either case $\frac{df}{dx}$ will have two equal roots; so that f will be of the form $(ax+by)^3+cy^3$, which obviously, for any real values of a, b, c, has only one real root.

(5) We may now pass to the case of $n=4$, and excluding for the moment the consideration of *zeros*, limit our attention to the criterion-series $-+-$.

Let $ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4=0$ be the equation for which the signs of the criteria b^2-ac, c^2-bd, d^2-ce are $-+-$. Call these criteria

(9) The theorem for the case of cubic equations may be also proved directly as follows:

Writing the equation $ax^3+3bx^2y+3cxy^2+dy^3=0$, the two criteria are $L=b^2-ac$, $M=c^2-bd$; and the discriminant is $a^2d^2+4ac^3+4db^3-3b^2c^2-6abcd=\Delta$.

1. Let L and M be of opposite signs, so that one and only one of them is negative. Then

$$\Delta=(ad-bc)^2-4(b^2-ac)(c^2-bd)=(ad-bc)^2-4LM,$$

and is therefore positive.

2. Let L and M be both negative. The equation may evidently, by writing x and y for $a^{\frac{1}{3}}x$, $d^{\frac{1}{3}}y$, be brought under the form

$$x^3+3\epsilon x^2y+3\eta xy^2+y^3=0,$$

with the conditions $\epsilon^2<\eta$, $\eta^2<\epsilon$; from which we may deduce that ϵ and η are both positive, and $\epsilon\eta<1$ and >0.

Also we have

$$\begin{aligned}\Delta&=1+4(\epsilon^3+\eta^3)-6\epsilon\eta-3\epsilon^2\eta^2\\&>1+4(\epsilon+\eta)\epsilon\eta-6\epsilon\eta-3\epsilon^2\eta^2\\&>1-6\epsilon\eta+8(\epsilon\eta)^{\frac{3}{2}}-3\epsilon^2\eta^2;\end{aligned}$$

or, writing $\epsilon\eta=q^2$,

$$\begin{aligned}\Delta&>1-6q^2+8q^3-3q^4,\\&>(1-q)^3(1+3q);\end{aligned}$$

but $1>q>0$. Hence Δ is positive.

Hence in either case two of the roots of the cubic are impossible. Or the same thing may be shown more immediately from the identities

$$a^2\Delta=(a^2d+2b^3-3abc)^2+4(ac-b^2)^3,$$
$$d^2\Delta=(ad^2+2c^3-3bcd)^2+4(bd-c^2)^3,$$

so that Δ must be positive, and therefore two roots imaginary, if either $bd>c^2$ or $ca>b^2$. It may be noticed that the square and cube in these identities are semi-invariants, being in the first of them unaffected by the change of x into $x+hy$, and in the second by the change of y into $y+hx$.

L, M, N respectively. It has to be proved that all four roots are imaginary, since there are two distinct negative sequences, each sequence consisting of a single $-$. Let x become $x+\epsilon y$ [10], where ϵ is an infinitesimal quantity, and the equation transformed into one between x and y; then we have obviously,

$$\delta a=0,\quad \delta b=a\epsilon,\quad \delta c=2b\epsilon,\quad \delta d=3c\epsilon,\quad \delta e=4d\epsilon,$$
$$\delta L=2b\delta b-a\delta c=0,\quad \delta M=2c\delta c-b\delta d-d\delta b=(bc-ad)\epsilon,$$
$$\delta^2M=(b\delta c+c\delta b-a\delta d)\epsilon=2(b^2-ac)\epsilon^2=2L\epsilon^2;$$

so that δ^2M is essentially negative, since L is so.

Hence, by continually augmenting x by an infinitesimal variation, we may, leaving L unaltered, so choose the sign of ϵ as to decrease M: nor can this process stop when $bc-ad$ becomes zero, by reason that δ^2M is *negative*. Hence we may reduce M to zero. Now, in the course of this reduction, either N retains its sign or changes it; and if the latter is the case, N must have passed through zero. If when M becomes zero N is still negative, the criteria of the linearly transformed equation become $-0-$; and it may be noticed that its first, middle, and last coefficients must have the same sign, by virtue of the negativity of the two last criteria, and the second and fourth the same signs, by virtue of the zero middle criterion; consequently the equation will take the form

$$(\lambda^2+e^4)x^4\pm4e^3\epsilon x^3y+6e^2\epsilon^2x^2y^2\pm4e\epsilon^3xy^3+(\mu^2+\epsilon^4)y^4=0,$$

or

$$\lambda^2x^4+\mu^2y^4+(ex\pm\epsilon y)^4=0,$$

which obviously has all its roots impossible. This being true of the transformed equation, will also, on the suppositions made, be equally so of the original equation.

Let us next suppose that N changes its sign either at the instant when, or before M becomes zero. If M and N both become zero together, so that the criteria of the transformed equation bear the signs $-\,0\,0$, calling the transformed equation $F=0$, $\frac{dF}{dy}$ will have all its roots equal, and F will therefore be of the form $(ax+by)^4+kx^4$, with the condition

$$(a^3b)^2-(a^4+k)(a^2b^2)<0.$$

Hence k is positive, and consequently $F=0$ has all its roots imaginary; and the same, as before, must hold good of the original equation $f=0$.

(10) This method of infinitesimal substitution is that which I applied* in my memoir "On the Theory of Forms," in the *Cambridge and Dublin Mathematical Journal*, to obtain the partial differential equations to every possible species of invariants (including covariants and contravariants) of forms, or systems of forms, with a single set or various sets of variables, proceeding upon the pregnant principle that every finite linear substitution may be regarded as the result of an indefinite number of *simple* and *separate* infinitesimal variations impressed upon the variables. M. Aronhold has erroneously ascribed to others the priority of the publication of these equations.

[* Volume I. of this Reprint, p. 356.]

It remains then only to consider the case when N becomes zero before M vanishes. When this is the case, as soon as N is reduced to zero, in lieu of the substitution of $x+\epsilon y$ for x, we must leave x unaltered, and continue substituting $y+\epsilon x$ for y. We thus start from the sequence $-+0$; N will then always remain zero, and we must either come to the series $-\,0\,0$, which we know, from what has been shown above, corresponds to four imaginary roots, or to the sequence $0+0$, which I shall proceed to consider.

Since the first and last coefficients must have the same sign, we may, by giving either variable a proper multiple (31), make these two coefficients alike,

(31) (*a*) The form $(1, e, e^2, e, 1 \between x, y)^4$ may be regarded as a new and, for many purposes, useful canonical form of a binary quartic. It may be made to comprise within its sphere of representation all forms corresponding to two or four imaginary factors, but excludes the case of four real factors. The ordinary canonical form $(1, 0, 6m, 0, 1 \between x, y)^4$ comprises within its spheres of representation those forms for which the factors are all real or all imaginary, but, so far as real transformations are concerned, excludes the case of two real and two imaginary factors [that case is met by the form $(1, 0, 6m, 0, -1 \between x, y)^4$], as may easily be established either by decomposing the form first named into its factors, or by the consideration that its discriminant Δ is $(1-9m^2)^2$, and is therefore always positive; whereas if a form which it is used to represent have two real and two unreal factors, its discriminant is negative. If now the determinant of transformation be D, and the discriminant corresponding thereto be called Δ', we have $\Delta' = D^6\Delta$, showing that D^2 is negative, and the transformation therefore unreal.

(*b*) The *reality* of m for each of these cases (usually assumed without proof) may be demonstrated as follows: Calling the cubic invariant and the discriminant of any quartic form T, D, we shall have, using the *ordinary* canonical form, $\frac{(m-m^3)^2}{(1-9m^2)^3} = \frac{T^2}{D}$, showing that when D is positive, which is the case of four real or unreal factors, there will be one real value of m, and when D is negative, a real value of im. The former case possesses over the latter a striking distinction, which is that *all* the roots of m will be real; for, as I have shown elsewhere*, if m is one root the complete system of roots will be $\pm m$, $\pm\frac{1-m}{1+3m}$, $\pm\frac{1+m}{1-3m}$: in the latter case the reality of the two values $\pm im$ does not seem necessarily to imply the reality of the other four values of the system.

(*c*) Analogy suggests the establishment of an analogous canonical form or forms for ternary cubics, of which, as is well known and is even dimly foreshadowed in Newton's Enumeration of Lines of the Third Order, the theory runs closely parallel to that of binary quartics. This will be effected by assuming the form

$$F(x, y, z) = \Sigma x^3 + 3e\Sigma x^2y + 6gxyz,$$

and assuming g so as to make the discriminants of

$$\frac{dF}{dx}, \frac{dF}{dy}, \frac{dF}{dz}$$

all zero. This gives rise to a quadratic equation in g, of which the roots are $g=e$, $g=2e^2-e$. When $g=e$, I find

$$S = e(1-e)^3, \quad T = (1-e)^4(1+4e-8e^2), \quad \Delta = T^2+64S^3 = (1+8e)(1-e)^8.$$

When $g=2e^2-e$, I find $\Delta = (1-e)^i(1-4e)^j(1+2e)^k$, where i, j, k are integers to be determined. These forms will, I think, be found important in the future perspective discussion of curves of the third degree. Whilst I yield to no one in admiration of the surpassing genius with which Newton has handled these curves, I cannot withhold the expression of my opinion that every theory of forms in which invariants are ignored must labour under an inherent imperfection, and that Newton, from want of acquaintance with the indelible characters which their invariants

[* Volume I. of this Reprint, p. 600.]

and with the first, second, and third, as well as the third, fourth, and fifth coefficients form geometrical series; hence it is obvious that the transformed equation may be reduced to one or the other of the two following forms, namely

$$x^4 + 4ex^3y + 6e^2x^2y^2 - 4exy^3 + y^4 = 0, \qquad (a)$$

or

$$x^4 + 4ex^3y + 6e^2x^2y^2 + 4exy^3 + y^4 = 0, \qquad (b)$$

with the condition in the latter case that $e^4 - e^2$ is positive, that is $e^2 > 1$.

It must be remembered that we know, from the form of the criteria-series to the derivatives in respect to either x or y (indifferently), that the equation must have *some* imaginary roots; and the question therefore lies between its having two or four. If the discriminant is negative, the former will be the case, if positive, the latter. I shall show that in each equation the discriminant is positive.

Let s, t represent in general the quartic invariants, then we have to show that $s^3 - 27t^2$ is positive.

In case (a),

$$s = 1 + 4e^2 + 3e^4 = (1+e^2)(1+3e^2), \qquad t = \begin{vmatrix} 1 & e & e^2 \\ e & e^2 & -e \\ e^2 & -e & 1 \end{vmatrix} = e^2 - e^4 - e^4 - e^2 - e^6 - e^2 = -e^2 - 2e^4 - e^6 = -e^2(1+e^2)^2,$$

stamp upon curves, has in the parallel which he has drawn between the generation by shadows of all conics from a common type, and of all cubic curves from a limited number of forms, either himself fallen into error of conception, or at least used language which could scarcely fail to lead others into such error. For no species whatever of cubic curve can be formed for which an infinite number of individuals cannot be found which defy linear or perspective transformation into each other; whereas all conics proper may be propagated as shadows from a single individual. It should be noticed in connexion with this subject, that the *indelible* characters of quartic binary, and cubic ternary forms are two in number, namely, the value of $\frac{s^3}{t^2}$ (where s, t are the two fundamental invariants in either case) and the *sign* of t. The indelibility of the sign of s being implied in the invariability of the value of $\frac{s^3}{t^2}$, does not constitute a distinct character. Of course all symmetrical invariants have an invariable sign; but this is not the case with skew invariants, as for example, M. Hermite's octodecimal invariant of a binary quintic, which will change its sign with that of the determinant of transformation.

(*d*) Whilst upon this subject of invariants, I may allow myself to make a remark, bearing upon what will be noticed further on in the text, about a case of equality between roots not necessarily being a mark of transition from real to imaginary roots. If a, b, c, d being the roots of a binary quartic we form a secondary cubic, of which the roots are $(a-b)(c-d)$, $(a-c)(d-b)$, $(a-d)(b-c)$, it may be easily shown that two of these quantities become equal, or, in other words, the roots of the original equation mark out a harmonic group of points, when t (the cubinvariant) is zero. Notwithstanding which a change of sign in t will not command a change of character in the above three roots of the secondary (nor consequently of the original equation), because it is not an odd but an even power of t, namely, t^2, which enters into the discriminant of the secondary.

so that

$$s^3 - 27t^2 = (1+e^2)^3 \{(1+3e^2)^3 - 27e^4(1+e^2)\} = (1+e^2)^3(1+9e^2),$$

and is positive.

In case (*b*),

$$s = (1 - 4e^2 + 3e^4) = (1-e^2)(1-3e^2),$$

$$t = \begin{vmatrix} 1 & e & e^2 \\ e & e^2 & e \\ e^2 & e & 1 \end{vmatrix} \begin{aligned} &= e^2 + e^4 + e^4 - e^2 - e^6 - e^2 \\ &= -e^2 + 2e^4 - e^6 = -e^2(1-e^2)^2, \end{aligned}$$

and

$$\begin{aligned} s^3 - 27t^2 &= (1-e^2)^3 \{(1-3e^2)^3 - 27e^4(1-e^2)\} \\ &= (1-e^2)^3(1-9e^2). \end{aligned}$$

The above can only be negative when e^2 lies between 1 and $\frac{1}{9}$; but in the case supposed $e > 1$. Hence the discriminant is positive, and the roots are all imaginary[12]. Thus, then, the theorem is established for $n = 4$, as well as for the cases where the criteria are zero (as will have been observed in the course of the demonstration), as for those where they are *plus* or *minus*; and it should be observed that the demonstration proceeds upon our being able to show that the quartic, in the case where it resists reduction to the case of the cubic, namely where the criteria are negative at the two extremes and positive in the middle, may by real linear transformations be changed into a form where either the middle criterion is zero and the two extremes negative, or the two extremes zero, and the middle one positive.

Observation.—To make the foregoing demonstration quite exact, it should be noticed that when the criteria L, M, N have been brought to the form $-+0$, and the series of substitutions of $y + \epsilon x$ for y has set in, we have

$$N = 0, \quad \delta N = 0, \quad \delta M = (cd - be)\epsilon, \quad \delta^2 M = N\epsilon = 0, \quad \delta^3 M = 0.$$

Consequently if $cd - be$ should become *zero*, we can no longer go on decreasing M. But as soon as $cd - be = 0$, since we have also $d^2 = ce$, b, c, d, e come to be in geometrical progression, and the transformed equation takes the form

$$ax^4 + 4\omega x^3 y + 6\omega^2 x^2 y^2 + 4\omega^3 x y^3 + \omega^4 x^4 = 0,$$

(12) The reader conversant only with ordinary algebra may easily verify this result. For writing $\frac{x}{y} + \frac{y}{x} = z$, the equation becomes $z^2 + 4ez + 6e^2 - 2 = 0$, and this will have its roots impossible unless $4e^2 > 6e^2 - 2$, or $2e^2 - 2$ negative, which it cannot be, since $e^2 > 1$, and consequently $x : y$ has all its roots impossible. Moreover the same conclusion would (as before shown) hold good unless e^2 lay between 1 and $\frac{1}{9}$; for on making $z = 2$, the function above written in z becomes $2 + 8e + 6e^2$, or $2(1+e)(1+3e)$; and making $z = -2$, it becomes $2 - 8e + 6e^2$, or $2(1-e)(1-3e)$, which two quantities evidently have both positive signs unless e lies between 1 and $\frac{1}{3}$, or between -1 and $-\frac{1}{3}$; so that the first and third Sturmian functions are (except on that supposition) respectively positive and negative for $z = 2$, and also for $z = -2$, showing that no root of z can lie between 2 and -2, and consequently that all the roots of $x : y$ remain impossible.

with the condition $\omega^2 - a\omega^2$ negative, or $a > 1$. Hence we have

$$q^2 x^4 + (x + \omega y)^4 = 0,$$

which obviously has all its roots impossible[13].

(6) We may now pass on to equations of the fifth degree, in which the case resisting reduction will be that where the criterion-series bears the signs

$$- + + -.$$

Let the criteria be called L, M, N, P, so that writing the equation

$$ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + fy^5 = 0,$$

$$L = b^2 - ac, \quad M = c^2 - bd, \quad N = d^2 - ce, \quad P = e^2 - df,$$

and writing for x, $x + \epsilon y$, we have, as before,

$$\delta L = 0, \quad \delta M = (bc - ad)\epsilon, \quad \delta^2 M = L\epsilon^2,$$

so that M may be continually diminished.

If M becomes zero before either N or P changes its sign, the criterion-series for the transformed equation becomes $-0+-$, and for its derivative in respect to x, the series is $0+-$, which proves the existence of four imaginary roots in the transformed, and consequently also in the given equation. In like manner, if N becomes zero before M or P have changed their signs, the criterion-series becomes $-+0-$, which obviously leads to the same result. So likewise the same inference may be drawn if L and M, or M and N, or L, M, N become zeros all at the same time, and we have only to consider the case when, L and M retaining their signs, N becomes zero. At this moment the order of the substitutions must be reversed, and for y must be written $y + \epsilon x$; we shall then have

$$P = 0, \quad \delta P = 0, \quad \delta N = (de - cf)\epsilon \ldots\ldots;$$

and reasoning as in the preceding case for $n = 4$ (with the sole difference, that if δN vanishes by virtue of $de - cf$ vanishing, we should have $P = 0$,

(13) From the first and third criteria it follows that in the form $(a, b, c, d, e)(x, y)^4$, a, c, e have the same sign and may be regarded as all positive; so that writing $a - \frac{b^2}{c} = h^2$, $e - \frac{d^2}{c} = k^2$, the form becomes $h^2x^2 + F + k^2y^2$, where

$$F = \frac{b^2}{c}x^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + \frac{d^2}{c}y^4,$$

and consequently the given form will have all its roots imaginary when this is true for F, so that we might have proceeded at once to deal with the forms marked (*a*), (*b*) at p. [387]; but as the method of homographic transformation by infinitesimal substitutions appears to be necessary in passing to the corresponding forms in the case of the fifth degree, and as in treating that case reference is made to what appears above, I have thought that no object would be gained by altering the text.

$N=0$, and the criterion-series $-+0\,0$, which at once indicates the existence of four imaginary roots), we see that there remains only to consider the case where the criterion-series takes the form $0++0$. It is scarcely necessary to observe that all the criteria can never vanish simultaneously; for that would indicate the equality of all the roots in the transformed, and therefore in the given equation, whose own criteria, contrary to hypothesis, would also be all zero. The zero values of the two extreme criteria indicate that the three first and the three last literal parts of the coefficients are in geometrical progression, from which it will immediately be seen that the equation to be considered may be thrown (by substituting in lieu of x and y suitable multiples of x and y, which will not affect the characters of the criteria) into the convenient form

$$x^5+5\epsilon x^4y+10\epsilon^2x^3y^2+10\eta^2x^2y^3+5\eta xy^4+y^5=0,$$

with the two conditions $\epsilon^4-\epsilon\eta^2$ *positive*, $\eta^4-\eta\epsilon^2$ *positive*.

The form of the criterion-series, apocopated from either end, shows that two of the roots must be imaginary; and consequently, in order to establish the existence of two imaginary pairs of roots, it is only necessary to show that the discriminant of the above equation, subject to the above conditions, must remain always positive. That discriminant I proceed to determine; but as a guide to the form under which it is to be expressed, the following observation is important. Let us take the more general form

$$ax^5+bx^4y+cx^3y^2+dx^2y^3+exy^4+fy^5=0,$$

where $\qquad a=1,\quad b=\lambda\epsilon,\quad c=\mu\epsilon^2,\quad d=\mu\eta^2,\quad e=\lambda\eta,\quad f=1,$

λ, μ being any numerical quantities.

The discriminant will evidently be a symmetrical function of ϵ and η.

Let $a^pb^qc^rd^se^t$ be the literal part of a term in the discriminant. By the *law of weight* we must have

$$q+2r+3s+4t=5\times 4=20.$$

But in the equation before us, $a^pb^qc^rd^se^t$ (to a numerical factor *près*) is $\epsilon^{q+2r}\eta^{2s+t}$, and

$$(q+2r)-(2s+t)=(q+2r+3s+4t)-5(s+t)$$
$$=5(4-s-t).$$

Hence the difference between the indices of ϵ and η in each term is a multiple of 5, and consequently, since the discriminant is a symmetrical function in ϵ and η, it will be a rational integral function of $\epsilon^5+\eta^5$ and $\epsilon\eta$. Moreover, as no such term as c^4d^4 can figure in the discriminant, which, as we know, must in all cases contain one or the other of the two final and of the two initial coefficients, we see that no term can be of higher than the

14th degree in ϵ, η, nor yet so high, for the only terms that could be of that degree would be bc^3d^3e; but making a and f each zero in the original form, it becomes obvious that all the terms free from a and f contain b^2e^2 as a factor [14]. Hence, in fact, the discriminant will be only of the twelfth degree in ϵ, η, and being therefore of only the second degree in $\epsilon^5+\eta^5$, will admit of comparatively easy treatment.

(7) Before proceeding to the calculation of this discriminant, it will be useful to investigate, as a Lemma ancillary to the subsequent discussion, under what conditions four of the roots of the supposed equation will become imaginary when $\epsilon=\eta$.

In this case writing $\frac{x}{y}+\frac{y}{x}=z$, the equation

$$\frac{1}{x+y}(1,\ \epsilon,\ \epsilon^2,\ \epsilon^2,\ \epsilon,\ 1\between x,\ y)^5=0$$

becomes

$$z^2-2-z+1+5\epsilon(z-1)+10\epsilon^2=z^2+(5\epsilon-1)z+10\epsilon^2-5\epsilon-1=0,$$

or say $f(z)=0$.

The determinant of $f(z)$ is thus

$$(5\epsilon-1)^2-40\epsilon^2+20\epsilon+4,$$

that is $5(1-\epsilon)(1+3\epsilon)$; and all the roots of z, and consequently of (x, y), will be impossible, unless z lies between 1 and $-\frac{1}{3}$.

Now $\qquad f(2)=1+5\epsilon+10\epsilon^2,$

$$f'(2)=3+5\epsilon;$$

so that when z has any real roots, that is when ϵ lies between 1 and $-\frac{1}{3}$, $f(2)$, $f'(2)$ are both positive, and the Sturmian functions are of the signs $+++$.

Again, $\qquad f(-2)=5-15\epsilon+10\epsilon^2=5(1-\epsilon)(1-2\epsilon),$

$$f'(-2)=-5+5\epsilon;$$

so that, on the same supposition as before, the Sturmian functions are $\pm-+$, namely

$$+-+ \text{ when } \tfrac{1}{2}>\epsilon>-\tfrac{1}{3},$$
$$--+ \text{ when } 1>\epsilon>\tfrac{1}{2}.$$

In the former case two real roots, in the latter one real root of z lies between $2, -2$. Hence in the former case no real roots of z lie between the limits ∞, 2, and the limits -2, $-\infty$, and in the latter case one real root lies between those limits. Hence x, y will have four imaginary roots, unless ϵ lies between 1 and $\frac{1}{2}$, and two such roots in every other case.

(14) For the discriminant of $xy\phi(x, y)$ = the discriminant of $\phi(x, y)$ multiplied by the square of the product of the resultant of (x, ϕ) and of (y, ϕ).

Thus the discriminant of $(1, \epsilon, \epsilon^2, \eta^2, \eta, 1 \between x, y)^5$, when $\epsilon = \eta$, is negative when ϵ lies between 1 and $\frac{1}{2}$, but for every other value of ϵ is positive, save that it vanishes when

$$\epsilon = 1, \text{ or } \epsilon = \tfrac{1}{2}\,^{(15)}, \text{ or } \epsilon = -\tfrac{1}{3}.$$

(8) I now proceed to calculate the discriminant of the form

$$x^5 + 5\epsilon x^4 y + 10\epsilon^2 x^3 y^2 + 10\eta^2 x^2 y^3 + 5\eta x y^4 + y^5$$

for general values of ϵ, η. This will be accomplished most expeditiously by taking the resultant of the two derivatives of the above form, say U and V, where

$$U = x^4 + 4\epsilon x^3 y + 6\epsilon^2 x^2 y^2 + 4\eta^2 x y^3 + \eta y^4,$$
$$V = \epsilon x^4 + 4\epsilon^2 x^3 y + 6\eta^2 x^2 y^2 + 4\eta x y^3 + y^4;$$

so that $\epsilon U - V = 6(\epsilon^3 - \eta^2) x^2 y^2 + 4(\epsilon\eta^2 - \eta) x y^3 + (\epsilon\eta - 1) y^4 = y^2 P,$

$$-U + \eta V = (\epsilon\eta - 1) x^4 + 4(\eta\epsilon^2 - \epsilon) x^3 y + 6(\eta^3 - \epsilon^2) x^2 y^2 = x^2 Q.$$

Hence

$$\textit{Resultant} \text{ of } (U, V) = \frac{1}{(\epsilon\eta - 1)^4} \times \textit{Resultant} \text{ of } (y^2 P, x^2 Q) = \textit{Resultant} \text{ of } (P, Q);$$

where $P = 6(\epsilon^3 - \eta^2) x^2 + 4(\epsilon\eta^2 - \eta) x y + (\epsilon\eta - 1) y^2$,

$$Q = (\epsilon\eta - 1) x^2 + 4(\eta\epsilon^2 - \epsilon) x y + 6(\eta^3 - \epsilon^2) y^2.$$

Hence, calling Δ the discriminant of the original form, we obtain by the well-known formula for the resultant of two binary quadratics, writing for the moment

$$P = (B, 2\eta A, A \between x, y)^2, \quad Q = (A, 2\epsilon A, B' \between x, y)^2,$$

$$\Delta = -(4\epsilon A^2 - 4\eta A B')(4\eta A^2 - 4\epsilon A B) + (A^2 - BB')^2$$
$$= (1 - 16\epsilon\eta) A^4 + 16(\epsilon^2 B + \eta^2 B') A^3 - 16\epsilon\eta BB' A^2 - 2BB' A^2 + B^2 B'^2.$$

Hence, writing $\epsilon\eta = q$, $\epsilon^5 + \eta^5 = S$,

$$\Delta = (1 - 16q)(q - 1)^4 + 96(S - 2q^2)(q - 1)^3 - 72(8q + 1)(q^3 + q^2 - S)(q - 1)^2$$
$$+ 36^2 (q^3 + q^2 - S)^2.$$

Let $S - q^2 - q^3 = \sigma$, $q - 1 = p$, so that

$$S - 2q^2 = \sigma - q^2 + q^3 = \sigma + (p + 1)^2 p.$$

Then

$$\Delta = 36^2\sigma^2 + 72(8p + 9) p^2 \sigma + 96 p^3 \sigma + 96(p + 1)^2 p^4 - (16p + 15) p$$
$$= 1296\sigma^2 + (648p^2 + 672p^3)\sigma + 96p^6 + 176p^5 + 81p^4$$
$$= \tfrac{1}{9}\{(108\sigma + 27p^2 + 28p^3)^2 + 729p^4 + 1584p^5 + 864p^6 - (27p^2 + 28p^3)^2\},$$

or

$$9\Delta = (108\sigma + 27p^2 + 28p^3)^2 + 72p^5 + 80p^6.$$

[15] When $\epsilon = \frac{1}{2}$ the discriminant of $f(z)$ does not vanish, but $z = -2$ satisfies the equation in z, and consequently $\frac{x}{y}$ has two equal roots -1, so that the discriminant of the original equation vanishes.

(9) Hence we see at once that Δ can be negative only when p lies between 0 and $-\frac{9}{10}$, that is when $\epsilon\eta$ (which is $p + 1$) lies between 1 and $\frac{1}{10}$. Accordingly when Δ is negative, ϵ and η must be both positive or both negative. The latter supposition may easily be disproved as follows: treating the equation $\Delta = 0$ as a quadratic equation in σ, in order that Δ may be capable of becoming negative, its discriminant in respect to σ must be negative, and its value when $\sigma = -\infty$ is positive. Now

$$S = \epsilon^5 + \eta^5, \quad p + 1 = \epsilon\eta, \quad \sigma = S - (p + 1)^2 - (p + 1)^3;$$

so that when ϵ and η are real we have

$$S > 2(p + 1)^{\frac{5}{2}}\,^{(16)}, \text{ that is } \sigma > -(p + 1)^2 + 2(p + 1)^{\frac{5}{2}} - (p + 1)^3$$

when ϵ, η are both positive, and

$$S < -2(p + 1)^{\frac{5}{2}}\,^{(16\,bis)}, \text{ that is } \sigma < (p + 1)^2 + (p + 1)^3 - 2(p + 1)^{\frac{5}{2}}$$

when ϵ, η are both negative.

If now we substitute $(p + 1)^2 + (p + 1)^3 - 2(p + 1)^{\frac{5}{2}}$ for σ in Δ, I say that the resulting value will be positive whatever positive value be given to $(p + 1)$; in fact, if we write $p = \nu^2 - 1$, and make $\sigma = -\nu^4 + 2\nu^5 - \nu^6$, so that Δ becomes a function of the twelfth degree in ν, this function is what the discriminant of the equation in x, y becomes when we have $\epsilon = \eta = \nu$; but in the antecedent Lemma it has been shown that this discriminant is only negative when the two equal quantities ϵ or η, or, which is the same thing, when ν lies between 1 and $\frac{1}{2}$; hence Δ is positive when ν is negative, and consequently when

$$\sigma = (p + 1)^2 + (p + 1)^3 - 2(p + 1)^{\frac{5}{2}}.$$

Thus Δ, a quadratic function in σ, and its discriminant are respectively $+$ and $-$ for this value of σ, as well as for $\sigma = -\infty$. Hence no real root of σ lies between such value of σ and $-\infty$, and consequently Δ must be always positive when ϵ and η are both negative. Hence, if Δ is negative, we must have $1 > \epsilon\eta > \frac{1}{10}$; $\epsilon > 0$; $\eta > 0$. But our *criteria* give $\epsilon^4 - \epsilon\eta^2 > 0$, $\eta^4 - \eta\epsilon^2 > 0$, which, when $\epsilon > 0$, $\eta > 0$, imply $\epsilon^3 > \eta^2$, $\eta^3 > \epsilon^2$, and consequently $\epsilon\eta > 1$, which is in contradiction to the inequality $1 > \epsilon\eta$. Hence when these criteria are satisfied the determinant is necessarily *positive*, and all the roots are imaginary, which completes the proof of Newton's rule for equations of the fifth degree.

(10) It follows as a corollary to the Lemma employed in the preceding investigation, that if in Δ we write $\sigma = -(\nu^2 - \nu^3)^2$ and $p = \nu^2 - 1$, and distinguish this particular value by the symbol (Δ), then (Δ) ought to break up into the product of odd powers of $\nu - 1$, $\nu - \frac{1}{2}$, of some even power of $(\nu + \frac{1}{3})$, and of a factor incapable of changing its sign, and remaining always positive. This may be easily verified; for dividing (Δ) by $(\nu - 1)^4$, we obtain

$$1296\nu^8 [648(\nu + 1)^2 + 24(\nu^2 - 1)(\nu + 1)^2]\nu^4 + 96(\nu^2 - 1)^2(\nu + 1)^4$$
$$+ 176(\nu^2 - 1)(\nu + 1)^4 + 81(\nu + 1)^4;$$

[16] It is of course understood that $(p + 1)^{\frac{5}{2}}$ is to be taken *positive*.

and collecting the terms $1296\nu^8 - 648\nu^6(\nu+1)^2 + 81(\nu+1)^4$ whose sum contains the factor $(\nu - 1)$, we have

$$\begin{aligned}
\frac{(\Delta)}{(\nu-1)^5} = \; & 648(\nu^7+\nu^6+\nu^5+\nu^4+\nu^3+\nu^2+\nu+1) \\
& -1296(\nu^6+\nu^5+\nu^4+\nu^3+\nu^2+\nu+1) \\
& -\;648(\nu^5+\nu^4+\nu^3+\nu^2+\nu+1) \\
& +\;81(\nu^3+5\nu^2+11\nu+15) \\
& -\;24(\nu^7+3\nu^6+3\nu^5+\nu^4) \\
& +\;96(\nu^7+5\nu^6+9\nu^5+5\nu^4-5\nu^3-9\nu^2-5\nu-1) \\
& +\;176(\nu^5+5\nu^4+10\nu^3+10\nu^2+5\nu+1) \\
= \; & 720\nu^7 - 240\nu^6 - 328\nu^5 + 40\nu^4 + 65\nu^3 + 5\nu^2 - 5\nu - 1.
\end{aligned}$$

Hence

$$\begin{aligned}
(\Delta) &= (\nu-1)^5(2\nu-1)^2\{90\nu^4 + 105\nu^3 + 49\nu^2 + 11\nu + 1\} \\
&= (\nu-1)^5(2\nu-1)^2(3\nu+1)^2\{10\nu^2 + 5\nu + 1\};
\end{aligned}$$

showing, agreeably with what was seen in the Lemma, that the discriminant of

$$(1,\ \epsilon,\ \epsilon^2,\ \epsilon^2,\ \epsilon,\ 1 \between x,\ y)^5$$

vanishes then, and then only, when

$$\epsilon = 1, \text{ or } \epsilon = \tfrac{1}{2}, \text{ or } \epsilon = -\tfrac{1}{3},$$

but does not *change its sign*, except as ϵ passes through the limits 1 and $\frac{1}{2}$, and only within those limits can become negative [17].

(11) Although the theory of the possibility of the roots of

$$(1,\ \epsilon,\ \epsilon^2,\ \eta^2,\ \eta,\ 1 \between x,\ y)^5 = 0$$

has now been completely investigated, so far as is necessary for the proof of Newton's theorem applied to equations of the fifth degree, it will be found that the labour will not be ill spent of considering more closely the real

(17) In *general* the case of equal roots of an equation is the state of transition of two real roots into imaginary, or *vice versâ*. But we see by the above instance that this is not necessarily the case *always*, for Δ vanishes on making $\epsilon = -\frac{1}{3}$, and two roots become equal without any change in the nature of the roots when ϵ passes from being greater to being less than $-\frac{1}{3}$. In such case, however, there is a sort of unstable equilibrium in the form of the equation, by which I mean that the effect of any general infinitesimal change performed upon the coefficients of the equation would be either to cause the real roots in the neighbourhood of $\epsilon = -\frac{1}{3}$ to disappear by the factor $(\epsilon + \frac{1}{3})^2$ becoming superseded by a quadratic function of ϵ with impossible roots, or else a region in the neighbourhood of $\epsilon = -\frac{1}{3}$ would reappear, for which the equation would acquire two real roots, owing to $(\epsilon + \frac{1}{3})^2$ becoming superseded by a quadratic function of ϵ with real roots, in which case there would be two values in the neighbourhood of $\epsilon = -\frac{1}{3}$, for *each* of which there would be a pair of equal roots in the equation. The above is probably the first instance distinctly noticed of this singular obliteration of the usual effect upon real and imaginary roots of a passage through equality, owing to the appearance of a square factor in the discriminant.

nature of the criteria which separate the case of one pair from that of two pairs of impossible roots in the above equation. Newton's *criteria* being constructed so as to cover every possible case for equations of every degree, will always be found to fit loosely, so to speak, upon each case treated *per se*; so that more precise conditions can be assigned in each particular case than those which are furnished by his rule. So, for example, it may be remembered that in the equation $(1, e, e^2, e, 1 \between x, y)^4 = 0$, Newton's rule implies only that when $e > 1$, the roots are all impossible; but we have found further that unless $1 > e > \frac{1}{3}$ (a much closer condition), the same thing takes place.

It is obvious from what has been demonstrated above, that if we treat p and σ, which are respectively $\epsilon\eta - 1$ and $\epsilon^5 + \eta^5 - \epsilon^2\eta^2 - \epsilon^3\eta^3$, as the abscissa and ordinate of a variable point in a plane, the curve $\Delta = 0$, that is $(108\sigma + 27p^2 + 28p^3)^2 + 72p^5 + 80p^6 = 0$ will be the line of demarcation between those values of ϵ, η which correspond to one pair, and those which correspond to two pairs of imaginary roots.

For all values of ϵ, η corresponding to internal points of the curve Δ there will be two imaginary and three distinct real roots; for all such as correspond to external points there will be four imaginary roots, and for points *on* the curve two imaginary and two equal roots.

The curve Δ is a curve of the 6th degree whose form* will presently be discussed. But there is an important remark to be made in the first instance. Not all the points within the curve Δ will correspond to *real* values of ϵ, η. In order that these quantities may be real, we must have

$$\epsilon^5 + \eta^5 > 2(\epsilon\eta)^{\frac{5}{2}},$$

that is $$\sigma + q^2 + q^3 > 2q^{\frac{5}{2}}, \text{ where } q = p + 1,$$

or $$\sigma^2 + 2(q^2 + q^3)\sigma + q^4 - 4q^5 + q^6 > 0.$$

Writing this inequality under the form $R > 0$, we see that the curve $R = 0$ will represent a second sextic curve intersecting the former. Δ may be called the curve of the discriminant or *discriminatrix*, and will be a closed curve, and R the curve of equal parameters or *equatrix*, and will consist of a single infinite branch. All points on the latter correspond to equal values of ϵ, η, those on one side of it to real values of ϵ, η, and those on the other side of it to conjugate values of the form $\lambda + i\mu$, $\lambda - i\mu$ respectively. Thus the area confined within the curve Δ will be divided into two portions by the equatrix, and it is impossible to shut one's eyes to the inquiry as to the meaning of the variable point lying in that portion which gives conjugate values to ϵ, η. It becomes clear by analogy that some kind of distinction must be capable of being drawn between the nature of the roots of the equation $(1, \epsilon, \epsilon^2, \eta^2, \eta, 1 \between x, y)^5 = 0$ when ϵ, η are conjugate, in some sense similar or parallel to that which we know to exist between them when ϵ, η are real; and obviously this inference

[* See the Figure, p. 478 below.]

cannot be confined to equations of the particular form and degree of that above written; in a word, equations whose coefficients are not real but conjugate, must have roots of two kinds, one analogous to the real, the other to the imaginary roots of equations with real coefficients. This inference will be justified in the sequel; but in the meanwhile it will be desirable to complete the investigation of the special equation under consideration, by a discussion of the forms and relations of the two curves Δ and R. These curves we know *à priori*, from what has been already demonstrated, can only meet in the three points corresponding to

$$\epsilon = \eta = 1, \quad \epsilon = \eta = \tfrac{1}{2}, \quad \epsilon = \eta = -\tfrac{1}{3};$$

and since $p = \epsilon\eta - 1$, the abscissæ of these three points will be $0, -\frac{3}{4}, -\frac{8}{9}$.

Moreover the 3rd point will be distinguished from the other two by the circumstance that Δ does not change its sign as p passes through the value $-\frac{8}{9}$. Consequently the two curves must touch each other at this point.

Since when $\Delta = 0$ p lies between 0 and $-\frac{9}{10}$, the curve Δ is confined to the negative side of the axis of σ. It is also confined to the negative side of the axis of p.

For between the limits $p = 0$, $p = -\frac{9}{10}$,

$648p^2 + 672p^3$, that is $24(27p^2 + 28p^3)$ is obviously positive,

and $96p^6 + 176p^5 + 81p^4 = \frac{p^4}{6}\{(24p + 22)^2 + 2\}$ is always positive.

Hence the two values of σ are both negative throughout the extent of the curve Δ.

