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THE COLLECTED MATHEMATICAL PAPERS OF

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PREFATORY NOTE.

AMONG the Papers contained in this Volume are, the Author's Lecture on Geometry, delivered before the Gresham Committee (No. 2), the Author's seven lectures on the Partition of Numbers, in outline (No. 26), the long memoir on Newton's Rule (No. 74), and the Presidential Address to the Mathematical and Physical Section of the British Association at Exeter (No. 100). The papers here numbered 87, 88, 89 and 94 were republished together with the title "Nugae Mathematicae," and are found in the British Museum Catalogue under that name.

As in the first Volume, save for obvious errors of algebraical formulae, the papers are reprinted unaltered, cross references, and—in a few cases—indications of correction, being enclosed in square brackets.

A Table of Contents is prefixed, a General Index being deferred to the last Volume.

H. F. BAKER.

ST JOHN'S COLLEGE, CAMBRIDGE.
2 *March* 1908.

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1.

ON THE DOUBLE SQUARE REPRESENTATION OF PRIME AND COMPOSITE NUMBERS.

[*York British Association Report*, (1844), Part II. p. 2.]

"THE author first alluded to what had been done by the French mathematicians; and then pointed out the manner in which he thought numbers might be conceived to be composed of squares; and concluded by mentioning some of the advantages which might be expected from this mode of considering them."

2.

A PROBATIONARY LECTURE ON GEOMETRY, DELIVERED BEFORE THE GRESHAM COMMITTEE AND THE MEMBERS OF THE COMMON COUNCIL OF THE CITY OF LONDON, 4 DECEMBER, 1854.

[PRECEDED BY A DEDICATION TO THE MEMBERS OF THE GRESHAM COMMITTEE, MENTIONED BY NAME.]

PREFACE.

To account for the manifold short-comings of the annexed Lecture, it may be excusable and is indeed needful to state the circumstances under which it was written and delivered.

The Author having declared himself a Candidate for the vacant Professorship of Geometry in Gresham College, received a notice of little more than eight and forty hours, that he would be required to deliver a Probationary Lecture on Monday the 4th inst., before the Trustees on the City side of the Gresham Trust.

Matters of pressing importance happening at that moment to absorb his whole attention, he addressed a letter to the Secretary of the Trust, containing an urgent request that he might have the delay of a week for preparation; but his application having been sent too late in the day to obtain a reply, the Author deemed it his duty (not knowing how far his absence might derange the intended proceedings, of the precise nature of which he was unaware) to arm himself with a lecture of some kind, and for better or for worse, to appear to his summons at the appointed place and time. Accordingly, under the necessity of the case, the following pages were commenced and finished at a single sitting of a few hours' duration; the Author being so pressed for time that he had not even enough of it at his disposal to write out a fair copy of the manuscript.

The Lecture, with unimportant exceptions, such as the insertion of the closing paragraph (which was felt but not spoken), the occasional retrenchment of an exuberant expression, or the toning down of an over-florid passage, is printed as it was composed and delivered.

It must not be regarded as a criterion of what the Author could produce, with sufficient leisure, and the usual aids to reflection and research at his command, and still less as a specimen of the kind of lecture which he would consider adapted to a professorial course; but as the hasty outpouring of some of the thoughts lying at the threshold of the subject, and happening at the moment of composition to be most present to his mind. However, with all its imperfections on its head, the Author has deemed that he would be wanting in

proper deference to his judges, especially to such of them as were unable to attend in their places on the day of probation, were he to fail to afford them an opportunity of considering it in print. The views put forth and the opinions expressed are, at all events, the result of the Author's own reflections and of the questioning of his own mind, and not of a foray upon the works of the standard writers on the subject.

If his example in printing his Lecture should be followed by any of the numerous body of gentlemen who lectured after him, he will have much gratification in feeling that he has been instrumental in causing the public to participate in the pleasure which he derived from the many excellent discourses which were pronounced on the occasion referred to, and which he hopes to experience again, in listening to the corps of gentlemen from the country, who remain to bring up the reserve of the little army of probationers, on the field day appointed for the 11th instant. He can say unaffectedly and knows that this opinion was shared by many of his fellow-listeners, that the marked variety in the views and manner of treatment adopted by the several lecturers following one another in rapid succession on the same subject, rendered the *concours*, held under the auspices of the Committee and in the presence of the Common Council of the City of London, on the 4th instant, in Gresham Hall, one of the most interesting exhibitions of character and mental differences at which he had ever the good fortune to be present; one, he believes, of most uncommon occurrence, if not altogether unprecedented and unique in this country.

The free and obviously improvised review of his opinions, to which each lecturer was subjected in turn at the hands of those who came after him, threw additional life and spirit into the scene. With scarcely an exception, these light arrows of criticism were untipped with venom and passed sportively to their destination, striking without wounding, or glanced harmlessly off from the impervious shield of good humour interposed to receive them. The usual right of reply under attack must, it is presumed, in this instance be reserved until a fresh vacancy occurs and the same parties re-assemble in the college*.

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* The Author will only so far forestall the arrival of the period (*quod longum absit!*) above alluded to, by protesting against the abuse of the word "practical" as employed by an ingenious lecturer who succeeded him at the desk.

To discourse fluently on things of practice is no sufficient evidence in itself of the possession of a practical mind. The first rule of practice is to do all things at the right time and in their proper places, to proportion the means to the ends and the ends to the means, above all to know what is possible, and to confine one's endeavours within the limits of the feasible.

The Author allows and has habitually acted on the principle that for the purpose of *illustrating* lectures on geometry or any other abstract science, the lecturer should lay his hands on the plough, the loom, the forge, the workshop, the mine, the sea, the stars, all things on earth or under heaven, which may help to arouse the attention or interest the imagination of his auditors. But to profess to make the mere applications of a science such as geometry, the staple of the matter to be taught within the walls of the college by the Gresham Lecturer, to undertake to comprise within a course of geometrical lectures systematic *instruction* in mechanics, astronomy and navigation, descriptive geometry, engineering and drawing, the method of interpolation, the theory of toothed wheels, the two kinds of perspective, machinery, mapping, the art of ship-building, rules for cutting the best form of screws, and for enabling the citizens of London to qualify themselves for being their own land-surveyors, is a suggestion which, with all due deference to its propounder, the author regards as one of the wildest and most visionary which ever entered into the mind or issued from the lips of a practical man.

A long life would not suffice to exhaust the circle of the applications of geometry. Sir Thomas Gresham, a true philosopher and man of practical wisdom, ordained that courses of lectures should be delivered on his Foundation, not upon the applications of the sciences but upon the sciences themselves; well knowing that he who has mastered the principles of a science will be capable of making for himself, whenever required, those specific applications of them, which the

The Author fully agrees with the sentiment expressed by one of the candidates, and emphatically assented to by several members of the Committee, a sentiment which he confidently anticipates will be found to actuate the whole honourable body of the electors, in the decision about to be pronounced by them, in the double and each so sacred character, of Trustees and Judges, which is, that without regard to any other claim than that of capacity and desert, the worthiest candidate may be the one preferred. With his whole heart he is ready to exclaim,

Detur Digniori.

LECTURE.

Mr Chairman and Gentlemen of the Gresham Committee*,

In compliance with your requisition, although upon a very brief and unexpected notice, I have felt it my duty not to shrink from appearing before you this day to deliver a lecture upon the science, of which the chair now stands vacant in Gresham College. A consciousness of want of sufficient opportunity for preparation on my part, and a consideration of the wearisome and laborious duty which you, Gentlemen, have undertaken to perform in listening to a succession of lecturers on the same subject, conspire to impress upon me the importance of condensation and brevity.

I do not propose to tax your patience by entering upon any elaborate discussion of the principles of geometry, its history, its methods, or its applications.

It would be a vain endeavour to seek to convey within the limits of a single lecture any adequate notion of the scope of a science which has engaged the attention and grown up amidst the accumulated labours and meditations of the greatest minds and most profound thinkers of ancient and modern times, which, for the last two or three thousand years, has pursued an almost unbroken course of development and progression, and, still flourishing in all the vigour and freshness of early youth, bids fair to furnish occupation to the reasoning and inventive faculties of mankind for ages yet to come.

I conceive that the purpose for which I have been summoned before you will be best attained and our time most profitably employed, if I confine myself to the suggestion of a

peculiar circumstances of his calling or his opportunities in life may render advisable, and that he who has surprised the citadel will have no difficulty in carrying the outworks.

In writing with reluctance and under a sense of duty, the above remarks, the author begs most distinctly to disclaim the intention of leading it to be inferred that he considers the statement of opposite views as constituting the slightest ground of disqualification in the candidate who gave expression to them. He cannot but surmise that they were hazarded under the exigency of the occasion and in the absence of sufficient time for mature reflection. It has appeared to him however to be not the less necessary on that account, seeing that they were put forth with considerable plausibility, and chime in with some confused notions of what is really useful and practical, which are too prevalent at the present day, and would if carried out be fatal to the cause of sound education, to express his own dissent from them, and to meet them with an immediate refutation. He would be doing pain to his feelings were he not to add that he entertains a high respect for the abilities of the gentleman whose opinions he has felt himself under the necessity of controverting, and that he considers him to occupy a place amongst the foremost rank of those whose election by the Committee would be received with satisfaction by the Public.

* In the place of the Lord Mayor, who was unavoidably absent, Mr Deputy Holt presided.

few general ideas, which will admit of being rendered easily intelligible to any sensible man who, without having made geometry his special study, may be disposed to form some conception of the nature of its subject-matter and the relation which it bears to the other mathematical sciences.

There are three ruling ideas, three so to say, spheres of thought, which pervade the whole body of mathematical science, to some one or other of which, or to two or all of them combined, every mathematical truth admits of being referred; these are the three cardinal notions, of Number, Space and Order*.

Arithmetic has for its object the properties of number in the abstract. In algebra, viewed as a science of operations, order is the predominating idea. The business of geometry is with the evolution of the properties and relations of Space, or of bodies viewed as existing in space; it is true that the ideas of quantity and order enter largely into the developments of the science, but its proper purpose and foundation, that which confers upon it its distinctive name and character, is the contemplation of the properties of space or of the relations of the parts of space to one another, and only through this, or the equivalent notion of infinite extension, can any approach be made to a just appreciation of its objects.

It is the province of the metaphysician to inquire into the nature of space as it exists in itself, or with relation to the human mind. The less aspiring but more satisfactory business of the geometer is to deal with space as an objective reality, and to view it in its relation to matter, and as the substratum or the condition necessary to the existence of our conception of form.

The first property which strikes the mind in dwelling upon the idea of space is its infinitude, its capacity of boundless extension. If we stretch our thoughts to the very verge of the universe we are still unable to conceive space as come to an end and are constrained to admit the existence of further space beyond.

We may next contemplate space with reference to its modes of extension. We frequently hear of space having three dimensions; that there exist subordinate forms of space in which one or more of these dimensions are wanting; we are all familiar with such forms of speech, with the ideas attaching to the terms solidity, surface, linear magnitude or direction; let us inquire how the notions which they convey may be conceived to arise.

If we imagine a solid figure indefinitely expanded and extended in all directions, we fall back upon the idea of infinite space. If this space be conceived to be subjected to an ideal separation into two parts, distinct but contiguous, the boundary of each such part will give rise to the notion of a superficies, which may be conceived as co-extensive with the space from which it is derived, and like it, infinite in extent. A continuation of this process, that is to say, the dichotomy of superficies in its turn into distinct contiguous parts, gives birth to the notion of an infinite line; a surface limits space, a line limits a surface, and is thus a limitation upon a limitation; now again, conceive a line to separate into two parts and we arrive at the notion of a point, the lowest term in the scale of geometrical being; for here our analysis comes to an end; we have arrived at a limitation of the third order and can go no further; the point admits of no division. Thus it is we become able to attach a distinct meaning to the well known axiom or definition of the old geometers,

* The subject-matter of the notion of abstract order or arrangement is undoubtedly time; thus number, space and time may be said to be the three mathematical categories giving birth to three pure mathematical theories, viz.: arithmetic, the most abstract of all, next tactic or the doctrine of aggregation, and finally geometry, or topic. Each of these again has a double aspect and admits of being pursued in a descriptive and in a quantitative direction.

that "a point is that which hath no parts." Three several times, as we have seen, the process of division, of dual separation may be applied; but we come at last to that which has no parts, and which consequently, or rather by the very force of the term, is insusceptible of further division.

To say that every solid has length, breadth and depth, and therefore that space has three dimensions, is to convey a very inadequate explanation of the fact to be accounted for; for if we consider a spherical or any other body bounded by a continuous surface without angles or edges, we see nothing to indicate the existence in it of three or any other specific number of directions of measurement; it is true that through the idea of quantity, we may compare any solid whatever with a cube of equal magnitude, which of course will possess three definite directions or dimensions of linear admeasurement; but this is at best a very indirect and imperfect mode of arriving at the notion of the property in question, the property of three-foldness, inherent as a quality in the conception of space under the most general and absolute point of view.

Having thus acquired a notion of surfaces and lines in general, it becomes important to limit our attention in the first instance to the study of the simplest forms of each, and here our intuitions evolved by the latent force of early and unremitting observation, experience and induction, present our minds with the plane and sphere, as the elementary forms of surface, and the right line and circle as the corresponding simplest forms of lines.

A plane surface should be always conceived for the purposes of the geometer as extending out indefinitely in all directions; it consists of parts capable of exact superposition each over every other, so that if two portions of a plane be supposed to be brought together they cannot contain a closed hollow between them. A plane may be folded down over itself, and if two planes coincide in three points, they must coincide throughout their whole infinite extent.

Different as a sphere and a plane surface may at first sight appear to be, they possess many properties in common; the most striking difference between them, but which turns out to be comparatively unimportant under a mathematical point of view, consists in the circumstance of the plane being free and unlimited in extent, whereas a sphere is a closed and bounded surface; but in the property of the parts of either being similar *inter se*, and capable of superposition upon one another, there is a perfect resemblance between the sphere and the plane. Nor is it at all necessary to consider the sphere as a result of the idea of the circle, or to define it as Euclid does, as produced by the revolution of a circle about its diameter; we may even form a complete notion of a sphere by regarding it as a simple whole without any express reference to a centre or radii, as a surface containing a solid figure and capable of moving in its own place, without encroaching upon the neighbouring parts of space exterior to itself.

As the plane and sphere are the simplest of all surfaces, so the right line and circle are the simplest of all lines. The right line and circle, like the plane and sphere, are each moveable in their own place, that is, they admit of their parts being shifted upon one another without any absolute change of place in the entire line. There is only one other line in nature, namely the screw line (well known as the helix or Archimedes' screw), which possesses this property of self-similarity, which is the final reason why all the simple mechanical powers exhibit only three sorts of motion, namely, rectilinear, circular, and helical; thus in the lever and toothed wheels circular motion is exemplified; in the pulley and inclined plane, rectilinear motion; and finally helical motion in the screw, such as is used in an ordinary press. I need hardly add that it is the screw of Archimedes which has lent a new power to steam navigation, and which imparts to the rifled barrel its sure and deadly aim.

The foundation of the ancient (indeed it may be said of all) geometry is laid in the contemplation of the properties of figures, capable of being drawn upon a plane, and especially of the simplest of these, namely, the right line and circle.

From the time of Thales, who flourished about 600 years before the birth of Christ, and is reputed to have been the first to bring geometry from the land of the Pharaohs to find a more genial home in Greece, down to the time of Plato, two centuries later, the attention of geometers appears to have been almost exclusively confined to the study of the properties of these simple species of form, and as derived from them, of the sphere and solid figures bounded by plane faces.

The discovery of the conic sections, attributed to Plato, first threw open the higher species of form to the contemplation of geometers*. But for this discovery, which was probably regarded in Plato's time and long after him, as the unprofitable amusement of a speculative brain, the whole course of practical philosophy at the present day, of the science of astronomy, of the theory of projectiles, of the art of navigation, might have run in a different channel; and the greatest discovery that has ever been made in the history of the world, the law of universal gravitation, with its innumerable direct and indirect consequences and applications to every department of human research and industry, might never to this hour have been elicited.

This law, as you are aware, is deduced from the motions of the heavenly bodies in their orbits; no correct system of physical astronomy, no knowledge of the forces binding together the distant parts of the universe was possible, until the form of their orbits had been correctly ascertained by observation.

It is to Kepler, Newton's precursor, that we are indebted for this important information. He it was who discovered that the motion of a planet is not circular nor derived from any combination of circular movements, as was previously supposed to be the case, from a perfection idly supposed to be inherent in that figure, which rendered it alone worthy to image the movements of the heavenly bodies. Kepler discovered that the true form of a planet's orbit is that of an oval perspective projection of a circle, familiar to the geometricians of the Platonic school under the name of an elliptic section of the cone; such also is the general form of the orbits of the moon, of the satellites to the other planets, and in a word of all the bodies in nature revolving about centres of force, subject only to deviations of more or less consequence, arising from disturbing forces for which geometry is perfectly able to account. Thus (as I have said) it was, that the way was laid open to the discovery of this great secret of nature. Little could Plato himself have imagined, when, indulging his instinctive love of the true and beautiful for their own sakes, he entered upon these refined speculations and revelled in a world of his own creation, that he was writing the grammar of the language in which it would be demonstrated in after ages that the pages of the universe are written.

As Plato and Pythagoras before him, the two greatest philosophers of ancient times, have stamped their names upon, and indissolubly associated their memories with the history of the geometry of their period, so the new geometry which has arisen in later days, and achieved still higher triumphs than its elder-born sister, may be said to have taken its origin in the methods invented by Descartes and Pascal, the great philosophical luminaries of modern times. It may be doubted whether Newton could have ever risen to the heights which he attained had not Descartes lived and written before him, and it may be difficult to pronounce the existence of which of the two, Kepler or Descartes, ought to be considered as the more essential link in the order of events prepared by

* Here the Lecturer with the aid of a model showed how the different species of plane conics may be obtained from the dissection of a solid cone.

providence, to furnish the materials to be elaborated by the genius of Newton and to fit it for its lofty appointed work.

But I must not allow myself to be tempted into the facile and seductive path of historical investigation or comparison, which would carry me far beyond the limits which I prescribed to myself at the outset of this discourse, or than I can hope to carry your indulgent attention along with me. For obvious reasons also I think it would be inexpedient to attempt in this place a description of the difference between the spirit and methods of the ancient and those of the modern schools of geometry. I shall prefer to occupy the short remaining period of the lecture with inviting your attention to a distinction which lies deeper in the subject-matter of the science itself, being drawn from its relation to the two leading attributes of space, namely, magnitude and direction. I allude to a distinction, which or the like to which, runs through every branch of mathematical speculation, and has its analogue even in the study of the natural sciences, such as chemistry, botany and anatomy.

When we have attained a certain elevation in our view of the subject, and can look down upon the territory which we have traversed to arrive there, we begin to perceive that geometry resolves itself naturally into two great divisions, geometry of position and geometry of measurement, geometry descriptive* or morphological and geometry quantitative or metrical. The ancients chiefly concerned themselves with the metrical properties of space; the more subtle and essential spirit of the science, however, probably resides in the purely descriptive part. A single proposition selected from each may serve to place the distinction between these two provinces of inquiry in a clearer light.

If we draw any two triangles upon the same base, say for instance along this floor where the wall meets it, terminating respectively in two points, (so chosen that their line of junction shall be parallel to the base line) as for instance to two points in the line running along the cornice of the room, it is easily proved that the two triangles so formed, will be of equal superficial magnitude; this would be true although the apex of one of them were taken anywhere along the actual line of the ceiling, but the other in a prolongation of the cornice stretching out a hundred miles away. Both triangles so obtained would contain the same number of square inches or square feet, although the measure of one round its periphery might be a thousand times greater than that of the other. This is an example of a metrical or quantitative proposition. Again, if we take a triangle and bisect each side and join each bisecting point with the opposite angle, it is a known property of the triangle that these three lines must meet one another, not as three lines taken at hazard would do, cutting out another triangle between them, but in one and the same point. This proposition is partly metrical and partly descriptive; it is descriptive so far as regards the property of the bisecting lines passing through the same point; quantitative in so far as the idea of a line being bisected implies a notion of the relative magnitudes of the equal parts.

Propositions however exist which are purely descriptive; as for instance, the celebrated theorem of Pascal known under the name of the Mystic Hexagram, which is, that if you take two straight lines in a plane, and draw at random other straight lines traversing in a zigzag fashion between them, from A in the first to B in the second, from B in the second to C in the first, from C in the first to D in the second, from D in the second to E in the first, from E in the first to F in the second and finally from F in the second back again to A the starting point in the first, so as to obtain $ABCDEF$ a twisted hexagon, or sort of cat's-cradle figure and if you arrange the six lines so drawn symmetrically in three couples: viz. the 1st and 4th in one couple, the 2nd and 5th in a second couple, the 3rd

* The word "descriptive" is here employed *out* of its technical sense.

and 6th in a third couple; then (no matter how the points ACE have been selected upon one of the given lines, and BDF upon the other) the three points through which these three couples of lines respectively pass, or to which they converge (as the case may be) will lie all in one and the same straight line.

This is a purely descriptive proposition, it refers solely to position, and neither invokes nor involves the idea of magnitude. The existence, I will not say of a class, but of a whole world of truths of this kind, truths undeniably geometrical in their nature, serves to show how imperfect is the definition once generally accepted of geometry (however conformable to the etymology of the word and the early history of the subject), which described it as the science of the measurement of magnitude, in a word as a science of mensuration, which is in fact only one and that a subordinate division of the science. Sciences, true sciences, spring from celestial seeds sown in a mortal soil, they outgrow the restrictions which human shortsightedness seeks to impose upon them, and spread themselves outwards and upwards to the heavens from whence they derive their birth. We may write learnedly upon the history of geometry, upon its origin, growth, and apparent tendencies; but there is that within it which baffles our predictions and sets at nought our calculations as to the uses to which it may hereafter be turned and the form which it may be finally destined to assume; that which, analogous to the vital principle in an organized being, resists the circumscription of language and defies mere verbal definition.

It has been said that to appreciate what virtue and morals mean, men must live virtuous and moral lives. It is equally true, that a knowledge of the objects of science is not to be attained by any scheme of definitions however carefully contrived. He who would know what geometry is, must venture boldly into its depths and learn to think and feel as a geometer. I believe that it is impossible to do this, to study geometry as it admits of being studied and am conscious it can be taught, without finding the reasoning invigorated, the invention quickened, the sentiment of the orderly and beautiful awakened and enhanced, and reverence for truth, the foundation of all integrity of character, converted into a fixed principle of the mental and moral constitution, according to the old and expressive adage "*abeunt studia in mores.*"

I have now only to thank you, Mr Chairman and Gentlemen, for the patient attention which you have accorded to me, and to assure you with perfect sincerity, that if I should have the honour of being selected by you for the permanent occupation of the chair which I this day fill upon trial, I shall not treat the appointment as a sinecure, nor content myself with discharging the mere duties of routine. Far otherwise! if accredited by you to teach publicly a science, the object of my passionate fondness and earliest predilection, to propagate a taste for which would be to me, not merely a labour of duty but of love, I should strive, both in and out of the lecture room, to respond to the intentions of your enlightened and munificent Founder, by imparting freely to all who might approach me for the purpose, advice, encouragement, and sympathy, in their pursuit of mathematical truth, and I should labour with unceasing diligence to evince myself a worthy successor of the many eminent men, who have previously occupied here the chair which it is my ambition to obtain.

As one who has given pledges to the world of an earnest devotion to science, who lays claim to the possession of faculties which would find or create here a fitting theatre for their development, I appeal to your public spirit. I seek, Gentlemen, at your hands to be placed in a position which shall entitle me to take a part in bringing this noble Institution into connection with the great movement of national education now in progress throughout the land, and as a professor in this place, to be permitted to dedicate the past and future labours of my life to the promotion of the general good. The privilege to be useful is the crown of honour which I covet, and which it is in your power to bestow.

3.

NOTE ON SIR JOHN WILSON'S THEOREM.

[*Cambridge and Dublin Mathematical Journal*, IX. (1854), pp. 84, 85.]

THE following is probably the best and the briefest mode of deducing Sir John Wilson's Theorem and its cognate Theorems from Fermat's. I can say nothing as to its originality.

p being any prime number, let

$$(x-1)(x-2)(x-3)\ldots\{x-(p-1)\} = x^{p-1} + A_1x^{p-2} + A_2x^{p-3} + \&c. + A_{p-1}.$$

Let x successively take the values $1, 2, 3, \ldots (p-1)$; then to modulus p, by Fermat's Theorem, we have

$$x^{p-1} + A_{p-1} \equiv 1 + A_{p-1}, \text{ say } A_0,$$

and we derive the $(p-1)$ congruences to modulus p:

$$A_0 + A_1 + A_2 + A_3 \ldots\ldots\ldots\ldots + A_{p-2} \equiv 0,$$
$$A_0 + 2^{p-2}A_1 + 2^{p-3}A_2 + 2^{p-4}A_3 \ldots + 2A_{p-2} \equiv 0,$$
$$A_0 + 3^{p-2}A_1 + 3^{p-3}A_2 + 3^{p-4}A_3 \ldots + 3A_{p-2} \equiv 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_0 + (p-1)^{p-2}A_1 + (p-1)^{p-3}A_2 + (p-1)^{p-4}A_3 \ldots + (p-1)A_{p-2} \equiv 0.$$

Now the determinant formed by the coefficients of

$$A_0, A_1, A_2, \ldots A_{p-2}$$

is $1.2.3\ldots(p-1)$ multiplied into the product of the differences of $1, 2, 3, \ldots (p-1)$, and is therefore incongruent to zero for the modulus p. Hence, there being $(p-1)$ *independent* homogeneous congruences between $(p-1)$ quantities, each of these quantities must be congruent to zero, that is

$$A_0 \equiv 0, \; A_1 \equiv 0, \ldots A_{p-2} \equiv 0 \text{ [mod. } p].$$

The congruence $A_0 \equiv 0$, that is $1 + 1.2.3\ldots(p-1) \equiv 0$ [mod. p], is evidently Sir John Wilson's Theorem. We see also (by virtue of the remaining equations) at the same time, that the sums of the binary, ternary, &c., up to the $(p-2)^{\text{ary}}$ combinations of the numbers $1, 2, 3, \ldots (p-1)$, are all severally congruent to zero to the modulus p; that is, are all divisible by that number.

4.

ON THE CALCULUS OF FORMS, OTHERWISE THE THEORY OF INVARIANTS.

[*Continued from* p. 422 *of Volume I. of this Reprint.*]

[*Cambridge and Dublin Mathematical Journal*, IX. (1854), pp. 85—103.]

SECTION VII. (*Continued.*)

BEFORE proceeding further I must guard against a misconception as to my meaning to which the modification of the title of this memoir might give birth; it is not to be understood that I regard the Theory of Invariants as coextensive with the Calculus of Forms, but only with a certain portion of that Calculus which is here exclusively treated of; the Calculus of Forms itself has for its subject-matter the whole theory of the Composition, Decomposition, and Comparison of Forms. In the theory of invariants the composition of single forms with sets of linear forms is alone considered, and the idea of invariance must be regarded as a transient idea arising out of an artificial mode of viewing the effects of composition, so as to ignore the presence in the result of factors which depend on the resultants of the linear forms employed, which resultants, although in this portion of the subject treated as mere moduli and as such generally supposed to be reduced to unity, yet in regard to the general theory are as important as the factors which are retained as the sole objects of contemplation; so that in fact the idea of invariance is but a special and it may be said accidental notion which merges in the more general notion of permanency of character in the resultant of forms compounded in a given manner out of given forms. Again, as to combinants, the idea contained in this word may, by a change in the mode of statement of the definition, be extended to functions of unlike degrees. A combinant of $U, V, W, \ldots$, all functions of the same system or systems of variables, is in fact only another name for invariants of the function $\lambda U + \mu V + \nu W + \&c.$, where, over and above the sets of variables contained in $U, V, W, \ldots$ there is a new correlated set of variables λ, μ, ν, &c. So now, more generally, if $U, V, W, \ldots$ are of $p, q, r, \ldots$ dimensions in one set of variables, of which the highest number is I, if λ is taken of $I-p$, μ of $I-q$, ν of $I-r$, &c. dimensions in the same, the functions λ, μ, ν, &c. being each the most general of their kind, any invariant of $\lambda U + \mu V + \nu W + \ldots$ which is such as well in respect to the coefficients in $\lambda, \mu, \nu, \ldots$ which must be considered as forming a set among themselves, as also in respect to the set of variables in $U, V, W, \ldots$ will be a combinant to the system $U, V, W, \ldots$; and so, more generally, if $U, V, W, \ldots$ contain several (say i) unrelated sets or systems of sets of variables, we must form in an analogous manner

$$\lambda_1\lambda_2 \ldots \lambda_i U + \mu_1\mu_2 \ldots \mu_i V + \nu_1\nu_2 \ldots \nu_i W + \&c.,$$

and then an invariant in respect to the i given sets in $U, V, W, \ldots$ and the i new sets contained in $(\lambda_1, \mu_1, \nu_1, \ldots)$, $(\lambda_2, \mu_2, \ldots \nu_2)$, &c. $(\lambda_i, \mu_i, \nu_i, \ldots)$ will be a combinant to the system $U, V, W, \ldots$. Perhaps, however, a more immediate extension of the idea of combinants to the case supposed of i unrelated sets or systems of sets would be to take, instead of $\lambda_1\lambda_2 \ldots \lambda_i$, $\mu_1\mu_2 \ldots \mu_i$, &c., the perfectly general forms of the same degrees in each set of the variables as these quantities are respectively of the same; to use these general forms, the coefficients of which will constitute not i new sets but a single new set of variables, as the syzygetic multipliers to $U, V, W, \ldots$, and then the invariant of the corresponding conjunctive in respect to the i original sets or systems of sets, and the one new set of variables thus obtained will be a combinant to the given system of functions*. As a matter of punctilio I may here take the opportunity of observing that the process for obtaining the relation between שׁ, ט (inadvertently written ט̇), and R, would have been more perfectly symmetrical to the eye had the equation for W [p. 416 of Vol. I.] been written $\tau(z^2 - y^2) = W$ in lieu of $\sigma(y^2 - z^2) = W$. I now return to take up the subject from the point where it was brought to a close in the last number of the *Journal*.

Let us consider what the equation (A)† becomes when U, V, W become the first partial derivatives (quâ x, y, z) of a single homogeneous cubic function ψ, so that

$$U = \frac{d\psi}{dx}, \quad V = \frac{d\psi}{dy}, \quad W = \frac{d\psi}{dz};$$

ϑ then becomes the Hessian of ψ, and the S of this (like every other invariant of ψ)‡ may be expressed, as is well known, as a rational integral function of

* I propose to append at the end of the next or some subsequent Section what ought to have been given in this or previous place, viz. the general differential equations for any concomitant to any congeries of forms, comprising amongst them any number of various distinct (that is unrelated) classes of systems of sets of variables, the relations between the sets belonging to any one system being supposed to be either simple or compound, and after the manner of either cogredience or contragredience; in fact, to do this only requires a slight extension of the formulæ given by me with that object in the fifth section of my paper in the *Philosophical Transactions* for the year 1853, Part III., which see. [Vol. I., p. 551.]

† *Vide* last number of this *Journal*, near the end of Author's paper therein. [Vol. I., p. 420.]

‡ I have given a perfectly rigid demonstration in the *Philosophical Magazine*, in the early part of 1853, that every invariant to a cubic function of three variables is a rational integral function of the two Aronholdian invariants S and T. [Vol. I., p. 599.]

the S and T of ψ. The relation between the S of the H and the S and T may readily be obtained from the canonical form

$$(\psi) = x^3 + y^3 + z^3 + 6mxyz.$$

The Hessian of this is

$$(1 + 2m^3)\, xyz - m^2 (x^3 + y^3 + z^3),$$

and making $-\dfrac{1 + 2m^3}{6m^2} = \mu$, the S of this Hessian will be

$$6^4 . m^8 \times (\mu - \mu^4),$$

which is $$(1 + 2m^3)\{(1 + 2m^3)^3 + 216m^6\}.$$

That is $$1 + 8m^3 + 240m^6 + 464m^9 + 16m^{12},$$
$$= (1 - 20m^3 - 8m^6)^2 + 48(m - m^4)^3,$$
$$= (S)^3 + 48(T)^2,$$

where (S) and (T) are respectively the S and T of (ψ). Hence we have in general

$$S . H . \psi = (S\psi)^3 + 48(T\psi)^2.$$

So that שׁ becomes $T^2 + 48S^3$, and ט evidently from Calculus of Forms [cf. Vol. I., p. 311] becomes

$$\tfrac{16}{8}(1 - 20m^3 - 8m^6),$$

that is $2T$, so that $$4\,\text{שׁ} - \tfrac{1}{4}\text{ט}^2 = 3T^2 + 192S^3;$$

so that equation (A) becomes

$$R = T^2 + 64S^3,$$

the Aronholdian representation of the Discriminant of ψ.

We see from this numerical calculation that it is not $\Sigma\Omega$ but $\frac{1}{2}\Sigma\Omega$, which ought to receive the appellation of ט, making which modification the general equation, written (A), becomes

$$\tfrac{1}{3}R = 4\,\text{שׁ} - \text{ט}^2.$$

The שׁ it will be observed is a compound combinant, being a biquadratic function of quantities all of which are invariants of the system U, V, W; the ט on the other hand is a simple combinant of the sixth degree.

The general dodecadic combinant שׁ may also in another manner be exhibited as a biquadratic function of cubic functions of the coefficients of the three given quadratics; but these cubic functions will no longer be invariants of the given quadratics. Thus, form the Jacobian of U, V, W, that is, the determinant

$$\begin{vmatrix} \frac{dU}{dx}, & \frac{dU}{dy}, & \frac{dU}{dz} \\ \frac{dV}{dx}, & \frac{dV}{dy}, & \frac{dV}{dz} \\ \frac{dW}{dx}, & \frac{dW}{dy}, & \frac{dW}{dz} \end{vmatrix},$$

which will be a cubic covariant to the system. The S of this will be another form of ש. So too, again, if we border the matrix to the Jacobian determinant above written vertically and horizontally with ξ, η, ζ, and call the determinant of the matrix thus formed I', I' will be quadratic in the system x, y, z, in the system ξ, η, ζ, and in the system formed by the coefficients of U, V, W, and the result of affecting this with the operator Σ will be the same as the result of the operation upon Ω with the same symbol; that is to say, $\frac{1}{24}E \,.\, I'$ will be equal to ט, this latter symbol being so taken (as last explained) in such a way that $3R$ shall equal 4ש $-$ ט^2, and each of the four lines in the operator Σ being supposed to go through their complete number (6) of permutations.

The terms sextic and dodecadic combinants will not be sufficient *per se* to characterize ש or ט (to a numerical factor *près*), supposing that there exist combinants of the 3rd and 9th degrees respectively in the coefficients, in which case the general sextic would contain two and the general dodecadic five arbitrary numerical parameters.

This makes so much the more remarkable and satisfactory the method above developed for finding ש and ט as undecompounded forms; the general dodecadic combinant at all events being rendered indeterminate by virtue of the existence of a sextic combinant above demonstrated.

It is interesting to evince the identity of the S of the Jacobian with that of the discriminant to the conjunctive of U, V, W, which latter has been called ש.

Starting with the canonical forms of the system U, V, W, and neglecting the ρ and σ, which cannot influence the result of the intended comparison, we have

$$J(U, V, W) = \begin{vmatrix} x, & -y, & 0 \\ 0, & -y, & z \\ gz+hy, & hx+y+fz, & gz+hy \end{vmatrix}$$
$$= fx(y^2+z^2) + gy(x^2+z^2) + hz(x^2+y^2) + xyz.$$

And multiplying by 6 and adopting the same notation as before (from the *Higher Plane Curves*, p. 182), we have

$$\begin{aligned} b_1 &= 2f, & b_2 &= 0, & b_3 &= 2h, \\ a_1 &= 0, & a_2 &= 2g, & a_3 &= 2h, \\ c_1 &= 2f, & c_2 &= 2y, & c_3 &= 0, \\ & & d &= 1. \end{aligned}$$

And the expression for S in *Higher Plane Curves*, p. 184, becomes, omitting every term containing a_1, b_2, or c_3,

$$d^4 - 2d^2(b_1c_1 + c_2a_2 + a_3b_3) + 3d(a_2b_3c_1 + a_3b_1c_2)$$
$$- (b_3c_2a_2a_3 + a_3c_1b_1b_3 + a_2b_1c_1c_2) + (b_1^2c_1^2 + c_2^2a_2^2 + a_3^2b_3^2),$$

that is

$$1 - 8(f^2+g^2+h^2) + 48fgh - 16(h^2g^2 + g^2f^2 + f^2h^2) + 16(f^4+g^4+h^4),$$

so that

$$\underset{x,\,y,\,z}{S}\ J(U, V, W) = \text{ש} = \underset{\lambda,\,\mu,\,\nu}{S}\ \ \underset{x,\,y,\,z}{\square}(\lambda U + \mu V + \nu W),$$

as was to be shown. As observed above, the form first found has the advantage over the one just obtained in disclosing the elements (cubic invariants to U, V, W) of which the ש is a biquadratic function. So, analogously, the resultant of two quadratic functions (P, Q) of x and y may be exhibited either under the form of the discriminant in respect to the coefficients of conjunction of the discriminant in respect to the original variables of the conjunctive of P, Q, or under the form of the discriminant of the Jacobian of P, Q. The former discloses the invariantive composition of the resultant which remains latent in the latter. As regards the ט, the proof of its being capable of the second mode of generation above indicated must, on account of the tediousness of the calculation, be for the present reserved; nor can I assert the fact with entire confidence until I have made a more complete investigation into the combinants of the system U, V, W, the remarks concerning which, in p. [416, Vol. I.], I wish to be considered as provisionally withdrawn.

The analogy between the invariants of a cubic form of three variables and a biquadratic of two has been frequently insisted upon in the foregoing pages; but we shall now see that this analogy has its foundation in the deeper-seated analogy which connects a ternary system of quadratics of three variables with a binary system of cubics of two variables.

We may suppose the two given functions so combined that the linear conjunctive $lP + mQ$ shall contain two equal roots, and so take the form x^2y; this may then be combined with either of the given functions so as to give a conjunctive of the form

$$ax^3 + 3xy^2 + dy^3,$$

and writing for x and y, $\frac{x}{\sqrt[3]{a}}$, $\frac{y}{\sqrt[3]{d}}$, respectively, and multiplying $lP + mQ$ by $\sqrt[3]{a^2d}$, we obtain for our standard form

$$P = 3x^2y,$$
$$Q = x^3 + 3exy^2 + y^3.$$

The resultant of this system rejecting an universally-irrelevant numerical factor is 1.

Again, write

$$\lambda P + \mu Q = 3\lambda x^2y + \mu x^3 + 3\mu exy^2 + \mu y^3;$$

and operate upon this with the commutator (say ω) [see Vol. I., p. 306].

$$\begin{vmatrix} \frac{d}{d\lambda}, & \frac{d}{d\mu} \\ \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{dx}, & \frac{d}{dy} \end{vmatrix}.$$

Keeping one of the lines (for example, the first) stationary, and, for greater brevity, writing $\delta_\lambda, \delta_\mu, \delta_x, \delta_y$ in place of $\frac{d}{d\lambda}, \frac{d}{d\mu}, \frac{d}{dx}, \frac{d}{dy}$, we obtain 8 positions, which, remembering that the order *in* the lines of these positions (and not the order *of* the lines) is the only thing to be attended to, are equivalent to

$$\begin{vmatrix} \delta_\lambda, & \delta_\mu \\ \delta_x, & \delta_y \\ \delta_x, & \delta_y \\ \delta_x, & \delta_y \end{vmatrix} - 3 \times \begin{vmatrix} \delta_\lambda, & \delta_\mu \\ \delta_x, & \delta_y \\ \delta_x, & \delta_y \\ \delta_y, & \delta_x \end{vmatrix} + 3 \times \begin{vmatrix} \delta_\lambda, & \delta_\mu \\ \delta_x, & \delta_y \\ \delta_y, & \delta_x \\ \delta_y, & \delta_x \end{vmatrix} - \begin{vmatrix} \delta_\lambda, & \delta_\mu \\ \delta_y, & \delta_x \\ \delta_y, & \delta_x \\ \delta_y, & \delta_x \end{vmatrix}.$$

Hence we have $\frac{1}{36}\omega(\lambda P + \mu Q) = -e$.

I need hardly observe, that in general for any two odd-degreed functions of the same degree in x, y, as

$$a_0 x^m + m a_1 x^{m-1} y + \tfrac{1}{2} m(m-1) a_2 x^{m-2} y^2 + \ldots + m(a_1) x y^{m-1} + (a_0) y^m,$$
$$b_0 x^m + m b_1 x^{m-1} y + \tfrac{1}{2} m(m-1) b_2 x^{m-2} y^2 + \ldots + m(b_1) x y^{m-1} + (b_0) y^m,$$

we may obtain, in an analogous manner, the combinant

$$a_0(b_0) - m a_1(b_1) + \tfrac{1}{2} m(m-1) a_2(b_2) + \&c.$$

Moreover it is easily shown that when m is an even integer the above expression will remain invariant, although of course it is no longer a combinant.

Again, the Hessian to $\lambda P + \mu Q$ will be

$$\begin{vmatrix} \mu x + \lambda y, & \lambda x + \mu e y \\ \lambda x + \mu e y, & \mu e x + \mu y \end{vmatrix},$$

which is equal to

$$e\mu^2 x^2 + \mu^2 xy - e^2\mu^2 y^2 + \lambda\mu y^2 - e\lambda\mu xy - \lambda^2 x^2,$$

which call $H.C$ (C meaning the conjunctive of P, Q). Let this be operated upon with the commutator

$$\begin{matrix} \delta_x^2, & \delta_x\delta_y, & \delta_y^2, \\ \delta_\lambda^2, & \delta_\lambda\delta_\mu, & \delta_\mu^2, \end{matrix}$$

which call Ω.

Since neither $y^2\lambda^2$ nor $xy\lambda^2$ enters in $H.C$, we have only to consider out of the full number 6 of positions the two effective positions

$$\begin{vmatrix} \delta_x^2, & \delta_x\delta_y, & \delta_y^2 \\ \delta_\lambda^2, & \delta_\lambda\delta_\mu, & \delta_\mu^2 \end{vmatrix} - \begin{vmatrix} \delta_x^2, & \delta_x\delta_y, & \delta_y^2 \\ \delta_\lambda^2, & \delta_\mu^2, & \delta_\lambda\delta_\mu \end{vmatrix}.$$

Hence $$\tfrac{1}{16} EHC(P, Q) = 1 - e^2.$$

So that $$\{-\tfrac{1}{36}\omega C(P, Q)\}^3 + \{\tfrac{1}{16} EHC(P, Q)\} = R(P, Q).$$

Thus R is expressed in terms of the cube of a simple quadratic combinant and a sextic compound combinant, which is made up of quadratic invariants. When P and Q become of the form $\frac{d\psi}{dx}, \frac{d\psi}{dy}$, respectively ($\psi$ being a quartic in x and y), these become respectively (to numerical factors *près*) the quadrinvariant of the given function and the cube invariant of its Hessian, which latter is a linear function of the cube of the quadrinvariant and the square of the cubinvariant of the given function, as we know *à priori* from the fact of the fundamental scale of the quartic consisting of the quadrinvariant and cubinvariant (for a rigid demonstration of which fact see the *Philosophical Magazine* in the early part of 1853 [Vol. I., p. 599]), and the expression for the resultant thus resolves itself into the known composite form of the sum of a square and cube.

The simple sextic combinant represented by $E.H.C(P, Q)$ may also, analogous to what has been observed concerning the ם, be expressed as a commutant (in fact the cubinvariant) of the Jacobian to P and Q, but then the form will no longer disclose its invariantive sub-composition. So too, if it were thought worth while to push the analogies to an extreme, the quadri-combinant to P, Q might have been found, first by bordering the Hessian to the conjunctive to P, Q with ξ, η horizontally and vertically, and operating upon the result with the commutator

$$\begin{vmatrix} \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{d\lambda}, & \frac{d}{d\mu} \\ \frac{d}{d\xi}, & \frac{d}{d\eta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta} \end{vmatrix},$$

or by bordering the Jacobian to P, Q with ξ, η, as before, and then operating upon the result with the commutator

$$\begin{vmatrix} \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{d\xi}, & \frac{d}{d\eta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta} \end{vmatrix}.$$

I propose hereafter to return to the consideration of the fundamental scale of combinants to the two systems, namely of three quadratics in x, y, z, and of two cubics in x, y, which have been treated of in this section.

Section VIII.

On the Reduction of a Sextic Function of Two Variables to its Canonical Form.

In the *London and Edinburgh Philosophical Magazine* for Nov. 1851, after giving a simple method for representing any function of two variables of an odd degree $(x, y)^{2m+1}$ under the form of

$$u_1^{2m+1} + u_2^{2m+1} + \dots + u_{m+1}^{2m+1},$$

where $u_1, u_2 \dots u_{m+1}$ are linear functions of x, y (which form, as appears from the method of obtaining it, is unique), I proceeded [Vol. I., p. 269] to show how by a certain method therein explained the biquadratic and octavic functions of $x, y, (x, y)^4, (x, y)^8$ could be thrown under the respective forms

$$u_1^4 + u_2^4 + mu_1^2u_2^2,$$

$$u_1^8 + u_2^8 + u_3^8 + u_4^8 + mu_1^2u_2^2u_3^2u_4^2,$$

the number of values of m in the first form being three and in the second form five, the quantity m in the one case depending on the solution of the equation

$$\begin{vmatrix} a_0, & a_1, & a_2 + \lambda \\ a_1, & a_2 - \frac{1}{2}\lambda, & a_3 \\ a_2 + \lambda, & a_3, & a_4 \end{vmatrix} = 0,$$

where a_0, a_1, a_2, a_3, a_4 are the coefficients of $(x, y)^4$ multiplied respectively by $1, \frac{1}{4}, \frac{1}{6}, \frac{1}{4}, 1$; and in the other case, on the solution of the equation

$$\begin{vmatrix} a_0, & a_1, & a_2, & a_3, & a_4 + \lambda \\ a_1, & a_2, & a_3, & a_4 - \frac{1}{4}\lambda, & a_5 \\ a_2, & a_3, & a_4 + \frac{1}{6}\lambda, & a_5, & a_6 \\ a_3, & a_4 - \frac{1}{4}\lambda, & a_5, & a_6, & a_7 \\ a_4 + \lambda, & a_5, & a_6, & a_7, & a_8 \end{vmatrix} = 0,$$

$$a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8,$$

being the coefficients of $(x, y)^8$ multiplied respectively by

$$1, \tfrac{1}{8}, \tfrac{1}{28}, \tfrac{1}{56}, \tfrac{1}{70}, \tfrac{1}{56}, \tfrac{1}{28}, \tfrac{1}{8}, 1.$$

Before proceeding to investigate the thèory of these methods of reduction under any more general point of view, it will be convenient to seek to obtain the representation of $(x, y)^6$ under some analogous form.