Thus $\epsilon^5 + \eta^5 - \epsilon^2\eta^2 - \epsilon^3\eta^3$ being negative, $\epsilon^3 - \eta^2$ and $\eta^3 - \epsilon^2$ have the same signs when ϵ, η are *real*, as should be the case; for in order that Δ may be capable of vanishing, $\epsilon(\epsilon^3 - \eta^2)$ and $\eta(\eta^3 - \epsilon^2)$ must, by Newton's rule, be *both* negative, which could not be the case if either ϵ or η were negative; so that $\epsilon^3 - \eta^2$ and $\eta^3 - \epsilon^2$ must have the same signs, in fact each must be negative.

The curve Δ under consideration has a multiple point of the 4th order of multiplicity at the origin, where it is touched by the axis of p. Its distance from the axis for the extreme value of p, namely $p = -\frac{9}{10}$, is $\frac{27}{2000}$.

It has three real maxima and minima, two belonging to its upper portion and one to the lower portion at the points, for which p has the *approximate* values $-\frac{9}{16}, -\frac{15}{22}$, and $-\frac{7}{8}$ [18].

(18) The large numbers which enter into Δ may be usefully reduced, and the equation $\Delta = 0$ made more manageable, by aid of the simple substitutions $\sigma = -\frac{27v}{64}$, $p = -\frac{9u}{4}$. The equation $\Delta = 0$ then becomes

$$(v - 3u^2 + 7u^3)^2 = 2u^5 - 5u^6,$$

The curve R, that is $\sigma = ((p+1) \pm (p+1)^{\frac{3}{2}})^2$, has the values 0 and -4 at the origin, a cusp at its extremity corresponding to $p = -1$, where both of its branches meet and touch the axis of p, and a negative maximum in its upper branch at the point where $p = -\frac{5}{9}$.

At all points within the curve R, ϵ and η are conjugate, and for the points outside real. Its lower branch will meet and touch the lower portion of Δ at the point where $p = -\frac{8}{9}$, and its upper branch will intersect and pass out of the upper branch of Δ at the point where $p = -\frac{3}{4}$. The only part of the area Δ therefore which corresponds to real values of ϵ, η, is that which is included between the upper segment of Δ and the upper branch of R, and extends only from $p = 0$ to $p = -\frac{3}{4}$, that is from $\epsilon\eta = 1$ to $\epsilon\eta = \frac{1}{4}$. Hence we may easily find an inferior limit to the values of ϵ and η when the equation $(\epsilon, \eta) = 0$ has two real roots; for we have in that case $\epsilon, \eta, \eta^2 - \epsilon^3, \epsilon^2 - \eta^3$ all positive. Hence

$$\eta^5 > \epsilon^3\eta^3 > q^3, \quad \eta^5 < \epsilon^2\eta^2 < q^2.$$

Consequently ϵ, η must each of them always lie between $q^{\frac{2}{5}}, q^{\frac{3}{5}}$; and since the least value of q is $\frac{1}{4}$, ϵ, η must each be always greater than $(\frac{1}{4})^{\frac{3}{5}}$, that is than ·33499 [19].

whose maxima and minima will be given by the equation

$$(v - 3u^2 + 7u^3)(-6u + 21u^2) = 5u^4 - 15u^5;$$

which, making $1 - 3u = \omega$, becomes

$$270\omega^3 - 46\omega^2 - 9\omega + 1 = 0,$$

whose roots are all real, and are one just a little greater than $-\frac{1}{6}$, another a little less than $\frac{1}{5}$, and the third a very little less than $\frac{1}{11}$ respectively; whence $p = \frac{3}{4}(\omega - 1)$ will have the approximate values given in the text.

(19) $\epsilon : \eta$ will have a maximum value, which can be found by writing $\delta\epsilon : \delta\eta :: \epsilon : \eta$; and consequently, remembering that $q = p + 1$, $S = \epsilon^5 + \eta^5$, $\sigma = S - q^2 - q^3$,

$$\delta S : \delta q :: 5S : 2q,$$

and therefore $\delta\sigma : \delta p :: 5\sigma + q^2 - q^3 : 2q :: 5\sigma + p(p+1)^2 : 2(p+1)$.

Substituting the values of $\delta\sigma : \delta p$ in $\delta\Delta = 0$, and combining the result with the equation $\Delta = 0$, p and σ may be found by the solution of a numerical equation of the 5th degree, and then ϵ and η may be found by the solution of a quadratic and the extraction of 5th roots. To find the maxima and minima values of ϵ and η themselves exactly would lead to the solution of an equation of a degree quite unmanageable.

But we may first find the greatest maximum and least minimum values of S, that is, $\epsilon^5 + \eta^5$, by making $\delta\sigma = (2q + 3q^2)\,\delta q$ in $\delta\Delta = 0$, which leads to an equation (I forget whether) of the 3rd or 5th degree (it is one of the two): calling this maximum and minimum m, μ respectively, and naming ρ (which of course must exceed unity) the greatest quotient of $\frac{\epsilon}{\eta}$ or $\frac{\eta}{\epsilon}$, we shall have

$$\sqrt[5]{\left(\frac{\rho^5}{1+\rho^5}\right)m} > \epsilon; \quad \eta > \sqrt[5]{\left(\frac{1}{1+\rho^5}\right)\mu}.$$

These limits will be tolerably near to the absolute maximum and minimum values of ϵ or η. It may be noticed that we know, from what has gone before, that ρ can never exceed $\left(\frac{1}{q}\right)^{\frac{1}{5}}$; and consequently ρ^5 cannot exceed 4, since q is always $> \frac{1}{4}$.

There is a third curve not undeserving of notice, of only the 3rd degree, which embodies the joint effect of the two middle criteria (the two extremes being supposed to be each zero) in the two cases where Newton's rule will prove all the roots of the equation under consideration to be impossible. These criteria are $c_1 = \epsilon^4 - \epsilon\eta^3$, $c_2 = \eta^4 - \eta\epsilon^3$. But

$$c_1\eta^4 + c_2\epsilon^4 = q(2q^3 - S) = q(2q^3 - q^3 - q^2 - \sigma) = q(q^3 - q^2 - \sigma),$$

which for all values of q on the positive side of the line $p = -1$ (that is $q = 0$) will have the same sign as $q^3 - q^2 - \sigma$, which we may call K[20]; and K positive will evidently imply that c_1, c_2 are one or both of them positive. The whole plane will be divided by the curve K into an *upper* region (commencing at $\sigma = \infty$), for which K is negative, and a lower region, in which K is positive. For any point of the curve K, $\sigma = q^3 - q^2$, which within the limits of q with which we are concerned, namely those within which Δ lies, is negative; for any point of the curve R, the smaller absolute value of σ is

$$-q^3 - q^2 + 2q^{\frac{5}{2}} = q^3 - q^2 + 2(q^{\frac{5}{2}} - q^3),$$

which $< q^3 - q^2$ within the limits in question. So that, remembering that each of these values of σ is negative, we see that the portion of the area Δ corresponding to real values of ϵ, η will be completely above the curve K, that is in the negative region of K, and that accordingly Δ for *real values* of ϵ, η can never vanish when K is positive, as should be the case. This remark does not, however, apply to the conjugate region of Δ; for the curve K will *pass through*[21] the lower or conjugate portion of the area Δ.

(12) I may now say a few words on the signification of that portion of Δ in which ϵ and η are conjugate imaginary quantities.

[20] I call K the Indicatrix, as exhibiting the joint effect of the *indicia* or criteria of the Rule.

[21] This may easily be verified; for at the point $p = -\frac{3}{4}$ it will be found that the ordinate in K and the lower ordinate in Δ are equal, and at the point $p = -\frac{9}{16}$ the lower ordinate in Δ is $-$[illegible], and in K is $-$[illegible]; which shows that the curve K entering the area Δ when at the lower half of the curve, at a point where $p = -\frac{3}{4}$, must pass through its upper contour in order to cut the line $p = -\frac{9}{16}$ as it does above the point where Δ is touched by that line.

The curve K has its negative maximum at the point $q = \frac{2}{3}$, that is, $p = -\frac{1}{3}$. It passes through the origin, and begins with sweeping under the curve Δ, which it enters exactly under the point where R quits Δ, and passes through Δ at a point very close indeed to the horizontal extremity of Δ. It may be noticed that when $p = -\frac{3}{4}$, the smaller ordinates of R and Δ are each $-$[illegible], the ordinate of K and the larger ordinate of Δ being each $-$[illegible].

I have found the points of contact of K with Δ by actually substituting $q^3 - q^2$, that is, $p(p+1)^2$ for σ in $\Delta = 0$. This gives the equation

$$2064p^4 + 7352p^3 + 9823p^2 + 5832p + 1296 = 0,$$

one factor of which is $4p + 3$, dividing out which we have

$$516p^3 + 1451p^2 + 1368p + 432 = 0.$$

The Newtonian criterion applied to the three first coefficients of the above gives -1362[illegible], showing that two of the roots are impossible; the remaining real root I find to be $\cdot 8946$, &c. It does not appear to be a rational number.

In general, let

$$(a + i\alpha,\ b + i\beta,\ c + i\gamma, \ldots\ldots,\ c - i\gamma,\ b - i\beta,\ a - i\alpha \between x,\ y)^n = 0$$

be an equation in which all the coefficients, reckoning simultaneously from the two ends, are conjugate to one another, and the central coefficient, if there is one, which can only be when n is even, *real*.

Let $\frac{x}{y} = p + iq$ satisfy this equation. Then evidently $\frac{y}{x} = p - iq$ will also satisfy it; or, which is the same thing, $\frac{x}{y} = \frac{p + iq}{p^2 + q^2}$ will satisfy it.

Now either this root will be identical with the former one, or a distinct root; in the former case we must have $p^2 + q^2 = 1$, and the root will be of the form $\cos\alpha + i\sin\alpha$; in the second case $p^2 + q^2$ will differ from unity, and there will be a pair of imaginary roots of the form $\rho(\cos\alpha + i\sin\alpha)$, $\frac{1}{\rho}(\cos\alpha + i\sin\alpha)$, in which the real parts ρ, $\frac{1}{\rho}$ are reciprocal to one another, and the directive parts $e^{i\alpha}$ identical. Moreover, if we write the given equation under the form $U + iV = 0$, and suppose, as can always be done, that U and V have been divested of any algebraical common factor, it may easily be shown that the equation so prepared, and which may be called a Conjugate Equation *proper*, can have no real roots and no *pairs of imaginary roots* in the sense in which that term is employed in the theory of equations with real coefficients; but the distinction between *simple* or solitary and *twin* or associated roots reappears in the theory of conjugate equations, under a different form. It will of course be understood that the class of simple roots for which the modulus is unity is quite as general as that of twin roots, for each of which the modulus may be anything different from unity, just as in the ordinary theory the case of real is quite as general as that of imaginary roots, although the former may be represented by points on a fixed straight line, whilst the points representing the latter may be anywhere in the plane, this liberty of displacement being balanced, so to say, by the constraint of coupling. The general geometrical representation of the roots of a real equation is a system of points in a line, and a system of pairs of points at equal distances on opposite sides of the line. So the general geometrical representation of the roots of a conjugate equation will be a system of points in the circumference of a circle of radius unity, and of points situated in pairs in the same radii at reciprocal distances from the centre. In a word, in each case we may say that the roots can be geometrically represented by points on a circle, and pairs of points electrical images of each other in respect to the circle, but the radius of the circle in the one case will be infinity, in the other unity. Conjugate like real equations will have all their invariants of an even degree real, and those of an odd degree will be pure imaginaries, or real quantities affected with the multiplier i.

Their morphological derivatives (covariants, contravariants, &c.) will be also conjugate forms. The whole doctrine of equations, as regards the separation of real from imaginary roots, and the determination of the limits within which the former lie, will reproduce itself with suitable modifications in the theory of conjugate equations, in which simple, on the one hand, and coupled or twin roots, on the other, will correspond respectively as analogues to the real and imaginary roots of the ordinary theory. Thus the following theorem may be demonstrated without difficulty, namely, in any conjugate equation the number of coupled roots is congruent to 0 in respect to the modulus 4 when the discriminant is positive, and to 2 in respect to the same modulus when the discriminant is negative [22]. We see now how to interpret the effect of the variable point whose coordinates are $\epsilon^5+\eta^5$ and $\epsilon\eta$ lying within the area Δ, in that portion of it for which ϵ, η become imaginary; namely it is that in such case the equation $(\epsilon, \eta)=0$, which then becomes of a conjugate form, will have three simple and two twin roots; and thus the unity of the interpretation is restored if we choose, as we very well may, to extend the use of these terms to the real roots and the paired imaginary roots of ordinary

[22] (*a*) A very simple linear transformation shows the immediate connexion between the solitary and associated roots of conjugate with the real and paired imaginary roots of ordinary equations. For if $f(x, y)=0$ be a conjugate equation, writing

$$y=v+iu, \quad x=v-iu,$$

$f(x, y)$ becomes $F(u, v)$, a real form in u, v.

When u, v are real, we have

$$\frac{y}{x}=\frac{v+iu}{v-iu}=\cos\left(2\tan^{-1}\frac{u}{v}\right)+i\sin\left(2\tan^{-1}\frac{u}{v}\right);$$

when $\frac{v}{u}=c\pm i\gamma$, the two values correspond to

$$\frac{y}{x}=\frac{c+i\gamma+i}{c+i\gamma-i}, \quad \left(\frac{y}{x}\right)'=\frac{c-i\gamma+i}{c-i\gamma-i}.$$

Thus $$\frac{y}{x}:\left(\frac{y}{x}\right)'::c^2+(\gamma+1)^2:c^2+(\gamma-1)^2;$$

also $$\frac{y}{x}\times\left(\frac{y}{x}\right)'=\frac{c^2-1+\gamma^2+2ci}{c^2-1+\gamma^2-2ci},$$

of which the modulus is obviously unity.

(*b*) Now it is known that if t be the number of real, and τ of imaginary roots in the real form, $(u, v)^n$, its discriminant, bears the sign $(-)^{\frac{t(t-1)}{2}}$. Hence the sign of the discriminant of the conjugate form $(x, y)^n$ (since the determinant of $v+iu$, $v-iu$ is $2i$) will be $(-)^q$, where

$$q=\frac{n(n-1)}{2}+\frac{t(t-1)}{2}=\frac{(t+\tau)(t-1+\tau)+t(t-1)}{2}=t(t-1)+t\tau+\frac{\tau(\tau-1)}{2}.$$

Hence since τ and $t(t-1)$ are both even, $(-)^q=(-)^{\frac{\tau(\tau-1)}{2}}$, and the sign of the discriminant of a conjugate form is + or − according as the number of imaginary roots does or does not contain 4 as a factor.

It must be remembered that the sign of the discriminant is not in general the same as that of the *zeta* or squared product of differences of the roots. The sign of the *zeta* for real equations follows precisely the same law as the sign of the *discriminant* for conjugate ones.

equations. We may neglect the curve of reality R altogether, and affirm that all over the area Δ, ϵ, η will have such values as will give rise to three simple and two coupled roots.

(13) That part of the theorem of Newton which had received a demonstration from Maclaurin and Campbell in the generalized form in which I have enunciated it in this paper, may be easily extended to the case of conjugate equations. It will, as applied to them, read thus: If the $(n-1)$ quadratic derivatives of a conjugate form of the nth degree, all whose roots are simple, be multiplied respectively by the coefficients of any other conjugate form, all whose roots are also *simple*, of the degree $(n-2)$, and the sum of these products be taken as a new quadratic form, the discriminant of this latter must be positive, or, which is the same thing, its determinant must be negative.

(14) So much for the case of $n=5$. If we were to proceed to the consideration of equations of the 6th degree, *two* cases of resistance would present themselves in the demonstration of Newton's rule, namely one in which the signs of the criteria are $-++\,+-$, the other $-+-+-$. In the latter it would only be necessary to show that the discriminant is necessarily negative, since we know from the derivatives that the equation must have four imaginary roots, and the choice would lie between the alternatives of there being four or six. In the former case the derivatives only indicate the necessary existence of two real roots, and it would become requisite to prove that there must be four or six—an alternative which depends not on the sign of one function of the coefficients, but on the nature of the signs of two such functions given by Sturm's or any equivalent theorem. It would thus become requisite to prove that two functions of the coefficients, say L, M, could not *both* be negative; and this might be shown by demonstrating the existence of two quantities, L', M', other functions of the coefficients incapable of assuming any but the positive sign such that $L'L+M'M$ would be necessarily positive.

PART II.—ON THE LIMIT TO THE NUMBER OF REAL ROOTS IN EQUATIONS OF THE FORM $\Sigma(ax+b)^n$.

(15) I shall now proceed to the consideration of a theorem relating to a particular class of ordinary equations, which occurred to me in the course of and in connexion with the preceding investigations. The theorem itself, but unaccompanied by proof, has appeared in the *Comptes Rendus* of the Academy for the month of March 1864 [above, p. 360].

Both as regards its nature and the processes involved in the proof, it stands in close relation to Newton's rule, my study of which in fact led me to its discovery. It will therefore take its place most appropriately in this paper.

Certain preliminary properties of circulation introducing some new notions of polarity must be first established, by way of Lemmas to the proof in question.

By a *type* let us understand a succession of symbols of any subject matter whatever susceptible of receiving the signs $+-$, or any suchlike indications of opposite polarity.

Let $a, b, c, \ldots i, k, l$ be any such type, where the *elements* $a, b, c, \ldots$ may be regarded either as points in a line or rays in a pencil affected respectively with the signs of $+$ and $-$.

Then by a *per-rotatory* circulation of such type, I mean the act of passing from the first element to the second, from the second to the third, &c., from the last but one to the last, and from the last to the first.

By a *trans-rotatory* circulation of the same, I mean the act of passing from the first to the second, the second to the third, &c., from the last but one to the last, and from the last to the first, *with its sign reversed.*

A type considered subject to per-rotatory circulation may be termed a Per-rotatory Type; one subject to the other sort of circulation, a Trans-rotatory Type.

If a, b, c, d, e be a per-rotatory type, its direct *phases* are

$$\begin{array}{ccccc} a, & b, & c, & d, & e, \\ b, & c, & d, & e, & a, \\ c, & d, & e, & a, & b, \\ d, & e, & a, & b, & c, \\ e, & a, & b, & c, & d, \end{array}$$

and its retrograde phases

$$\begin{array}{ccccc} a, & e, & d, & c, & b, \\ e, & d, & c, & b, & a, \\ d, & c, & b, & a, & e, \\ c, & b, & a, & e, & d, \\ b, & a, & e, & d, & c. \end{array}$$

If, on the other hand, a, b, c, d, e be a trans-rotatory type, its direct *phases* will be

$$\begin{array}{ccccc} a, & b, & c, & d, & \bar{e}, \\ b, & c, & d, & \bar{e}, & \bar{a}, \\ c, & d, & \bar{e}, & \bar{a}, & \bar{b}, \\ d, & \bar{e}, & \bar{a}, & \bar{b}, & \bar{c}, \\ \bar{e}, & \bar{a}, & \bar{b}, & \bar{c}, & \bar{d}, \end{array}$$

and its retrograde phases

$$\begin{array}{ccccc} a, & \bar{e}, & \bar{d}, & \bar{c}, & \bar{b}, \\ \bar{e}, & \bar{d}, & \bar{c}, & \bar{b}, & \bar{a}, \\ \bar{d}, & \bar{c}, & \bar{b}, & \bar{a}, & e, \\ \bar{c}, & \bar{b}, & \bar{a}, & e, & d, \\ \bar{b}, & \bar{a}, & e, & d, & c, \end{array}$$

where the sign $(-)$ is, for greater convenience of writing, placed over instead of before the elements which it affects; and so on in general a type of n elements, whether per-rotatory or trans-rotatory, will admit of n direct and n retrograde phases.

If we count the number of variations of sign in the circulations of any phase of a per-rotatory type, this number will be the same for all the phases, and will be an even number; this even number may be termed the variation-index of the type.

So, again if, whatever be the original signs of the element in a trans-rotatory type, we count the number of variations in the circulation of any of its phases, this number also will be constant and will be odd, and this odd number may then be termed the variation-index of the type.

(16) Let any phase be taken of a per-rotatory type, and out of such phase let any element be *suppressed*; then we obtain a type one degree lower in the elements, which, if we please, we may consider as a trans-rotatory type, and such trans-rotatory type may be termed a derivative of the original per-rotatory one.

In like manner any phase being taken of a trans-rotatory type, one element may be suppressed, and the reduced type treated as a per-rotatory one, and termed a derivative of the original trans-rotatory one.

We may now enunciate the following important general proposition, namely:

Any trans-rotatory type or any per-rotatory type whose variation-index is different from zero being given, a per-rotatory derivative of the one and a trans-rotatory derivative of the other may be found such that the variation-index of the derived types in either case shall be less by a unit than the variation-index of the types from which they are derived.

Case (1). Let the given type be per-rotatory. Then by hypothesis, since it has some variations, we may find a phase of it beginning with $+$ and ending with $-$, by which I mean beginning with an element that is positive and ending with one that is negative. This gives rise to two sub-cases.

T, the phase in question, will be $+ \ldots\ldots + -$

Θ, the phase in question, will be $+ \ldots\ldots - -$.

In either sub-case let the last sign be suppressed, and the result treated as a trans-rotatory type; then T, Θ become respectively T', Θ', where

$$T' \text{ is } + \ldots\ldots +$$

and

$$\Theta' \text{ is } + \ldots\ldots -$$

and evidently the variation-index of T − variation-index of T' = number of changes of sign in $+-+$ less changes of sign in $+- = 2-1=1$; and again variation-index of Θ − variation-index of Θ' = number of changes of sign in $--+$ less changes of sign in $-- = 1-0=1$. Hence the theorem is proved for the case where the given type is per-rotatory.

Case (2). Let the given type be *trans-rotatory*.

Then, again, there must either be a phase of the form P, or one of the form Φ, where P represents a *continual succession* of signs of the same name as $++\ldots+$ or $--\ldots-$, and Φ represents a succession beginning with one sign as + and ending with one or more signs −, or else beginning with − and ending with a succession of signs +. Essentially, then, as a change of signs throughout a whole succession does not affect the variation-index, we may suppose

$$P = + \ldots\ldots ++,$$

$$\Phi = - \ldots - + \ldots +,$$

the signs intervening between the two expressed signs − in Φ being filled up in any manner whatever, and those between the two signs + with signs exclusively +.

Let now that phase of Φ be taken which commences with the first sign of the final succession of +. Then Φ becomes

$$(\Phi) = + \ldots ++ \ldots . +,$$

which is of the form

$$+ \ldots\ldots ++,$$

so that P is only a particular case of (Φ). If the last sign in (Φ) be suppressed and the result treated as a per-rotatory type be called $(\Phi)'$, so that $(\Phi)' = + \ldots\ldots +$, we have variation-index in (Φ) − variation-index in $(\Phi)'$ = changes of sign in $-+$ less changes of sign in $++ = 1-0=1$.

Hence the proposition is established for both cases.

(17) The theorem to which this Lemma-proposition is to be applied concerns equations of the form

$$\epsilon_1 u_1^m + \epsilon_2 u_2^m + \ldots + \epsilon_n u_n^m = 0,$$

where $u_1, u_2, \ldots, u_n$ are any linear functions of x, y; m is any positive integer, and $\epsilon_1, \epsilon_2, \ldots \epsilon_n$ are each respectively and separately, either *plus unity* or *minus unity*.

Such an equation for convenience of reference may be termed a superlinear equation, and the function equated to zero a superlinear function.

Every superlinear function may be conceived as having attached to it a pencil of rays constructed in a manner about to be explained.

We may conceive the function to be prepared in such a manner, that, supposing $ax+by$ to be any one of the n linear elements u, every b shall be positive. If m is even, this can be effected by writing when required for $ax+by$, $-ax-by$ without further change. If m is odd, we may write when required $-ax-by$ in place of $ax+by$, changing at the same time the factor ϵ, which appertains to $(ax+by)^m$ from $+1$ to -1, or, *vice versâ*, from -1 to $+1$.

Now take in a plane any two axes of coordinates $O\xi$, $O\eta$, and consider a, b as the ξ and η coordinates of a point. All the n points thus obtained, on account of every b being positive, will lie on the same side of the axis $O\xi$, and thus the entire n linear functions will be represented by a pencil of n rays, the two extreme rays of which make an angle less than two right angles with each other; but each term of the superlinear function contains, besides $(ax+by)^n$, a definite multiple $+1$, or -1, and we must accordingly, to completely express such term, conceive every ray affected with a distinct sign + or −. A pencil thus drawn with its rays so polarized will give a complete representation of any given superlinear function, and may be called its type-pencil [23].

I am now able to state the following proposition:

(18) *The number of real roots in a superlinear equation cannot exceed the variation-index of its type-pencil, regarded as a per-rotatory type, if the degree of the equation be even, and as a trans-rotatory type if the degree of the equation be odd.* I prove this inductively as follows.

[23] Let a circle be imagined pierced by a pencil containing any number of rays protracted in both directions, say in the opposite points a, α; b, β; c, γ; d, δ; and let these points, taken in order of natural succession from left to right, or right to left, be $a, b, c, d, \alpha, \beta, \gamma, \delta$. Then, commencing with any point c, a *complete* circulation will be represented by the succession of transits

$$c \text{ to } d,\quad d \text{ to } \alpha,\quad \alpha \text{ to } \beta,\quad \beta \text{ to } \gamma,\quad \gamma \text{ to } \delta,\quad \delta \text{ to } a,\quad a \text{ to } b,\quad b \text{ to } c.$$

But whether $\alpha, \beta, \gamma, \delta$ bear respectively the same signs or signs contrary to those of a, b, c, d, the transit between any two points β to γ will be of the same nature, as regards continuance or change of sign, as the transit from b to c, and thus we see that the complete cycle or total revolution above indicated is only a reduplication of, and may be fully designated by the hemicyclic succession c to d, d to α, α to β, β to γ, for which the number of variations therefore will be the same as for any similar succession obtained by commencing with any other element in the original system of points instead of c. If the opposite points bear like signs, the above succession of transits may be indicated by the order c, d, a, b, c; if they bear contrary signs by the order $c, d, \bar{a}, \bar{b}, \bar{c}$, and thus it is that the idea arises of the two kinds of so-called circulation, but which are in fact only more or less disguised species of semicirculation.

Suppose the theorem to be true when the variation-index of the type-pencil is not greater than the even number ν, and consider an equation of the odd degree $(2i+1)$, for which the type-pencil viewed as trans-rotatory has the variation-index $\nu+1$.

Let a *phase* of this type be taken, say corresponding to the rays $\rho_n, \rho_{n-1} \ldots \rho_2, \rho_1$, such that the per-rotatory type obtained by striking out the term ρ_1 has the variation-index ν (as we know may be done by virtue of the Lemma).

Take for new axes $O\xi', O\eta'$, when $O\xi'$ coincides with ρ_1; then it is clear that the pencil $\rho_n, \rho_{n-1} \ldots \rho_2, \rho_1$ will still serve as a type-pencil to the given function, the only change being that some of the rays, namely those that did lie on one side of ρ_1, have been inverted in direction and changed in sign (corresponding to a change in the coefficients a, b, accompanied with a change in the sign of the corresponding ϵ), whilst the rays on the other side of ρ_1 have been left unaltered.

The points $(a_1, b_1), (a_2, b_2) \ldots (a_n, b_n)$ corresponding to the rays $\rho_1, \rho_2, \ldots \rho_n$ will, with respect to the new axes, change their values, becoming converted into $(\alpha_1, 0), (\alpha_2, \beta_2), (\alpha_3, \beta_3), \ldots (\alpha_n, \beta_n)$, where $\beta_2, \beta_3, \ldots \beta_n$ will still all be positive, the angle between ρ_1 and ρ_n being the same as between the two extreme rays in the original figure of the type-pencil, and the superlinear equation may now be written in the form

$$F(u, v) = \epsilon_1(\alpha_1 u)^{2i+1} + \epsilon_2(\alpha_2 u + \beta_2 v)^{2i+1} + \epsilon_3(\alpha_3 u + \beta_3 v)^{2i+1} \ldots + \epsilon_n(\alpha_n u + \beta_n v)^{2i+1} = 0,$$

where u, v are real linear functions of x, y.

Let the derivative of this function be taken in regard to v, and we have

$$\frac{1}{2i+1} F'(u, v) = \beta_2\epsilon_2(\alpha_2 u + \beta_2 v)^{2i} + \beta_3\epsilon_3(\alpha_3 u + \beta_3 v)^{2i} \ldots + \beta_n\epsilon_n(\alpha_n u + \beta_n v)^{2i},$$

where $\beta_2\epsilon_2, \beta_3\epsilon_3 \ldots \beta_n\epsilon_n$ have the same signs as $\epsilon_2, \epsilon_3, \ldots \epsilon_n$ respectively.

Now the pencil-type of $F'(u, v)$ will be the per-rotatory type $\rho_n, \rho_{n-1}, \ldots \rho_2$, of which by construction the variation-index is ν. Hence by hypothesis $F'(u, v)$ has not more than ν real roots, that is, at least $2i - \nu$ imaginary roots. Hence $F(u, v)$ has at least that number of imaginary roots, that is, at most $(2i+1) - (2i - \nu)$, that is, $\nu + 1$ real roots. Hence if the theorem is true for ν an even number, it is true for $\nu + 1$.

In like manner let us proceed to show that when it is true for ν an odd number, it would remain true for $\nu + 1$.

The reasoning will be precisely similar to that followed in the antecedent case. We must find a phase of the *per-rotatory* type $\rho_n, \rho_{n-1}, \ldots \rho_2, \rho_1$ having the variation-index $\nu + 1$ such that the trans-rotatory reduced type $\rho_n, \rho_{n-1}, \ldots \rho_2$ shall have the variation-index ν; the new pencil will still continue to be a type-pencil of the given superlinear function, the change of direction in the bunch of rays on one side of ρ_1 being now unaccompanied with change of sign, such change corresponding to $\epsilon(ax+by)^{2i}$ becoming changed into $\epsilon(-ax-by)^{2i}$ without ϵ undergoing a change of sign.

As before, the axes of coordinates are transformed from ξ, η into ξ', η', and we obtain

$$F(u, v) = \epsilon_1(\alpha_1 u)^{2i} + \epsilon_2(\alpha_2 u + \beta_2 v)^{2i} + \ldots + \epsilon_n(\alpha_n u + \beta_n v)^{2i},$$

$$\frac{1}{2i} F'(u, v) = \beta_2\epsilon_2(\alpha_2 u + \beta_2 v)^{2i-1} + \ldots + \beta_n\epsilon_n(\alpha_n u + \beta_n v)^{2i-1},$$

for which the type-pencil is the trans-rotatory type $\rho_n, \rho_{n-1}, \ldots \rho_2$, of which by construction the variation-index is ν, so that its number of imaginary roots is $2i - 1 - \nu$, and consequently the number of real roots of $F(u, v)$ will be $\nu + 1$.

Thus, then, if the theorem be true for ν, whether ν be even or odd, it will be true for $\nu + 1$.

But when $\nu = 0$, the superlinear function becomes a sum of even powers of linear functions of x, y, all taken with the same sign, of which the number of roots is evidently 0. Hence, being true for this case, the proposition is true universally.

It will be noticed that the algebraical part (as distinguished from the purely polartactic part of the above demonstration) depends on the principle of which such abundant use has been made in the former part of this dissertation, namely that the number of imaginary roots in any ordinary algebraical equation in x cannot be increased when we operate any homographic substitution upon x, and take the derivative of the equation thus transformed in lieu of the original [24].

([24]) For greater clearness I present in an inverted order of arrangement a summary of the foregoing argument.

By an ith derivative of $f(x, y)$ is meant any derived form

$$\left(\lambda_1 \frac{d}{dx} + \mu_1 \frac{d}{dy}\right)\left(\lambda_2 \frac{d}{dx} + \mu_2 \frac{d}{dy}\right) \cdots \left(\lambda_i \frac{d}{dx} + \mu_i \frac{d}{dy}\right) f(x, y),$$

the λ, μ quantities being any real quantities whatever. Then I say—

1. If T is the type-pencil (per-rotatory or trans-rotatory) of any superlinear form F, every derivative of T of the contrary name is the type-pencil of some first derivative of F, as shown in art. (18).

2. A derivative of T of contrary name may be found such that its variation-index shall be less by a unit than that of T itself, as shown in art. (16).

3. Hence if i is the variation-index of the type-pencil of F, an ith derivative of F may be found such that its variation-index shall be zero, and consequently having no real roots.

Hence, finally, since the number of real roots of any rational integral homogeneous function in x, y cannot exceed by more than i the number of the real roots in any of its ith derivatives, F cannot have more real roots than there are units in the variation-index of its type-pencil.

The subtle point of the argument, it will be noticed, lies in forming the conception of the variation-index to a trans-rotatory pencil, in which the singular phenomenon occurs of a reversal of *relative polarity* in passing from the last ray to the first, whereas in a per-rotatory pencil any ray indifferently may be regarded as the initial ray, no such reversal in that case taking place.

(19) The proposition above established leads immediately to the theorem and corollary following, namely:

THEOREM. If $c_1, c_2, \ldots c_n$ be a series of ascending or descending magnitudes, and m any positive integer, the equation

$$\lambda_1(x+c_1)^m + \lambda_2(x+c_2)^m + \ldots + \lambda_n(x+c_n)^m = 0$$

cannot have more real roots than there are changes of sign in the sequence $\lambda_1, \lambda_2, \ldots \lambda_n, (-)^m \lambda_1$.

For obviously $(1, c_1), (1, c_2), \ldots (1, c_n)$ will be points corresponding to rays within a semirevolution, and therefore forming a type-pencil.

Corollary. If the above equation be transformed by any real homographic substitution into the form

$$\mu_1(y+\gamma_1)^m + \mu_2(y+\gamma_2)^m + \ldots + \mu_n(y+\gamma_n)^m = 0,$$

where $\gamma_1, \gamma_2, \ldots \gamma_n$ are taken in ascending or descending order, the number of changes of sign in the series $\mu_1, \mu_2, \ldots \mu_n, (-)^m \mu_1$ is *invariable* (25); for the effect of any such formation will be to leave the type-pencil unaltered except in its *phase*.

(20) If we look to the undeveloped form of the superlinear function

$$S = \epsilon_1 u_1^m + \epsilon_2 u_2^m + \ldots + \epsilon_n u_n^m,$$

and are supposed to possess no knowledge of the coefficients which enter into the linear elements u, we may still draw some general inferences as to the limit of the number of real roots in $S=0$. Thus if the number of positive units ϵ is j, and of the negative units k, and j is not greater than k, it is obvious that, whatever may be the form of the type-pencil to S, its variation-index cannot be more than $2j$ when m is even, nor more than $2j+1$ when m is odd; for the arrangement the most favourable to the largeness of the number of the real roots is that where every two rays with the signs belonging to the j group of ϵ are separated by one or more of the rays with a contrary sign to themselves. Thus it appears that when only the units $\epsilon_1, \epsilon_2, \ldots \epsilon_n$ are given, we may impose a maximum upon the number of real roots in the superlinear equation; this limit may be called the *absolute maximum*, being the double of the inferior number of like signs in the series $\epsilon_1, \epsilon_2, \ldots \epsilon_n$ when the degree is even, and one more than such double when the degree is odd (26).

(25) It may be noticed that, contrariwise, the limit to the number of real roots given by Newton's criteria is *not* an invariant; it fluctuates with the homographic transformations operated upon the equation; and a question suggests itself as to the maximum value the number of imaginaries indicated by the rule can attain. I presume this maximum is not in all cases necessarily the actual number of the imaginary roots possessed by the equation.

(26) (*a*) If a superlinear form of an odd degree contains an odd number of terms, say $2k+1$,

The *specific maximum*, on the other hand, will depend on the form of the type-pencil, and cannot be ascertained until the coefficients of the linear elements are given. It can never exceed, but may be less than the absolute maximum. It may, indeed, be easily proved that *in general* the specific maximum will be less than the absolute maximum. Thus, by way of example, suppose the degree to be even, and the inferior number of like signs to be 2; the absolute maximum number of real roots will be four, but the specific maximum will more generally be only two. For let the number of linear terms in the superlinear function be $2+n$, n being 2 or any greater number; and first, to fix the ideas, suppose $n=2$. The type-pencil, which is to be read per-rotatorily, consists of four rays, say a, b, c, d, following each other in uninterrupted circular order, of which two are to bear positive and two negative signs. If the two negative signs fall on a, c or on b, d, the variation-index will be 4, but in the other four cases of incidence such index will be only 2. Consequently the chance is 2 to 1 (27) that the specific maximum, which may be 4, is not greater than 2; and consequently the chance that there will be four real roots in the equation will be only a chance (too difficult to be calculated, but which is a function of the degree of the equation) *of* the chance $\frac{1}{3}$ that there will be as many as four real roots in the equation $u_1^m + u_2^m - u_3^m - u_4^m = 0$, where u_1, u_2, u_3, u_4 are unknown linear functions of x: thus we are entitled to say that *in general* the number of real roots in such an equation is *not* the maximum four, but a less number. This remark is of importance, as showing that on this subject it is possible to speak with scientific

the greatest value of the *inferior* number of like signs is k, and the extreme limit to the number of real roots will be $2k+1$.

If it contain an even number of terms, say $2k$, the greatest value of the inferior index is k; but for this particular case it will readily be seen that a limit may be assigned to the variation-index closer than that given by the rule in the text; in fact the variation-index cannot in that case exceed $2k-1$, which will therefore be the extreme limit to the number of real roots. Now suppose the canonizant of an odd-degreed function of x, y to have all its roots real, then it may be expressed by a superlinear form of which the number of terms will be $2i+1$ or $2i$, according as the degree is $4i+1$ or $4i-1$. In the one case the number of real roots cannot exceed $2i+1$, in the other $2i-1$. Hence the following somewhat curious theorem:

(*b*) *If the canonizant of an odd-degreed quantic in* x, y, *of the degree* $4i \pm 1$, *has* no *imaginary roots, the quantic itself must have at least i pairs of imaginary roots.* From the fact that when the roots of the canonizant of a quintic are all real there must be one pair at least of imaginary roots, we can infer that when the discriminant of a quintic is positive and that of its canonizant is negative, the equation has one real and four imaginary roots. This observation has led to a long train of reflections, which will be found embodied in the 3rd part of the memoir.

(27) This, in fact, is identical in substance with the noted problem of determining the chance that two straight lines drawn on a black board will cross. Mr Cayley, of whom it may be so truly said, whether the matter he takes in hand be great or small, "nihil tetigit quod non ornavit," suggests the following independent proof of this. Taking unity as the length of the contour, fixing the extremity of one of the lines, and calling s the distance of its other end from it measured on the contour, the chance of the second line crossing this is easily seen to be $2s(1-s)$, which, integrated between $s=0$, $s=1$, gives $\frac{1}{3}$, as before obtained.

certainty, and on other than empirical grounds, of what may *in general* be expected to take place. Thus we find Newton declaring twice over in the chapter quoted, that *in general* his rule will give not merely the maximum, but the actual number of the imaginary roots in an equation. I am strongly inclined to doubt the truth of this assertion; but it is important to be satisfied by analogy that such an assertion may rest on a scientific and demonstrative basis, and not on the utterly fallacious foundation of arithmetical empiricism [28].