It might at first be supposed that the corresponding form should be

$$u_1^6 + u_2^6 + u_3^6 + mu_1^2u_2^2u_3^2;$$

if, however, the method which succeeds for the quartic and octavic functions be attempted to be applied to this it will be found entirely to fail. Here, however, considerations of a purely morphological character step in to our aid and immediately lead to the true canonical representation of the sextic function. Algebraically speaking, the only connexion between two identical forms F and F is through the equation $F = \psi^{-1}\psi F$; but, morphologically considered, a *form* F may admit of being derived by a series of entirely heterogeneous operations from *itself*. In general, supposing

$$F(x, y) = ax^n + nbx^{n-1}y + \&c. \dots + n(b)xy^{n-1} + (a)y^n,$$

the form
$$\xi^n \frac{d}{da} + \xi^{n-1}\eta \frac{d}{db} + \dots + \xi\eta^{n-1}\frac{d}{d(b)} + \eta^n \frac{d}{d(a)},$$

operating upon any concomitant to F will, we know (from the law of reciprocity in Section IV.), produce another concomitant. The operative form above written is termed the evector, and the result of operating therewith upon a concomitant is termed the evectant of the latter, which is said, when so operated upon, to be evected*. The polar reciprocal of the evector may be termed the contravector, and for two variables is of course of the form

$$y^n \frac{d}{da} - y^{n-1}x \frac{d}{db} \pm \&c.$$

If we suppose n to be even, $F(x, y)$ will have the well-known quadrinvariant

$$a(a) - nb(b) + \tfrac{1}{2}n(n-1)c(c) \mp \&c.,$$

and if this be operated upon with the contravector, or if we like so to say, be contravected, we recover the original function F, so that any function of two variables of an even degree is the contravect of its quadrinvariant.

* These terms "evector, evectant, contravectant, to evect and contravect," will of course admit of an immediate extension to functions of any number of variables. Evection gives rise to contravariants, contravection to covariants; but on this account to interchange the meanings respectively attached to the terms evector and contravector, and their respective allied terms, would be a simplification too dearly purchased at the expense of contravening the principle that the *word for the base should be the base for the word.*

If now we return to the representation of $(x, y)^4$ under the form

$$u_1^4 + u_2^4 + m(u_1 u_2)^2,$$

and make

$$u_1 u_2 = F_2(x, y);$$

or to that of $(x, y)^8$ under the form

$$u_1^4 + u_2^4 + u_3^4 + u_4^4 + m(u_1 u_2 u_3 u_4)^2,$$

and make

$$u_1 u_2 u_3 u_4 = F_4(x, y),$$

the outstanding terms multiplied by the parameter m may be regarded in each of these two cases as the squared contravects of the quadrinvariants F_2 and F_4 respectively. Under this point of view we at once see a ground for the proved fact of $(x, y)^6$ not being capable of being thrown under the form

$$u_1^6 + u_2^6 + u_3^6 + m\{F_3(x, y)\}^2,$$

where

$$u_1 u_2 u_3 = F_3(x, y),$$

because there exists no quadrinvariant to $F_3(x, y)$, the only invariant which it possesses being the discriminant which is of the fourth degree; if however instead of $m\{F_3(x, y)\}^2$ we write $mF_3(x, y)\, G_3(x, y)$, where $G_3(x, y)$ is the contravect of the discriminant of F_3, we shall find that the method applied to the reduction of $(x, y)^4$ and to $(x, y)^8$ will perfectly well succeed for $(x, y)^6$, as I proceed to demonstrate.

Let this function be written under the form

$$a_0 x^6 + 6a_1 x^5 y + 15a_2 x^4 y^2 + 20a_3 x^3 y^3 + 15a_4 x^2 y^4 + 6a_5 x y^5 + a_6 y^6,$$

which suppose made equal to

$$(p_1 x + q_1 y)^6 + (p_2 x + q_2 y)^6 + (p_3 x + q_3 y)^6$$
$$+ (Ax^3 + 3Bx^2 y + 3Cxy^2 + Dy^3)(Lx^3 + Mx^2 y + Nxy^2 + Py^3);$$

where

$$(p_1 x + q_1 y)(p_2 x + q_2 y)(p_3 x + q_3 y) = Ax^3 + 3Bx^2 y + 3Cxy^2 + Dy^3;$$

the discriminant of this will be, as is well known,

$$A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD,$$

and contravecting this with the operator

$$-y^3 \frac{d}{dA} + y^2 x \frac{d}{dB} - yx^2 \frac{d}{dC} + x^3 \frac{d}{dD},$$

and identifying the result with $Lx^3 + Mx^2 y + Nxy^2 + Py^3$, we have

$$L = -6ABC + 2A^2D + 4B^3,$$
$$M = 6ABD - 12AC^2 + 6B^2C,$$
$$N = -6ACD + 12DB^2 - 6BC^2,$$
$$P = 6BCD - 2AD^2 - 4C^3.$$

A, B, C, D are known functions of $p_1, p_2, p_3; q_1, q_2, q_3$, and we shall have seven equations for determining these six unknown quantities and the unknown parameter m.

Let

$$q_1 = p_1 \lambda_1, \quad q_2 = p_2 \lambda_2^4, \quad q_3 = p_3 \lambda_3,$$
$$\lambda_1 + \lambda_2 + \lambda_3 = 3s_1, \quad \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2 = 3s_2, \quad \lambda_1\lambda_2\lambda_3 = s_3,$$
$$p_1 p_2 p_3 = m.$$

Then

$$A = m, \quad 3B = 3ms_1, \quad 3C = 3ms_2, \quad D = s_3.$$
$$L = m^3(4s_1^3 - 6s_1 s_2 + 2s_3),$$
$$M = m^3(6s_1^2 s_2 + 6s_1 s_3 - 12s_2^2),$$
$$N = m^3(12s_1^2 s_3 - 6s_1^2 s_2 - 6s_2 s_3),$$
$$P = m^3(+6s_1 s_2 s_3 - 4s_2^3 - 2s_3^2).$$

Let

$$(Ax^3 + 3Bx^2 y + 3Cxy^2 + Dy^3)(Lx^3 + Mx^2 y + Nxy^2 + Py^3)$$
$$= K_0 x^6 + K_1 x^5 y + K_2 x^4 y^2 + K_3 x^3 y^3 + K_4 x^2 y^4 + K_5 x y^5 + K_6 y^6 = T.$$

Then, equating this term for term with

$$\lambda_1^6 (x + p_1 y)^6 + \lambda_2^6 (x + p_2 y)^6 + \lambda_3^6 (x + p_3 y)^6 + \mu T,$$

we obtain the seven equations following:

$$p_1^6 + p_2^6 + p_3^6 + \mu K_0 = a_0, \tag{1}$$
$$p_1^6\lambda_1 + p_2^6\lambda_2 + p_3^6\lambda_3 + \frac{\mu}{6} K_1 = a_1, \tag{2}$$
$$p_1^6\lambda_1^2 + p_2^6\lambda_2^2 + p_3^6\lambda_3^2 + \frac{\mu}{15} K_2 = a_2, \tag{3}$$
$$p_1^6\lambda_1^3 + p_2^6\lambda_2^3 + p_3^6\lambda_3^3 + \frac{\mu}{20} K_3 = a_3, \tag{4}$$
$$p_1^6\lambda_1^4 + p_2^6\lambda_2^4 + p_3^6\lambda_3^4 + \frac{\mu}{15} K_4 = a_4, \tag{5}$$
$$p_1^6\lambda_1^5 + p_2^6\lambda_2^5 + p_3^6\lambda_3^5 + \frac{\mu}{6} K_5 = a_5, \tag{6}$$
$$p_1^6\lambda_1^6 + p_2^6\lambda_2^6 + p_3^6\lambda_3^6 + \mu K_6 = a_6. \tag{7}$$

Eliminating linearly

p_1^6,	p_2^6,	p_3^6	between	equations	1, 2, 3, 4,
$\lambda_1 p_1^6$,	$\lambda_2 p_2^6$,	$\lambda_3 p_3^6$	„	„	2, 3, 4, 5,
$\lambda_1^2 p_1^6$,	$\lambda_2^2 p_2^6$,	$\lambda_3^2 p_3^6$	„	„	3, 4, 5, 6,
$\lambda_1^3 p_1^6$,	$\lambda_2^3 p_2^6$,	$\lambda_3^3 p_3^6$	„	„	4, 5, 6, 7,

we obtain the four equations following, namely,

$$
\begin{aligned}
a_0 s_3 - 3a_1 s_2 + 3a_2 s_1 - a_3 &= \mu \vartheta_0, \\
a_1 s_3 - 3a_2 s_2 + 3a_3 s_1 - a_4 &= \mu \vartheta_1, \\
a_2 s_3 - 3a_3 s_2 + 3a_4 s_1 - a_5 &= \mu \vartheta_2, \\
a_3 s_3 - 3a_4 s_2 + 3a_5 s_1 - a_6 &= \mu \vartheta_3,
\end{aligned}
$$

where

$$
\begin{aligned}
\vartheta_0 &= K_0 s_3 - \tfrac{3}{6} K_1 s_2 + \tfrac{3}{15} K_2 s_1 - \tfrac{1}{20} K_3 \\
&= \tfrac{1}{60}(60K_0 s_3 - 30K_1 s_2 + 12K_2 s_1 - 3K_3), \\
\vartheta_1 &= \tfrac{1}{6} K_1 s_3 - \tfrac{3}{15} K_2 s_2 + \tfrac{3}{20} K_3 s_1 - \tfrac{1}{15} K_4 \\
&= \tfrac{1}{60}(10K_1 s_3 - 12K_2 s_2 + 9K_3 s_1 - 4K_4), \\
\vartheta_2 &= \tfrac{1}{15} K_2 s_3 - \tfrac{3}{20} K_3 s_2 + \tfrac{3}{15} K_4 s_1 - \tfrac{1}{6} K_5 \\
&= \tfrac{1}{60}(4K_2 s_3 - 9K_3 s_2 + 12K_4 s_1 - 10K_5), \\
\vartheta_3 &= \tfrac{1}{20} K_3 s_3 - \tfrac{3}{15} K_4 s_2 + \tfrac{3}{6} K_5 s_1 - K_6 \\
&= \tfrac{1}{60}(3K_3 s_3 - 12K_4 s_2 + 30K_5 s_1 - 60K_6),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{m^4} K_0 &= \frac{A}{m} \cdot \frac{L}{m^3} = 4s_1^3 - 6s_1 s_2 + 2s_3, \\
\frac{1}{m^4} K_1 &= \frac{AM + 3BL}{m^4} = (6s_1^2 s_2 + 6s_1 s_3 - 12s_2^2) + 3s_1(4s_1^3 - 6s_1 s_2 + 2s_3) \\
&= 12s_1^4 - 12s_1^2 s_2 + 12s_1 s_3 - 12s_2^2, \\
\frac{1}{m^4} K_2 &= \frac{AN + 3BM + 3CL}{m^4} = (12s_1^2 s_3 - 6s_1 s_2^2 - 6s_2 s_3) \\
&\quad + (18s_1^3 s_2 + 18s_1^2 s_3 - 36s_1 s_2^2) \\
&\quad + (12s_1^3 s_2 - 18s_1 s_2^2 + 6s_2 s_3) \\
&= 30s_1^3 s_2 + 30s_1^2 s_3 - 60s_1 s_2^2, \\
\frac{1}{m^4} K_3 &= \frac{AP + 3BN + 3CM + DL}{m^4} = (6s_1 s_2 s_3 - 4s_2^3 - 2s_3^2) \\
&\quad + (36s_1^3 s_3 - 18s_1^2 s_2^2 - 18s_1 s_2 s_3) \\
&\quad + (18s_1^2 s_2^2 + 18s_1 s_2 s_3 - 36s_2^3) \\
&\quad + (4s_1^3 s_3 - 6s_1 s_2 s_3 + 2s_3^2) \\
&= 40s_1^3 s_3 - 40s_2^3, \\
\frac{1}{m^4} K_4 &= \frac{3BP + 3CN + DM}{m^4} = 18s_1^2 s_2 s_3 - 12s_1 s_2^3 - 6s_1 s_3^2 \\
&\quad + 36s_1^2 s_2 s_3 - 18s_1 s_2^3 - 18s_2^2 s_3 \\
&\quad + 6s_1^2 s_2 s_3 + 6s_1 s_3^2 - 12s_2^2 s_3 \\
&= 60s_1^2 s_2 s_3 - 30s_1 s_2^3 - 30s_2^2 s_3,
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{m^4} K_5 &= \frac{3CP + DN}{m^4} = 18s_1 s_2^2 s_3^2 - 12s_2^4 - 6s_2 s_3^2 \\
&\quad + 12s_1^2 s_3^2 - 6s_1 s_2^2 s_3 - 6s_2 s_3^2 \\
&= 12s_1^2 s_3^2 + 12s_1 s_2^2 s_3 - 12s_2^4 - 12s_2 s_3^2, \\
\frac{1}{m^4} K_6 &= \frac{DP}{m^4} = 6s_1 s_2 s_3^2 - 4s_2^3 s_3 - 2s_3^3;
\end{aligned}
$$

therefore

$$
\begin{aligned}
\frac{60\vartheta_0}{\mu m^4} &= 240s_1^3 s_3 - 360s_1 s_2 s_3 + 120s_3^2 \\
&\quad - 360s_1^4 s_2 + 360s_1^2 s_2^2 - 360s_1 s_2 s_3 + 360s_2^3 \\
&\quad + 360s_1^4 s_2 + 360s_1^3 s_3 - 720s_1^2 s_2^2 \\
&\quad - 120s_1^3 s_3 + 120s_2^3 \\
&= 120(s_3^2 + 4s_2^3 + 4s_1^3 s_3 - 3s_1^2 s_2^2 - 6s_1 s_2 s_3),
\end{aligned}
$$

that is $\vartheta_0 = 2\mu(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD).$

Again,

$$
\begin{aligned}
\frac{\vartheta_1}{\mu m^4} &= 120s_1^4 s_3 - 120s_1^2 s_2 s_3 + 120s_1 s_3^2 - 120s_2^2 s_3 \\
&\quad - 360s_1^3 s_2^2 - 360s_1^2 s_2 s_3 + 720s_1 s_2^3 \\
&\quad + 360s_1^4 s_3 - 360s_1 s_2^3 \\
&\quad - 240s_1^2 s_2 s_3 + 120s_1 s_2^3 + 120s_1 s_2^2 s_3 \\
&= 120(s_1 s_3^2 + 4s_1 s_2^3 + 4s_1^4 s_3 - 3s_1^3 s_2^2 - 6s_1^2 s_2 s_3);
\end{aligned}
$$

therefore $\vartheta_1 = 2\mu(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD)s_1.$

Again,

$$
\begin{aligned}
\frac{60\vartheta_2}{\mu m^4} &= 120s_1^3 s_2 s_3 + 120s_1^2 s_3^2 - 240s_1 s_2^2 s_3 \\
&\quad - 360s_1^3 s_2 s_3 + 360s_2^4 \\
&\quad + 720s_1^3 s_2 s_3 - 360s_1^2 s_2^3 - 360s_1 s_2^2 s_3 \\
&\quad - 120s_1^2 s_3^2 - 120s_1 s_2^2 s_3 + 120s_2^4 + 120s_2 s_3^2 \\
&= 120(s_2 s_3^2 + 4s_2^4 + 4s_1^3 s_2 s_3 - 3s_1^2 s_2^3 - 6s_1 s_2^2 s_3),
\end{aligned}
$$

therefore $\vartheta_2 = 2\mu(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD)s_2.$

Finally,

$$
\begin{aligned}
\frac{60\vartheta_3}{\mu m^4} &= 120s_1^3 s_3^2 - 120s_2^3 s_3 \\
&\quad - 720s_1^2 s_2^2 s_3 + 360s_1 s_2^4 + 360s_2^3 s_3 \\
&\quad + 360s_1^3 s_3^2 + 360s_1^2 s_2^2 s_3 - 360s_1 s_2^4 - 360s_1 s_2 s_3^2 \\
&\quad - 360s_1^2 s_2 s_3^2 + 240s_2^3 s_3 + 120s_3^3 \\
&= 120(s_3^3 + 4s_2^3 s_3 + 4s_1^3 s_3^2 - 3s_1^2 s_2^2 s_3 - 6s_1 s_2 s_3^2),
\end{aligned}
$$

therefore $\vartheta_3 = 2\mu(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD)s_3$.

Hence, writing

$$2\mu(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD) = \rho,$$

the four equations connecting a_0, a_1, a_2, a_3 with $\vartheta_0, \vartheta_1, \vartheta_2, \vartheta_3$, take the form

$$a_0s_3 - 3a_1s_2 + 3a_2s_1 - (a_3 + \rho) = 0,$$

$$a_1s_3 - 3a_2s_2 + 3(a_3 - \tfrac{1}{3}\rho)s_1 - a_4 = 0,$$

$$a_2s_3 - 3(a_3 + \tfrac{1}{3}\rho)s_2 + 3a_4s_1 - a_5 = 0,$$

$$(a_3 - \rho) - 3a_4s_2 + 3a_5s_1 - a_6 = 0.$$

Hence we derive the equation involving only the known coefficients of the given function for finding ρ, namely, the determinant

$$\begin{vmatrix} a_0, & a_1, & a_2, & a_3 + \rho \\ a_1, & a_2, & a_3 - \tfrac{1}{3}\rho, & a_4 \\ a_2, & a_3 + \tfrac{1}{3}\rho, & a_4, & a_5 \\ a_3 - \rho, & a_4, & a_5, & a_6 \end{vmatrix} = 0. \qquad \text{(R)}$$

If in this matrix ρ be changed into $-\rho$, the determinant evidently remains unaltered in value; hence the odd powers of ρ disappear from the equation, and ρ may be found by the solution of a double quadratic only. In fact the above equation for finding ρ, expanded out, becomes

$$\frac{\rho^4}{9} + \left(\frac{2}{3}\begin{vmatrix} a_2, & a_3 \\ a_3, & a_4 \end{vmatrix} - \frac{2}{3}\begin{vmatrix} a_1, & a_3 \\ a_3, & a_5 \end{vmatrix} + \begin{vmatrix} a_2, & a_3 \\ a_3, & a_4 \end{vmatrix} + \frac{1}{9}\begin{vmatrix} a_0, & a_3 \\ a_3, & a_6 \end{vmatrix}\right)\rho^2,$$

$$- \begin{vmatrix} a_0, & a_1, & a_2, & a_3 \\ a_1, & a_2, & a_3, & a_4 \\ a_2, & a_3, & a_4, & a_5 \\ a_3, & a_4, & a_5, & a_6 \end{vmatrix} = 0;$$

that is
$$\rho^4 + (15a_2a_4 - 6a_1a_5 - 10a_3^2 + a_0a_6)\rho^2$$

$$+ \begin{vmatrix} a_3, & a_2, & a_1, & a_0 \\ a_4, & a_3, & a_2, & a_1 \\ a_5, & a_4, & a_3, & a_2 \\ a_6, & a_5, & a_4, & a_3 \end{vmatrix} = 0;$$

the coefficient of ρ^2 being the well-known quadrinvariant, and the final term the meiocatalecticizant of the given function. There will consequently be four different values of ρ and four different systems of values of s_1, s_2, s_3, expressible for each system respectively in terms of ρ by means of any three out of the four equations (R), and consequently there will be four systems of

values of $\lambda_1, \lambda_2, \lambda_3$, each of which may be found separately by solving the cubic equation

$$\lambda^3 - 3s_1\lambda^2 + 3s_2\lambda - s_3 = 0;$$

also $K_0, K_1, K_2, K_3, K_4, K_5, K_6$ become known multiples of m^4, and finally, the values of any λ and K system being thus determined, we may then, by means of the identity

$$p_1^6(x + \lambda_1y)^6 + p_2^6(x + \lambda_2y)^6 + p_3^6(x + \lambda_3y)^6$$

$$+ \mu m^4\left(\frac{K_0}{m^4}x^6 + \frac{K_1}{m^4}x^5y + \&\text{c.} + \frac{K_6}{m^4}y^6\right) = a_0x^6 + 6a_1x^5y + \&\text{c.} + a_6y^6,$$

write down at will any four equations out of the seven equations therefrom resulting, and these will serve to determine linearly the values of $p_1^6, p_2^6, p_3^6, \mu m^4$; and consequently, by means of the equations

$$q_1 = p_1\rho_1, \quad q_2 = p_2\rho_2, \quad q_3 = p_3\rho_3,$$

q_1, q_2, q_3 are known, and consequently every coefficient in

$$(p_1x + q_1y)^6 + (p_2x + q_2y)^6 + (p_3x + q_3y)^6 + \mu M$$

is completely determined. But we shall hereafter return to this theory, and seek for a direct method of finding the four values of the functions

$$(p_1x + q_1y), \quad (p_2x + q_2y), \quad (p_3x + q_3y).$$

It appears from the above investigation that there are four modes of throwing $(x; y)^6$ under the assumed form which possess the remarkable property of separating into two pairs of modes, as is obvious from the fact of the resolving equation in ρ having two pairs of roots, those of the same pair being equal but of contrary signs. As this form will be of extreme value in studying the invariants of $(x, y)^6$, it may be well to consider the simplest shape to which it admits of being reduced.

We may suppose $(p_1x + q_1y)(p_2x + q_2y)(p_3x + q_3y)$ thrown under the form of $u^3 + v^3$, the contravectant of the discriminant to which in respect to u and v is $v^3 - u^3$, so that we may use for the canonical form the expression

$$a(u + v)^6 + b(u + \rho v)^6 + c(u + \rho^2v)^6 + \mu(u^6 - v^6),$$

where $\rho^3 = 1$; or if we please, more simply

$$a(u + v)^6 + b(u + \rho v)^6 + c(u + \rho^2v)^6 + u^6 - v^6.$$

I may take this occasion to observe that there are generally two modes of a distinct kind for obtaining any simple concomitant; the difference (a most important practical one) consisting in the circumstance that in the one mode there are differentiations to be performed in respect to the coefficients, the consequence of which is that the whole of the operations must be gone through for obtaining the concomitant with the primitive in its most general form, and no advantage can be taken in the course of these operations of the simplification resulting from the absence of any terms in the primitive or of any other speciality therein; whereas in the other mode of derivation, where all the differentiations have to be performed quâ the variables only, the partial form may be operated with throughout. Thus, for instance, to find the contravectant to the discriminant of a cubic function the general form of the cubic must be employed, and then the special values of the coefficient corresponding to a specific form of the cubic substituted at the close of the operations; but this same concomitant may also be obtained by taking the resultant of the first emanant of the given cubic and of the first emanant of its Hessian in respect to the variables of emanation, and consequently the specific form may, after this mode, be retained from the first. Thus, if we start with u^3+v^3, the Hessian is uv, and the two emanants in question will be $u^2u'+v^2v'$ and $vu'+uv'$, the resultant of which in respect to u' and v' is u^3-v^3; or, again, if we commence with uvw subject to the relation that $u+v+w=0$, the Hessian will be

$$\begin{vmatrix} 0, & w, & v, & 1 \\ w, & 0, & u, & 1 \\ v, & u, & 0, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix},$$

that is to say, $$u^2+v^2+w^2-2uv-2uw-2vw.$$

The two emanants will then be

$$vwu'+wuv'+uvw'$$
$$(u-v-w)\,u'+(v-w-u)\,v'+(w-u-v)\,w',$$

subject to the relation

$$u'+v'+w'=0;$$

and taking the resultant of these three equations, or, which is the same thing, of

$$vwu'+wuv'+uvw',$$
$$uu'+\ vv'\ +ww',$$
$$u'+\ v'\ +w',$$

we obtain the determinant

$$\begin{vmatrix} vw, & wu, & uv \\ u, & v, & w \\ 1, & 1, & 1 \end{vmatrix},$$

which is equal to

$$vw\,(v-w)+wu\,(w-u)+uv\,(u-v),$$

that is to say $$(u-v)\,(v-w)\,(w-u).$$

Hence another variety of the external shape to which the canonical form for the sextic function of x, y may be reduced will be

$$au^6+bv^6+cw^6+\mu uvw\,(u-v)\,(v-w)\,(w-u).$$

I shall presently revert to the theory of the corresponding mode of reducing to their canonical forms the biquadratic and octavic functions of x, y, the number of solutions for which will be respectively three and five, and the discovery of which, as shown by me in the Number of the *Philosophical Magazine* before adverted to, depends upon the solution of equations of the third and fifth degrees in ρ expressed by means of determinants of the third and fifth orders formed in precise correspondence with that of the fourth order, upon which, as we have found above, the reduction of the sextic function to its canonical form depends.

5.

THÉORÈME SUR LES DÉTERMINANTS.

[*Nouvelles Annales de Mathématiques*, XIII. (1854), p. 305.]

SOIENT les déterminants

$$\begin{vmatrix}\lambda\end{vmatrix}, \quad \begin{vmatrix}\lambda, & 1 \\ 1, & \lambda\end{vmatrix}, \quad \begin{vmatrix}\lambda, & 1, & 0 \\ 2, & \lambda, & 2 \\ 0, & 1, & \lambda\end{vmatrix}, \quad \begin{vmatrix}\lambda, & 1, & 0, & 0 \\ 3, & \lambda, & 2, & 0 \\ 0, & 2, & \lambda, & 3 \\ 0, & 0, & 1, & \lambda\end{vmatrix}, \quad \begin{vmatrix}\lambda, & 1, & 0, & 0, & 0 \\ 4, & \lambda, & 2, & 0, & 0 \\ 0, & 3, & \lambda, & 3, & 0 \\ 0, & 0, & 2, & \lambda, & 4 \\ 0, & 0, & 0, & 1, & \lambda\end{vmatrix}, \quad \begin{vmatrix}\lambda, & 1, & 0, & 0, & 0, & 0 \\ 5, & \lambda, & 2, & 0, & 0, & 0 \\ 0, & 4, & \lambda, & 3, & 0, & 0 \\ 0, & 0, & 3, & \lambda, & 4, & 0 \\ 0, & 0, & 0, & 2, & \lambda, & 5 \\ 0, & 0, & 0, & 0, & 1, & \lambda\end{vmatrix};$$

la loi de formation est évidente ; effectuant, on trouve

$$\lambda, \quad \lambda^2 - 1^2, \quad \lambda(\lambda^2 - 2^2), \quad (\lambda^2 - 1^2)(\lambda^2 - 3^2), \quad \lambda(\lambda^2 - 2^2)(\lambda^2 - 4^2),$$
$$(\lambda^2 - 1^2)(\lambda^2 - 3^2)(\lambda^2 - 5^2), \quad \lambda(\lambda^2 - 2^2)(\lambda^2 - 4^2)(\lambda^2 - 6^2),$$

et ainsi de suite.

6.

NOTE ON A POINT OF NOTATION.

[*Philosophical Magazine*, VII. (1854), pp. 50—51.]

IT frequently becomes important in algebraical investigations, and in the representation of results, to have a means of expressing that the sign + or − is to be affixed to an algebraical expression, according as certain indices $\theta_1, \theta_2, \theta_3 \ldots \theta_n$ which occur therein, and which represent the natural numbers from 1 to n in some regular or irregular order, can be derived from the fundamental arrangement $1, 2, 3 \ldots n$ by an even or by an odd number of interchanges. An example of this occurred in my short paper in the last Number of the *Philosophical Magazine*, on the extension of Lagrange's Rule of Interpolation [Vol. I., p. 646], where I used to denote that such a choice of signs was to be made, the awkward and unsuggestive symbol "?" There exists, however, a very simple algebraical mode of denoting the presence of the factor +1 or −1, according to the order of the natural numbers in the scale $\theta_1, \theta_2, \theta_3 \ldots \theta_n$.

ζ has been always consecrated by me to the purpose of signifying that the product of the squared differences is to be taken of the elements with which it is in regimen; and in the paper adverted to I introduced the highly convenient new symbol $\zeta^{\frac{1}{2}}$ to denote that the product is to be taken of the simple differences obtained by subtracting from each element in regimen therewith every subsequent element in the arrangement of the elements as set down. By aid of this new symbol $\zeta^{\frac{1}{2}}$, the positive or negative character of any permutation, as $\theta_1, \theta_2 \ldots \theta_n$, can be completely expressed; for

$$\zeta^{\frac{1}{2}}(\theta_1, \theta_2, \theta_3 \ldots \theta_n) \div \zeta^{\frac{1}{2}}(1, 2, 3 \ldots n)$$

will be +1 or −1 according as $1, 2, 3 \ldots n$ and $\theta_1, \theta_2, \theta_3 \ldots \theta_n$ belong to the same group, or to opposite groups in the natural dichotomous separation of the permutations of the n symbols in question, and thereby the desired object of giving a functional representation of the ambiguous sign is perfectly attained.

7.

NOTE ON THE "ENUMERATION OF THE CONTACTS OF LINES AND SURFACES OF THE SECOND ORDER."

[*Philosophical Magazine*, VII. (1854), pp. 331—334.]

IN the month of February, 1851, I gave in this *Magazine* an *à priori* and exhaustive process, founded upon the method of determinants, for determining every different kind of simple or collective contact capable of happening between lines and surfaces, and in general between all loci (whether intraspatial or extraspatial) of the second order [Vol. I., p. 219]. The question was shown to resolve itself into that of determining the number of singular relations capable of existing between two quadratic homogeneous functions of any given degree. My object in the paper referred to was actually to calculate the geometrical and analytical characters of these contents and singularities for intraspatial loci, that is loci representable by homogeneous quadratics of two, three, and four variables; but I incidentally appended [Vol. I., p. 239] a statement of the number of such for loci of five, six, seven, and eight variables, without, however, dwelling upon the means of representing the general law. This statement is, however, affected with certain inaccuracies of computation which will be presently pointed out.

It will be at once apparent, from an inspection of the principle of my method, that it remains equally applicable (*mutatis mutandis*) to the more general question of determining the relative singularities (in character and amount) of two functions, each linear in respect of two systems of variables $x_1, x_2 \ldots x_n$; $x_1', x_2' \ldots x_n'$; which species of functions degenerate into quadratic forms, when the two systems of variables become identical so as to coalesce into a single system. Some researches of Mr Cayley into the autometamorphic substitutions of quadratic forms (meaning thereby the linear substitutions which leave the form unaltered) required him to consider the nature of the singular relations capable of existing between two linear substitutions, which is precisely the question, differently stated, of the singular relations

connecting two lineo-linear functions above adverted to; accordingly, I am indebted to Mr Cayley for making an observation on the effect of my rule for finding such singularities, which leads to a most elegant formulization of the number of singularities in question, and which I proceed to introduce to the notice of my readers.

If U and V be two quadratic functions, each of n variables, and if we call D the discriminant of $U+\lambda V = D(\lambda)$, $D(\lambda)$ will be a function of λ of the nth degree. Now, first, I have observed that if any of these n roots be repeated any number of times, there will be a corresponding degree of singularity about one of the points of intersection of the loci represented by $U=0$, $V=0$; so that if the n roots of $D(\lambda)$ be made up of r_1 roots a_1, r_2 roots a_2, r_3 roots a_3, &c., there will be an *inclusive singularity* r_1 at one point, r_2 at another, r_3 at a third, and so on—by *inclusive singularity* meaning a number one unit greater than the index of singularity properly so termed; the inclusive-singularity at an *ordinary* intersection being called 1, at a point of simple singularity 2, of double singularity 3, and in general at a point of the $(r-1)$th degree of singularity r.

Hence the total-inclusive singularity (which is an unit greater than the total-singularity, properly so called) may be broken up into as many partial heaps of inclusive-singularity as there are modes of decomposing n into integers. We may now confine our attention exclusively to the different modes in which a given amount of inclusive-singularity at a single point admits of subdivision into distinct species of singularity, for which I have given in my paper referred to the following rule: The minor systems of determinants corresponding to the matrix of $U+\lambda V$ are to be considered in succession; and if a be any root of the complete determinant of the matrix occurring r times, every hypothesis is to be exhausted as regards the number of times in which $(\lambda - a)$ may be conceived to enter as a factor into each of the system of 1st minors, into each minor of the system of 2nd minors, into each minor of the system of 3rd minors, and so on; the number of such hypotheses being limited by the condition that, if *quoad* the root a, $(\lambda-a)^{k_1}$, $(\lambda-a)^{k_2}$, $(\lambda-a)^{k_3}$ be the greatest common factors respectively to three consecutive systems of minor determinants, k_1 must be not less than $2k_2 - k_3$. Here steps in the beautiful observation of Mr Cayley, that the question of assigning the different species of singularities respondent to the factor a supposed to occur r times, is, by virtue of the above condition, tantamount precisely to that of assigning the total number of decreasing* series of positive integers, commencing with a given number r, subject to the condition that the second differences shall be all positive; which (he adds), calling the successive second differences $\delta, \delta', \delta''$, &c., is tantamount to finding

* Such a series must, from its very nature, be *always* decreasing or increasing in the same direction.

the number of ways that the equation $r = \delta + 2\delta' + 3\delta'' +$ &c., admits of being solved by positive integers, which is obviously the same as the number of modes in which r admits of being decomposed into positive integer parts. Thus the idea of partition, which arises naturally in the first part of the process (that, namely, of the decomposition of the collective inclusive-singularity in every possible way into modes of distributive inclusive-singularity), reappears quite unexpectedly (it may almost be said miraculously), and as the result of an analytical transformation in the second part of the same.

It should be observed that the case of complete coincidence between U and V, which, supposing them to be functions of n variables, corresponds to the supposition of the same factor occurring respectively n times, $(n-1)$ times, $(n-2)$ times, &c., 2 times and 1 time in the complete determinant, the 1st minor system, the 2nd minor system, &c., the $(n-2)$th minor system and the $(n-1)$th minor system respectively, is here taken as the highest case of singularity; this and the case of non-singularity, which also adds a unit to the index of singularity, properly so called, will together make a difference of two units in the numbers given by me in the paper referred to, which numbers will accordingly be 3, 6, 14, &c., in lieu of 1, 2, 12*, &c. We are now enabled to give the following simple statement of the law for determining the total number of singularities which can exist between two quadratic forms of n variables (or if we like so to say, more generally between two linear substitution-systems of the nth order), namely the number of the singularities (including absolute unrelatedness and entire coincidence within the purview of the term) is the index of double decomposition into parts of the number n. To raise up in the mind a clear conception of the idea of double decomposition, we may proceed as follows: First. Suppose a state of things in which a body is supposed to be determined completely, provided that the number of molecules which it contains, and the different number of atoms in each molecule are given, the index of simple decomposition, that is of ordinary partitionment of the number of n, will be the number of different bodies which are capable of being formed out of n atoms. Now imagine that, for the complete determination of a body, another step in the hierarchy of aggregation is to be taken into account, and that we must know for this purpose not only the number of molecules in the body and the number of atoms in each molecule, but also the number of monads in each atom; the number of bodies (differing by definition) capable of being formed out of n monads will then represent what I mean by the index of double decomposition of (or if we like so to say), to the modulus, n. And it is obvious that this idea admits of indefinite extension, and that we may speak of the index of decomposition of any order of multiplicity (single, double, treble, &c.) of, or to the modulus, n.

* These numbers refer to quadratic homogeneous functions, containing respectively 2, 3, 4, &c. variables. For the case of functions containing but one variable there is no distinction between coincidence and unrelatedness, and the number of modes of relation is a single unit.

For single decomposition it is well known and immediately obvious, that the indices to the successive moduli given by the rational numbers in regular progression will be the coefficients of x, x^2, x^3, &c. in the continued product

$$(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1} \text{ \&c. } ad\ inf.;$$

calling these n_1, n_2, n_3, &c., it is of course obvious, as Mr Cayley has observed, that the indices of double decomposition to the same successive moduli will be the coefficients of the same arguments x, x^2, x^3, &c., in the continued product

$$(1-x)^{-n_1}(1-x^2)^{-n_2}(1-x^3)^{-n_3} \text{ \&c. } ad\ inf.;$$

and by aid of this formula he has calculated (with extreme facility) the indices in question up to the modulus 11, and found that they form the series 1, 3, 6, 14, 27, 58, 111, 223, 424, 817, 1527, which accordingly is the series representing the number of singularities capable of existing between quadratic loci commencing with 1 and ending with 11 variables.

The values of $n_1, n_2, n_3, \ldots n_{11}$, &c. themselves are given in Euler's introduction, and are respectively

1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, &c.,

which numbers will accordingly represent to their respectively corresponding moduli the number of *classes* of singularity, whether these classes be defined with reference to the different modes of distribution of the total collective singularity about different points, or with reference to the degree of the lowest system of minor determinants of the matrix to the determinant to $U + \lambda V$ having one or more factors in common, which latter is the mode of forming the classes adopted by me in the "Enumeration."

Let me be permitted to express the satisfaction which I have felt in finding this theory, which appeared to be doomed to hopeless oblivion, thus unexpectedly, after three years of interment, coming back to life, and at once filling a desired place in analytical researches pursued with apparently a totally different aim.

8.

NOTE ON A FORMULA BY AID OF WHICH AND OF A TABLE OF SINGLE ENTRY THE CONTINUED PRODUCT OF ANY SET OF NUMBERS (OR AT LEAST A GIVEN CONSTANT MULTIPLE THEREOF) MAY BE EFFECTED BY ADDITIONS AND SUBTRACTIONS ONLY WITHOUT THE USE OF LOGARITHMS.

[*Philosophical Magazine*, VII. (1854), pp. 430—436.]

INTRODUCTION.

THE remark to which this note refers is not new; it has been well observed somewhere in Gergonne's *Annales* (Mr Cayley being my informant), that by aid of the formula $4ab=(a+b)^2-(a-b)^2$ the question of finding the product of two numbers is virtually reduced to a process of addition and subtraction, and of finding the values of two squares out of a table of squares. If the two factors a and b are both even or both odd, the formula ought to be changed into

$$ab=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2;$$

if one of them is odd and the other even, we may employ the formula

$$ab=\left(\frac{a+b-1}{2}\right)^2-\left(\frac{a-b+1}{2}\right)^2+a.$$

So, again, for the product of three numbers, there exists the analogous formula

$$24abc=(a+b+c)^3-(a+b-c)^3-(b+c-a)^3-(c+a-b)^3.$$

OBJECT OF THE PAPER.

The object of this brief note is to exhibit and demonstrate the generalization of the above formulæ, that is, to express the product of any n quantities $a_1, a_2, a_3, \ldots a_n$ under the form of the sum of powers of simple linear functions of $a_1, a_2, \ldots a_n$. This may be done as follows:

GENERAL FORMULA.

Let $\theta_1, \theta_2, \theta_3, \ldots \theta_n$

be disjunctively equal to $1, 2, 3, \ldots n$,

then

$$2\,.\,4\,.\,6 \ldots (2n)\,a_1a_2\ldots a_n$$
$$=(a_{\theta_1}+a_{\theta_2}+a_{\theta_3}+\ldots+a_{\theta_n})^n-\Sigma(-a_{\theta_1}+a_{\theta_2}+\ldots+a_{\theta_n})^n$$
$$+\Sigma(-a_{\theta_1}-a_{\theta_2}+a_{\theta_3}+\ldots+a_{\theta_n})^n+\&c.$$
$$+(-)^n(-a_{\theta_1}-a_{\theta_2}-\ldots-a_{\theta_n})^n,$$

which I call the principal equation.

DEMONSTRATION OF THE PRINCIPAL EQUATION.

Let $\phi_1, \phi_2, \phi_3, \ldots \phi_{n-1}$

be disjunctively equal to

$$1, 2, 3, \ldots (n-1),$$

then it is easily seen that

$$(a_{\theta_1}+a_{\theta_2}+\ldots+a_{\theta_n})^n=(a_{\phi_1}+a_{\phi_2}+\ldots+a_{\phi_{n-1}}+a_n)^n$$
$$\Sigma(-a_{\theta_1}+a_{\theta_2}+\ldots+a_{\theta_n})^n=(a_{\phi_1}+a_{\phi_2}+\ldots+a_{\phi_{n-1}}-a_n)^n$$
$$+\Sigma(-a_{\phi_1}+a_{\phi_2}+\ldots+a_{\phi_{n-1}}+a_n)^n$$
$$\Sigma(-a_{\theta_1}-a_{\theta_2}+\ldots+a_{\theta_n})^n=\Sigma(-a_{\phi_1}+a_{\phi_2}+\ldots+a_{\phi_{n-1}}-a_n)^n$$
$$+\Sigma(-a_{\phi_1}-a_{\phi_2}+\ldots+a_{\phi_{n-1}}+a_n)^n$$
$$\&c. \quad = \quad \&c.$$
$$\Sigma(-a_{\theta_1}-a_{\theta_2}\ldots-a_{\theta_{n-1}}+a_{\theta_n})^n=\Sigma(-a_{\phi_1}-a_{\phi_2}\ldots-a_{\phi_{n-2}}+a_{\phi_{n-1}}-a_n)^n$$
$$+(-a_{\phi_1}-a_{\phi_2}\ldots-a_{\phi_{n-1}}+a_n)^n$$
$$(-a_{\theta_1}-a_{\theta_2}\ldots-a_{\theta_{n-1}}-a_{\theta_n})^n=(-a_{\phi_1}-a_{\phi_2}\ldots-a_{\phi_{n-1}}-a_n)^n.$$

Hence it is apparent that when $a_n=0$, the right-hand side of the so-called principal equation spontaneously vanishes; it will therefore always contain a_n as a factor, and by parity of reasoning it will contain every one of the quantities $a_1, a_2, \ldots a_n$ as a factor, and will consequently be equal to the product $a_1a_2\ldots a_n$ multiplied by a numerical factor, which, by making $a_1, a_2, \ldots a_n$ each equal to unity, is readily seen to be

$$2^n\times(1\,.\,2\,.\,3\ldots n),$$

or if we please so to say, $2\,.\,4\,.\,6\ldots(2n)$. Q. E. D.

Conclusion.

If n is odd and be called $2m+1$, we have

$$4.6.8\ldots(2n)\,a_1a_2\ldots a_n$$
$$=(a_{\theta_1}+a_{\theta_2}+\ldots+a_{\theta_n})^n-\Sigma(-a_{\theta_1}+a_{\theta_2}+\ldots+a_{\theta_n})^n$$
$$+\Sigma(-a_{\theta_1}-a_{\theta_2}+a_{\theta_3}+\ldots+a_{\theta_n})^n\mp\&c.$$
$$+(-)^m(-a_{\theta_1}-a_{\theta_2}\ldots-a_{\theta_m}+a_{\theta_{m+1}}+a_{\theta_{m+2}}+\ldots+a_{\theta_n})^n;$$

and if n be even and be called $2m$, we have

$$4.6.8\ldots(2n)\,a_1a_2\ldots a_n$$
$$=(a_{\theta_1}+a_{\theta_2}+\ldots+a_{\theta_n})^n-\Sigma(-a_{\theta_1}+a_{\theta_2}+\ldots+a_{\theta_n})^n$$
$$+\Sigma(-a_{\theta_1}-a_{\theta_2}+a_{\theta_3}+\ldots+a_{\theta_n})^n\mp\&c.$$
$$+\tfrac{1}{2}(-)^m\Sigma(-a_{\theta_1}-a_{\theta_2}\ldots-a_{\theta_m}+a_{\theta_{m+1}}+a_{\theta_{m+2}}+\ldots+a_{\theta_n})^n;$$

where, it should be observed, that the last term is made up of integer parts, notwithstanding the presence of the factor $\frac{1}{2}$, which factor may be construed as only serving to denote that, of any pair of complementary linear functions of those which enter into this term, such as

$$-a_{q_1}-a_{q_2}\cdots-a_{q_m}+a_{q_{m+1}}+a_{q_{m+2}}+\cdots+a_{q_n}$$

and

$$-a_{q_{m+1}}-a_{q_{m+2}}\cdots-a_{q_n}+a_{q_1}+a_{q_2}+\cdots+a_{q_m},$$

one only is to be retained. The entire term is of course made up exclusively of such pairs.

Corollary.

If $R(a_1, a_2, \ldots a_n)$ denote any symmetrical algebraic function whatever of $a_1, a_2, \ldots a_n$,

$$\Sigma_n^{\,i}{}^{0}\,\Sigma_{\nu_i}^{\,\theta}{}^{0}\,(-)^i R(-a_{\theta_1}, -a_{\theta_2}, \ldots -a_{\theta_i}, a_{\theta_{i+1}}, a_{\theta_{i+2}}, \ldots a_{\theta_n})$$

will contain $a_1a_2a_3\ldots a_n$ as a factor. In this formula ν_i denotes the number of combinations of n things taken i together.

Postscript.

In constructing a table of single entry for applying the formula

$$4ab=(a+b)^2-(a-b)^2,$$

that is,

$$ab=\frac{1}{4}(a+b)^2-\frac{1}{4}(a-b)^2,$$

it is only necessary to retain the integer part of the quarters of the squares of all the numbers from 2 to the sum of the highest of the values of a and b to which the application of the table is proposed to be restricted, because the

fractional parts of $\left(\frac{a+b}{2}\right)^2$ and $\left(\frac{a-b}{2}\right)^2$ will always destroy one another. A table for the multiplication of a ternary set of factors by means of the formula

$$abc=\frac{1}{24}(a+b+c)^3-\frac{1}{24}(a+b-c)^3-\frac{1}{24}(a-b+c)^3-\frac{1}{24}(-a+b+c)^3$$

will imply the registration of the values of the 24th parts of all numbers up to the highest value of $(a+b+c)$, and it becomes a question of some practical interest to determine in what way the fractional remainders of these 24th parts are to be dealt with.

The formula last written may give rise to either of the two subjoined cases, according as the numbers a, b, c correspond or not to the lengths of a possible triangle, namely:

$$(1)\quad abc=\frac{1}{24}N_1^3-\frac{1}{24}N_2^3-\frac{1}{24}N_3^3-\frac{1}{24}N_4^3,$$

or

$$(2)\quad abc=\frac{1}{24}N_1^3+\frac{1}{24}N_2^3-\frac{1}{24}N_3^3-\frac{1}{24}N_4^3,$$

the quantities N_1, N_2, N_3, N_4 being all supposed to represent positive integers.

A very little consideration will show, that if we neglect fractions in the table there may be entailed an error of 2, 1, 0, or -1. Whether the error is, on the one hand, an error of an even order (namely, 0 or 2), or, on the other hand, of an odd order (namely, 1 or -1), would be at once obvious by looking to see whether the formula, after neglecting the fractions, gave an odd result when the result ought to be odd, and an even result when the result ought to be even, or *vice versâ*. And the nature of the result as to whether it *ought* to be odd or even could be immediately inferred from observing whether a, b, c were or were not all of them odd numbers. But there would still remain an ambiguity in the correction to be applied in either case, arising from the doubt whether it should be zero or 2 in the one case, or whether it should be $+1$ or -1 in the other case.

This ambiguity might of course be removed by inserting in the table employed the first decimal place of $\frac{N^3}{24}$, and increasing the decimal part in the final result to unity, or lowering it to zero, according as its value might be greater or less than $\frac{1}{2}$; and it would be easy to ascertain the limits within which the decimal digit in the result must lie, and the range of values (of which 5 is one) from which it is excluded. The same end may, however, be gained much more elegantly and expeditiously, and by a method more closely analogous to that employed for the evolution of binary products, by the intervention of a very simple expedient.

The cubic residues in respect to the modulus 24 are easily verified to be as follows: 0, 1, 3, 5, 7, 8, 9, 11, 13, 15, 16, 17, 19, 21, 23. Let the tabular value of $\frac{N^3}{24}$ be made $\left[\frac{N^3}{24}\right] + K_N$, where $\left[\frac{N^3}{24}\right]$ means the integer part of the quantity within the brackets, and K_N may have any one of the three values 0, $\frac{1}{2}$, 1, namely:

$K_N = 0$ when the remainder of N^3 to the divisor 24 is 0, 1, 3 or 5;

$K_N = \frac{1}{2}$ when the said remainder is 7, 8, 9, 11, 13, 15, 16 or 17;

and

$K_N = 1$ when the remainder is 19, 21 or 23;

and let $\left[\frac{N^3}{24}\right] + K_N$ be called the cubic respondent to N, and be denoted by $R(N)$;

and let the exact value of $\frac{N^3}{24}$ be called $R'(N)$.

Let

$$R'(a+b+c) - R'(a+b-c) - R'(a-b+c) - R'(-a+b+c)$$
$$= R(a+b+c) - R(a+b-c) - R(a-b+c) - R(-a+b+c) + \Delta.$$

If in general we write $R'(n) - R(n) = E(n)$, Δ must be of one or the other of the two forms

$$E(n_1) - E(n_2) - E(n_3) - E(n_4),$$

or

$$E(n_1) + E(n_2) - E(n_3) - E(n_4),$$

where n_1, n_2, n_3, n_4 are supposed to be all positive integers. Now it is easily seen that $E(n)$ always lies within the limits $\pm\frac{5}{24}$; that is to say, it may reach up to $\frac{5}{24}$ or down to $-\frac{5}{24}$, but can never transgress these values in either direction. Hence it is obvious that Δ, which is made up of four terms, each of the form $E(n)$, can never be so great as $+1$ or so small as -1, and consequently Δ can only have one of the three values $+\frac{1}{2}$, 0, $-\frac{1}{2}$.

Hence, then, we may work with the tabular cubic respondents in lieu of the exact cubic respondents; if the result is an integer, it is good without any correction; if it is a fraction, $\frac{1}{2}$ must be added to, or taken away from it. And to ascertain which of these processes is to be applied, it is only necessary to consider whether the three factors to be multiplied are or are not all of them odd.

In practically constructing a table of cubic respondents, it would not be necessary actually to insert the fraction $\frac{1}{2}$ in any case; a dot over, or a stroke through the last integer, would serve to denote that this fraction was to be understood.

A table of quadratic respondents (that is, of the integer parts of the fourths of the square numbers) up to the base 20,000, has been actually constructed and published by a M. Antoine Voisin, under the title "Tables des Multiplications ou Logarithmes de Nombres entiers depuis 1 jusqu'à 20,000, au moyen desquelles on peut multiplier tous les nombres qui n'excèdent pas 20,000 par 20,000," &c. 12mo. à Paris, Firmin Didot, 1817. A copy of this is in Mr J. T. Graves's valuable mathematical library at Cheltenham.

By logarithms the author intends the same quantities as I term respondents, certainly a less objectionable and safer term to employ. There appears to be an error in the title in affirming that any two numbers, not separately exceeding 20,000, may be multiplied by aid of these tables, as the sum of the two factors ought not to exceed 20,000. Mr Peter Gray, so favourably known to an important section of the public as the author of many useful tables, has informed me that Major Shortredd, now in India, has computed a table of quadratic respondents extending to the argument 200,000, which he is taking measures to have published. Such tables would be very useful to computers, as they would serve for the multiplication of any two numbers whatever not containing more than five figures each. I should like to see a table of cubic respondents up to 30,000 appended to this work*.

* The best practical mode of using and arranging such a table I find, after much thought and consideration, would be as follows. It is easy to add two quantities and subtract their sum from a third by a single operation. If, then, a, b, c are the three numbers whose product it is required to find, they should be written under one another; and against (a) should be set the value of $a-b-c$; against (b), that of $b-a-c$; and against (c), that of $c-a-b$; under these three last results should be written the value of $a+b+c$; of the three former, two at least must be, all *may* be negative; their values arithmetically expressed will be of the form $K(10,000)+N$, where K is 0, 1 or 2. In order that the final process of combining the 4 cubes may be made purely additive, the tables should show the values of $(10,000)^3$ less the respondent to $K(10,000)+N$, when K is 1 or 2 for all values of N from 1 to 9999. These complements to the respondents of the simple or augmented complements of N may be termed respectively the simply and doubly affected respondents of N, but in using the tables no distinction need be drawn between the respondents and the affected respondents. The arrangement of the tables will be as follows. In each page there will be a column for the arguments, which will extend from 1 to 9999, and five other columns containing respondents and bearing respectively for their headings the numbers $\bar{2}$, $\bar{1}$, 0, 1, 2. The four quantities formed by addition, or by addition and subtraction, from a, b, c, will all be of the form $K\nu_1\nu_2\nu_3\nu_4$ ($\nu_1, \nu_2, \nu_3, \nu_4$ denoting respectively some one or other of the digits from 0 to 9), and K being one of the five symbols $\bar{2}$, $\bar{1}$, 0, 1, 2; the value corresponding to $\nu_1\nu_2\nu_3\nu_4$ will then be sought for in its proper column (according to the value of the guiding figure K), and the sum of the four values so found will be taken (the last figure to the left, which will be 2 or 3, being rejected). This result, affected, if necessary, with the proper correction of $\pm\frac{1}{2}$, will express the value of abc.

9.

ON SOME NEW THEOREMS IN ARITHMETIC.

[*Philosophical Magazine*, VIII. (1854), pp. 187—190.]

LET $S_i\,(a, b, c, \ldots k, l)$ denote, as is not unusual, the complete sum of the products of the elements (n in number) $a, b, c, \ldots k, l$, combined in every possible way i together. Let $\ddot{S}_i\,(a, b, c, \ldots k, l)$ denote the sum of the products of the same elements combined i together, but so that all combinations are excluded in which any two consecutive elements as a and b, or b and c, ... or k and l, appear simultaneously. S_i may be termed a complete sum of ith products, and $\ddot{S}_i$ a sum of products of *anakolouthic* elements, or briefly an anakolouthic sum of ith products. If we expand the continued fraction

$$\frac{1}{\rho +}\,\frac{a}{\rho +}\,\frac{b}{\rho +} \cdots \frac{k}{\rho +}\,\frac{l}{\rho},$$

it will be easily found to take the form

$$\frac{\rho^{n-1} + \ddot{S}_1{}'\rho^{n-3} + \ddot{S}_2{}'\rho^{n-5} + \&c.}{\rho^n + \ddot{S}_1\rho^{n-2} + \ddot{S}_2\rho^{n-4} + \&c.};$$

where $\ddot{S}_i{}'$ is intended to denote the anakolouthic sum of the ith products of $b, c, \ldots l$, and $\ddot{S}_i$ the anakolouthic sum of the ith products of $a, b, c, \ldots l$.

It is this fact, and the close relation of reciprocity in which the generating continued fraction for anakolouthic sums stands to ordinary continued fractions (a reciprocity which becomes more apparent when ρ is made unity), which gives a peculiar importance to the theory of anakolouthic sums of the kind denoted by $\ddot{S}$; otherwise we might be tempted to embark upon a premature generalization, extending the force of the term anakolouthic so as to denote by S a sum of products in which no *three* consecutive elements came together, $\ddot{S}$ a sum of products in which no four consecutive elements came together, and so on; these more general forms of anakolouthic sums may hereafter merit and reward attention, but my present business will be exclusively with a statement of some remarkable properties which have accidentally fallen under my observation, of anakolouthic sums of the kind first mentioned, and referring to elements formed in a manner presently to be

explained, from the natural progression of numbers. In order to familiarize the reader with the construction of anakolouthic series, I subjoin the following examples:

$$\ddot{S}_1\,(abcde) = a + b + c + d + e,$$
$$\ddot{S}_2\,(abcde) = ac + ad + ae + bd + be + ce,$$
$$\ddot{S}_3\,(abcde) = ace,$$
$$\ddot{S}_4\,(abcde) = 0,$$
$$\ddot{S}_4\,(abcdef) = 0,$$
$$\ddot{S}_4\,(abcdefg) = aceg,$$
$$\ddot{S}_4\,(abcdefgh) = aceg + aceh + bdfh.$$

First Theorem. Let n be any odd number; form the $\frac{1}{2}(n-1)$ elements

$$n,\ 2\,(n-1),\ 3\,(n-2) \ldots \frac{n-1}{2} \cdot \frac{n+3}{2};$$

the anakolouthic sum of the ith products of these elements is equal to the ith power of negative unity into the complete sum of the $2i$th products of the elements $n, -(n-2), (n-4), \ldots \pm 1$. Thus suppose $n = 7$, the elements for the anakolouthic sums will be

$$7,\ 12,\ 15;$$

and for the complete sums,

$$7,\ -5,\ 3,\ -1;$$

and we find

$$\ddot{S}_1\,(7, 12, 15) = 7 + 12 + 15 = 34, \quad S_2\,(7, -5, 3, -1) = -7.3 - 5.2 - 3 = -34,$$
$$\ddot{S}_2\,(7, 12, 15) = 7.15 = 105, \quad S_4\,(7, -5, 3, -1) = 1.3.5.7 = 105.$$

Or, again, if $n = 9$, the one set of elements will be

$$9,\ 16,\ 21,\ 24,$$

and the other set

$$9,\ -7,\ 5,\ -3,\ 1;$$

and we have

$$-(9 + 16 + 21 + 24) = -70 = 9 \times (-4) + 7\,(-3) + 5\,(-2) + 3\,(-1),$$
$$9.21 + 9.24 + 16.24$$
$$= 789 = 9.7.5.3 + 9.7.3.1 - 9.7.5.1 - 9.5.3.1 + 7.5.3.1.$$

Second Theorem. Take away the last element belonging to the anakolouthic group above written, so as to reduce the elements to the following sequence:

$$n,\ 2\,(n-1),\ 3\,(n-2) \ldots \frac{n-3}{2} \cdot \frac{n+5}{2};$$

$\frac{1}{2}(n+1)$ times the anakolouthic sum of ith products of this sequence will be equal to $(-1)^i$ multiplied by the complete sum of the $(2i+1)$th products of the series $n, -(n-2), (n-4), \dots \pm 1$. Thus if $n=9$, the two series of elements are respectively

$$9,\ 16,\ 21; \qquad 9,\ -7,\ 5,\ -3,\ 1;$$

and we find

$$5 \,.\, 1 = 9 - 7 + 5 - 3 + 1,$$

$$5 \,.\, (9 + 16 + 21) = 230 = 9.7.5 - 9.7.3 + 9.7.1 + 9.5.3 - 9.5.1 + 9.3.1 - 7.5.3 + 7.5.1 - 7.3.1 + 5.3.1,$$

$$5 \,.\, (9 \,.\, 21) = 9.7.5.3.1.$$

I now pass on to the cases where n is an even number.

Third Theorem. Let n be of the form $4m+k$, where k is zero or 2; construct the sequence

$$1.n,\ 2(n-1),\ 3(n-2) \dots \left(\frac{n}{2}-1\right)\left(\frac{n}{2}+2\right);$$

the ith anakolouthic series of products formed out of these elements is equal to the ith complete series of products formed out of the elements $(n-2)^2, (n-6)^2, \dots (k+2)^2$.

Ex. Let $n=10$, the two sequences will be

$$10,\ 18,\ 24,\ 28,$$

$$64,\ 16,$$

and we have

$$10 + 18 + 24 + 28 = 80 = 64 + 16,$$

$$10.24 + 10.28 + 18.28 = 1024 = 64.16.$$

So, if $n=12$, the two sequences will be

$$12,\ 22,\ 30,\ 36,\ 40,$$

$$100,\ 36,\ 4;$$

and we have

$$12 + 22 + 30 + 36 + 40 = 140 = 100 + 36 + 4,$$

$$12.(30 + 36 + 40) + 22.(36 + 40) + 30.40 = 4144 = 100.36 + 100.4 + 36.4,$$

$$12.30.40 = 4.36.100.$$

Fourth Theorem. If n be any even number, and we form the three sequences

$$1.n,\quad 2(n-1),\quad 3(n-2) \dots \frac{n}{2}\left(\frac{n}{2}+1\right),$$

$$1.(n+2),\ 2(n+1),\ 3(n) \dots \frac{n}{2}\left(\frac{n}{2}+3\right),$$

$$1.n,\quad 2(n-1),\quad 3(n-2) \dots \left(\frac{n}{2}-2\right)\left(\frac{n}{2}+3\right),$$

the ith anakolouthic sum in respect to the second sequence less the ith anakolouthic sum in respect to the first sequence is equal to $\frac{n}{2}\left(\frac{n}{2}+1\right)$ into the $(i-1)$th anakolouthic sum in respect to the third sequence.

Ex. Take the three sequences

$$1.10,\ 2.9,\ 3.8,\ 4.7,\ 5.6,$$

$$1.12,\ 2.11,\ 3.10,\ 4.9,\ 5.8,$$

$$1.10,\ 2.9,\ 3.8.$$

These, written out with simple elements, are as follows:

$$10\quad 18\quad 24\quad 28\quad 30,$$

$$12\quad 22\quad 30\quad 36\quad 40,$$

$$10\quad 18\quad 24;$$

and we have

$$(12 + 22 + 30 + 36 + 40) - (10 + 18 + 24 + 28 + 30) = 30 \,.\, 1,$$

$$\{12.(30 + 36 + 40) + 22.(36 + 40) + 30.40\} - \{10.(24 + 28 + 30) + 18.(28 + 30) + 24.30\} = 4144 - 2584 = 1560 = 30.(10 + 18 + 24),$$

$$12.30.40 - 10.24.30 = 14400 - 7200 = 7200 = 30.(10.24).$$

These four theorems are only particular cases of one much more general relating to a determinant, to which I was led by my method of integrating the system of two partial differential equations to the general invariant of a function or system of functions of two variables. In like manner the integration of the system of t partial differential equations to the general invariant of a function or system of functions of t variables conducts to a determinant*, of which *à priori* we know the constitution, and which will (save as to the periodic occurrence of a single factor λ) resolve itself into factors of the form $\lambda^t \pm m^t$, m being an integer; and thus promises to lay open a road to the discovery of a new genus of theorems relating to the powers of the natural progression of integer numbers, destined apparently to occupy a sort of neutral ground between the formal and quantitative arithmetics.

* The integration of this system of equations always depends essentially upon the integration of one homogeneous equation which is doubly linear, that is of the first degree in the variables, and also of the first degree in respect to the order of the differentiations; such an equation can always be integrated, and the integral will depend upon the solution of an algebraical equation expressed by equating a certain determinant to zero.

10.

NOTE ON BURMAN'S LAW FOR THE INVERSION OF THE INDEPENDENT VARIABLE.

[*Philosophical Magazine*, VIII. (1854), pp. 535—540.]

THIS Note refers to the development of the nth differential coefficient of u in respect to x in terms of the nth and lower differential coefficients of x in respect to u.

The late Mr Gregory, in his very valuable book of examples on the Calculus, in alluding to this development, speaks of it as "extremely complicated, and involving so much preliminary matter for its demonstration," that he contents himself "with referring to a memoir by Mr Murphy on the subject in the *Philosophical Transactions*, 1837, p. 210." The development there given is of course essentially no other than that included in Burman's general formula. I recently have had occasion (as a preliminary step to the investigation of the laws of inverse transformation between two systems of t variables each, instead of between two single variables only, an investigation in which I have already made such progress that I expect shortly to be in possession of the general formula for the purpose) to reconsider what I shall term Burman's law, and have been somewhat surprised to find that, so far from affording a complicated expression, it does, when properly stated, give rise to an expression of the very simplest form that could be conceived or desired, and one that admits of an easy and elementary proof.

To fix the ideas, let us take the case of $\dfrac{d^7u}{dx^7}$, where $x = \phi u$. For greater brevity write $\dfrac{d^rx}{du^r}$ as x_r. The most cursory consideration will suffice to show, irrespective of all calculation, that we should have the following form of expansion, namely,

$$\begin{aligned}\frac{d^7u}{dx^7} = &- x_7 \div x_1^8\\ &+ \{(2,6)\,x_2x_6 + (3,5)\,x_3x_5 + (4,4)\,x_4x_4\} \div x_1^9\\ &- \{(2,2,5)\,x_2x_2x_5 + (2,3,4)\,x_2x_3x_4 + (3,3,3)\,x_3x_3x_3\} \div x_1^{10}\\ &+ \{(2,2,2,4)\,x_2x_2x_2x_4 + (2,2,3,3)\,x_2x_2x_3x_3\} \div x_1^{11}\\ &- \{(2,2,2,2,3)\,x_2x_2x_2x_2x_3\} \div x_1^{12}\\ &+ (2,2,2,2,2,2) \div x_1^{13}.\end{aligned}$$

In the first group of a single term, 7 is taken in one part, in the second group of 3 terms, 8 is taken in every possible way of partition in two parts, in the third group of 3 terms, 9 is taken in every possible way of partition in three parts, and so on, until finally 12, that is, the double of the number next inferior to the given index 7, is taken in the sole possible way in which it can be taken of six parts; I ought to add, that in the groups of indices, *unity* is always understood to be inadmissible.

The groups of indices in the parentheses indicate numerical coefficients to be determined, and the whole and sole real difficulty (if any) of the question consists in determining the value of these numerical symbols. Now the law which furnishes these values would be seen on the most perfunctory examination to be the very simplest law that could possibly be stated, namely, any such symbol as $(r, s, t, \ldots)$ is to be understood to denote the number of distinct ways in which a number of things equal to the sum of the indices r, s, t, &c. admit of being thrown into combination groups of r, s, t, &c.!

Thus, for example,

$$(2,6) = \frac{8!}{2!\,6!} = 28, \qquad (3,5) = \frac{8!}{3!\,5!} = 56, \qquad (4,4) = \frac{1}{2}\frac{8!}{(4!)^2} = 35,$$

$$(2,2,5) = \frac{1}{2}\frac{9!}{(2!)^2\,5!}, \qquad (2,3,4) = \frac{9!}{2!\,3!\,4!}, \qquad (3,3,3) = \frac{1}{6}\frac{9!}{(3!)^3},$$

$$(2,2,2,4) = \frac{1}{3!}\frac{10!}{(2!)^3\,4!}, \qquad (2,2,3,3) = \frac{1}{(2!)^2}\frac{10!}{(2!)^2\,(3!)^2},$$

and so on. The general law is obvious; and to prove its applicability in general, we have only to show that if it be true for the case of $\dfrac{d^ru}{dx^r}$, it is true for $\dfrac{d^{r+1}u}{dx^{r+1}}$. The proof is as follows. Let in general $[l, m, n, \&c.]$ indicate the value of

$$\frac{1\,.\,2\,.\,3 \ldots\ldots (l + m + n + \&c.)}{1\,.\,2 \ldots l \times 1\,.\,2 \ldots m \times 1\,.\,2 \ldots n \times \&c.},$$

without reference to l, m, n, &c. being equal or unequal *inter se*.

Lemma 1. It is very easily seen that

$$[l, m, n, \&c.] = [l-1, m, n, \&c.] + [l, m-1, n, \&c.] + [l, m, n-1, \&c.] + \&c.$$

If now we use the notation $[\rho^r, \sigma^s, \tau^t, \ldots]$ as an abbreviated form of the notation $[\rho, \rho, \rho \ldots$ to r terms, $\sigma, \sigma, \sigma \ldots$ to s terms, $\tau, \tau \ldots$ to t terms, &c.], it is obvious that the equation last written becomes

$$[\rho^r, \sigma^s, \tau^t, \ldots] = r[\rho - 1, \rho^{r-1}, \sigma^s, \tau^t, \ldots] + s[\rho^r, \sigma - 1, \sigma^{s-1}, \tau^t, \ldots] \\ + t[\rho^r, \sigma^s, \tau - 1, \tau^{t-1}, \ldots] + \ldots .$$

Lemma 2. Let $C(\rho^r, \sigma^s, \tau^t, \ldots)$ denote the number of ways in which $r\rho + s\sigma + t\tau + \ldots$ can be taken in combinations of $\rho, \rho \ldots$ to r places, $\sigma, \sigma \ldots$ to s places, &c., then upon the supposition that ρ, σ, τ, &c., which are to be understood as arranged in an ascending order of magnitude, are all unequal, we shall have

$$C(\rho^r, \sigma^s, \tau^t, \ldots) = [\rho^r, \sigma^s, \tau^t, \ldots]/r!\,s!\,t!\ldots,$$

which by Lemma 1

$$= \frac{[\rho-1, \rho^{r-1}, \sigma^s, \tau^t, \ldots]}{(r-1)!\,s!\,t!\ldots} + \frac{[\rho^r, \sigma-1, \sigma^{s-1}, \tau^t, \ldots]}{r!\,(s-1)!\,t!\ldots} + \frac{[\rho^r, \sigma^s, \tau-1, \tau^{t-1}, \ldots]}{r!\,s!\,(t-1)!\ldots} + \ldots$$

$$= C(\rho-1, \rho^{r-1}, \sigma^s, \tau^t, \ldots) + \{1 + rF(\sigma-\rho)\}\, C(\rho^r, \sigma-1, \sigma^{s-1}, \tau^t, \ldots) \\ + \{1 + sF(\tau-\sigma)\}\, C(\rho^r, \sigma^s, \tau-1, \tau^{t-1}, \ldots) + \ldots .$$

$F(\sigma-\rho)$, $F(\tau-\sigma)$, &c. meaning quantities which are respectively zero when $\sigma - 1 > \rho$, $\tau - 1 > \sigma$, &c., and respectively units when $(\sigma - 1) = \rho$, $\tau - 1 = \sigma$, &c.; for it will be obvious that if $\sigma - 1 = \rho$, the quantity

$$[\rho^r, \sigma-1, \sigma^{s-1}, \tau^t, \ldots]$$

becomes $$[\rho^{r+1}, \sigma^{s-1}, \tau^t, \ldots],$$

and consequently when divided by $r!\,(s-1)!\,t!\ldots$ does not give

$$C(\rho^{r+1}, \sigma^{s-1}, \tau^t, \ldots),$$

but $$(r+1)\,C(\rho^{r+1}, \sigma^{s-1}, \tau^t, \ldots),$$

and so similarly for the cases of $\tau - 1 = \sigma$, &c.

Now let us suppose that we are considering any group ($\rho, \rho \ldots$ to r places, $\sigma, \sigma \ldots$ to s places, &c.), or more briefly $(\rho^r, \sigma^s, \tau^t, \ldots)$, the numerical coefficient of the term $x_\rho{}^r x_\sigma{}^s x_\tau{}^t \ldots$ in the inverse development of $\frac{d^\mu x}{du^\mu}$.

And first, suppose that ρ is not 2.

The coefficient in question will evidently be made up exclusively of the following parts, each, however, affected with the factor $(-)^{N-1}$, derived from

the expansion of $\frac{d^{\mu-1}x}{du^{\mu-1}}$, for which the law to be established is supposed to hold, namely,

$$\left.\begin{aligned} & C(\rho-1, \rho^{r-1}, \sigma^s, \tau^t, \ldots) \\ + \{1 + rF(\sigma-\rho)\}\ & C(\rho^r, \sigma-1, \sigma^{s-1}, \tau^t, \ldots) \\ + \{1 + sF(\tau-\sigma)\}\ & C(\rho^r, \sigma^s, \tau-1, \tau^{t-1}, \ldots) \\ + \&c. & \end{aligned}\right\} \quad (\vartheta),$$

each part being affected with the factor $(-1)^{N-1}$, derived from the differentiations performed upon

$$x_{\rho-1} x_\rho{}^{r-1} x_\sigma{}^s x_\tau{}^t \ldots \div x_1{}^N,$$

$$x_\rho{}^r x_{\sigma-1} x_\sigma{}^{s-1} x_\tau{}^t \ldots \div x_1{}^N,$$

$$x_\rho{}^r x_\sigma{}^s x_{\tau-1} x_\tau{}^{t-1} \ldots \div x_1{}^N,$$

&c.

Secondly, suppose ρ, the lowest index, is 2, then the term

$$x_{\rho-1} x_\rho{}^r x_\sigma{}^s x_\tau{}^t \ldots$$

must be rejected, because $x_{\rho-1}$ becomes x_1, which is excluded from appearing in any numerator. But then, *per contra*, in this case there will be a portion of the coefficient derivable from the differentiation of the denominator of the term

$$(-)^{N-2} \cdot \frac{(2^{r-1}, \sigma^s, \tau^t \ldots)\, x_2{}^{r-1} x_\sigma{}^s x_\tau{}^t \ldots}{x_1{}^{N-1}},$$

where $$(N-1) = 1 + (r-1)\,2 + s\sigma + t\tau + \&c.$$

This portion will be

$$(-)^{N-1}(N-1)\, C(2^{r-1}, \sigma^s, \tau^t, \ldots),$$

or, which is the same thing,

$$C(1, 2^{r-1}, \sigma^s, \tau^t, \ldots),$$

and therefore the portion of the coefficient corresponding to $x_{\rho-1} x_\rho{}^r x_\sigma{}^s x_\tau{}^t \ldots$, &c. is supplied from another source, and the expression (ϑ) remains good for *all* values of ρ, σ, τ, &c., and consequently, by virtue of the second lemma, is equal to $C(\rho^r, \sigma^s, \tau^t, \ldots)$; and thus we see that if the law assumed is true for $\frac{d^r u}{dx^r}$ it remains true for $\frac{d^{r+1} u}{dx^{r+1}}$, as was to be shown. And as it is evidently true for $r = 1$, it is true generally.

POSTSCRIPT.

The formula expressing Burman's law may be exhibited as follows: x_r will still be understood to denote $\frac{d^r x}{du^r}$, and $C\{p, q, \ldots m\}$ will, as before, denote the number of distinct modes of combining $p+q+\ldots+m$ things in sets of $p, q, \ldots m$ at a time; so that, for example, $C\{2, 2, 4, 4, 4\}$ will denote

$$\frac{1 \times 2 \times 3 \ldots \times 16}{(1.2)^2.(1.2.3.4)^3} \cdot \frac{1}{1.2} \cdot \frac{1}{1.2.3}.$$

Let now $n-1$ be broken up *without restriction in every possible way* into parts, and let $r, s, t \ldots l$ denote one such system of parts so that

$$r+s+t+\ldots+l=n-1,$$

r, s, &c. being all *actual* positive integers. Then is $\frac{d^n u}{dx^n}$ equal to

$$\Sigma C\{(1+r), (1+s), (1+t) \ldots (1+l)\} \cdot \frac{1}{x_1^n} \cdot \left\{\frac{-x_{1+r}}{x_1} \cdot \frac{-x_{1+s}}{x_1} \cdot \frac{-x_{1+t}}{x_1} \ldots \frac{-x_{1+l}}{x_1}\right\},$$

than which nothing more clear and simple can be desired or imagined. And so more generally, if we make, as before, $r+s+t+\ldots+l=n-g$, and give g in succession every different value from 1 to n, we shall have $\frac{d^n \vartheta}{dx^n}$ equal to

$$\Sigma\Sigma\left[[\{(1+r), (1+s), \ldots (1+l)\}, (g-1)] \frac{d^g \vartheta}{du^g} \cdot \frac{1}{x_1^n} \left(\frac{-x_{1+r}}{x_1} \cdot \frac{-x_{1+s}}{x_1} \ldots \frac{-x_{1+l}}{x_1}\right)\right],$$

where $[\{(1+r), (1+s), \ldots (1+l)\}, (g-1)]$ means the number of ways in which $(1+r)+(1+s)+\ldots+(1+l)+(g-1)$ elements can be partitioned off into groups of one kind containing respectively $(1+r), (1+s), \ldots (1+l)$ of the elements, and into a group of another kind containing the remainder $(g-1)$ of the elements. This distinction of the groups into two kinds has no effect upon the result except when $g-1$ is equal to any of the numbers $(1+r)$, $(1+s), \ldots (1+l)$. If we write, according to the notation above employed, $(1+r), (1+s), \ldots (1+l)$ under the form $(\alpha^a, \beta^b, \ldots \gamma^c)$, then

$$[\{(1+r), (1+s), \ldots (1+l), (g-1)\}]$$

will represent

$$\frac{(a\alpha+b\beta+\ldots+c\gamma+g-1)!}{a!(\alpha!)^a b!(\beta!)^b \ldots c!(\gamma!)^c (g-1)!}.$$

This more general theorem may of course be demonstrated by a similar method to that employed in the text for the case of $\vartheta=u$, for which all the terms in the expansion vanish except those in which $g=1$.

I have, since this paper was sent to the press, obtained a new solution of the far more difficult and interesting question of the change from one *system* of independent variables to another system*. I say a new solution, because one has already been *virtually* effected, but under a form leaving much to be desired, by the great Jacobi in his Memoir *De Resolutione Æquationum per series infinitas*, Crelle, Vol. VI. 1830. In my solution, a remarkable species of quantities, to which I give the name of Arborescent Functions, make their appearance in analysis for the first time.

[* p. 65 below.]

11.

ON DIFFERENTIAL TRANSFORMATION AND THE REVERSION OF SERIESES*.

[*Proceedings of the Royal Society of London*, VII. (1856), pp. 219—223.]

[Also *Philosophical Magazine*, IX. (1855), pp. 391—394.]

WITH a view to its publication in the *Proceedings* of the Society, I take occasion to communicate the result of my investigations, as far as they have yet extended, into the general theory of differential transformations, containing a complete and general solution of the important problem of expanding a given partial differential coefficient of a function in respect of one system of independent variables in terms of the partial differential coefficients thereof, in respect to a second system of independent variables, each respectively given as explicit functions of the first set.

This question may be shown to be exactly coincident with that of the reversion of simultaneous serieses proposed by Jacobi, which may be thus stated: given $(n+1)$ quantities, each expressed by rational infinite serieses as functions of n others; required to express any one of the first set in a rational infinite series in terms of the other n of the same set. This question has only been resolved by Jacobi for a particular case; the result hereunder given for the transformation of differential coefficients contains the solution of the general question. My method of investigation is entirely different from that adopted by the great Jacobi, and I hope in a short time to be able to lay it in a complete form before the Society, and probably to add a solution of the still more general question comprising the reversion of serieses as a particular case, namely, the question of expressing any one of n quantities connected by m equations in terms of any $(n-m)$ others of the same.

Let there be any number of variables, say u, v, w, of which x, y, z, ϑ are given functions, it is required to expand

$$\left(\frac{d}{dx}\right)^f \left(\frac{d}{dy}\right)^g \left(\frac{d}{dz}\right)^h \vartheta$$

in terms of the partial differential coefficients of ϑ, x, y, z in respect of u, v, w.

[* See p. 65, below.]

Form the determinant

$$\begin{vmatrix} \frac{dx}{du}, & \frac{dx}{dv}, & \frac{dx}{dw} \\ \frac{dy}{du}, & \frac{dy}{dv}, & \frac{dy}{dw} \\ \frac{dz}{du}, & \frac{dz}{dv}, & \frac{dz}{dw} \end{vmatrix},$$

which call J.

The required expansion will contain in each term an integer numerical coefficient, a power of $\frac{1}{J}$, one factor of the form

$$\left(\frac{d}{du}\right)^p \left(\frac{d}{dv}\right)^q \left(\frac{d}{dw}\right)^r \vartheta,$$

and other factors of the form

$$\left(\frac{d}{du}\right)^l \left(\frac{d}{dv}\right)^m \left(\frac{d}{dw}\right)^n x,$$

$$\left(\frac{d}{du}\right)^{l'} \left(\frac{d}{dv}\right)^{m'} \left(\frac{d}{dw}\right)^{n'} y,$$

$$\left(\frac{d}{du}\right)^{l''} \left(\frac{d}{dv}\right)^{m''} \left(\frac{d}{dw}\right)^{n''} z.$$

Let the latter class of factors be distinguished into two sets, those where $l+m+n=1$,

$$\left(\begin{matrix} & l=1 & m=0 & n=0 \\ \text{or} & l=0 & m=1 & n=0 \\ \text{or} & l=0 & m=0 & n=1 \end{matrix}\right),$$

which I shall call uni-differential factors, and those in which $l+m+n>1$, which I shall call pluri-differential factors.

First, then, as to the form of the general term abstracting from the numerical coefficient and the uni-differential factors (except of course so far as they enter into J). This will be as follows:

$$\left(\frac{d}{du}\right)^{{}^1l_1} \left(\frac{d}{dv}\right)^{{}^1m_1} \left(\frac{d}{dw}\right)^{{}^1n_1} x \times \left(\frac{d}{du}\right)^{{}^2l_1} \left(\frac{d}{dv}\right)^{{}^2m_1} \left(\frac{d}{dw}\right)^{{}^2n_1} x \times \ldots \left(\frac{d}{du}\right)^{{}^{e_1}l_1} \left(\frac{d}{dv}\right)^{{}^{e_1}m_1} \left(\frac{d}{dw}\right)^{{}^{e_1}n_1} x$$

$$\times \left(\frac{d}{du}\right)^{{}^1l_2} \left(\frac{d}{dv}\right)^{{}^1m_2} \left(\frac{d}{dw}\right)^{{}^1n_2} y \times \ldots\ldots \times \left(\frac{d}{du}\right)^{{}^{e_2}l_2} \left(\frac{d}{dv}\right)^{{}^{e_2}m_2} \left(\frac{d}{dw}\right)^{{}^{e_2}n_2} y$$

$$\times \left(\frac{d}{du}\right)^{{}^1l_3} \left(\frac{d}{dv}\right)^{{}^1m_3} \left(\frac{d}{dw}\right)^{{}^1n_3} z \times \ldots\ldots \times \left(\frac{d}{du}\right)^{{}^{e_3}l_3} \left(\frac{d}{dv}\right)^{{}^{e_3}m_3} \left(\frac{d}{dw}\right)^{{}^{e_3}n_3} z$$

$$\times \left(\frac{d}{du}\right)^p \left(\frac{d}{dv}\right)^q \left(\frac{d}{dw}\right)^r \vartheta \times \frac{1}{J^\omega},$$

subject to the limitations about to be expressed.

Call
$$ {}^1l_1 + {}^2l_1 + \ldots + {}^{e_1}l_1 = L_1, $$
$$ {}^1l_2 + {}^2l_2 + \ldots + {}^{e_2}l_2 = L_2, $$
$$ {}^1l_3 + {}^2l_3 + \ldots + {}^{e_3}l_3 = L_3, $$
and form the analogous quantities M_1, M_2, M_3; N_1, N_2, N_3. Then we must have
$$ L_1 + L_2 + L_3 + M_1 + M_2 + M_3 + N_1 + N_2 + N_3 + p + q + r = f + g + h + e_1 + e_2 + e_3; $$
and as the sum of any group of indices l, m, n must not be less than 2, we have
$$ f + g + h + e_1 + e_2 + e_3 + p + q + r, \text{ not less than } 2e_1 + 2e_2 + 2e_3, $$
so that $e_1 + e_2 + e_3$ must not exceed $f + g + h + p + q + r$; furthermore, $p+q+r$ must not exceed $f + g + h$; and finally,
$$ \omega = f + g + h + e_1 + e_2 + e_3. $$

1. We may first take $e_1 + e_2 + e_3 = E$, giving to E in succession all integer values from $f + g + h$ to $2f + 2g + 2h$, and find all possible solutions of this equation with permutations between the values of e_1, e_2, e_3.

2. We may then take $p + q + r = s$, giving s in succession all integer values from 1 to $f + g + h$, and find all possible solutions of this equation with permutations between f, g, h.

3. We may then take $L + M + N = f + g + h + E - s$, and find all the values of L, M, N, with permutations allowable between the values of L, M, N.

4. We may then take
$$ L_1 + L_2 + L_3 = L, $$
$$ M_1 + M_2 + M_3 = M, $$
$$ N_1 + N_2 + N_3 = N, $$
and solve these several equations in every way possible, with permutations as before.

5. We must take
$$ {}^1l_1 + {}^2l_1 + \ldots + {}^{e_1}l_1 = L_1, \quad {}^1m_1 + {}^2m_1 + \ldots + {}^{e_1}m_1 = M_1, \quad {}^1n_1 + {}^2n_1 + \ldots + {}^{e_1}n_1 = N_1, $$
$$ {}^1l_2 + {}^2l_2 \quad \ldots \quad {}^{e_2}l_2 = L_2, \quad {}^1m_2 + {}^2m_2 \quad \ldots \quad {}^{e_2}m_2 = M_2, \quad {}^1n_2 + {}^2n_2 \quad \ldots \quad {}^{e}m_2 = N_2, $$
$$ {}^1l_3 + {}^2l_3 \quad \ldots \quad {}^{e_3}l_3 = L_3, \quad {}^1m_3 + {}^2m_3 \quad \ldots \quad {}^{e_3}m_3 = M_3, \quad {}^1n_3 + {}^2n_3 \quad \ldots \quad {}^{e}m_3 = N_3, $$
and solve in every possible manner these equations, but without admitting permutations between the values of ${}^1l_1, {}^2l_1 \ldots {}^{e_1}l_1$, or between the values of the members of the other of the third sets taken each *per se*, and subject to the

condition that every such sum as ${}^rl_i + {}^rm_i + {}^rn_i$ must be greater than unity. Every possible system of values of these nine sets will furnish a corresponding pluri-differential part to the general term.

Next, as to the uni-differential part, we may form the quantity
$$ \left(\frac{dy}{dv}\frac{dz}{dw} - \frac{dy}{dw}\frac{dz}{dv}\right)^{\lambda_1} \left(\frac{dy}{dw}\frac{dz}{du} - \frac{dy}{du}\frac{dz}{dw}\right)^{\mu_1} \left(\frac{dy}{du}\frac{dz}{dv} - \frac{dy}{dv}\frac{dz}{du}\right)^{\nu_1}, $$
$$ \left(\frac{dz}{dv}\frac{dx}{dw} - \frac{dz}{dw}\frac{dx}{dv}\right)^{\lambda_2} \left(\frac{dz}{dw}\frac{dx}{du} - \frac{dz}{du}\frac{dx}{dw}\right)^{\mu_2} \left(\frac{dz}{du}\frac{dx}{dv} - \frac{dz}{dv}\frac{dx}{du}\right)^{\nu_2}, $$
$$ \left(\frac{dx}{dv}\frac{dy}{dw} - \frac{dx}{dw}\frac{dy}{dv}\right)^{\lambda_3} \left(\frac{dx}{dw}\frac{dy}{du} - \frac{dx}{du}\frac{dy}{dw}\right)^{\mu_3} \left(\frac{dx}{du}\frac{dy}{dv} - \frac{dx}{dv}\frac{dy}{du}\right)^{\nu_3}, $$
where
$$ \lambda_1 + \lambda_2 + \lambda_3 = L + p, $$
$$ \mu_1 + \mu_2 + \mu_3 = M + q, $$
$$ \nu_1 + \nu_2 + \nu_3 = N + r. $$
These equations are to be solved in every possible manner with permutations between the members of the λ set, the μ set, and the ν set. Finally, we have to consider the numerical coefficient. To give a perfect representation of this, we must ascertain what identities exist in the factors of the pluri-differential part. Let us suppose that one set of operators upon x is repeated θ_1 times, another θ_2 times, and so on, giving rise to the powers $\theta_1, \theta_2, \ldots\ldots \theta_a$ in the x line. Similarly, form $\phi_1, \phi_2, \ldots \phi_\beta$ from the y line, and $\psi_1, \psi_2, \ldots \psi_\gamma$ from the z line. Then the numerical part of the general term will be
$$ \frac{\Pi(\lambda_1 + \mu_1 + \nu_1)\,\Pi(\lambda_2 + \mu_2 + \nu_2)\,\Pi(\lambda_3 + \mu_3 + \nu_3)}{\Pi\lambda_1\Pi\mu_1\Pi\nu_1\Pi\lambda_2\Pi\mu_2\Pi\nu_2\Pi\lambda_3\Pi\mu_3\Pi\nu_3} $$
$$ \times \frac{\Pi(L+p)\,\Pi(M+q)\,\Pi(N+r)}{\left\{\begin{matrix} \Pi\,{}^1l_1\Pi\,{}^1m_1\Pi\,{}^1n_1\Pi\,{}^2l_1\Pi\,{}^2m_1\Pi\,{}^2n_1 \ldots\ldots \\ \Pi\,{}^1l_2\Pi\,{}^1m_2\Pi\,{}^1n_2\Pi\,{}^2l_2\Pi\,{}^2m_2\Pi\,{}^2n_2 \ldots\ldots \\ \Pi\,{}^1l_3\Pi\,{}^1m_3\Pi\,{}^1n_3\Pi\,{}^2l_3\Pi\,{}^2m_3\Pi\,{}^2n_3 \ldots\ldots \end{matrix}\right\}} $$
$$ \times \frac{D}{\Pi\theta_1\Pi\theta_2\ldots\Pi\theta_a\Pi\phi_1\Pi\phi_2\ldots\Pi\phi_\beta\Pi\psi_1\Pi\psi_2\ldots\Pi\psi_\gamma}, $$
where in general Πm means $1.2.3 \ldots m$: as regards D, it is the following determinant, namely,
$$ \begin{vmatrix} \lambda_1+\mu_1+\nu_1 & \nu & \nu & L_3 & M_3 & N_3 \\ \nu & \lambda_2+\mu_2+\nu_2 & \nu & L_2 & M_2 & N_2 \\ \nu & \nu & \lambda_3+\mu_3+\nu_3 & L_1 & M_1 & N_1 \\ \lambda_1 & \lambda_2 & \lambda_3 & L_1+L_2+L_3+p & \nu & \nu \\ \mu_1 & \mu_2 & \mu_3 & \nu & M_1+M_2+M_3+q & \nu \\ \nu_1 & \nu_2 & \nu_3 & \nu & \nu & N_1+N_2+N_3+r \end{vmatrix}. $$

The result, for greater brevity, has been set out in the above pages for the case of ϑ, a function of three variables, but the reader can have no difficulty in extending the statement to any number. In the case of a single variable, the formula can easily be identified with that given by Burman's law. It is noticeable that the determinant written is of the form

$$Apqr + Bpq + Cqr + Drp + Ep + Fq + Gr,$$

the part independent of p, q, r being easily seen to vanish. Moreover, the coefficients $A, B, C, \ldots$ are all essentially positive, so that the determinant can only vanish (except for $p=0$, $r=0$, $q=0$) by virtue of one condition at least more than the number of the variables.

12.

A TRIFLE ON PROJECTILES.

[*Philosophical Magazine*, XI. (1856), pp. 450—453.]

IN teaching the subject of projectiles *in vacuo*, the following solution has presented itself to me of a question not wholly without practical interest, namely, of determining the angle of projection to give the best range in the most general case, namely, when a gun is fired upon a slope at a given vertical height above the slope. The solution is not wholly either without theoretical interest in point of method, as leading to a result of some little complexity in maxima and minima by very simple calculations, and without the aid of the differential calculus. Therefore I venture to submit it to the readers of the *Philosophical Magazine*. In the next number of the Magazine I hope to have leisure to lay before them a subject of much greater interest, also belonging to the theory of projectiles, showing how, by the oblique action of gravity combined with the earth's rotation, a pendulum suitably adjusted may be caused to advance in a westerly direction, and so the earth be made the means of impelling a light carriage without any visible motive force, or any influence of magnetism.

To this pendulum I give, for reasons which will be apparent when the matter is more clearly set forth, and in contradistinction to the ordinary fixed or circular pendulum on the one hand, and to Foucault's free or spherical pendulum on the other, the name of the *Cylindrical* or *Travelling* Pendulum. But to resume the business of this present communication: let us begin with determining the angle of projection to give the maximum range when a gun is fired from a point *in* a plane sloping at an angle i from the horizon.

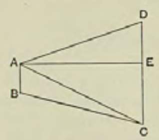
This question is most simply solved (the result itself is of course familiar to all who will read this paper) by resolving the velocity V, supposed to make an angle θ with the horizon, as also g, the accelerating force of gravity, each into two parts, V into $V\cos(\theta+i)$ and $V\sin(\theta+i)$, and g into $g\sin i$ and $g\cos i$, respectively parallel and perpendicular to the plane of the slope.

The time of flight is of course found by looking to the perpendicular part of the velocity and of gravity alone, and is evidently $2\,\dfrac{V\sin(\theta+i)}{g\cos i}$, which call τ; the range will evidently be

$$\frac{V\cos\theta\,.\,\tau}{\cos i},\ \text{that is, } \frac{V^2}{g\cos i}\,[\sin(2\theta+i)+\sin i].$$

Hence the best angle of range for this case is found by making $2\theta+i=90^\circ$, $\theta=\frac{1}{2}(90^\circ-i)$.

Now let us proceed to apply this result to the general case, as in the figure below, where BC is the slope upon which the range is to be measured, A the point of projection, AD the direction which gives the maximum range upon the slope, and BC the actual extent of this range; then I say AD is the direction which would give also the best range upon the slope AC. Since if, with the given velocity of projection, any other direction than AD would give a better range upon AC, the path corresponding to such direction must evidently cut BC at a point beyond C in that line in order to strike a point beyond C in the line AC.