[28] A few additional words on this question of probability may not be unacceptable. In order to meet the case of the degree of the superlinear form or equation being odd as well as even, let it be supposed known under the form

$$\sum_{i=1}^{n} \lambda_i (x+c_i)^m,$$

the value of the quantities c being supposed to be left wholly indeterminate, and only the signs of the quantities λ to be given. Let ω be the inferior number of like signs in the λ series, meaning thereby that the number of signs of one sort is ω, and of the other sort ω, or more than ω.

Let the probability of the specific maximum of real roots being $2k$ when m is even, be represented by p_{2k}, and of its being $2k+1$ when m is odd by π_{2k+1}; also let s_{2k}, σ_{2k+1} represent the number of cases when ω and n are given which correspond to the specific maximum being $2k$, $2k+1$ respectively. Suppose $\omega=1$, then obviously, when m is even, we have $s_2=n$, $p_2=1$. But when n is odd $\sigma_1=2$ (for when either extreme element alone is negative the trans-rotatory cycle has the variation-index unity), and $\sigma_3=n-2$, so that

$$\pi_1=\frac{2}{n}, \quad \pi_3=\frac{n-2}{n}.$$

Again, suppose $\omega=2$, m being even; then obviously s_2 is the number of contiguous duads in a cycle of n elements, and s_4 is the remaining number of duads; hence

$$s_2=n, \quad s_4=n\frac{n-1}{2}-n=n\frac{n-3}{2};$$

so that

$$p_2=\frac{2}{n-1}, \quad p_4=\frac{n-3}{n-1}.$$

2nd. Suppose $\omega=2$, m being odd, so that σ_1, σ_3, σ_5 will have to be separately estimated. To fix the ideas, let the λ series be termed a, b, c, d, e, f, g, in which two of the elements are supposed of one sign, say negative, and the rest of the opposite sign, say positive; then the only dispositions of sign which correspond to the specific maximum being 1 are those in which a, b or else f, g are both negative. Hence $\sigma_1=2$. Again, the dispositions of sign which make the specific maximum equal to 3 are those in which a, g are both negative, those in which a and c, d, e, or f are negative, those in which g and e, d, c, or b are negative, and, finally, those in which any two contiguous elements except the a and g are negative. Hence $\sigma_3=1+2(n-3)+(n-3)=3n-8$; and it should be observed that this result cannot be prejudiced in its generality by the supposition of any of the components of σ_3 becoming negative, since $\omega=2$ implies that n is at least 4. Hence, finally,

$$\sigma_5=\frac{n^2-n}{2}-(3n-8)-2=\frac{n^2-7n+12}{2}=\frac{(n-3)(n-4)}{2};$$

so that

$$\pi_1=\frac{4}{n^2-n}, \quad \pi_3=\frac{6n-20}{n^2-n}, \quad \pi_5=\frac{n^2-7n+16}{n^2-n}.$$

This example serves to show how much more difficult is the computation of the respective probabilities when m is odd than when m is even, owing to the break of continuity in the cycle of readings on passing from the last to the first term.

It seems hardly worth while to pursue this subject in greater detail. I will only notice that when m is even the chance of the specific maximum attaining the absolute maximum, that is, becoming 2ω, will depend on the proportion of the ways in which in a cycle of n elements ω of them may be marked with a distinctive sign in such a way that no two of such signs shall come together. Accordingly I find by a computation of no great difficulty,

$$s_{2\omega}=\frac{n(n-\omega-1)!}{\omega!(n-2\omega)!},$$

and hence, since the total number of combinations of n elements ω and ω together is $\frac{n!}{\omega!(n-\omega)!}$, I deduce

$$p_{2\omega}=\frac{(n-\omega)!(n-\omega-1)!}{(n-1)!(n-2\omega)!}.$$

Thus when n has its minimum value, namely, 2ω, $p_{2\omega}=\frac{\omega!(\omega-1)!}{(2\omega-1)!}$, and becomes very small as ω increases. When again n increases towards infinity $p_{2\omega}$ approaches indefinitely near to unity, and the chance approaches near to certainty of the specific not becoming less than the absolute maximum of real roots.

Notes to Part II.

On the probability of the specific superior limit to the number of real roots in a superlinear equation equalling any assigned integer.

(21) The question comes to that of determining the probability of a per-rotatory or trans-rotatory pencil with a definite number of rays of each kind possessing a given variation-index.

Since the footnote below was written, a method has occurred to me of obtaining the probability in question in general terms, as follows.

For a *per-rotatory* pencil of μ positive and ν negative rays. Let $[\mu, \nu, g]$ be the probability of the rays being so disposed as to give rise to $2g$ variations of sign in making a complete revolution. Then there will be g distinct groups of positive, and g of negative rays. The number of partitions with permutations of the parcels *inter se* of μ elements in g parcels is

$$\frac{(\mu-1)(\mu-2)..(\mu-g+1)}{1.2..(g-1)},$$

and of ν elements into g parcels is

$$\frac{(\nu-1)(\nu-2)..(\nu-g+1)}{1.2..(g-1)}.$$

If we combine each parcel with each in every possible way, and then imagine the combined parcels let into a circle containing $m+n$ places and shifted round in the circle through a complete revolution, we shall obtain

$$(\mu+\nu).\frac{(\mu-1)(\mu-2)..(\mu-g+1)}{1.2..(g-1)}.\frac{(\nu-1)(\nu-2)..(\nu-g+1)}{1.2..(g-1)}$$

arrangements; but on examination it will be found that every arrangement so produced will be repeated g times; moreover it is obvious that no other arrangement giving rise to g groups of each sort can be found. Hence the true number of distinct groupings of the sort in question is

$$\frac{(\mu+\nu)}{g} \cdot \frac{(\mu-1)(\mu-2)..(\mu-g+1)}{1.2..(g-1)} \cdot \frac{(\nu-1)(\nu-2)..(\nu-g+1)}{1.2..(g-1)}.$$

And the total number of arrangements, which is the number of ways in which μ things can be distributed over $(\mu+\nu)$ places, is $\frac{(\mu+\nu)!}{\mu!\,\nu!}$. Hence we obtain

$$[\mu, \nu, g] = \frac{\mu!\,\nu!}{(\mu+\nu-1)!}\left\{\frac{(\mu-1)(\mu-2)..(\mu-g+1)\times(\nu-1)(\nu-2)..(\nu-g+1)}{1.2.(g-1)(1.2..g)}\right\}$$

$$= \frac{\mu!\,(\mu-1)!\,\nu!\,(\nu-1)!}{g!\,(g-1)!\,(\mu-g)!\,(\nu-g)!\,(\mu+\nu-1)!}.$$

If there should appear any obscurity in the statement of the method by which has been obtained the number of distinct distributions of the μ, ν elements into g groups of each, the reader is referred to the equation in differences obtained further on in this Note, by which all doubt of the correctness of the result will be removed.

(22) For a *trans-rotatory* pencil of rays, to ascertain the probability of the variation-index being $2g+1$.

Imagine a circular arrangement of μ positive elements and ν negative elements containing 2γ variations.

Let this circle be supposed opened out at any point and the variations of the open pencil so formed to be reckoned according to the trans-rotatory law, which is that in passing from one extremity to the other a change is to be seen as a variation, and a variation as a change. If the break is made between two negative or between two positive elements, the number of variations obviously becomes *increased* by one unit; but if between a positive and a negative element, that number becomes decreased by one unit. The number of these latter intervals is 2γ, and of the former $\mu+\nu-2\gamma$.

Hence the probability of the index becoming $2\gamma+1$ is $\frac{\mu+\nu-2\gamma}{\mu+\nu}$, and of its becoming $2\gamma-1$ is $\frac{2\gamma}{\mu+\nu}$.

If, then, we denote the probability to be calculated by $[\mu, \nu, g+\frac{1}{2}]$, it is obvious that we shall have

$$[\mu, \nu, g+\tfrac{1}{2}] = \frac{\mu+\nu-2g}{\mu+\nu}[\mu, \nu, g] + \frac{2(g+1)}{\mu+\nu}[\mu, \nu, g+1].$$

But by the formula previously obtained it will easily be seen that

$$[\mu, \nu, g+1] = \frac{(\mu-g)(\nu-g)}{g(g+1)}[\mu, \nu, g].$$

Hence

$$[\mu, \nu, g+\tfrac{1}{2}] = \frac{[\mu, \nu, g]}{\mu+\nu}\left\{(\mu+\nu-2g) + \frac{2(\mu-g)(\nu-g)}{g}\right\}$$

$$= \left(\frac{2\mu\nu}{g(\mu+\nu)} - 1\right)[\mu, \nu, g] \qquad (*)$$

$$= \frac{2(\mu!)^2(\nu!)^2}{(g-1)!\,(g+1)!\,(\mu+\nu)!\,(\mu-g)!\,(\nu-g)!}$$

$$- \frac{\mu!\,(\mu-1)!\,\nu!\,(\nu-1)!}{(g-1)!\,g!\,(\mu+\nu-1)!\,(\mu-g)!\,(\nu-g)!}.$$

When $g=0$ the above expression fails; but reverting to the equation from which it is derived, we obtain

$$(\mu, \nu, \tfrac{1}{2}) = \frac{2}{\mu+\nu}[\mu, \nu, 1] = \frac{2.\mu!\,\nu!}{(\mu+\nu)!}.$$

(23) These combined results admit of an easy corroboration, for

$$\Sigma_\infty^0[\mu, \nu, g+\tfrac{1}{2}] = 1, \text{ and } \Sigma_\infty^0[\mu, \nu, g] = 1.$$

Hence the equation marked * gives

$$1 = [\mu, \nu, \tfrac{1}{2}] + \frac{2\mu\nu}{\mu+\nu}\Sigma\frac{[\mu, \nu, g]}{g} - 1.$$

Hence we ought to have

$$\frac{\mu!\,\nu!}{(\mu+\nu)!} + \frac{\mu\nu}{\mu+\nu}\Sigma\frac{[\mu, \nu, g]}{g} = 1,$$

that is

$$1 + \Sigma\frac{\mu!\,\nu!}{(\mu-g)!\,g!\,(\nu-g)!\,g!} = \frac{(\mu+\nu)!}{\mu!\,\nu!};$$

which is true, since the left-hand side of the equation is

$$1 + \mu\nu + \mu\frac{\mu-1}{2}.\nu\frac{\nu-1}{2} + \ldots,$$

which is obviously the coefficient of x^ν in $(1+x)^\mu(x+1)^\nu$, that is, in $(1+x)^{\mu+\nu}$.

(24) If we wish to find the chance of the specific superior limit becoming equal to the absolute superior limit, we must write g in the above formulæ

equal to ν, that one of the two quantities μ, ν which is not greater than the other, and we shall obtain

$$[\mu, \nu, \nu] \quad = \frac{\mu!(\mu-1)!}{(\mu+\nu-1)!(\mu-\nu)!},$$

$$[\mu, \nu, \nu+\tfrac{1}{2}] = \frac{\mu!(\mu-1)!}{(\mu+\nu)!(\mu-\nu-1)!};$$

so that, in fact, $[\mu, \nu, \nu+\frac{1}{2}] = [\mu, \nu+1, \nu+1]$, which relation may also be obtained by *à priori* considerations.

(25) With reference to the remark made concerning the mode of obtaining the value of $[\mu, \nu, g]$, I proceed to show how it may be obtained directly by the integration of an equation in differences, and by a method analogous in idea to that by which $[\mu, \nu, g+\frac{1}{2}]$ was made to depend on $[\mu, \nu, g]$. For as in that case we conceived an open pencil to be closed and then reopened, so we may imagine one of the rays to be withdrawn and then reinserted. In this way, observing that the effect of introducing a negative sign into a circle of μ positive and n negative signs consisting of ν distinct groups of each is to produce no change in the number of the groups if inserted between two negative signs, but to increase that number by unity if inserted between two positive signs, we may infer that the probability of ν becoming $\nu+1$, in consequence of such insertion, is $\frac{\mu-\nu}{\mu+\nu}$, and of ν remaining unaltered, is $\frac{n+\nu}{\mu+n}$.

Hence we obtain the equation in differences,

$$[\mu, \nu, g] = \frac{\nu-1+g}{\mu+\nu-1}[\mu, \nu-1, g] + \frac{\mu-g+1}{\mu+\nu-1}[\mu, \nu-1, g-1],$$

in which μ may be considered constant, and ν and g to vary.

The integral must satisfy the further condition that $[\mu, 1, g]$ shall be unity when g is 1, and zero for all values of g greater than 1.

Assume the value of $[\mu, 1, g]$ obtained by the method given in art. (21). This obviously satisfies the initial conditions corresponding to $g=1$. Moreover we may easily deduce from it the equalities

$$[\mu, \nu-1, g-1] = \frac{(g-1)g}{(\mu-g+1)(\nu-g)}[\mu, \nu-1, g],$$

and

$$[\mu, \nu, g] = \frac{(\nu-1)\nu}{(\mu+\nu-1)(\nu-g)}[\mu, \nu-1, g].$$

Hence the equation in differences will be satisfied if it be true that

$$\frac{(\nu-1)\nu}{\nu-g} = (\nu-1+g) + \frac{(g-1)g}{\nu-g},$$

which is obviously the case, since $\nu^2-\nu-g^2+g = (\nu-g)(\nu+g-1)$.

Since, then, the assumed value of $[\mu, \nu, g]$ is correctly determined when $\nu=1$, it is obvious, from the form of the equation, that it holds good for all other values of ν, as was to be shown.

(26) From the equation

$$\frac{[\mu, \nu, g+1]}{[\mu, \nu, g]} = \frac{(\mu-g)(\nu-g)}{g(g+1)}$$

making $(\mu-g)(\nu-g) = g(g+1)$ or $g = \frac{\mu\nu}{\mu+\nu+1}$, we may readily infer that the value of g for which the probability $[\mu, \nu, g]$ is greatest is the integer part of $\frac{\mu\nu}{\mu+\nu+1}$, if that quantity is non-integer, or the quantity itself and the number next below it (indifferently) if it is an integer.

(27) If we apply a similar method to $[\mu, \nu, g+\frac{1}{2}]$, we obtain by aid of the formula above given,

$$\frac{[\mu, \nu, g+\frac{1}{2}]}{[\mu, \nu, g-\frac{1}{2}]} = \frac{2\mu\nu-(\mu+\nu)\gamma}{2\mu\nu+\mu+\nu-(\mu+\nu)\gamma} \cdot \frac{(\mu+1)-\nu(\nu+1-\gamma)}{\gamma^2};$$

and equating this ratio to unity, we obtain

$$\frac{2\mu\nu-(\mu+\nu)\gamma}{2\mu\nu+\mu+\nu-(\mu+\nu)\gamma} = \frac{\gamma^2}{(\mu+1)(\nu+1)-(\mu+\nu+2)\gamma};$$

or writing $\mu+\nu=p$, $\mu\nu=q$,

$$(p^2+p)\gamma^2-(3pq+4q+p^2+p)\gamma+2q(q+p+1)=0.$$

The roots of this equation will be both of them real, for its *determinant* is

$$p^2q^2+16pq^2+16q^2+(p^2+p^3)(\mu^2+\nu^2),$$

which is necessarily positive. Hence it follows that there are two positive roots of the equation. Whether there will exist values of g which give actual maxima or minima values, or one and the other to $[\mu, \nu, g+\frac{1}{2}]$, depends on the further condition being satisfied that the values of g in the above equation shall come out, one or both of them, not greater than either of the two numbers μ, ν. The inquiry connected with the satisfaction of this condition may be conducted by means of repeated applications of the processes of Sturm's theorem; but I shall not enter upon it, as it appears to lead to calculations of complexity disproportionate to the interest of the result.

(28) It may be noticed that the *average* value of $[\mu, \nu, g]$ can be calculated without any difficulty. This will be $\Sigma(g[\mu, \nu, g])$, or

$$\frac{\mu!\nu!}{(\mu+\nu-1)!}\left[1+\frac{(\mu-1)(\nu-1)}{1}+\frac{(\mu-1)(\mu-2)(\nu-1)(\nu-2)}{1.2^2}+\ldots\right]$$

$$= \frac{\mu!\nu!}{(\mu+\nu-1)!} \cdot \frac{(\mu+\nu-2)!}{(\mu-1)!(\nu-1)!} = \frac{\mu\nu}{(\mu+\nu-1)};$$

so that the average number of variations of sign in a per-rotatory pencil with μ positive and ν negative signs is $\dfrac{2\mu\nu}{\mu+\nu-1}$, or a little more than the harmonic mean between μ, ν.

In like manner, for a trans-rotatory pencil this number will be

$$\Sigma\,(2g+1)\,[\mu,\nu,g+\tfrac{1}{2}] = [\mu,\nu,\tfrac{1}{2}] + \Sigma\left\{(2g+1)\left(\frac{2\mu\nu}{g(\mu+\nu)}-1\right)[\mu,\nu,g]\right\},$$

which, observing that $\Sigma\,[\mu,\nu,g]=1$, and $[\mu,\nu,\tfrac{1}{2}]+\dfrac{2\mu\nu}{\mu+\nu}\Sigma\dfrac{[\mu,\nu,g]}{g}=2$, gives as the average number of variations of sign $\dfrac{4\mu\nu}{\mu+\nu}-\dfrac{2\mu\nu}{\mu+\nu-1}+1$.

(29) The simplest mode of calculating the value of $[\mu,\nu,g]$ is the following:

Let $[\mu,\nu,g)$, $[\mu,\nu,g-\tfrac{1}{2})$ denote the probabilities that an arrangement in open line (in which, as is the case in applying Des Cartes's rule of signs, no account is taken of the relation of the extreme signs to each other) shall contain respectively $2g$ and $2g-1$ variations. Conceive a circular arrangement of γ groups of positive and γ groups of negative signs. If this circle be opened out into a line at an interval between a positive and a negative sign (of which there are 2γ), one variation will be lost; but if at any of the remaining $\mu+\nu-\gamma$ intervals, the number of variations remains unaltered. Hence we derive immediately

$$[\mu,\nu,g) = \frac{\mu+\nu-2g}{\mu+\nu}[\mu,\nu,g] \quad\text{and}\quad [\mu,\nu,g-\tfrac{1}{2}) = \frac{2g}{\mu+\nu}[\mu,\nu,g].$$

But we may find $[\mu,\nu,g-\tfrac{1}{2})$ by counting the arrangements which give μ, ν, $2g-1$ variations of sign. These may be all obtained, and without repetition, by intercalating every distribution of μ into g groups with every distribution of ν into the same; and the intercalation may be performed in *two* ways, according as the parcels of the μ signs, or those of the ν signs, are taken first in order. Hence we have

$$[\mu,\nu,g-\tfrac{1}{2}) = \frac{2(\mu-1)(\mu-2)\ldots(\mu-g+1)}{1.2\ \ldots\ (g-1)} \cdot \frac{(\nu-1)(\nu-2)\ldots(\nu-g+1)}{1.2\ \ldots\ (g-1)} \cdot \frac{\mu!\,\nu!}{(\mu+\nu)!}$$

$$= \frac{2(\mu-1)!\,\mu!\,(\nu-1)!\,\nu!}{(\mu+\nu)!\,(g-1)!\,(g-1)!\,(\mu-g)!\,(\nu-g)!};$$

and thus

$$[\mu,\nu,g] = \frac{\mu+\nu}{2g}[\mu,\nu,g-\tfrac{1}{2}] = \frac{\mu!\,(\mu-1)!\,\nu!\,(\nu-1)!}{(\mu+\nu-1)!\,g!\,(g-1)!\,(\mu-g)!\,(\nu-g)!}$$

as previously found; also

$$[\mu,\nu,g) = \frac{(\mu+\nu-2g)\,[\mu!\,(\mu-1)!\,\nu!\,(\nu-1)!]}{(\mu+\nu)!\,g!\,(g-1)!\,(\mu-g)!\,(\nu-g)!}.$$

(30) Moreover, we thus see that the average number of variations in an open line with μ positive and ν negative signs, which is

$$\Sigma\,(2g-1)\,[\mu,\nu,g-\tfrac{1}{2}) + \Sigma 2g\,[\mu,\nu,g),$$

or

$$\Sigma 2g\,\{[\mu,\nu,g-\tfrac{1}{2})+[\mu,\nu,g)\} - \Sigma\,[\mu,\nu,g-\tfrac{1}{2})$$

will be equal to

$$\Sigma 2g\,[\mu,\nu,g] - \Sigma\frac{2g}{\mu+\nu}[\mu,\nu,g] = \frac{\mu+\nu-1}{\mu+\nu}\Sigma 2g\,[\mu,\nu,g]$$

$$= \frac{\mu+\nu-1}{\mu+\nu}\cdot\frac{2\mu\nu}{\mu+\nu-1} = \frac{2\mu\nu}{\mu+\nu}.$$

The total number of variations and continuations together is $\mu+\nu-1$. Hence the difference between the two is

$$\frac{4\mu\nu}{\mu+\nu}-(\mu+\nu-1),$$

or

$$\frac{(\mu+\nu)-(\mu-\nu)^2}{\mu+\nu};$$

so that the average number of variations is greater than, equal to, or less than that of the continuations, according as the difference between the numbers of the two sets is less than, equal to, or greater than the square root of the entire number of signs. Obviously the average should be the same for the variations as for the continuations if the number of signs, say $n+1$, is given, and each is supposed equally likely to be positive or negative. This is easily verified; for multiplying the probable value of each distribution of signs by the probable value of the number of variations corresponding thereto, we obtain the series

$$\frac{1}{(n+1)2^n}\left\{1.n.(n+1)+2(n-1)(n+1)\frac{n}{2}+3(n-2)\frac{(n+1)n.(n-1)}{1.2.3}+\ldots\right\}$$

$$= \frac{n(n+1)2^{n-1}}{(n+1)2^n} = \frac{n}{2}.$$

This is the final average of the number of variations of sign, and will be equal to that of the continuations, since the entire number of the two together is n.

Part III.—On the Nature of the Roots of the General Equation of the Fifth Degree.

(31) In a footnote, Part II. of this memoir, [p. 409 above] I have shown that when the discriminant of the canonizant (constituting an invariant of the twelfth order) of an equation of the fifth degree bears a particular sign, the character of the roots becomes completely determined by the sign of the discriminant of that equation.

This has naturally led me to investigate *de novo* the whole question of the character of the roots of an equation of that degree; and I have succeeded in obtaining under a form of striking and unexpected simplicity the invariantive criteria which serve to ascertain in all cases the nature of the equation as regards the number of real and imaginary roots which it contains; then passing to the expression for these criteria in terms of the roots themselves, I obtain expressions which exhibit the intimate connexion between this subject and a former theory of my own relative to the construction of the conditions for the existence of a given number and grouping of equal roots, which can hardly fail to lead eventually to the extension of the results herein obtained to equations of any odd degree whatever. It is the more needful that these results in a question of so high moment to the advancement of algebraical science should be made public, inasmuch as they do not seem to accord with those obtained by my eminent friend M. Hermite, who has preceded me in this inquiry in a classic memoir, published in the year 1854 in the ninth volume of the *Cambridge and Dublin Mathematical Journal*, since which time I am not aware that the subject has been resumed by any other writer. The discrepancy between our conclusions may be only apparent; but there can be no doubt of the superiority of the form in which they are herein presented, inasmuch as only three functions of the coefficients are required by my method, and five by M. Hermite's. The solution offered by M. Hermite is confessedly incomplete, but to this great analyst none the less will always belong the honour, not only of having initiated the inquiry, but of having emitted the fundamental conceptions through which it would seem best to admit of successful treatment. The arrow from my hand may have been the first to hit the mark, but it was his hand which had previously shaped, bent, and strung the bow.

Our methods of procedure, however, are widely dissimilar, and by employing my well-known canonical form for odd-degreed binary quantics, long since given to the world, I have succeeded in evading all necessity for the colossal labours of computation required in M. Hermite's method, and am able to impart to my conclusions the clearness and certainty of any

elementary proposition in geometry, not scrupling to avail myself for such purpose of that copious and inexhaustible well-spring of notions of continuity which is contained in our conception of space, and which renders it so valuable an auxiliary to Mathematic, whose sole proper business seems to me to be the development of the three germinal ideas—of which continuity is one and order and number the other two*.

Section I.—*Preparation of the General Binary Quantic of the Fifth Degree.*

(32) Let $$(a, b, c, d, e, i \between x, y)^5 = F(x, y);$$
a cubic covariant of F is the canonizant C, where C represents the determinant

$$C = \begin{vmatrix} a & b & c & d \\ b & c & d & e \\ c & d & e & i \\ y^3 & -y^2x & yx^2 & -x^3 \end{vmatrix}.$$

Let us first suppose that this form does not vanish identically, and has at least two distinct factors ξ, η, linear functions of x, y, where of course ξ, η are each of them determinate to a constant factor *près*; giving any value to the constant factor for either of them, we may write $F(x, y) = \Phi(\xi, \eta) = (\alpha, \beta, \gamma, \delta, \epsilon, \iota \between \xi, \eta)^5$, and the canonizant of Φ with respect to ξ, η becomes the determinant T, where T represents

$$T = \begin{vmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \gamma & \delta & \epsilon \\ \gamma & \delta & \epsilon & \iota \\ \eta^3 & -\eta^2\xi & \eta\xi^2 & -\xi^3 \end{vmatrix}.$$

Hence since T to a constant factor *près* is identical with C, the coefficients of η^3 and ξ^3 in the above determinant must vanish in order that $\xi\eta$ may be contained in T.

Hence the two determinants

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \delta \\ \gamma & \delta & \epsilon \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \beta & \gamma & \delta \\ \gamma & \delta & \epsilon \\ \delta & \epsilon & \iota \end{vmatrix}$$

both vanish.

* Herein I think one clearly discerns the internal grounds of the coincidence or parallelism, which observation has long made familiar, between the mathematical and musical ἔθος. May not Music be described as the Mathematic of sense, Mathematic as Music of the reason? the soul of each the same! Thus the musician *feels* Mathematic, the mathematician *thinks* Music,—Music the dream, Mathematic the working life—each to receive its consummation from the other when the human intelligence, elevated to its perfect type, shall shine forth glorified in some future Mozart-Dirichlet or Beethoven-Gauss—a union already not indistinctly foreshadowed in the genius and labours of a Helmholtz!

Hence either β, γ, δ, or otherwise γ, δ, ϵ, or else the first minors of

$$\begin{vmatrix} \beta & \gamma \\ \gamma & \delta \\ \delta & \epsilon \end{vmatrix}$$

are each zero.

The first two suppositions must be excluded, since either of them would lead to the conclusion of T, and therefore C, being a perfect cube, contrary to hypothesis. The last supposition implies either that β, γ, δ, or otherwise that γ, δ, ϵ, or else that $\beta\delta - \gamma^2$ and $\gamma\epsilon - \delta^2$ are each zero.

If β, γ, δ are each zero, T becomes a multiple of $\eta^2\xi$; if γ, δ, ϵ are each zero, T becomes a multiple of $\eta\xi^2$; that is to say, T, and consequently C, contains a square factor; and obviously the converse is true, so that when C contains a square factor F is reducible to the form $au^5 + 5euv^4 + fv^5$. When this is not the case $\delta = \frac{\gamma^2}{\beta}$, $\epsilon = \frac{\delta^2}{\gamma} = \frac{\gamma^3}{\beta^2}$. Hence

$$F = \left(\alpha - \frac{\beta^2}{\gamma}\right)\xi^5 + \frac{\beta^2}{\gamma}\left(\xi + \frac{\gamma}{\beta}\eta\right)^5 + \left(\iota - \frac{\epsilon^2}{\delta}\right)\eta^5,$$

which is of the form $\omega^5 + \phi^5 + \psi^5$, ω, ϕ, ψ being linear functions of x, y.

(33) We have supposed C not to be a perfect cube. When it is a perfect cube, say ξ^3, we may assume η any second linear function of x, y; and expressing F in the same manner as before in terms of ξ, η, it is clear that all the first minors of

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \delta \\ \beta, & \gamma, & \delta, & \epsilon \\ \gamma, & \delta, & \epsilon, & \iota \end{vmatrix},$$

except the one obtained by cancelling the last column in the above matrix, must vanish; consequently δ, ϵ, ι must all vanish, so that Φ, and consequently F, must contain a cube factor identical with the canonizant itself.

Lastly, if the canonizant vanish entirely, every first minor in the above matrix, when we write again a, b, c, d, e, i in lieu of $\alpha, \beta, \gamma, \delta, \epsilon, \iota$, will be zero. Hence either a, b, c, d, or b, c, d, e, or c, d, e, i must each vanish or else that must be the case with the first minors of

$$\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \end{vmatrix},$$

or of

$$\begin{vmatrix} b, & c, & d, & e \\ c, & d, & e, & i \end{vmatrix},$$

or of

$$\begin{vmatrix} a, & b, & c, & d \\ c, & d, & e, & i \end{vmatrix}.$$

Under the first or third supposition F must contain four equal factors; under the second Φ becomes $a\xi^5 + i\eta^5$; under the fourth or fifth it is readily seen that the form becomes

$$a\left(\xi + \frac{b}{a}\eta\right)^5 + \left(i - \frac{e^2}{d}\right)\eta^5, \text{ or } \left(a - \frac{b^2}{c}\right)\xi^5 + i\left(\eta + \frac{e}{i}\xi\right)^5$$

respectively, so that the second, fourth, and fifth suppositions conduct alike to the form $\omega^5 + \phi^5$, a particular case of the preceding one.

It remains only to consider the sixth supposition, namely that the first minors of

$$\begin{vmatrix} a, & b, & c, & d \\ c, & d, & e, & i \end{vmatrix}$$

are all zero.

In this case if we write

$$\sqrt{(a)}\,x + \sqrt{(c)}\,y = u,$$

$$\sqrt{(a)}\,x - \sqrt{(c)}\,y = v,$$

$$A + B = \frac{1}{a^{\frac{1}{2}}},$$

$$A - B = \frac{b}{a^2 c^{\frac{1}{2}}},$$

and if neither a nor c is zero, it will readily be seen that $F(x, y)$ becomes $Au^5 + Bv^5$ by virtue of the relations

$$d = \frac{c}{a}b, \quad e = \left(\frac{c}{a}\right)^2 a, \quad i = \left(\frac{c}{a}\right)^2 b\ ^{(29)}.$$

If $a = 0$ or $c = 0$, the preceding transformation fails.

But unless also $i = 0$ or $e = 0$ at the same time as $a = 0$ or $c = 0$, a legitimate transformation similar to the above may be performed by interchanging a, c, x, y with i, a, y, x.

If now

$a = 0$, it will easily be seen that a, b, c, d or else a, c, e are each zero.

Similarly, if

$i = 0$, it will easily be seen that i, e, d, c or else i, d, b are each zero.

Again, if

$c = 0$, it will easily be seen that a, b, c, d or else c, e are each zero;

and if

$d = 0$, it will easily be seen that c, d, e, i or else d, b are each zero.

(29) Thus we see that the equation $ax^5 + 5bx^4 + 10acx^3 + 10bcx^2 + 5ac^2x + bc^2 = 0$ belongs to the class of soluble forms.

Thus, then, if $a=0$ and $i=0$, all the coefficients, or else all except one, namely b or e, are zero;
if $a=0$ and $d=0$, all the coefficients, or else only not e and i or only not b or only not i are zero;
so if $i=0$ and $c=0$, all must be zero except b and a or e or a;
if $c=0$ and $d=0$, only e and i or else a and b or else a and i will differ from zero.

Hence, then, in any case there will be at least four equal roots, or else F is of the form ax^5+iy^5.

Thus, then, for the first time has been here rigorously demonstrated, free from all doubt and subject to no exceptions, the following important proposition:

Every binary quintic function *not containing three or more equal roots* is reducible to one or the other of the two following forms,

$$u^5+v^5+w^5, \text{ or } au^5+5euv^4+fv^5.$$

The former is the case when the discriminant of the canonizant is different from zero, the latter when it is equal to zero; for it will be observed that, whether the canonizant has equal roots or totally disappears, its discriminant in both cases alike is zero.

(34) It has been seen that when the quintic has three equal roots the canonizant becomes a perfect cube; and it may not be out of place here to point out what the conditions (necessary and sufficient) are to ensure the quintic having four equal roots. These are all comprised in that of the quadratic covariant vanishing. To prove this, let η be a factor of $F(x, y)$, so that

$$F(x, y)=\Phi(x, \eta)=(\alpha, \beta, \gamma, \delta, \epsilon, 0 \between x, \eta)^5.$$

Then, since the similar covariant *quoad* x, y must also vanish, we have

$$\alpha\epsilon-4\beta\delta+\gamma^2=0, \quad -3\beta\epsilon+2\gamma\delta=0, \quad -4\gamma\epsilon+3\delta^2=0.$$

If $\epsilon=0$, then $\delta=0$, $\gamma=0$ by virtue of the two extreme equations, and Φ, and therefore F, contains four equal factors. If ϵ is not zero,

$$\gamma=\frac{3\delta^2}{4\epsilon}, \quad \beta=\frac{\delta^3}{2\epsilon^2}, \quad \alpha=\frac{5\delta^4}{16\epsilon^3}, \text{ and } \Phi \text{ becomes } \frac{5\epsilon}{16}x\left(\frac{\delta}{\epsilon}x+2\eta\right)^4;$$

so that, as before, there are four equal factors. Conversely, it is obvious that if there are four equal factors u, so that $\Phi=au^5+5bu^4v$, the quadratic covariant of Φ disappears.

(35) The quadratic covariant also it was which led me to perceive the transformation applied in the antecedent article. For when the first minors of

$$\begin{vmatrix} a, & b, & c, & d \\ c, & d, & e, & f \end{vmatrix}$$

are all zeros, the quadratic covariant becomes

$$4(c^2-bd)x^2+4(d^2-ce)y^2.$$

Supposing neither of those coefficients to vanish, and calling its two factors u and v, and making

$$F(x, y)=\Phi(u, v)=(\alpha, \beta, \gamma, \delta, \epsilon, \iota \between u, v)^5,$$

it is clear that the minors of

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \delta \\ \gamma, & \delta, & \epsilon, & \iota \end{vmatrix}$$

can no longer all be zero, since in that case we should have

$$4(\gamma^2-\beta\delta)u^2+4(\delta^2-\gamma\epsilon)v^2$$

containing u, v as factors. Consequently the canonizant of Φ must vanish under one or the other of those remaining suppositions which had been previously shown to conduct to the form au^5+bv^5, or else to the case of three or more equal roots. When the quadratic covariant vanishes, we know that there must be four equal roots; and when it becomes a perfect square but does not vanish, it will be found on examination that the equation has three equal roots.

(36) Returning to the general case, where $\Phi=u^5+v^5+w^5$, and making $\frac{u}{r^{\frac15}}+\frac{v}{s^{\frac15}}+\frac{w}{t^{\frac15}}$ identically zero, and writing u', v', w' for $\frac{u}{r^{\frac15}}, \frac{v}{s^{\frac15}}, \frac{w}{t^{\frac15}}$ respectively, Φ becomes $ru'^5+sv'^5+tw'^5$, or, if we please, $ru^5+sv^5+tw^5$, with the condition $u+v+w=0$.

Moreover u, v, w will all three be factors of the canonizant of F. For taking the canonizant of F with respect to u, v, it becomes

$$\begin{vmatrix} r-t, & -t, & -t, & -t \\ -t, & -t, & -t, & -t \\ -t, & -t, & -t, & s-t \\ v^3, & -v^2u, & vu^2, & -u^3 \end{vmatrix}, \text{ or } rt \begin{vmatrix} 1, & 0, & 0, & 0 \\ -1, & -1, & -1, & -1 \\ 0, & 0, & 0, & s \\ v^3, & -v^2u, & vu^2, & -u^3 \end{vmatrix}$$

or $rst(uv^2+vu^2)$, that is, $-rst\,uvw$.

Hence if $x+ey$, $x+fy$, $x+gy$ are three distinct factors of the canonizant of F with respect to x, y, if we choose the ratios $\lambda:\mu:\nu$ so that $\lambda+\mu+\nu=0$, $e\lambda+f\mu+g\nu=0$, we may make $u=\lambda(x+ey)$, $v=\mu(x+fy)$, $w=\nu(x+gy)$, and shall then have $F(x, y)=ru^5+sv^5+tw^5$, with the condition $u+v+w=0$, where r, s, t may be found from three equations obtained by identifying any three of the six terms in F with the corresponding terms $ru^5+sv^5+tw^5$ expressed as a function of x, y. These equations being linear, it follows that ru^5, sv^5, tw^5 form a *single and unique* system of functions of x, y.

So when the canonizant has two equal roots and is of the form

$$C(x+py)(x+qy)^2,$$

in which case the reduced form is $au^5+5euv^4+fv^5$, the canonizant in respect to u, v becomes

$$\begin{vmatrix} a, & 0, & 0, & 0 \\ 0, & 0, & 0, & e \\ 0, & 0, & e, & f \\ v^3, & -v^2u, & vu^2, & -u^3 \end{vmatrix},$$

that is, ae^2uv^3. Hence, writing

$$u=x+py, \quad v=x+qy, \quad F=au^5+5euv^4+fv^5,$$

a, e, f may be obtained, as before, by means of three linear equations, and the terms au^5, $5euv^4$, fv^5 form a single and unique system.

Finally, when the canonizant vanishes entirely, so that the form becomes au^5+fv^5, the quadratic covariant will take the form $C(x+ey)(x+fy)$; and making $u=x+py$, $v=x+qy$, a, f become determined by means of two linear equations, so that au^5, fv^5 form a single and unique system, as in the preceding cases.

(37) When the canonizant has three distinct roots, they may be all real, or one real and the other two imaginary. In the former case, in the expression $ru^5+sv^5+tw^5$, u, v, w may be considered as all real functions of x, y, and r, s, t will then also all of them be real. In the latter case w may be taken as a real function of x, y and u, v as conjugate imaginary functions; and consequently it is easy to see that, except when r, s are equal to each other, they will constitute a pair of conjugate imaginary quantities: in this case we may take for our canonizant form

$$r\left(\frac{-u+iv}{2}\right)^5+s\left(\frac{-u-iv}{2}\right)^5+tu^5,$$

or, if we please,

$$ru_,^5+sv_,^5+tu^5,$$

understanding by $u_,$, $v_,$ respectively $\dfrac{-u+iv}{2}$ and $\dfrac{-u-iv}{2}$. And it should be noticed that the determinant of $u_,$, $v_,$ in respect to u, v will be

$$\begin{vmatrix} -\frac{1}{2} & \frac{i}{2} \\ -\frac{1}{2} & \frac{-i}{2} \end{vmatrix}.$$

which is i.

(38) Let us proceed briefly to express the invariants of $ru^5+sv^5+tw^5$, which call Φ, with respect to u, v; the corresponding ones of $ru_,^5+sv_,^5+tw^5$,

which call $\Phi_,$, in respect to the same variables u, v, will be found by attaching to these suitable powers of i.

$$\Phi=(r-t,\ -t,\ -t,\ -t,\ -t,\ s-t\between u, v)^5.$$

Hence its quadratic covariant is the quadratic invariant of

$$((r-t)u-tv,\ -tu-tv,\ -tu-tv,\ -tu-tv,\ -tu+(s-t)v\between u',\ v')^4,$$

which is obviously

$$-rtu^2-stv^2+(rs-rt-st)uv.$$

Of this the quadratic invariant is

$$rt\,.\,st-\tfrac{1}{4}(rs-rt-st)^2;$$

or writing $\rho=st$, $\sigma=tr$, $\tau=rs$, and calling this invariant $-\frac{1}{4}(J)$,

$$(J)=\rho^2+\sigma^2+\tau^2-2\rho\sigma-2\sigma\tau-2\tau\rho.$$

Again, the cubic covariant or canonizant has been already shown to be $rst(u^2v+uv^2)$. Calling the discriminant of this $-\frac{1}{27}(L)$, we have

$$(L)=r^4s^4t^4\ {}^{(30)}=\rho^2\sigma^2\tau^2.$$

Again, to find the discriminant (D) in respect to u, v.