Hence if we draw the horizontal line AE, we know by the preceding case that the angle $DAE=\frac{1}{2}CAB$*.

Let $CAB=\phi$, which is to be found; also let $AB=h$, and the inclination of BC to $AE=i$, h and i being given; and let $t=$ time of flight, then

$$CAD=(90^\circ-\phi)+\frac{\phi}{2}$$

$$=90^\circ-\frac{\phi}{2}.$$

Hence also $$ADC=180^\circ-\phi-\left(90^\circ-\frac{\phi}{2}\right)=90^\circ-\frac{\phi}{2}.$$

Hence $$\tfrac{1}{2}gt^2=CD=AC=h\frac{\sin ABC}{\sin ACB}$$

$$=\frac{h\cos i}{\cos(i+\phi)};$$

* This equation, and the isoscelism of the principal triangle of the figure to which it leads, would not readily present themselves to notice in the direct method of seeking the maximum range. It is for the sake of this pleasing geometrical relation, not unmixed perhaps with a desire of exhibiting the simple yet delicate turn of reasoning, the agreeable little point of method (a fly embalmed in amber) contained in the immediately preceding paragraph, that I have thought this trifle worth preserving in the pages of the Magazine.

and $$v\cos\frac{\phi}{2}t=AE=\frac{h\sin\phi\cos i}{\cos(i+\phi)}.$$

Hence eliminating t, we have

$$\frac{v^2}{gh\cos i}=\frac{(\sin\phi)^2}{1+\cos\phi}\cdot\frac{1}{\cos(i+\phi)}=\frac{1-\cos\phi}{\cos(i+\phi)}.$$

If $i=0$, that is, if the gun is fired from the top of a battery commanding a level plain, we have simply

$$\sec\phi=1+\frac{v^2}{gh},$$

which gives ϕ the double of the angle of elevation.

In other cases we may make $\phi+i=\psi$, we have then

$$\frac{1-\cos(\psi-i)}{\cos\psi}=\frac{1}{\cos\psi}-\frac{\sin\psi}{\cos\psi}\sin i-\cos i=\frac{v^2}{gh}\sec i.$$

Let $$\left(1+\frac{v^2}{gh}\sec^2 i\right)\cot i=\cot\epsilon;$$

then $$\frac{\sin i}{\sin\epsilon}\cos(\psi-\epsilon)=1,$$

$$\cos(\psi-\epsilon)=\frac{\sin\epsilon}{\sin i},$$

or $$\cos(\phi+i-\epsilon)=\frac{\sin\epsilon}{\sin i},$$

from which ϕ, the double of the angle of elevation, may be determined.

Calling $\dfrac{\sin\epsilon}{\sin i}=\cos\mu$, and taking ϕ_1, ϕ_2 as the two values of ϕ, we have

$$\phi_1+i-\epsilon=\mu,$$

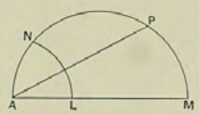
$$\phi_2+i-\epsilon=360^\circ-\mu.$$

ϕ_1, ϕ_2 correspond to the angles of projection *down* *and* *up* the slope respectively, the one affording what in an *algebraical* sense is a maximum, and the other a minimum, but of course, *arithmetically* speaking, both giving maximum values of the range.

Thus when $h=0$, so that $\sin\epsilon=0$, $\mu=90^\circ$, and $\frac{1}{2}(\phi_2-\phi_1)$ is a right angle, as may easily be verified.

It may be worth while to exhibit the geometrical construction for the case of firing from a gun in position commanding a *horizontal plane*.

Let A be the position of the gun, LN a portion of a circle of radius AL which represents the height of the gun above the plain, LM twice the height due to the velocity of projection, ANM a semicircle on AM, P the point in it bisecting the arc MN, then (abstraction made of the resistance of the air) AP is the elevation at which the gun must be pointed to give the greatest range on the plain below, for $\sec 2PAM$ obviously $= 1 + \frac{(\text{velocity of ball})^2}{g \,.\, AL}$.



Suppose a sea battery as much as 300 feet* above the water, and a cannon-ball projected at the low rate of 1200 feet per second (which is less than that of a common musket-ball), we should have twice the height due to the velocity of projection equal to 44720, and therefore

$$\sec 2\alpha = \frac{44720}{1200} + 1$$

$$= 38{,}2666,$$

and consequently $$2\alpha = 88^\circ\, 30'\, 9''$$

or $$\alpha = 44^\circ\, 15'\, 5'',$$

differing very little from 45°; showing that certainly in a non-resisting medium, and in all probability in air, the height of the point of fire above the plane which it commands will very little indeed influence, under any conceivable circumstances of practice, the angle of elevation which gives the best range.

[* The succeeding calculation uses 1200.]

13.

NOTE ON AN INTUITIVE PROOF OF THE EXISTENCE OF TWENTY-SEVEN CONICS OF CLOSEST CONTACT WITH A CURVE OF THE THIRD DEGREE.

[*Philosophical Magazine*, XI. (1856), pp. 463, 464.]

IN general a conic can only be made to have five coincident points with a curve, and if the curve be of the third degree, the conic will of course cut it in a remaining sixth point; but at certain points of the cubic all these six points may come together. How many of these are there, and where are they? This question, which originated with Steiner, who stated the number, and was subsequently treated by Plücker, who assigned the position of the points, may be resolved by very simple considerations and without calculation. For if we can succeed in putting the characteristic of the curve (I mean what is commonly, but not altogether commodiously, called "the-left-hand-side-of-the-equation-to-the-curve-when-the-right-hand-side-of-it-is-made-equal-to-zero") under the form $u^3 + v(uw + \omega^2)$, it is obvious that the conic $uw + \omega^2$ will intersect the cubic curve in the six coincident points $u^3 = 0$, $\omega^2 = 0$.

If now we take for our cubic the reduced form $x^3 + y^3 + z^3 - 6mxyz$, and make $x + y + 2mz = p$, $\rho x + \rho^2 y + 2mz = q$, $\rho^2 x + \rho y + 2mz = r$ [where ρ is an imaginary cube root of unity], it may be written under the form

$$(1 - 8m^3)\, z^3 + pqr, \text{ say } -\mu z^3 + pqr\,;$$

or, if we please, under the form

$$-\mu\,(z + kp)^3 + p\,(qr + \mu k^3 p^2 + 3\mu k^2 pz + 3\mu kz^2).$$

And if we assume k properly, $z + kp$ may be made to touch the multiplier of p, that is, the cubic may be made to take the form

$$-\mu\,(z + kp)^3 + p\,\{(z + kp)\,v + \omega^2\}.$$

From the symmetry which reigns between x and y, it is obvious *à priori* that any value of k which is rightly assumed for the object in view will make

ω (when z is eliminated from it by means of the equation $z+kp=0$) a multiple either of $x-y$ or $x+y$; the latter obviously cannot be true, since such values would make the given cubic a function of $x+y$ and z; the proper values of k will therefore make $x-y=0$, from which, combined with the equation $2x^3+z^3+6mx^2z=0$, the values of $x:y:z$ may be determined. These will be three in number; and as we may write, instead of x and y, ρx, $\rho^2 y$, or ρy, $\rho^2 x$, we obtain three sets of three points, corresponding to p being taken $x+y+2mz$; and consequently, by interchanging z with x and with y successively, we obtain altogether three systems of three sets of three points each; any such factor as $x+y+2mz$ is a tangent to a point of inflexion, and it is clear *à priori* that if the cubic is put under the form $u^3+v\,(uw+\omega^2)$, since $v=0$ makes $u^3=0$, v *can only be* a tangent at an inflexion. Hence the nine sets of three points just assigned are *all* that can be found enjoying the property in question, and it is readily seen that $x-y$ is the straight line containing the three points of intersection in which the second emanant,

$$\left(x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}\right)^2(x^3+y^3+z^3-6mxyz),$$

at the point of inflexion ($x+y=0$, $z=0$) cuts the given cubic over and above the three coincident points $x+y=0$, $z=0$. In other words, each ternary group of the twenty-seven points in question consists of the three points in which the curve is met by the tangents drawn from a point of inflexion, which agrees with the geometrical construction given by Plücker in *Crelle's Journal.*

14.

LETTER ON PROFESSOR GALBRAITH'S CONSTRUCTION FOR THE RANGE OF PROJECTILES.

[*Philosophical Magazine*, XII. (1856), pp. 112—114.]

To the Editors of the Philosophical Magazine and Journal.

GENTLEMEN,

Professor Galbraith's geometrical construction for finding the elevations of a projectile corresponding to any given velocity and given range in a plane, horizontal or sloping, is truly elegant, and, if new, constitutes a real acquisition to the subject. It might be worth while for its accomplished author to see if some analogous construction can be found extending to the more general case where the field is a portion of a circle. I need hardly add that the *isoscelism* referred to is, except for some extreme suppositions (impossible to occur in practice), absolutely independent of the form of the field.

As well-constructed names are, in fact, condensed lessons, lending an aid to the memory and imagination, of which modern mathematicians are only beginning to appreciate the importance, I suggest the following designations.

The point of *projection* and point of *impact* speak for themselves; the point vertically over the point of impact in the direction of projection may be called the point of *aim*. The line joining the point of aim and the point of impact is the *drop* or *fall*; the line joining the point of projection and the point of impact may be called the *excursion*; and that joining the point of projection and the point of aim, the *length of aim*.

A vertical section of the ground (plane or curved) through the axis of the gun may be called the *field*. We may then say, that, for the maximum range, the fall is always equal to the excursion, whatever the form of the field; and that in general the locus of the point of aim, for a rectilinear field, when the point of the projection and the velocity are given, is a circle to which, in the case of the angle of best elevation, the line of fall is of course a tangent. It would not be surprising if a good deal of elegant geometry (like ivy twining round an old wall) should hereafter associate itself with Mr Galbraith's "circle of aim": *à propos* of projectiles, it is not unworthy of observation, that the velocities at any two points P and Q of the parabolic path are as the lines PT, QT which the tangents at P and Q mutually cut off from one another, a remark which of course is easily seen to extend itself to the case of an elliptic orbit with the force in the centre.

Ever, Gentlemen,

Your faithful friend and reader,

J. J. SYLVESTER.

WOOLWICH COMMON,
July 3, 1856.

P.S. The value of Mr Galbraith's method consists simply in the *act of conception* of the locus of the point of aim; it was scarcely worth while (at this time of day) to append a synthetical proof of so simple a proposition, which may be got at immediately by calling the length of aim ρ, its inclination to the vertical, θ, and that of the field to the vertical, i; when by similar triangles (if H denote the quantity $\frac{2v^2}{g}$, and η the vertical distance of the point of projection from the field) we obtain the equation

$$\frac{\frac{\rho^2}{H} - \eta}{\rho} = \frac{\sin(i-\theta)}{\sin i},$$

or

$$\rho^2 - H\frac{\sin(i-\theta)}{\sin i}\rho - H\eta = 0;$$

which obviously corresponds to the circle of Professor Galbraith. I imagine this circle has been long known for the case of the point of projection being in the field, but it may have escaped notice for the more general case. The equality between the *fall* and the *excursion* for the angle of maximum range subsists, not merely for a rectilinear or curved section, but for the ground itself (whatever its form of surface) when the gun is supposed to admit of being laid to any angle, as well as at any elevation.

15.

RECHERCHES SUR LES SOLUTIONS EN NOMBRES ENTIERS POSITIFS OU NÉGATIFS DE L'ÉQUATION CUBIQUE HOMOGÈNE À TROIS VARIABLES.

[*Annali di Scienze Matematiche e Fisiche* (Tortolini), VII. (1856), pp. 398—400.]

J'AI l'honneur de vous envoyer pour être inséré dans votre journal estimable, si vous les jugéz dignes, les énoncés de quelques théorèmes que j'ai trouvé dans mes recherches sur les solutions en nombres entiers positifs ou négatifs de l'équation cubique homogène à trois variables.

On sait selon *Fermat* que l'équation

$$x^3 + y^3 + z^3 = 0$$

n'est pas résoluble en nombres entiers.

On peut ajouter la même chose pour les équations

$$x^3 + y^3 + 2z^3 = 0,$$
$$x^3 + y^3 + 3z^3 = 0;$$

j'ajoute que l'équation

$$x^3 + y^3 + z^3 + 6xyz = 0$$

est irrésoluble: aussi l'équation

$$2(x^3 + y^3 + z^3 + 6xyz) = 27nxyz,$$

quand

$$27n^2 - 8n + 4$$

est un nombre premier, est irrésoluble: aussi l'équation

$$4(x^3 + y^3 + z^3 + 6xyz) = 27vxyz$$

est irrésoluble quand

$$27v^2 - 36v + 16$$

est un nombre premier.

De plus l'équation

$$x^3 + y^3 + Az^3 = Mxyz$$

est irrésoluble dans les circonstances suivantes.

Posons $$M^3 - 27A = \Delta^3\Delta',$$

où Δ' ne contient nul facteur cubique. Alors si Δ' est pair et ne contient nul nombre de la forme

$$f^2 + 3g^2,$$

et si A est un nombre premier, l'équation est irrésoluble, excepté dans les cas que $\sqrt{\frac{-M}{A}}$ soit un nombre entier, et dans ce cas-là on peut donner la solution général de l'équation.

La même chose a lieu quand, Δ' restant assujétie aux mêmes conditions qu'auparavant, A est une puissance d'un nombre premier de la forme $p^{3\omega \pm 1}$.

La même chose a aussi lieu sans que Δ' soit pair, pourvu qu'il ne contient nul facteur $f^2 + 3g^2$, et que

$$A = 2^{3\omega \pm 1}.$$

La même chose a lieu encore pourvu que Δ' ne contient nul nombre de la forme $f^2 + 3g^2$ avec les conditions suivantes:

$$\begin{cases} \dfrac{A}{2} = \text{un nombre premier de la forme } qi \pm 4, \\[2mm] \dfrac{M}{9} = \text{un nombre entier,} \end{cases}$$

ou

$$\begin{cases} \dfrac{A}{4} = \text{un nombre premier de la forme } qi \pm 2, \\[2mm] \dfrac{M}{18} = \text{un nombre entier,} \end{cases}$$

ou si A étant un nombre premier on a A, B respectivement de la forme

$$qn + 2, \quad qn + 6,$$

ou bien de la forme $\quad qn - 2, \quad qn - 6,$

ou bien de la forme $\quad qn + 4, \quad qn + 3,$

ou de la forme $\quad qn - 4, \quad qn - 3,$

ou de la forme $\quad qn \pm 3, \quad qn.$

16.

ON THE CHANGE OF SYSTEMS OF INDEPENDENT VARIABLES.

[*Quarterly Journal of Mathematics*, I. (1857), pp. 42—56, 126—134.]

(1) THE theorem contained in the subjoined pages having been printed *, with many typographical and other errors †, in the *Proceedings of the Royal Society*, Vol. VII. No. 8, I think, on account of its importance to the direct march of the differential calculus, of which, as an instrument of expansion, it may be said to complete the processes, that the reissue of it in a more correct form may be acceptable and useful to the readers of this journal.

The purpose of the theorem is to effect for any number of variables the same end which has been accomplished by Burmann and others for a single variable; that is to say, ϑ being supposed to be a function of the variables, $x, y, \ldots z$, each of which is a given function of $u, v, \ldots w$, and $\alpha, \beta, \ldots \gamma$, being any positive integers, the theorem gives the complete development of $\left(\frac{d}{dx}\right)^\alpha \left(\frac{d}{dy}\right)^\beta \cdots \left(\frac{d}{dz}\right)^\gamma \vartheta$ in terms of $\frac{d}{du}, \frac{d}{dv}, \cdots \frac{d}{dw}, x, y, \ldots z, \vartheta$. Such, I say, is the primary form of the theorem; but it enables us, as will hereafter be shown in this paper, in fact, and as a consequence, to do much more than this, namely, to solve the question of differential transformation, under its most general aspect. The question so proposed may be stated as follows:

Given $\phi_1 = 0, \phi_2 = 0, \ldots \phi_n = 0$, where each ϕ is a function of $x_1, x_2, \ldots x_{n+i}$, it is required to pass from an expression in which the differentiations have respect to $x_1, x_2, \ldots x_i$ to an equivalent expression, in each of the terms of which the differentiations have respect to $x_{\theta_1}, x_{\theta_2}, \ldots x_{\theta_i}$, these last-written quantities being any i arbitrarily chosen terms out of the given set of $n + i$ variables, $x_1, x_2, \ldots x_{n+i}$. Through the medium of the reversion of series, the solution of this problem for the case contemplated in the theorem about to be enunciated (where $x_1, x_2, \ldots x_i$ are given *explicitly* in terms of

* Want of leisure prevented me then, and still prevents me, from producing the proof of the theorem, or the investigation by which I arrived at it. It must, however, be understood, that the theorem was not obtained tentatively, but that the proof of it is in my possession.

[† p. 50 above.]

$u_1, u_2, \dots u_i$), enables us to write down the solution for the case where these two systems of variables are connected by equations in the more general manner just above supposed. It may then be asked whether it is meant to affirm that Burmann's law for passing from one independent variable x to another y, of which the first is a known function, conducts immediately to the law for effecting such change, when x and y are connected through the intervention of one equation between x and y, or several equations between x, y, and other connecting variables. The answer to this question is in the negative; for even if we take the simpler case where x and y are connected by a single equation, it will be found that to solve the problem for this case in the manner indicated, we shall need to know the solution of the problem, how to pass to *two* variables, u, v, from two others, x, y, given explicitly as functions of the former two; and so in general it is the fact, that the theorem applicable to the case of implicit connections between any number of variables, is always a corollary to the theorem applicable to the case of explicit connection between a *greater* number of variables. Thus it comes to pass, that Burmann's law for one variable explicitly connected with another, does not contain within itself the law for one variable implicitly connected with another; but the general law which I have discovered for a system of *any number* of variables explicitly connected with another such system, does contain within itself the general law for systems implicitly so connected*.

As the theorem is one of considerable complexity, it will be rendered most easily intelligible by taking separately and successively the cases of two and of three variables; the reader will then not experience any difficulty in seeing how it is to be extended to any greater number.

Problem for Two Variables.

(2) Let x, y be given functions of u, v, it is required to express $\left(\frac{d}{dx}\right)^f \left(\frac{d}{dy}\right)^g \vartheta$ in terms of the partial differential coefficients of x, y, ϑ in respect of u and v.

Solution.

Form the Jacobian determinant

$$\begin{vmatrix} \frac{dx}{du}, & \frac{dx}{dv} \\ \frac{dy}{du}, & \frac{dy}{dv} \end{vmatrix},$$

* Any linear function of infinity is still infinity, and all infinity is one, but not so of a finite integer; thus it is that the particular does not carry with it the particular, although the general does the general.

which call J; the required expression will be made up of terms, each of which will have for its components; 1°, a power of $(-)$; 2°, a positive integer numerical multiplier; 3°, a negative power of J; 4° and 5° (subject to a subsequent distinction into two sets), factors of the form

$$\left(\frac{d}{du}\right)^p \left(\frac{d}{dv}\right)^q x, \quad \left(\frac{d}{du}\right)^{p'} \left(\frac{d}{dv}\right)^{q'} y;$$

and 6°, a factor of the form $\quad \left(\frac{d}{du}\right)^A \left(\frac{d}{dv}\right)^B \vartheta.$

The distinction of the factors under the combined headings 4 and 5 into two sets, referring to these headings separately taken, is dependent upon the values of p, q; p', q'. The 4th heading is intended to comprise the factors, for which $p=1$ and $q=0$ or $p=0$, $q=1$, and similarly for p' and q', that is, factors for which $p+q$ or $p'+q'$ is unity. The 5th heading comprises those factors in which $p+q$ or $p'+q'$ (as the case may be), exceeds unity. These two sets require to be carefully distinguished and considered apart: those values of p, q; p', q' belonging to the second set will be distinguished by the letters a, b; a', b', so that it is to be understood that $a+b>1$, $a'+b'>1$.

The general term may thus be put under the form

$$(-)^i N \frac{1}{J^\omega} \left(\frac{dy}{dv}\right)^\alpha \times \left(-\frac{dy}{du}\right)^\beta \times \left(-\frac{dx}{dv}\right)^{\alpha'} \times \left(\frac{dx}{du}\right)^{\beta'}$$
$$\times \left\{\left(\frac{d}{du}\right)^a \left(\frac{d}{dv}\right)^b x\right\}^l \times \&c.$$
$$\times \left\{\left(\frac{d}{du}\right)^{a'} \left(\frac{d}{dv}\right)^{b'} y\right\}^{l'} \times \&c.$$
$$\times \left(\frac{d}{du}\right)^A \left(\frac{d}{dv}\right)^B \vartheta.$$

The negative signs are employed with $\frac{dy}{du}, \frac{dx}{dv}$ in the first line of factors, because, as will be seen when we pass to the case of more than two variables, it is the first minors of J which give rise to these factors, and these first minors are respectively

$$\frac{dy}{dv}; \quad -\frac{dy}{du}; \quad -\frac{dx}{dv}; \quad \frac{dx}{du}.$$

The &c. in the second line of factors refers to a, b, l becoming changed into a_1, b_1, l_1; a_2, b_2, l_2 &c.; and indicates that the product is to be taken of all the factors thus formed upon the type of

$$\left\{\left(\frac{d}{du}\right)^a \left(\frac{d}{dv}\right)^b x\right\}^l.$$

Similarly, the &c. in the third line of factors refers to a', b', l' becoming changed into a_1', b_1', l_1' &c.; and the product taken of all such factors so formed upon the type of

$$\left\{\left(\frac{d}{du}\right)^{a'}\left(\frac{d}{dv}\right)^{b'} y\right\}^{l'}.$$

[We may of course, if we please, write the first line under the form

$$(-)^{i'}\frac{N}{J^{\omega}}\left(\frac{dy}{dv}\right)^{\alpha}\left(\frac{dy}{du}\right)^{\beta}\left(\frac{dx}{dv}\right)^{\alpha'}\left(\frac{dx}{du}\right)^{\beta'}$$

by making $i' = i + \beta + \alpha'$.]

In the first place,

$$i = l + \&c. + l' + \&c.$$

In the second place,

$$\omega = \alpha + \beta + \alpha' + \beta'.$$

In the third place,

$$\alpha + \alpha' = la + \&c. + l'a' + \&c. + A, \qquad (1)$$

$$\beta + \beta' = lb + \&c. + l'b' + \&c. + B, \qquad (2)$$

and

$$\alpha + \beta = f + \Sigma l, \qquad (3)$$

$$\alpha' + \beta' = g + \Sigma l', \qquad (4)$$

which two systems of equations of course imply the existence of the equation

$$\Sigma l\,(a+b-1) + \Sigma l'\,(a'+b'-1) = (f+g) - (A+B). \qquad (5)$$

And finally:

$$N = D \times \frac{\Pi\,(\alpha+\beta-1)\,\Pi\,(\alpha'+\beta'-1)}{\Pi\alpha\,\Pi\beta\,\Pi\alpha'\,\Pi\beta'}$$

$$\times \frac{\Pi\,(\alpha+\alpha'-1)\,\Pi\,(\beta+\beta'-1)}{\{\Pi l\,(\Pi a\,\Pi b)^l\} \times \&c. \times \{\Pi l'\,(\Pi a'\,\Pi b')^{l'}\} \times \&c. \times \Pi A\,\Pi B},$$

Πn for any value of the integer n indicating the factorial $1.2.3 \ldots n$, and D denoting the determinant hereunder written, namely:

$$\begin{vmatrix} \alpha+\beta, & 0, & la + \&c., & lb + \&c. \\ 0, & \alpha'+\beta', & l'a' + \&c., & l'b' + \&c. \\ \alpha, & \alpha', & \alpha+\alpha', & 0 \\ \beta, & \beta', & 0, & \beta+\beta' \end{vmatrix},$$

which writing $la + \&c. = \Sigma la$, $l'a' + \&c. = \Sigma l'a'$, and substituting for $\alpha + \alpha'$, $\Sigma la + \Sigma l'a' + A$, and for $\beta + \beta'$, $\Sigma lb + \Sigma l'b' + B$, becomes when developed

$$(\alpha+\beta)(\alpha'+\beta')\,AB$$
$$+ \{\beta\,(\alpha'+\beta')\,\Sigma la + \beta'\,(\alpha+\beta)\,\Sigma l'a'\}\,B$$
$$+ \{\alpha\,(\alpha'+\beta')\,\Sigma lb + \alpha'\,(\alpha+\beta)\,\Sigma l'b'\}\,A.$$

D being essentially positive, N can only vanish when the following equations (or the analogues to them obtained by the interchange of a, α, A with b, β, B) are fulfilled, namely:

$$A = 0, \quad B = 0,$$

or $\qquad A = 0, \quad \beta = 0, \quad \beta' = 0,$

or $\qquad A = 0, \quad \beta = 0, \quad \alpha = 0,$

or $\qquad A = 0, \quad \alpha' = 0, \quad \beta' = 0,$

or $\qquad A = 0, \quad \beta = 0, \quad \Sigma l'a' = 0,$

or $\qquad A = 0, \quad \beta' = 0, \quad \Sigma la = 0,$

or $\qquad A = 0, \quad \Sigma la = 0, \quad \Sigma l'a' = 0.$

(3) By way of illustration, let us suppose $f = 2, g = 0$, so that the expression to be developed is $\frac{d^2\vartheta}{dx^2}$, which is to be expressed in terms of $\frac{d}{du}, \frac{d}{dv}, x, y, \vartheta$. It will be the simpler mode of proceeding to find this development by actual expansion, and compare the result with that given by the theorem in the text.

We shall find without difficulty by the ordinary process

$$\frac{d^2\vartheta}{dx^2} = \frac{1}{J^2}\left(\frac{dy}{dv}\right)^2\frac{d^2\vartheta}{du^2} - \frac{2}{J^2}\frac{dy}{du}\frac{dy}{dv}\frac{d^2\vartheta}{du\,dv} + \frac{1}{J^2}\left(\frac{dy}{du}\right)^2\frac{d^2\vartheta}{dv^2}$$
$$+ \frac{1}{J^3}\frac{dx}{dv}\left(\frac{dy}{dv}\right)^2\frac{d^2y}{du^2}\frac{d\vartheta}{du} - \frac{2}{J^3}\frac{dx}{dv}\frac{dy}{du}\frac{dy}{dv}\frac{d^2y}{du\,dv}\frac{d\vartheta}{du}$$
$$+ \frac{1}{J^3}\frac{dx}{dv}\left(\frac{dy}{du}\right)^2\frac{d^2y}{dv^2}\frac{d\vartheta}{du}$$
$$- \frac{1}{J^3}\frac{dy}{du}\left(\frac{dy}{dv}\right)^2\frac{d^2y}{du^2}\frac{d\vartheta}{du} + \frac{2}{J^3}\left(\frac{dy}{dv}\right)^2\frac{dy}{du}\frac{d^2x}{du\,dv}\frac{d\vartheta}{du}$$
$$- \frac{1}{J^3}\frac{dy}{dv}\left(\frac{dy}{du}\right)^2\frac{d^2x}{dv^2}\frac{d\vartheta}{du}$$
$$- \frac{1}{J^3}\frac{dx}{dv}\left(\frac{dy}{dv}\right)^2\frac{d^2y}{du^2}\frac{d\vartheta}{dv} + \frac{2}{J^3}\frac{dx}{du}\frac{dy}{du}\frac{dy}{dv}\frac{d^2y}{du\,dv}\frac{d\vartheta}{dv}$$
$$- \frac{1}{J^3}\frac{dx}{dv}\left(\frac{dy}{du}\right)^2\frac{d^2y}{dv^2}\frac{d\vartheta}{dv}$$
$$+ \frac{1}{J^3}\frac{dy}{du}\left(\frac{dy}{dv}\right)^2\frac{d^2x}{du^2}\frac{d\vartheta}{dv} - \frac{2}{J^3}\frac{dy}{du}\frac{dy}{du}\frac{dy}{dv}\frac{d^2x}{du\,dv}\frac{d\vartheta}{dv}$$
$$+ \frac{1}{J^3}\frac{dx}{du}\left(\frac{dy}{du}\right)^2\frac{d^2y}{dv^2}\frac{d\vartheta}{dv}.$$

(4) In the first term

$$\begin{aligned} &\alpha=2, && \beta=0, && \alpha'=0, && \beta'=0, \\ &a=0, && b=0, && \&c.=0, && \Sigma l=0, \\ &a'=0, && b'=0, && \&c.=0, && \Sigma l'=0, \\ &A=2, && B=0, \end{aligned}$$

and we have, as indicated by the theorem,

$$i=\Sigma l+\Sigma l'=0, \quad \omega=\alpha+\beta+\alpha'+\beta'=2,$$

$$\begin{aligned} \alpha+\alpha' &= \Sigma la+\Sigma l'a'+A=2, \\ \beta+\beta' &= \Sigma lb+\Sigma l'b'+B=0, \\ \alpha+\beta &= f=2, \\ \alpha'+\beta' &= g=0. \end{aligned}$$

N becomes

$$\frac{\Pi(\alpha+\beta-1)\,\Pi(\alpha'+\beta'-1)}{\Pi\alpha\,\Pi\beta\,\Pi\alpha'\,\Pi\beta'} \times \frac{\Pi(\alpha+\alpha'-1)\,\Pi(\beta+\beta'-1)}{\Pi A} \times (\alpha+\beta)(\alpha'+\beta')\,AB;$$

it is easily seen that

$$\begin{aligned} &\Pi(\beta+\beta'-1)\times B \\ &=\Pi(\beta+\beta'-1)\times(\beta+\beta') \\ &=\Pi 0=1, \end{aligned}$$

$$(\alpha'+\beta')\,\Pi(\alpha'+\beta'-1)=\Pi(\alpha'+\beta')=\Pi 0=1,$$

so that the value of the fraction above written is in fact

$$\frac{\Pi(\alpha+\beta)\,A}{\Pi\alpha\,\Pi A}=\frac{(\Pi 2)^2}{(\Pi 2)^2}=1.$$

In the second term,

$$\alpha=1, \quad \beta=1, \quad \alpha'=0, \quad \beta'=0;$$

everything else remains as before, except that the numerical factor is $(-)^\beta N$, that is, $-N$, where $N=2$.

(5) If we take the eighth term (the second one of the fourth line) we have

$$\begin{aligned} &\alpha=2, && \beta=1, && \alpha'=0, && \beta'=0, \\ &a=1, && b=1, && a'=0, && b'=0, && \Sigma l=l=1, \\ &A=1, && B=0, \end{aligned}$$

and we have

$$i=l+\beta+\alpha'=\Sigma l+\beta+\alpha'=2,$$

$$\omega=\alpha+\beta+\alpha'+\beta'=3,$$

$$\begin{aligned} \alpha+\alpha' &= la+A=2, \\ \beta+\beta' &= lb+B=1, \\ \alpha+\beta &= f+l=3, \\ \alpha'+\beta' &= g=0, \end{aligned}$$

and N becomes

$$\frac{\Pi(\alpha+\beta-1)\,\Pi(\alpha'+\beta'-1)}{\Pi\alpha\,\Pi\beta\,\Pi\alpha'\,\Pi\beta'} \times \frac{\Pi(\alpha+\alpha'-1)\,\Pi(\beta+\beta'-1)}{\Pi a\,\Pi b\,\Pi A\,\Pi B} \times \{\alpha(\alpha'+\beta')\}\,bA,$$

which, since

$$\Pi(\alpha'+\beta'-1)\times(\alpha'+\beta')=\Pi 0=1,$$

$$=\frac{\Pi(\alpha+\beta-1)\,A}{\Pi A}=\frac{\Pi 2\,.\,2}{\Pi 2}=2.$$

(6) The above examples, although taken from the simplest terms, are in a certain sense exceptional cases, inasmuch as N for these cases involves one or more fractions of the form $\frac{0}{0}$; but this is a mere accident, resulting from the peculiar form of representation which I choose to employ, as being in general the most convenient to operate with.

If we take the fifth term (that is, the second term of the second line), this exception does not apply. We have for this term

$$\begin{aligned} &\alpha=1, && \beta=1, && \alpha'=1, && \beta'=0, \\ &a=0, && b=0, && a'=1, && b'=1, && \Sigma l'=l'=1, \\ & && && A=1, && B=0, \end{aligned}$$

and we find

$$N=\frac{\Pi 1\times\Pi 0}{\Pi 1\times\Pi 1}\times\frac{\Pi 1\,\Pi 0}{\Pi 1\times\Pi 1\times\Pi 1}\times D=D$$

$$=\left\{\begin{array}{l} \quad 2\times 0\times 1\times 0 \\ +(1\times 1+0+0\times 1\times 1)\,0 \\ +\;1\times 1\times 0+1\times 2\times 1 \end{array}\right\}=2.$$

(7) In general, to form all the terms in $\left(\frac{d}{dx}\right)^f\left(\frac{d}{dy}\right)^g \vartheta$, that is, to find all the systems of indices, we may begin by taking $A+B=\mu$, and giving to μ in succession, every value from 1 to $f+g$, and calling $f+g=n$, and writing

$$\Sigma l\,(a+b-1)=L,$$

$$\Sigma l'\,(a'+b'-1)=L',$$

we have to combine each solution of the equation $A+B=\mu$ with each of the equation $L+L'=n-\mu$, that is, we may assume for A in succession each value from 1 to μ, and for L, from 1 to $n-\mu$.

It will be convenient to denote in general an integer which may be anything from 1 to p by $[p]$. We have then

$$\mu=[n],$$

$$A=\Big[[n]\Big], \quad B=[n]-\Big[n-[n]\Big],$$

$$L=\Big[n-[n]\Big], \quad L'=n-[n]-\Big[n-[n]\Big].$$

We have then to break up L in every possible way into parts which will give by combining equal parts into groups all the values of l, $(a+b-1)$. In like manner, the partitionment of L' will give all the values of l', $(a'+b'-1)$.

Any of the values of $a+b-1$ and of $a'+b'-1$ respectively being called c and c', we have

$$a=[c+1],\quad b=c+1-[c+1],\quad a'=[c'+1],\quad b'=c'+1-[c'+1].$$

Hence every system of $\quad l_1,\ a_1,\ b_1;\ \ l_2,\ a_2,\ b_2;\ \dots$

and of $\quad l_1',\ a_1',\ b_1';\ \ l_2',\ a_2',\ b_2';\ \dots$

satisfying the equations of condition may be found. To find the corresponding values of α, β; α', β' we must observe that one combination of the equations (1), (2), (3), (4), having been employed to obtain the quantities already found, only three of these equations are independent; we shall accordingly have

$$\alpha=[f+\Sigma l],\quad \beta=f+\Sigma l-[f+\Sigma l],$$
$$\alpha'=\Sigma la+\Sigma l'a'+A-\alpha,$$
$$\beta'=\Sigma lb+\Sigma l'b'+B-\beta,$$

and the problem is completely resolved.

(8) If now we pass to the case of three variables x, x', x'', given explicitly as functions of u, u', u'', we must take

$$J=\begin{vmatrix} \frac{dx}{du}, & \frac{dx}{du'}, & \frac{dx}{du''} \\ \frac{dx'}{du}, & \frac{dx'}{du'}, & \frac{dx'}{du''} \\ \frac{dx''}{du}, & \frac{dx''}{du'}, & \frac{dx''}{du''} \end{vmatrix},$$

which, for greater brevity, using $\bar{u}, \bar{u}', \bar{u}''$, to denote $\frac{d}{du}, \frac{d}{du'}, \frac{d}{du''}$, may be written

$$\begin{vmatrix} \bar{u}x, & \bar{u}'x, & \bar{u}''x \\ \bar{u}x', & \bar{u}'x', & \bar{u}''x' \\ \bar{u}x'', & \bar{u}'x'', & \bar{u}''x'' \end{vmatrix}.$$

The nine first minor determinants may then be expressed under the respective forms

$$\frac{dJ}{d\bar{u}x},\quad \frac{dJ}{d\bar{u}'x},\quad \frac{dJ}{d\bar{u}''x},$$
$$\frac{dJ}{d\bar{u}x'},\quad \frac{dJ}{d\bar{u}'x'},\quad \frac{dJ}{d\bar{u}''x'};$$
$$\frac{dJ}{d\bar{u}x''},\quad \frac{dJ}{d\bar{u}'x''},\quad \frac{dJ}{d\bar{u}''x''}.$$

The general term in $\left(\frac{d}{dx}\right)^f \left(\frac{d}{dx'}\right)^{f'} \left(\frac{d}{dx''}\right)^{f''} \vartheta$ will then be

$$(-)^i \frac{N}{J^\omega} \left(\frac{dJ}{d\bar{u}x}\right)^\alpha \left(\frac{dJ}{d\bar{u}'x}\right)^\beta \left(\frac{dJ}{d\bar{u}''x}\right)^\gamma$$
$$\times \left(\frac{dJ}{d\bar{u}x'}\right)^{\alpha'} \left(\frac{dJ}{d\bar{u}'x'}\right)^{\beta'} \left(\frac{dJ}{d\bar{u}''x'}\right)^{\gamma'}$$
$$\times \left(\frac{dJ}{d\bar{u}x''}\right)^{\alpha''} \left(\frac{dJ}{d\bar{u}'x''}\right)^{\beta''} \left(\frac{dJ}{d\bar{u}''x''}\right)^{\gamma''}$$
$$\times \left\{\left(\frac{d}{du}\right)^a \left(\frac{d}{du'}\right)^b \left(\frac{d}{du''}\right)^c x\right\}^l \times \text{\&c.}$$
$$\times \left\{\left(\frac{d}{du}\right)^{a'} \left(\frac{d}{du'}\right)^{b'} \left(\frac{d}{du''}\right)^{c'} x'\right\}^{l'} \times \text{\&c.}$$
$$\times \left\{\left(\frac{d}{du}\right)^{a''} \left(\frac{d}{du'}\right)^{b''} \left(\frac{d}{du''}\right)^{c''} x''\right\}^{l''} \times \text{\&c.}$$
$$\times \left(\frac{d}{du}\right)^A \left(\frac{d}{du'}\right)^B \left(\frac{d}{du''}\right)^C \vartheta;$$

and similarly to the last case

$$i=\Sigma l+\Sigma l'+\Sigma l'',$$
$$\omega=\alpha+\beta+\gamma$$
$$+\alpha'+\beta'+\gamma'$$
$$+\alpha''+\beta''+\gamma'',$$
$$\alpha+\alpha'+\alpha''=\Sigma la+\Sigma l'a'+\Sigma l''a''+A$$
$$\beta+\beta'+\beta''=\Sigma lb+\Sigma l'b'+\Sigma l''b''+B$$
$$\gamma+\gamma'+\gamma''=\Sigma lc+\Sigma l'c'+\Sigma l''c''+C,$$
$$\alpha+\beta+\gamma=f+\Sigma l$$
$$\alpha'+\beta'+\gamma'=f'+\Sigma l'$$
$$\alpha''+\beta''+\gamma''=f''+\Sigma l'',$$

from which six equations we may deduce

$$\Sigma l(a+b+c-1)+\Sigma l'(a'+b'+c'-1)+\Sigma l''(a''+b''+c''-1)$$
$$=f+g+h-(A+B+C).$$

(9) And the six equations first written may be solved in a manner analogous to the four equations in the preceding case.

We have finally

$$N=\frac{\Pi(\alpha+\beta+\gamma-1)\,\Pi(\alpha'+\beta'+\gamma'-1)\,\Pi(\alpha''+\beta''+\gamma''-1)}{\Pi\alpha\,\Pi\beta\,\Pi\gamma\,\Pi\alpha'\,\Pi\beta'\,\Pi\gamma'\,\Pi\alpha''\,\Pi\beta''\,\Pi\gamma''}$$
$$\times\frac{\Pi(\alpha+\alpha'+\alpha''-1)\,\Pi(\beta+\beta'+\beta''-1)\,\Pi(\gamma+\gamma'+\gamma''-1)}{\Pi l\,(\Pi a\,\Pi b\,\Pi c)^l\times\text{\&c.}\times\Pi l'\,(\Pi a'\,\Pi b'\,\Pi c')^{l'}\times\text{\&c.}\times\Pi l''\,(\Pi a''\,\Pi b''\,\Pi c'')^{l''}\times\text{\&c.}}$$
$$\times D\div(\Pi A\,\Pi B\,\Pi C).$$

where D = the determinant following, namely,

$$\begin{vmatrix} \alpha+\beta+\gamma, & 0, & 0, & \Sigma la, & \Sigma lb, & \Sigma lc \\ 0, & \alpha'+\beta'+\gamma', & 0, & \Sigma l'a', & \Sigma l'b', & \Sigma l'c' \\ 0, & 0, & \alpha''+\beta''+\gamma'', & \Sigma l''a'', & \Sigma l''b'', & \Sigma l''c'' \\ \alpha, & \alpha', & \alpha'', & \alpha+\alpha'+\alpha'', & 0, & 0 \\ \beta, & \beta', & \beta'', & 0, & \beta+\beta'+\beta'', & 0 \\ \gamma, & \gamma', & \gamma'', & 0, & 0, & \gamma+\gamma'+\gamma'' \end{vmatrix},$$

which, employing the equations

$$\alpha+\alpha'+\alpha'' = \Sigma la + \Sigma l'a' + \Sigma l''a'' + A$$
$$\beta+\beta'+\beta'' = \Sigma lb + \Sigma l'b' + \Sigma l''b'' + B$$
$$\gamma+\gamma'+\gamma'' = \Sigma lc + \Sigma l'c' + \Sigma l''c'' + C,$$

may be expressed under the forms

$$\lambda ABC + \mu BC + \mu' CA + \mu'' AB + \nu A + \nu' B + \nu'' C,$$

where all the coefficients λ, μ, ν, are essentially positive functions of α, β, γ, &c., Σla, Σlb, Σlc, &c.

The general form of D is apparent, as is also the reason why there is no term in which one of the indices, A, B, C ... does not appear, namely, that the sum of the lines in the lower half of the square, minus the sum of the lines in its upper half, gives rise to the line of terms following, which may be substituted in place of any one of the existing lines

$$0, 0, 0 \ldots A, B, C \ldots$$

so that one of the letters A, B, C ... must appear in every actual term of the development.

(10) Let us return for a moment to show what the theorem becomes for the case of a single variable x, from which the transition is to be made to u.

For this case $$J = \frac{dx}{du},$$

and the 1st minor which is a determinant of zero places, as is well known to those conversant with determinants, must be taken $+1$. The formula then becomes

$$(-)^i (1)^a \frac{N}{J^\omega} \left\{\left(\frac{d}{du}\right)^{a_1} x\right\}^{l_1} \left\{\left(\frac{d}{du}\right)^{a_2} x\right\}^{l_2} \cdots \left\{\left(\frac{d}{du}\right)^{a_e} x\right\}^{l_e} \cdot \left(\frac{d}{du}\right)^A \vartheta,$$

where $i = l_1 + l_2 + \ldots + l_e$, $\omega = \alpha = l_1 a_1 + l_2 a_2 + \ldots + l_e a_e + A$,

and $$N = \frac{\Pi(\alpha-1)}{\Pi\alpha} \frac{\Pi(\alpha-1)}{\Pi l_1 (\Pi a_1)^{l_1} \Pi l_2 (\Pi a_2)^{l_2} \ldots \Pi l_e (\Pi a_e)^{l_e} \Pi A} \cdot D,$$

where $$D = \begin{vmatrix} \alpha, & \alpha - A \\ \alpha, & \alpha \end{vmatrix} = \alpha A.$$

Hence $$N = \frac{\Pi(\alpha-1)}{\Pi l_1 (\Pi a_1)^{l_1} \times \&c. \times \Pi l_e (\Pi a_e)^{l_e} \times \Pi(A-1)},$$

agreeing, as it ought, with Burmann's Law.

(11) For particular classes of terms N admits of a reduction to a simpler form.

Thus, in the case of three variables, suppose that the matrix

$$\begin{matrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{matrix} \quad \text{assumes the form} \quad \begin{matrix} \alpha, & 0, & 0, \\ 0, & \beta', & 0, \\ 0, & 0, & \gamma'', \end{matrix}$$

by which I mean that

$$\beta = 0, \quad \gamma = 0,$$
$$\alpha' = 0, \quad \gamma' = 0,$$
$$\alpha'' = 0, \quad \beta'' = 0.$$

Then by substituting for the 4th, 5th, and 6th lines in D the differences between the 4th and 1st, the 5th and 2nd, the 6th and 3rd, respectively, D assumes the form

$$\begin{vmatrix} \alpha, & 0, & 0, & \Sigma la, & \Sigma lb, & \Sigma lc \\ 0, & \beta', & 0, & \Sigma l'a', & \Sigma l'b', & \Sigma l'c' \\ 0, & 0, & \gamma'', & \Sigma l''a'', & \Sigma l''b'', & \Sigma l''c'' \\ 0, & 0, & 0, & \begin{matrix}\Sigma l'a' \\ +\Sigma l''a'', \\ +A\end{matrix} & -\Sigma lb, & -\Sigma lc \\ 0, & 0, & 0, & -\Sigma l'a', & \begin{matrix}\Sigma lb \\ +\Sigma l''b'', \\ +B,\end{matrix} & -\Sigma l'c' \\ 0, & 0, & 0, & -\Sigma l''a'', & -\Sigma l''b'', & \begin{matrix}\Sigma lc \\ +\Sigma l'c' \\ +C\end{matrix} \end{vmatrix},$$

which

$$= \alpha\beta'\gamma'' \times \begin{vmatrix} \Sigma l'a' + \Sigma l''a'' + A, & -\Sigma l'a', & -\Sigma l''a'' \\ -\Sigma lb, & \Sigma lb + \Sigma l''b'' + B, & -\Sigma l''b'' \\ -\Sigma lc, & -\Sigma l'c', & \Sigma lc + \Sigma l'c' + C \end{vmatrix},$$

which we may call $\alpha\beta'\gamma''D'$.

The entire value of N is consequently

$$\alpha\beta'\gamma''\frac{\Pi(\alpha-1)\,\Pi(\beta'-1)\,\Pi(\gamma''-1)}{\Pi\alpha\,\Pi\beta'\,\Pi\gamma''}$$

$$\times\frac{\Pi(\alpha-1)\,\Pi(\beta'-1)\,\Pi(\gamma''-1)}{\Pi l\,(\Pi a\,\Pi b\,\Pi c)^{l}\times\&c.\times\Pi l'\,(\Pi a'\,\Pi b'\,\Pi c')^{l'}\times\&c.\times\Pi l''\,(\Pi a''\,\Pi b''\,\Pi c'')^{l''}\times\&c.}$$

$$\times\frac{D}{\Pi A\,\Pi B\,\Pi C}$$

$$=\frac{\Pi\alpha\,\Pi\beta'\,\Pi\gamma''}{\Pi l\,(\Pi a\,\Pi b\,\Pi c)^{l}\times\&c.\times\Pi l'\,(\Pi a'\,\Pi b'\,\Pi c')^{l'}\times\&c.\times\Pi l''\,(\Pi a''\,\Pi b''\,\Pi c'')^{l''}\times\&c.}$$

$$\times\frac{D'}{\Pi A\,\Pi B\,\Pi C}.$$

(12) The form of D' is deserving of consideration on its own account.

Call

$$\Sigma l'a' = A_b,\quad \Sigma l''a'' = A_c,$$
$$\Sigma lb = B_a,\quad \Sigma l''b'' = B_c,$$
$$\Sigma lc = C_a,\quad \Sigma l'c' = C_b.$$

Then

$$D' = ABC + (A_b + A_c)\,BC + (B_c + B_a)\,CA + (C_a + C_b)\,AB$$
$$+ (B_cC_a + B_aC_b + B_aC_a)\,A + (C_aA_b + C_bA_c + C_bA_b)\,B$$
$$+ (A_bB_c + A_cB_a + A_cB_c)\,C.$$

The entire number of terms is 16. In general, for m variables the corresponding number will be $(m+1)^{m-1}$, as may easily be shown*.

* The number of terms in D', since each of them has positive unity for its numerical coefficient, is evidently the value of a determinant, which, for three variables, is

$$\begin{vmatrix} 3, & -1, & -1 \\ -1, & 3, & -1 \\ -1, & -1, & 3 \end{vmatrix}.$$

To find in general the value of such a determinant in its more general form

$$\begin{vmatrix} a, & -1, & -1 \\ -1, & a, & -1 \\ -1, & -1, & a \end{vmatrix},$$

which is the discriminant of $a(x^2+y^2+z^2) - 2yz - 2zx - 2xy$, we may observe that this latter formula becomes a perfect square, that is, loses two orders when $a = -1$. Hence $(a+1)^2$ is a factor of the determinant. Again, when $a = 2$ the sum of all the terms in each column is zero. Hence $(a-2)$ is also contained in it as a factor; the complete value of the determinant is therefore $(a-2)(a+1)^2$, that is, 4^2, when $a = 3$; and so for a determinant of the mth order we obtain $\{a-(m-1)\}(a+1)^{m-1}$, which becomes $(m+1)^{m-1}$ when $a = m$.

The same result may also be obtained directly by the integration of a linear equation of differences of the second order of the form given in the example at the foot of page 14, in Mr Cohen's paper in this *Journal*.

If we take D, which also, like D', consists exclusively of positive terms, only with unit coefficients, the number of these terms for the case of 1, 2, 3 variables I find to be 1, 12, 432; and for the general case of m variables I presume that the law is $m^m(m+1)^{m-1}$.

The terms themselves may be found without calculation by means of a simple rule.

Suppose that there are four variables, we may then find D' for the case corresponding to the one just treated of for three variables by taking the product of

$$A_b + A_c + A_d + A,$$
$$B_a + B_c + B_d + B,$$
$$C_a + C_b + C_d + C,$$
$$D_a + D_b + D_c + D,$$

and *rejecting* every term in such product in which any group of the letters forms a cycle.

Thus, for example, every term in which $A_b \times B_a$ enters must be rejected, because AB, BA is a cycle.

So, again, every term in which $A_b \times B_c \times C_a$ enters must be rejected, because AB, BC, CA forms a cycle.

We might take the product of $A_a + A_b + A_c + A_d + A$, and the quantities similarly formed, and proceed as above; for since AA is a cycle, as is also BB, CC, DD, therefore $A_aB_bC_cD_d$ will not appear in the final result.

Applying the method of rejection, we find without difficulty D', which represents the determinant

$$\begin{vmatrix} A + A_b + A_c + A_d, & -A_b, & -A_c, & -A_d \\ -B_a, & B + B_a + B_c + B_d, & -B_c, & -B_d \\ -C_a, & -C_b, & C + C_a + C_b + C_d, & -C_d \\ -D_a, & -D_b, & -D_c, & D + D_a + D_b + D_c \end{vmatrix},$$

$$= ABCD + \Sigma\,(A_b + A_c + A_d)\,BCD + \Sigma\,(A_bB_c + A_bB_d + A_cB_a$$
$$+ A_cB_c + A_cB_d + A_dB_a + A_dB_c + A_dB_d)\,CD$$
$$+ \Sigma\begin{pmatrix} A_dB_dC_d + (A_b + A_c)\,B_dC_d + (B_c + B_a)\,C_dA_d + (C_a + C_b)\,A_dB_d \\ \{B_a(C_a + C_b) + B_cC_a\}\,C_d + \{C_b(A_b + A_c) + C_aA_b\}\,A_d \\ + \{A_c(B_c + B_a) + A_bB_c\}\,B_d \end{pmatrix} D.$$

The total number of terms being

$$1 + 4\times 3 + 8\times 6 + 4\times 16 = 125 = 5^3,$$

as it ought to be.

Other cases of simplification will readily suggest themselves; and, of course, when $\gamma = 0$, $\gamma' = 0$, $\gamma'' = 0$, which equations imply also $\Sigma lc = 0$, $\Sigma l'c' = 0$, $\Sigma l''c'' = 0$, and $C = 0$, the value of N will reduce as it ought to the form corresponding to the case of only two variables, and so in general (the value of the coefficient of any term in the development of the transformed value of any differential coefficient of a function of several variables depending only upon such of them as appear in the term itself, and in no way upon the other variables not so appearing).

(13) To indicate the method of passing from the theory of transformation of systems explicitly to that of systems of variables implicitly connected, let us suppose $\phi(x, y) = 0$ and that $\frac{d^f\vartheta}{dx^f}$ is to be expressed in terms of $\frac{d}{dy}, \phi, \vartheta$.

We may make this transformation depend upon our being able to solve the following question in the reversion of series, namely:

Given
$$\xi = a\rho + b\sigma + \frac{1}{1.2}(c\rho^2 + 2d\rho\sigma + e\sigma^2) + \&c.,$$
$$\eta = a'\rho + b'\sigma + \frac{1}{1.2}(c'\rho^2 + 2d'\rho\sigma + e'\sigma^2) + \&c.,$$

to express $\rho^h\sigma^k$ *in terms of* ξ, η. The solution of this question, when $b=0$, $a'=0$, has been given by Jacobi, *Crelle*, t. VI. 1830; and as is obvious and pointed out by Jacobi, the general case, by either of two methods, namely, combination of the equations or linear transformations effected in the variables ρ, σ contained in them, may be made to depend on the particular case for which $b=0$, $a'=0$; but Jacobi has not followed out the effects of these processes, and apparently was not aware of the results being (as we may now see is the case) capable of an explicit representation, which mode of representation is essential for the purpose we have in view.

Let x, y be functions of u, v; and suppose x, y, ϑ to become $x+\xi$, $y+\eta$, $\vartheta+\tau$, when u and v become $u+h$ and $v+k$ respectively; then we shall have
$$\xi = \frac{dx}{du}h + \frac{dx}{dv}k + \&c. + \left\{\&c. + \frac{1}{\Pi A\,\Pi B}\frac{d^{A+B}x}{du^A dv^B}h^A k^B + \&c.\right\} + \&c.,$$
$$\eta = \frac{dy}{du}h + \frac{dy}{dv}k + \&c. + \left\{\&c. + \frac{1}{\Pi A\,\Pi B}\frac{d^{A+B}y}{du^A dv^B}h^A k^B + \&c.\right\} + \&c.,$$
$$\tau = \frac{d\vartheta}{du}h + \frac{d\vartheta}{dv}k + \&c. + \left\{\&c. + \frac{1}{\Pi A\,\Pi B}\frac{d^{A+B}\vartheta}{du^A dv^B}h^A k^B + \&c.\right\} + \&c.;$$
but treating τ as a function of ξ, η, we have also
$$\tau = \&c. + \left\{\&c. + \frac{1}{\Pi f\,\Pi g}\cdot\frac{d^{f+g}\vartheta}{dx^f dy^g}\xi^f\eta^g + \&c.\right\} + \&c.$$

Hence $\frac{1}{\Pi f\,\Pi g}\cdot\frac{d^{f+g}\vartheta}{dx^f dy^g}$ being expanded by means of our theorem in terms of $\frac{d}{du}, \frac{d}{dv}, x, y, \vartheta$, the coefficient in such expansion of $\frac{d^{A+B}\vartheta}{du^A dv^B}$ will exhibit the value of the coefficient of $\xi^f\eta^g$ in the expansion of $\frac{1}{\Pi A\,\Pi B}h^A k^B$ in terms of ξ and η.

(14) As there are no quantitative relations between the coefficients in the equations above written which express ξ and η, we are therefore now able to express the value of $h^A k^B$ in terms of ξ and η when ξ and η are respectively expressed as rational integral functions of (and vanishing with) h and k. Thus, let us write in general
$$\xi = \Sigma p_{r,s}h^r k^s,$$
$$\eta = \Sigma q_{r,s}h^r k^s,$$
where $p_{0,0}$ and $q_{0,0}$ are each zero, but all the other values of p and q absolutely arbitrary. We have now $p_{r,s}$, $q_{r,s}$ respectively replacing
$$\frac{1}{\Pi r\cdot\Pi s}\frac{d^{r+s}x}{du^r dv^s},\quad \frac{1}{\Pi r\cdot\Pi s}\frac{d^{r+s}y}{du^r dv^s}$$
and consequently the general term in the expansion of $h^A k^B$ as a function of ξ and η will be $I_{f,g}\xi^f\eta^g$, where
$$I_{f,g} = \frac{\Pi A\,\Pi B}{\Pi f\,\Pi g}\Sigma(-)^{l+\beta+\alpha'}\frac{N}{J^{\alpha+\beta+\alpha'+\beta'}}q^{\alpha}{}_{0,1}\,p^{\alpha'}{}_{0,1}\,q^{\beta}{}_{1,0}\,p^{\beta'}{}_{1,0}$$
$$\times(\Pi a\,\Pi b\,p_{a,b})^l \times \&c. \times (\Pi a'\,\Pi b'\,q_{a',b'})^{l'} \times \&c.,$$
where
$$N = \frac{\Pi(\alpha+\beta-1)\,\Pi(\alpha'+\beta'-1)}{\Pi\alpha\,\Pi\beta\,\Pi\alpha'\,\Pi\beta'}$$
$$\times\frac{\Pi(\alpha+\alpha'-1)\,\Pi(\beta+\beta'-1)}{\Pi l(\Pi a\,\Pi b)^l \times \&c. \times \Pi l'(\Pi a'\,\Pi b')^{l'} \times \&c.} \times \frac{D}{\Pi A\,\Pi B},$$
and
$$J = \begin{vmatrix} p_{1,0}, & p_{0,1} \\ q_{1,0}, & q_{0,1} \end{vmatrix}.$$
Hence
$$I_{f,g} = \Sigma\left\{(-)^{\Sigma l+\Sigma l'+\beta+\alpha'}\frac{\Pi(\alpha+\beta-1)\,\Pi(\alpha'+\beta'-1)}{\Pi\alpha\,\Pi\beta\,\Pi\alpha'\,\Pi\beta'}\right.$$
$$\times\frac{\Pi(\alpha+\alpha'-1)\,\Pi(\beta+\beta'-1)}{\Pi f\,\Pi g}\times\frac{q^{\alpha}{}_{0,1}\,q^{\beta}{}_{1,0}\,p^{\alpha'}{}_{0,1}\,p^{\beta'}{}_{1,0}}{(p_{1,0}\,q_{0,1} - p_{0,1}\,q_{1,0})^{\alpha+\beta+\alpha'+\beta'}}$$
$$\times\begin{vmatrix} \alpha+\beta, & & \Sigma la, & \Sigma lb \\ & \alpha'+\beta', & \Sigma l'a', & \Sigma l'b' \\ \alpha, & \alpha', & \alpha+\alpha', & \\ \beta, & \beta', & & \beta+\beta' \end{vmatrix}$$
$$\left.\times\left(\frac{p^l{}_{a,b}}{\Pi l}\times\&c.\times\frac{q^{l'}{}_{a',b'}}{\Pi l'}\times\&c.\right)\right\},$$

$\alpha, \beta; \alpha', \beta'; l, l'$, &c., being any system of positive integers which are capable of satisfying the equations

$$\alpha+\beta=\Sigma l+f,$$
$$\alpha'+\beta'=\Sigma l'+f',$$
$$\alpha+\alpha'=\Sigma la+\Sigma l'a'+A,$$
$$\beta+\beta'=\Sigma lb+\Sigma l'b'+B.$$

Hence the value of $h^A k^B$, which $=\Sigma I_{f,g}\xi^f\eta^g$, is completely determined as an explicit function of ξ, η, and the coefficients p, q, of the equations by which ξ, η, are given in terms of h and k.

(15) So for three variables, supposing

$$\xi=\Sigma m_{r,s,t}\,h^r k^s l^t,$$
$$\eta=\Sigma n_{r,s,t}\,h^r k^s l^t,$$
$$\zeta=\Sigma p_{r,s,t}\,h^r k^s l^t,$$

where $m_{0,0,0}$, $n_{0,0,0}$ and $p_{0,0,0}$, are each zero, but all other values of m, n, p absolutely arbitrary, making

$$\begin{vmatrix} m_{1,0,0}, & m_{0,1,0}, & m_{0,0,1} \\ n_{1,0,0}, & n_{0,1,0}, & n_{0,0,1} \\ p_{1,0,0}, & p_{0,1,0}, & p_{0,0,1} \end{vmatrix}=J,$$

and writing in general

$$\frac{d\log J}{dm_{i,i',i''}}=\mu_{i,i',i''},$$
$$\frac{d\log J}{dn_{i,i',i''}}=\nu_{i,i',i''},$$
$$\frac{d\log J}{dp_{i,i',i''}}=\phi_{i,i',i''},$$

we shall find $$h^A k^B l^C=\Sigma I_{f,g,h}\xi^f\eta^g\zeta^h,$$

where

$$I_{f,g,h}=\Sigma\left\{\left((-)^{\Sigma l+\Sigma l'+\Sigma l''}\frac{\Pi(\alpha+\beta+\gamma-1)\,\Pi(\alpha'+\beta'+\gamma'-1)\,\Pi(\alpha''+\beta''+\gamma''-1)}{\Pi\alpha\,\Pi\beta\,\Pi\gamma\,\Pi\alpha'\,\Pi\beta'\,\Pi\gamma'\,\Pi\alpha''\,\Pi\beta''\,\Pi\gamma''}\right.\right.$$
$$\left.\frac{\Pi(\alpha+\alpha'+\alpha''-1)\,\Pi(\beta+\beta'+\beta''-1)\,\Pi(\gamma+\gamma'+\gamma''-1)}{\Pi f\,\Pi g\,\Pi h}\right)$$
$$\times D\times\mu^{\alpha}{}_{1,0,0}\,\mu^{\beta}{}_{0,1,0}\,\mu^{\gamma}{}_{0,0,1}\,\nu^{\alpha'}{}_{1,0,0}\,\nu^{\beta'}{}_{0,1,0}\,\nu^{\gamma'}{}_{0,0,1}\,\phi^{\alpha''}{}_{1,0,0}\,\phi^{\beta''}{}_{0,1,0}\,\phi^{\gamma''}{}_{0,0,1}$$
$$\left.\times\frac{m^l{}_{a,b,c}}{\Pi l}\times\&\text{c.}\times\frac{n^{l'}{}_{a',b',c'}}{\Pi l'}\times\&\text{c.}\times\frac{p^{l''}{}_{a'',b'',c''}}{\Pi l''}\times\&\text{c.}\right\},$$

where D is a known determinant of the sixth order expressible in terms of

$\alpha, \beta, \gamma; \alpha', \beta', \gamma'; \alpha'', \beta'', \gamma''; \Sigma la, \Sigma lb, \Sigma lc; \Sigma l'a', \Sigma l'b', \Sigma l'c'; \Sigma l''a'', \Sigma l''b''$, $\Sigma l''c''$, and where

$$\alpha+\beta+\gamma=\Sigma l+f,$$
$$\alpha'+\beta'+\gamma'=\Sigma l'+g,$$
$$\alpha''+\beta''+\gamma''=\Sigma l''+h,$$
$$\alpha+\alpha'+\alpha''=\Sigma la+\Sigma l'a'+\Sigma l''a''+A,$$
$$\beta+\beta'+\beta''=\Sigma lb+\Sigma l'b'+\Sigma l''b''+B,$$
$$\gamma+\gamma'+\gamma''=\Sigma lc+\Sigma l'c'+\Sigma l''c''+C.$$

(16) Suppose now that we wish from the equation $0=\Sigma p_{r,s}h^r k^s$ to deduce the value of k^β in terms of h.

We may put $$\xi=\Sigma p_{r,s}h^r k^s,$$
$$\eta=h,$$

and then apply the formula of reversion for finding k^β in terms of ξ and η; but since $\xi=0$, we may reject all the terms out of $\Sigma I_{f,g}\xi^f\eta^g$, except those in which $f=0$; moreover, in adapting the formula applicable to this case, we must put $q_{r,s}=0$ for all values of the system r, s, except 1, 0, and for that system $q_{1,0}=1$; we have, therefore, to retain such terms only in $I_{0,g}$ for which $\alpha=0$, $\Sigma l'a'=0$, $\Sigma l'b'=0$;

D consequently becomes $$\begin{vmatrix} \beta, & 0, & \Sigma la, & \Sigma lb \\ 0, & \alpha'+\beta', & 0, & 0 \\ 0, & \alpha', & \alpha', & 0 \\ \beta, & \beta', & 0, & \beta+\beta' \end{vmatrix}$$

$$=\alpha'(\alpha'+\beta')\begin{vmatrix} \beta, & \Sigma lb \\ \beta, & \beta+\beta' \end{vmatrix}=\alpha'(\alpha'+\beta')\beta\begin{vmatrix} 1, & \beta+\beta'-B \\ 1, & \beta+\beta' \end{vmatrix}$$
$$=\alpha'(\alpha'+\beta')\beta B;$$

hence

$$I_{0,g}=\Sigma\left\{(-)^{\Sigma l+\beta+\alpha'}\frac{\Pi\beta\,\Pi(\alpha'+\beta')}{\Pi\beta\,\Pi\alpha'\,\Pi\beta'}\times\frac{\Pi\alpha'\,\Pi(\beta+\beta'-1)B}{\Pi g}\right.$$
$$\left.\times\frac{p^{\alpha'}{}_{0,1}\,p^{\beta'}{}_{1,0}}{p_{0,1}{}^{\beta+\alpha'+\beta'}}\times\frac{p^l{}_{a,b}}{\Pi l}\times\&\text{c.}\right\},$$

with the conditions

$$\left.\begin{array}{ll} \alpha'=\Sigma la, & \beta=\Sigma l \\ \beta+\beta'=\Sigma lb+B, & \alpha'+\beta'=g \end{array}\right\}; \qquad (\omega)$$

we have, therefore, finally

$$k^B = B\Sigma\left(\left\{\Sigma(-)^{\Sigma la}\frac{\Pi\{\Sigma(lb)+B-1\}}{\Pi\Sigma\{l(b-1)+B\}}\cdot\frac{p_{1,0}{}^{\Sigma\{l(b-1)\}+B}}{p_{0,1}{}^{\Sigma(lb)+B}}\cdot\frac{(p^{l}{}_{a_1,b_1})}{\Pi l}\cdots\frac{(p^{l_e}{}_{a_e,b_e})}{\Pi l_e}\right\}h\right)$$

with the sole condition deduced from the system (ω),

$$l_1(a_1+b_1-1)+l_2(a_2+b_2-1)+\ldots+l_e(a_e+b_e-1)=g-B.$$

Suppose, now, that $\phi(x, y)=0$, and that we wish to express $\frac{d^g\vartheta}{dx^g}$ (where, for greater simplicity, I consider ϑ a function only of y) in terms of x, y, without solving the equation $\phi=0$; we know that if we write

$$\frac{d\phi}{dx}h+\frac{d\phi}{dy}k+\&c.+\left\{\&c.+\frac{\left(\frac{d}{dx}\right)^{\epsilon}\left(\frac{d}{dy}\right)^{\omega}}{\Pi\epsilon\,\Pi\omega}h^{\epsilon}k^{\omega}+\&c.\right\}+\&c.,$$

then $\frac{1}{\Pi g}\frac{d^g\vartheta}{dx^g}$ will be the coefficient of h^g in the expansion of

$$\frac{d\vartheta}{dy}k+\frac{d^2\vartheta}{dy^2}\frac{k^2}{1.2}+\frac{d^3\vartheta}{dy^3}\frac{k^3}{1.2.3}\ \&c.\ \text{in terms of } h.$$

Consequently, if we make

$$\frac{d^g\vartheta}{dx^g}=\Sigma E_B\frac{d^B\vartheta}{dy^B},$$

$$E_B=\frac{\Pi g}{\Pi(B-1)}\Sigma(-)^{\Sigma la}\frac{\Pi\{\Sigma(lb)+B-1\}}{\{\Pi\Sigma(lb)+B-\Sigma l\}}$$

$$\times\frac{\left(\frac{d\phi}{dx}\right)^{\Sigma lb+B-\Sigma l}}{\left(\frac{d\phi}{dy}\right)^{\Sigma lb+B}}\frac{\left\{\left(\frac{d}{dx}\right)^{a}\left(\frac{d}{dy}\right)^{b}\phi\right\}^{l}}{\Pi l(\Pi a\,\Pi b)^{l}}\cdots\frac{\left\{\left(\frac{d}{dx}\right)^{a_e}\left(\frac{d}{dy}\right)^{b_e}\phi\right\}^{l_e}}{\Pi l_e(\Pi a_e\,\Pi b_e)^{l_e}},$$

where, as before, writing in general $a_k+b_k-1=c_k$,

$$l_1c_1+l_2c_2+\ldots+l_ec_e=g-B,$$

g being now given, and B variable and subject to assume in succession every value from 1 up to B.

(17) By way of verifying the above formula, and as a protection against accidental errors of calculation, suppose $\phi=-x+\psi(y)$,

so that $$\frac{d\phi}{dx}=-1,\quad \frac{d\phi}{dy}=\frac{d\psi}{dy};$$

the only terms to be retained are those in which no (a) index appears.

We have, therefore, for this case $\Sigma l(b-1)=g-B$,

that is, $$\Sigma lb+B-\Sigma l=g,$$

and $$E_B=\frac{\Pi\{\Sigma(lb)+B-1\}}{\Pi l(\Pi b_1)^{l}\ldots\Pi l_e(\Pi b_e)^{l_e}}\frac{(-)^g}{\left(\frac{d\psi}{dy}\right)^{g+\Sigma l}}\left\{\left(\frac{d}{dy}\right)^{b}\psi\right\}^{l}\cdots\left\{\left(\frac{d}{dy}\right)^{b_e}\psi\right\}^{l_e}$$

agreeing, as required, with Burmann's law.

(18) As another example, in illustration of the fact that our general theorem embraces the whole theory of reversion, suppose we have the equation $\Sigma m_{r,s,t}q^rk^sl^t=0$, and that it is required from this equation to deduce l^C as a function of h and k.

We may write $$\xi=\Sigma m_{r,s,t}q^rk^sl^t,$$

$$\eta=q,$$

$$\zeta=k.$$

We have then $$l^C=\Sigma I_{0,g,h}q^gk^h,$$

and in assigning the value of $I_{0,g,h}$, we need, moreover, to retain in $I_{0,g,h}$ only those terms in which the a', b', c' and a'', b'', c'' systems of indices are wanting; for

$$\Sigma l'a'=0,\quad \Sigma l'b'=0,\quad \Sigma l'c'=0,$$

$$\Sigma l''a''=0,\quad \Sigma l''b''=0,\quad \Sigma l''c''=0.$$

Moreover α, β, γ; α', β', γ'; $\alpha'', \beta'', \gamma''$; being the indices respectively of the minor determinants of the matrix

$$\begin{matrix} m_{1,0,0}; & m_{0,1,0}; & m_{0,0,1}; \\ 1\ ; & 0\ ; & 0\ ; \\ 0\ ; & 1\ ; & 0\ ; \end{matrix}$$

we may consider $\alpha=0$, $\beta=0$, $\beta'=0$, $\alpha''=0$, since the minor determinants which have these indices are all zero.

Hence, for the actual terms in $I_{0,g,h}$, D becomes

$$\begin{vmatrix} \gamma, & \ldots, & \ldots, & \Sigma la, & \Sigma lb, & \Sigma lc \\ \ldots, & \alpha'+\gamma', & \ldots, & \ldots, & \ldots, & \ldots \\ \ldots, & \ldots, & \beta''+\gamma'', & \ldots, & \ldots, & \ldots \\ \ldots, & \alpha', & \ldots, & \alpha', & \ldots, & \ldots \\ \ldots, & \ldots, & \beta'', & \ldots, & \beta'', & \ldots \\ \gamma, & \gamma', & \gamma'', & \ldots, & \ldots, & \gamma+\gamma'+\gamma'' \end{vmatrix}$$

which obviously reduces to the form

$$\alpha'\beta''(\alpha'+\gamma')(\beta''+\gamma'')\begin{vmatrix}\gamma, & \Sigma lc \\ \gamma, & \Sigma lc + C\end{vmatrix}$$
$$=\alpha'\beta''(\alpha'+\gamma')(\beta''+\gamma'')\gamma C;$$

also the equations of condition between the indices become

$$\gamma=\Sigma l, \qquad \alpha'=\Sigma la,$$
$$\alpha'+\gamma'=g, \qquad \beta''=\Sigma lb,$$
$$\beta''+\gamma''=h, \quad \gamma+\gamma'+\gamma''=\Sigma lc+C,$$

in addition to the special equations

$$\alpha=0, \quad \beta=0, \quad \alpha''=0, \quad \beta'=0.$$

Hence $l^C=\Sigma I_{g,h}q^g k^h$, where $I_{g,h}$ represents

$$(-)^{\Sigma l}\frac{\Pi g\,\Pi h}{\Pi\gamma\,\Pi\alpha'\,\Pi\gamma'\,\Pi\beta''\,\Pi\gamma''}\cdot\frac{\Pi\alpha'\,\Pi\beta''\,\Pi(\gamma+\gamma'+\gamma''-1)\,C}{\Pi g\,\Pi h}\,.\,\&c.$$
$$=(-)^{\Sigma l}C\,\frac{\Pi(\gamma+\gamma'+\gamma''-1)}{\Pi\gamma\,\Pi\gamma'\,\Pi\gamma''}\cdot\frac{m_{0,0,1}{}^{\alpha'+\beta''}(-m_{0,1,0})^{\gamma''}(-m_{1,0,0})^{\gamma'}}{m_{0,0,1}{}^{\gamma+\gamma'+\gamma''+\alpha'+\beta''}}\times\&c.$$
$$=\Sigma(-)^{\gamma+\gamma'+\gamma''}C\,\frac{\Pi(\gamma+\gamma'+\gamma''-1)}{\Pi\gamma\,\Pi\gamma'\,\Pi\gamma''}\,\frac{m^{\gamma''}{}_{0,1,0}\,m^{\gamma'}{}_{1,0,0}}{m_{0,0,1}{}^{\gamma+\gamma'+\gamma''}}$$
$$\times\frac{m^{l}{}_{abc}}{\Pi l}\times\frac{m^{l_2}{}_{a_2b_2c_2}}{\Pi l_2}\ldots\times\frac{m^{l_e}{}_{a_eb_ec_e}}{\Pi l_e},$$

where $l_1(a_1+b_1+c_1-1)+\ldots+l_e(a_e+b_e+c_e-1)=g+h-C$, g and h being assumed of any values respectively, such that their sum is not less than C: the partitionment of $g+h-C$, gives every possible system

$$l_1\ldots l_e; \quad (a+b+c-1)\ldots(a_e+b_e+c_e-1);$$

and to every such system correspond known systems of values of $a, b, c;\ \ldots;\ a_e, b_e, c_e$. We have then $\gamma=\Sigma l$, $\gamma'+\gamma''=\Sigma l(c-1)+C$, which latter equation, for each value of c, gives $\Sigma l(c-1)+C+1$ systems of values of γ' and γ''. Thus we have the complete solution of the equation $\Sigma m_{r,s,t}q^r k^s l^t=0$.

In like manner, if we suppose i variables $q_1, q_2, \ldots q_i$, and for greater simplicity, in addition to the condition always supposed of the constant term being zero, likewise conceive that the coefficient shall be unity in each linear term of the equation

$$\Sigma m_{r_1r_2\ldots r_i}q_1^{r_1}q_2^{r_2}\ldots q_i^{r_i}=0,$$

we shall find

$$q_i^{A_i}=\Sigma(-)^{\gamma_1+\gamma_2+\ldots+\gamma_i}A_i\,\frac{\Pi(\gamma_1+\gamma_2+\ldots+\gamma_i-1)}{\Pi\gamma_1\,\Pi\gamma_2\ldots\Pi\gamma_i}$$
$$\times\frac{(m^{l}{}_{{}^1a\,{}^2a\ldots{}^ia})^{l}}{\Pi l}\times\&c.\times\frac{(m_{{}^1a_e\,{}^2a_e\ldots{}^ia_e})^{l_e}}{\Pi l_e}\,q_1^{f_1}q_2^{f_2}\ldots q_{i-1}^{f_{i-1}},$$

with the conditions following for finding the (γ) and ${}^1a, {}^2a\ldots{}^ia\ldots{}^1a_e, {}^2a_e\ldots{}^ia_e$ systems, namely,

$$\Sigma l({}^1a+{}^2a+\ldots+{}^ia-1)=f_1+f_2+\ldots+f_{i-1}-A_i,$$
$$\gamma_1=\Sigma l, \quad \gamma_2+\gamma_3+\ldots+\gamma_i=\Sigma l_i(a_i-1)+A_i+1.$$

(19) In like manner we may without difficulty assign the general law for solving with like generality any number of simultaneous equations between any greater number of variables, the functions equivalent to zero being all supposed to be without a constant term, and to be expressed as rational integral functions of the variables; and we can consequently pass from one system of independent variables to any new system in whatever way, whether explicitly or implicitly, through any number of equations and any number of connecting variables, the two systems may be supposed to be related.

17.

ON A DISCOVERY IN THE PARTITION OF NUMBERS*.

[*Quarterly Journal of Mathematics*, I. (1857), pp. 81—84.]

LET $a_1, a_2, \ldots a_r$ be any given system of integer elements; I call the number of ways of composing the number n with these elements the *quotity*† of n in respect to the given elements. Let the least common multiple of $a_1, a_2, a_3, \ldots a_r$ be called p and let the roots of $\frac{x^p-1}{x-1}=0$ be called ρ, then we may express the quotity in question under the form

$$A+U,$$

* From the last foot-note at p. 87, it follows that the non-periodical part of the analytical expression for the number of ways in which n can be composed of the r elements $a, b, c \ldots l$, is the coefficient of $\frac{1}{t}$ in the expansion, in a series of ascending powers of t, of the fraction $\frac{e^{nt}}{(1-e^{-at})(1-e^{-bt})\ldots(1-e^{-lt})}$. Moreover, if we suppose $\frac{1}{(1-x^a)(1-x^b)\ldots(1-x^l)}=\frac{P}{(1-x)^r}+$ fractions not containing $(1-x)$ in the denominator, it further follows that, for values of n not less than r, the coefficient of x^n in P will be the coefficient of $\frac{1}{t}$ in the expression

$$\frac{e^{(n-r)t}\cdot(e^t-1)^r}{(1-e^{-at})(1-e^{-bt})\ldots(1-e^{-lt})},$$

which is evidently zero, as it ought to be.

† Thus the quotity of n in respect to a and 1 is the integer next greater than $\frac{n}{a}$; the complete expression for this quantity, it may be mentioned, is

$$\frac{1}{a}(n+\tfrac{1}{2})-\frac{1}{a^2}\Sigma\{(a-1)+(a-2)\rho+\ldots+\rho^{a-2}\}\rho^{n+1},$$

where ρ is a prime root of $\frac{\rho^a-1}{\rho-1}=0$.

The quotity of n in respect to the consecutive elements 1, 2, 3 ... r is equal to the number of ways of partitioning $n+r$ into r parts.

where A is an algebraical function of n and the elements, which is clear of all exponential expressions, and U is of the form

$$\Sigma(A_0+A_1\rho+A_2\rho^2+\&c.+A_{p-1}\rho^{p-1})\rho^n\text{*}.$$

I call U the quot-undulant†; A the quot-additant‡. I shall say nothing at present about the former, although I can express its value completely for any system of elements which are prime each to each, or of which the relations of identity existing between the prime factors are given: my theorem, for present purposes, is confined to the quot-additant, which may be written under the form following, namely,

$$\frac{1}{a_1a_2\ldots a_r}\left\{B_1+B_2n+B_3\frac{n^2}{1.2}+\&c.\ldots+B_{r-1}\frac{n^{r-2}}{1.2\ldots(r-2)}\right.$$
$$\left.+B_r\frac{n^{r-1}}{1.2\ldots(r-1)}\right\},$$

where $$B_\omega=\Sigma C_{\omega_{\theta_1}}C_{\omega_{\theta_2}}\ldots C_{\omega_{\theta_r}}S(a_1^{\omega_{\theta_1}}a_2^{\omega_{\theta_2}}\ldots a_r^{\omega_{\theta_r}}),$$

S denoting as usual that a symmetrical function is to be formed, of which the quantity which follows it is the type of the general term, and the symbol Σ referring to a summation to be effected in respect to all the distinct systems of integer values (zeros included in the number) of $\omega_{\theta_1}, \omega_{\theta_2}, \ldots \omega_{\theta_r}$, whose sum is $r-\omega$, and where, in general, C_m denotes the coefficient of t^m in

$$\frac{te^t}{e^t-1}\text{§}.$$

* The coefficients A_0, A_1, &c., are, in general, algebraical functions of n and of the elements whose degree in n is one unit inferior to the greatest number of the elements having the same common measure.

† The quot-undulant, although for present purposes presented as a single whole, is in fact a collective quantity made up, and most simply and naturally expressed by means of the sum of a series of analogous periodic or periodico-progressive functions, whose number is the same as that of the distinct elements, and whose periods are respectively measured by the number of units in each such element; it may be compared with a great wave, composed of a number of wavelets, whose lengths are either the same as or submultiples of its own. This is the view first taken by Mr Cayley, who, in his researches, has followed in the footsteps of Euler, but to which, also, I have been independently and unavoidably conducted by the method of investigation peculiar to myself. Sir John Herschel and Mr Kirkman have not taken this view, and accordingly there is an unnecessary complexity in their statements of results.

‡ If we suppose the fraction $\frac{1}{(1-x^{a_1})(1-x^{a_2})\ldots(1-x^{a_r})}$ thrown under the form

$$\frac{P}{(1-x)^r}+\frac{Q}{(1-x^{a_1})(1-x^{a_2})\ldots(1-x^{a_r})\div(1-x)^r},$$

the quot-additant of n is the coefficient of x^n in $\frac{P}{(1-x)^r}$ which gives the means of expressing P, and consequently also Q. Compare Note iv. in M. Serret's excellent *Cours d'Algèbre supérieure*, 2nd edition.

§ The theorem may also be stated as follows. Let A_0, A_1, A_2, &c., denote the successive coefficients in the expansion in a series of ascending powers of x of the reciprocal of the product of $1-e^{-ax}, 1-e^{-bx}, \ldots 1-e^{-lx}$, then will the quot-additant of n in respect of the r elements $a, b, c \ldots l$, be expressed by $A_{r-1}+A_{r-2}.n+A_{r-3}.\frac{n^2}{1.2}+\ldots+A_0.\frac{n^{r-1}}{1.2.3\ldots(r-1)}$. [See note * of p. 86.]

Examples. The quot-additants of n, in respect to the systems a; a, b; a, b, c; a, b, c, d, &c., respectively, are as follows:

$$\frac{1}{a},\ \frac{1}{ab}\left(\frac{a+b}{2}+n\right),$$

$$\frac{1}{abc}\left(\frac{a^2+b^2+c^2+3ab+3ac+3bc}{12}+\frac{a+b+c}{2}n+\frac{n^2}{1.2}\right),$$

$$\frac{1}{abcd}\left(\frac{\Sigma a^2b+3\Sigma abc}{24}+\frac{(\Sigma a)^2+3\Sigma ab}{12}n+\frac{\Sigma a}{4}n^2+\frac{n^3}{1.2.3}\right)^*.$$

So, again, the constant term in the quot-additant to the system of elements a, b, c, d, e will be

$$\frac{1}{abcde}\Sigma\left\{-\frac{a^4}{720}+\frac{a^2b^2}{144}+\frac{a^2bc}{48}+\frac{abcd}{16}\right\},$$

and to the system of six elements it will be

$$\frac{1}{abcdef}\Sigma\left\{-\frac{a^4b}{1440}+\frac{a^2b^2c}{288}+\frac{a^2bcd}{96}+\frac{abcde}{32}\right\};$$

it will be seen that the latter quantity under the sign of summation is obtained term for term from the one above by introducing a new element with the index unity in the numerator and doubling each denominator; this law is general, and is an immediate consequence of the fact that for a coefficient of i dimensions in the elements the only partitionments of i which appear in the groups of indices are those which are made up of the elements 1, 2, 4, 6, &c., all the odd elements except 1 from the nature of Bernouilli's numbers giving rise to the coefficient zero, so that, consequently, the partitionments of $2\iota+1$ which enter into the expression in question, are all derived from those of 2ι by the addition of a single distinct unit.

The series of fractions $\frac{1}{2}, \frac{1}{12}, 0, \frac{1}{120}, 0$, &c., arise in my method as the results of substituting $\frac{1}{\omega+1}$ in place of $\frac{\delta^\omega\phi}{\phi}$ in the expansion of the successive variations of $\log\phi$.

Thus, $\quad \delta\log\phi=\frac{\phi'}{\phi}=\frac{1}{2},\ \delta^2\log\phi=\frac{\phi''}{\phi}-\left(\frac{\phi'}{\phi}\right)^2=\frac{1}{3}-\frac{1}{4}=\frac{1}{12},$

$$\delta^3\log\phi=\frac{\phi'''}{\phi}-3\frac{\phi''\phi'}{\phi^2}+2\frac{\phi'^3}{\phi^3}=\frac{1}{4}-\frac{3}{6}+\frac{2}{8}=0, \text{ \&c. \&c.}$$

* When the elements $a, b, c \ldots l$ are prime each to each, the quot-undulant will not contain n, that is, will be strictly periodic. For this case, therefore, the difference between the quot-additant of n and that of $n-(a.b.c \ldots l)$ will represent the difference between the entire quotity of n and that of $n-(a.b.c \ldots l)$ in respect to the system supposed. We have consequently an easy method of verifying, by actual decompositions of numbers, the general expression for the additant part without knowing the value of the undulant part in the complete expression for the quotity. For the case in question the quot-additant may also be defined and calculated as the algebraical expression whose mean value is the same as the mean value of the quotity when the partible number passes through a period of $a.b.c \ldots l$ consecutive integer values.

Hence it may easily be collected, that if we write

$$\frac{te^t}{e^t-1}=1+K_1t-2K_2t^2+3K_3t^3\mp \text{\&c.},$$

we ought to have

$$K_i=\frac{1}{i}E_0-\frac{1}{i-1}E_1+\frac{1}{i-2}E_2 \text{ \&c. } \pm E_{i-1},$$

where E_ω denotes, in general, the coefficient of h^ω in

$$\left(\frac{1}{2}+\frac{h}{2.3}+\frac{h^2}{2.3.4}+\text{\&c.}\right)^{i-\omega},$$

or, which is the same thing, K_i ought to be equal to the coefficient of t^i in $\log\frac{e^t-1}{t}$ as is easily demonstrable to be the case by Maclaurin's Theorem.

In general, if the quot-additant of n, in respect to the roots of

$$x^r+p_1x^{r-1}+p_2x^{r-2}+\text{\&c.}+p_r,$$

be expressed as a function of $p_1, p_2, \ldots p_r$ and n, and be called $\frac{1}{p_r}Q_r$, we have the following equations existing, namely,

$$Q_r=\int dn\, Q_{r-1} \text{ and } \left(\frac{d}{dp_1}+p_1\frac{d}{dp_2}+\ldots+p_{r-1}\frac{d}{dp_r}\right)Q_r=-\frac{1}{2}Q_{r-1}.$$

Observation. My method which has led me to the preceding theorem reposes upon the axiom, which I believe is quite new, that the mean value of the $a.b.c \ldots l$ sums of homogeneous powers and products (all affected with the coefficient unity) of n dimensions in $\alpha, \beta, \gamma, \ldots, \lambda$, where $\alpha^a=1$, $\beta^b=1, \ldots \lambda^l=1$, is equal to the quotity of n in respect to $a, b, c, \ldots l$.

18.

ON THE PARTITION OF NUMBERS.

[*Quarterly Journal of Mathematics*, I. (1857), pp. 141—152. *Also printed, Tortolini's Annali di Matematiche*, VIII. (1857), pp. 12—21.]

I MUST reluctantly content myself for the present (unexpected events, which have robbed me of the leisure and calm of mind necessary for composition, and the due evolution and embodiment of ideas on any extensive scale, forbid me to do more) with a brief statement of the general solution of this important question, which (as known to my thrice-distinguished friend, Mr Cayley) I succeeded in completing almost immediately after the appearance of the last number of the Journal.

It must be clearly understood that the methods of Euler, De Morgan, Herschel, Kirkman, and Cayley (the last a great advance upon all that went before) have only afforded the means (with more or less generality) of determining the *quotity* of a number in respect of given elements in any particular case; the existence of a universal algebraical representation of this quotity seems not even to have been suspected. Moreover, it will be found that the general formula, which I am about to give, possesses an immense practical advantage in point of facility of computation over the methods previously employed. Thus, for example, I have been able to compute by it, in a moderate space of time, the number of ways of partitioning n into nine parts; the enormous complexity of the calculations required by the methods of Herschel and Cayley had induced those distinguished authors to rest satisfied with stopping short at the formula for only five parts.

My result has been erected upon a completely independent basis, and deduced by an equally original method, namely, the axiom contained in the observation at the end of my former paper combined with a simple theorem for expressing, by means of partial fractions, the sum of the homogeneous

powers and products of any number of quantities, not merely for the *special* case of these quantities being all unlike, but for the *general* case of their being made up of any sets of equals. MM. Cayley and Terquem have both suggested, what is no doubt true, the possibility of obtaining my result otherwise, and perhaps a little more simply, by aid of M. Cauchy's Theory of Residues.

I now proceed to enunciate the theorem.

$a_1, a_2, \dots a_r$ (all positive integers) are supposed to be the elements, n the partible number, and the object in view is the expression of the quotity of n quâ the elements $a_1, a_2, \dots a_r$, that is, of the number of solutions of the equation in integers $a_1x_1 + a_2x_2 + \&c. + a_rx_r = n$, in which equation it may be observed no further condition is imposed upon the coefficients $a_1, a_2, \dots a_r$ than that of their being positive integers. There is no restriction upon their being equal in any manner *inter se*. Call Q the quotity in question: then we may consider Q as made up of an infinite number of waves, of which, however (as it will immediately be seen), only a finite number have an *actual* existence, the rest will be *abortive*.

Let $\frac{p}{q}$ be any rational numerical fraction whatever, not exceeding unity, in its lowest terms, and use $w_{\frac{p}{q}}$ to denote the coefficient of $\frac{1}{t}$ in the development of the expression

$$e^{nw}(1-e^{a_1u})^{-1}(1-e^{a_2u})^{-1}\dots(1-e^{a_ru})^{-1},$$

where
$$w = \frac{2\pi ip}{q} + t, \quad u = \frac{2\pi ip}{q} - t, \quad i = (-1)^{\frac{1}{2}};$$

then
$$Q = \Sigma w_{\frac{p_i}{q}}.$$

If $p_1, p_2, \dots p_i$ be all the numbers (unity included) less than q, and prime to it, and if we write

$$w_{\frac{p_1}{q}} + w_{\frac{p_2}{q}} + \dots + w_{\frac{p_i}{q}} = W_q,$$

we shall have more simply
$$Q = \Sigma W_q.$$

W_q again may be expressed* under a more easily intelligible form as the coefficient of $\frac{1}{t}$ in the development in ascending powers of t of

$$\Sigma \frac{\rho^n e^{nt}}{(1-\rho^{a_1}e^{-a_1t})(1-\rho^{a_2}e^{-a_2t})\dots(1-\rho^{a_r}e^{-a_rt})},$$

where ρ is in succession each of the roots of the *prime factor* of $\rho^q - 1$; and

[* Cf. p. 157 below.]

consequently since this development will contain no term where $\frac{1}{t}$ enters, unless some one at least of the quantities $\rho^{a_1}, \rho^{a_2}, \ldots \rho^{a_r}$ is unity, it follows that $W_q = 0$, except for those values of q which are contained in some one or more of the elements $a_1, a_2, \ldots a_r$. The number of *actual* waves expressing a given quotity is consequently the number of distinct integers, unity included, which enter into the composition of the elements to which the quotity has reference.

It will readily be seen that on making $q = 1$ we shall obtain the expression for the so-called quot-additant (a name only adopted for provisional purposes, and which I now discard) given [p. 87 above] in the preceding number of the Journal. The generating function for this part of the quotity becomes, from the general formula,

$$\frac{e^{nt}}{(1-e^{-a_1 t})(1-e^{-a_2 t})\ldots(1-e^{-a_r t})}.$$

The coefficient of $\frac{1}{t}$ in this expression will give the formulæ contained in the body of the paper referred to; but far more expeditious formulæ of computation may be substituted in lieu of these. For we may write

$$W_1 = \text{coefficient of } \frac{1}{t} \text{ in}$$

$$e^{nt - \{\log(1-e^{-a_1 t}) + \log(1-e^{-a_2 t}) + \ldots + \log(1-e^{-a_r t})\}}.$$

But in general

$$\log(1-e^{-t}) = \log t - \frac{t}{2} + \frac{t^2}{24} + \&\text{c.}$$

$$= \log t - \frac{t}{2} + \frac{B_1}{1.2^2}t^2 - \frac{B_2}{1.2.3.4^2}t^4 + \&\text{c.},$$

B_1, B_2, &c., denoting Bernouilli's numbers, namely,

$$\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \&\text{c.}$$

Hence

$$W_1 = \frac{1}{a_1 a_2 \ldots a_r} \times \text{coefficient of } t^{r-1} \text{ in } e^{(n+\frac{1}{2}s_1)t - \frac{B_1 s_2}{1.2^2}t^2 + \frac{B_2 s_4}{1.2.3.4^2}t^4 \,\&\text{c.}},$$

where s_ω in general denotes the sum of the ωth powers of the elements $a_1, a_2, \ldots a_r$.