When $ru^5+sv^5+tw^5=0$ has two equal roots, and $u+v+w=0$, it is easy to see that we have $ru^4+\lambda=0$, $sv^4+\lambda=0$, $tw^4+\lambda=0$.

Hence to a constant factor *près* (D) will be the *Norm* of

$$(st)^{\frac{1}{4}}+(tr)^{\frac{1}{4}}+(rs)^{\frac{1}{4}}, \text{ that is of } \rho^{\frac{1}{4}}+\sigma^{\frac{1}{4}}+\tau^{\frac{1}{4}}\ (^{31}).$$

To find the value of this norm, suppose $\rho^{\frac{1}{4}}+\sigma^{\frac{1}{4}}+\tau^{\frac{1}{4}}=0$, then

$$\rho+\sigma+\tau=2(\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}}+\sigma^{\frac{1}{2}}\tau^{\frac{1}{2}}+\tau^{\frac{1}{2}}\rho^{\frac{1}{2}}),$$

and

$$\rho^2+\sigma^2+\tau^2-2\rho\sigma-2\rho\tau-2\sigma\tau=8\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}}\tau^{\frac{1}{2}}(\rho^{\frac{1}{2}}+\sigma^{\frac{1}{2}}+\tau^{\frac{1}{2}}).$$

Hence

$$\begin{aligned}(\rho^2+\sigma^2+\tau^2-2\rho\sigma-2\rho\tau-2\sigma\tau)^2&=64\rho\sigma\tau\{(\rho+\sigma+\tau)+2(\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}}+\sigma^{\frac{1}{2}}\tau^{\frac{1}{2}}+\tau^{\frac{1}{2}}\rho^{\frac{1}{2}})\}\\&=128\rho\sigma\tau(\rho+\sigma+\tau).\end{aligned}$$

Hence (D) must contain $(J)^2-128\rho\sigma\tau(\rho+\sigma+\tau)$ as a factor; and since when $t=0$, $\rho=0$, $\sigma=0$, and $(D)=\tau^4=(J)^2$, it is clear that $(D)=(J)^2-128(K)$, where $(K)=\rho\sigma\tau(\rho+\sigma+\tau)$.

(30) For this is $\left(0, \frac{rst}{3}, \frac{rst}{3}, 0\between u, v\right)^3$, and the discriminant of $(a, b, c, d\between u, v)^3$ is

$$a^2d^2+4ac^3+4db^3-3b^2c^2-6abcd.$$

(31) It is worthy of observation that (J) is also a Norm, namely, of $\rho^{\frac{1}{2}}+\sigma^{\frac{1}{2}}+\tau^{\frac{1}{2}}$, so that (J) is the discriminant of $ru^3+sv^3+tw^3$. I have not been able to perceive the morphological significancy of this relation.

(39) Although in the investigation in view (K) will only figure as an abbreviation of $\frac{(J)^2-(D)}{128}$, it may not be amiss to indicate a direct process for finding it. Let us for this purpose act upon the Hessian of Φ, treated as a function of u, v, twice with the canonizant of Φ converted into an operator by substituting $\frac{d}{dv}$, $-\frac{d}{du}$ in place of u and v.

The Hessian of Φ may be obtained without difficulty under the form

$$rsu^3v^3 + stv^3w^3 + trw^3u^3 \text{ or } \tau u^3v^3 + \rho v^3w^3 + \sigma w^3u^3$$ [32].

Operating upon this with

$$r^2s^2t^2\left(\frac{d}{dv}\cdot\frac{d}{du}\left(\frac{d}{du}-\frac{d}{dv}\right)\right)^2,$$

we obtain $\rho\sigma\tau(A\tau + B\rho + C\sigma)$, where

$$A = -2\left(\frac{d}{du}\right)^3\left(\frac{d}{dv}\right)^3 u^3v^3 = -72;$$

and as we know that this quantity must be of the form $\lambda(K) + \mu(J)^2$, we have $\mu = 0$, $\lambda = -72$; so that, denoting the *operator* corresponding to the canonizant by T, and the Hessian by H, we have $(K) = -\frac{1}{72}T^2H\Phi$ [33]. This gives a ready practical method for finding the discriminant of a general quintic F by means of the identity $D = J^2 + \frac{16}{9}T^2H$, where D is the discriminant, H the Hessian, T the canonizantive operator, and J the quadratic invariant of F in respect to its own variables.

(40) If now we suppose the determinant of u, v in respect to x, y to be μ, where μ is by hypothesis a real quantity, and if we call the

Quadratic invariant in respect to x, y	. .	$-\frac{1}{4}J$,
Discriminant of primitive „ „	. .	D,
Discriminant of the canonizant „	. .	$-\frac{1}{27}L$,

we have obviously

$$\left.\begin{aligned} J &= \mu^{10}(\rho^2+\sigma^2+\tau^2-2\rho\sigma-2\rho\tau-2\sigma\tau), \\ K &= \mu^{20}\rho\sigma\tau(\rho+\sigma+\tau), \qquad D = J^2 - 128K, \\ L &= \mu^{30}\rho^2\sigma^2\tau^2, \end{aligned}\right\} \text{invariants of } \Phi.$$

This applies to the case where the reduced form is Φ, that is, where the roots of the canonizant are all real, and consequently where $-L$ is negative, that is, L positive.

[32] It will be the quadratic invariant of $ru^3\xi^2+sv^3\eta^2+tw^3\zeta^2$ with respect to ξ, η, $\xi+\eta+\zeta$ being zero; just as the quadratic covariant of Φ is the quadratic invariant of $ru\xi^4+sv\eta^4+tw\zeta^4$ with regard to the same variables. This latter is in fact $rsuv+stvw+trwu$.

[33] The intervening covariantic form of degree 3 in the variables and 5 in the coefficients, namely, $TH\Phi$, will easily be seen to be

$$rst^2(u^2v-uv^2)+str^2(v^2w-vw^2)+trs^2(w^2u-wu^2).$$

When L is negative and the reduced form is $\Phi_{,}$, then, since the determinant of $u_{,}$, $v_{,}$, in respect to u, v, is i, we have

$$\left.\begin{aligned} J &= -\mu^{10}(\rho^2+\sigma^2+\tau^2-2\rho\sigma-2\rho\tau-2\sigma\tau), \\ K &= \quad \mu^{20}\rho\sigma\tau(\rho+\sigma+\tau), \qquad D = J^2 - 128K, \\ L &= -\mu^{30}\rho^2\sigma^2\tau^2, \end{aligned}\right\} \text{invariants of } \Phi_{,}.$$

By means of the ratios $\frac{L}{J^3}$, $\frac{K}{J^2}$, it is obvious that in either case alike the ratios of ρ, σ, τ become determinable by means of the same cubic equation, namely

$$\theta^3 - K\theta^2 + \frac{K^2-JL}{4}\theta - L^2 = 0;$$

ρ, σ, τ will be to each other as the roots of this equation [34].

(41) Since $ru^5+sv^5+tw^5$ represents a function in x, y with real coefficients, it follows that when L is positive, u, v as well as w being real, $\alpha:\beta:\gamma$ are ratios of real quantities, and the roots of the preceding cubic will be real; when L is negative, u, v becoming conjugate imaginary functions of x, y, whilst w remains real, r, s, unless they are equal, must become conjugate imaginary constants. When r, s, t are all real, ρ, σ, τ will be so too; and when r, s are imaginary and t real, ρ, σ will be imaginary and τ real. Thus according as L is positive or negative the roots of θ are or are not all real. Hence understanding by Δ the discriminant of the preceding equation with respect to θ and 1, Δ/L must be always either zero or negative. We see *à priori* that Δ/L must be integer, because when $L=0$ the cubic has two equal roots, $\frac{1}{2}K$. To compute its value more conveniently, write $K=6k$, $J=12j$. Then the equation becomes

$$(1,\ 2k,\ 3k^2-jL,\ L^2 \between \theta,\ -1)^3,$$

[34] For since the absolute values of ρ, σ, τ are not in question, we may consider ρ, σ, τ as the roots of $\theta^3-K\theta^2+q\theta-r$, so that $\rho+\sigma+\tau=K$. We have then

$$\frac{\rho^4\sigma^4\tau^4}{(\rho\sigma\tau)^3(\rho+\sigma+\tau)^3}=\frac{L^2}{K^3}, \text{ or } \frac{r}{K^3}=\frac{L^2}{K^3},$$

which gives $r=L^2$. Again,

$$\frac{\rho\sigma\tau K^2}{(K^2-4q)^2}=\frac{K^2}{J^2}, \text{ or } \frac{(K^2-4q)^2}{r}=J^2, \text{ or } (K^2-4q)^2=L^2J^2, \text{ or } q=\frac{K^2\mp JL}{4}.$$

As regards the sign to be given to JL in q, since

$$\frac{J^3}{L}=\frac{(K^2-4q)^3}{r^2}=\frac{(K^2-4q)^3}{L^4},$$

we have $(K^2-4q)^3=J^3L^3$. Hence

$$q=\frac{K^2-1^{\frac{1}{3}}JL}{4}.$$

Consequently

$$q=\frac{K^2-JL}{4}, \text{ and } not \ \frac{K^2+JL}{4}.$$

of which the discriminant is

$$L^4 + 4(3k^2 - jL)^3 + 32k^3L^2 - 12k^2(3k^2 - jL)^2 - 12kL^2(3k^2 - jL).$$

Hence
$$\frac{\Delta}{L} = L^3 - 108k^4j + 36k^2j^2L - 4j^3L^2 + 32k^3L$$
$$+\ 72k^4j - 12k^2j^2L \qquad - 36k^3L + 12jkL^2$$
$$= L^3 - 36k^4j + 24k^2j^2L - 4j^3L^2 - 4k^3L + 12jkL^2.$$

Accordingly, multiplying the above equation by -3×12^3 in order to avoid fractions, replacing k, j by their values in terms of K, J, and naming G the quantity $-432\Delta/L$, positive, or to speak more strictly non-negative, we have

$$G = JK^4 + 8LK^3 - 2J^2LK^2 - 72JL^2K - 432L^3 + J^3L^2\ [35].$$

It is evident that G must be identical to a positive numerical factor *près* with the function which M. Hermite denotes by I^2 [36].

[35] It will be observed that when $J=0$ and $L=0$, G vanishes. This is easily verifiable *à priori*; for when $J=0$ and $L=0$, the reduced form has been seen to be ax^5+5exy^4, of which the canonizant is

$$\begin{vmatrix} a, & 0, & 0, & 0 \\ 0, & 0, & 0, & e \\ 0, & 0, & e, & 0 \\ y^3, & -y^2x, & yx^2, & -x^3 \end{vmatrix}$$

which equals $aexy^2$.

Hence the form and its canonizant have a common factor x, and consequently their resultant vanishes; hence $I=0$ and $G=I^2=0$. G also vanishes when $K=0$ and $L=0$, which is also easily verifiable; for then the reduced form becomes u^5+v^5, of which the canonizant vanishes, and consequently the resultant of the form and its canonizant becomes intensely zero; which accounts for the high power of K in (JK^4), the sole term of G in which L does not appear.

[36] (*a*) Compare expression for $16I^2$, *Cambridge and Dublin Journal*, p. 203. This will be found to contain nine terms, and to rise as high as the fifth power in Δ (which to a constant factor *près* is identical with my J); whereas in $-\Delta/L$ there are only six terms, and no power of J beyond the third. This seems to indicate that the K and L are more fortunately chosen than M. Hermite's J_2, J_3, which are invariants of the like degrees 8 and 12. It is of course evident that the following relations exist between M. Hermite's Δ, J_2, J_3 and the J, K, L of this paper,

$$\Delta = lJ,$$
$$J_2 = mJ^2 + nK,$$
$$J_3 = pJ^3 + qJK + rL,$$

where l, m, n, p, q, r are certain numerical quantities. Until these are ascertained, it is impossible to confront M. Hermite's results with my own, to ascertain whether or not they are identical in substance, and, if not, wherein the difference consists. I therefore subjoin the necessary calculations for effecting this important object.

Let us first take the form $x^5+5exy^4+y^5$. The quadratic covariant of this is $x(ex+y)$.

Accordingly, to obtain M. Hermite's A, B, C, C', B', A' (*Cambridge and Dublin Journal*, vol. IX. p. 179), we must make

$$x = X; \quad ex + y = Y,$$

(42) In fact M. Hermite's octodecimal invariant is most simply obtained as the resultant of the primitive quintic and its canonizant. Using the

which gives (vide *Cambridge and Dublin Journal*, p. 180)

$$F = X^5 + 5eX(Y-eX)^4 + (Y-eX)^5$$
$$= (A, B, C, C', B', A' \between X, Y)^5,$$

where $A = 1+4e^5,\ B = -3e^4,\ C = 2e^3,\ C' = -e^2,\ B' = 0,\ A' = 1.$

Accordingly (vide *Cambridge and Dublin Journal*, p. 184),

$$AA' - 3BB' + 2CC' = 1 + 4e^5 - 4e^5 = 1 \qquad = \sqrt{\Delta},$$
$$AA' + BB' - 2CC' = 1 + 4e^5 + 4e^5 = 1 + 8e^5 = \frac{I_1}{2\sqrt{\Delta^3}},$$
$$AA' + 5BB' + 10CC' = 1 + 4e^5 - 20e^5 = 1 - 16e^5 = \frac{I_2}{2\sqrt{\Delta^5}}.$$

Hence $\Delta = 1,\ I_1 = 2 + 16e^5,\ I_2 = 2 - 32e^5.$

Again (vide *Cambridge and Dublin Journal*, p. 186, § vii.),

$$8J_1 = I_1 - \Delta^2 = 1 + 16e^5, \quad 24J_2 = I_2 - 2I_1\Delta + \Delta^3 = -1 - 64e^5;$$

but J_1, J_2 are subsequently *without warning* (compare expressions for AA', BB', CC', pp. 186, 192) renamed J_2, J_3; so that

$$8J_2 = 1 + 16e^5, \quad 24J_3 = -1 - 64e^5.$$

The corresponding values of J, K, L have been already calculated, and we have found

$$J = 1, \quad K = -2e^5, \quad L = 0.$$

Hence $A = 1,\ \frac{1}{8} + 2e^5 = B - 2Ce^5,\ \frac{-1}{24} - \frac{64}{24}e^5 = D - 2Ee^5.$

Thus $A = 1,\ B = \frac{1}{8},\ C = -1,\ D = -\frac{1}{24},\ E = \frac{4}{3}.$

To find F, take another form convenient for the purpose, as $x^5 + 10dx^2y^3 + y^5$.

Taking the emanant of this $(x, 0, dy, dx, y \between x', y')^4$, the quadratic covariant is obviously $xy + 3d^2y^2$, so that $J = 1$.

Also its discriminant is

$$\begin{vmatrix} 1, & 0, & 0, & d \\ 0, & 0, & d, & 0 \\ 0, & d, & 0, & 1 \\ y^3, & -y^2x, & yx^2, & -x^3 \end{vmatrix}$$

namely, $d^3y^3 - d(-dx^3 + y^2x) = d^3y^3 - dy^2x + d^2x^3,$

of which the discriminant is

$$d^{10} + 4d^2\left(\frac{-d}{3}\right)^3 = d^{10} - \tfrac{4}{27}d^5.$$

Hence by definition

$$L = c - \tfrac{27}{4}d^{10} + d^5.$$

Again, to find A, B, C, C', B', A', we must write

$$x + 3d^2y = X,$$
$$y = Y,$$

and we have then

$$(X - 3d^2Y)^5 + 10d(X - 3d^2Y)^2Y^3 + Y^5 = (A, B, C, C', B', A' \between X, Y)^5.$$

Since $J = 1$ and K is of the eighth order only in the coefficients, it is obvious that neither J^3 nor JK can contain a term involving d^{10}. In order therefore to find F, it will be sufficient to compare the coefficient of d^{10} in J_3 and in L.

Now $A = 1,\ B = -3d^2,\ C = 9d^4,\ C' = 27d^5 + d,\ B' = 81d^8 - 12d^3,\ A' = 243d^{10} + 90d^5 + 1.$

Also $\Delta = J = 1$. Hence neglecting all but the terms which bring in d^{10}, $24J_3$ (p. 186, Memoir) is tantamount to I_2, and I_2 (p. 186) is tantamount to

$$2(243d^{10} - 5.3.81d^{10} + 10.9.27d^{10}),$$

which is $$12 \times 243d^{10}.$$

reduced forms for these two functions,

$$ru^5 + sv^5 - t(u+v)^5, \quad rstuv(u+v),$$

Hence in J_3 the term containing d^{10} is $\frac{243}{2}$.

Hence $\quad -\frac{27}{4}F = \frac{243}{2}$, or $F = -18$.

Hence we have, finally *,

$$\Delta = J,$$
$$J_2 = -K + \tfrac{1}{8}J^2,$$
$$J_3 = -18L + \tfrac{4}{3}JK - \tfrac{1}{24}J^3;$$

and conversely,

$$J = \Delta,$$
$$K = -\frac{J}{2} + \frac{1}{8}\Delta^2,$$
$$L = -\frac{J_3}{18} - \frac{2}{27}\Delta J_2 + \frac{1}{8}\Delta^3.$$

Unhappily a further step is wanting to bring M. Hermite's results to the final test of comparison; for the value of AA' (p. 192) does not agree with that given for AA' (p. 186) by simply changing J_1, J_2 into J_2, J_3 respectively; a further change of Δ into 2Δ becomes necessary to make the ratios of AA', BB', CC' (p. 192) accord with the ratios of the same quantities at p. 186. Finally, even after making this change the expression for $16I^2$ (p. 203) does not accord (even to a constant coefficient *près*) with that with which it is meant to be identical, namely, $16I_3{}^2$ (p. 187); so that after great labour I am still baffled in my attempt to ascertain the agreement or discrepancy of my conclusions with those of my precursor in the inquiry. As will appear hereafter, the two sets of conclusions are undoubtedly discrepant in form; but whether they are so in substance or not, or rather whether they are not in contradiction to each other, requires a close examination to discover, the more especially because, as will hereafter be shown, there is a certain necessary element of indeterminateness in the scheme of invariantive conditions which serve to fix the character of the roots. It is greatly to be lamented that so valuable a paper as M. Hermite's should be to some extent marred, in respect of the important end it would serve as a term of comparison, by the existence of these numerical and notational inaccuracies. I have spent hours upon hours in endeavouring to reconcile these several texts of the same memoir, and, after all my labour, the work is left unperformed without which the truth as between the two methods cannot be elicited. I feel, however, as confident of the correctness of my own conclusions as of the truth of any proposition in Euclid.

(*b*) It is worthy of notice that there is a failing case in M. Hermite's process for finding I^2 in terms of Δ, J_2, J_3, just as there is one in mine for finding G in terms of J, K, L,—the failure of the process, however, in neither case entailing any corresponding defect in the results obtained. The process employed in this memoir fails when $L=0$: for then the general form $ru^5+sv^5+tw^5$ is superseded by the supplementary one, $au^5+5euv^4+fv^5$. M. Hermite's fails when J (the J of *this* memoir) $=0$; for then the quadratic invariant becomes a perfect square, and the substitution of its factors in place of the original variables becomes inadmissible, since the two former coincide.

(*c*) It may be as well here to notice the form which M. Hermite's two linear covariants assume when referred to the canonical form above written. The quadratic covariant being $rsuv+stvw+trwu$, if we operate with the correlative of this, obtained by writing in it

$$\frac{d}{dv}, \quad -\frac{d}{du}, \quad \frac{d}{du}-\frac{d}{dv}$$

in lieu of u, v, w, namely with

$$-rs\frac{d}{du}\frac{d}{dv} - st\frac{d}{du}\left(\frac{d}{du}-\frac{d}{dv}\right) + tr\frac{d}{dv}\left(\frac{d}{du}-\frac{d}{dv}\right),$$

upon the primitive, we obtain to a factor *près* the canonizant $rstuvw$, which has been already

[* After Salmon, *Higher Algebra*, 1885, p. 250, $\Delta=J$, $J_2=-K$, $J_3=JK+9L$. Cayley's examination of Hermite's criteria is given, *Coll. Papers*, VI., p. 170.]

their resultant in respect to u, v is obviously

$$(rst)^5(r-s)(s-t)(t-r)\ ^{(37)},$$

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obtained; repeating the process, it is easy to see that the first linear covariant of the fifth degree in the coefficient assumes the simple form $rst(stu+trv+rsw)$, or $rst(\rho u+\sigma v+\tau w)$. Taking again the correlative of this, namely,

$$rst\left\{\rho\frac{d}{dv} - \sigma\frac{d}{du} + \tau\left(\frac{d}{du}-\frac{d}{dv}\right)\right\},$$

and operating with it upon $rsuv+stvw+trwu$, it will be found without difficulty that the second linear covariant of the seventh degree in the coefficients becomes

$$rst\{(\sigma-\tau)(\sigma+\tau-\rho)u+(\tau-\rho)(\tau+\rho-\sigma)v+(\rho-\sigma)(\rho+\sigma-\tau)w\},$$

which is distinguishable in species from the former one by its symmetry being only of the hemihedral kind.

(*d*) It may not be out of place to notice here that the Hessian of the canonical form will be found to be

$$\rho v^3w^3+\sigma w^3u^3+\tau u^3v^3.$$

(*e*) Again, if we write

$$rst(\rho u+\sigma v+\tau w)=\xi,$$
$$rst\{(\sigma-\tau)(\sigma+\tau-\rho)u+(\tau-\rho)(\tau+\rho-\sigma)v+(\rho-\sigma)(\rho+\sigma-\tau)w\}=\eta,$$
$$u+v+w=0,$$

and from these equations deduce the values of u, v, w, and substitute them in $ru^5+sv^5+tw^5$, we shall obtain M. Hermite's "forme-type" expressed in terms of the parameters of the reduced form, and every coefficient therein will be invariantive.

The resultant of the equations above written (on making $\xi=0$, $\zeta=0$) will appear in the denominator of each such coefficient. Hence it appears, from M. Hermite's expressions (*Cambridge and Dublin Mathematical Journal*, vol. IX. p. 193), where J_3 will be seen to enter into the denominator of A, B, C, C', B', A', that this resultant to a factor *près* is his J_3. Its value may easily be calculated, and will be found to be

$$\rho\sigma\tau[(\rho+\sigma+\tau)^3-4(\rho+\sigma+\tau)(\rho\sigma+\rho\tau+\sigma\tau)+9\rho\sigma\tau]=JK+9L.$$

Accordingly as L (to use Dr Salmon's convenient elliptical expression) is the condition of the failure of my *general* reduced form, so is $9L+JK$ the condition of the failure of M. Hermite's "forme-type." As particular cases of this last failure, we may suppose $J=0$, $L=0$, or $K=0$, $L=0$. In the former case the reduced form is ax^5+5ex^4y, of which the simplest quadratic and cubic covariants are respectively aex^2; ae^2y^2x. Thus to find L, the first linear covariant, we have to operate upon ae^2y^2x with $ae\left(\frac{d}{dy}\right)^2$, which gives a^2e^3x; and to find L_2, we have to operate on $(aex^2)^2$ with $ae^2\left(\frac{d}{dx}\right)^2\frac{d}{dy}$, or, if we please (according to M. Hermite's method), with $\left(a^2e^3\frac{d}{dy}\right)$, on aex^2, showing that L_2 vanishes, but L_1 continues to subsist. When, secondly, $K=0$, $L=0$, the reduced form is ax^5+ey^5, and the canonizant disappears entirely, so that the first, and consequently also the second, linear covariants, each of them becomes a *null*.

(37) By aid of the reduced forms of the invariants J, K, L, I given in the text, it is easy to prove that every other invariant, say Ω of a quintic, is a rational integral function of these four. In what follows, let a parenthesis enclosing the symbol of any invariant signify its value when any two of the quantities u, v, w in the reduced form $ru^5+sv^5+tw^5$, where $u+v+w=0$, are taken as the independent variables. We have then

$(J)=\rho^2+\sigma^2+\tau^2-2\rho\sigma-2\rho\tau-2\sigma\tau$, $(K)=\rho\sigma\tau(\rho+\sigma+\tau)$, $(L)=\rho^2\sigma^2\tau^2$, $(I)=\rho^2\sigma^2\tau^2(\rho-\sigma)(\sigma-\tau)(\tau-\rho)$, ρ, σ, τ meaning st, tr, rs.

The degree of Ω must be of the form $4m$ or $4m+2$. (1) Let it be of the form $4m$. Then, since the interchange of any two of the variables, u, v, w must leave (Ω) unaltered, (Ω) will be and consequently, if we call I the resultant in respect to x, y, we have

$$\pm I = \mu^{40}\rho^2\sigma^2\tau^2(\sigma-\rho)(\tau-\sigma)(\rho-\tau),$$

and

$$I^2 = \mu^{80}\rho^4\sigma^4\tau^4(\sigma-\rho)^2(\tau-\sigma)^2(\rho-\tau)^2$$
$$= \mu^{30}(\sigma-\rho)^2(\tau-\sigma)^2(\rho-\tau)^2L^2.$$

(43) Thus we see that the two quantities G, I^2, which are both rational integral functions of the degree 36 in the coefficients of $F(x, y)$, cannot one vanish without the other, at all events when L is not equal to zero. This is sufficient to show that they are identical to a numerical factor *près*, whatever L may be, zero or not zero [38], and consequently that the quantity called G, proved to be positive upon the supposition of L not being zero, must also remain positive when L is zero, because it is in fact the square of a rational function of the coefficients. But we may also prove this independently by virtue of the supplementary reduced form $au^5 + 5euv^4 + fv^5$ applicable to the case of L zero.

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unaltered by the interchange of any two of the letters r, s, t, and is consequently a symmetric function of ρ, σ, τ, the roots of the equation

$$\theta^3 - \frac{(K)}{(L)^{\frac{1}{3}}}\theta^2 + \frac{(K)^2-(J)(L)}{(L)} - (L^{\frac{1}{3}}) = 0.$$

Hence

$$(\Omega) = \frac{F\{(J), (K), (L)\}}{(L)^{2m}},$$

F denoting a rational integral function-form of the quantities it affects. Consequently

$$\Omega = \frac{F(J, K, L)}{L^{2m}}.$$

Hence since Ω cannot become infinite when $L=0$, which merely implies that the general form reduces to

$$(a, 0, 0, 0, e, i\between x, y)^5,$$

$\Omega = \Phi(J, K, L)$, a rational integral function of J, K, L.

(2) If the degree of Ω is of the form $4m+2$, (Ω) will be a function of r, s, t, which changes its sign when u and v or any two of the quantities u, v, w, are interchanged, such interchange having the effect of introducing as a multiplier the $5(2m+1)$th power of the determinant of substitution (-1). Hence (Ω) is of the form

$$(\rho-\sigma)(\sigma-\tau)(\tau-\rho)F(\rho, \sigma, \tau), \text{ that is } \frac{(I).F(\rho, \sigma, \tau)}{(L)^{\frac{4}{3}}},$$

which again is of the form

$$\frac{(I).F\{(J), (K), (L)\}}{(L)^{2m-8}},$$

so that Ω is of the form

$$\frac{I.F(J, K, L)}{L^{2m-8}}.$$

Hence since, as before, Ω cannot become infinite when $L=0$, and since, furthermore, I does not vanish (for if so then G, which is I^2, would vanish) when $L=0$, Ω must be of the form

$$I\Phi(J, K, L). \quad \text{Q.E.D.}$$

([38]) For if $Q^2 = KI^2$ for an indefinite number of systems of values of a, b, c, d, e, f, of which Q, I are rational integral functions, Q^2 and KI^2 must be *absolutely* identical; this of course is the case when Q^2 and KI^2, as proved in the text, are known to be identical for all values of a, b, c, d, e, f which do not make L zero.

For when $L=0$, G becomes JK^4; so that the condition "G not negative" implies simply that J is positive unless K vanishes.

Now the canonizant, when it does not vanish, that is when e is not zero, contains v^2u as a factor, and, its coefficients being real, u, v are both of them necessarily real functions of x, y. Consequently J, which by definition is $-4 \times$ discriminant of quadratic covariant, becomes $-4\mu^{10} \times$ discriminant of $au(eu+fv)$ in respect to u, v, which $= \mu^{10}a^2f^2$, μ being real. Consequently J is positive, since the reality of u, v implies that of a, e, f, when e is not zero. When e is zero u, v may be either real or imaginary; for $u^5 + v^5$ may be real whether u, v be real or conjugate imaginary functions of x, y; but in that case K, which is found by operating twice upon the Hessian with a canonizant turned into an operator, vanishes, since then all the coefficients of the canonizant vanish [39]. Hence the rule that G cannot be negative is seen to be true, whatever L may be.

([39]) (*a*) In the more general form $au^5+5euv^4+fv^5$, taking $\mu=1$, the canonizant is ae^2uv^2; this squared and turned into an operator becomes $a^2e^4\left(\frac{d}{dv}\right)^2\left(\frac{d}{du}\right)^4$, which, applied to the Hessian, namely $3aeu^4v^2+afu^3v^3-e^2v^6$, after multiplying by $-\frac{1}{72}$, gives $K=-2a^3e^5$, so that $D=J^2-128K=a^4f^4+256a^3e^5$, which is capable of easy verification. In fact D becomes the resultant of au^4+ev^4 and $v^3(4eu+fv)$; v^3 introduces the factor a^3 into D; and further, making $u:v::-f:4e$ and substituting in au^4+ev^4, we obtain the other factor af^4+256e^5.

If we adopt $u^5+5euv^4+v^5$ as the reduced form for the failing case (a form analogous to the well-known one, $u^4+6eu^2v^2+v^4$, for the general quartic), to find e we have $J=\mu^{10}$, $K=-2\mu^{20}e^5$. Hence $e^5=-\frac{K}{2J^2}$; thus when $K=0$, $e=0$.

(*b*) By a linear transformation we may always take away any two (except the two first or last) coefficients of a given quintic, but the vanishing of more than two coefficients always corresponds to some invariantive condition. Thus, for example, in the form

$ax^5+5exy^4+fy^5$	$L=0$		
ax^5+fy^5	$L=0$	$K=0$	
ax^5+5exy^4	$L=0$	$J=0$	
$ax^5+10dx^2y^3$	$J=0$	$K=0$*	
$ax^5+5bx^4y+10cx^3y^2$	$L=0$	$J=0$	$K=0$.

(*c*) The condition for the existence of four equal roots in a quintic is the vanishing of the quadratic covariant; that is to say, we must have

$$ae-4bd+3c^2=0, \quad af-3be+2cd=0, \quad bf-4ce+3d^2=0.$$

The three quantities equated to zero are not separately invariants, but constitute in their *ensemble* an invariantive plexus.

(*d*) [It may here be noticed incidentally that the conditions for equal roots in the biquadratic form are as follows. For two equal roots, of course, the discriminant is zero, for three equal roots the two lowest invariants are each zero, and for two pairs of equal roots the Hessian $(A, B, C, D, E\between x, y)^4$ becomes to a factor *près* identical with the primitive $(a, b, c, d, e\between x, y)^4$, so that all the first minors of the matrix

$$\begin{vmatrix} a, & b, & c, & d, & e, & f \\ A, & B, & C, & D, & E, & F \end{vmatrix}$$

vanish. *Quære*, whether the character of the five-rayed pencil (centre at origin), in which a, A; b, B; c, C; d, D; e, E mark points, may not serve to distinguish between the case of four real and four imaginary roots.]

[* Or, if $d=m^3$, $\omega^3=1$, $(x+my)^5+\omega(\omega x+m\omega^2y)^5+\omega^2(\omega^2x+m\omega y)^5$.]

It may be said that the case of three or more equal roots existing in $F(x, y)$ has been lost sight of; but we know, and it is capable of immediate verification by taking as the reduced form $au^5 + 5bu^4v + 10cu^3v^2$, that on such

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(*e*) When $J=0$ and $K=0$, but *not* $L=0$, it is obvious that $\rho:\sigma:\tau::1:\iota:\iota^2$, ι being any imaginary cube root of unity, and the reduced form is $u^5+\iota v^5+\iota^2 w^5$, with the relation $u+v+w=0$.

J and K being zero, D will be so too, and accordingly the equation $u^5+\iota v^5+\iota^2 w^5=0$ will have two equal roots. It will easily be found that these equal roots correspond to the system of ratios $u=1$, $v=\iota^2$, $w=\iota$. In fact, if we write $u=1+\rho$, $v=\iota^2+\iota\rho$, $w=\iota+\iota^2\rho$, the equation becomes $u^5+\iota v^5+\iota^2 w^5=\rho^2(30\rho+3\rho^3)=0$.

Hence, understanding by ϵ either of the two prime sixth roots of unity, the complete system of ratios of u, v, w may be expressed as follows:—

$$\begin{array}{lll} u=1 & v=\iota^2 & w=\iota \\ u=1 & v=\iota^2 & w=\iota \\ u=1-\sqrt[3]{(10)} & v=\iota^2-\sqrt[3]{(10)} & w=\iota-\iota^2\sqrt[3]{(10)} \\ u=1+\sqrt[3]{(10)}\,\epsilon & v=\epsilon^4-\sqrt[3]{(10)} & w=\epsilon^2+\sqrt[3]{(10)}\,\epsilon^5 \\ u=1+\sqrt[3]{(10)}\,\epsilon^5 & v=\epsilon^4+\sqrt[3]{(10)}\,\epsilon & w=\epsilon^2-\sqrt[3]{(10)}. \end{array}$$

Thus, when $J=0$ and $K=0$, u, v, w (with the relation $u+v+w=0$) may first be found, in terms of x, y, by solving the cubic equation, obtained by equating to zero the canonizant of $(a, b, c, d, e, f \between x, y)$, and then x, y will be known from the above system of values for any two of the quantities u, v, w.

(*f*) It is obvious that the form $ax^5+10dx^2y^3$ gives $J=0$ and $K=0$; but it seems desirable to prove the converse, namely that when $J=0$ and $K=0$, but not $L=0$, the form is always reducible to $\alpha u^5+10\delta u^2v^3$; which may be done as follows. Since $J=0$ and $K=0$ the discriminant is zero, and we may assume

$$F=ax^5+5bx^4y+10cx^3y^2+10dx^2y^3,$$

and we have $J=$ discriminant of

$$(-4bd+3c^2)\,\xi^2+2cd\xi\eta+3d^2\eta^2.$$

Hence

$$3d^2(3c^2-4bd)-c^2d^2=0;$$

d cannot be zero, for then we should have $J=0$, $K=0$, $L=0$, contrary to hypothesis. Hence $8c^2-12bd=0$.

If $b=0$ and $c=0$, F is already reduced to the desired form; but if not, $d=\frac{2c^2}{3b}$, and F becomes

$$ax^5+\frac{5b}{6}x^2\left(6x^2y+\frac{12c}{b}xy^2+\frac{8c^2}{b^2}y^3\right);$$

or, making

$$a-\frac{5b^2}{6c}=\alpha,\quad \frac{b^3}{6c}=2\delta,\quad x+\frac{2cy}{b}=v,$$

$F=\alpha x^5+10\delta x^2v^3$, as was to be shown.

The corresponding converses for the case of $J=0$, $L=0$, and of $K=0$, $L=0$ have been already established.

(*g*) It will be observed that under a certain point of view L for binary quintics is the analogue of Δ the *discriminant* for binary quartics, the condition of failure in the *general* reduced form in the two cases being $L=0$ and $\Delta=0$ respectively. The mere vanishing of the discriminant in the case of the quintic function, unattended by any other condition, does not affect the nature of the reduced form.

(*h*) It has been shown previously in the text that when $L=0$ the primitive is reducible to the form

$$(a, 0, 0, 0, e, f \between x, y)^5.$$

Hence if I_{12} is any duodecimal invariant which vanishes when $b=0$, $c=0$, $d=0$, I_{12} must vanish whenever L vanishes, and consequently, since L is of as high a degree as I_{12}, I_{12} must be a numerical multiple of L. In Mr Cayley's Third Memoir on Quintics, "No. 29" represents

hypothesis all the invariants J, K, L must vanish, so that JK^4 is still non-negative [40].

(44) It is most important to notice that G can only become zero by virtue of two of the quantities ρ, σ, τ, and therefore of r, s, t becoming equal. When u, v are imaginary, it is the coefficients r, s which must become equal, as otherwise the reduced form would not be a real function of x, y. By equating r to s, and using as an auxiliary variable the ratio $\frac{r}{t}$ or $\frac{s}{t}$, we shall be able to study the composition and inward nature of G with the utmost clearness and facility.

SECTION II.—*On the Criteria which decide the Number of Real and Imaginary Roots.*

(45) Since in the preceding section we have supposed that u, v are always real linear functions of x, y, it is obvious that the character of the roots of the given quintic in x, y is completely identical with that of the roots in the reduced form, and it has been shown that only one reduced form corresponds to a given system of values of J, D, L [41].

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a duodecimal invariant calculated by M. Faà de Bruno, and characterized* morphologically by Mr Cayley as being that duodecimal invariant in which "the leading coefficient a does not rise above the fourth degree." On examining No. 29 it will be found to contain no term in which b, c, d are all simultaneously absent. Hence it is, by virtue of the above observation, a multiple of my L: to determine the numerical factor, let all the coefficients in the primitive except a, d be supposed zero; then the canonizant becomes

$$\begin{vmatrix} a & 0 & 0 & d \\ 0 & 0 & d & 0 \\ 0 & d & 0 & 0 \\ y^3 & -y^2x & yx^2 & -x^3 \end{vmatrix} = d^3y^3+ad^2x^3.$$

Hence L becomes $-27a^2d^{10}$, but "No. 29" becomes $27a^2d^{10}$. Hence we have the important relation "No. 29" $=-L$, so that No. 29 is a discriminant, an *intrinsic* property of the calculated invariant, which, I believe, was not suspected.

(*i*) It will at once be recognized that "No. 19" given in Mr Cayley's Second Memoir upon Quantics is identical with the J of this memoir, whence it follows from† Mr Cayley's equation (No. 26) = (No. 19)^2 − 1152 (No. 25), that $K=9$ (No. 25). Thus abstraction made of a mere numerical factor, Mr Cayley and myself agree upon perfectly distinct grounds in recognizing K and L as the true simplest invariants of their respective degrees, an accordance as satisfactory as it was unexpected, and which must be considered as setting at rest the question of what should be deemed the, so to say, *staple* invariants of the Binary Quintic.

(40) When the form is $au^5+5euv^4+fv^5$ so that $L=0$, the canonizant, as has been seen before, is ae^2v^2u; the resultant of these two is $a^5e^{10}a^2f=a^7e^{10}f$. Again, $J=a^2f^2$, $K=-2a^3e^5$; thus the square of the resultant $=\frac{1}{16}JK^4$; so that if we call this resultant, which we may take as the definition of the Octodecimal Invariant I, we have $G=16I^2$.

(41) It should be well noticed that the mere ratios $\frac{D}{J^2}$, $\frac{L}{J^3}$ do not suffice to determine the character of the roots. When these ratios are given, it is true that the ratios r, s, t in the

[* Cayley's *Coll. Papers*, II., pp. 294, 314.] [† *Ibid.*, II., p. 313 and VI., p. 148.]