Hence, writing $n + \frac{1}{2}s_1 = \nu$, we have

$$W_1 = \text{coefficient of } t^{r-1} \text{ in}$$

$$(a_1 a_2 \ldots a_r)^{-1}\left(1 + \nu t + \nu^2\frac{t^2}{1.2} + \nu^3\frac{t^3}{1.2.3} + \&\text{c.}\right)$$
$$\times\left(1 - \frac{1}{24}s_2 t^2 + \frac{1}{1152}s_2^2 t^4 + \&\text{c.}\right)$$
$$\times\left(1 + \frac{1}{2880}s_4 t^4 + \frac{1}{165888}s_4^2 t^8 + \&\text{c.}\right)$$
$$\times\left(1 - \frac{1}{181440}s_6 t^6 + \&\text{c.}\right)$$
$$\times \&\text{c.}^*$$

The wave W_2 is also deserving of particular notice, on account of it also being free from the sign of summation, and involving only the Bernouillian numbers. To find this wave we have to take ρ, the root of the prime factor of $\rho^2 - 1$, that is, we have simply $\rho = -1$.

And if we distinguish the elements $a_1, a_2 \ldots a_r$ into two groups, say $\alpha_1, \alpha_2 \ldots \alpha_l$, all even, and $\beta_1, \beta_2 \ldots \beta_m$ all odd, we have

$$W_2 = \text{coefficient of } \frac{1}{t} \text{ in the generator}$$

$$e^{nt}(-)^n \frac{1}{(1-e^{-\alpha_1 t})(1-e^{-\alpha_2 t})\ldots(1-e^{-\alpha_l t})}\frac{1}{(1+e^{-\beta_1 t})(1+e^{-\beta_2 t})\ldots(1+e^{-\beta_m t})},$$

which $\qquad = (-)^n e^{nt-R},$

where $\qquad R = \Sigma\log(1-e^{-\alpha_i t}) + \Sigma\log(1+e^{-\beta_i t}).$

But $\log(1-e^{-t})$ has been already expressed, and

$$\log(1+e^{-t}) = \log 2 - \frac{t}{2} + \frac{1}{8}t^2 + \&\text{c.}$$

$$= \log 2 - \frac{t}{2} + \frac{3}{4}B_1 t^2 - \frac{15}{16}\frac{B_2}{1.2.3}t^4 + \&\text{c.}$$

* To save circumlocution, I have not expressed in the text the general value of the coefficient of t_i, but of course there is not the slightest difficulty in so doing; let i be thrown in every possible way under the form $K_1 + 2K_2 + 4K_4 + 6K_6 + \&\text{c.}$ (that is to say, all the partitions of i quâ the elements 1, 2, 4, 6, &c., are to be written down), then the coefficient in question is

$$\Sigma\phi(K_1, K_2, K_4, \&\text{c.})\, \nu^{K_1} . s_2^{K_2} . s_4^{K_4} . s_6^{K_6} \,\&\text{c.},$$

where $\quad \phi(K_1, K_2, K_4, \&\text{c.}) = \frac{\pm 1}{\Pi K_1\, \Pi K_2\, \Pi K_4 \,\&\text{c.}}\left(\frac{B_1}{1.2^2}\right)^{K_2}\left(\frac{B_2}{1.2.3.4^2}\right)^{K_4}\left(\frac{B_3}{1.2.3.4.5.6^2}\right)^{K_6}\&\text{c.}$

It is deeply interesting to observe how, in the very formula for expressing partitions, a class of partitions reappears,—in fact, partitions constitute the sphere in which analysis lives, moves, and has its being; and no power of language can exaggerate or paint too forcibly the importance of this till recently almost neglected, but vast, subtle, and universally permeating element of algebraical thought and expression.

Happy ought I to feel in the reflection of having been the appointed instrument to make so great an advance in a doctrine which contains a large part of the future of pure analysis, and to have impressed upon it a form which must inevitably give rise to an illimitable host of the most important applications and consequences.

Hence, using $s_1, s_2 \ldots$ to denote the sums of the 1st, 2nd ... powers of $\alpha_1, \alpha_2 \ldots$ and $\sigma_1, \sigma_2 \ldots$ to denote the sums of the 1st, 2nd ... powers of $\beta_1, \beta_2 \ldots$ and writing $n + \frac{1}{2}(s_1 + \sigma_1) = \nu$, we have

$$nt - R = -\log(2^m \alpha_1 \alpha_2 \ldots \alpha_l) + l \log t + \nu t - \frac{B_1}{1 \,.\, 2^2}(s_2 + 3\sigma_2)\, t^2$$

$$+ \frac{B_2}{1 \,.\, 2 \,.\, 3 \,.\, 4^2}(s_4 + 15\sigma_4)\, t^4 + \&c.$$

and consequently, $W_2 = \dfrac{1}{2^r \left(\frac{\alpha_1}{2} \cdot \frac{\alpha_2}{2} \cdots \frac{\alpha_l}{2}\right)} \times$ coefficient of t^{l-1} in

$$\left(1 + \nu t + \nu^2 \frac{t^2}{1 \,.\, 2} + \nu^3 \frac{t^3}{1 \,.\, 2 \,.\, 3} + \&c.\right)$$

$$\times \left(1 - \frac{1}{24}(s_2 + 3\sigma_2)\, t^2 + \frac{1}{1152}(s_2 + 3\sigma_2)^2\, t^4 + \&c.\right)$$

$$\times \left(1 + \frac{1}{2880} s_4 t^4 + \&c.\right)$$

$$\times \&c.$$

So in general if we wish to find the wave W_q, we must distinguish the elements into two groups,

$\alpha_1, \alpha_2 \ldots \alpha_l$, all exactly divisible by q,

and $\beta_1, \beta_2 \ldots \beta_m$ not so divisible;

W_q will then be the coefficient of $\dfrac{1}{t}$ in

$$\Sigma \frac{\rho^n e^{nt}}{(1 - \rho^{\alpha_1} e^{-\alpha_1 t})(1 - \rho^{\alpha_2} e^{-\alpha_2 t}) \ldots (1 - \rho^{\alpha_l} e^{-\alpha_l t}) \times (1 - \rho^{\beta_1} e^{-\beta_1 t})(1 - \rho^{\beta_2} e^{-\beta_2 t}) \ldots (1 - \rho^{\beta_m} e^{-\beta_m t})}$$

where the sign of summation indicates that all the values are to be taken in succession of the prime roots of $\rho^q - 1 = 0$; this, again, may be expressed as the coefficient of t^{l-1} in a quantity of the form $\rho^{nt - R}$ where R may be expressed by means of the prime qth roots of unity and the known numerical coefficients which enter into the expansion in ascending powers of t of the quantity $\dfrac{1}{1 - c\rho^{-t}}$; but I do not propose here to enter into the details of the method. It will be enough for present purposes to illustrate it by an example. Suppose, then, that we take the elements 1, 2, 3, 4, 5, 6; in other words, that we propose to express algebraically the number of ways in which n can be divided into six or a smaller number of parts.

The expression will here consist of six parts, which I shall reckon in inverse order, beginning with W_6.

W_6 will be the coefficient of $\dfrac{1}{t}$ in

$$\Sigma \frac{e^{nt}\rho^n}{(1 - e^{-6t}) \times (1 - \rho e^{-t})(1 - \rho^2 e^{-2t})(1 - \rho^3 e^{-3t})(1 - \rho^4 e^{-4t})(1 - \rho^5 e^{-5t})},$$

where ρ is either root of $\rho^2 - \rho + 1 = 0$;

this is evidently equal to

$$\frac{1}{6} \times \Sigma \rho^n \frac{1}{(1 - \rho)(1 - \rho^2)(1 - \rho^3)(1 - \rho^4)(1 - \rho^5)}$$

$$= \Sigma \frac{1}{6} \rho^n \frac{1}{(1 - \rho)(1 - \rho^2)\, 2\, (1 + \rho)(1 + \rho^2)}$$

$$= \Sigma \frac{\rho^n}{12} \frac{1}{(1 - \rho^2)(1 - \rho^4)}$$

$$= \Sigma \frac{\rho^n}{12} \frac{1}{(2 - \rho)(1 + \rho)} = \Sigma \frac{\rho^n}{36}.$$

In like manner,

$$W_5 = \frac{1}{5} \cdot \Sigma \frac{\rho^n}{(1 - \rho)(1 - \rho^2)(1 - \rho^3)(1 - \rho^4)(1 - \rho)},$$

where ρ is any root of

$$\rho^4 + \rho^3 + \rho^2 + \rho + 1 = 0.$$

Hence $$W_5 = \frac{1}{25} \Sigma \frac{\rho^n}{1 - \rho}$$

$$= \frac{1}{125} \Sigma \rho^n (4 + 3\rho + 2\rho^2 + \rho^3)$$

$$= \frac{1}{125} \Sigma \rho^n (2 + \rho - \rho^3 - 2\rho^4).$$

Again,

$$W_4 = \frac{1}{4} \Sigma \frac{\rho^n}{(1 - \rho)(1 - \rho^2)(1 - \rho^3)(1 - \rho)(1 - \rho^2)},$$

where ρ is either root of $\rho^2 + 1 = 0$.

Hence $$W_4 = \frac{1}{16} \Sigma \frac{\rho^n}{(1 - \rho)^2 (1 + \rho)}$$

$$= \frac{1}{64} \Sigma \frac{\rho^n (1 - \rho)}{-\rho}$$

$$= \frac{1}{64} \Sigma (\rho^n - \rho^{n-1}).$$

Again, $W_3 =$ coefficient of $\dfrac{1}{t}$ in

$$\Sigma \frac{\rho^n e^{nt}}{(1 - e^{-3t})(1 - e^{-6t}) \times \&c.},$$

where $$\rho^2+\rho+1=0,$$

$$=\frac{1}{18}\Sigma\frac{\rho^n\nu}{(1-\rho)(1-\rho^2)(1-\rho^4)(1-\rho^5)}=\frac{1}{18}\frac{\Sigma\rho^n\nu}{\{(1-\rho)(1-\rho^2)\}^2}$$

$$=\frac{\Sigma\rho^n\nu}{162},$$

where $$\nu=n+\frac{3}{2}+\frac{6}{2}+\frac{\rho}{\rho-1}+\frac{2\rho^2}{\rho^2-1}+\frac{4\rho^4}{\rho^4-1}+\frac{5\rho^5}{\rho^5-1}$$

$$=n+\frac{9}{2}+\frac{\rho}{\rho-1}+\frac{2\rho^2}{\rho^2-1}+\frac{4\rho}{\rho-1}+\frac{5\rho^2}{\rho^2-1}$$

$$=n+\frac{9}{2}-\frac{1}{3}\{5(2+\rho)+7(2+\rho^2)\}$$

$$=n+\frac{9}{2}-\frac{1}{3}(24+5\rho+7\rho^2)$$

$$=n-\frac{1}{6}(21+5\rho+7\rho^2)=n-\frac{1}{3}(7-\rho).$$

$$W_2=\frac{(-)^n}{2^6(1.2.3)}\times\text{coefficient of } t^2 \text{ in } \left(1+\nu t+\frac{\nu^2t^2}{2}\right)\left(1-\frac{1}{24}(s_2+3\sigma_2)t^2\right),$$

where $$\nu=n+\frac{1}{2}(1+2+3+4+5+6)$$

$$=n+\frac{21}{2},$$

$$s_2=2^2+4^2+6^2=56;\quad \sigma_2=1^2+3^2+5^2=35;\quad 3\sigma_2=105.$$

Hence $$W_2=(-)^n\frac{1}{384}\times\left(\frac{\nu^2}{2}-\frac{161}{24}\right)$$

$$=(-)^n\left(\frac{\nu^2}{768}-\frac{161}{9216}\right).$$

Finally $$W_1=\frac{1}{1.2.3.4.5.6}\times\text{coefficient of } t^5 \text{ in}$$

$$\left(1+\nu t+\nu^2\frac{t^2}{2}+\nu^3\frac{t^3}{6}+\nu^4\frac{t^4}{24}+\nu^5\frac{t^5}{120}\right)$$

$$\times\left(1-\frac{1}{24}s_2t^2+\frac{1}{1152}s_2^2t^4\right)$$

$$\times\left(1+\frac{1}{2880}s_4t^4\right)$$

$$=\frac{1}{720}\left\{\frac{\nu^5}{120}-\frac{\nu^3}{144}s_2+\nu\left(\frac{s_2^2}{1152}+\frac{s_4}{2880}\right)\right\},$$

where $$\nu=n+\frac{21}{2}$$

$$s_2=1^2+2^2+3^2+4^2+5^2+6^2=91$$

$$s_4=1+2^4+3^4+4^4+5^4+6^4=2275$$

$$s_2^2=\qquad 8281.$$

Hence $$\frac{s_2^2}{1152}+\frac{s_4}{2880}=\frac{1}{1152}(8281+910)$$

$$=\frac{1}{1152}(9191),$$

and $$W_1=\frac{\nu}{720}\left(\frac{\nu^4}{120}-\frac{91\nu^2}{144}+\frac{9191}{1152}\right).$$

As no useful object would be attained by substituting for ν its value $n+\frac{21}{2}$, I leave the expressions for W_1, W_2 in their present form as explicit functions of ν.

It is well worthy of observation that the exponent of the generating function of W_1, namely, $nt-R$, when the elements are taken the consecutive numbers 1, 2, 3 ... r, consists exclusively of Bernouillian numbers and sums of powers of 1, 2, 3 ... r; but as these latter are themselves expressible by Euler's theorem, in terms of powers of r and the Bernouillians, R is for this case a quadratic function of the numbers of Bernouilli.

If we express the quantities of the form $\Sigma\rho^{w}$, which occur in the different modes in terms of Herschel's circulating functions, then our expression assumes the very same form in which Mr Cayley had observed it was the most advantageous to express the quotity, and to which he has given the name of "prime radical circulators*."

* Thus what with Mr Cayley was an invention, with me becomes a theorem. Mr Cayley was led to the use of prime circulators from a perception of their affording the best analytical means of giving determinateness to the representation of the results; in my method they offer themselves spontaneously, and cannot be rejected.

Supposing that Mr Cayley could claim a right to the exclusive use of these forms, we should have an instructive instance of one of the mischiefs ascribed to the general system of patent law, namely, of blocking up the necessary march of invention. For the benefit of foreign readers of the *Journal*, I should add that r_n is used by Herschel to denote a quantity which is unity when n contains r as a factor, and is otherwise zero; and that any function of $r_n, r_{n-1}, r_{n-2}\dots r_{n-r+1}$ is called a circulating function, and may, of course, be expressed as a linear function of the above quantities.

Suppose ρ to be any factor whatever of r, i to be less than ρ, and $r=\rho\sigma$. Then if for all admissible values of ρ and i

$$A_{n-i}+A_{n-i-\rho}+A_{n-i-2\rho}+\&c.+A_{n-i-(\sigma-1)\rho}=0,$$

$A_n.r_n+A_{n-1}.r_{n-1}+\&c.+A_{n-r+1}$ (where A_n, A_{n-1} ... are ordinary constants) is, according to Cayley, a prime radical circulator (prime circulator would be quite as specific and more convenient).

I shall conclude this very brief notice of my theory by converting the W waves in the example above treated into the form of these prime circulators.

For W_6 ρ is any root of $\rho^2 - \rho + 1 = 0$.

Hence

$$\Sigma\rho^0 = 2;\ \Sigma\rho = 1;\ \Sigma\rho^2 = -1;\ \Sigma\rho^3 = -2;\ \Sigma\rho^4 = -1;\ \Sigma\rho^5 = 1.$$

Hence in the notation of Herschel

$$W_6 = \frac{6_n}{18} + \frac{6_{n-1}}{36} - \frac{6_{n-2}}{36} - \frac{6_{n-3}}{18} - \frac{6_{n-4}}{36} + \frac{6_{n-5}}{36}.$$

For W_5 ρ is any root of $\rho^4 + \rho^3 + \rho^2 + \rho + 1 = 0$.

Hence

$$\Sigma\rho^0 = 4;\ \Sigma\rho = -1;\ \Sigma\rho^2 = -1;\ \Sigma\rho^3 = -1;\ \Sigma\rho^4 = -1.$$

Hence

$$W_5 = \frac{1}{125}\{(8 - 1 + 1 + 2)\,5_n + (-2 - 1 + 1 - 8)\,5_{n-1}$$
$$+ (-2 - 1 - 4 + 2)\,5_{n-2} + (-2 - 1 + 1 + 2)\,5_{n-3}$$
$$+ (-2 + 4 + 1 + 2)\,5_{n-4}\}$$
$$= \frac{2}{25}\,5_n - \frac{2}{25}\,5_{n-1} - \frac{1}{25}\,5_{n-2} + \frac{1}{25}\,5_{n-3}.$$

In W_4 ρ is either root of $\rho^2 + 1 = 0$.

Hence $\Sigma\rho^0 = 2;\ \Sigma\rho = 0;\ \Sigma\rho^2 = -2;\ \Sigma\rho^3 = 0$,

and hence

$$W_4 = \frac{1}{16}(2\,.\,4_n + 2\,.\,4_{n-1} - 2\,.\,4_{n-2} - 2\,.\,4_{n-3})$$
$$= \frac{4_n}{8} + \frac{4_{n-1}}{8} - \frac{4_{n-2}}{8} - \frac{4_{n-3}}{8}.$$

In W_3 $\rho^2 + \rho + 1 = 0$.

Hence $\Sigma\rho^0 = 2;\ \Sigma\rho = -1;\ \Sigma\rho^2 = -1$.

Hence

$$W_3 = \left(\frac{1}{81}\,3_n - \frac{1}{162}\,3_{n-1} - \frac{1}{162}\,3_{n-2}\right)\left(n - \frac{7}{3}\right)$$
$$- \frac{1}{486}\,3_n - \frac{1}{486}\,3_{n-1} + \frac{1}{243}\,3_{n-2}$$
$$= \left(\frac{1}{81}\,3_n - \frac{1}{162}\,3_{n-1} - \frac{1}{162}\,3_{n-2}\right) n$$
$$- \left(\frac{5}{162}\,3_n - \frac{1}{81}\,3_{n-1} - \frac{3}{162}\,3_{n-2}\right).$$

Finally

$$W_2 = \left(\frac{\nu^2}{768} - \frac{161}{9216}\right)(2_n - 2_{n-1}).$$

W_1 has already been expressed in its simplest terms; and the solution of the question of the partition of n into six parts is now complete.

The same causes which have interposed to prevent my setting forth at length the method above sketched out, have also interposed to preclude me from extending the exposition which I had intended of the application of the principles contained in my paper on Differential Transformations to the general question of the solution of equations, or systems of equations, containing any number of variables, thereby entirely superseding the necessity of all special considerations whatever in obtaining Lagrange's and Laplace's Theorems, either in their developed or ordinary form; and theorems to many degrees of infinity more general than these; for their method being, by the aid of this most desiderated discovery, now capable of being substituted for artifice, which, in general, may be said to stand in the same relation to method as what instinct is to reason, or the craft of the savage to the wisdom of the civilised man.

19.

NOTE ON A FORMAL PROPERTY OF A LATENT INTEGER.

[*Quarterly Journal of Mathematics*, I. (1857), p. 185.]

THE following was proposed some years ago by an author, whose name I do not recollect, among the mathematical questions in the *Educational Times**.

"Required to prove that the integer part of $(1+\sqrt{3})^{2m+1}$ contains 2^{m+1} as a factor."

The proof probably ran, or at all events might have run, as follows:

$(1-\sqrt{3})^{2m+1}$ being a negative fraction less than unity, the integer part of $(1+\sqrt{3})^{2m+1}$ is evidently

$$(1+\sqrt{3})^{2m+1}+(1-\sqrt{3})^{2m+1},$$

or is the sum of the $(2m+1)$th powers of the roots of the equation

$$x^2-2x-2=0,$$

from which the truth of the proposition is manifest.

We may add the remark that it may easily be shown in like manner that the integer next *above* the fraction $(1+\sqrt{3})^{2m}$, will also contain 2^{m+1} as a factor; and more generally, if we suppose that a is that integer congruent *quâ* the modulus 2 with n, which is next above or next below $\sqrt{n}$, then in the former case the two integers next above $(\sqrt{n}+a)^{2m+1}$ and $(\sqrt{n}+a)^{2m}$ respectively, and in the latter case the two integers next below the first and next above the second respectively, will each of them contain the factor 2^{m+1}.

The student is invited to ascertain whether any analogous theorem exists for latent integers expressed by means of higher surd forms.

* Questions of a similar nature, I am informed by Mr Ferrers, appeared in the Cambridge Senate House problems for the years 1847 and 1848.

20.

NOTE ON A PRINCIPLE IN THE THEORY OF NUMBERS AND THE RESOLUBILITY OF ANY NUMBER INTO THE SUM OF FOUR SQUARES.

[*Quarterly Journal of Mathematics*, I. (1857), pp. 196, 197.]

PROBABLY no one who has had any experience in the properties of numbers, could be found seriously doubting the truth of the proposition that any function of a variable integer, not algebraically decomposable into factors, must, among the infinite number of positive integers which it represents, be capable of affording prime as well as composite numbers, except in the case of its being of such a form as to admit of a constant divisor for all values of the variable. To prove this generally is probably a task reserved for remote generations, and for a more advanced development of the cerebral organization, but it is to my mind and conviction, and probably to that of most others, no less certain than the equally undemonstrable theorem which lies at the basis of the ordinary empirical geometry, that two parallels to the same line cannot be drawn through the same point. It would certainly be interesting to be able to deduce a connected body of doctrine as consequences flowing from the assumption of this principle, nor could the rigour of mathematical demonstration be in any degree prejudiced by its use, provided that every such consequence were stated only as a contingent truth, until either the principle itself had been as far as necessary apodictically established, or some other mode of demonstration substituted in its place. If this plan were followed out, it is not unlikely that the path would ultimately be discovered leading back to the demonstration of the fundamental principle, and, in the meanwhile, the *à priori* probability of its truth (if supposed to be inferior to moral certainty) would be confirmed by each additional experience of the correctness of the results to which it might be found to conduct.

Under this point of view it may not be uninteresting to show how the principle in question affords an almost instantaneous demonstration of the celebrated theorem of the resolubility of every integer into the sum of four squares.

Lemma. If M be any integer, and $3M = p^2 + q^2 + r^2 + s^2$, M may be expressed under the form $p'^2 + q'^2 + r'^2 + s'^2$.

For it may be observed, that of the four quantities p, q, r, s, either all are divisible by 3, or else one will be so divisible, and each of the others not; in either case, let p be divisible by 3, and give to the absolute values of $\sqrt{q^2}$, $\sqrt{r^2}$, $\sqrt{s^2}$ respectively such signs (if they do not all contain 3) as will make them congruent to one another *quâ* the modulus 3, then

$$M = \frac{p^2 + q^2 + r^2 + s^2}{3} = p'^2 + q'^2 + r'^2 + s'^2,$$

where
$$p' = \frac{\sqrt{q^2} + \sqrt{r^2} + \sqrt{s^2}}{3},$$

$$q' = \frac{\sqrt{r^2} - \sqrt{s^2} + p}{3},$$

$$r' = \frac{\sqrt{s^2} - \sqrt{q^2} + p}{3},$$

$$s' = \frac{\sqrt{q^2} - \sqrt{r^2} + p}{3};$$

consequently p', q', r', s' in either case, are all of them integers.

Suppose N to be odd, and of the form $4\mu + 1$; take the expression $3^{2x+1}N - 2$, and let T be one of the primes which, by virtue of our *principle* (since obviously in this case there is no constant factor), we assume it must contain. Then T is a prime number of the form $4\mu + 1$, that is, the sum of two squares, and consequently $T + 2$, that is, $3^{2x+1}N$ is the sum of four squares, whence, by the Lemma, it follows that N is the same.

In like manner if N is of the form $4\mu + 3$, we take T, one of the primes contained in the form $3^{2x}N - 2$, and as before $T + 2$, that is, $3^{2x}N$, and consequently N will be the sum of four squares.

If N be even, we may obviously consider it to be of the form $4\mu + 2$ (since the theorem, if true for N, will be so for $4N$), and then if T be a prime contained in the form $3^x N - 1$, $T + 1$ will be the sum of three squares, or which is the same thing of four squares, of which zero is one, and the reasoning is the same as before. Hence, in all cases, N is the sum of four squares; and the same result might be obtained with equal or greater facility by the application of the *principle* to various other forms.

21.

DEVELOPMENT OF AN IDEA OF EISENSTEIN.

[*Quarterly Journal of Mathematics*, I. (1857), pp. 199—203.]

EISENSTEIN has remarked, in a note among his collected works, that the expansion of any negative power of a series of ascending powers of x may be made to depend upon the expansions of the positive powers of the same series. The following method which reposes upon the most elementary principles of algebra serves to establish this practically important proposition.

Let
$$u = 1 + A_1 x + A_2 x^2 + \&c.$$

Then

$$\frac{1}{u^i} = \frac{1}{\{1 - (1 - u)\}^i} = 1 + i(1 - u) + \frac{i(i+1)}{2}(1 - u)^2 + \frac{i(i+1)(i+2)}{2 \,.\, 3}(1 - u)^3 + \&c.$$

If now we wish to express the nth power of x or, in fact, any power of x lower than the nth by means of this series, it is obvious that we may stop at the term containing the nth power of $1 - u$. In general, then, denoting by $C_{i,\nu}$ the coefficient of x^ν in u^i, provided ν is greater than unity and not greater than n, we have

$$\begin{aligned} C_{-i,\nu} = &- i\left(1 + (i+1) + \frac{(i+1)(i+2)}{2} + \&c. + \frac{(i+1)(i+2)\ldots(i+n-1)}{1 \,.\, 2 \ldots (n-1)}\right) C_{1,\nu} \\ &+ i \,.\, \frac{i+1}{2}\left(1 + (i+2) + \&c. + \frac{(i+2)(i+3)\ldots(i+n-1)}{1 \,.\, 2 \ldots (n-2)}\right) C_{2,\nu} \\ &\quad \&c., \&c., \&c. \\ &\pm \frac{i(i+1)\ldots(i+n-1)}{1 \,.\, 2 \ldots n} C_{n,\nu} \\ = &- i \,.\, \frac{(i+2)(i+3)\ldots(i+n)}{1 \,.\, 2 \ldots (n-1)} C_{1,\nu} + i\,\frac{i+1}{2} \,.\, \frac{(i+3)\ldots(i+n)}{1 \,.\, 2 \ldots (n-2)} C_{2,\nu} \\ &\mp \&c. \pm \frac{i(i+1)\ldots(i+n-1)}{1 \,.\, 2 \ldots n} C_{n,\nu}. \end{aligned}$$

In practice it will, of course, be always most expedient (on the score of brevity of expression) to assume $n=\nu$.

If $$u = 1 + A_\omega x^\omega + A_{\omega+1}x^{\omega+1} + \&c.$$

the condition for the truth of the above equation will be that ν shall not exceed ωn; and then in practice it will be expedient to take n equal to $\frac{\nu}{\omega}$, if that be an integer, or, if not, to take n, the integer next above $\frac{\nu}{\omega}$.

We may with propriety denote by $C_{0,n}$ the coefficient of x^n in $\log u$*; and since

$$\log u = -\{(1-u) + \tfrac{1}{2}(1-u)^2 + \tfrac{1}{3}(1-u)^3 + \&c.\}$$

we shall have, subject to the same conditions as before,

$$\begin{aligned} C_{0,\nu} &= (1+1+\&c.\text{ to } n \text{ terms})\, C_{1,\nu} \\ &\quad - \tfrac{1}{2}(1+2+\&c. + (n-1))\, C_{2,\nu} \\ &\quad + \tfrac{1}{3}\left(1+3+6+\&c. + \frac{(n-2)(n-1)}{2}\right) C_{3,\nu} \\ &\quad \&c., \&c. \\ &= nC_{1,\nu} - \frac{(n-1)n}{2^2}C_{2,\nu} + \frac{(n-2)(n-1)n}{2\,.\,3^2}C_{3,\nu}\ \&c. + (-)^{n-1}\frac{C_{n,\nu}}{n}. \end{aligned}$$

In the general theorem suppose $i=1$, $\nu=n$. Then

$$C_{-1,n} = -\frac{(n+1)n}{1\,.\,2}C_{1,n} + \frac{(n+1)n(n-1)}{1\,.\,2\,.\,3}C_{2,n}\ \&c. \pm C_{n,n}.$$

Thus, to take the example alluded to by Eisenstein, suppose

$$u = \frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + B_1\frac{x^2}{2\,!} - B_2\frac{x^4}{4\,!} + B_3\frac{x^6}{6\,!}\ \&c., \&c.,$$

so that by the formula

$$\frac{(-)^{n+1}B_n}{(2n)\,!} = -\frac{(2n+1)\,2n}{1\,.\,2}C_{1,2n} + \frac{(2n+1)(2n)(2n-1)}{1\,.\,2\,.\,3}C_{2,2n}\ \&c. + C_{2n,2n}.$$

Here

$$\begin{aligned} C_{u,2n} &= \text{coefficient of } x^{2n} \text{ in } \left(\frac{1-e^{-x}}{x}\right)^u \\ &= \text{coefficient of } x^{2n+u} \text{ in } 1 - ue^{-x} + u\frac{u-1}{2}e^{-2x}\ \&c. + (-)^u e^{-ux} \\ &= \frac{(-)^u}{(2n+u)\,!}\left\{-u + u\frac{u-1}{2}2^{2n+u} \mp \&c. + (-)^u u^{2n+u}\right\} \\ &= \frac{\Delta^u 0^{2n+u}}{(2n+u)\,!}. \end{aligned}$$

* The rule for any power, positive or negative, of $\log u$ deserves investigation; the case of $C_{0,n}$ (using that symbol in a more extended sense than in the text above) containing, as it were, a microcosmical reiteration of the whole theory under discussion.

Hence

$$\begin{aligned} (-)^{n+1}B_n &= (2n\,!)\left\{-\frac{(2n+1)\,2n}{2\,!}\frac{\Delta 0^{2n+1}}{(2n+1)\,!}\right. \\ &\qquad \left. + \frac{(2n+1)(2n)(2n-1)}{3\,!}\frac{\Delta^2 0^{2n+2}}{(2n+2)\,!} + \&c.\right\} \\ &= 2n\left\{\frac{-\Delta 0^{2n+1}}{2\,!} + \frac{2n-1}{2n+2}\frac{\Delta^2 0^{2n+2}}{3\,!} - \frac{(2n-1)(2n-2)}{(2n+2)(2n+3)}\frac{\Delta^3 0^{2n+3}}{4\,!} + \&c.\right. \\ &\qquad \left. + \frac{(2n-1)(2n-2)\dots 1}{(2n+2)(2n+3)\dots 4n}\frac{\Delta^{2n}0^{4n}}{(2n+1)\,!}\right\}. \end{aligned}$$

Thus, if $n=1$,

$$\begin{aligned} B_1 &= 2\left\{-\frac{\Delta 0^3}{2} + \frac{1}{4}\frac{\Delta^2 0^4}{6}\right\} \\ &= 2\left\{-\frac{1}{2} + \frac{1}{24}(2^4 - 2\,.\,1^4)\right\} \\ &= 2\left\{-\frac{1}{2} + \frac{14}{24}\right\} = \frac{1}{6}. \end{aligned}$$

If $n=2$,

$$\begin{aligned} -B_2 &= 4\left\{\frac{-\Delta 0^5}{2} + \frac{1}{2}\frac{\Delta^2 0^6}{6} - \frac{1}{7}\frac{\Delta^3 0^7}{24} + \frac{1}{56}\frac{\Delta^4 0}{120}\right\} \\ &= 4\left\{-\frac{1}{2} + \frac{1}{12}(2^6 - 2) - \frac{1}{168}(3^7 - 3\,.\,2^7 + 3)\right. \\ &\qquad \left. + \frac{1}{6720}(4^8 - 4\,.\,3^8 + 6\,.\,2^8 - 4)\right\} \\ &= 4\left\{-\frac{1}{2} + \frac{31}{6} - \frac{43}{4} + \frac{243}{40}\right\} \\ &= 4\left\{-\frac{1}{2} + \frac{1}{6} + \frac{1}{4} + \frac{3}{40}\right\} \\ &= \frac{1}{30}\{-60 + 20 + 30 + 9\} \\ &= -\frac{1}{30}, \text{ or } B_2 = \frac{1}{30}, \text{ and so on.} \end{aligned}$$

The annexed independent demonstration of the formula for $\frac{1}{fu}$ appertains to Mr Cayley.

Write $$x = u + hfx,$$

then, by Lagrange's theorem,

$$Fx = Fu + \frac{h}{1}F'u\,fu + \frac{h^2}{1\,.\,2}\frac{d}{du}F'u\,(fu)^2 + \frac{h^3}{1\,.\,2\,.\,3}\left(\frac{d}{du}\right)^2 F'u\,(fu)^3 + \dots,$$

or, differentiating with respect to u,

$$\frac{F'x}{1-hf'x}=F'u+\frac{h}{1}\frac{d}{du}F'ufu+\frac{h^2}{1\,.\,2}\left(\frac{d}{du}\right)^2F'u\,(fu)^2+\ldots;$$

whence, putting $h=\dfrac{x-u}{fx}$,

$$\frac{F'x}{1-(x-u)\dfrac{f'x}{fx}}=F'u+\frac{x-u}{fx}\frac{d}{du}F'ufu+\frac{1}{1\,.\,2}\left(\frac{x-u}{fx}\right)^2\frac{d^2}{du^2}F'u\,(fu)^2+\ldots,$$

which is true identically.

Suppose now $F'u=\dfrac{u}{fu}$, we have

$$\frac{x}{fx-(x-u)f'x}=\frac{u}{fu}+\frac{x-u}{fx}\frac{d}{du}u+\frac{1}{1\,.\,2}\left(\frac{x-u}{fx}\right)^2\frac{d^2}{du^2}ufu+\&\text{c.},$$

or, if $\qquad fu=1+bu+cu^2+du^3+\&\text{c.},\quad fx=1+bx+cx^2+\&\text{c.},$

and $x=0$, that is, $fx=1$, the formula becomes

$$0=\frac{u}{fu}-u+\frac{u^2}{1\,.\,2}\left(\frac{d}{du}\right)^2ufu-\frac{u^3}{1\,.\,2\,.\,3}\left(\frac{d}{du}\right)^3u\,(fu)^2+\ldots.$$

Whence

$$\frac{1}{fu}=1-\frac{u}{1\,.\,2}\left(\frac{d}{du}\right)^2ufu+\frac{u^2}{1\,.\,2\,.\,3}\left(\frac{d}{du}\right)^3u\,(fu)^2$$
$$-\frac{u^3}{1\,.\,2\,.\,3\,.\,4}\left(\frac{d}{du}\right)^4u\,(fu)^3+\&\text{c.},$$

which gives the expansion of $\dfrac{1}{fu}$ when the expansions of the positive powers $(fu)^2$, $(fu)^3$, &c. are known.

22.

NOTE ON THE ALGEBRAICAL THEORY OF DERIVATIVE POINTS OF CURVES OF THE THIRD DEGREE.

[*Philosophical Magazine*, XVI. (1858), pp. 116—119.]

Two years and upwards have elapsed since I discovered the extraordinary theorem in the doctrine of cubic forms which I am about to state, but which has never yet been published by me, although communicated in confidence to a few friends, including Mr Cayley. It arose out of purely arithmetical speculations relating to such forms, to some of which I may make a brief allusion in the course of this note.

If we suppose the general homogeneous equation of the third degree in x, y, z reduced to the canonical form

$$x^3+y^3+z^3+mxyz=0,$$

any solution $x=a$, $y=b$, $z=c$ of this equation is of course one of a group of six obtained by the permutations of the three letters a, b, c, and having an obvious relation to one another through the medium of the points of inflexion. So, too, it is manifest if we take the equation to the curve in its most general form, from any given solution, a group of six, including the given one, may be formed, *the characteristics* of each of which will be linear functions of one another. For the purpose of the theorem about to be enunciated, such a group of solutions will be treated as a single solution; and then we can affirm the proposition following, in which a solvent system means a system of values of the variables x, y, z satisfying the equation $f(x, y, z)=0$, and free from any common factor.

Let a, b, c *be any solvent system to a cubic homogeneous equation in* x, y, z; *then from* a, b, c *we may derive a new solvent system,* a′, b′, c′, *where* a′, b′, c′ *are each of them functions of the fourth degree of* a, b, c, *and another system* a″, b″, c″ *of the ninth degree in* a, b, c, *and another* a‴, b‴, c‴ *of the sixteenth* *degree, and so in general a new solvent system of the degree* n^2 *in* a, b, c. *One such derivative system, and only one, of the degree* n^2 *can be formed, and none of any intermediate degree.*

Thus, for instance, the coordinates of the tangential (the name adopted from me by Mr Cayley to express the point of intersection of a tangent to a cubic curve at any point with the curve) being called a', b', c', these last letters are *biquadratic* functions of a, b, c*.

So again, as I also suggested to Mr Cayley, the point in which the conic of closest contact with a cubic curve cuts the curve will necessarily have a derivative system of coordinates of a square-numbered degree in respect of the original ones, which by actual trial Mr Cayley has found to be the 25th. Mr Salmon, I believe, has obtained in certain geometrical investigations derivatives of the 49th degree.

I am in possession of the equations by means of which the successive systems of the fourth, ninth, &c. degrees, which I incline to call the first or primary, the second, third, &c. derivative systems, may be formed explicitly by successive derivation from one another; so that, for instance, as soon as I am informed that the system investigated by Mr Cayley is of the twenty-fifth degree or fifth order, I can find them without any reference to the geometry of the question, the quantities belonging to the nth derivative being in fact a known algebraical function of n! I was led to the discovery of this surprising and unique law by a statement of a friend, *not since verified, and which, for aught that has yet been shown, may or may not be true*, that the number 5 *could be* divided into two rational cubes: assuming this to be the fact, it necessitated (by virtue of my investigations) the coincidence to a factor *près* of two functions obtained by apparently independent algebraical processes, which coincidence by actual comparison of the functions I found to obtain.

With reference to the connexion of this theory of derivation with the arithmetic of equations of the third degree between three variables with integer coefficients, it is after this kind. Fermat has taught us that a certain class of such equations, viz. the equation $x^3 + y^3 + z^3 = 0$, is absolutely insoluble in integers (abstraction made of the trivial solutions of the type $x = 0$, $y + z = 0$). I have greatly multiplied the classes of such known insoluble equations, as may be seen by a communication from me to Tortolini's *Annali* in 1856 [p. 63 above]. But over and above such equations I have ascertained the existence of a large class of equations, soluble, or possibly so, it is true, but enjoying the property that all their solutions in integers, when they exist, are *monobasic*; that is to say, all their solutions are known functions of one

* This derivative solution (though not as corresponding to the *tangential*) was known also to Euler for a particular case, as will be seen by reference to his *Algebra*.

of them, which I term the *base*, and which is characterized by this property—that of all the solutions possible it is the one for which the *greatest* of the three variables is the *smallest* number possible. If this solution be laid down as a point in the curve corresponding to the given cubic, all the other solutions possible in integers will be represented by points in this curve, which are derivatives (in the sense previously employed in this note) to the given point, having coordinates respectively of the 4th, 9th, 16th, &c. degrees, in respect of the coordinates of the *basic* point*.

If my memory serves me truly, I have found (as a particular case) that all cubic equations in numbers of the form

$$x^3 + y^3 + z^3 = imxyz,$$

where i is 1 or 3 or 6 (I cannot at the moment remember which), are either *insoluble* or *monobasic*. The case of $im = 3$ must of course be exceptional, being satisfied by $x + y + z = 0$. This doctrine of derivation evidently conducts to a new branch of the grand doctrine of invariance. I hope to have tranquillity of mind ere long to give to the world my memoir, or a fragment of it, "On an Arithmetical Theory of Homogeneous and the Cubic Forms," the germ of which, now, alas! many weary years ago, first dawned upon my mind on the summit of the Righi, during a vacation ramble.

* This theorem is analogous to that relating to the integer solutions of $x^2 - Ay^2 = 1$, in so far as there is a *basic* solution to this equation in integers of which all the other solutions are derivatives, and not more than one such derivative exists of any given degree, but with the difference that there does exist one of every degree, and not merely (as in my theorem for cubic forms) of every square degree.

23.

NOTE ON THE EQUATION IN NUMBERS OF THE FIRST DEGREE BETWEEN ANY NUMBER OF VARIABLES WITH POSITIVE COEFFICIENTS.

[*Philosophical Magazine*, XVI. (1858), 369—371.]

I PROPOSE to show that all the systems of values $(x, y, z \dots w)$ which satisfy a given equation in integers,

$$ax + by + cz + \dots + lw = n,$$

($(a, b, c \dots l)$ being all positive, and the number of systems therefore definite), may be made to depend on algebraical equations whose coefficients are known functions of $a, b, c \dots l$ and n. The fact is somewhat surprising, the proof easy, being an immediate consequence of the theorem I have given* in the *Quarterly Journal of Mathematics*, and also in Tortolini's *Annali* for January 1857, of the problem of the partition of numbers.

For my present purpose, this theorem may be with advantage presented under a somewhat modified form as follows:—Let $\Theta(Ft)$ be used to denote the coefficient of $\frac{1}{t}$ in the expansion of Ft in ascending powers of t. Let N stand for the number of solutions of the equation

$$ax + by + cz + \dots + lw = n;$$

let m be the least common multiple of $a, b, c, \dots l$,

ρ be any *primitive* root of $\rho^m = 1$,

and ρe^{-pt} be called Λp; then

$$N = \Sigma\Theta \left\{\frac{\Lambda(-n)}{(1-\Lambda a)(1-\Lambda b)\dots(1-\Lambda l)}\right\}.$$

If now we call N' what N becomes when, in lieu of the equation

$$ax + by + cz + \dots + lw = n, \qquad (1)$$

we write

$$ax' + ax'' + by + cz + \dots + lw = n, \qquad (2)$$

[* p. 90 above.]

it is clear that

$$N' = \Sigma\Theta \left\{\frac{\Lambda(-n)}{(1-\Lambda a)^2(1-\Lambda b)\dots(1-\Lambda l)}\right\}.$$

But it is also clear that all the solutions of equation (2) may be derived from those of equation (1), by writing for each value of x

$$x' + x'' = x; \qquad (3)$$

and as the number of solutions of equation (3) is evidently $x+1$, we have $N' = \Sigma x + N$, or

$$\Sigma x = \Sigma\Theta \left\{\frac{\Lambda a \,.\, \Lambda(-n)}{(1-\Lambda a)^2(1-\Lambda b)\dots(1-\Lambda l)}\right\}.$$

In like manner, if we write

$$ax' + ax'' + ax''' + by + cz + \dots + lw = n,$$

the solutions of this equation spring from those of equation (1) by making $x' + x'' + x''' = x$, the number of solutions of which equality is $\frac{1}{2}(x+1)(x+2)$: wherefore

$$\Sigma \frac{x^2 + 3x + 2}{2} = \Sigma\Theta \left\{\frac{\Lambda(-n)}{(1-\Lambda a)^3(1-\Lambda b)\dots(1-\Lambda l)}\right\};$$

from which we may readily deduce, by aid of what has been already shown,

$$\Sigma x^2 = \Sigma\Theta \frac{(\Lambda a)(1+\Lambda a)\Lambda(-n)}{(1-\Lambda a)^3(1-\Lambda b)\dots(1-\Delta l)};$$

and so in general,

$$\Sigma x^i = \Sigma\Theta \frac{\Lambda(a)(1+\Lambda a)\dots\{(i-1)+\Lambda a\}}{(1-\Lambda a)^{i+1}(1-\Lambda b)\dots(1-\Lambda l)}.$$

Again, if we write

$$ax + by_1 + by_2 + \dots + by_\epsilon + cz + \dots + lw = n, \qquad (4)$$

we shall find by parity of reasoning (seeing that in this last equation the solutions may be derived from those of equation (1) by keeping $x, z, \dots w$ all unaltered, whilst we give to $y_1, y_2 \dots y_\epsilon$ all the values compatible with $y_1 + y_2 + \dots + y_\epsilon = y$), the value of Σx^i in equation (4) will be the same as that of

$$\Sigma x^i \,.\, \frac{(y+1)(y+2)\dots(y+\epsilon)}{1\,.\,2\dots\epsilon}$$

in equation (1). Wherefore we shall evidently obtain

$$\Sigma x^i \,.\, y^\epsilon = \Sigma\Theta \frac{\Lambda a(1+\Lambda a)\dots\{(i-1)+\Lambda a\} \times \Lambda b(1+\Lambda b)\dots\{(\epsilon-1)+\Lambda b\}}{(1-\Lambda a)^{i+1}(1-\Lambda b)^{\epsilon+1}(1-\Lambda c)\dots(1-\Lambda l)};$$

the extension of the theorem to $\Sigma x^i \,.\, y^\epsilon \,.\, z^\omega \dots$ is too obvious to need further allusion.

Thus, then, to find $x_1, x_2 \dots x_N$, we may begin by forming an equation of the Nth degree, whose coefficients are known, because the sums of the powers of the roots are given. Supposing these roots to consist of N_1 values x_1, N_2 values $x_2, \dots N_\mu$ values x_μ, the solution of μ simple equations will enable us to find the sum of the N_1 values of y corresponding to x_1, the sum of the N_2 values of y corresponding to $x_2 \dots$, and the sum of the N_μ values of y corresponding to x_μ. To effect this, we have only to write down the values of $\Sigma xy, \Sigma x^2y, \dots \Sigma x^\mu y$. In like manner we may find the sum of the N_1 values of y^2 corresponding to x_1, the N_2 values of y^2 corresponding to x_2, &c., and so in general for y^μ. Thus, then, we may obtain the requisite number of sets of equations for determining independently by means of equations of the degrees $N_1, N_2, \dots N_\mu$ respectively the values of y corresponding to each of the distinct values of x; and in like manner for all the other variables. The principal interest of this note consists, however, in the appreciation of the fact that we can represent algebraically, as has been shown above, the value of $\Sigma x^\alpha . y^\beta . z^\gamma \dots$, where the sign of summation extends over all the simultaneous solutions of

$$ax + by + cz + \&c. = n.$$

This is a considerable advance upon the conception (itself before my discovery entirely unrecognized*) of the explicit representability of the mere number of the solving systems $x, y, z \dots$ by general algebraical formulæ. By this new theorem we pass, as it were, from the shadow to the substance.