Let us suppose J, D, L to be taken as coordinates of a point in space; when J, D, L are so related that the condition G non-negative is satisfied, the point will correspond to an equation with real coefficients, and may be termed a *facultative* point. But when G is negative it will correspond to an equation of the kind alluded to in the recent section of this paper, and there called conjugate: such a point may be termed non-facultative. Thus the whole of space will be divided into two parts, separated by the surface $G=0$, which may be termed respectively facultative and non-facultative (as being made up of facultative or non-facultative points [42]). It is clear that these two portions will be exactly equal, similar, and symmetrical with regard to the axis of D; by which I mean that, if two points be taken in any line perpendicular to the axis of D at equal distances from that axis, one will be facultative and the other non-facultative, as is evident from the fact that when J, L become $-J, -L$ (K, and therefore D or $J^2 - 128K$, remaining unaltered), G is converted into $-G$. Thus by a semirevolution round the axis of D the facultative and non-facultative portions may be made to exchange places.

(46) The axis of D itself lies on the surface of G, and like every other portion of this surface is facultative, for there is no reason for disallowing G to become zero. Conversely, if instead of a real equation, we take one of the conjugate class (described in the second section), the whole of the facultative portion of space (except the separating surface G) becomes non-facultative, and the non-facultative part becomes facultative, but G itself remains facultative. When the invariants, or any of them, become imaginary, we are put out of space altogether, and the system can belong neither to a real nor to a conjugate family, but to one with coefficients at the same time imaginary and non-conjugate. $G=0$ [43], it may be remarked, will in all cases be the condition of an equation capable of linear transformation into one of recurrent [44] form; for the reduced form then in general becomes

$$ru^5 + rv^5 - t(u+v)^5.$$

reduced form are given, but according as L is positive or negative, the arguments u, v in $ru^5 + sv^5 + tw^5$ (supposing w to be the real linear function of x, y) will be real or imaginary. When J, L, D are all given *absolutely*, then the character of the roots is completely determined. The *indelible* marks of a quintic function are three in number, namely the ratios $\frac{K}{J^2}, \frac{L}{J^3}$, and the sign of L or J, as for a quartic function they are two in number, namely $\frac{s^3}{t^2}$ and the sign of s.

(42) It will also be convenient to call the coordinates J, D, L corresponding to any facultative point a facultative system of invariants, and $\frac{D}{J^2}, \frac{L}{J^3}$ corresponding to the same (for a *given sign* of J) a facultative system of invariantive ratios.

(43) I shall hereafter allude to the surface denoted by $G=0$ under the name of the Amphigenous Surface, as being the locus of the points which give birth to real and conjugate forms indifferently.

(44) The roots of recurring equations, geometrically represented, in general go in quadruplets,

The case when G becomes zero by virtue of $J=0$ and $L=0$, that is to say when the function is reducible by real or imaginary linear substitutions (see footnote [39] (f)) to the form $u(u^4 \pm v^4)$, is the one which might for a moment be supposed to offer an exception to the rule; but the exception is only apparent, since $u(u^4 - v^4)$, on writing $u = p+q$, $v = p-q$, becomes

$$8(p+q)pq(p^2+q^2).$$

(47) To every point in space, it has been remarked, will correspond one particular family of equations all of the same character as regards the number they contain of real or imaginary roots, because capable of being derived from one another by real linear substitutions, such family consisting of an infinite number of ordinary or conjugate equations according as the point is facultative or non-facultative; but it may be well to notice that, conversely, every point does not correspond to a distinct family. In fact the equations $D = pJ^2$, $L = qJ^3$ (p, q being constants) will denote a curve divided into two branches by the origin of coordinates, one of which will be facultative and the other non-facultative; but in each separate branch every point will represent the same family. Any such separate branch may be termed an isomorphic line; and we see that the whole of space may be conceived as permeated by and made up of such lines radiating out from the origin in all directions.

(48) The origin at which $J=0$, $D=0$, $L=0$, as already noticed, corresponds to the case of three equal roots. The theorem that, when more than half as many roots are equal to each other as there are units in the degree of any binary form, all the invariants vanish, was remarked by myself originally in the very infancy of the subject, before Mr Cayley's paper, alluded to by M. Hermite, appeared in Crelle. The method of proof which then occurred to me is the simplest that can be given. For instance, in the case before us, if the quintic have three equal roots, we may reduce it to the form

$$ax^5 + 5bx^4y + 10cx^3y^2.$$

Suppose now, if possible, an invariant of the degree m; the *weight* of each term therein, say $a^r b^s c^t$, in respect to x or y would be the same (namely $5m/2$), so that we should have

$$5r + 4s + 3t = \frac{5m}{2} = s + 2t, \text{ or } 5r + 3s + t = 0,$$

and therefore $r=0$, $s=0$, $t=0$, $m=0$. So for a sextic with four equal

A, A'; B, B', where A and B, as also A', B', are mutual optical images of each other in respect to a fixed line, and A, A', as also B, B', are electrical images of each other in respect to a circle of which the fixed line is a diameter—with liberty, of course, for the images taken in either mode of combination to coalesce so as to reduce the quadruplet to a simple pair.

roots reduced to the form $(a, b, c, 0, 0, 0, 0 \between x, y)^6$. Supposing any term in one of its invariants to be $a^r b^s c^t$, we should have

$$6r + 5s + 4t = \frac{6m}{2} = s + 2t, \text{ or } 6r + 4s + 2t = 0,$$

which is absurd, unless $r = 0$, $s = 0$, $t = 0$, $m = 0$, and so in general for a binary form of any degree. If in the above example for the degree m only three roots were equal *inter se*, the form assumed being $(a, b, c, d, 0, 0, 0 \between x, y)^6$, any term in a supposed invariant being $a^r b^s c^t d^u$, where $r + s + t + u = m$, we should have

$$6r + 5s + 4t + 3u = 3m = s + 2t + 3u,$$

and, as before,

$$6r + 4s + 2t = 0, \quad r = 0, \quad s = 0, \quad t = 0;$$

no longer, however, $m = 0$, but $m = u$, which is left undetermined.

(49) Before proceeding further it will be proper to consider under what circumstances a variation (in the coefficients of any equation) arbitrary, except that the coefficients are to remain real, can affect the character of the roots.

Let $F(x) = 0$ be any algebraical equation with real coefficients, and let $\delta F(x)$ be the variation of F due to the variation of the coefficients, $dF(x)$ the variation due to the change of x into $x + dx$. If, now, r be a root of $F(x) = 0$, and $r + dr$ the corresponding root of $F(x) + \delta F(x) = 0$, we have

$$F(r) = 0, \quad F(r + dr) + \delta F(r) = 0,$$

or

$$\delta F(r) + \frac{d}{dr} F(r)\, dr + \frac{1}{1 \,.\, 2} \left(\frac{d}{dr}\right)^2 F(r)\, dr^2 + \&\text{c.} = 0.$$

Hence, unless $\dfrac{\delta F}{dr} = 0$, that is, unless there are two equal roots r, we shall have $dr = -\dfrac{\delta F(r)}{\dfrac{d}{dr} F(r)} =$ a real quantity; so that the character of the root $r + dr$ will be the same as that of r.

But if

$$\frac{dF}{dr} = 0, \quad \frac{d^2F}{dr^2} = 0, \ \dots \left(\frac{d}{dr}\right)^{i-1} F = 0,$$

so that there are i roots r, i being any integer greater than zero, then to find dr we have the equation

$$(dr)^i + \frac{\Pi(i)\, \delta Fr}{\left(\dfrac{d}{dr}\right)^i F(r)} = 0.$$

Thus dr will have i distinct values; of these, if i is odd, all but one will be imaginary, but if i is even they will be all imaginary, or only all but two

imaginary and the remaining two real, according as the sign of $\delta F(r)$ is the same as or the contrary to that of $\left(\dfrac{d}{dr}\right)^i F(r)$. Accordingly, if r is real [45] and i even, the nature of the *ensemble* of the i roots $r + dr$ will not be the same when $\delta F(r)$ is positive as when $\delta F(r)$ is negative.

(50) So, further, if $F(x) = 0$ have $2m$ equal roots r, $2n$ equal roots s, and so on, the deduced corresponding groups of roots in $F(x) + \delta F(x) = 0$ will, or may at least each of them, undergo a change of character to the extent of one pair of the r group changing their nature with the sign of $\delta F(r)$, one pair of the s group changing their nature with the sign of $\delta F(s)$, and so on; but in no case, except $F(x)$ possess some equal roots (that is unless its discriminant be zero), can an infinitesimal variation in the constants affect the character of the roots [46].

(51) To every facultative point corresponds a certain set of values of J, D, L; and when these are given, it has been shown that the equation $(a, b, c, d, e, f \between x, y)^5$ is reducible to the form $ru^5 + sv^5 + tw^5$, where

$$u + v + w = 0,$$

or to the form $ru_1{}^5 + sv_1{}^5 + tw^5$, where

$$u_1 + v_1 + w = 0, \text{ and } u_1 = \frac{-w + iv}{2}, \quad v_1 = \frac{-w - iv}{2},$$

or to the form $au^5 + 5euv^4 + fv^5$, u, v, w being always real linear functions of x, y, with the sole exception that when $J = 0$, $K = 0$, $L = 0$, the reduced form is

$$au^5 + 5bu^4v + 10cu^3v^2.$$

When these three invariants are not all zero, the coefficients in the reduced form r, s, t or a, e, f are known functions of J, D, L, and the character of the roots is perfectly determinate; so that to every facultative point corresponds an infinite family of equations with real linear coefficients all deducible from each other by real linear substitutions. Thus then, with the sole exception of the origin, every facultative point corresponds to a determinate character of equation, namely to an equation with four, or two, or no imaginary roots; so that by a bold figure of speech we may be permitted to speak of

(45) r, although supposed to be one of a group of equal roots, is not *necessarily* real, for it may belong to a factor $(x^2 + 2xe \cos\theta + e^2)^2$.

(46) Compare this statement with the corresponding one given by M. Hermite, *Camb. and Dub. Journal*, vol. IX. p. 204, where only one parameter is supposed to undergo a change. I think that greater breadth and at the same time greater precision and clearness are gained by the mode of exposition employed in the text above. It will be observed that for a change of character to be possible when the function passes through a phase of equal roots, it is not enough that there shall exist a group of equal roots r, but there must be an even number of such roots in the group, and, furthermore, the equal roots must be *real*; when this last supposition is not satisfied, no change in the character of dr will affect the character of $r + dr$: an instructive exemplification of this remark will occur in the sequel.

every point but one in facultative space having a determinate quality, as masculine, feminine, or neuter. The origin alone is exempt from this law, and may be considered to be of epicene gender, since the factor

$$au^2 + 5buv + 10v^2$$

may have its roots real or imaginary. As we travel continuously from point to point in the facultative portion of space we pass from family to family, or, if we please, from an individual of one family to an individual of another family, differing from the former individual by an infinitesimal variation of the constants.

(52) If, then, we insulate any portion of facultative space, and in the block so insulated it is possible to pass from one point to any other—that is to say, if we can draw a *continuous* curve of any sort from one point to another without passing out of the block, and without cutting or touching the plane $D = 0$, then by virtue of the principle just laid down, we see that all the points in such block have the same character, and the nature of the roots will be the same in the infinite number of families, each containing an infinite number of individuals which the points in that block severally represent. Now imagine a block taken so extensive as to admit of no further augmentation, except accompanied with a violation of the condition of the capability of free communication between point and point without cutting or touching the surface D; such a block may be termed a *region*, and the whole of facultative space will be capable of subdivision into a certain number of these regions. This being supposed effected, the character of each region will be known when we know the character of a single point in it; that is to say, every region will have a determinate character of positive, negative, or neuter. It will presently be shown that the number of such regions is only three [(47)] (the least number it could be to meet the three cases of four, two, or no imaginary roots), one masculine, one feminine, one neuter; and consequently there will be but three cases to consider when the invariantive coordinates J, D, L are given; according as J, D, L belong to one or the other of these three regions, the equation to which they belong will have all its roots real, or only one real, or three real and two imaginary. The origin, it need hardly be added, constitutes a region *per se*, in which, so to say, the characters of masculine and feminine are blended.

(53) Let it be observed that we can see *à priori* that, were it not for the distinction between facultative and non-facultative portions of space, it would be impossible for each point corresponding to a given system of invariants to possess an unequivocal character; for in such case there would necessarily

(47) It is clear from the definition, that a *region* can only be bounded by G the amphigenous surface*, and D the plane of the discriminant: and granted (as will be shown hereafter) that G and D *touch* each other in only one continuous line, it becomes obvious *à priori* that there can be but two regions on one side of D and a single region on the other. [* p. 436, Footnote (43).]

be free continuous communication possible between all the points on each side of D *inter se*, and consequently we should be landed in the absurdity of conceiving the general equation of the fifth degree not to admit of division into cases of four, two, or no imaginary roots; D being negative, we know, would imply two roots, and not more than two, being imaginary; and accordingly D positive would imply either that four roots are imaginary or none—not sometimes one and sometimes the other, but in all cases alike four imaginary, to the exclusion of the supposition of the roots being all real—or else that all the roots are real and never four imaginary. Thus we see that the mere fact of a given system of invariants communicating a definite character to the roots, implies the necessity of the invariants exercising a restraining action over each other's limits, and that where this restraint does not exist it is impossible that the character of the roots can be determined by the values of the invariants.

(54) This is precisely what happens in biquadratic equations. In such we know the fundamental invariants t, s, or, if we please,

$$t,\ \Delta \text{ (where } \Delta = s^3 + 27t^2\text{)},$$

are perfectly independent and subject to no equation of condition; so that if we consider t, Δ as the coordinates of points in a plane, the whole of the plane will be made up of facultative points. When Δ is negative, that is for representative points lying on one side of the line Δ, it is true we know that there is just one pair of imaginary roots constituting what may be termed the neuter case; but when the representative points lie on the other side of this plane, they cannot be said to be either masculine or feminine, but will every one of them possess that epicene character which is peculiar to the origin alone in the case of quintic forms. A single example will make this clear.

Take the two reduced forms

$$u^4 + 6(1+\epsilon)\, u^2v^2 + v^4,$$

$$\omega^4 + 6(1-\epsilon)\, \omega^2\theta^2 + \theta^4,$$

where u, v are real linear functions of x, y, and ω, θ conjugate imaginary ones of the same; and suppose s, the quadrinvariant in respect to x, y, to be the same for both forms. For greater convenience of computation consider ϵ to be infinitesimal.

Then in the one case the t is of the same sign as

$$(1+\epsilon)(1-(1+\epsilon)^2), \text{ that is, } -2\epsilon,$$

and in the other the t is of the contrary sign to

$$(1-\epsilon)(1-(1-\epsilon)^2), \text{ that is, } \quad 2\epsilon,$$

so that t is of the same *sign* (namely negative) in each case.

Again, in the two cases respectively

$$\frac{t^2}{s^3} = \frac{4\epsilon^2}{1 + 3(1 \pm \epsilon)^2} = 4\epsilon^2.$$

Hence t as well as s, and consequently t and Δ are alike for both forms.

But in the one first written the roots are of the same nature as those of $u^4 + 6u^2v^2 + v^4$, that is, are all impossible, and in the other of the same nature as in

$$\left(\frac{u+iv}{2}\right)^4 + 6\left(\frac{u+iv}{2}\right)^2\left(\frac{u-iv}{2}\right)^2 + \left(\frac{u-iv}{2}\right)^4 = 0,$$

where u, v are real linear functions of x, y and $i = \sqrt{-1}$, in which case the roots are all possible. Thus we see that the very same values of t, Δ may correspond either to the case of four real or four imaginary roots, showing that the point t, Δ is what we have termed *epicene*. If we choose to take s, t as the coordinates, the same remarks would apply, except that Δ instead of a straight line would become a semicubical parabola. All the points on one side of this curve would have a definite neuter character, but those on the opposite side would be neither masculine nor feminine, but epicene.

(55) With a view to its subsequent distribution into regions, I now proceed to ascertain the form of that moiety of space which I have termed facultative.

Let $J^2 = qK$, $J^3 = \nu L$. Then

$$\frac{G}{J^3} = \frac{1}{q^4} + \frac{8}{\nu q^3} - \frac{2}{\nu q^2} - \frac{72}{\nu^2 q} - \frac{432}{\nu^3} + \frac{1}{\nu^2}, \text{ and } \frac{D}{J^2} = 1 - \frac{128}{q}.$$

We may for the moment make abstraction of the section of G made by the plane of L; that being done, and J, K, L being referred to the form

$$ru^5 + sv^5 + tw^5 \text{ or } ru_1^5 + sv_1^5 + tw^5,$$

calling μ^{10}, M, and, as before, using ρ, σ, τ to denote st, tr, rs, we have [pp. 426–7]

$$\pm J = M(\rho^2 + \sigma^2 + \tau^2 - 2\rho\sigma - 2\rho\tau - 2\sigma\tau),$$
$$K = M^2\rho\sigma\tau(\rho + \sigma + \tau),$$
$$\pm L = M^3\rho^2\sigma^2\tau^2.$$

Now when $G = 0$, we may suppose $\rho = \sigma$, $\frac{\tau}{\rho} = \frac{\tau}{\sigma} = \theta + 4$, θ being a new auxiliary variable [real. Cf. § 44]. We have then

$$\pm J = M(\tau^2 - 4\rho\tau) = M\rho\tau\theta,$$
$$K = M^2\rho^2\tau(2\rho + \tau) = M^2\rho^2\tau^2\left(1 + \frac{2}{\theta + 4}\right),$$
$$\pm L = M^3\rho^4\tau^2 \qquad = M^3\rho^3\tau^3\frac{1}{\theta + 4},$$

and consequently

$$\nu = \frac{J^3}{L} = \theta^4 + 4\theta^3,$$
$$q = \frac{J^2}{K} = \frac{\theta^2(\theta + 4)}{\theta + 6}.$$

(56) In general we have $\theta^4 + 4\theta^3 - \nu = 0$.

By a well-known corollary to Descartes' rule this equation can never have more than two real roots; when ν is positive there will always be two real roots of opposite signs; but when ν is negative and inferior to a certain negative limit, *all the roots become imaginary*. When ν lies between zero and that limit, two roots of θ will be real and both negative. To find that limit we may make $4\theta^3 + 12\theta^2 = 0$, or $\theta = -3$, which gives $\nu = 81 - 108 = -27$.

(57) When $\quad D = 0, \; q = \frac{J^2}{K} = 128,$

that is, $\quad \theta^3 + 4\theta^2 - 128\theta - 768 = 0, \text{ or } (\theta + 8)^2(\theta - 12) = 0;$

thus the roots of θ, when $D = 0$, are -8, -8, $+12$, and the corresponding values of ν are 2^{11}, 2^{11}, $27 \cdot 2^{10}$.

If now we make $\theta^4 + 4\theta^3 = 2^{11}$, one of the real values of θ we know is -8, and the other will be the real root of the cubic equation

$$\theta^3 - 4\theta^2 + 32\theta - 256 = 0.$$

When $\theta = 5$, the left-hand side of the equation

$$= 125 + 160 - 100 - 256 = -71.$$

When $\theta = 6$, the left-hand side of the equation

$$= 216 + 192 - 144 - 256 = \quad 8.$$

Hence the real root lies between 5 and 6, and q lies between $\frac{225}{11}$ and $\frac{360}{12}$. Thus $q < 30$ and $\frac{D}{J^2} = 1 - \frac{128}{q}$ is *negative*.

Again, if we take $\theta^4 + 4\theta^3 = 27 \cdot 2^{10}$, and take out the root $\theta = 12$, the resulting cubic becomes

$$\theta^3 + 16\theta^2 + 192\theta + 2304 = 0,$$

where it will easily be seen the real root lies between -12 and -16.

When $\theta = -12$,

$$q = \theta^2\frac{\theta + 4}{\theta + 6} = 144 \times \frac{8}{6} = 192;$$

and when $\theta = -16$,

$$q = 256 \times \frac{12}{10} = 307\tfrac{1}{5}.$$

Moreover, when q is a maximum or minimum, it will readily be found that $\theta^2 + 11\theta + 24 = 0$; so that $\theta = -3$, or $\theta = -8$. Hence for the value of θ found from the above cubic $q < 192$ and $\frac{D}{J^2} = 1 - \frac{128}{q}$ is *positive*.

(58) When $J=0$, $\nu=0$; and when $L=0$, $\nu=\infty$.

For these two cases it will be more simple to dispense with the auxiliary variable θ, and to revert to the original equation between J, K, L.

Accordingly, when $J=0$, we find $8LK^3-432L^3=0$. Hence

$$J=0, \text{ or } K^3=54L^2, \text{ that is } \left(\frac{-D}{128}\right)^3=54L^2;$$

thus the complete section of G made by the coordinate plane J becomes a straight line, namely the axis of D, and a semicubical parabola whose axis is the negative part of D.

When J is very nearly zero, ν becomes a positive or negative infinitesimal in the equation $\theta^4+4\theta^3=\nu$.

One real root of this equation is $\theta=\left(\frac{\nu}{4}\right)^{\frac{1}{3}}$.

The other is $-4+\delta$, where $[4(-4)^3+12(-4)^2]\,\delta=\nu$,

or
$$\delta=-\frac{\nu}{64}.$$

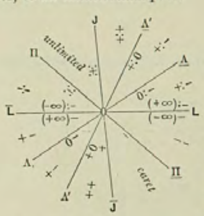
Now
$$\frac{K^3}{L^2}=\left(\frac{\theta+6}{\theta+4}\right)^3(\theta+4)^2=\frac{(\theta+6)^3}{(\theta+4)}.$$

The first value of θ gives $K^3=54L^2$ to an infinitesimal *près*; the other value gives

$$K^3=-\frac{512}{\nu}L^2,$$

or, to an infinitesimal *près*,

$$\left(\frac{D}{128}\right)^3=\frac{512}{\nu}L^2;$$



so that D passes from $+\infty$ to $-\infty$, that is, $\frac{J^3}{L}$ passes through zero.

(59) In the annexed figure [48], the plane of the paper represents the plane of D, that is, the plane for which $D=0$; $JO\bar{J}$ is the axis of J, OJ being the positive and $O\bar{J}$ the negative direction; $LO\bar{L}$ is the axis of L, OL being the positive and $O\bar{L}$ the negative direction. In order to avoid any appearance of an attempt at a practicably impossible accuracy of drawing, I use straight lines to denote cubical parabolas, and pay no attention whatever to relative magnitudes, but only to the order or progression of magnitudes, using the lines which are drawn in the figure

([48]) I shall refer, when I have occasion to do so, to this figure, which contains a synopsis of the whole theory, under the name of the Dial figure.

not as *copies* but as *symbols* of the actual curves which are to be mentally imagined.

Thus the line $JO\bar{J}$ is used to represent the straight line $L=0$; $\Lambda'O'\underline{\Lambda}'$ the cubical parabola $J^3=27\cdot2^{10}L$; $\Lambda O\underline{\Lambda}$ the cubical parabola $J^3=2^{11}L$; $\Pi O\underline{\Pi}$ the cubical parabola $J^3=-27L$ [49].

It will be observed that certain combinations of *plus*, *zero*, *minus*, positive and negative *infinity* are placed along the lines and inside the sectorial spaces. The meaning of these will be sufficiently obvious from what has preceded. They refer to the signs of the two values of D in the surface G for each point in the line or sector along or within which they are placed. At every point along the line $O\bar{J}$, $\frac{D}{J^2}$ has only one value, and that positive; along $\Lambda'O\underline{\Lambda}'$, $\frac{D}{J^2}$ has two values, one positive and the other zero. Along $\Lambda O\underline{\Lambda}$, $\frac{D}{J^2}$ has two values, one zero, the other negative. Immediately below $\bar{L}OL$ two values, one $+\infty$, the other finite and negative. Immediately above $\bar{L}OL$ two values, one $-\infty$, the other finite and negative. Along $\Pi O\underline{\Pi}$ one value, finite and negative.

Moreover D has been shown to be never zero, except along $\Lambda'O\underline{\Lambda}'$, $\Lambda O\underline{\Lambda}$. Hence it is obvious that *inside* $\Lambda'O\bar{J}$ and the opposite sector D has two values, both *plus*; inside the next pair of opposite sectors two values, one *plus*, the

([49]) It has been shown in the preceding articles that corresponding to the line $\bar{J}OJ$ and to the line $\Pi O\underline{\Pi}$, the vertical ordinate D of the amphigenous surface ($G=0$) has only one value, positive for the former, negative for the latter; along the line $\Lambda'O\underline{\Lambda}'$ two values, one positive the other zero; for the space between $\Lambda O\underline{\Lambda}$, $\bar{L}OL$ indefinitely near to the latter two values, one positively infinite, the other negative; and for the space indefinitely near to the same on the opposite of it, two values, one negatively infinite, the other negative. These results are collected and represented symbolically in the Table annexed.

$\bar{J}$	Λ'	Λ	$\bar{L}$	Π
+	+	0	$(+\infty)$ −	−
	0	−	− $(-\infty)$	

Thus, corresponding to the upper sheet of G, we have the succession

+	+	0	$(+\infty)$	−	−

and to the lower sheet

+	0	−	−	$(-\infty)$	−

the two sheets coming together at a cuspidal edge above $\bar{J}OJ$ and below $\Pi O\underline{\Pi}$.

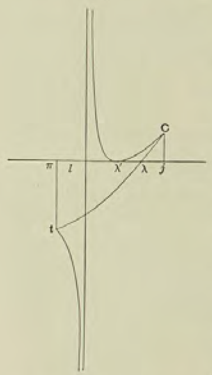
Moreover these are the only positions of the line revolving in the plane of D corresponding to which a change in the nature of D can take place, and thus we can without further examination fill up the Table, giving the nature of D for the intervening spaces, and may thus obtain the Table embodied in the *dial-figure* above, that is,

$\bar{J}$		Λ'		Λ		$\bar{L}$		Π
+	+	+	+	0	+	$(+\infty)$ −	−	−
	+	0	−	−	−	− $(-\infty)$	−	

other *minus*; inside the next pair of sectors also two values, one *plus*, the other *minus*; inside the next pair of sectors two values both *minus*, and in the pair of sectors left vacant, for which $\nu < -27$, it has been shown that D becomes impossible.

(60) Thus it will be seen that the surface G consists of two opposite portions precisely similar and symmetrical in respect to the axis of D.

Let us trace that one of these whose ground-plan is comprised within the sector $\Pi O\bar{J}$. It will consist of two sheets coming to a cuspidal edge (a common parabola) in the superior part of the plane of L. The upper sheet will touch the plane of D in $O\Lambda$ [50], and, remaining above the plane of D, approach continually to the plane of J as an asymptotic plane. The lower sheet will cut the plane of D in $O\Lambda'$, pass under the plane of D, cut the plane of J, progress to a maximum distance from it, and then approach indefinitely to J as its asymptotic plane. This will become apparent by taking a vertical section of this portion, cutting the lines $O\bar{L}$, $O\bar{J}$; for the nature of the flow of the two branches of the section will evidently be as figured here, where j, λ, λ', l, π represent the points in which the lines $O\bar{J}$, $O\Lambda'$, $O\Lambda$, $O\bar{L}$, $O\Pi$ are cut by the secant plane. [It should be particularly noticed that this figure is only intended to exhibit, under its most general aspect, the nature of the flow of the two branches of the curve; it is drawn in other respects almost at random, and makes no pretension whatever to giving a representation of the actual form of the curve.]



No part of the surface G lies under or above the sector ΠOJ, except the axis of D. The cusp C, where the two branches meet, is the intersection of the cutting plane with the parabola $J = D^2$ lying in the plane of L, and there will be another cusp at t, the point of maximum recession from the plane of J.

(61) I now proceed to discriminate, by aid of this surface, the facultative from the non-facultative portion of space.

If in the expression for G as a function of J, K, L we substitute for K its value $-\frac{D}{128} + \frac{J^2}{128}$, we obtain $G = \frac{J}{(128)^4} D^4 +$ terms involving only lower powers

(50) For the value of D for this sheet is zero all along $O\Lambda$, and positive on either side of it.

of D; so that, calling D_1, D_2 the two real values of D in the upper and lower sheets of G respectively corresponding to any point J, L,

$$G = J(D - D_1)(D - D_2)Q,$$

Q being a quantity essentially positive.

Hence when J is negative the *facultative* points in any line parallel to D will be those for which D lies between D_1, D_2, but when J is positive, the facultative points must be exterior to the segment D_1D_2; I denote this difference in the figure by placing a colon between the signs in each sector for which J is positive, indicating thereby that the facultative points lie between $+\infty$ and D_1, and between D_2 and $-\infty$; but where no colon is interposed, then it is to be understood that the facultative points lie between D_1 and D_2. Thus, if we turn back for a moment to the section of G last drawn, the whole of the space included between the two branches and the asymptote is facultative, because up to the asymptote J is negative, and beyond the asymptote the whole of the space not included between the asymptote and the lower branch is facultative, because beyond the asymptote J becomes positive. Thus, then, we see that the whole of that portion of the plane which lies on the left-hand side of the entire curve is facultative, and the portion on the right-hand side of the same non-facultative; the curve separating facultative from non-facultative space as a coast-line, indefinitely extended, separates land from water; so that there is, as of course we might have anticipated, no break of continuity in passing through the plane J.

If we take a corresponding section of the opposite portion of space corresponding to the ground-plan $JL\Pi$, it is obvious that precisely the contrary takes place, because the sign of J is opposite in the opposite sectors; so that what was facultative becomes non-facultative, and *vice versâ*.

(62) It is now clear that the whole of the facultative part of space is divided into three, and only three of the *regions* previously defined. One region will consist of that portion of it which is entirely under the plane of D: the second region will be so much of the upper portion as stands upon the acute sector $\bar{J}O\Lambda$; and the third of so much of the remainder of this portion as stands on the sector $\Lambda OJJO\underline{\Pi}$ [51]. Again, as regards the second region, the line $O\Lambda'$ is quite inoperative against its unity, because we have vertical ordinates above $O\Lambda'$ through which free communication can take place between the blocks over $JO\Lambda'$ and $\Lambda'O\Lambda$; but when we come

(51) It will be borne in mind that the whole of the infinite prism, both above and below, standing on ΠOJ belongs to *facultative* space: the prism standing on the opposite sector $\bar{J}O\bar{\Pi}$, or, to speak more strictly, on the *inside* of this last-named sector, is wholly unfacultative. The facultative line D which passes through O is completely isolated from the facultative portion which stands over $\Lambda O\bar{J}$, except at the point O (which we are forbidden to pass through if we would remain in the same region), and is of course a rectilinear edge to the facultative prism above referred to.

to $O\Lambda$, where G touches the plane of D, there we have an effective line of demarcation between the adjoining blocks *above* the plane of D; for it is impossible to pass from one into the other without going under D and coming up again through that plane, or else descending to the line $O\Lambda$ and so meeting the plane of D [52].

(63) It remains only to fix the characters of the several regions; but this requires no calculation to effect, for we know that when D is negative there is one and only one pair of imaginary roots. This disposes of the first of the regions above enumerated. Again, we know that when L is positive so that the reduced form is the superlinear equation $ru^5 + sv^5 + tw^5 = 0$, u, v, w being *real* functions, D being also positive, there must be four imaginary roots, as follows from the theory of the second section. Hence the third region has for its character two pairs of imaginary roots; and consequently the only remaining region, the second described, must correspond to the case of no imaginary roots, since otherwise we should be absurdly assuming the impossibility in any case of a quintic equation having all its roots real.

(64) It may, however, be an additional satisfaction to see how the change of character comes to pass at the critical line $O\Lambda$ from one to five real roots.

Along the line $O\Lambda$ [with $G = 0$] we have found [p. 443] that, calling the reduced form

$$ru_1^5 + sv_1^5 + tw^5,$$

$$r = s, \quad \frac{\tau}{\rho} = \frac{rs}{st} = \frac{r}{t} = \theta + 4 = -4.$$

Hence the equation becomes

$$4u_1^5 + 4v_1^5 + (u_1 + v_1)^5 = 0,$$

u_1, v_1 being of the form $\dfrac{-u + iv}{2}$, $\dfrac{-u - iv}{2}$, because L is negative.

Hence, beside $u_1 + v_1 = 0$,

$$4(u_1^4 - u_1^3 v_1 + u_1^2 v_1^2 - u_1 v_1^3 + v_1^4) + (u_1 + v_1)^4 = 0,$$

that is
$$5u_1^4 + 10u_1^2 v_1^2 + 5v_1^4 = 0,$$

that is
$$(u_1^2 + v_1^2)^2 = 0;$$

[52] Two superior regions we know *à priori* must exist to correspond respectively to the two cases of five and of one real root. Moreover we know *à priori* that two regions can only meet on the plane of D, and an inspection of the *dial-figure* shows that only $O\Lambda$ can be such line. Thus without completely making out the geometry of the question as regards the remarkable line ($J = 0$, $L = 0$) (the axis of D) which lies on the surface G, we may feel assured that the upper part of this line (which is easily found to belong to the 1-real-root region) cannot have any point except the origin in common with the 5-real-roots region, since otherwise these two regions would communicate along this line and merge into one. When it is considered that G is a surface of the ninth order in J, D, L, it will not appear surprising that some difficulty arises in forming a mental conception of certain of its local properties; on the contrary, the subject of wonder rather is that enough can be ascertained about it in a very brief compass to shed all the needful light upon the analytical problem which it illustrates.

so that there are two pairs of equal roots of $\dfrac{u_1}{v_1}$, namely $\pm i$; to these values of $\dfrac{u_1}{v_1}$ correspond

$$\frac{u - iv}{u + iv} = i, \quad \frac{u - iv}{u + iv} = -i.$$

Hence
$$(1 - i)u = (i - 1)v, \text{ or } (1 + i)u = (i + 1)v;$$

so that the two pairs of equal roots of u/v are ± 1, the outstanding root corresponding to $u_1 + v_1 = 0$ being $u/v = 0$.

Now, *still keeping upon the surface* G, which we know is facultative, let θ become $-8 + 4\epsilon$, where ϵ is an infinitesimal, then

$$\delta\left(\frac{J^3}{L}\right) = \delta\nu = (4\theta^3 + 12\theta^2)\,\delta\theta = -5120\epsilon;$$

also the supposed equation becomes

$$(4 - 4\epsilon)(u_1^5 + v_1^5) + (u_1 + v_1)^5 = 0,$$

or
$$(iv - u)^5 - (iv + u)^5 - 8(1 + \epsilon)u^5 = 0;$$

or, calling $v/u = \rho$,

$$(i\rho - 1)^5 - (i\rho + 1)^5 - 8(1 + \epsilon) = 0.$$

Let $\rho = \pm 1 + \sigma$, where σ is an infinitesimal. Hence

$$[-10(\pm i - 1)^3 + 10(\pm i + 1)^3]\sigma^2 - 8\epsilon = 0,$$

or
$$20(-3 + 1)\sigma^2 - 8\epsilon = 0,$$

or
$$\sigma^2 = \frac{-\epsilon}{5} = +\frac{1}{25600}\,\delta\left(\frac{J^3}{L}\right).$$

Hence calling σ_1, σ_2 the two values of σ, the four roots that at $O\Lambda$ were 1, 1, -1, -1 become $1 + \sigma_1$, $1 + \sigma_2$, $-1 + \sigma_1$, $-1 + \sigma_2$, when J^3/L becomes varied by $\delta(J^3/L)$, and consequently become all real if J^3/L is increased, and all imaginary if J^3/L is decreased, that is, become real or imaginary according as the line $O\Lambda$ sways towards or away from $O\bar{J}$, conformably with what has been shown on other grounds.

It will be noticed that in the line $O\Lambda$ produced in the opposite direction, that is, along the line $O\underline{\Lambda}$, L being positive, the reduced form is

$$4(u^5 + v^5) + (u + v)^5 = 0,$$

and the roots of $\dfrac{u}{v}$ become $\dfrac{u}{v} = -1$, $\dfrac{u}{v} = \pm i$, $\dfrac{u}{v} = \pm i$; so that, according to the canon laid down at the commencement of this discussion (see foot-note [46]), no change in the character of the roots can possibly take place along $O\Lambda$, and accordingly we have seen that this curved line does not correspond to any demarcation of regions.

(65) It is easy to express the conditions to be satisfied by the coordinates

of a point according as it lies in one or another of the three regions which have now been mapped out, and it is clear that we have the following rule:

When D is negative the equation has two imaginary roots.

When D is positive the equation has *no* imaginary roots, provided the two criteria J and $2^{11}L - J^3$ are both* negative [53]; but if either of these is zero or positive, there are two pairs of imaginary roots [54].

The duodecimal criterion-invariant, $2^{11}L - J^3$, and the invariants of the like order, $27 \cdot 2^{10}L - J^3$, $-27L - J^3$, I shall henceforth call Λ, Λ', Π respectively. It has been shown just above that the three invariants J, D, Λ of the 4th, 8th, and 12th orders respectively are sufficient for ascertaining the character of the roots of the quintic to which they appertain.

(66) The assertion that the *whole* of facultative space is divisible into three regions, in strictness requires a slight modification. It is obvious that the plane of D itself cannot be said to belong to any of the regions; and in order to make our theory quite complete, so as to furnish criteria applicable to equations having equal roots, and to enable us to distinguish between the case of the unequal roots being all three real, or two imaginary and one real, we must examine what takes place in this plane, and under what circumstances a passage from one point of it to another will or may be accompanied with a change of character in the roots.

If the roots of $f(x) = 0$ are supposed to be a, a, c, d, e, where c, d, e are unequal, on varying the constants of $f(x)$ in such a manner that the variation of the discriminant D is zero, the two equal roots a, a will remain equal. Now *in general* we have $\delta f(a) + \frac{1}{2} f''(a)\, da^2 = 0$; if this, under the particular supposition made, continued to obtain, da would have two distinct values, and the two equal roots would cease to continue to be equal, contrary to hypothesis. Hence we see that $D = 0$, $\delta D = 0$ necessarily implies $\delta f(a) = 0$ [55],

(53) Observe that this implies L also being negative; so that $2^{11} - \frac{J^3}{L}$ is positive and $\frac{J^3}{L} < 2^{11}$.

(54) (*a*) Observe that in general when $2^{11}L - J^3$ is zero there are no facultative points above the plane of D, but when J and $2^{11}L - J^3$, and consequently L and J are both simultaneously zero, a facultative right line springs into existence, namely, the axis of D extending both above and below the plane of D. The reduced form of equation (as previously demonstrated) corresponding to this singular line is $u^5 \pm uv^4 = 0$.

(*b*) It may further be noticed that on each side of the line $O\Lambda$ the limits of D are between positive infinity and a positive quantity, and between negative infinity and a negative quantity; so that as we pass from $O\Lambda$ to either side of it no facultative point can be found lying in the plane of D, showing that we cannot pass by a real infinitesimal variation of coefficients from an equation with two pairs of equal imaginary roots to an equation with a single pair of equal roots, as is apparent also on purely analytical grounds.

(55) (*a*) This is a somewhat curious theorem (whether new or otherwise I know not) thus incidentally established in the text, namely, that if $D(f)$ represent the discriminant of f, and if

[* $2^{11}L - J^3$ should be positive, and L negative. Cf. p. 372 above.]

and consequently $\delta f(a + da)$ is no longer $\delta f(a)$, but $\delta f'(a)\, da$; so that we obtain $da = 0$, or $da = -\frac{2\delta f'(a)}{f''(a)}$, and no change of character in the five roots results. If, however, the original roots are a, a, c, c, e, then, as shown in the general case, δc will have two distinct values, which will be both real or both imaginary. Accordingly we see that in the plane of D no change can possibly take place except in crossing the line which corresponds to a family of *two pairs* of equal roots.

(67) It has already been pointed out, in a foot-note, that we cannot pass facultatively from $O\Lambda$ to either side of this curve line. Hence the separation of the plane of D into subregions can only take place along the line $O\Lambda$, and it remains but to ascertain the character of the points on either side of this line, which we know, therefore, *à priori*, must possess opposite characters, since otherwise we should be admitting the absurd proposition of its being impossible to construct an equation of the fifth degree having two equal roots without the remaining three being always of *one character*, either all real or all not real. Let us, then, ascertain the character of the points in OJ for which $D = 0$, $L = 0$, and J is positive [56].

$D(f) = 0$ and $\delta D(f) = 0$, then when $f = 0$ we must have $\delta(f) = 0$. The very simplest example that can be chosen will serve to illustrate this proposition. Let

$$f = ax^2 + 2bxy + cy^2.$$

Suppose $$D(f) = ac - b^2 = 0,$$

and also $$\delta D(f) = a\delta c + c\delta a - 2b\delta b = 0,$$

we have $$\delta(f) = x^2\delta a + 2xy\delta b + y^2\delta c.$$

Now if $f = 0$ we may write $x = b$, $y = -a$, and δf becomes

$$\begin{aligned} &b^2\delta a - 2ab\delta b + a^2\delta c \\ &= b^2\delta a - 2ab\delta b + 2ab\delta b - ac\delta a \\ &= (b^2 - ac)\,\delta a = 0, \end{aligned}$$

according to the theorem.

If we make $f = (x, 1)^n$, D we know becomes a syzygetic function of f and f' or df/dx.

Hence since δD vanishes when $f(x) = 0$, $D = 0$, and $\delta f(x) = 0$, we learn that $\delta(D)$ is a syzygetic function of $(f, f', \delta f)$.

The theorem thus stated easily admits of extension to the higher variations of D, and so extended takes, I believe, the following form:

$$\delta^i(D) = \text{a syzygetic function of } (f, f', f'', \ldots f^i, \delta f).$$

(*b*) Professor Cayley has since informed me that the theorem in (55) (*a*), about whose originality I was in doubt, will be found in Schläfli's *De Eliminatione*. This is not the first unconscious plagiarism I have been guilty of towards this eminent man, whose friendship I am proud to claim. A much more glaring case occurs in a note by me [? p. 242 above] in the *Comptes Rendus*, on the twenty-seven straight lines of cubic surfaces, where I believe I have followed (like one walking in his sleep), down to the very nomenclature and notation, the substance of a portion of a paper inserted by Schläfli in the *Mathematical Journal*, which bears my name as one of the editors upon its face!

(56) We could not take J negative, for the facultative points of D in J are two positive quantities. See dial figure.

Since $L=0$, the reduced form is $u^5+5euv^4+v^5$.

This equation, by Descartes's rule, must contain imaginary roots. Hence in the sector $\Lambda O\bar{J}$ the roots are all real, and in the remainder of the facultative portion of the plane (from which it may be noticed the sector $\underline{\Lambda}O\bar{J}$ is excluded) two of the roots are imaginary.

Along $O\Lambda$ itself there are, as already observed, two pairs of real equal roots, and along $O\underline{\Lambda}$ two pairs of imaginary equal roots. Thus, finally, we have the *complete rule**.

If D is negative,	2 roots imaginary.	
If D is positive.		
When J, Λ are both negative,	0 roots imaginary.	
„ J, Λ are *not* both negative,	4 roots imaginary.	
If D is zero.		
When J, Λ are both negative,	0 roots imaginary	1 pair of equal roots.
„ J, Λ are *not* both negative,	2 roots imaginary	
„ J is negative, Λ zero,	0 roots imaginary	2 pairs of equal roots.
„ J is positive, Λ zero,	4 roots imaginary	
„ J is zero, Λ zero,		3 equal roots [56 bis].

Thus we see our space referred to an arbitrary origin, and with the invariants J, D, Λ for the coordinates, has been first divided into facultative and non-facultative space. The former has then been resolved prismatically into two regions above and one below the plane of D. The plane of D itself, or the facultative part of it, into two planar regions on opposite sides of the line $\Lambda O\underline{\Lambda}$; and again this line into two linear regions on either side of the origin O, which last corresponds to the case of three equal roots, and constitutes a region or microcosm in itself.

(68) It may as well be noticed here that the ambiguity of character in the points representing the different families of biquadratic forms when t and D are taken as the coordinates (and the same would be true if s and D were

(56 bis) When $D=0$, $\Lambda=0$, there are two pairs of equal roots. If J is negative these pairs are both real. If J is positive they are both imaginary. When J is zero there are no longer two pairs, but a single triad of equal roots. This perfectly explains what at first sight has the air of a paradox, namely, that the discrimination between the two kinds of double equality of an apparently equal order of generality that may subsist between the roots of an equation, depends on the fulfilment or failure of an algebraical equality. The fact is, as shown above, that there are not, as commonly supposed, two, but three kinds of double equality, according as there are two pairs of real, two pairs of imaginary, or one triad of equal roots; and the last is a sort of transition case between the other two.

[* For $D=0$, the sector $\bar{J}O\Lambda'$ of the dial figure is non-facultative, as follows from the diagram of p. 446. Thus the rule is: $D<0$, 2 roots imaginary; $D>0$, 5 roots real when $L<0$, $\Lambda>0$, 1 root real when one or both of these is reversed; $D=0$, 5 roots real when $\Lambda>0$, $\Lambda'<0$, 3 roots real when $\Lambda<0$, $\Lambda'<0$; $D=0$, $\Lambda=0$, 5 roots or 1 root real according as $L<0$ or $L>0$.]

employed), which prevails when these points lie above the line $D=0$, equally obtains along this line itself. For the reduced form, when $D=0$, is

$$ax^4+4bx^3y+6cx^2y^2.$$

In that case, calling the determinant of transformation μ, we have

$$s=3\mu^{12}c^2, \quad D=-\mu^{24}c^3;$$

and thus, whatever s and D may be, the character of the unequal roots is left undecided.

It may also be noticed that the blending of characters at the *origin* for the quintic form is not precisely of the same nature as that for the points above the line D in the biquadratic form; for at these points it is the cases of 4 and 0 imaginaries which become undistinguishable invariantively; whereas at the origin for quintics the reduced form becomes $ax^5+5bx^4y+10x^3y^2$, and the characters left undistinguished are those of 4 and of 2 imaginary roots—unless, indeed, we consider equal real roots as belonging indifferently to the class of real and imaginary; on which supposition all the three genders (so to say), masculine, feminine and neuter, become blended together at that point. But if we consider equal real roots as exclusively of the real class, then the *origin* for quartics ceases to be epicene; for when there are three equal roots all of them must be real. Thus the origin in quintics is the only epicene point, and in quartics the only non-epicene point—understanding by epicene the blending of the masculine (4 *imaginary roots*) and feminine (*no imaginary roots*) characters.

(69) We may draw some further important inferences from an inspection of the "dial figure," or the section of facultative space which follows it.

Within the prism $\bar{J}O\Lambda'$ [57] it will be observed D is always positive [58]. Hence, when J is negative and Λ' is negative, all the roots *must* be real, and the necessity for using the criterion D is done away with.

Again, when J and L are both negative, D is always negative, so that just two of the roots must be imaginary; and in this case also it becomes unnecessary to apply the criterion D.

Again, since there is no facultative prism corresponding to $\underline{\Pi}OJ$, the combination of L and D, both negative, can never occur unless $\underline{\Pi}$ is negative.

When L is negative, but J not negative, there may be two or four imaginary roots, according to the sign of D; but all the roots cannot be real.

(57) By which I mean within the facultative prism of which $\bar{J}O\Lambda'$ is the section made by the plane of D.

(58) The vertical section of facultative space in this supposition (see figure) is the area $\lambda C\lambda'$, which lies wholly *above* the plane of D.

(70) M. Hermite's rule is as follows. For remarks on the relation between his Δ, J_2, J_3 and the J, K, L of this paper, see [p. 430, above] footnote [36]. D is still the discriminant.

If D is negative (of course) two roots are imaginary.

If D is positive.

When Δ is negative, $25\Delta^3 - 3 \cdot 2^{10} J_3$ negative and J_3 positive, no roots are imaginary.

Δ is negative, $25\Delta^3 - 3 \cdot 2^{10} J_3$ positive, $25\Delta^3 - 2^{11} J_3$ negative, no roots are imaginary.

Δ is positive, $25\Delta^3 - 3 \cdot 2^{10} J_3$ positive, $25\Delta^3 - 2^{11} J_3$ negative, four roots are imaginary.

Δ is negative, $25\Delta^3 - 3 \cdot 2^{10} J_3$ positive, $25\Delta^3 - 2^{11} J_3$ positive, four roots are imaginary [59].

(71) What is the effect of the condition "Λ *positive* or *negative*," as the case may be? or rather, how does this condition arise? The ground of it is simply this, that $\Lambda = 0$ represents a cylindrical surface passing through the curve $O\Lambda$ (see dial figure), which curve is the *edge* of separation between two regions of opposite characters above the plane of D; the cylinder in question cuts the facultative portion of space below the plane of D, but above this plane (except along the vertical line $J = 0$, $L = 0$, that is, the axis of D)

[59] (*a*) The last four conditions ought to tally (and be in effect coextensive) with the two given by me for the case of D positive. The third of them, namely the case of D positive Δ positive, I have already noticed, as inferences from the dial figure; for M. Hermite's Δ, if not identical with my J, is at all events a positive multiple of it. I do not see how the case of Δ negative, $25\Delta^3 - 3 \cdot 2^{10} J_3$ negative with D positive, is met by this system of criteria, since J_3, as well as Δ, may be negative consistently with the second condition. I have not been able to ascertain whether in the memoir such a combination is shown to be impossible. M. Hermite admits, and indeed has been always aware of, the existence of a *lacuna* in the conditions above stated, which, I understand from him, it is his intention at some future time to fill up, and thus to complete his original solution. In the meanwhile he has been led to study the question from a different point of view, and has succeeded in obtaining a new set of criteria adequate to a complete solution of the question without calling in the aid of the principle of continuity. In this new system my Λ criterion is replaced by an invariant of the twenty-fourth degree, which is of course an objection as far as it goes, but in no wise diminishes the extraordinary interest that attaches to this altered mode of approaching the question, which bears to his original method and my own the same relation as the proof of Sturm's theorem by the law of inertia for quadratic forms bears to that given by Sturm himself.

(*b*) It is apparent from the fact that when $D = 0$, G (M. Hermite's I^2) becomes

$$(25\Delta^3 - 3 \cdot 2^{10} J_3)(25\Delta^3 - 2^{11} J_3)^2$$

(*Camb. and Dub. Journal*, vol. IX. p. 206), that the factors of this product are respectively of the form $a\Lambda' + bJD$, $c\Lambda + eJD$, a, b, c, e being certain numerical quantities. This gives rise to a singular reflection, *to wit*, that my own criteria for the case of D positive may be varied by the addition of a term λDJ to Λ (λ being a numerical coefficient), provided λ lies within certain limits, the form of the criteria in all other respects remaining unchanged. This proposition, fraught with the most important consequences, and not unlikely to lead to an entire revolution in the mode of attacking the general problem of criteria, I proceed to establish in the text.

it passes exclusively through non-facultative space, never again cutting or meeting the surface G (the facultative boundary). Now it is clear that any surface whatever which passes through $O\Lambda$ and never meets the surface G above the plane $D = 0$, except along the axis of D (that is, the line $J = 0$, $L = 0$), may be substituted for Λ [60] and will serve equally well with Λ to distinguish between the masculine and feminine regions of space. $\Lambda - \rho JD$ will fulfil the condition of passing through the line $O\Lambda$, whose equation is $\Lambda = 0$, $D = 0$, and obviously is the only invariant not exceeding the twelfth order capable of so doing; it only remains to ascertain within what limits the numerical coefficient ρ must be taken so as to fulfil the condition that the combined equations $\Lambda - \rho JD = 0$, $G = 0$ shall be incapable of being satisfied by any positive value of D.

(72) Substituting for Λ and D their values, the equation to be combined with $G = 0$ becomes

$$J^3 - 2^{11} L + \rho J (J^2 - 128K) = 0.$$

Returning to the notation of Art. (55) [p. 442], and dividing by JK, this equation, when $G = 0$, becomes

$$q - 2^{11}\frac{q}{\nu} + \rho(q - 128) = 0,$$

or

$$(1 + \rho) q\nu - 2^{11} q = 128\rho\nu,$$

which, substituting for q, ν in terms of θ, gives

$$\frac{(1 + \rho)\theta^3(\theta + 4)^2}{\theta + 6} = 2^{11}\frac{\theta^3 + 4\theta^2}{\theta + 6} - 128\rho\theta^3(\theta + 4),$$

or

$$(\theta + 4)\theta^2(\theta + 8)[(\theta^3 - 4\theta^2 + 32\theta - 256) + (\theta^3 - 4\theta^2 - 96\theta)\rho] = 0.$$

When $\theta + 8 = 0$, $D = 0$, see Art. (57); neglecting, then, this factor, the condition to be satisfied is that when from the equation

$$(\theta + 4)\theta^2[(\theta^3 - 4\theta^2 + 32\theta - 256) + (\theta^3 - 4\theta^2 - 96\theta)\rho] = 0$$

a value of θ has been deduced, the value of D corresponding thereto shall not be a positive finite quantity.

(73) Now

$$\frac{D}{J^2} = 1 - \frac{128(\theta + 6)}{\theta^2(\theta + 4)} = \frac{\theta^3 + 4\theta^2 - 128(\theta + 6)}{\theta^2(\theta + 4)} = \frac{(\theta + 8)^2(\theta - 12)}{\theta^2(\theta + 4)}.$$

If $\theta = 0$, or $\theta + 4 = 0$, since D cannot be infinite, we have $J = 0$, so that $\Lambda - \rho JD$ becomes identical with the original criterion Λ. Hence the factor

[60] The surface to be employed will be $\Lambda - \rho JD$, which call M. Λ and M (or at least their upper portions above the plane of D) may then be regarded as the two sides of a sack, of infinite dimensions, open at the top, and seamed together at the bottom, along the curved line $D = 0$, $\Lambda = 0$, and in the vertical direction along the straight line $J = 0$, $L = 0$. The surface Λ serving as a screen of separation between the two upper regions, it is clear that M will serve equally well as such screen, provided no superior facultative points lie in the interior of the sack.

$(\theta+4)\theta^2$ in the quantity just above equated to zero may be neglected, and the condition to be fulfilled by ρ is that if θ be any root of the equation

$$\frac{-\theta^3+4\theta^2-32\theta+256}{\theta^3-4\theta^2-96\theta}=\rho,$$

θ shall be between -4 and 12; this equation on making $\theta=-4\phi$, so that $1>\phi>-3$, becomes

$$-\rho=\frac{\phi^3+\phi^2+2\phi+4}{\phi^3+\phi^2-6\phi},$$

or, writing $\sigma=\frac{-1-\rho}{4}$,

$$\sigma=\frac{2\phi+1}{\phi^3+\phi^2-6\phi}=\frac{2\phi+1}{(\phi-2)\phi(\phi+3)}.$$

(74) We wish to ascertain what values of σ will be incompatible with the violation of the limits just assigned to ϕ, and accordingly we must inquire what is the range of values assumed by σ when $\phi>1$ or $\phi<-3$; any values of σ *not* included within this range will be admissible for the purpose in view.

When $\phi<-3$, σ is always positive, and proceeds continuously from ∞ to 0 as ϕ passes from $-3-\epsilon$ (ϵ being infinitesimal) to $-\infty$. Consequently σ must not be allowed to have any positive value. When $\phi=\infty$, $\sigma=0$, and when $\phi=1$, $\sigma=-\frac{3}{4}$.

Hence, if no minimum value of σ (that is, no maximum value of $-\sigma$) occurs between $\phi=1$, $\phi=\infty$, σ may have any value between 0 and $-\frac{3}{4}$; but if such a minimum value, $-M$, where $M>\frac{3}{4}$, should exist, the admissible values of σ would become more enlarged, and might be taken between 0 and $-M$.

Making then $\delta\sigma=0$, we have

$$\frac{2}{2\phi+1}=\frac{3\phi^2+2\phi-6}{\phi^3+\phi^2-6\phi},$$

or

$$4\phi^3+5\phi^2+2\phi-6=0;$$

which, substituting $1+\psi$ for ϕ, becomes

$$4\psi^3+17\psi^2+24\psi+5=0;$$

so that there can be no real root of the equation in ϕ greater than unity.

Hence the admissible values of σ are defined by the inequalities

$$0>\sigma>-\tfrac{3}{4},$$

that is, $\quad 0>-\frac{1+\rho}{4}>-\frac{3}{4},\quad$ or $0>-(1+\rho)>-3,\quad$ or $2>\rho>-1$.

(75) We have thus obtained the complete solution of the problem of assigning invariantive criteria, such that their signs (positive, negative, or zero) shall serve to fix the nature of the roots. These criteria we now see are

$$J,\ D,\ \Lambda+\mu JD,$$

where μ (the negative, it must be noticed, of ρ) is any numerical quantity intermediate between 1 and -2 [61].

(76) This important modification of the original criteria J, D, Λ I proceed to apply to the problem of obtaining the *simplest* and *most symmetrical* expression for the criteria in terms of the roots of the equation. Let a, b, c, d, e be the roots, and write

$$Z=\Sigma\{(a-b)^2(a-c)^2(b-c)^2(a-d)^4(a-e)^4(b-d)^4(b-e)^4(c-d)^4(c-e)^4\},$$

or say

$$Z=\Sigma\left\{\zeta(a,b,c)\begin{pmatrix}a&b&c\\&d&e\end{pmatrix}\right\}\ [62].$$

Then, since each letter occurs the same number of times (12) in each term, Z will be an invariant.

(77) Again, suppose any two roots to become equal, say that e becomes d, then Z reduces to the single term $\zeta(a,b,c)\begin{pmatrix}a&b&c\\&d&d\end{pmatrix}$; for any such factor as $\zeta(a,b,d)$ will be accompanied with the factor $\begin{pmatrix}a&b&d\\&c&d\end{pmatrix}$ which vanishes.

If, further, we suppose any two of the letters a, b, c to become equal, then Z disappears entirely, since on that supposition $\zeta(a,b,c)$ vanishes. Hence Z is an invariant of the twelfth order, possessing the property of vanishing when the equation to which it belongs has two pairs of equal roots. Hence Z is of the form $p\Lambda+qJD$, and it becomes of importance to ascertain the value of the ratio $\frac{q}{p}$.

To do this let us suppose $e=0$, $a=-b$, $c=-d$.

The ten terms in Z correspond to the following ten partitions:—

(1)	(2)	(3)	(4)
abc	*abd*	*acd*	*bcd*
de	*ce*	*be*	*ae*
	(5)	(6)	
	abe	*cde*	
	cd	*ab*	
(7)	(8)	(9)	(10)
ace	*bde*	*ade*	*bce*
bd	*ac*	*bc*	*ad*

[61] Strictly it has only been proved that the surface $\Lambda+\mu JD$, which passes through the line Λ, D, contains no superior facultative points except those comprised in the line $L=0$, $J=0$. It is, I think, not difficult to see from this, that, if in the "*sack*" formed between Λ and $\Lambda+\mu JD$ any such points were contained, $L=0$, $J=0$, that is the axis of D would be a double or multiple line on the surface G, which is easily disproved by examining the algebraical form of G in Art. 41, where K represents $\frac{-D+J^2}{128}$; any obscurity, however, which may be supposed to cling to this view is immaterial, as a demonstration capable of being followed *in plano* and leaving nothing to be desired in point of perspicuity, will be found in the Note appended to this Part.

[62] Agreeable to the meaning assigned to ζ and to a couple of rows of letters in my memoir on Syzygetic Relations in the *Philosophical Transactions*. [Vol. I. of this Reprint, p. 429.]

(78) The corresponding values of the terms will be

$$4a^2(a^2-c^2)^2\,16a^8c^8(a^2-c^2)^4;\ 4a^2(a^2-c^2)^2\,16a^8c^8(a^2-c^2)^4;$$
$$4c^2(a^2-c^2)^2\,16a^8c^8(a^2-c^2)^4;\ 4c^2(a^2-c^2)^2\,16a^8c^8(a^2-c^2)^4;$$
$$4a^6c^8(a^2-c^2)^8;\ 4c^6a^8(a^2-c^2)^8;\ (a-c)^2\,256a^{10}c^{10}(a+c)^8;\ (a-c)^2\,256a^{10}c^{10}(a+c)^8;$$
$$(a+c)^2\,256a^{10}c^{10}(a-c)^8;\ (a+c)^2\,256a^{10}c^{10}(a-c)^8.$$

Collecting and simplifying these terms, and observing that

$$(a-c)^2(a+c)^8+(a+c)^2(a-c)^8=(a^2-c^2)^2[(a+c)^6+(a-c)^6]$$
$$=4(a^2-c^2)^2(a^2+c^2)(a^4+14a^2c^2+c^4),$$

we find $$Z=128(a^2+c^2)a^8c^8(a^2-c^2)^6+4(a^2+c^2)a^6c^6(a^2-c^2)^8$$
$$+1024(a^2+c^2)(a^4+14a^2c^2+c^4)(a^2-c^2)^2a^{10}c^{10}.$$

Let $(a^2-c^2)^2=p$, $a^2c^2=q$, and let $Z_1=\frac{Z}{(a^2+c^2)q^3}$. Then

$$Z_1=16384pq^3+1024p^2q^2+128p^3q+4p^4$$
$$=2^{14}pq^3+2^{10}p^2q^2+2^7p^3q+2^2p^4.$$

(79) We must now calculate J, D, L:

$$D=\frac{1}{5^5}\zeta(a,-a,c,-c,0)$$
$$=\frac{1}{5^5}4a^6c^6(a^2-c^2)^4;$$

or writing $$D_1=\frac{D}{q^3},$$
$$D_1=\frac{4}{5^5}p^2.$$

Again, for J. The form to which it belongs is

$$x^5-(a^2+c^2)x^3y^2+a^2c^2xy^4,$$

or $$\left(1,0,-\frac{a^2+c^2}{10},0,\frac{a^2c^2}{5},0\between x,y\right)^5;$$

so that the coefficients of the biquadratic Emanant are

$$x;\quad -\frac{a^2+c^2}{10}y;\quad -\frac{a^2+c^2}{10}x;\quad \frac{a^2c^2}{5}y;\quad \frac{a^2c^2}{5}x.$$

Hence the quadratic covariant becomes

$$\frac{a^2c^2}{5}x^2+\frac{2}{25}(a^2+c^2)a^2c^2y^2+\frac{3}{100}(a^2+c^2)^2x^2$$
$$=\frac{20a^2c^2+3(a^2+c^2)^2}{100}x^2+\frac{2}{25}(a^2+c^2)(a^2c^2)y^2.$$

Hence, by definition, J (which $=-4\times$ Discriminant of the Quadratic Covariant)

$$=-\frac{4}{1250}(a^2c^2)(a^2+c^2)[3(a^2-c^2)^2+32a^2c^2];$$

and making $$J_1=\frac{J}{(a^2+c^2)q},$$
$$J_1=-\frac{6}{625}p-\frac{64}{625}q=-\frac{6}{5^4}p-\frac{2^6}{5^5}q.$$

Finally, to calculate L. The canonizant of the form,

$$\begin{vmatrix} 1 & 0 & A & 0 \\ 0 & A & 0 & B \\ A & 0 & B & 0 \\ y^3; & -xy^2; & x^2y; & -x^3 \end{vmatrix},$$

is $$(A^3-AB)x^3+(B^2-A^2B)xy^2,$$

of which the discriminant is

$$-4\frac{AB^3}{27}(A^2-B)^4,$$

where $$A=-\frac{a^2+c^2}{10},\quad B=\frac{a^2c^2}{5}.$$

Hence, by definition,

$$L=AB^3(A^2-B)^4=-\frac{1}{125.10^9}(a^2+b^2)(a^6b^6)[(a^2-b^2)^2-16a^2b^2];$$

and making

$$L_1=-\frac{L}{(a^2+c^2)q^3},$$
$$L_1=\frac{1}{125.10^9}(p-16q)^4=-\frac{1}{5^{12}.2^7}(p^2-16q)^4.$$

(80) Now let us write

$$\frac{1}{5^{12}}Z=\eta L+eJD^{(63)}+\epsilon J^3.$$

This gives $$\frac{1}{5^{12}}Z_1=eqJ_1D_1+\epsilon(p+4q)J_1^3+\eta L_1$$

or $$4p^4+128qp^3+1024q^2p^2+16384pq^3$$
$$=125(256p^2q^2+24p^3q)e+(p+4q)(6p+64q)^3\epsilon+\frac{1}{2^7}(p-16q)^4\eta,$$

by means of which identity we can obtain linear equations for finding the values of e, ϵ, η.

(63) Since Z has been proved to be of the form $p\Lambda+qJD$, we know *à priori* the value of $\frac{\epsilon}{\eta}$; but I have thought it safer to determine ϵ, η independently, as an additional check upon the accuracy of the computations.

Thus, equating the coefficients of p^4, q^4, p^3q respectively, we obtain

$$4 = 216\epsilon + \frac{1}{2^7}\eta,$$

$$4 \,.\, 64^3\epsilon + \frac{16^4}{2^7}\eta = 0,$$

which gives $\eta = -2^{11}\epsilon$ (as it ought to do),

$$128 = (24 \times 125)e + (4 \times 216 + 108 \times 64)\epsilon + 64 \,.\, 2^{11}\epsilon$$
$$= 3000e + 8800\epsilon.$$

Hence $200\epsilon = 4, \quad \epsilon = \frac{1}{50}, \quad \eta = -\frac{2^{10}}{25},$

$$3000e = 128 - 176 = -48, \quad e = -\frac{2}{125} \text{ and } \frac{e}{\epsilon} = -\frac{4}{5}.$$

In order to verify the value of e, let $p = -4$, $q = 1$; then, assuming the correctness of the above determinations, we ought to find

$$4^5 - 128 \,.\, 4^3 + 1024 \,.\, 16 + 16384$$
$$= 125(256 \,.\, 16 - 24 \,.\, 64) \,.\, \frac{-2}{125} + \frac{1}{128} \,.\, 160000 \,.\, -2^{11} \,.\, \frac{1}{50},$$

or $2^{10}(1 - 8 + 16 - 64) = (-32 \,.\, 256 + 48 \,.\, 64) - \frac{8}{25} \times 160000,$

or $2^{10}(-55) = -5120 - 25 \,.\, 2048 = 2^{10}(-5 - 50),$

which is right.

(81) Thus
$$-Z = \frac{5^{10}}{2}\left(2^{11}L - J^2 + \frac{4}{5}JD\right)$$
$$= \frac{5^{10}}{2}\left(\Lambda + \frac{4}{5}JD\right);$$

and accordingly we have proved that $-Z$ is of the form $(\Lambda + \frac{4}{5}JD)$; and consequently, since $\frac{4}{5}$ lies within the allowed limits 1 and -2, $-Z$ may be used to replace Λ in the system of criteria.

(82) On examining the composition of Z, it will be found to have a remarkable relation to the lower criterion J.

J we know is, to a numerical factor *près*, of the form

$$\Sigma\,[(d - e)^4 \zeta(a, b, c)],$$

ζ denoting, as usual, the squared product of the differences of the quantities which it affects; and Z, it will readily be seen, is of the form

$$\{\zeta(a, b, c, d, e)\}^2\, \Sigma \frac{1}{\zeta(a, b, c)(d - e)^4};$$

and the squared factor is always positive whatever the roots may be, for ζ is always real.

Hence the essential part of our rule thus transformed comes to this, that if

$$\Sigma\,\{\zeta(a, b, c) \times (d - e)^4\} \quad\text{and}\quad \Sigma\,\{(\zeta(a, b, c))^{-1}(d - e)^{-4}\}$$

are both of them positive, then when the discriminant is positive, so that the case of two of the five quantities a, b, c, d, e being conjugate and the other three real is excluded, and the choice lies between supposing all or only one of them real, we are able to affirm that they will all be real. Nothing could be easier than to multiply tests expressed by simple symmetric functions of differences of the roots, any infringement of which would contradict the hypothesis of all the five letters denoting real quantities; the difficulty consists in discovering a system of the least number that will suffice of decisive tests, such that not only their infringement shall contradict the hypothesis of imaginary roots, but whose fulfilment shall ensure the roots being all real. This is what has been proved to be effected by means of the invariants J, D, $\Lambda + \frac{4}{5}JD$.

In the case before us it is clear that when the roots are all real, each of the sums above written must be positive and greater than zero. That their being both positive and greater than zero is inconsistent with four of the letters a, b, c, d, e being imaginary would probably not admit of an easy direct demonstration.

Z we have seen is only a particular value of the general invariant $\Lambda + \mu JD$, which may be called M, where μ is an arbitrary constant limited to lie between 1 and -2.

(83) It may be well to notice the effect of using *as a criterion*, in conjunction with J and D, the value of M corresponding to either extreme value of μ. In such case, supposing M to become zero, it might for a moment appear doubtful to which region that point representing the family of forms is to be referred. But since the doubt can only arise when J is negative and D positive, and since by hypothesis we have $\Lambda = -\mu JD$, we see that Λ takes the sign of μ; and consequently the sign of M, when it becomes zero, is to be understood as following the sign of μ, that is, as positive when μ is 1 and negative when μ is -2.

(84) The method above given for ascertaining the nature of the roots of a quintic involves the use of only three criteria. It may be inquired how many would become needful in applying Sturm's method. In the case of a cubic equation only the last of the two Sturmian criteria comes into use; and it seems therefore desirable to ascertain whether all four of the Sturmian criteria applicable to that case are required, or whether a smaller number are sufficient. I speak of four criteria, inasmuch as the leading terms fx and $f'x$ cannot be considered as such, their signs being fixed; so that we are at liberty to consider them both positive. Suppose all six Sturmian functions to be written down, including fx (a function of x of the fifth degree) and $f'x$,

and let us characterize by the index (r, s) any succession of signs of the leading coefficients which contain r continuations and s variations, and which therefore will correspond to the case of $(r-s)$ roots.

The total number of cases to be considered are the sixteen following:

(5, 0)	+	+	+	+	+	+
(4, 1)	+	+	+	+	+	−
	+	+	+	+	−	−
	+	+	+	−	−	−
	+	+	−	−	−	−
(3, 2)	+	+	+	+	−	+
	+	+	+	−	+	+
	+	+	+	−	−	+
	+	+	−	+	+	+
	+	+	−	−	+	+
	+	+	−	−	−	+
(2, 3)	+	+	+	−	+	−
	+	+	−	+	+	−
	+	+	−	+	−	−
	+	+	−	−	+	−
(1, 4)	+	+	−	+	−	+

the successions corresponding to the indices (2, 3), (1, 4) will become impossible, as corresponding to a *negative* number of real roots. An inspection of the eleven cases corresponding to the indices (5, 0), (4, 1), (3, 2) will show that no *ternary* combination of signs in the third, fourth, and sixth columns belongs to any of the three characters (5, 0), (4, 1), (3, 2) exclusively, and consequently all four signs must be used; and therefore, if the method of Sturm is employed, four criteria are indispensable for determining effectually the character of the roots in an equation of the fifth degree [64]; whereas in the symmetrical and invariantive method which I have employed three have been seen to suffice.

[64] (*a*) For an equation of the nth degree there are $n-1$ variable criteria, each capable of being + or −, and thus giving rise to 2^{n-1} conceivable diversities of combination. The actual number possible, however, is considerably less than this; and I find by an easy method that this number, when n is odd, is $2^{n-2}+\dfrac{\Pi(n-1)}{2\left(\Pi\frac{n-1}{2}\right)^2}$, and when n is even, is $2^{n-2}+\dfrac{\Pi(n-1)}{\Pi\frac{n}{2}\,\Pi\left(\frac{n}{2}-1\right)}$.

(*b*) Not quite foreign to this subject is the inquiry as to the comparative probability of each different succession or each different family of successions possessing equivalent characters; and, as connected therewith, the comparative probability of a certain specified number of the roots of an equation of a given degree being real and the remainder imaginary. In the simplest case of a quadratic equation of which the coefficients are real but otherwise arbitrary, I find that upon the particular hypothesis of the squares of the three coefficients being limited by one and the same quantity, the probability of the roots being imaginary is $\frac{31}{72}-\frac{\log 2}{12}$, or ·3727932, a little less than $\frac{3}{8}$, this being the value of the integral $\int_0^{\frac{1}{2}} db \int_{b^2}^{1} da\left(1-\frac{b^2}{a}\right)$; but we are not at liberty to infer from this the value of the probability in question when the coefficients are left absolutely unlimited. A case in point, as illustrating the effect of imposing a limit in questions of this kind, occurs in the problem (which I raised in my lectures on Partitions) of finding the probability that four points placed at hazard in a plane will form the angles of a reentrant quadrilateral, which Professor Cayley has shown is exactly $\frac{1}{4}$ in the absence of any limit. For if $ABCD$ be the four points, and ABC the greatest of the four triangles of which they may be regarded as the angular points, and if through A, B, C be drawn lines parallel to BC, CA, AB respectively, the triangle $\alpha\beta\gamma$ so formed will be four times as great as ABC, and the point D must be somewhere within $\alpha\beta\gamma$, otherwise ABC would not be less than each of the three other triangles ABD, BCD, CAD; and consequently, since D must lie within ABC when the quadrilateral is reentrant, the probability in question is $\frac{ABC}{\alpha\beta\gamma}$, or $\frac{1}{4}$. Now it is easy to see, by using the very same construction, that if any contour whatever be imposed as a limit upon the positions of the four points, the probability referred to will exceed $\frac{1}{4}$ by a finite quantity—a result somewhat paradoxical, since *à priori* one would have supposed that the value of it for the case of *no limit* would be the *mean* of the values corresponding to the respective suppositions of every possible form of limit.

In an equation of the seventh degree the case of 0 or 4 will be distinguishable from that of 2 or 6 imaginary roots by the sign of the discriminant, and then again the case of 0 from that of 4, and of 2 from that of 6, by other invariantive criterion-systems. So for an equation of the ninth degree, the first separation will be that of the 0, 4, or 8 case from that of 2 or 6; then it may be conjectured the 2 case will be invariantively separated from the 6, and the 0 or 8 from that of 4, and, finally, 0 and 8 from each other—the reduction of cases apparently depending upon the relation of the index of the equation to the powers of the number 2. This much we know (from Art. 49) as matter of certainty, that no single criterion other than the discriminant can ever serve to distinguish one form of roots from another so that all other criteria must accompany each other in groups; and accordingly the scheme of criteria established in the foregoing investigation is in kind the very simplest *à priori* conceivable.

Note on the arbitrary constant which appears in one of the criteria for distinguishing the case of four from that of no imaginary roots, and on the curve whose coordinates express the limiting relations of all the octodecimal invariants of a binary quintic, &c.

(85) The appearance of an arbitrary constant in a criterion is a circumstance so unexampled and remarkable that I have thought it desirable to give a more complete, or at least a more palpable proof of the validity of the substitution of $\Lambda+\mu JD$ for Λ than that furnished in the foregoing text, where some indistinctness arises from the difficulty of raising up in the mind a clear conception of the form of the amphigenous surface, and the two portions of space which it separates. That difficulty is entirely obviated, and

the theory rendered palpable to the senses by the following investigation, where the problem is so handled as to involve the contemplation of two dimensions only of space. We have in general

$$D = J^2 - 128K, \quad \Lambda = 2048L - J^3,$$

and at the amphigenous surface (see Art. 57)

$$\frac{K}{J^2} = \frac{\theta + 6}{(\theta + 4)\theta^2}, \quad \frac{L}{J^3} = \frac{1}{(\theta + 4)\theta^3}.$$

Let

$$\theta = 4\phi, \quad y = \frac{D}{J^2}, \quad x = \frac{\Lambda}{J^3}.$$

Then

$$y = 1 - 128\frac{\theta + 6}{(\theta + 4)\theta^2} = 1 - \frac{8\phi + 12}{\phi^2(\phi + 1)} = \frac{(\phi + 2)^2(\phi - 3)}{\phi^2(\phi + 1)},$$

$$x = -1 + \frac{2048}{(\theta + 4)\theta^3} = -1 + \frac{8}{\phi^4 + \phi^3} = \frac{-(\phi + 2)(\phi^3 - \phi^2 + 2\phi - 4)}{\phi^3(\phi + 1)};$$

and consequently

$$\delta y = \frac{4(\phi + 2)(4\phi + 3)}{\phi^3(\phi + 1)^2}\delta\phi, \quad \delta x = -\frac{8(4\phi + 3)}{\phi^4(\phi + 1)^2}\delta\phi, \quad \frac{\delta y}{\delta x} = -\frac{\phi^2 + 2\phi}{2}:$$

x, y may be considered as the coordinates (inclined to each other at any angle) of a curve of the fourth order, whose form, so far as is essential to the object in view, I proceed to determine. It is obvious, furthermore, that this curve will be a section of the amphigenous surface made by the plane $J = 1$.

(86) This curve will be seen to consist of four branches, coming together in pairs or two cusps, so as to form two distinct horns[(65)]. For when $\phi = \infty$, or $\phi = -\frac{3}{4}$, δy, δx will each of them be zero. Hence there is a cusp at the point where $x = -1$, $y = 1$[(66)], and again at the point where

$$x = -1 + \frac{8 \times 256}{81 - 108} = -76\tfrac{23}{27}, \quad y = \frac{(\frac{5}{4})^2(-\frac{9}{4})}{(\frac{3}{4})^2\frac{1}{4}} = -25.$$

(87) When $\phi = 0$, and also when $\phi = -1$, x and y each become infinite; when $\phi = \pm\infty$, x and y each become unity.

[(65)] (*a*) Since

$$\phi^4 + \phi^3 - \frac{J^3}{256L} = 0,$$

we see at once, from Descartes's rule, that ϕ can never have more than two real values to one of $\frac{L}{J^3}$, or consequently of x, and consequently there can only be two values of y to each of x.

(*b*) When $J = 0$, the cusp of the left-hand horn and the two points of intersection of the dexter horn with the axis of L coincide at the origin; the upper branch of the latter and the lower of the former become the lower and upper parts of the axis of D, whilst the lower and upper branches of the same respectively become the left and right-hand branches of the semi-cubical parabola $27 . 2^{22}L^2 = -D^3$.

[(66)] Where this branch cuts the axis of y we have $\phi^3 - \phi^2 + 2\phi - 4 = 0$, of which the real root will be a trifle less than $\frac{3}{2}$.

As ϕ passes from $+\infty$ to 0, δy is always negative, and x always positive; so that there will be one branch of the curve (*CMP* in Figure [p. 479]) extending from $x = -1$ to $x = +\infty$, for which y commences at $y = 1$, which cuts the axis of x when $\phi = 3$, that is $x = -\frac{32}{27}$[(67)], and which, for the remaining part of its course, lies completely under the axis of x, becoming infinite when x becomes indefinitely great.