* As witness the comparatively unfructuous labours of Paoli, Herschel, Kirkman, and even of Cayley. But as honest labour is seldom entirely wasted, so in the present case it was my valued friend Mr Kirkman's Manchester memoir on partitions which first drew and fixed my attention on the subject.

24.

ON THE PROBLEM OF THE VIRGINS, AND THE GENERAL THEORY OF COMPOUND PARTITION.

[*Philosophical Magazine*, XVI. (1858), pp. 371—376.]

In the *Opera Minora* of the great Euler, in the last page of his *second* memoir on the partition of numbers (Vol. I. p. 400), occur these words:—"Ex hoc principio definiri potest quot solutiones problemata quæ ab arithmeticis *ad regulam virginum* referri solent, admittunt; hujusmodi problemata huc redeunt ut inveniri debeant numeri p, q, r, s, &c., ita ut his duabus conditionibus satisfiat,

$$ap + bq + cr + ds + \&c. = n, \text{ et } \alpha p + \beta q + \gamma r + \delta s + \&c. = \nu;$$

et jam quæstio est quot solutiones in numeris integris positivis locum sint habituræ ubi quidem tenendum est numeros $a, b, c, d \dots n$ et $\alpha, \beta, \gamma, \delta \dots \nu$ esse integros"; and he then proceeds to observe that the number in question is the coefficient of $x^m y^n$ in the expansion of the expression

$$\frac{1}{(1 - x^a y^\alpha)(1 - x^b y^\beta)(1 - x^c y^\gamma) \dots}$$

in terms of ascending positive powers of x and y.

Why the solution in integers of two simultaneous equations with an indefinite number of variables should be referred to "the rule of the Virgins" I am at a loss to conjecture, unless indeed it be supposed to have some mystical reference to the alligation or coupling of the coefficients of the two equations*. The problem in question may be otherwise stated as having

* Professor De Morgan has kindly furnished me with the following information as to the use of this singular phrase:—

"I have seen this process cited as the rule of—Ceres, Series, Verginum, Virginum, Ceres and Virginum, Series and Virginum, Ceres and Verginum, Series and Verginum. I do not think any one of the eight is missing. I cannot find that Ceres is attended by any maidens, and I cannot guess who the ladies were. It is applied by the arithmeticians to the rule of *alligation* when of an indeterminate number of solutions—just Euler's problem which you quote." Mr De Morgan subsequently writes, "I forget whether they wrote Series or Ceries; I think the latter"; and adds a pleasant caution against indulging a passion for one of these algebraical virgins; "for that though Jupiter did once animate a statue maiden at the prayer of an enamoured sculptor, yet even Jupiter himself could not impart a body to an algebraical abstraction."

for its object to discover the number of modes in which the couple m, n may be made up of the couples a, α; b, β; c, γ &c.

I need hardly remark that Euler's form of representation is no solution, but merely a transformation of the question. The problem in its most general form is to determine the number of modes in which a given set of conjoint partible numbers $l_1, l_2, \ldots l_r$ can be made up simultaneously of the compound elements,

$$a_1, a_2, \ldots a_r;\quad b_1, b_2, \ldots b_r;\quad c_1, c_2, \ldots c_r;\ \text{\&c.}$$

The problem of simple partition has been already completely resolved by the author of this notice; but the resolution of the problem of double, and still more of multiple decomposition in general, seemed to be fenced round with insurmountable difficulties.

Let the reader imagine then with what surprise and joyful emotion, within a few days of despatching my previous paper on Partitions to this present Number of the Magazine, following out a train of thought suggested by the simple idea in that paper contained, I found myself led, as by a higher hand, to the marvellous discovery that the problem of compound partition in its utmost generality is capable of a complete solution—in a word, that this problem may in all cases be made to depend on that of simple partition. The theorem by which this is effected has been already confided to the great mathematical genius of England, and will be shortly committed to the 'Transactions' of one of our learned societies; for the present I shall confine myself to a disclosure of the general character of the theorem without going into any details. Thus, then, may the theorem be stated in general terms:—

Any given system of simultaneous simple equations to be solved in positive integers being proposed, the determination of the number of solutions of which they admit may in all cases be made to depend upon the like determination for one or more systems of equations of a certain fixed standard form. When a system of r *equations between* n *variables of the aforesaid standard form is given, the determination of the number of solutions in positive integers of which it admits may be made to depend on the like determination for*

$$\frac{n(n-1)\ldots(n-r+2)}{1 \cdot 2\ \ldots\ (r-1)}$$

single independent *equations derived from those of the given system by the ordinary process of elimination, with a slight modification; the final result being obtained by taking the sum of certain numerical multiples (some positive, others negative) of the numbers corresponding to those independent determinations. This process admits of being applied in a variety of modes, the resulting*

sum of course remaining unaltered in value whichever mode is employed, only appearing for each such mode made up of a different set of component parts.*

In the Problem of the Virgins, where but two equations are concerned, the equations are reduced to the standard form when the two coefficients of every the same variable in the two equations are prime to one another, and when no two pairs of coefficients have the same ratio; and for this problem the process is always limited to only two modes of application. The method, however, in a very important class of cases admits of being applied in *one*, and only *one* mode when these conditions are not strictly fulfilled.

Thus the virgins who appeared to Euler, but with their forms muffled and their faces veiled, have not disdained to reveal themselves to me under their natural aspect. Wonderful indeed has been the history of this theory of partitions. Notwithstanding that the immortal Euler had written two elaborate memoirs on the subject, that Paoli, and I believe other Italian mathematicians, had taken it up from another but less advantageous point of view, so completely had it fallen into oblivion, as far as the mathematicians of this country are concerned, that Sir John Herschel has written a memoir upon it, inserted in the *Philosophical Transactions*, without any reference to, and evidently in complete unconsciousness of, the labours of his predecessors, and subsequently Professor De Morgan, so justly celebrated for his mathematical erudition, in a paper in the *Cambridge and Dublin Mathematical Journal*, refers to the doctrine of partitions as being of quite recent creation. The importance of the subject in these later times has been vastly augmented by the magnificent applications which our great mathematical luminary has made of it to the doctrine of invariants.

* Since the above was in print, I have discovered a much more specific theorem, which, indeed, is to be regarded as the fundamental theorem in the doctrine of compound partition, and the basis of that given in the text. It is as follows:—*If there be* r *simultaneous simple equations between* n *variables (in which the coefficients are all positive or negative integers) forming a definite system (that is, one in which no variable can become indefinitely great in the positive direction without one or more of the others becoming negative), and if the* r *coefficients belonging to each of the same variable are exempt from a factor common to them all, and if not more than* $r-1$ *of the variables can be eliminated simultaneously between the* r *equations, then the determination of the number of positive integer solutions of the given system may be made to depend on like determinations for each of* n *derived independent systems, in each of which the number of variables and equations is one less than in the original system.*

This reduction in general can be effected in a great but limited variety of modes. When only two equations, however, are concerned, the number of modes is always two, neither more nor less. So that in fact we are still navigating in the narrows, and have not fairly entered upon the wide ocean of the theory of compound partitions until we have passed the case of double partition. When the given system supposed definite is one of three equations between four variables, the number of modes of reduction is twelve or sixteen, according to that type out of two (to one or the other of which it must of necessity belong) under which the system falls. The theory of types applicable to any system of simultaneous simple equations with rational coefficients, here faintly shadowed forth, constitutes, I apprehend, a new and important branch in the theory of inequalities.

Postscript. In the first instance I discovered the theorem above given by a method of induction, aided by an effort of imagination, and confirmed by numerous trials; but I have since obtained a very simple, although somewhat subtle general proof of it. Mr Cayley on his part, and independently, has also laid the foundation of a most ingenious and instructive method of demonstration entirely distinct from my own. I reason upon the equations, Mr Cayley upon the Eulerian generating function; but it was by operations performed upon this function that I was myself originally led to a perception of the transcendental analogies out of which I was enabled to evolve the law.

The very interesting case of the composition of a proposed integer out of elements given both in number and species, to which Euler has called particular attention, falls without preparation under the standard form; for this question is in fact merely that of determining the number of solutions of the binary system of equations,

$$ax + by + cz + \dots + lw = m,$$

$$x + y + z + \dots + w = \mu,$$

$a, b, c, \dots l$ being supposed to be all different.

Thus, by way of very simple illustration, suppose it required to find in how many ways the number m can be made up of μ elements, limited to consist of the numbers 1, 2, 3. My method gives me at once the following solution. Call ν the number required. Then m must be not less than μ, and not greater than 3μ, or there will be no solutions. For all values of m between μ and 2μ, both inclusive,

$$\nu = \frac{m-\mu}{2} + \frac{3}{4} + (-1)^{\frac{m-\mu}{2}};$$

for all values of m between 2μ and 3μ, *still both inclusive*,

$$\nu = \frac{3\mu - m}{2} + \frac{3}{4} + (-1)^{\frac{m-\mu}{2}}.$$

It will be observed that when $m = 2\mu$, the two formulæ give the same value, so that either may be employed. Again, suppose we wish to express the number of modes of composition of m with the four elements 1, 2, 3, 4, the number of parts being μ, $\frac{m}{\mu}$ must be not less than 1 nor greater than 4, or there will be no solutions possible.

For all values of m from μ to 2μ inclusive,

$$\nu = \tfrac{1}{12}\{(m-\mu+3)^2 - \tfrac{7}{6}\} + \tfrac{1}{8}(-1)^{m-\mu} + \tfrac{1}{9}(\rho^{m-\mu} + \rho'^{m-\mu}),$$

ρ, ρ' being the prime cube roots of unity.

For all values of m from 2μ to 3μ inclusive,

$$\nu = \frac{(m-\mu+3)^2}{12} - \frac{(m-2\mu+3)^2}{4} + \frac{73}{36}$$

$$+ \tfrac{1}{8}[(-1)^{m-\mu} + (-1)^m]$$

$$+ \tfrac{1}{9}\{\rho^{m-\mu} + \rho'^{m-\mu}\}.$$

Finally, for all values of m from 3μ to 4μ inclusive,

$$\nu = \tfrac{1}{12}\{(4\mu - m - 3)^2 - \tfrac{7}{6}\} + \tfrac{1}{8}(-1)^m + \tfrac{1}{9}(\rho^{m-\mu} + \rho'^{m-\mu}).$$

At the joining points (so to say) between the successive cases, viz. where $m = 2\mu$ or $m = 3\mu$, the contiguous formulæ give like results whichever of them is applied, so that the discontinuity in the form of the solution resembles that arising from the juxtaposition of different curves*. This discontinuity (in itself a remarkable phænomenon to be brought to light), far from being a reproach to the method employed, is to be regarded as a quality inherent in the subject matter under representation, and inexpugnable, as such, in the very nature of things.

* The connexion between the contiguous formulæ is always closer than what is symbolized by the phrase used above. The curves must be regarded as not merely placed end to end, but to be, as it were, knit or spliced together through a certain finite portion of the extent of each of them. Thus the first and second formulæ in the text coincide [?] in value, not merely for $m = 2\mu$, but also for $m = 2\mu - 1$ and $m = 2\mu - 2$; and the second and third formulæ coincide, not merely for $m = 3\mu$, but also for $m = 3\mu + 1$ and $m = 3\mu + 2$. The adjacent curves have, so to say, in the instance above, the same tangents and circles of curvature at the points of union, so that we may be said to *modulate* from one formula into another. The *raison raisonnée* of this fact is easily explicable on *à priori* analytical principles.

25.

ON A GENERALIZATION OF PONCELET'S THEOREMS FOR THE LINEAR REPRESENTATION OF QUADRATIC RADICALS.

[*Oxford British Association Report*, Pt. II. (1860), p. 7.]

THE author explained the application of Poncelet's theorems to practical questions of mechanics in the case of forces acting in a single plane as in the theory of bridges.

He next referred to the mode of extension of this theorem, suggested by Poncelet, applicable to the case of forces in space, and pointed out its insufficiency, and, in a certain sense, its incorrectness.

The essential preliminary question to be resolved in the first instance (after which the matter became one of easy calculation), was shown to be that of cutting off by a plane the smallest possible segment of a sphere that should contain the whole of a given set of points lying on the sphere's surface. Some years ago Prof. Sylvester had proposed in the *Quarterly Mathematical Journal*, without any suspicion of its having any practical applications, the following question:—"Given a set of points in a plane, to draw the smallest possible circle that should contain them all." By a singular coincidence, Professor Peirce, of Cambridge University, U.S., had studied this question and obtained a complete solution of it, which he had communicated to the author during the present meeting of the British Association. A slight consideration served to show that precisely the same solution as Professor Peirce had found for the problem of points in a plane was applicable with a merely nominal change to the sphere also; and thus the solution of a question set almost in sport was found to supply an essential link for the complete development of a method of considerable importance in practical mechanics. The author stated that it would be easy to draw up tables of the values of the constants appearing in the linear function, representing the resultant of three forces at right angles to one another, for the principal cases likely to occur in practice, the values of these constants depending solely upon the condition of relative magnitude to which the component forces are supposed to be subjected.

26.

OUTLINES OF SEVEN LECTURES ON THE PARTITIONS OF NUMBERS.

[*Proceedings of the London Mathematical Society*, XXVIII. (1897), pp. 33—96.]

PREFACE.

THESE outlines appertain to lectures delivered by Prof. Sylvester at King's College, London, during the year 1859. The outline of each lecture was printed shortly before its delivery and handed to those in attendance, and a few copies also were privately circulated. They are now published for the first time. The Professor's attention was called away shortly afterwards to another department of mathematics, with the result that his researches on compound partitions were never published. As the lectures constitute the only serious attempt that has ever been made to deal with the subject, and as copies of the outlines are very scarce, Prof. Sylvester has yielded to the suggestion made to him in regard thereto by the Council of the London Mathematical Society, so far as to assent to their publication in the *Proceedings*, with all their imperfections on their heads. The present state of his health and the long lapse of time combine to render any revision upon the part of the Professor impossible. He desires it to be known that he cannot vouch for the correctness of all that appears in the notes, and that they were prepared in a hand-to-mouth manner during the process of investigation between the lectures, and that it is only on the opinion of the Council urgently expressed to him that the work should not entirely perish that he has consented at this late hour to the publication.

The Council desires to acknowledge the assistance it has derived from Prof. H. W. Lloyd Tanner, of University College, Cardiff, who kindly placed his annotated copy of the outlines at its disposal, and also to Mr R. F. Scott, of St John's College, Cambridge, who presented a copy to the Society.

FIRST LECTURE*.

Introductory Remarks.

Resolution of an integer into parts.

Resolution of an integer into parts limited in number.

Resolution of a number into parts limited in magnitude.

Euler's law of reciprocity, viz.,

As many ways as an integer n can be resolved into parts not exceeding m in number, so many ways can it be resolved into parts not exceeding m in amount.

Ferrers' Proof. Example. $n=5$, $m=3$,

$$\left.\begin{array}{cccc} 111 & 11 & 11 & 1 \\ 11 & 11 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{array}\right\} \begin{array}{l} \text{may be read as} \\ 3,2;\quad 2,2,1;\quad 2,1,1,1;\quad 1,1,1,1,1; \\ \text{or } 2,2,1;\quad 3,2;\quad 4,1;\quad 5. \end{array}$$

Cayley's application of this law to the calculation of groups of symmetric functions.

Example. To find Σx^5, Σx^4y, Σx^3y^2, Σx^2y^2z, where x, y, z are roots of $x^3-p_1x^2+p_2x-p_3=0$, $p_1\cdot p_1\cdot p_1\cdot p_1\cdot p_1$, $p_2\cdot p_1\cdot p_1\cdot p_1$, $p_2\cdot p_2\cdot p_1$, $p_3\cdot p_2$ will be linear functions of the quantities to be found.

Euler, Waring, Paoli, De Morgan, Warburton, Herschel, Kirkman, Ferrers, Cayley, in connexion with question of resolution.

The resolution of a number into parts is the problem of ascertaining the different modes of composing n with the elements

$$1, 2, 3, \ldots \text{ up to } n.$$

General problem of *simple* partition is to find in how many ways a given number n can be composed of given elements $a, b, c, \ldots k$.

General problem of *binary* partition is to find in how many ways the couple m, m' can be composed of the couples $a, a', b, b', c, c', k, k'$.

Statement of problem under form of equations.

Denumeration and denumerant defined.

Denumerant of $U=0$ same as that of $kU=0$.

Denumerant of $U=0$, $V=0$ same as that of

$$kU+lV=0,\quad k'U+l'V=0.$$

* Delivered at King's College, London, on the 6th June, 1859.

Coefficient groups and constant group defined.

How the resolution of an equation or system of equations with any *real* coefficients may be made to depend on the inverse problem of the centre of gravity of a system of points.

Example. A system of two equations.

Origin, coefficient points, primary defined.

Total of coefficient points is called a cluster.

Coefficient points may be denoted by the variables to which they belong.

Weighted cluster.

Weight of primary assumed to be positive unity.

If primary and cluster balance about the origin, the weights at the several points of cluster will satisfy the given system of equations.

Linear cluster; plane cluster; solid cluster.

The cluster origin and primary may be considered apart from the axes used in the construction.

Ray cluster; axis of cluster defined.

Derivative of an equation-system. An equation-system really consists of the universe of its derivatives.

How this universe is contained in the geometrical representation of the system.

A principal derivative of a binary system is the equation resulting from the elimination of any *one* of its variables.

A principal derivative of a ternary system is the equation resulting from the elimination of any *two* of its variables.

Universe or Plexus of Principal Derivatives.

How to construct geometrically the principal derivatives by aid of the cluster, primary, and origin.

(1) For binary system.

(2) For ternary system.

We can thus perform the process of elimination geometrically.

If more than the regular number of variables can be eliminated simultaneously out of the system, this will be evidenced in the plane cluster by three or more points lying in a line, and in the solid cluster by four or more points lying in a plane*.

* The general polyhedron *in solido* analogous to the polygon *in plano* is a polyhedron with triangular faces exclusively.

An equation is said to be homonymous when the coefficients of the variables are all positive or all negative.

It may be congruous or incongruous.

Example. $2x+3y+4z = 10$ congruous,

$2x+3y+4z = -10$ incongruous.

An *omni-positive* solution of an equation or system means a solution in which the variables are all positive.

An *omni-negative* solution is one in which the variables are all negative.

A *homonymous* solution is one which is either omni-positive or omni-negative.

An equation or equation-system may be *definite* or *indefinite*.

Indefinite when homonymous solutions can be found wherein the variables may be made indefinitely great.

Definite when the variables cannot be made indefinitely great in any homonymous solution.

The equations $ax-by=m$ and $ax+by-cz=m$, where $a, b, c, \ldots m$ are any real positive quantities whatever, are indefinite.

The character as to definite or indefinite depends only on the coefficients, and not on the constant term.

A single equation to be definite must be homonymous.

A system of equations to be definite must admit of a homonymous derivative.

If it admit of one, it must admit of an infinite number of such.

Definiteness and indefiniteness of systems depend only on the relative values of coefficients, and not on the constant terms.

Hence, the relative position of origin and cluster must suffice geometrically to determine this character.

Definition of boundary of a plane or solid cluster of points.

Lemma. The centre of gravity of any weighted cluster is contained inside the boundary, and may be made to lie at any point within it by a due adjustment of the relative magnitudes of the weights at the several points.

Theorem. If the origin lies within the cluster, the system is indefinite; if outside, definite.

In Fig. 1 the centre of gravity of the cluster may be brought to the position g or g' as near as we please to O on either side of it in a line with PO, and, the sum of the weights

$$x+y+z+t+u+v+w+\omega$$

being $\frac{PO}{gO}$ or $-\frac{PO}{g'O}$, may be made indefinitely great either on the positive or negative side of zero.

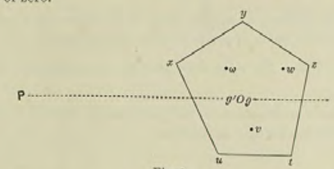


Fig. 1.

In Fig. 2, if origin is at O, Σx will lie between $\frac{PO}{gO}$ and $\frac{PO}{g'O}$; if origin is at O', Σx will lie between $-\frac{PO'}{gO'}$ and $\frac{PO'}{g'O'}$ *; if at O'', the system cannot by any system of weights, all positive or all negative, be made to balance the weight at P about the origin.

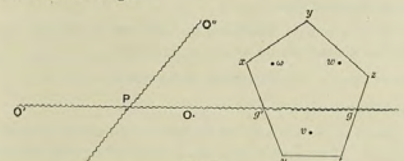


Fig. 2.

The same method is applicable to points *in solido*.

Indefinite systems in general admit of homonymous solutions of both kinds.

The only case of exception is when the origin is in the contour of cluster.

Definite systems admit only of solutions of one kind.

* Hence it may easily be shown that the greatest and least values of Σx in any definite system of equations

$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots = m,$$
$$a_1'x_1 + a_2'x_2 + a_3'x_3 + \ldots = m',$$
$$a_1''x_1 + a_2''x_2 + a_3''x_3 + \ldots = m'',$$

will be the greatest and least values of ρ deduced successively from all the equations that can be formed after the type of the following:—

$$\begin{vmatrix} a_1, & a_2, & a_3, & 1 \\ a_1', & a_2', & a_3', & 1 \\ a_1'', & a_2'', & a_3'', & 1 \\ m, & m', & m'', & \rho \end{vmatrix} = 0,$$

and in like manner we may derive from the geometrical method a simple rule for determining algebraically the maxima and minima values of each separate variable.

Three Species of Definite Systems.

The system is *positive* or *negative* when the axis cuts the cluster according as primary or cluster lie on opposite or same side of origin.

It is *neuter* when the axis does not cut the cluster. (For example, origin O''.)

An analytical determination of the genus and species of a system may be deduced from the preceding construction.

Binary System.

In the indefinite case, if we draw lines from origin to every point in cluster, each such ray divides the cluster into two parts.

In the definite case there are two extreme rays leaving all the points in the cluster on the same side.

Hence, if a system is indefinite, the universe or plexus of principal derivatives will contain no homonymous equations.

If it be definite, it will contain two homonymous equations.

Again, as regards species—

If the system is positive definite, the two homonymous derivatives will be both congruous. If the system is negative, they will be both incongruous.

If the system be neuter, the homonymous will be one congruous, the other incongruous.

Ternary System.

If the system is indefinite, all the planes through the origin and any two points of the cluster divide the cluster into two parts.

If it be definite, the bounding planes of the pyramid formed by joining the origin with each point of the cluster will leave the other points of cluster all on one side.

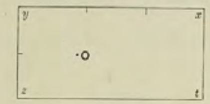
Hence, when the system is indefinite, the plexus of principal derivatives will contain no homonymous equations; when it is definite, there will be some homonymous derivatives, and the number cannot be less than three or greater than the number of variables. For, if we project the cluster from the origin on a plane cutting the rays all on the same side of origin, the number of sides in contour of this projection will be the number of planes in

the pyramid, and n points in a plane cannot form a figure bounded by less than three nor more than n sides*.

The different species of definite will be distinguishable by the homonymous principal derivatives being all congruous, all incongruous, or partly congruous, partly incongruous.

Examples of *indefinite* two-equation systems:

Let the cluster of coefficient points be at the four angles of a parallelogram x, y, z, t, the origin O being at the distance of one unit from yz, yx, zt, and two units from xt.



The system of equations will be

$$2x - y - z + 2t = n,$$
$$x + y - z - t = m.$$

The universe of principal derivatives will be

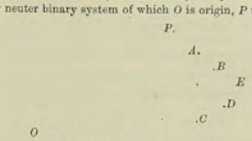
$$3y - z - 4t = 2m - n,$$
$$3x - 2z + t = m + n,$$
$$-x + 2y - 3t = m - n,$$
$$4x + y - 3z = n + 2m,$$

all of which are heteronymous or indefinite, showing that the system is indefinite.

* Thus we see in like manner that the number of homonymous principal derivatives to a definite quaternary system of n variables is some number intermediate to 4 and q (where q is the number of faces in a triangular polyhedron with n summits), that is, $2n - 4$.

This would be difficult to prove by a direct analytical process.

N.B. In any neuter binary system of which O is origin, P the primary and $ABCDE$ the



cluster, all the triangles OPA, OPB, &c., following the same order of rotation will represent determinants of the same sign.

This cannot be the case for definite positive or negative, or for indefinite systems.

Hence the neuter case may be recognised by the determinants, obtained by conjugating *in situ* each coefficient group in succession with the constant group, never changing sign.

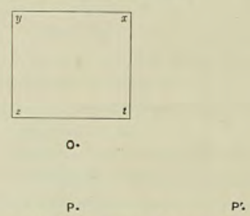
If y, z, x, t were a square, y, t as well as x, z would be brought into line with O; equations would become

$$x - y - z + t = m,$$
$$x + y - z - t = n,$$

and, on account of these two syzygies, there would be only two principal derivatives, viz.:

$$2y - 2t = n - m,$$
$$2x - 2z = n + m.$$

Examples of Definite Systems.*



Positive Case. Take x, y, z, t at the angles of a square, two units each way (breadth and depth).

Let the origin O be at an equal distance from z and t, and from x and y and *the primary P in a line with xt.* The system referred to OP and OQ at right angles to OP as axes of moment gives rise to the equations

$$x - y - z + t = 0,$$
$$(1 + c)x + (1 + c)y + cz + ct = m,$$

* In order that a system may be definite the points of the cluster, whether in line, plane, or solid, must be all *in front* to an eye at the origin. In the last two cases accordingly, a line or a plane may be drawn through the origin, leaving the cluster entirely on one side. Now, as a line in a plane will cut three out of any four quadrants in the plane made by two intersecting lines, and a plane *in solido* will cut seven out of any eight octants made by three intersecting planes, it follows that a binary system *may* be definite when of the four possible combinations of signs affecting the terms of the several coefficient groups, that is, $\begin{smallmatrix} + & + & - & - \\ + & - & + & - \end{smallmatrix}$, three are found among the several groups, and so a ternary system may remain definite even when out of the 8 possible combinations of signs

$$\begin{matrix} ++++ & ---- \\ ++-- & ++-- \\ +-+- & +-+- \end{matrix}$$

all but one are found among the several coefficient groups.

We may then safely infer that, in general, all but one of the possible combinations of sign may occur in the coefficient groups of any system without the system necessarily ceasing to be definite. But, if all possible combinations occur, the system will be necessarily indefinite.

c being the distance of O from zt, and m its distance from P. The extreme rays being Oz and Ot, the two homonymous principal derivatives will be the two resultants in respect to z and t, that is,

$$(1 + 2c)x + y + 2ct = m,$$
$$x + (1 + 2c)y + 2cz = m,$$

both of which are congruous.

Negative Case. Figure the same as the preceding, but *position of P reversed* (that is, *passed through origin to an equal distance from it on the other side*). The equations will be as above, with the exception of m becoming negative, so that the two homonyms will be incongruous.

Neuter Case. Same figure as above, but *the primary P moved horizontally through d to P′ lying to the right of zO produced.* This condition implies that $\frac{m}{d} < c$ or $m < cd$.

The two equations now become

$$x - y - z + t = d,$$
$$(1 + c)x + (1 + c)y + cz + ct = m,$$

and the two homonyms are

$$(1 + 2c)x + y + 2ct = m + cd,$$
$$x + (1 + 2c)y + 2cz = -(cd - m),$$

of which the first is congruous, the second incongruous, thereby indicating that the system is neuter.

The determinants

$$\begin{vmatrix} 1 & d \\ 1+c & m \end{vmatrix} \quad \begin{vmatrix} -1 & d \\ 1+c & m \end{vmatrix} \quad \begin{vmatrix} -1 & d \\ c & m \end{vmatrix} \quad \begin{vmatrix} 1 & d \\ c & m \end{vmatrix}$$

that is $\quad m - cd - d, \quad -m - d - cd, \quad -m - cd, \quad m - cd,$

being all negative, would also have served to prove the system to be neuter.

Scholium. If our equation or equation-system be now supposed to be integer equations, we see that the denumerant will be in all cases zero, if the system be negative or neuter. If it be indefinite, the denumerant in general will be infinite (according to the known theory of numbers), but it may be zero, namely, in the case where the coefficients of any equation or derived equation of the given system have a common factor which is not a factor of the constant term.

The plexus of principal derivatives affords an absolute criterion for determining whether the denumerant of a given indefinite system of equations is infinite or zero.

SECOND LECTURE*.

Definition of denumerant recalled,

$$\alpha U, \quad \alpha(U, V), \quad \alpha(U, V, W),$$

used as implicit symbols of denumeration.

$$\alpha U \text{ in its explicit form } \frac{n;}{a, b, c, \ldots l;}$$

$$\alpha(U, V) \quad \text{"} \quad \text{"} \quad \frac{n, n';}{a, a'; b, b'; c, c'; \ldots; l, l';}$$

&c. &c.

Numeratives and denominatives defined.

Herschel's symbol r_n explained.

Its value as a linear function of nth powers of the rth roots of unity.

r_n in the new theory will be replaced by $\frac{n;}{r;}$

Observation.

$$\frac{n;}{r;} + \frac{n-1;}{r;} + \ldots \frac{n-(r-1);}{r;} = 1.$$

More generally,

$$\frac{n;}{r'r;} + \frac{n-r;}{r'r;} + \frac{n-2r;}{r'r;} + \ldots \frac{n-(r'-1)r;}{r'r;} = \frac{n;}{r;}$$

for, if $\frac{n}{r}$ is fractional, so is $\frac{n}{r}-1, \frac{n}{r}-2, \ldots,$

and *a fortiori*,

$$\frac{1}{r'} \cdot \frac{n}{r}, \; \frac{1}{r'}\left(\frac{n}{r}-1\right), \; \frac{1}{r'}\left(\frac{n}{r}-2\right), \ldots \frac{1}{r'}\left(\frac{n}{r}-r'-1\right),$$

and, if $\frac{n}{r}$ is integer, one and only one of the above quantities will be an integer.

What $E\left(\frac{n}{r}\right)$ is commonly used to denote; Herschel's notation $\frac{n}{r}$.

$$E\left(\frac{n}{r}\right) = \frac{(n-r);}{1, r;}$$

Examples.

$$x + 2y = 7, \quad E\left(\frac{7}{2}\right) = 3,$$

$$x + 3y = 8, \quad E\left(\frac{8}{3}\right) = 2,$$

$$x + 5y = 15, \quad E\left(\frac{15}{5}\right) = 3.$$

* Delivered at King's College, London, on the 9th June, 1859.

N.B. In the partition theory, zero always counts as a *positive* integer*.

Of course the residue of n to modulus r is

$$n - r \times \frac{(n-r);}{1, r;};$$

but it may also be expressed as a binary denumerant.

Simplest class of indeterminate equations

$$x_1 + x_2 + \ldots + x_r - n = 0,$$

$$\frac{n}{1} = 1, \quad \frac{n}{1, 1} = n + 1, \quad \frac{n}{1, 1, 1} = \frac{(n+1)(n+2)}{2}, \text{ \&c.}$$

Generally αU in above equation is coefficient of t^n in $\frac{1}{(1-t)^r}$.

The denumerant of the equation

$$ax_1 + ax_2 + \ldots + ax_r - n = 0,$$

$$\frac{n;}{a;} \times \left\{\frac{n+a}{a} \cdot \frac{n+2a}{2a} \cdots \frac{n+(r-1)a}{(r-1)a}\right\}.$$

Provisional Method of Simple Denumeration.

Any simple denumerant may be expressed in terms of denumerants of the class last treated of.

Example 1. $x + 2y = n;$

x must be of the form 2ξ or $2\xi + 1$, two suppositions mutually exclusive.

Hence the denumerant of the given equation is the sum of those of the two equations,

$$2\xi + 2y = n \text{ and } 2\xi + 2y = n - 1.$$

Hence

$$\frac{n;}{1, 2;} = \frac{n;}{2, 2;} + \frac{n-1;}{2, 2;}$$

$$= \frac{n;}{2;} \times \frac{n+2}{2} + \frac{n-1;}{2;} \times \frac{n+1}{2}.$$

But

$$\frac{n;}{2;} + \frac{n-1;}{2;} = 1.$$

Hence

$$\frac{n;}{1, 2;} = \frac{2n+3}{4} + \left\{\frac{n;}{2;} - \frac{(n-1);}{2;}\right\}\frac{1}{4}.$$

Observe that

$$\left\{\frac{n;}{2;} - \frac{(n-1);}{2;}\right\}\frac{1}{4} = (-)^n \frac{1}{4}.$$

* Consequently, the equation $ax + by + cz \ldots = 0$ has the denumerant *unity*, and is not neuter; there being in fact no neuter cases for *simple* partition.

Example 2. $$x + 3y = n;$$

x must be of the form 3ξ, or $3\xi + 1$, or $3\xi + 2$.

Hence
$$\frac{n;}{1, 3;} = \frac{n;}{3, 3;} + \frac{(n-1);}{3, 3;} + \frac{(n-2);}{3, 3;}$$
$$= \frac{n;}{3;} \cdot \frac{n+3}{3} + \frac{(n-1);}{3;} \cdot \frac{n+2}{3} + \frac{(n-2);}{3;} \cdot \frac{n+1}{3}$$
$$= \frac{1}{3}\left[(n+2) + \left\{\frac{n;}{3;} - \frac{(n-2);}{3;}\right\}\right].$$

Example 3. To find the denumerant of
$$x + 2y + 4z = n.$$

4 is the least common multiple of 1, 2, 4.

x is either 4ξ, $4\xi + 1$, $4\xi + 2$, or $4\xi + 3$,

$2y$ is either 4η, or $4\eta + 2$,

$4z$ is $4z$.

Thus there are eight cases, each giving rise to an equation of the form
$$4\xi + 4\eta + 4z + c = n.$$

I combine together those in which the constant on the left side is either the same, or leaves the same residue when divided by 4.

Thus, we obtain
$$\frac{n;}{1, 2, 4;} = \frac{n;}{4, 4, 4;} + \frac{n-4;}{4, 4, 4;}$$
$$+ \frac{n-1;}{4, 4, 4;} + \frac{n-5;}{4, 4, 4;}$$
$$+ 2\frac{n-2;}{4, 4, 4;} + 2\frac{n-3;}{4, 4, 4;}$$
and observing that
$$\frac{n-4;}{4;} = \frac{n;}{4;}, \quad \frac{n-5;}{4;} = \frac{n-1;}{4;}$$
we obtain
$$\frac{n;}{1, 2, 4;} = \frac{n;}{4;}\left\{\frac{(n+4)(n+8)}{4 . 8} + \frac{n(n+4)}{4 . 8}\right\}$$
$$+ \frac{n-1;}{4;} \cdot \left\{\frac{(n+3)(n+7)}{4 . 8} + \frac{(n-1)(n+3)}{4 . 8}\right\}$$
$$+ \frac{n-2;}{4;} \cdot \frac{(n+2)(n+6)}{4 . 4}$$
$$+ \frac{n-3;}{4;} \cdot \frac{(n+1)(n+5)}{4 . 4}$$
$$= \frac{1}{16}\left\{\frac{n;}{4;}(n^2 + 8n + 16) + \frac{n-1;}{4;}(n^2 + 6n + 9) + \frac{n-2;}{4;}(n^2 + 8n + 12) + \frac{n-3;}{4;}(n^2 + 6n + 5)\right\}.$$

By aid of the identities,
$$\frac{n;}{4;} + \frac{n-1;}{4;} + \frac{n-2;}{4;} + \frac{n-3;}{4;} = 1,$$
$$\frac{n;}{4;} + \frac{n-2;}{4;} = \frac{n;}{2;}, \quad \frac{n-1;}{4;} + \frac{n-3;}{4;} = \frac{n-1;}{2;}$$
the above equation becomes
$$\frac{n;}{1, 2, 4;} = F + \left(\frac{n;}{2;} - \frac{n-1;}{2;}\right)G + \left(\frac{n;}{4;} + \frac{n-1;}{4;} - \frac{n-2;}{4;} - \frac{n-3;}{4;}\right)H,$$
where
$$F = \frac{1}{16}\left(n^2 + 7n + \frac{21}{2}\right); \quad G = \frac{2n+7}{32}, \quad H = \frac{1}{8}.$$

F will be the mean value of the transcendental function $\frac{n;}{1, 2, 4;}$.

The mean value of any simple denumerant, by virtue of the theorem discovered by the lecturer, is always expressible directly as an algebraical function of n, and of the quantities $a_1, a_2, \ldots a_r$ left perfectly indefinite.

Observe that in the multipliers of G and H the sums of the coefficients are all zero.

General direct method of expressing every simple denumerant under a simple form is furnished by theorem above referred to.

The method above given substantially consists in making the denumeration of $a_1x_1 + a_2x_2 \ldots + a_rx_r = n$ depend on finding all the solutions of the congruence
$$a_1u_1 + a_2u_2 + a_3u_3 \ldots + a_ru_r - u_{r+1} = 0 \text{ to modulus } K,$$
K being the least common multiple of $a_1, a_2, \ldots a_r$, and $u_1, u_2, \ldots u_r$ being all limited to be positive integers less than K, but u_{r+1} being left indefinite.

Thus the numbering of the solutions in positive integers of an equation can be brought to depend upon finding the solutions themselves of a congruence in positive integers.

Euler's Method of Generating Fractions.

The denumerant of
$$ax + by + cz + \ldots + lt = n$$
is the coefficient of t^n in the expansion of
$$\frac{1}{(1-t^a)(1-t^b)\ldots(1-t^l)}$$
expanded in ascending powers of t.

Proof that the product of the series generated by $\frac{1}{1-t^a}\cdot\frac{1}{1-t^b}$, &c., gives $\frac{n;}{a, b, c, \ldots l;}$ as the coefficient of t^n.

Note that when $n=0$ the coefficient of t^n is 1.

Thus we see that the denumerant of $x_1+x_2+\ldots+x_r=n$ is the coefficient of t^n in $\frac{1}{(1-t)^r}$ as already found.

Necessity of attending to the *order* of terms in the denominators of generating fractions; $\frac{1}{p-q}$ and $\frac{1}{-q+p}$ distinguished; $\frac{1}{p \sim q}$ may be used to signify one or the other of the two previous forms, the choice being left subject to ulterior determination.

Euler's generating fraction continues to hold good even when any of the coefficients become negative, the expansion becoming *indefinite*.

Example. The denumerant of $x-y=n$ is generated by the product of $\frac{1}{1-t}$ by that of $\frac{1}{1-t^{-1}}$, that is to say, of the series

$$1+t\ \ +t^2\ \ +t^3\ \ +\ldots \textit{ad inf.},$$

by the series

$$1+t^{-1}+t^{-2}+t^{-3}+\ldots \textit{ad inf.}$$

This product will consist of an ascending and descending branch, and the coefficients of every term in each branch will be infinite, showing that the denumerant of $x-y=n$ is infinite for all integer values of n whether positive or negative.

The cognate forms to a generating fraction defined.

Their number, if there are r factors in the denominator, is 2^r.

In above example $\frac{1}{(1-t)(1-t^{-1})}$ generates a double indefinite development, but the cognate form $\frac{1}{(1-t)(-t^{-1}+1)}$ will generate a series in which the indices of t ascend from 1 to ∞, and the coefficients for any finite value of an index remain finite.

So in general for $\frac{1}{(1-t^a)(1-t^{-b})}$; the coefficient of t^n in a cognate form to this is the coefficient of t^{n-b} in $\frac{1}{(1-t^a)(1-t^b)}$ with the sign changed. So again the coefficient of t^n in a cognate form to

$$\frac{1}{(1-x^a)\ldots(1-x^b)(1-x^{-c})(1-x^{-d})}$$

will be the coefficient of t^{n-c-d} in

$$\frac{1}{(1-x^a)\ldots(1-x^b)(1-x^c)(1-x^d)},$$

and so on.

Every generating fraction to a single equation contains two cognate forms (of which itself may be one), which admit of development in series with *finite* coefficients. One of these will be purely an ascending, the other purely a descending, series.

The coefficient of t^n in the ascending development I call the connumerant of the equation.

The connumerant is always finite; it may be positive or negative; when the coefficients are all positive the connumerant and denumerant are identical.

The meaning of the symbol $\frac{n;}{a_1, a_2, \ldots a_r;}$ extended and modified.

Rule for transforming a connumerant with some or all of its denominatives negative into one with all its denominatives positive.

Why connumerants are necessary.

When the numerative is a negative quantity the connumerant by virtue of the definition is always zero.

The denumerant of a binary system of equations

$$ax+by+cz+\ldots=m,$$
$$a'x+b'y+c'z+\ldots=m',$$

is the coefficient of $t^m t^{m'}$ in

$$\frac{1}{(1-t^a.t'^{a'})(1-t^b.t'^{b'})(1-t^c.t'^{c'})\ldots}.$$

Unnecessariness of the limitation imposed by Euler upon the signs of the coefficients.

How to exhibit geometrically, the limiting ratios to the values of the indices of t and t' which can appear in the development of the Eulerian fraction containing t and t'.

Hence we see that the series generated by such an Eulerian may consist of a single branch, or of two branches, or of three branches.

So the Eulerian of a definite ternary system developed may have any number of branches from one to seven inclusive.

Definition. A determinate series is one in which none of the coefficients of terms at a finite distance from the origin become infinite in value. A determinate generating function is one which generates a determinate series.

Then $\frac{1}{(1-t)^2}$ is determinate, but $\frac{1}{(1-t)(t-1)}$ indeterminate.

So, again, $\frac{1}{(1-t)(u-1)(ut-1)}$ is determinate, but

$$\frac{1}{(1-t)(u-1)(t-u)} \text{ indeterminate.}$$

Denumerative function distinguished from denumerant.

Reversal of a point in a cluster defined.

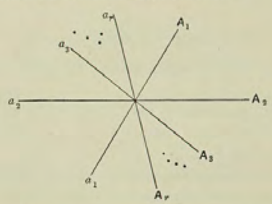
If we change any factor of an Eulerian of the second order

$$\frac{1}{1-t^a.t'^{a'}} \text{ into } \frac{1}{-t^a.t'^{a'}+1},$$

the equation-system by the denumeration of which the coefficient of $t^m t'^{m'}$ may be calculated undergoes a change in its constant terms as well as in its coefficients; but it is only the change in the latter which can influence the character of the system as to being definite or indefinite, and consequently the character of the coefficients of the developed Eulerian as to being finite or infinite.

Hence, it is easy to show geometrically that $2r$ out of the 2^r cognate forms to an Eulerian fraction of the 2nd order will give rise to series with finite coefficients.

For if there be r variables the total number of cognate forms will correspond to the 2^r clusters consisting of A_1 or a_1 (its reverse), combined with A_2 or a_2 its reverse, with A_3 or a_3 its reverse, and so on.



Now of all these clusters the only ones which do not enclose the origin are the pairs

$$\left.\begin{matrix} A_1A_2A_3 \ldots\ldots A_r \\ a_1A_r \ldots\ldots A_3A_2 \end{matrix}\right\},$$

$$\left.\begin{matrix} A_2A_3 \ldots\ldots A_ra_1 \\ a_2a_1A_r \ldots\ldots A_3 \end{matrix}\right\},$$

and so on, there being as many pairs of clusters outside the origin as there are points $A_1, A_2, \ldots A_r$.

In like manner an Eulerian fraction of the 3rd order and with r factors in its denominator will admit of as many cognate pairs of forms generating series with finite coefficients as there are combinations of r elements, 2 and 2 together, that is, $\frac{r.r-1}{2}$ pairs, and so on, for any order whatever.

Hence it would not be possible without further specification to extend the definition of connumerants (if it were wished to do so) from simple equations to equation-systems.

Happily the necessity for the consideration of such does not arise, as it will be shown that denumerants of all orders may be expressed in terms of *simple* connumerants.

By the connumerant to

$$-ax-by-cz+dt+eu+\&c.=K,$$

I shall understand the expression

$$\frac{K;}{-a,-b,-c,d,e,\ldots;}.$$

This connumerant will be the same save as to sign (which is or is not to be changed, according as the number of negative coefficients $-a, -b, -c$ is odd or even) as the denumerant of

$$a(x+1)+b(y+1)+c(z+1)+dt+eu+\&c.=K.$$

THIRD LECTURE*.

Reduction.

Reduction explained.

Reduction in partitions analogous to elimination in equations.

A prime group defined. Examples.

Syzygy of variables; predicable also (elliptically) of groups.

In a plane cluster, syzygy is evinced by two or more points being in a line with the origin.

In a solid cluster, by three or more points being in the same plane with the origin.

* Delivered at King's College, London, on June 16th, 1859.

Analytical condition of two groups in a binary system being in syzygy is that the determinant formed by their coefficients vanishes.

Analytical condition of three variables in a ternary system being in syzygy is same as above; and so in general.

If $ab' - a'b = 0$,

$$\frac{a}{b} = \frac{a'}{b'};$$

and, if a, b is a prime group, and also a', b', either

$$\begin{matrix} a = a' \\ b = b' \end{matrix} \quad \text{or} \quad \begin{matrix} a = -a' \\ b = -b'. \end{matrix}$$

On the latter supposition, the system would be indefinite (for the origin would either lie *on* the contour of the cluster or *within* it).

Hence two non-identical prime groups cannot be in syzygy.

The same will be true of three non-identical prime groups in a ternary system.

If, in a definite binary system, each of a certain set of groups is a prime group, and no two of the groups the same, the system will be asyzygetic so far as this set of groups or their variables is concerned.

Importance of the case of equal, that is, identical, coefficients or coefficient groups.

The symmetric functions of the roots of indeterminate equations may be expressed as denumerants to equations or equation-systems with equal coefficients or coefficient groups.

Scheme: its definition as collective name for cluster and primary.

Scheme: linear, plane, or solid.

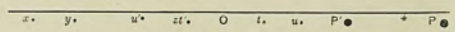


Fig. 1.

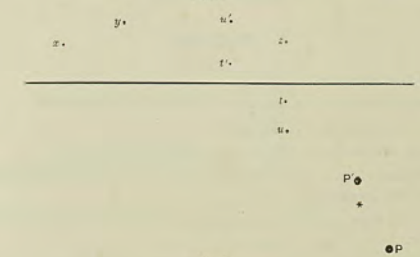


Fig. 2.