Again, as ϕ passes from $-\infty$ to -1, δx remains always positive, but δy is negative so long as $\phi < -2$ vanishes when $\phi = 2$, and ever afterwards continues positive. Thus there is a second branch, *COQ*, which starts from the cusp *C*, touches the axis of x at the origin, ever afterwards remaining positive, and increasing up to positive infinity.

Since when $\phi = \infty$, $\frac{\delta y}{\delta x} = \infty$, the tangent at *C* is parallel to the axis of y, and consequently the two branches which start from *C* lie on the same side of the tangent, so that the cusp at this point is of the second or ramphoidal kind; in Professor Cayley's nomenclature a cusp-node, and equivalent to the union of a double point and a cusp of the first kind.

There remains to account for the values of ϕ in the interval between 0 and -1. Throughout this interval y and x remain both of them negative, and $\frac{\delta y}{\delta x} = -\frac{\phi(\phi + 2)}{2}$[(68, 69)] is always positive.

[(67)] From this it is easily seen that, whatever may be supposed to be the inclination of the axes x, y, the curve in question is rectifiable by means of elliptic functions; for $\frac{ds}{d\phi}$ will be expressible as a rational function of ϕ and the square root of a quartic function of ϕ. The same conclusion will hold for the curve obtained by making J constant when J, together with any invariant of the eighth and any of the twelfth order, are taken as the coordinates of the amphigenous surface.

[(68)] To ascertain which range of ϕ gives the superior and which the inferior outline of the sinister horn, let $\phi = \epsilon$, an infinitesimal; then $\phi^4 + \phi^3 = \epsilon^3$, and the other value of ϕ is $-1 - \eta$, where $\eta = \epsilon^3$. Hence the two values of y corresponding to ϕ nearly zero and ϕ nearly -1 respectively will be

$$y_1 = -\frac{12\epsilon}{\epsilon^3} = -\frac{12}{\epsilon^2} \text{ and } y_2 = \frac{-4(-1 - \eta)}{\epsilon^3} = \frac{4}{\epsilon^3}.$$

Thus y_1 is negative for ϵ positive or negative, but y_2 is positive in the one case and negative in the other, as already seen for the dexter horn. We see also that y_2 becomes indefinitely greater than y_1, so that it is the value of ϕ near to -1 which gives the inferior branch; and consequently the superior branch of the sinister horn belongs to the range from $-\frac{3}{4}$ to 0, and the inferior to the range from $-\frac{3}{4}$ to -1.

[(69)] It may further be noticed that each horn so called is a true horn, being destitute of any point of contrary flexure, except at infinity; for otherwise we should have

$$\frac{d^2y}{dx^2} = \frac{d\phi}{dx} \cdot \frac{d\frac{dy}{dx}}{d\phi} = -\frac{d\phi}{dx}(\phi + 1) = \frac{(\phi + 1)^3\phi^4}{8(4\phi + 3)} = 0,$$

which implies $\phi = 0$ or $\phi = -1$, for each of which values of ϕ, x and y become infinite. It will be seen hereafter that it is only for the value corresponding to $\phi = 0$ that there does exist at infinity a point of inflexion.

There will thus be two branches, in each of which x and y increase simultaneously in the negative direction, coming to a cusp necessarily of the first kind at the point $x=-76\frac{23}{27}$, $y=-25$, one branch corresponding to the values of ϕ from $-\frac{3}{4}$ to 0, the other to the values of ϕ from $-\frac{3}{4}$ to -1, both of them lying completely under the axis of x, and becoming respectively infinite at the extreme values of ϕ (0 and -1).

Again,

$$2y-x+5=\frac{\phi+2}{\phi^4+\phi^3}\{(2\phi^3-2\phi^2+2\phi)+(\phi^3-\phi^2+2\phi-4)\}+5$$
$$=\frac{\phi+2}{\phi^3}(3\phi^2-6\phi-4)+5=\frac{8\phi^3-16\phi-8}{\phi^3}.$$

Hence when $\phi=-1$, for which value of ϕ x and y both become infinite, $2y-x+5=0$; hence the straight line $2y-x+5=0$, represented by AN in the diagram, will be an asymptote to the curve [70].

If now we draw the straight line $2y-x=0$, represented by OB in the figure and join OC, the curvilinear triangle OCM will be completely under OC, and the curvilinear infinite sector XOP completely under OB.

(88) What we have to prove is, that so long as μ lies between -2 and 1, so long may $\Lambda+\mu JD$ be substituted as a criterion in lieu of Λ, it being remembered that Λ only plays the part of a criterion when D is positive and J is not positive. Hence, since when $J=0$, $\Lambda+\mu JD$ and Λ coincide, we have only to show that, so long as D is positive and J is negative, $\Lambda+\mu JD$ and Λ will bear the same sign for all such values of J, D, L as constitute a facultative system, that is coordinates to a facultative point in space.

Now at any facultative point G (the function of the amphigenous surface), or say rather $G(J, K, L)>0$, or $\frac{1}{J^3}G\left(1, \frac{D}{J^2}, \frac{L}{J^3}\right)>0$, and consequently considering $\frac{D}{J^2}$, $\frac{L}{J^3}$ as the coordinates of a plane curve, the line $G\left(1, \frac{D}{J^2}, \frac{L}{J^3}\right)=0$ (the sign of J being fixed) will separate those points for which J, K, L constitute a facultative system from those in which

(70) The two points where the asymptote cuts the curve will be found by writing

$$\frac{\phi^3-2\phi-1}{\phi+1}=\phi^2-\phi-1=0,$$

which gives

$$\phi=\frac{1\pm\sqrt{5}}{2}.$$

The superior sign corresponds to a point x, y in the inferior branch of the dexter horn, and the lower sign, for which $\phi>-\frac{3}{4}$, to the superior branch of the sinister horn. It is easy to see that there can be no other asymptote; for x, y only become infinite when $\phi=-1$, or $\phi=0$; so that if $\lambda x+\mu y+\nu$ is an asymptote, it must contain $(\phi+1)^2$, or ϕ^2 as a factor. The first condition is only satisfied when $\lambda:\mu:\nu::-1:2:5$; and the latter cannot be satisfied at all.

J, K, L constitute a non-facultative one. But the curve above traced is obviously a homographic derivative of that line

$$\left(\text{for } G \text{ is the resultant of } \frac{K}{J^2}=\frac{\theta+6}{(\theta+4)\theta^2},\ \frac{L}{J^3}=\frac{I}{(\theta+4)\theta^3}\right).$$

Hence this latter curve will also separate systems of values of J, D, Λ corresponding to facultative from those corresponding to non-facultative points. Moreover when J is negative and D positive, it has been shown (see dial figure) that the values of D (in facultative systems) corresponding to finite values of J are *limited* in magnitude; hence, upon the same suppositions, facultative systems of J, D, Λ will correspond to the interior and contour of the curve we have been considering.

(89) Accordingly, since D is supposed positive, our sole concern will be with the curvilinear triangle CMO and the infinite sector QOX, and we have to show that for all points not exterior to those areas Λ and $\Lambda+\mu JD$ have the same sign; that is to say, $1+\mu\frac{JD}{\Lambda}$, or $1+\mu\frac{y}{x}$ is *positive*.

When y and x have opposite signs (as is the case in the triangle CMO), all negative values of μ, and when y and x have the same signs (as is the case in the sector XOQ), all positive values of μ obviously make $1+\mu\frac{y}{x}$ positive. But furthermore $\frac{y}{x}$, which is -1 for the line OC, is greater than -1 for all points in the triangle just named; and again, $\frac{y}{x}$, which is $\frac{1}{2}$ for OB (the parallel to the asymptote through O), will be less than $\frac{1}{2}$ for all points in the sector QOX. Thus, then, as regards points either in the triangle or in the sector, $\frac{y}{x}$ is always intermediate between -1 and $\frac{1}{2}$; so that when μ lies between 1 and -2, $1+\mu\frac{y}{x}$ will be always positive, and Λ and $\Lambda+\mu JD$ will bear the same sign, so that $\Lambda+\mu JD$ may be used to replace Λ as a criterion. Q.E.D.

(90) It is apparent from the nature of the preceding demonstration that Λ may be replaced by an invariant containing not one merely, but an infinite number of arbitrary constants (limited), provided we are indifferent to the degree which the substitute for Λ may assume. To this end we have only to draw any algebraical curve $F(x, y)=0$ passing through the origin, and with its parameter subject to such conditions of inequality as will ensure the mixtilinear triangle and sector COM, XOQ lying on opposite sides of the curve. If its degree be n, the number of parameters in F left arbitrary within limits will be $\frac{n^2+3n-2}{2}$, and $\epsilon F(\Lambda, JD)$, where ϵ means one of the

two quantities $+1$ or -1, may be used as a criterion in lieu of Λ. For instance, a common parabola with its axis coincident with that of x and passing through O will obviously serve as a screen between these figures; its equation will be $y^2 - x = 0$, and the invariant $D^2 - J\Lambda$, which is of the sixteenth degree in the coefficients, will serve together with J and D to fix the nature of the roots; so in general we may obtain invariants of any degree of the form $4i$ from twelve upwards. Thus M. Hermite, by a method not introducing the notion of continuity, has found one of the twenty-fourth degree, which he has been so obliging as to communicate to me, namely $(D_1 - 5\Delta J_2)^2 + (9D - 25\Delta^2) J_2^2$, where $D_1 = 16J_3 + 25\Delta J_2$; and D is his discriminant, which I cannot safely attempt to express in terms of x, y for want of a certain knowledge of the arithmetical relations between his Δ, J_2, J_3, D, and my own J, K, L; but were this transformation effected, the curve so represented must, *ex necessitate*, pass clear between the triangle and sector above referred to, or else the invariant in question could not be a criterion. I have ascertained without difficulty that it passes through the origin and represents one of the principal species of Newton's diverging parabolas.

(91) The curve which we have been discussing will, on reference to Plücker's *Algebraischen Curven*, p. 193, be seen to belong to his sixteenth species of curves of the fourth order having two double points; but as in reality one of these is tantamount to the union of two, it may be considered as having three, the maximum possible number of such points, and consequently comes under the operation of Clebsch's rule, given in the last Number of Crelle's *Journal*, and accordingly its coordinates have been seen to be rational functions of a single variable. The equation connecting x, y may of course be obtained by means of a simple and obvious substitution operated upon the G of Art. 41, or it may be found directly by writing

$$\frac{x+1}{8} = \xi = \frac{1}{\phi^4 + \phi^3}, \quad \frac{y-1}{4} = \eta = -\frac{2\phi + 3}{\phi^3 + \phi^2},$$

whence we obtain

$$\phi^4 + \phi^3 - \frac{1}{\xi} = 0, \qquad (1)$$

$$2\phi^2 + 3\phi + \frac{\eta}{\xi} = 0. \qquad (2)$$

Calling ϕ_1, ϕ_2 the two roots of equation (2), making

$$\left(\phi_1^4 + \phi_1^3 - \frac{1}{\xi}\right)\left(\phi_2^4 + \phi_2^3 - \frac{1}{\xi}\right) = 0,$$

and substituting the values of the symmetric functions of ϕ_1, ϕ_2 found from the same equation, we obtain without difficulty

$$\eta^4 - \xi\eta^3 - 8\xi\eta^2 + 36\xi^2\eta + 16\xi^2 - 27\xi^3 = 0$$

for the equation in question. The curve thus denoted I propose to call the Bicorn. Its figure is given below [p. 479], in which ξ, η are taken at right angles, but they may of course be supposed to be inclined at any angle whatever. If we now assume at pleasure any two new axes U, V in the plane of the Bicorn, the coordinates u, v will be always respectively proportional to two invariants of the twelfth order of the given quintic, whose particular forms will depend upon the positions of the two new axes so taken. If one of these axes, say that of u, be made coincident with the axis of ξ, v will be proportional to JD, and u to some other invariant of the twelfth degree. When this is the case, then in general v, as u travels from one end of infinity to the other, will sometimes have four, and sometimes two, or else sometimes two and sometimes no real values, as will be obvious by inspection of the figure. There is, however, one direction of the axis of v which will cause v in all cases to have two, and only two real values. This direction is that of the line joining the two cusps. At the node-cusp, for which $\phi = \infty$, $\xi = 0$, $\eta = 0$; at the other cusp, for which $\phi = -\frac{3}{4}$, $\xi = -\frac{256}{27}$, $\eta = -\frac{32}{3}$. Hence the equation of the joining line is $9\xi - 8\eta = 0$. Now

$$\frac{K}{J^2} = -\frac{\eta}{32}, \quad \frac{L}{J^3} = \frac{\xi}{256}.$$

Hence along this line $9L + JK = 0$; and consequently, if the axis of v be taken parallel to this line and passing through the origin, whilst u is proportional to $9L + JK$, v will be proportional to JD; and thus we see that for every value of $9L + JK$, which is Hermite's J_3 (see footnote [36] (*e*)) [p. 431], D at the amphigenous surface (that is when $G = 0$, and therefore when Hermite's $I = 0$) will always have two, and only two real values. This perfectly agrees with M. Hermite's conclusion [71], and in an unexpected manner confirms the correctness of the concordance established, in the footnote cited, between his J_3 and my J, K, L. Had M. Hermite employed any duodecimal invariant whatever other than J_3, a mere inspection of the Bicorn shows that a similar conclusion could not have obtained.

(92) The intersections of the curve whose equation is written in the preceding article with infinity evidently lie in the lines $\eta^3 = 0$, $\eta - \xi = 0$. This latter is the equation to a line parallel to the asymptote which touches the highest and lowest of the four branches of the curve, and corresponds to the value -1 of ϕ. Thus we see that there is a point of inflexion corresponding to the point at infinity at which the second and third branches of the Bicorn may be conceived to unite. It is easy to show that the Bicorn has no double tangent; for we have seen that

$$\frac{dy}{dx} = -\frac{\phi^2 + 2\phi}{2},$$

([71]) Lemma 3, p. 202, *Cambridge and Dublin Journal*, vol. IX.

and consequently the values of ϕ corresponding to the two supposed points of contact may be regarded as the two roots ϕ_1, ϕ_2 of the equation $\phi^2 + 2\phi + 2\lambda = 0$, and we shall have

$$-\frac{2\phi_1+3}{\phi_1^3+\phi_1^2} + \frac{2\phi_2+3}{\phi_2^3+\phi_2^2} = \lambda\left(\frac{2}{\phi_1^4+\phi_1^3} - \frac{2}{\phi_2^4+\phi_2^3}\right),$$

that is

$$-(2\phi_1+3)(\phi_2^3+\phi_2^2) + (2\phi_2+3)(\phi_1^3+\phi_1^2) = (\phi_2^4+\phi_2^3) - (\phi_1^4+\phi_1^3),$$

or $$4\lambda.(-2) + 4\lambda + 3(4-2\lambda) + 6\{-2(4-4\lambda) + (4-2\lambda)\} = 0,$$

or $$(-8+4-6+8-2)\lambda + 12 - 6 - 8 + 4 = 0,$$

that is $$-4\lambda + 2 = 0, \quad \lambda = \tfrac{1}{2}, \quad \phi^2 + 2\phi + 1 = 0,$$

and the two values of ϕ coincide, contrary to hypothesis.

It is also easy to find its class; for when $\frac{d\eta}{d\xi}$ corresponds to any point in which the curve is met by a tangent drawn from the point whose ξ, η coordinates are a, b, we have

$$\left(\frac{2\phi+3}{\phi^3+\phi^2} + b\right) + \frac{d\eta}{d\xi}\left(\frac{1}{\phi^4+\phi^3} - a\right) = 0;$$

but $$\frac{d\eta}{d\xi} = 2\frac{dy}{dx} = -(\phi^2 + 2\phi);$$

hence $$\frac{(2\phi+3)-(\phi+2)}{\phi^3+\phi^2} + (\phi^2+2\phi)a + b = 0;$$

hence $$a\phi^4 + 2a\phi^3 + b\phi^2 + 1 = 0;$$

and ϕ having four values, four tangents (real or imaginary) can be drawn to the Bicorn from every point in its plane. It is thus of the fourth order, fourth class, possesses a common cusp and a cusp-node, no double tangent, and one point of inflexion at infinity. These results accord with those given by Plücker (*Algebraischen Curven*, p. 222).

(93) The canonical form of the equation to the Bicorn is

$$(pr + q^2)^2 + pq^3 = 0,$$

as seen in Plücker, p. 193, where $p = 0$, $r = 0$, $q = 0$ will obviously be the equations to the tangent at the node-cusp, to the tangent at the common cusp, and to the line of junction of the two cusps. This leads to a remarkable transformation of the invariant G of Art. 41. Thus we may write $p = \xi$, $q = \mu(9\xi - 8\eta)$; and to find r, we must draw the tangent to the lower cusp, for which $\phi = -\frac{3}{4}$, which gives

$$\xi = -\frac{256}{27}, \quad \eta = -\frac{32}{3}, \quad \frac{d\eta}{d\xi} = -\frac{15}{16} \text{ (72)};$$

(72) I find, by a calculation which offers no difficulty, that the value of ϕ at the point where this tangent cuts the curve will be given by the equation

$$-256\phi^4 - 256\phi^3 + 288\phi^2 + 432\phi + 135 = 0;$$

and taking away the factor $(4\phi+3)^3$ which belongs to the cusp, there remains $\phi = \frac{5}{4}$, which corresponds to a point in the lower branch of the superior horn.

consequently we may write $r = \lambda(144\eta - 135\xi + 256)$, and then proceed to satisfy, by assigning suitable values to λ, μ, ν, the identity

$$\{\lambda(144\eta\xi - 135\xi^2 + 256\xi) + \mu^2(8\eta - 9\xi)^2\}^2 + \mu^3\xi(8\eta - 9\xi)^3$$
$$= \nu(\eta^4 - \eta^3\xi - 8\eta^2\xi + 36\eta\xi^2 + 16\xi^2 - 27\xi^3) = \nu . 2^{20} G.$$

On performing the necessary calculations it will be found that

$$\lambda = -\frac{1}{2^{12}}, \quad \mu = \frac{1}{2^6}, \quad \nu = \frac{1}{2^{12}}.$$

Hence we see that J^3G may be expressed under the form $(LL_1 + cJ_3^2)^2 + eLJ_3^3$, where L_1 is a new duodecimal invariant, and c, e are two known numbers; in fact

$$J^3G = \{L(18JK + 135L^2 - J^3L) + (JK + 9L)^2\}^2 + 64L(JK + 9L)^3.$$

I am indebted to my friend Dr Hirst for these references to the immortal work of Plücker.

(94) The existence has been demonstrated of a linear asymptote which is a tangent at infinity to the first and fourth branch. A cubic asymptote touches the intermediate branches in the point at infinity corresponding to $\phi = 0$. For we have

$$\xi = \frac{1}{\phi^2(1+\phi)} = \frac{1}{\phi^2}(1 - \phi + \phi^2 - \phi^3 \ldots);$$

and writing v for $-\eta$,

$$v = \frac{3 + 2\phi}{\phi^2(1+\phi)} = \frac{1}{\phi^2}(3 - \phi + \phi^2 - \phi^3 \ldots),$$

$$v^{\frac{1}{2}} = \frac{3^{\frac{1}{2}}}{\phi^3}\left(\phi^2 - \frac{1}{6}\phi^3 + \ldots\right), \quad v^{\frac{3}{2}} = \frac{3^{\frac{1}{2}}}{\phi^3}\left(3 - \frac{3}{2}\phi + \frac{13}{8}\phi^2 - \frac{27}{16}\phi^3 \ldots\right).$$

Hence we may determine A, B, C, D so that

$$Av^{\frac{3}{2}} + Bv + Cv^{\frac{1}{2}} + D - \xi \text{ shall} = \lambda\omega^n + \mu\omega^x + \ldots,$$

and I find $$A = \frac{1}{3^{\frac{1}{2}}}, \quad B = -\frac{1}{6}, \quad C = \frac{7}{72}, \quad D = -\frac{2}{9}.$$

Thus the cubic asymptote will have for its equation

$$\left(\xi + \frac{1}{6}v + \frac{2}{9}\right)^2 = 3v\left(\frac{v}{9} + \frac{7}{72}\right)^2,$$

which is a divergent cubic parabola with a conjugate point, namely the point for which

$$v = -\frac{7}{8}, \quad \xi + \frac{1}{6}v + \frac{2}{9} = 0, \quad \text{or } \eta = \frac{7}{8}, \quad \xi = -\frac{9}{128}.$$

(95) It is obvious from the preceding article, that we may expand ξ in terms of v by the descending series

$$\xi = Av^{\frac{3}{2}} + Bv + Cv^{\frac{1}{2}} + D + \frac{E}{v^{\frac{1}{2}}} + \dots.$$

But we may also obtain an ascending series for ξ in terms of v which will exhibit the nature of the curve of the cusp-node at which point $\phi = \infty$. Let $\phi = \frac{1}{\omega}$, then

$$\xi = \frac{1}{\phi^3(\phi+1)} = \frac{\omega^4}{1+\omega} = \omega^4(1 - \omega + \omega^2 - \omega^3 \dots \quad),$$

$$v = \frac{2\phi+3}{\phi^3(\phi+1)} = \omega^2\left(\frac{2+3\omega}{1+\omega}\right) = \omega^2(2 + \omega - \omega^2 + \omega^3 \dots).$$

Hence

$$v^2 = \omega^4(4 + 4\omega - 3\omega^2 + 2\omega^3 \dots \quad),$$

$$v^{\frac{5}{2}} = \omega^4(\quad 4\sqrt{(2)}\,\omega + 5\sqrt{(2)}\,\omega^2 - \frac{25}{8}\sqrt{(2)}\,\omega^3 \dots \quad),$$

$$v^3 = \omega^4(\quad 8\omega^2 + 12\omega^3 \dots \quad),$$

$$v^{\frac{7}{2}} = \omega^4(\quad \sqrt{(2)}\,\omega^3 \dots \quad),$$

&c. = &c.,

from which we may easily deduce

$$\xi = 2\left(\frac{v}{2}\right)^2 - \left(\frac{v}{2}\right)^{\frac{5}{2}} + \frac{7}{4}\left(\frac{v}{2}\right)^3 - \frac{109}{32}\left(\frac{v}{2}\right)^{\frac{7}{2}}, \text{ \&c.,}$$

in which it will be observed that the indices of the powers of v are precisely complementary to those in the preceding expansion, the two series of indices together comprising all multiples of $\frac{1}{2}$ from positive to negative infinity.

(96) We now see how, supposing the curve to be given with ξ and η at any angle, the axes corresponding to $\frac{K}{J^2}, \frac{L}{J^3}$ may be defined: namely, the origin of coordinates will be at the cusp-node; η, along which $\frac{K}{J^2}$ is reckoned, will be in the direction of the tangent at that point; and ξ, along which $\frac{L}{J^3}$ is reckoned, will be the axis of that common parabola which at the same point has the closest contact with the given curve.

It seems desirable, with a view to a more complete comprehension of the form of the amphigenous surface, that is the *limiting surface* of invariantive parameters, to ascertain the nature of the systems of plane sections of it, parallel to each of the three coordinate planes. The sections parallel to J, which are curves of the fourth order, have been already satisfactorily elucidated. It remains to consider briefly the sections parallel to J and D, which will be curves of the ninth order.

(97) When L is constant, writing $J = z$, $D = y$, where for facility of reference we may conceive y horizontal and z vertical, and making $L = \frac{k^3}{256}$, we have

$$z^3 = k^3\phi^3(\phi+1), \quad y = z^2\frac{(\phi+2)^2(\phi-3)}{\phi^2(1+\phi)} = k^2\frac{(\phi-3)(\phi+2)^2}{(1+\phi)^{\frac{1}{3}}},$$

$$\frac{\delta y}{y} = \frac{2}{3}\frac{(\phi-1)(4\phi+3)}{(\phi+2)(\phi-3)(\phi+1)}\delta\phi, \quad \frac{\delta z}{z} = \frac{1}{3}\frac{4\phi+3}{\phi(\phi+1)}\delta\phi,$$

$$\frac{\delta z}{\delta y} = \frac{1}{2k}\frac{(\phi+1)^{\frac{2}{3}}}{(\phi-1)(\phi+2)}\delta\phi,$$

when	$\phi = -1$,	$z = 0$,	$y = \infty$,
„	$\phi = -\frac{3}{4}$,	$\delta y = 0$,	$\delta z = 0$,
„	$\phi = 0$,	$z = 0$,	$y = -12k^2$,
„	$\phi = 1$,	$\frac{\delta y}{\delta z} = 0$,	
„	$\phi = +\infty$,	$z = +\infty$,	$y = +\infty$,
„	$\phi = -2$,	$y = 0$,	$\frac{\delta z}{\delta y} = \infty$,
„	$\phi = -\infty$,	$z = +\infty$,	$y = +\infty$.

Hence it appears that the curve consists of three branches, two coming together at an ordinary cusp at the point corresponding to $\phi = -\frac{3}{4}$, and the third completely separate. The nature of the sign of k does not affect the nature of the curve. If, for greater clearness, k be supposed positive, the first branch, having the negative part of the axis of y for its asymptote, lies entirely in the $-y, -z$ quadrant, and is always convex to the axis of y; the second branch, joining the first at a cusp corresponding to $\phi = -\frac{3}{4}$, is concave to the origin, cuts the axis of y negatively and of z positively, and goes off to infinity; the third branch, having the positive part of the axis of y for its asymptote, lies in the $+y, +z$ quadrant, is always convex to the axis of z, which it touches at a point below that where it is cut by the second branch, and also goes off to infinity, lying entirely under the second branch. A straight line, according to the direction in which it is drawn, may cut the curve in one, three, or five real points.

(98) When D is constant, writing $J = z$, $L = x$, we have

$$z^2 = D\frac{\phi^2(\phi+1)}{(\phi+2)^2(\phi-3)}, \quad x = \frac{Dz}{(\phi-3)\phi(\phi+2)^2}.$$

The form of the curve changes with the sign of D. For sections parallel to and above the plane of D, we may make

$$D = c^2, \quad \tau^2 = \frac{\phi+1}{\phi-3}, \quad \text{or} \quad \phi = \frac{3\tau^2+1}{\tau^2-1};$$

then the complete equation-system to the curve will be

$$z = c\tau\,\frac{3\tau^2+1}{5\tau^2-1}, \quad x = c^3\tau\,\frac{(\tau^2-1)^4}{4(5\tau^2-1)^3},$$

it being unnecessary to affect c with a double sign, since z and x change their signs with that of τ.

Also

$$\frac{\delta x}{x} = \frac{(\tau^2+1)(15\tau^2+1)\,\delta\tau}{\tau(\tau^2-1)(5\tau^2-1)}, \qquad \frac{\delta z}{z} = \frac{(\tau^2-1)(15\tau^2+1)\,\delta\tau}{\tau(3\tau^2+1)(5\tau^2-1)},$$

$$\delta x = \frac{c^3}{4}\,\frac{(\tau^2+1)(15\tau^2+1)(\tau^2-1)^3}{(5\tau^2-1)^4}\,\delta\tau, \quad \delta z = c\,\frac{(15\tau^2+1)(\tau^2-1)}{(5\tau^2-1)^2}\,\delta\tau,$$

$$\frac{\delta x}{\delta z} = \frac{c^2}{4}\,\frac{(\tau^2+1)(\tau^2-1)^2}{(5\tau^2-1)^2}.$$

To the values of τ included between $+\sqrt{(\frac{1}{5})}$ and $-\sqrt{(\frac{1}{5})}$ will correspond one branch of the curve passing through the origin, where it has a point of contrary flexure, and extending to infinity in both directions.

When $(5\tau^2-1)$ is positive $\frac{x}{\tau}$ is always positive; and when $\tau^2=1$,

$$\delta x = 0, \quad \delta z = 0, \quad \frac{\delta x}{\delta z} = 0.$$

Hence there will be a cusp of the second kind when $x=0$, $z=\pm c$, the axis of z being a tangent to the curve at each cusp. One pair of branches has its cusp at the point $x=0$, $z=c$, and the values of x, z increase indefinitely in the respective branches as τ passes from 1 to $+\infty$ and from 1 to $\sqrt{(\frac{1}{5})}$. This pair lies in the $+x$, $+z$ quadrant, and there will be a precisely similar and similarly situated pair in the $-x$, $-z$ quadrant. Thus there will be in all one infinite ∫-formed branch passing through the origin, and two detached pairs of infinite branches lying in opposite quadrants[78]. The value $\frac{1}{5}$ for τ^2, it will of course be seen, corresponds to -2 for ϕ, and gives, as it ought to do, the position of the cusp.

[78] Let ϵ be an infinitesimal, and $\theta^2=1+\epsilon$; then

$$\delta z = \frac{4}{c^3}\,\frac{(4+5\epsilon)^2}{(2\mp\epsilon)\epsilon^2}\,\delta x = \frac{32}{c^2}(1+2\epsilon)\,\frac{\delta x}{\epsilon^2}.$$

Hence at either cusp the branch the further removed from the axis of x corresponds to the values of θ^2 between 1 and ∞, and the inferior branch to its values between 1 and $\frac{1}{5}$; so that the order of continuity of the five branches of the curve may be read as follows:—from the infinite point in the higher branch of the upper pair to its cusp; thence to the infinite point in the connected branch, which is contiguous to the infinite point in the opposite extremity of the middle branch; thence along this branch to its contrary infinite extremity; thence to the infinite point in the upper branch of the inferior pair; along that branch to its cusp; and thence, finally, along the lower branch to infinity.

(99) Finally, for sections parallel to the plane of the discriminant and lying below it, making $D=-k^2$, $t^2=\frac{1+\phi}{3-\phi}$, we obtain in like manner

$$z = kt\,\frac{3t^2-1}{5t^2+1}, \quad x = k^3t\,\frac{(t^2+1)^4}{4(5t^2+1)^3}, \quad \frac{\delta x}{x} = \frac{(t^2-1)(15t^2-1)}{t(t^2+1)(5t^2+1)}\,\delta t,$$

$$\frac{\delta z}{z} = \frac{(t^2+1)(15t^2-1)}{t(3t^2-1)(5t^2+1)},$$

$$\delta x = \frac{k^3}{4}\,\frac{(t^2-1)(15t^2-1)(t^2+1)^3}{(5t^2+1)^4}, \quad \delta z = k\,\frac{(15t^2-1)(t^2+1)}{(5t^2+1)^2},$$

$$\frac{\delta x}{\delta z} = \frac{k^2}{4}\,\frac{(t^2-1)(t^2+1)^2}{(5t^2+1)^2}.$$

When $t^2=\frac{1}{15}$ there will be an ordinary cusp, and when $t^2=1$, $\frac{\delta x}{\delta z}=0$.

There will therefore be three branches,—one corresponding to the values of t between $-\sqrt{(\frac{1}{15})}$ and $+\sqrt{(\frac{1}{15})}$, the other two to values of t between these limits and $-$ and $+$ infinity respectively. The middle branch passes through the origin, where it undergoes an inflexion, and comes to a cusp at a finite distance from the origin in two opposite quadrants. The connected branch at each cusp crosses the axis of x, sweeps convexly towards the axis of z, arrives at a minimum distance from it, and then goes off to infinity.

The value $\frac{1}{15}$ for t^2 corresponds to $-\frac{3}{4}$ for ϕ, and gives, as it ought to do, the cusp-node. In fact the values $\phi=-\frac{3}{4}$, $\phi=-2$ correspond respectively to a cuspidal and to a cusp-nodal line in the limiting surface whose sections we have been considering.

When the cutting plane is that of D itself, the section becomes a double cubic parabola and a single cubical parabola crossing each other at the origin.

DESCRIPTION OF THE FIGURES [pp. 478, 479].

FIGURE I. [see p. 395 above].

The (ϵ, η) equation is $(1, \epsilon, \epsilon^2, \eta^2, \eta, 1 \between x, y)^5 = 0$, of which two roots are always imaginary; its extreme criteria are 0, 0; its middle criteria $\epsilon^4 - \epsilon\eta^2$, $\eta^4 - \eta\epsilon^2$,

$$p = \epsilon\eta - 1, \quad \sigma = (\epsilon^3 - \eta^2)(\epsilon^2 - \eta^3).$$

Points (p, σ) above the discriminatrix indicate 2 pairs of associated roots in the (ϵ, η) equation.

Points (p, σ) on the discriminatrix indicate 2 equal roots in the (ϵ, η) equation.

Points (p, σ) under the discriminatrix indicate 3 solitary roots in the (ϵ, η) equation.

Points (p, σ) above the equatrix indicate ϵ, η real and unequal.

Points (p, σ) on the equatrix indicate ϵ, η equal.

Points (p, σ) under the equatrix indicate ϵ, η imaginary and conjugate.

Points (p, σ) above the loop of the indicatrix indicate middle criteria not *both* positive.

Points (p, σ) on the loop of the indicatrix indicate middle criteria of opposite signs.

Points (p, σ) under the loop of the indicatrix indicate middle criteria not *both* negative.

The discriminatrix is a closed curve, the *whole* of which is figured, and is shaped somewhat like a harp: it has a cusp of the fourth order at the origin.

The equatrix consists of two branches coming together at a cusp at the distance 1 from the origin; the upper branch touches the horizontal axis at the origin; the lower branch, after touching the discriminant at a single point, sweeps out from and round it, cutting the vertical axis at the distance 4 below the origin. Both branches go off to infinity to the right, and lie completely under the horizontal axis. Where the lower branch touches the discriminatrix, the discriminant of the (ϵ, η) equation passes through zero without changing its sign.

The indicatrix consists of a single branch extending indefinitely in both directions. It passes from infinity below and to the left until, at the distance 1 from the origin, it touches the axis, which at the origin it crosses at an angle of 45°, after which it goes off to infinity in the positive direction. Its *loop* extends from $p = 0$ to $p = -1$. The two portions of it figured join on together, coming to a maximum at a great distance below the horizontal axis. The narrow tract marked "Region of Real parameters" is that portion of the harp-shaped space for which alone, ϵ, η being *real*, the (ϵ, η) equation can have more than one real root. The areas of each of the three regions into which the discriminatrix is divided by the equatrix and indicatrix may readily be expressed numerically in terms of algebraic and inverse circular functions only.

I am indebted to Gentleman Cadet S. L. Jacob, of the Royal Military Academy, for the tracing of the curves of which the Figure is a somewhat imperfect reproduction.

FIGURE II.

Described in text, p. 465, etc.

CONTENTS.

		ARTS.
1.	Proof (up to fifth degree inclusive) of Newton's Rule for obtaining an inferior limit to the number of real roots in an equation	1—9
2.	Theory of the equation $(1, \epsilon, \epsilon^2, \eta^2, \eta, 1 \between x, y)^5$ and conjugate equations	10—14
3.	Theory of per-rotatory and trans-rotatory circulation	15—16
4.	On an inferior limit to the number of real roots in superlinear equations	17—20
5.	On the probable value of the above limit	21—30
6.	On the reduction of the general equation of the fifth degree to its canonical form	31—44
7.	Geometrical representation of the mutual limitations of the basic invariants of Quintic forms, and of the cause of the absence of the same for Quartic forms	45—54
8.	On the invariantive criteria for determining the nature of the roots of such equation	55—74
9.	On an endoscopic representation of the above criteria	75—83
10.	Geometrical determination of the arbitrary constant (limited) of the third criterion by means of one of the principal sections of the limiting surface of invariants	84—88
11.	On the forms of the other principal sections of the same	89 to end

SUPPLEMENTAL REFERENCES.

Proposed new reduced forms for binary quartics and ternary cubics (note [11]).

Theorem on the imaginary roots of odd-degreed equations (note [26]).

Concordance between Hermite's invariants and those of the memoir (note [34]).

Identification of the latter with the corresponding numbered Tables of Professor Cayley (note [39] (*h*) (*i*)).

Proof that every invariant of a quintic is a rational integral function of the four basic invariants (note [35]).

Invariantive conditions for certain special forms of quintics (note [37]).

Conditions necessary in order that an infinitesimal variation of the coefficients of an equation may be accompanied with a change of character in the roots (note [43]).

Schläfli's theorem (proof and extension of) (note [52]).

On a number of cases capable of arising under Sturm's theorem, and on certain questions of probability (note [61]).

All the invariants of a binary form vanish when more than half the roots are equal to one another, art. 48.

Identification of section of limiting surface of invariants as a variety of the sixteenth species in Plücker's enumeration of quartic curves with two multiple points, art. 92.

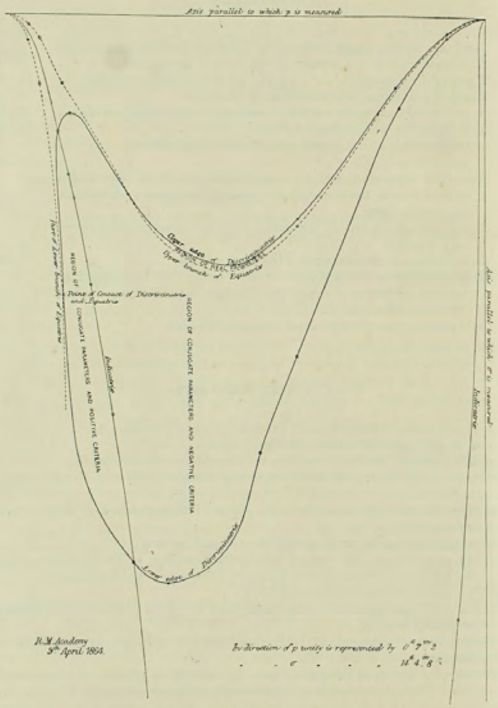


FIGURE I.

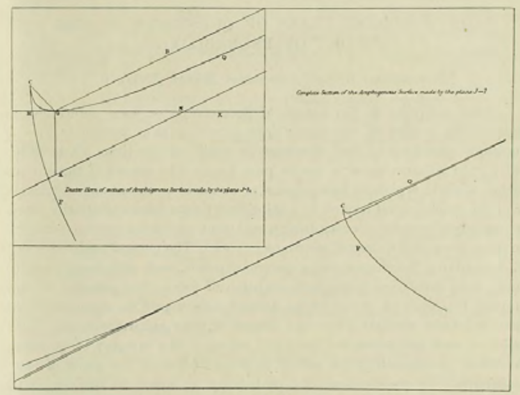


FIGURE II.

75.

ON A SPECIAL CLASS OF QUESTIONS ON THE THEORY OF PROBABILITIES.

[*Birmingham British Association Report* (1865), p. 8.]

AFTER referring to the nature of geometrical or local probability in general, the author of the paper drew attention to a particular class of questions partaking of that character in which the condition whose probability is to be ascertained is one of pure form. The chance of three points within a circle or sphere being apices of an acute or obtuse-angled triangle, or of the quadrilateral formed by joining four points, taken arbitrarily within any assigned boundary, constituting a reentrant or convex quadrilateral, will serve as types of the class of questions in view. The general problem is that of determining the chance that a system of points, each with its own specific range, shall satisfy any prescribed condition of form. For instance, we may suppose two pairs of points to be limited respectively to segments of the same indefinite straight line: the chance of their anharmonic ratio being under or over any prescribed limit will belong to this category of questions, to which, provisionally, the author proposed to attach the name of form-probability. In questions of form-probability, in which all the ranges are either collinear segments or coplanar areas, or defined portions of space, rules may be given for transforming the data, so as to make the required probability depend on one or more probabilities of a simpler kind, leading to summations of an order inferior by two degrees to those required by the methods in ordinary use. Thus Mr Woolhouse's question relating to the chance of a triangle within a circle or sphere being acute can be made to depend upon an easy simple integration, the solutions heretofore given of this problem involving complicated triple integrals. It was shown, as a further illustration, that the form-probability of a group of points all ranging over the same triangle remains unaltered when the range of one of them is limited to any side of the triangle chosen at will, and, again, (for convenience of expression distinguishing the contour into a base and two sides) will be the mean of the two probabilities resulting from limiting one

point to range over either side with uniform probability, and simultaneously therewith a second point of the group over the base, with a probability varying as its distance from that end of the base in which it is met by the side. An analogous rule can be given for transforming the form-probability of a group limited to any the same parallelogram. So again for a group of points ranging over a plane figure bounded by any curvilinear contour. The problem may be transformed by supposing two of the points of the group to range on the contour itself, according to a law which may be expressed by saying that the probability of their being found on any arc shall vary as the product of the segment included between the arc and its chord, multiplied by the time of describing the arc about any centre of force arbitrarily chosen within or upon the contour,—a theorem which, accepting the idea of negative probability, admits also of extension to the case of a centre of force exterior to the contour.

Among other problems which the author readily resolves by aid of his principle of transformation, may be mentioned that of determining the mean value of a triangle whose angles are taken at random anywhere within a given triangle, parallelogram, ellipse, or ellipsoid. In this description of questions a peculiar difficulty arises, from the fact that the figure which is to be integrated in order to determine the numerator of the fraction which gives its mean value must always be taken positive, whereas its algebraical expression will repeatedly change its sign, according to a more or less complicated law. This quality of the analytical exponent of the arithmetical value of the figure constitutes, in fact, a sort of polarization which has to be got rid of; and the depolarizing process is effected with great ease by virtue of the simplified form impressed upon the data by the method set forth in the paper.

The author further took occasion briefly to allude to the form in which his own problem of four and Mr Woolhouse's problem of three points were originally proposed, viz. in each case without a specified boundary, and to express his opinion that the principle which had been applied to them, and in which he had formerly acquiesced, was erroneous, as it could be made to lead to contradictory conclusions, and must be abandoned. He was strongly inclined to believe that, under their original form, these questions do not admit of a determinate solution.

76.

NOTE SUR LES CONDITIONS NÉCESSAIRES ET SUFFISANTES POUR DISTINGUER LE CAS QUAND TOUTES LES RACINES D'UNE ÉQUATION DU CINQUIÈME DEGRÉ SONT RÉELLES.

[*Comptes Rendus de l'Académie des Sciences*, LX. (1865), pp. 759—761.]

DANS une communication [p. 371 above] que j'ai eu l'honneur de faire précédemment à l'Académie, j'ai donné la définition de trois invariants appartenant à une forme binaire du cinquième degré, que j'ai nommés J, D, L, D étant le discriminant.

Quant à J et L, il y a une autre méthode très-nette qui suffit pour les définir.

Si on suppose la fonction donnée mise sous la forme

$$(ax+by)^5+(cx+dy)^5+(ex+fy)^5,$$

et si on écrit

$$(ad-bc)^5=A,\quad (cf-de)^5=B,\quad (eb-fa)^5=C,$$

on aura

$$J=A^2+B^2+C^2-2AB-2AC-2BC,$$
$$L=A^2B^2C^2.$$

Avec J et L on forme un nouvel invariant que j'ai nommé Λ, tel que $\Lambda=2^{11}L-J^3$.

Alors, quand D est positif, on sait que les conditions nécessaires et suffisantes pour que toutes les racines soient réelles, sont* que J et $\Lambda+\mu JD$ soient tous les deux négatifs, μ étant une quantité numérique choisie à volonté, pourvu qu'elle ne sorte pas de l'intervalle compris entre les deux limites 1 et -2.

On voit donc (chose jusqu'ici inouïe dans les recherches de cette nature) que l'un des trois *criteria* est variable entre des limites fixes.

Mais on se forme une idée beaucoup trop restreinte de la nature de cette indétermination en se bornant aux invariants (tels que Λ) du douzième degré par rapport aux coefficients de la fonction donnée pour servir ainsi comme troisième critérium.

Au lieu de $\Lambda+\mu JD$, on peut substituer une fonction rationnelle et entière quelconque de J, K, L, K étant la quantité $A^2BC+AB^2C+ABC^2$

[* Cf. footnote, p. 452.]

homogène par rapport à J, $K^{\frac{1}{2}}$, $L^{\frac{1}{3}}$, à savoir $F(J, K, L)$, pourvu que certaines conditions soient satisfaites que je vais donner. Écrivons

$$J=\theta^2-4\theta,\quad K=\theta^3+2\theta,\quad L=\theta^2,$$

alors F devient une fonction de θ, et les conditions nécessaires et suffisantes pour que F (pris avec le signe convenable) soit un bon troisième critérium (comme remplaçant de Λ) sont les suivantes, qu'en écartant toutes les racines de F, qui se répètent un nombre pair de fois, une des restantes est égale à -4, mais nulle autre ne sort des limites 0 et 12.

Ainsi, par exemple, on peut se servir (comme critérium) d'un invariant du seizième degré par rapport aux coefficients, dans lequel il entrera deux paramètres variables, et on tombe sur une question très-intéressante d'Algèbre pour trouver les conditions auxquelles ces deux paramètres doivent être assujettis pour que l'invariant soit bon comme critérium, problème qui se résout par des considérations géométriques et sur lequel je prendrai quelque autre occasion de revenir. Comme exemple de la manière de mettre à l'épreuve un critérium quelconque donné, je prendrai la fonction trouvée par M. Hermite par une méthode particulière à lui qu'il a eu la grande bonté de me communiquer.

Cette fonction, exprimée dans ma notation, est

$$18L^2-JKL-K^3;$$

en faisant les substitutions dont j'ai parlé, cette quantité devient

$$-2\theta^3(\theta^3+2\theta^2-7\theta+4)=-2(\theta+4)(\theta-1)^2\theta^3,$$

où on voit qu'il existe une racine -4 et que les autres racines d'une multiplicité impaire, c'est-à-dire celles qui appartiennent au facteur θ^3, ne sortent pas des limites 0, 12.

De même, on peut démontrer plus généralement que

$$(2L^2-K^3)+\mu(16L^2-JKL)$$

sera bon comme critérium, pourvu que $\mu>-\frac{1}{9}$.

Par exemple, en mettant $\mu=1$, on retombe sur le critérium de M. Hermite: en mettant $\mu=0$, on trouve comme critérium $2L^2-K^3$, et en mettant $\mu=\infty$, on trouve $16L^2-JKL$, équivalant au seul facteur $16L-JK$, qui à son tour peut s'exprimer sous la forme

$$\frac{\Lambda+JD}{128}\quad(\text{car } D=J^2-128K);$$

on reconnaît immédiatement que 1 étant compris, comme cas extrême, entre les limites 1 et -2, $\Lambda+JD$ et conséquemment $16L-JK$ doit être bon comme critérium.

On comprend aisément que la forme $(2L^2-K^3)+\mu(16L^2-JKL)$, avec $\mu>-\frac{1}{9}$, n'est qu'une solution particulière du problème de trouver le critérium le plus général du degré 24 dans les coefficients, lequel contiendra 5 paramètres variables, c'est-à-dire 2 moins que le nombre des compositions du nombre 6 qu'on peut effectuer avec les éléments 1, 2, 3.

77.

THÉORÈME D'ARITHMÉTIQUE.

[*Comptes Rendus de l'Académie des Sciences*, LX. (1865), pp. 1011—1012.]

SOIT $F(a, b, c, d)$ le représentant de la quantité

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd;$$

soient b, c deux quantités *positives* qui satisfont à l'équation

$$F(a, b, c, d) = 0;$$

écrivons l'équation cubique en x

$$F(a, x, c, d) = 0,$$

et soient (b, b_1) les deux racines positives de cette équation. De même écrivons

$$F(a, b_1, x, d) = 0,$$

et soient (c, c_1) ses deux racines positives ; posons semblablement

$$F(a, x, c_1, d) = 0,$$

dont (b_1, b_2) sont les deux racines positives, et ainsi de suite ; on obtiendra de cette façon deux séries infinies $b, b_1, b_2, b_3 \ldots$; $c, c_1, c_2 \ldots$.

Or je dis : 1° que si b est plus grand que b_1, chacune des deux séries sera constamment décroissante, et si au contraire b est moindre que b_1, chacune sera constamment croissante. De plus, je dis : 2° que dans ces deux cas les quantités b tendront vers $\sqrt[3]{(a^2d)}$, et les quantités c vers $\sqrt[3]{(ad^2)}$ comme limite. Nommons $\sqrt[3]{(a^2d)} - b_n = \beta_n$, $\sqrt[3]{(ad^2 - c_n)} = \gamma_n$. Je dis : 3° qu'en même temps que β_n et γ_n deviennent infiniment petits quand n est infini, les différences $\beta_n - \gamma_n$, $\beta_n - \beta_{n-1}$, $\gamma_n - \gamma_{n-1}$ deviendront infiniment petites par rapport à β_n et γ_n.

On remarquera que $F(a, b, c, d)$ est un discriminant binaire du troisième ordre. Il y a un théorème général analogue pour le discriminant binaire d'un ordre quelconque.

78.

RECTIFICATION ET DÉMONSTRATION D'UN THÉORÈME D'ARITHMÉTIQUE DONNÉ DANS LE COMPTE RENDU DU 15 MAI.

[*Comptes Rendus de l'Académie des Sciences*, LX. (1865), pp. 1121—1125.]

UNE erreur s'est glissée dans l'énoncé que j'ai eu l'honneur de donner tout récemment dans les *Comptes rendus* ; je me hâte de la corriger en ajoutant en même temps la démonstration du théorème auquel il se rapporte.

Considérons l'équation cubique

$$\phi u = au^3 + 3bu^2 + 3cu + d = 0.$$

Supposons a, b, c, d tous positifs. Il est évident que si les racines sont toutes réelles et distinctes, on peut faire varier à volonté d'une quantité infinitésimale ou b ou c, sans que les racines cessent d'être réelles. Mais quand ϕ possède deux racines égales ρ, en faisant

$$\phi u + 3\delta b \,.\, u^2 = 0,$$

pour déterminer si les racines sont ou non toutes réelles, il faut considérer l'équation

$$\phi'' \rho \,.\, \frac{(d\rho)^2}{2} + 3\delta b \,.\, \rho^2 = 0,$$

et les racines resteront réelles ou non, selon que $-\phi''\rho$ et δb auront les mêmes signes ou des signes contraires. De même, la réalité des racines de l'équation

$$\phi u + 3\delta c \,.\, u = 0$$

dépend de la circonstance que $-\phi''\rho$ et $\delta b \,.\, \rho$ aient ou non les mêmes signes, c'est-à-dire, puisque ρ est nécessairement négatif, dans le cas où ϕu possède deux racines égales, il sera toujours possible, ou en diminuant infiniment peu b ou en diminuant infiniment peu c, de conserver la réalité des trois racines. Si en diminuant b cela a lieu, il n'en sera pas de même quand on diminue c, et *vice versâ*, c'est-à-dire en diminuant une des quantités b, c, par exemple b, et en augmentant l'autre c, les racines restent réelles ; au contraire, en augmentant b et en diminuant c, deux des racines deviennent imaginaires.

J'ai supposé que deux seulement des racines de ϕ sont égales ; si toutes trois sont égales, la chose marche autrement, car dans ce cas on aura nonseulement $\phi\rho = 0$, $\phi'\rho = 0$, mais aussi $\phi''\rho = 0$, et en faisant varier en même temps b et c, on trouve

$$\phi'''\rho \cdot \frac{(\delta\rho)^3}{6} + 3\delta b \cdot \rho^2 + 3\delta c \cdot \rho = 0.$$

Donc, en faisant ou $\delta b = 0$ ou $\delta c = 0$, il serait impossible d'empêcher que deux des trois racines deviennent imaginaires. Afin de conserver la réalité de ces trois racines, il faut prendre $\delta b \cdot \rho + \delta c = \theta$, où θ est ou zéro ou une quantité infinitésimale d'un certain degré i *au moins*, par rapport à δb ou δc; alors la réalité de ces racines dépendra de la circonstance que $2\delta b \cdot \rho + \delta c$ soit du signe contraire à $\phi'''\rho$, c'est-à-dire négatif, ainsi donc δc, et conséquemment δb sera positif, et en même temps $\frac{\delta b}{\delta c}\rho + 1$ infiniment près de zéro.

Or, commençons avec l'équation $\phi u = 0$, en possession de deux racines réelles, et supposons que c'est b qu'on peut diminuer sans introduire des racines imaginaires: allons toujours en diminuant b tant que cela sera possible, c'est-à-dire jusqu'à ce que ϕu ait deux racines égales; à cet instant, on ne peut plus diminuer b, mais on peut diminuer c sans perdre de racines réelles, et le diminuer jusqu'à ce que deux des racines deviennent égales; alors il faut recommencer avec b, et ainsi de suite pour c et b tour à tour. Je dis qu'en continuant ces opérations, ni b ni c ne peut devenir zéro, car dans ce cas on sait que l'équation en u ne pourrait avoir qu'une seule racine réelle, il faut en effet se rappeler que quand b deviendrait zéro, c serait positif, et *vice versâ*. De plus, il est évident, les variations de b et c n'étant pas simultanées, qu'on ne peut pas tomber *exactement* sur le cas de trois racines réelles. Donc, en commençant avec b et c, on tombe sur une série double prolongée à l'infini bc, b_1c_1, b_2c_2, $b_3c_3, \ldots$, telle, que tous les b décroissent et tous les c décroissent, mais sans que ou b ou c dépasse jamais une certaine limite fixe pour l'une et pour l'autre. J'ai supposé que c'était b qui commençait à décroître; si b ne peut pas être diminué, on sera nécessairement en droit de commencer avec c, et on trouvera la série double

$$cb, \quad {}_1c_1b, \quad {}_2c_2b, \ldots.$$

Ainsi on voit qu'on peut toujours former deux paires de séries

$$b, b_1, b_2, b_3, \ldots, \quad c, c_1, c_2, \ldots,$$
$$c, {}_1c, {}_2c, {}_3c, \ldots, \quad b, {}_1b, {}_2b, \ldots,$$

et que l'équation

$$au^3 + 3xu^2 + 3yu + d = 0$$

aura deux racines réelles quand

$$x = b_i, \qquad y = c_i,$$

ou bien

$$x = b_{i+1}, \qquad y = c_{i+1},$$

et aussi quand $x = {}_ib$, $y = {}_ic$,

ou bien $x = {}_ib$, $y = {}_{i+1}c$.

Dans l'une des deux paires de séries les b et les c croîtront, comme il est facile de démontrer, sans limite; dans l'autre paire, il y aura une limite pour les b et une limite pour les c, vers lesquelles ces quantités tendent continuellement.

La condition que $au^3 + 3xu^2 + 3yu + d = 0$ ait deux racines égales sera

$$a^2d^2 + 4ay^3 + 4dx^3 - 3x^2y^2 - 6adxy = 0,$$

disons $F(x, y) = 0$.

En supposant cette équation satisfaite par les valeurs positives $x = b$, $y = c$, pour obtenir la première paire de séries, on écrit:

$F(x, c) = 0$, qui donnera $x = b$, $x = b_1$, b_1 étant positif,

$F(b_1, y) = 0$, qui donnera $y = c$, $y = c_1$, c_1 étant positif,

$F(x, c_1) = 0$, qui donnera $x = b_1$, $x = b_2$, b_2 étant positif,

et ainsi de suite. De cette manière, on peut trouver les séries b, b_1, $b_2, \ldots$, c, $c_1, \ldots$, et semblablement l'autre paire.

De plus, on remarquera que ces séries se développent par le moyen de la solution d'équations quadratiques, car dans les équations cubiques $F(x, A) = 0$, ou $F(B, y) = 0$, dont il est question, une des racines est toujours connue d'avance.

Il reste seulement à fixer la valeur de la limite pour chaque série décroissante, ce qui est bien facile. Car si b_n diffère infiniment peu de b_{n+1}, c'est que deux racines de $F(x, c_n)$ seront infiniment près l'une de l'autre, c'est-à-dire que le discriminant de F sera infiniment voisin de zéro.

Or le discriminant du discriminant

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd$$

par rapport à b, on le trouve facilement (à un facteur positif numérique près) égal à $(ad^2 - c^3)^3$; donc la limite de c_n, quand n devient infini, sera nécessairement $\sqrt[3]{(ad^2)}$; de même, la valeur limite de b_∞ sera $\sqrt[3]{(a^2d)}$, de sorte que, comme on aurait pu le deviner *à priori*, la fonction limite de ϕu est la forme pour laquelle toutes ces trois racines deviennent égales.

Ainsi on voit que les valeurs limites des b et des c sont indépendantes de la valeur initiale de l'une ou de l'autre. On voit aussi que par ce théorème on se trouve approcher continuellement de la racine cubique d'un nombre quelconque donné et de son carré *sans tâtonnement* et sans autre procédé que l'extraction de la racine positive d'une suite infinie d'équations *quadratiques*. Pour cela, tout ce qui est nécessaire est de commencer avec l'équation

$$\phi u = (u + \lambda)^2 \left(u + \frac{D}{\lambda^2}\right),$$

λ étant arbitraire. Cela donnera

$$a=1, \quad d=D, \quad b=2\lambda+\frac{D}{\lambda^2}, \quad c=\lambda^2+\frac{2D}{\lambda}.$$

Alors l'une ou l'autre des deux paires de séries, commençant avec les valeurs données pour b, c, aura nécessairement $\sqrt[3]{(D)}$, $\sqrt[3]{(D^2)}$ pour limites respectives.

Puisque b et c décroissent continuellement vers leurs limites respectives, on voit que le théorème suppose que quand l'équation

$$a^2d^2+4ac^3+4db^3-6abcd-3b^2c^2=0$$

est satisfaite par des valeurs positives a, b, c, d, on aura nécessairement

$$b>\sqrt[3]{(a^2d)}, \quad c>\sqrt[3]{(ad^2)}.$$

Cela se confirme très-simplement. Car en traitant cette équation comme une équation en b, puisqu'une racine positive existe, toutes les racines seront réelles; donc le discriminant par rapport à b sera négatif, c'est-à-dire $(ad^2-c^3)^3$ sera négatif; conséquemment $c^3>ad^2$, et de même on démontre que $b^3>a^2d$.

A l'aide des principes expliqués plus haut, on démontre sans difficulté qu'en supposant $\sqrt[3]{(a^2d)}$, $\sqrt[3]{(ad^2)}$ les limites de b_n et c_n, quand on écrit

$$\beta_n=\sqrt[3]{(a^2d)}-b_n, \qquad \gamma_n=\sqrt[3]{(ad^2)}-c_n,$$

$\frac{\beta_{n+1}-\beta_n}{\beta_n}$, $\frac{\gamma_{n+1}-\gamma_n}{\gamma_n}$ seront tous les deux infiniment petits quand n devient infini; et, de plus, $\frac{\sqrt[3]{(d)}\beta_n-\sqrt[3]{(a)}\gamma_n}{\beta_n \text{ ou } \gamma_n}$ sera infiniment petit sous la même supposition.

Je prends la liberté d'ajouter que le théorème ici donné ressort tout naturellement d'une étude approfondie que j'ai eu récemment occasion de faire sur les conditions que la variation d'une fonction rationnelle doit remplir pour qu'elle n'amène pas une perte de racines réelles. C'est M. Hermite qui, à ce qu'il me paraît, a été le premier à se servir du grand principe de la variation des coefficients pour l'étude de la nature des formes algébriques. En poursuivant cette théorie dans ses détails, j'ai déjà réussi avec son aide à établir le théorème de Newton pour la découverte de racines imaginaires jusqu'au septième degré inclusivement, et il est bien probable que dans un court délai on réussira (moi ou quelque autre) à établir ce grand théorème dans toute sa généralité pour les équations d'un degré quelconque.

79.

SUR LES LIMITES DU NOMBRE DES RACINES RÉELLES DES ÉQUATIONS ALGÉBRIQUES.

[*Comptes Rendus de l'Académie des Sciences*, LX. (1865), pp. 1261—1263.]

J'AI l'honneur de soumettre à l'Académie un théorème que j'ai tout récemment réussi à établir par une analyse des plus simples. On verra qu'il comprend comme cas particulier le célèbre théorème de Newton qui, donné sans preuve par son auteur, n'a pas été démontré jusqu'à ce jour, nonobstant les efforts des Maclaurin, des Waring et des Euler. Soit fx une fonction rationnelle et entière de x. Soit c_0, nc_1, $\frac{1}{2}n(n-1)c_2$, …, c_n les coefficients des puissances successives de x dans $f(x+p)$. Écrivons

$$C_0=c_0^2, \quad C_1=c_1^2-c_0c_2, \quad C_2=c_2^2-c_1c_3, \ \ldots, \ C_n=c_n^2.$$

Alors on peut dire qu'à chaque petite lettre c_r est associée une grande lettre C_r, et de même à chaque succession c_r, c_{r+1} de petites lettres est associée une succession de grandes lettres C_r, C_{r+1}. Quand ces successions forment toutes deux des permanences, c'est-à-dire quand les produits $c_r . c_{r+1}$ et $C_r . C_{r+1}$ sont tous les deux positifs, on peut dire que la succession composée $\begin{pmatrix} c_r c_{r+1} \\ C_r C_{r+1} \end{pmatrix}$ forme une double permanence; et en prenant de cette sorte toutes les successions simultanées fournies par ces deux suites, il y aura un certain nombre de ces permanences qu'on peut nommer le nombre de permanences doubles propres à p.

Or, je dis qu'en supposant p plus grand que q, la différence entre le nombre des permanences doubles propres à p et le nombre de ces permanences propres à q ne sera jamais négative, et de plus elle fournira une limite supérieure au nombre de racines réelles comprises entre p et q.

Si l'on prend p égal à zéro et q égal à $-\infty$, il est évident que le nombre de permanences doubles propre à $-\infty$ est zéro, car toutes les successions simples dans $f(-\infty)$ sont des variations.

Ainsi, en donnant aux coefficients de fx, disons $c_0, c_1, c_2, \ldots, c_n$, le nom de *suite cartésienne*, et à $C_0, C_1, C_2, \ldots, C_n$, formés de la manière décrite plus haut, celui de *suite newtonienne* appartenant à fx, on peut affirmer que le nombre des racines négatives dans une équation a pour limite supérieure le nombre des permanences doubles fournies par la combinaison de la suite cartésienne avec la suite newtonienne; et conséquemment, en changeant x en $-x$, on voit également que le nombre des racines positives de la même équation aura pour limite supérieure le nombre des successions simultanées composées d'une permanence newtonienne associée à une variation cartésienne. C'est là, en d'autres termes, le théorème complet de Newton, comme on peut le vérifier en consultant l'*Arithmétique universelle*.

On voit facilement que, pour la forme $f(x+p)$, les éléments $c_0, c_1, \ldots, c_n$, au moyen desquels on forme $C_0, C_1, \ldots, C_n$, ne sont autre chose (pris en ordre inverse) que les quantités

$$fp,\ \frac{f'p}{n},\ \frac{f''p}{n(n-1)},\ \frac{f'''p}{n(n-1)(n-2)},\ \ldots,\ \frac{f^{(n)}p}{n(n-1)\ldots1};$$

mais on n'est nullement borné à cette suite déterminée de valeurs pour les éléments. Je trouve qu'on peut prendre pour éléments un système de multiples numériques de $fp, f'p, f''p, \ldots$ dans lesquels il entre deux paramètres arbitraires, dont l'un cependant est limité par la grandeur de n. Par exemple, on peut prendre tout simplement pour les deux séries

$$\begin{array}{lllll} fp, & f'p, \ldots, & f^{(n-1)}p, & f^{(n)}p, \\ Tp, & T_1p, \ldots, & T_{n-1}p, & T_np, \end{array}$$

où T_rp signifie

$$(f^{(r)}p)^2 - f^{(r-1)}p \,.\, f^{(r+1)}p.$$

Alors le nombre de permanences double dans ces deux suites, moins le nombre semblable quand on écrit q pour p, donnera comme auparavant une limite supérieure au nombre des racines réelles de fx compris entre p et q: et l'on doit remarquer que quelquefois l'une des méthodes et quelquefois l'autre donnera la meilleure limite, excepté pour les cas de $n=2$ et $n=3$, cas où la première méthode est toujours préférable.

Ainsi l'on voit qu'on peut substituer à la règle de Fourier une règle où les fonctions qu'il emploie sont associées à des combinaisons quadratiques d'elles-mêmes, formant deux systèmes dont l'un est effectivement fixe, l'autre variable. Je n'entre pas dans les détails sur la loi de variabilité, parce que mon seul but, en faisant cette communication, est de faire connaître les principes sur lesquels repose la démonstration du théorème de Newton, démonstration qui a, depuis près deux siècles, échappé aux recherches des géomètres.

80.

THÉORÈME D'ALGÈBRE ÉLEMENTAIRE.

[*Comptes Rendus de l'Académie des Sciences*, LXI. (1865), pp. 282—283.]

Je demande la permission d'ajouter l'énoncé exact du théorème général auquel (sans le préciser) allusion a été faite dans les *Comptes rendus* de la séance du 19 juin dernier.

Désignons une quelconque des quatre combinaisons de signes

$$\begin{array}{cccc} +\,+ & +\,+ & -\,- & -\,- \\ +\,+ & -\,- & +\,+ & -\,- \end{array}$$

par le mot *double permanence*, et une quelconque des combinaisons

$$\begin{array}{cccc} +\,- & +\,- & -\,+ & -\,+ \\ +\,+ & -\,- & +\,+ & -\,- \end{array}$$

par le mot *varia permanence*.

Soit $fx=0$ une équation algébrique du degré n; ν une quantité réelle qui n'est pas comprise *en dedans* des limites $0, -n$ (bien entendu que les limites elles-mêmes ne sont pas exclues). Que f^rx représente la quantité $\frac{d^r}{dx^r}fx$; G_rx la quantité

$$(f^rx)^2 - \frac{\nu+r-1}{\nu+r} f^{r-1}x \,.\, f^{r+1}x.$$

Formons la progression simultanée

$$\left.\begin{array}{llll} fx, & f^1x, & f^2x, \ldots, & f^nx; \\ Gx, & G_1x, & G_2x, \ldots, & G_nx. \end{array}\right\} \qquad (P)$$

Alors je dis: 1° qu'en faisant x croître de λ jusqu'à μ, le nombre de doubles permanences dans (P) ne peut pas décroître, et que le nombre de *varia* permanences ne peut pas croître;

2° Que le nombre des racines réelles de fx comprises entre λ et μ ne peut excéder ni le nombre des doubles permanences gagnées, ni le nombre des *varia* permanences perdues par (P) quand x passe de λ à μ.

3° On peut ajouter que la différence entre le premier et le second ou entre le premier et le troisième de ces nombres sera toujours un nombre *pair*.

Pour retrouver le théorème de Newton donné dans le chapitre intitulé *De formâ æquationis*, dans l'*Arithmétique universelle*, en tant qu'il se rapporte à la limite du nombre des racines négatives de fx, on prend $\nu = -n$, $\lambda = -\infty$, $\mu = 0$, et on fait le compte des doubles permanences gagnées; en tant qu'il se rapporte à la limite du nombre des racines positives, on prend $\nu = -n$, $\lambda = 0$, $\mu = \infty$, et on fait le compte des *varia* permanences perdues. Ainsi on obtient une règle qui est en effet identique avec celle de Newton, savoir: qu'en écrivant

$$fx = ax^n + nbx^{n-1} + \tfrac{1}{2}n(n-1)cx^{n-2} + \ldots,$$

la progression simultanée

$$\left.\begin{matrix} a, & b, & c, \ldots, & l, \\ a^2, & b^2 - ac, & c^2 - bd, \ldots, & l^2 \end{matrix}\right\} \qquad \text{(Q)}$$

fournit, par ses doubles permanences et par ses *varia* permanences, des limites au nombre des racines négatives et positives respectivement de fx.

J'ajoute qu'en écrivant fx dans la forme beaucoup plus générale

$$ax^n + \frac{\nu}{i+1}bx^{n-1} + \frac{\nu(\nu+1)}{(i+1)(i+2)}cx^{n-2} + \frac{\nu(\nu+1)(\nu+2)}{(i+1)(i+2)(i+3)}dx^{n-3} + \ldots,$$

ou bien sous la forme

$$ax^n - \frac{\nu}{i+1}bx^{n-1} + \frac{\nu(\nu+1)}{(i+1)(i+2)}cx^{n-2} - \frac{\nu(\nu+1)(\nu+2)}{(i+1)(i+2)(i+3)}dx^{n-3} + \ldots,$$

selon que ν est positif ou négatif, alors, pourvu que i soit un entier positif et ν une quantité réelle quelconque qui n'est pas comprise *en dedans* des limites i, $-n$, la progression (Q) sert toujours à limiter, comme auparavant, le nombre total des racines négatives et positives de fx.

Comme corollaire particulier on déduit que, sous les conditions supposées, la fonction hypergéométrique*

$$x^n + \frac{\nu}{i+1}x^{n-1} + \frac{\nu(\nu+1)}{(i+1)(i+2)}x^{n-2} + \frac{\nu(\nu+1)(\nu+2)}{(i+1)(i+2)(i+3)}x^{n-3} + \ldots$$

(sauf le cas où, ν étant $-n$ et i étant 0, cette fonction devient une puissance exacte) ne peut jamais avoir plus d'une seule racine réelle.

[* Cf. p. 513.]

81.

ON NEWTON'S RULE FOR THE DISCOVERY OF IMAGINARY ROOTS OF EQUATIONS.

[*Proceedings of the Royal Society of London*, XIV. (1865), pp. 268—270.]

IN the first part of my "Trilogy of Algebraical Researches," printed in the *Philosophical Transactions*, will be found a proof of Newton's Rule for the discovery of imaginary roots carried as far as equations of the 5th degree inclusive. The method, however, therein employed offered no prospect of success as applied to equations of the higher degrees. I take this opportunity, therefore, of announcing that I have recently hit upon a more refined and subtle method and idea, by means of which the demonstration has been already extended to the 6th degree, and which lends itself with equal readiness to equations of all degrees. Ere long I trust to be able to lay before the Society a complete and universal proof of this rule—so long the wonder and opprobrium of algebraists. For the present I content myself with stating that the new method consists essentially, first, in the discerption of the question as applied to an equation of any specified degree into distinct cases, corresponding to the various combinations of signs that can be attached to the coefficients; secondly, in the application of the fecund principle of variation of constants, laid down in the third part of my "Trilogy," and, in particular, of the theorem that if a rational function of a variable undergoes a continuous variation flowing in one direction through any prescribed channel, then at the moment when it is on the point of losing real roots, not only must it possess two equal roots (a fact familiar to mathematicians as the light of day), but also its second differential, and the variation, when for the variable is substituted the value of such equal roots, must assume the same algebraical sign*. By aid of the processes afforded by this principle, which admits of an infinite variety of modes of application, according to the form imparted to the channel of variation, and constitutes in effect for the examination of algebraical forms an instrument of analysis as powerful as the

* The above is on the supposition that there is no ternary or higher group of equal roots.

microscope for objects of natural history, or the blowpipe for those of chemical research, the problem in view is resolved with a surprising degree of simplicity; so much so that, as far as I have hitherto proceeded with the inquiry, the computations, algebraical and arithmetical, which I have had occasion to employ may be contained within the compass of a single line. The new method, moreover, enjoys the prerogative of yielding a proof of the theorem in the complete form in which it came from the hands of its author (but which has been totally lost sight of by all writers, without exception, who have subsequently handled the question), namely, in combination with, and as supplemental to, the Rule of Descartes. On my mind the internal evidence is now forcible that Newton was in possession of a proof of this theorem (a point which he has left in doubt and which has often been called into question), and that, by singular good fortune, whilst I have been enabled to unriddle the secret which has baffled the efforts of mathematicians to discover during the last two centuries, I have struck into the very path which Newton himself followed to arrive at his conclusions.

Since the above note was sent in to the Society, I have completed the demonstration for the 7th degree, and in the course of the inquiry have had occasion to consider the conditions to be satisfied in order that a rational function of x, with r equal roots a, may undergo no loss of real roots for any assigned variation imparted to the function: for the theory of the 7th degree the case of three equal roots has to be considered, and the conditions in question are that the variation itself may contain the equal root a, and that its first differential coefficient may have the contrary sign to that of the third differential coefficient of the function which it varies when a is substituted for x—a theorem which is, of course, capable of extension to the case of an equation passing through a phase of any number of equal roots*.

* The above is on the supposition that one of the three equal roots remains unaffected in magnitude by the variation, whilst the other two change. If all three are to change simultaneously, infinitesimals beyond the first order and with fractional indices have to be brought into consideration; in that case, on making $x=a$, the variation need not become absolutely zero, but must contain no infinitesimal of the first order. And a further limitation becomes necessary in addition to the conditions stated in the text, in order that no loss of real roots may be incurred in consequence of the variation.

82.

ON A THEOREM CONCERNING DISCRIMINANTS.

[*Proceedings of the Royal Society of London*, XIV. (1865), pp. 336—337.]

LET $F(a, b, c, d) = a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd$, and let a, b, c, d be four quantities all greater than zero, which make this function vanish.

(1) The cubic equation in x, $F(a, x, c, d) = 0$, will have two positive roots (b, b_1); so $F(a, b_1, x, d)$ will have two such roots (c, c_1), $F(a, x, c_1, d)$ two such (b_1, b_2), $F(a, b_2, x, d)$ two such (c_1, c_2), and so on *ad infinitum*; we may thus generate the infinite series $b_1c_1b_2c_2$.......

Similarly, beginning with the equation $F(a, b, x, d)$, and proceeding as above, we shall obtain a similar series, c', b', c'', b'' ...; and combining the two together, and with the initial quantities b, c, we obtain a series proceeding to infinity in both directions ...... $b''\,c''\,b'\,c'\,b\,c\,b_1\,c_1\,b_2\,c_2$ ....

(2) The four quantities

$$\frac{\delta F}{\delta a},\ \frac{\delta F}{\delta b},\ \frac{\delta F}{\delta c},\ \frac{\delta F}{\delta d},$$

where F represents $F(a, b, c, d)$, will present one or the other of the three following successions of sign,

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ 0 & 0 & 0 & 0 \end{matrix}$$

(3) When the last is the case, that is, when the differential derivatives all vanish, the quantities b, c remain stationary in the above double infinite series; in the two other cases, the b quantities and c quantities *continually* increase in one direction and *continually* decrease in the other, the increase taking place in that direction in which we must read the successions of sign of the derivatives of F so as to begin with passing from plus to minus.

(4) To the increase of b and c there is no limit, but to the decrease of each there is a limit, namely $a^{\frac{2}{3}}\,d^{\frac{1}{3}}$ and $a^{\frac{1}{3}}\,d^{\frac{2}{3}}$ are the limits towards which the b and the c terms respectively converge.

I conclude with remarking that the above theorem is only a particular illustration, and the most simple that can be given, of a very wide theory relating to discriminants of all orders which springs as an immediate consequence from the principles involved in the theory of variation of algebraical forms referred to in the note which I had recently the honour of laying before the Society.

83.

ON LAMBERT'S THEOREM FOR ELLIPTIC MOTION.

[*Monthly Notices of the Royal Astronomical Society*, XXVI. (1865), pp. 27—29.]

THE original demonstration by Lambert of the celebrated theorem which bears his name was a geometrical one, see *Monthly Notices*, vol. XXII. p. 238, where this demonstration is reproduced by Mr Cayley. Lagrange has given no less than three distinct demonstrations of the same: one a sort of verification by aid of trigonometrical formulae, another founded on a property of integrals, and a third, perhaps the most remarkable of all, derived from the general expressions for the time in an orbit described about two centres of force varying according to the law of nature by supposing one of them to be situated in the orbit itself, and to become zero. Notwithstanding this plethora of demonstration, the following direct algebraical method of proving from the ordinary formulae for the time of a planet passing from one point to another, that, when the period is given, the time is a function only of the sum of the distances of these points from the centre of force, and of their distance from one another, may be deemed not wholly undeserving of notice.

Let ρ, ρ' be the distances of the two positions from the Sun, c their distance from one another, v, v' the true, u, u' the excentric, m, m' the mean anomalies thereunto corresponding, e the excentricity,

$$\omega = m - m',\ s = \rho + \rho',\ \Delta = \tfrac{1}{2}(s^2 - c^2):$$

then

$$\rho = 1 - e\cos u,\ \rho' = 1 - e\cos u',\ m = u - e\sin u,\ m' = u' - e\sin u',$$

$$\rho\cos v = \cos u - e, \quad \rho\sin v = \sqrt{(1-e^2)}\sin u,$$

$$\rho'\cos v' = \cos u' - e,\ \rho'\sin v' = \sqrt{(1-e^2)}\sin u',$$

$$c^2 = \rho^2 + \rho'^2 - 2\rho\rho'\cos(v' - v).$$

Writing for brevity c, c', s, s', for $\cos u$, $\cos u'$, $\sin u$, $\sin u'$, and to avoid confusion putting also for the moment $\bar{s}, \bar{c}$ in place of the original s and c, we have

$$\bar{s} = 2 - ec - ec',\ \omega = u - u' - es + es',$$

$$\Delta = \rho\rho' + \rho\rho'\cos(v' - v) = 1 + cc' + ss' - 2e(c + c') + e^2(1 + cc' - ss').$$

Let $J = \dfrac{d(\Delta, \bar{s}, \omega)}{d(e, u, u')}$; then J is the determinant

$$\begin{vmatrix} -2(c+c')+2e(1+cc'-ss'); & cs'-c's+2es-e^2(cs'+c's); & c's-cs'+2es'-e^2(cs'+c's) \\ -c-c' & ; \quad es & ; \quad es' \\ -s+s' & ; \quad 1-ec & ; \quad -1+ec' \end{vmatrix}.$$

Denoting this determinant by

$$\begin{vmatrix} A, & B, & C \\ D, & E, & F \\ G, & H, & K \end{vmatrix},$$

we find

$$(A, B, C) - 2H(D, E, F) + 2E(G, H, K) = (0, B, -B),$$

$$(A, B, C) - 2K(D, E, F) + 2F(G, H, K) = (0, -C, C),$$

so that

$$J = \begin{vmatrix} A, & B, & C \\ 0, & B, & -B \\ 0, & -C, & C \end{vmatrix} = 0.$$

Hence restoring s, c, instead of $\bar{s}$, $\bar{c}$, it appears that $d\omega$ is a linear function of ds and $d\Delta$; that is, ω is a function of s and Δ, or what is the same thing of s and c, independent of e. If then, when $e = 1$, the corresponding values of ρ, ρ', v, v', u, u' are r, r', θ, θ', ϕ, ϕ', we have $\cos\theta = -1$, $\cos\theta' = -1$, $\sin\theta = 0$, $\sin\theta' = 0$, $r - r' = c$, $r + r' = s$, whence writing

$$1 - \cos\phi = \frac{s+c}{2},\ 1 - \cos\phi' = \frac{s-c}{2},$$

we have finally $\omega = \phi - \phi' - \sin\phi + \sin\phi'$ as was to be proved.

Essentially this demonstration is of the same value as the first of Lagrange's three methods of proof above referred to, but with the difference that it leads up to and accounts beforehand for the success of the transformations therein employed.