Centre: Axis: Balancing plane of scheme: denumerant of a linear scheme in respect to a given centre; of a plane scheme in respect to a given centre or axis; of a solid scheme in respect to a centre, axis, or plane.

Connumerant of a linear scheme in respect to a given centre; of a plane scheme in respect to a given axis*; of a solid scheme in respect to a given plane.

Notice the algebraical sign of the *connumerant*, which is positive or negative, according as an even number (including *zero* as one) or an odd number of transpositions of cluster-points is *transposed*.

Definitions of rays and planes of cluster recalled and applied to schemes.

The term beam substituted for ray of the primary.

The theory of the reduction of binary systems of equations may be geometrically stated.

Network. In line, plane, or solid.

Nodes and nodal lines.

Prime *point* or prime *ray* in network corresponds to prime groups of coefficients.

Prime *couples* or prime *planes* in network correspond to prime double-groups of coefficients, meaning a pair of groups whose minor determinants form a prime group.

Anticipatory statement, namely—

The denumerant of a plane scheme in respect to a given centre is the sum of its connumerants with respect to each in succession of the rays which lie on either side (chosen at will) of the beam, provided that all the rays on the side so chosen are prime rays, and the points to which they are drawn are no two of them coincident. If these conditions are satisfied on both sides of the beam, each of the segments of the ray-cluster into which it is divided by the beam will give a distinct solution, and the two sums of connumerants appertaining respectively to the rays in either cluster will be equal to one another.

Observe that, if the system be neuter, all the rays will be on one side of the beam, and there will be but one solution.

The denumerant of a solid scheme in respect to a given centre (corresponding to a ternary system of equations), which satisfies analogous conditions to the preceding, will be shown later on in the course to be expressible in very similar terms (cluster-planes being substituted for cluster-rays), with this remarkable difference, however, that, in lieu of a single dichotomous

* If the motion of P should carry it to P' on the opposite side of the axis, the transformed centre and primary will be brought to lie on one side of the axis, and consequently the cluster must have contrary signs to the primary, in order to balance about the axis; and, as there will thus be no omni-positive solution, the connumerant in that case will become zero. If P is sufficiently remote, this cannot take place.

division of the planes, there will be a considerable number of such, each of which will or may furnish a distinct pair of solutions.

The formation of these dichotomies involves the consideration of the doctrine of *normal orders*, or orders of *perspective sequence*—a branch of the doctrine of free geometry to which allusion was made in the opening address.

The problem of partitions stated as a problem in plane or solid network.

A system of equations in $x, y, z, \ldots u$ may be denoted by $S(x, y, z, \ldots u)$, or, when more convenient, by S alone, with implied reference to $x, y, z, \ldots u$.

Resultants of systems. R_xS, where S is binary, defined. $R_{x,y}S$, where S is ternary, defined.

R_xS is the equation which expresses that the coefficient cluster and primary of S balance about the axis Ox. This will remain good for a ternary system, so that R_xS will then denote a specific binary system, that which corresponds to *projection* of cluster and primary on a plane through the origin perpendicular to O_x. $R_{x,y}S$ will denote that the centre of gravity of the cluster and primary of S is in the plane xy.

Interpretation of $\mathfrak{D}R_xS$ when S is binary. Interpretation of the same when S is ternary.

$\mathfrak{D}R_xS$ in the latter case is perfectly definite just as much as in the former, although the modes of expressing R_xS are infinitely varied.

If S' is what S becomes when we write in S, $fx+g$, or more generally ϕx, in place of x we may denote S' symbolically by $\frac{\phi x}{x}S$.

Note that
$$R_xS = R_x\left(\frac{\phi x}{x}S\right)$$
$$\left(\frac{\phi x}{x}\cdot\frac{\phi y}{y}\cdot\frac{\phi z}{z}\ldots S\right) \text{ explained.}$$

Order of operative symbols $\frac{\phi x}{x}, \frac{\phi y}{y}$, &c. is indifferent. The denumerant of any principal derivative R_xS, if homogeneous, will furnish a superior limit to the denumerant of S; for all the solutions of F must be solutions of R_xS.

Hence, to the denumerant of a definite binary system we can always, by *simple* denumeration, obtain two superior limits; to the denumerant of a ternary system, some number of superior limits, between 3 and n inclusive, such number depending upon the morphological character of the system (as will hereafter be explained).

Examples of superior limits to binary denumerants.

Examples of superior limits to ternary denumerants:—

1. By means of principal *simple* derivatives.
2. By means of principal derivative *binary systems*.

Hereafter we shall find that when the coefficient groups are all prime groups, and none of them alike, these two limits are the respective first terms of two distinct finite series of connumerants, each of which expresses the value of the denumerant of the given binary systems.

Lemma. If the x group in any system is a prime group, any omni-positive integer solution of R_xS is in general an omni-positive integer solution either of S or of $\frac{-x}{x}S$.

Proof in case of binary system.

Proof in case of ternary or ultra-ternary system.

How an exception arises when the solution of R_xS, substituted in S, makes $x=0$.

Were it not for this exception, the equation following would always subsist for any variable x corresponding to a prime group, namely,
$$\mathfrak{D}S + \mathfrak{D}\left(\frac{-x}{x}\cdot S\right) = \mathfrak{D}RS.$$

The number of omni-positive solutions of system $S(x, y, z, \ldots v)$, subject to the condition $x > k$, is the denumerant of
$$S\{(x+k), y, z, \ldots v\}.$$

Thus, if $x>0$, for x we must substitute $1+x$. Hence the true equation which connects the denumerants referred to *without* exception is
$$\mathfrak{D}S + \mathfrak{D}\frac{-1-x}{x}(S) = \mathfrak{D}R_xS;$$
or, if we please,
$$= \mathfrak{D}R_x\frac{-1-x}{x}S.$$

In future I shall denote
$$-x-1 \text{ by } \bar{x},$$
$$-y-1 \text{ by } \bar{y},$$
and so on.

We may therefore write
$$\mathfrak{D}S = \mathfrak{D}R_x\frac{\bar{x}}{x}S - \mathfrak{D}\left(\frac{\bar{x}}{x}.S\right).$$

Now let the y group be also a prime group; we shall have
$$D\frac{\bar{x}}{x}S = S\mathfrak{D}R_y\left(\frac{\bar{x}}{x}\frac{\bar{y}}{y}S\right) - \mathfrak{D}\left(\frac{\bar{x}}{x}\frac{\bar{y}}{y}S\right);$$
therefore
$$\mathfrak{D}S = \mathfrak{D}R_x\frac{\bar{x}}{x}S - \mathfrak{D}R_y\left(\frac{\bar{x}}{x}\frac{\bar{y}}{y}S\right) + \mathfrak{D}\left(\frac{\bar{x}}{x}\frac{\bar{y}}{y}S\right);$$

and so, if the z group be a prime group,

$$\mathcal{Q}S=\mathcal{Q}R_x\frac{\bar{x}}{x}S-\mathcal{Q}R_y\left(\frac{\bar{x}}{x}\frac{\bar{y}}{y}S\right)+\mathcal{Q}\left(R_z\frac{\bar{x}}{x}\frac{\bar{y}}{y}\frac{\bar{z}}{z}S\right)-\mathcal{Q}\left(\frac{\bar{x}}{x}\frac{\bar{y}}{y}\frac{\bar{z}}{z}S\right),$$

and so on to any extent.

Extensions of this Equation to Systems of Equations of a Higher Order than the first indicated.

This I call the process of eduction. The question above indicated is always *true*, amounting in fact to the assertion of identity as regards the *solutions* themselves (not merely their number) of the systems on one side of the equation and those on the other. But, although true, it will be *nugatory* if any of the systems become indefinite, for then in general their denumerants will be infinite in magnitude.

The above equation applies to systems of any order. Its application will be first studied in respect to binary systems.

By continuing the process of eduction through a sufficient number of steps, we shall find that the equation $R_t\left(\frac{\bar{x}}{x}\frac{\bar{y}}{y}\dots\frac{\bar{z}}{z}\frac{\bar{t}}{t}\right)S$ will become at length *incongruous*. Its denumerant will then vanish.

When this is the case, *a fortiori*, the denumerant of the system $\frac{\bar{x}}{x}\frac{\bar{y}}{y}\dots\frac{\bar{z}}{z}$ will vanish. And thus the series is brought to a close, and the denumerant of S expressed entirely in terms of *simple* denumerants.

FOURTH LECTURE*.

Theory of Eduction (*continued*).

Process of eduction exemplified. Suppose the system $S(x, y, z, t)$; then

$$\mathcal{Q}S=\mathcal{Q}R_xS-\mathcal{Q}\frac{\bar{x}}{x}R_yS+\mathcal{Q}\frac{\bar{y}}{y}.\frac{\bar{x}}{x}R_zS-\mathcal{Q}\frac{\bar{z}}{z}.\frac{\bar{y}}{y}.\frac{\bar{x}}{x}R_tS+\mathcal{Q}\left(\frac{\bar{t}}{t}.\frac{\bar{z}}{z}.\frac{\bar{y}}{y}.\frac{\bar{x}}{x}\right)S;$$

but, if S is definite positive, $\frac{\bar{t}}{t}.\frac{\bar{z}}{z}.\frac{\bar{y}}{y}.\frac{\bar{x}}{x}S$ is definite negative. Hence its denumerant is zero, and

$$\mathcal{Q}S=\mathcal{Q}R_xS-\mathcal{Q}\frac{\bar{x}}{x}R_yS+\mathcal{Q}\frac{\bar{y}}{y}\frac{\bar{x}}{x}R_zS-\mathcal{Q}\frac{\bar{z}}{z}\frac{\bar{y}}{y}\frac{\bar{x}}{x}R_tS.$$

The same equation will subsist if S be definite neuter, but not if S be definite negative or indefinite.

* Delivered at King's College, London, on June 20th, 1859.

It is not necessary in general that *all* the coefficient groups should be prime groups, or *all* of them distinct from one another. Great importance of this observation.

Depression of order of denumerants by one degree.

Depression by several degrees:—(1) By successive eductions. (2) By one compound eduction.

Observe that successive eduction can only finally conduct to equations which are simple resultants of the original system, being resultants of its resultants.

Allusion to fundamental theorem for depression by two degrees, namely—

$$\mathcal{Q}R_{x,y}.S=\mathcal{Q}S+\mathcal{Q}.\frac{\bar{x}}{x}S+\mathcal{Q}.\frac{\bar{y}}{y}S+\mathcal{Q}.\frac{\bar{x}}{x}.\frac{\bar{y}}{y}S.$$

This equation is subject to the condition that the minor determinants of the matrix formed by the x and y coefficient groups conjoined shall form a prime group.

Observe the singular symbolical equations—

$$R_x=-\frac{1}{x},\quad R_{x,y}=R_x.R_y=\frac{1}{x}\times\frac{1}{y}.$$

Notice that the lemma at p. [139] is true for systems of any order.

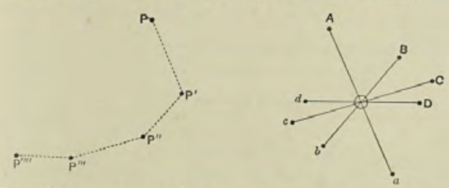
Problem of normal sequences stated. Its geometrical solution by means of the perspective to the cluster, (if a plane cluster) upon a line, (if a solid cluster) upon a plane, or (if a hypersolid cluster) upon a space.

Binary Systems.

Geometrical representation of the successive systems

$$S.\frac{\bar{x}}{x}S,\quad \frac{\bar{y}}{y}\frac{\bar{x}}{x}S,\ \&c.$$

in which S is supposed to be definite and positive. (See figure.)



Reversal of successive points in cluster, accompanied with parallel motion of primary.

The systems $\frac{\bar{x}}{x}.S$, $\frac{\bar{y}}{y}\frac{\bar{x}}{x}.S$, &c. may be regarded as successive deformations of S, and then we may say the system, as it undergoes deformation, tends more and more to lose its positive character, until at length it becomes *neutral*, and immediately after changes into and continues *negative*.

The deformation may be commenced from either side. There are thus two courses of deformation.

If the primary were stationary, the deformed system in either course would become neuter after as many deformations as there are rays in S outside the beam, on that side of it from which the deformation proceeds.

The effect of the motion of the primary is to *accelerate* the tendency of the deformed system to become neuter.

If the primary be moved along the beam to a sufficient distance from the origin, the effect of such tendency will become at length insensible for such and for all greater distances.

The number of deformations in either course which may take place before the primary ray becomes denuded gives the number of terms in the development (by the eduction process) corresponding to that course.

When the primary is sufficiently remote, the sum of these numbers corresponding to the two courses of deformation will be the number of rays, that is, the number of variables in the system.

On account of the motion of the primary, the united sum may be *less* than this number; it cannot be greater.

Corollary. If all the groups are prime groups, the denumerant of a binary system may be expressed by a number of simple denumerants, *not greater* than half the number of variables in the system.

The question of partitions, given in number and species, is expressed by a binary system satisfying these conditions.

Example. To find the number of ways in which n can be made up of r values each limited to be either 1, 2, 3, or 4. This requires the denumeration of the system

$$x + 2y + 3z + 4t = n,$$
$$x + y + z + t = r.$$

The *skew-matrix* of elimination becomes

x	y	z	t	
0	1	2	3	$n - r$
-1	0	1	2	$n - 2r$
-2	-1	0	1	$n - 3r$
-3	-2	-1	0	$n - 4r$.

If $n < r$, the system is neuter, and the number required is 0.

If $n > r$ but $< 2r$, the number required is the connumerant $\frac{n-r;}{1, 2, 3;}$ which is the same in value as the sum of the three complementary connumerants.

If $n > 2r < 3r$, the number required is the sum of the connumerants

$$\frac{n-r;}{1, 2, 3;} + \frac{n-2r;}{-1, 1, 2;}, \text{ or, if we please, } \frac{4r-n;}{1, 2, 3;} + \frac{3r-n;}{-1, 1, 2;}.$$

If $n > 3r$ but $< 4r$, the number required is $\frac{4r-n;}{1, 2, 3;}$, which is the same in value as the sum of the three complementary denumerants.

If $n > 4r$, the system is again neuter, and its denumerant is zero.

By aid of the above expressions we may give the general analytical representation of the number of partitions of the kind proposed.

These expressions, it will be observed, are discontinuous, the particular one to be employed depending on the comparative values of n and r. It may be shown, however, that they, as it were, melt or modulate into one another, so not only at the mere limiting values of n, which separate the several formulæ, but also for a short distance beyond these limits, either of the conterminous expressions may be used indifferently.

The proof of this depends upon the proposition that the coefficient of t^n in the *ascending* expansion of

$$\frac{1}{(1-t^a)(1-t^b)(1-t^c)\dots(1-t^k)}$$

treated *as a function of* n remains fixed at zero, when n is made to become any negative number whose absolute value is inferior to

$$a + b + c + \dots + k.$$

The truth of this proposition follows immediately from a theorem of Mr Cayley relating to the development of any rational fraction $\frac{1}{\phi t}$ in its two forms. If $\theta(n)$ is the fractional value of the coefficient of t^n in the positive, and $\varpi(n)$ the fractional value of the coefficient of t^n in the negative, expansion of $\frac{1}{\phi t}$, $\theta(n) \pm \varpi(n) = 0$, the sign + or − being employed according as the degree of the rational function ϖt is an odd or an even number.

If the coefficient groups be arranged in natural order, the transformed systems will throughout remain definite.

The natural order for binary systems is indicated by the order in either direction in which the rays of the cluster succeed each other.

If the coefficients are all positive, this order will correspond with the order in which the variables must succeed each other, so that the ratios in each group of the coefficient out of one given equation with the corresponding coefficient out of the other may continually increase or decrease.

But, if the coefficients are positive and negative intermingled, the rule is that the determinants formed by the combination of any group with each in succession of those which follow must all bear the same sign; or, as we may express it, the algebraical sign arising from the contact of the groups, with due regard to antecedence, must be always the same.

If the system were indefinite, such a uniformity of signs could not be established by any arrangement of the groups whatever.

Example (1):—

$$\left.\begin{array}{l} x+\ y+\ z=\ 50 \\ x+2y+3z=120 \end{array}\right\}.$$ In this system all the groups are prime groups.

First solution.

x	y	z	
0	1	2	70
−1	0	1	20
−2	−1	0	30.

Observe that the coefficients form a skew-matrix.

$$R_x.\frac{\bar{x}}{x}.S \text{ is } y+2z=70.$$

$$R_y.\frac{\bar{y}}{y}.\frac{\bar{x}}{x}.S \text{ is } (x+1)+z=20, \text{ or } x+z=19.$$

$$R_z.\frac{\bar{z}}{z}.\frac{\bar{y}}{y}.\frac{\bar{x}}{x}.S \text{ is } 2(x+1)+(y+1)=-30, \text{ or } 2x+y=-33.$$

The desired denumerant will be $\frac{70;}{1,2;}-\frac{19;}{1,1;}$ which is

$$36-20=16.$$

Second solution.

z	y	x	
0	1	2	30
−1	0	1	−20
−2	−1	0	−70.

$$R_z.\frac{\bar{z}}{z}.S \text{ is } y+2x=30.$$

$$R_y.\frac{\bar{y}}{y}.\frac{\bar{z}}{z}.S \text{ is } z+x=-21.$$

$$R_x.\frac{\bar{x}}{x}.\frac{\bar{y}}{y}.\frac{\bar{z}}{z}.S \text{ is } 2z+y=-73.$$

The desired denumerant will therefore be $\frac{30;}{1,2;}=16$, as before.

Observe, that $\mathfrak{G}S$ is the *sum* of the connumerants of R_xS, R_yS, R_zS, when these equations are written in the form in which they appear when the process of successive elimination is conducted by a uniform course of operations. This may be done in two ways, so as to give rise to two sets of equations differing from each other only in the signs. Observe, that the connumerant $L=\pm c$ is not the same as of $-L=\mp c$; of that one of these equations, in which the constant term is negative, the connumerant being zero, but of the other the connumerant being generally different from zero.

Corollary. The coefficient of $t^m\tau^\mu$ in

$$\frac{1}{(1-t^a\tau^\alpha)(1-t^b\tau^\beta)(1-t^c\tau^\gamma)}$$

is the sum of the coefficients of

$$\rho^{a\mu-\alpha m} \text{ in } \frac{1}{(1\sim\rho^{a\beta-b\alpha})(1\sim\rho^{a\gamma-c\alpha})},$$

of

$$\rho^{b\mu-\beta m} \text{ in } \frac{1}{(1\sim\rho^{b\alpha-a\beta})(1\sim\rho^{b\gamma-c\beta})},$$

and of

$$\rho^{c\mu-\gamma m} \text{ in } \frac{1}{(1\sim\rho^{c\alpha-a\gamma})(1\sim\rho^{c\beta-b\gamma})},$$

subject to the interpretation that the preceding fractions are to be expanded all in terms of ascending, or all in terms of descending, powers of ρ, *provided* that the system

$$\left.\begin{array}{l} ax+by+cz+dt=m \\ \alpha x+\beta y+\gamma z+\delta t=\mu \end{array}\right\}$$

is a definite non-negative system.

Allusion to Mr Cayley's proof of the proposition in this form. The proposition is, of course, general, that the denumerant of a definite binary system with r variables, in which the groups are all prime groups, admits of a double mode of representation as the sum of the connumerants of its principal derivatives. In the one mode of representation, only a certain number, at *utmost*, of these connumerants, say p, can differ from zero; in the other mode, only a certain number, at utmost, say q, can differ from zero; and, as we shall have $p+q=r$, these two modes of representation may be termed complementary to each other.

The denumerant of the system

$$\left.\begin{array}{l} ax+by+cz+\ldots+dt=m \\ x+\ y+\ z+\ldots+\ t=\mu \end{array}\right\}$$

may therefore be represented in two modes as the sum of simple denumerants.

This is the system the denumeration of which constitutes the problem of the resolution of integers into parts given in number and species. See Euler's *Second Memoir on Partitions.*

The theorem of eduction may be put under the following form :—

$$\text{ɑ}S = \text{ɑ}R_x \frac{\overline{x}}{x} S - \text{ɑ}R_y \frac{\overline{y}}{y}\frac{\overline{x}}{x} S + \text{ɑ}R_z \frac{\overline{z}}{z}\frac{\overline{y}}{y}\frac{\overline{x}}{x} S \mp \&c.,$$

provided that S is a definite system in which the groups of coefficients appertaining to the several variables, or at least to so many of them as are included between x and the last of them which appears as a suffix in a non-zero term of the above expression, are all distinct prime groups.

2nd Example:—

$$\left.\begin{array}{l} x + y + z = 10 \\ x + 2y + 3z = 40 \end{array}\right\}.$$

$$\begin{array}{ccc|c} x & y & z & \\ \hline 0 & 1 & 2 & 30 \\ -1 & 0 & 1 & 20 \\ -2 & -1 & 0 & 10 \end{array} \qquad \begin{array}{c|ccc} & x & y & z \\ \hline -10 & 2 & 1 & 0 \\ -20 & 1 & 0 & -1 \\ -30 & 0 & -1 & -2. \end{array}$$

The solution corresponding to the right-hand matrix is evidently 0, the system in fact being neuter. The left-hand matrix gives the solution—

$$\frac{30\,;}{1,\,2\,;} + \frac{20\,;}{-1,\,1\,;} + \frac{10\,;}{-2,\,-1\,;}$$
$$= \frac{30\,;}{1,\,2\,;} - \frac{19\,;}{1,\,1\,;} + \frac{7\,;}{2,\,1\,;}$$
$$= 16 - 20 + 4 = 0, \text{ as before.}$$

Illustrate effect of taking the variables in abnormal order.

3rd Example:—
$$2x + 5y + 2z - t = m,$$
$$x + y - 2z - 2t = m,$$

the system will also be neuter, and we shall have

$$\frac{m\,;}{6,\,9,\,3\,;} + \frac{2m\,;}{-3,\,6,\,3\,;} + \frac{4m\,;}{-9,\,-12,\,3\,;} + \frac{m\,;}{-3,\,-6,\,-3\,;} = 0,$$

or

$$\frac{m\,;}{3,\,6,\,9\,;} - \frac{2m-3\,;}{3,\,3,\,6\,;} + \frac{4m-21}{3,\,9,\,12\,;} - \frac{m-12}{3,\,3,\,6\,;} = 0.$$

Allusion to importance and fertility of theory of neuter systems.

Example of a denumeration of a binary system containing unprime groups:—

$$x + y + 4z + 3t = m,$$
$$3x + 2y + 6z + 3t = n,$$

$\frac{m}{n}$ being supposed to be intermediate between $\frac{1}{2}$ and $\frac{2}{3}$. The denumerant will be

$$\frac{3m-n\,;}{1,\,6,\,6\,;} - \frac{2m-n\,;}{-1\,;\,2\,;\,3\,;}.$$

Cæsura (definition of, and how determined), accidental and universal, distinguished.

FIFTH LECTURE*.

Eduction and Reduction.

The cæsura for equation-systems generally falls after that coefficient group subsequent to the introduction of which, in the eduction process, the depressed systems whose denumerants are to be taken *must* cease to be positive, so that they may be neglected. It is determined for binary systems by the relation of the ratios of the terms in the coefficient groups to that of the terms in the constant group; the determinant formed by the apposition of the constant group with any group on one side of the cæsura being positive, on the other side negative.

The point after which the terms in an eduction process can be neglected *may* (if the constant terms are sufficiently small) be attained before the cæsura is reached. Such a point may be termed a turning-point, or pause. There may thus (in the case of binary systems) be two turning-points or pauses on each side of the cæsura corresponding to the two courses of eduction, but either or both of them may, and in general will, coincide with the cæsura.

For greater simplicity, we may suppose the constant terms given in ratio only, and not in magnitude, so as to obviate the necessity of paying any attention to the accidental pauses as distinguishable from the cæsura. The cases where they are so distinguishable are always exceedingly limited in number. Their existence arises solely from the fact of the introduction of $-x-1$, $-y-1$, &c., and not $-x$, $-y$, &c., in lieu of x, y, in applying the method of eduction.

A *per-reducible* binary system is one in which *all* the coefficient groups are *prime* groups *distinct* from each other.

Being prime and distinct, none of them can be in syzygy. Such a system admits of a double process of eduction, giving rise in general to two distinct forms of solution. But it may happen, in some very special cases, that these two solutions are identical in form as well as in value.

Example. The system

$$\left.\begin{array}{l} x + 3y + 7z + 9t = 5i \\ x + 2y + 4z + 5t = 3i \end{array}\right\}$$

gives rise to the bordered matrix

$$\begin{array}{cccc|c} 0 & 1 & 3 & 4 & 2i \\ -1 & 0 & 2 & 3 & i \\ -3 & -2 & 0 & 1 & -i \\ -4 & -3 & -1 & 0 & -2i \\ \hline -2i & -i & i & 2i. & \end{array}$$

* Delivered at King's College, London, June, 1859.

Here each solution is the same, namely, $\frac{2i;}{1,3,4;} + \frac{i;}{-1,2,3;}$, meaning

$$\frac{2i;}{1,3,4;} - \frac{i-1;}{1,2,3}.$$

But, if the constant terms in the above system were $11i$ and $7i$ respectively, the bordered matrix would be

$$\begin{array}{|cccc|c} \hline 0 & 1 & 3 & 4 & 4i \\ -1 & 0 & 2 & 3 & i \\ -3 & -2 & 0 & 1 & -5i \\ -4 & -3 & -1 & 0 & -8i \\ \hline \multicolumn{1}{c}{-4i} & -i & 5i & \multicolumn{1}{c}{8i} & \end{array}$$

giving rise to the two equal sums of connumerants,

$$\frac{4i;}{1,3,4;} + \frac{i;}{-1,2,3;} \text{ and } \frac{8i;}{1,3,4;} + \frac{5i;}{-1,2,3;}.$$

In this example the matrix happens to be *persymmetrical*, which is the reason of the denumeratives being the same in each solution.

This is avoided in the example below of the system

$$\left.\begin{array}{r} x + 2y + z + t = 7i \\ x + 3y + 2z + 4t = 12i \end{array}\right\},$$

for which the bordered matrix is

$$\begin{array}{cccc|c} 0 & 1 & 1 & 3 & 5i \\ -1 & 0 & 1 & 5 & 3i \\ -1 & -1 & 0 & 2 & -2i \\ -3 & -5 & -2 & 0 & -16i \\ \cline{1-4} -5i & -3i & 2i & 16i & \end{array}$$

giving rise to the two equivalent solutions

$$\frac{5i;}{1,1,3;} + \frac{3i;}{-1,1,5;} \text{ and } \frac{16i;}{2,5,3;} + \frac{2i;}{-2,1,1;},$$

meaning
$$\frac{5i;}{1,1,3;} - \frac{3i-1;}{1,1,5;} \text{ and } \frac{16i;}{2,3,5;} - \frac{2i-2;}{1,1,2;}.$$

A *simply-reducible* system is one for which the coefficient groups are prime and distinct on *one* side of the cæsura only.

Example:—
$$\left.\begin{array}{r} x + 2y = 4m \\ x + 4y = 5m \end{array}\right\}.$$

The eduction from the x side gives rise to the equation of $2y = m$, of which the denumerant is $\frac{m;}{2;}$. This is the true solution, whereas the eduction

from the y side gives rise to the denumerant of $2x = 6m$, that is, 1, which is a false solution, owing to the group (2, 4) being a non-prime group.

If in a binary system the groups, which are either non-prime or repeated, or non-prime and repeated, represent ratios (between quantities given in algebraical sign) which are all less or all greater than the corresponding ratio of the constant terms, the system is still depressible by eduction commenced from that side of the system on which the groups of the kind mentioned do not fall.

Corollary. A single non-prime group, or a single sequence of any number of identical groups, can in no case hinder a binary system from being soluble by eduction.

The above remark is true also *à fortiori* for ultra-binary systems.

It should be noticed that $\left.\begin{array}{c} a \\ 0 \end{array}\right\}$ is a non-prime group unless $a = \pm 1$. (For non-prime we may in future use the term composite.)

Example:—
$$\left.\begin{array}{r} 3x + 2y + z + t = i \\ 2z + 3t = i \end{array}\right\}.$$

Here the coefficient groups of x and y are both of them composite; but, the cæsura falling between y and z, the denumerant required will be the sum of the connumerants of the two resultants in respect to t and z, that is,

$$\frac{2i;}{1,6,9;} + \frac{i;}{-1,4,6;},$$

meaning
$$\frac{2i;}{1,6,9;} - \frac{i-1;}{1,4,6;}.$$

If a system is affected with composite or repeated groups on *each* side of the cæsura, its denumeration may be made to depend on systems where such groups exist on only one side of their respective cæsuras*.

Example:—
$$\left.\begin{array}{r} 10x + 2y + 3z = 5i \\ 15x + 4y + 9z = 11i \end{array}\right\}, \text{ which call } S.$$

If we form a ternary system as follows:—

$$\left.\begin{array}{r} 10x + 2y + 3z \quad = 5i \\ 15x + 4y + 9z \quad = 11i \\ px + qy + rz - t = -m \end{array}\right\}, \text{ which call } S',$$

* In certain special cases the composite groups may be reduced in number by substituting a connective of the equations in lieu of one of them, as in the example

$$\left.\begin{array}{r} 10t - 7z - 8y = 5i \\ 9z + 2y = 7i \end{array}\right\},$$

which is apparently irreducible, but which, put under the equivalent form

$$\left.\begin{array}{r} 10t - 7z - 8y = 5i \\ 15t - 6z - 11y = 11i \end{array}\right\},$$

becomes simply-reducible.

where p, q, r, m are any positive integers whatever, it is apparent that the omni-positive solutions of S' may be found from the omni-positive solutions of S, and to each of the latter will correspond one, and only one, of the former. Hence the denumerant of S is the same as the denumerant of S', and, if p, q, r are so chosen that $10, 15, p$; $2, 4, q$; $3, 9, r$ are all prime groups, $\mathfrak{C}S'$, and therefore $\mathfrak{C}S$, may be made to depend on the denumerants of a certain set of new binary systems obtained by the eduction of S'. Thus let $p=1, q=1, r=1, m=0$, so that the auxiliary equation becomes

$$x+y+z-t=0.$$

R_xS' may be represented by

$$\left.\begin{aligned}10t-\ 8y-7z&=\ 5i\\15t-11y-6z&=11i\end{aligned}\right\},$$

R_yS' by

$$\left.\begin{aligned}2t+\ 8x+\ z&=\ 5i\\4t+11x+5z&=11i\end{aligned}\right\},$$

R_zS' by

$$\left.\begin{aligned}3t+\ 7x-\ y&=\ 5i\\9t+\ 6x-5y&=11i\end{aligned}\right\},$$

R_tS' by

$$\left.\begin{aligned}10x+\ 2y+3z&=\ 5i\\15x+\ 4y+9z&=11i\end{aligned}\right\}, \text{ being the original system } S.$$

It will be seen therefore that

$$R_xS';\quad \overset{\overline{x}}{\underset{x}{=}}R_yS';\quad \overset{\overline{y}}{\underset{y}{=}}\,.\,\overset{\overline{x}}{\underset{x}{=}}R_zS';\quad\text{and}\quad \overset{\overline{z}}{\underset{z}{=}}\,.\,\overset{\overline{y}}{\underset{y}{=}}\,.\,\overset{\overline{x}}{\underset{x}{=}}R_tS'$$

respectively represent the systems following:—

$$\left.\begin{aligned}10t-\ 8y-7z&=\ 5i\\15t-11y-6z&=11i\end{aligned}\right\}, \qquad (1)$$

$$\left.\begin{aligned}2t+\ z-\ 8x&=\ 5i+\ 8\\4t+5z-11x&=11i+11\end{aligned}\right\}, \qquad (2)$$

$$\left.\begin{aligned}3t+\ y-7x&=\ 5i+6\\9t+5y-6x&=11i+1\end{aligned}\right\}, \qquad (3)$$

$$\left.\begin{aligned}-10x-2y-3z&=\ 5i+15\\-15x-4y-9z&=11i+28\end{aligned}\right\}. \qquad (4)$$

All these four systems are definite: in the first of them the natural order of the groups is

$$\begin{pmatrix}10 & -7 & -\ 8\\15 & -6 & -11\end{pmatrix};$$

in the others the natural order is that in which they are written. The first two only will be definite-positive, the last will be definite-negative, and the

last but one neuter if $i>0$, negative if $i=0$, and the denumerant required will be the difference between the denumerants of the two systems

$$\left.\begin{aligned}10p-7q-\ 8r&=\ 5i\\15p-6q-11r&=11i\end{aligned}\right\},$$

$$\left.\begin{aligned}2p+\ q-\ 8r&=\ 5i+\ 8\\4p+5q-11r&=11i+11\end{aligned}\right\}.$$

In each of these systems there is one, but only one, of the original non-prime groups, and no new ones have been introduced.

Consequently they admit of being depressed, and the final result will be an aggregate of simple denumerants.

If we had applied to S' a different course of eduction as follows:—

$$R_zS',\quad \overset{\overline{z}}{\underset{z}{=}}R_yS',\quad \overset{\overline{z}}{\underset{z}{=}}\overset{\overline{y}}{\underset{y}{=}}R_xS',\quad \overset{\overline{z}}{\underset{z}{=}}\overset{\overline{y}}{\underset{y}{=}}\overset{\overline{x}}{\underset{x}{=}}R_tS',$$

it may easily be seen that all these systems likewise would be definite, and only the first of them definite-positive. Hence a second solution of the question will be $\mathfrak{C}R_zS'$, that is,

$$\frac{5i,\ 11i;}{3, 9;\ 7, 6;\ -1, -5;}\,,$$

which is of a depressible form, there being only one *affected* group (3, 9), and may be educed into a linear function of simple denumerants.

Dispersion process defined.

Cases which resist its application.

Theorem. The denumeration of any equation-system whatever may be made to depend upon the denumeration of systems that shall contain no composite groups, and at most only one set of repeated groups, and which will consequently be depressible.

Proof of this theorem in case of binary systems.

Definition of meaning of kG, and of $kG \pm lG'$, where G, G' represent any two coefficient groups of a system, and k, l are any two integers*.

Lemma 1. A system S, containing the coefficient group G, may be made to depend for its denumeration upon systems in each of which the coefficient groups are the same as in S, except that kG takes the place of G.

Lemma 2. A system S, containing the groups G and G', may be made to depend upon two systems, in one of which the coefficient groups are the same as in G, with the exception that H replaces G, and in the other the

* In a definite ternary system, where all the coefficient groups are prime groups, it may be shown that the only possible cases of syzygy are where $F=G$ or $F+G=H$ (F, G, H denoting coefficient groups of the system).

same as in G, with the exception that H replaces G', where H is $G-G'$ or $G'-G$.

Note that, if, instead of the coefficient group G in any definite system, first any other group H and then $-H$ be substituted, one *at least* of these substitutions must leave the deformed system definite.

Lemma 3 (*Corollary to Lemma* 1). Any equation-system may be made to depend for its denumeration on equation-systems in each of which one of the equations has all its coefficients positive units.

It follows from this lemma that, if a binary equation-system is free from syzygy (that is, from equalities of ratios between the coefficients of different variables), its denumeration may be made to depend upon that of systems which [their coefficient groups being all different and of the form $(a, 1)$] are per-reducible. But, if there be e sets of syzygies in the given system, there will be e sets of repetitions in the groups $(a, 1)$ in each of the deduced systems.

Lemma 4. In the case immediately above supposed, the e sets of syzygetic groups in the deduced systems may be replaced by e other syzygetic sets of groups of which all but one are of the form $(a, 1), (a, 1), \ldots; (b, 1), (b, 1), (b, 1), \ldots$, &c., and that one of the form $(\sigma, 0); (\sigma, 0); (\sigma, 0), \ldots$.

Lemma 5. Any system of the form last supposed may (by virtue of Lemma 2) be replaced by two, in one of which $\pm(a-k\sigma, 1)$ takes the place of $(a, 1)$, and in the other $\pm(a-k\sigma, 1)$ takes the place of $(\sigma, 0)$, k being so chosen that $(a-k\sigma, 1)$ is distinct from every other coefficient group associated with it in the same system.

Lemma 6. Hence, by *repeated* application of this last process of replacement, the number of syzygetic groups in the deduced systems may be continually reduced until we arrive at systems in one class of which all the groups $(\sigma, 0)$ have disappeared, and in the other class of which all the syzygetic groups except those of the form $(\sigma, 0)$ have disappeared.

Lemma 7. Hence, so long as e is greater than 1, the deduced systems will eventually none of them contain more than $(e-1)$ sets of groups in syzygy, and thus we must eventually arrive at systems in none of which will be found more than a single set of groups in syzygy, which may be taken indifferently of the form $(a, 1)$ or $(a, 0)$.

Consequently the denumeration of every binary system, if free of syzygies, may be made to depend on the denumeration of per-reducible systems; and, if not free of syzygies, on the denumeration of simply reducible systems.

A similar demonstration may be extended to systems of a higher order than the second. Consequently every denumerant of an order higher than the first may be made to depend on denumerants of a lower order, and eventually upon simple denumerants.

Examples of Reduction of Persyzygetic Systems.

Let the given system S be

$$\left.\begin{aligned} x+y \qquad &= m \\ z+t &= n \end{aligned}\right\},$$

and suppose m not less than n. Making

$$x+z-u=0,$$

we obtain the systems

$$\left.\begin{aligned} u+y-z \qquad &= m \\ z+t &= n \end{aligned}\right\},$$

which is $R_z S'$, say T; and

$$\left.\begin{aligned} u+t+x \qquad &= (n-1) \\ x+y &= m \end{aligned}\right\},$$

which is $\overset{\bar{x}}{\underset{x}{=}} R_z S'$, say U.

Since $\overset{\bar{x}}{\underset{x}{=}}\overset{\bar{z}}{\underset{z}{=}} S'$ contains the equation

$$-x-z-n=2,$$

the eduction is complete, and the required denumerant $= \text{Ɑ}T - \text{Ɑ}U$.

T arranged in natural order becomes

$$\begin{aligned} z+t \qquad\quad &= n, \\ -z \qquad +u+y &= m, \end{aligned}$$

the cæsura falling between t and u. Accordingly we obtain

$$\text{Ɑ}T = \frac{n+m;}{1, 1, 1;} + \frac{m;}{-1, 1, 1;} = \frac{n+m;}{1, 1, 1;} - \frac{m-1;}{1, 1, 1;}$$

$$= \frac{(n+m+1)(n+m+2)}{2} - \frac{m(m+1)}{2}.$$

In like manner

$$\text{Ɑ}U = \frac{n-1;}{1, 1, 1;} - \frac{n-1-m;}{-1, 1, 1;} = \frac{n-1;}{1, 1, 1;} = \frac{n(n+1)}{2},$$

for $n-1-m$ is negative. And we have

$$\text{Ɑ}S = \frac{(n+m+1)(n+m+2)}{2} - \frac{m(m+1)}{2} - \frac{n(n+1)}{2}$$

$$= \frac{2nm+2m+2n+2}{2} = (m+1)(n+1),$$

which is evidently the correct answer, being the product of the denumerants of the two given equations taken independently.

Second Example:— $$\left.\begin{array}{r} x+y+\theta \qquad\quad = m \\ \theta+z+t=n \end{array}\right\}.$$

By the same method as above, we obtain

$$\text{Ɑ}S = \text{Ɑ}T - \text{Ɑ}U,$$

where T is $$\left.\begin{array}{r} z+t+\theta \qquad\quad = n \\ -z \quad +\theta+u+y=m \end{array}\right\},$$

and U is $$\left.\begin{array}{l} u+t+\theta+x \quad = n-1 \\ \qquad \theta+x+y=m \end{array}\right\},$$

the cæsura in T falling between θ and u, and in U between x and y. Hence

$$\text{Ɑ}T = \frac{m+n\,;}{1,1,1,1\,;} - \frac{m-1\,;}{1,1,1,1\,;} + \frac{m-n-3\,;}{2,1,1,1\,;},$$

and $$\text{Ɑ}U = \frac{n-1\,;}{1,1,1,1\,;}.$$

Thus $$\text{Ɑ}S = \frac{m+n\,;}{1,1,1,1\,;} + \frac{m-n-3\,;}{1,1,1,1\,;} - \frac{m-1\,;}{1,1,1,1\,;} - \frac{n-1\,;}{1,1,1,1\,;}\,^{*}.$$

Example of a composite group and a syzygy falling on opposite sides of the cæsura. *Problem*:—To express the residue of q in respect to p as a linear function of simple denumerants.

If we call x the required residue, we have

$$x+py=q, \quad x<p,$$

or

$$x+py=q,$$
$$x+z \;=p-1.$$

Hence the required residue is the denumerant of the system

$$py+t+u \quad = q-1,$$
$$t+u+z=p-2,$$

in which the coefficient groups are in natural sequence.

In its present form the system is irreducible, because the cæsura falls between y and t (observe that p, 0 is a non-prime group); but, by the method above given, the denumerant of this system, by virtue of the subsidiary equation

$$y+t=v,$$

becomes the difference between the denumerants of

$$\left.\begin{array}{l} (1-p)\,t \quad +u+pv=q-1 \\ \qquad t+z+u \quad\; = p-2 \end{array}\right\}$$

* The value of this expression will evidently be the sum

$$(m+1)(n+1)+mn+(m-1)(n-1)+\&c.+(m-n+1),$$

which is $$\left(m+1-\frac{n}{3}\right)\frac{(n+1)(n+2)}{2}.$$

and of $$\left.\begin{array}{l} (1-p)\,y+v+u+t=(q-1)+(p-1)=q+p-2 \\ \qquad\quad y+z+u+t=(p-2)-1 \qquad\;\; = p-3 \end{array}\right\}.$$

The second system is neuter, for all the coefficient groups put in apposition with the constant group give determinants with negative values.

Hence the required expression is simply the denumerant of the first system, in which the cæsura falls between u and v. $(p, 0)$ being a composite group, the eduction must be commenced from the t side, and accordingly we obtain the series

$$\frac{(q-1)+(p-1)(p-2)\,;}{(p-1),\,p,\,p\,;} + \frac{(q-1)\,;}{-(p-1),\,1,\,p\,;} + \frac{q-p+1\,;}{-p,\,-1,\,p\,;}$$
$$= \frac{p^2-3p+q+1\,;}{p-1,\,p,\,p\,;} - \frac{q-p\,;}{1,\,p-1,\,p\,;} + \frac{q-2p\,;}{1,\,p,\,p\,;}$$

as the expression required for the residue of q in respect to p.

SIXTH LECTURE*.

SIMPLE PARTITION.

Resolution of an integer into a defined number of parts.

With or without repetition.

$\dfrac{n-r\,;}{1,2,3,\ldots r\,;}$ expresses the r-ary partibility of n when repetitions are allowed;

$\dfrac{n-r\frac{r+1}{2}\,;}{1,2,3,\ldots r\,;}$ the same when repetitions are excluded.

Example: $n=7$, $r=3$.

Proof of the above formulæ by Ferrers' method.

When n is great compared with r, these two functions approach to a ratio of equality.

The generating function for partitions without repetition.

Indefinite resolution of numbers with and without repetition.

Generating functions for both these kinds of indefinite resolutions.

Euler's *Series Mirabilis*, and its application.

Remark on indefinite *partition* with the elements 1, 2, 4, 8, &c.

Partition or composition. Partible number. Elements.

* Delivered at King's College, London, July 4th, 1859.

Construction of equation-system whose denumerant is the number of compositions of n with unrepeated elements.

The negation of the *possibility* (for integers) of the equation $x^i + y^i = z^i$ capable of being transformed into the affirmation of an analytical *identity* by the method of partitions.

Resolution of integers into a given number of parts, how treated by Sir John Herschel and others.

Formula of reduction $\dfrac{n;}{1, 2, \ldots k;} = \dfrac{n-k;}{1, 2, \ldots k;} + \dfrac{n-k;}{1, 2, \ldots (k-1);}$.

Objections to this method:—(1) As inductive instead of direct (besides being limited to a mere special case of partition). (2) As excessively prolix and unmanageable. (3) As leading to an amorphous result.

Mr Cayley's improved method. The true form of representation.

The lecturer's discovery of the general analytical solution.

The provisional method superseded.

Fundamental Theorem in Simple Partition.

Axiom:— $$\frac{n;}{a, b, c \ldots l;} = \frac{1}{abc \ldots l} \Sigma H_n(\alpha, \beta, \gamma, \ldots \lambda),$$

where $H_n(\alpha, \beta, \gamma, \ldots \lambda)$ indicates the sum of the homogeneous powers and products of $\alpha, \beta, \gamma, \ldots \lambda$ of the nth degree, and $\alpha, \beta, \gamma, \ldots \lambda$ are respectively roots of

$$x^a = 1,\ y^b = 1,\ z^c = 1, \ldots w^l = 1.$$

Example:— $$\frac{7;}{2, 3;} = \tfrac{1}{6}\Sigma(\rho^7 + \rho^6\sigma + \ldots + \sigma^7),$$

where Σ includes six sums corresponding to the following six systems of values ρ, σ; namely,

$$1, 1;\ -1, 1;\ 1, \rho;\ -1, \rho;\ 1, \rho^2;\ -1, \rho^2,$$

ρ meaning a root of $$\rho^2 + \rho + 1 = 0.$$

In general,

$$H_n(p, q) = \frac{p^{n+1}}{p-q} + \frac{q^{n+1}}{q-p},$$

$$H_n(p, q, r) = \frac{p^{n+2}}{(p-q)(p-r)} + \frac{q^{n+2}}{(q-p)(q-r)} + \frac{r^{n+2}}{(r-p)(r-q)},$$

$$H_n(p, q, r, s) = \frac{p^{n+3}}{(p-q)(p-r)(p-s)} + \&c. + \&c. + \&c.$$

In applying this formula to the preceding axiom, several or all of the quantities p, q, r, &c., will become equal *inter se*, because the equations $a^\alpha = 1$, $b^\beta = 1$, $c^\gamma = 1, \ldots$ have the root unity in common, and will have other roots in common unless $\alpha, \beta, \gamma, \ldots$ are all prime to each other.

The value of $$\Sigma \frac{\phi p_1}{(p_1 - p_2)(p_1 - p_3) \ldots (p_1 - p_{e+1})},$$

when $$p_1 = p_2 = \ldots p_{e+1},$$

is $$\frac{1}{1.2.3 \ldots e}\left(\frac{d}{dp}\right)^e \phi p.$$

Every distinct root of $x^m = 1$, where m is the least common multiple of $a, b, c, \ldots l$, furnishes a distinct expression to the sum and gives rise to a separate term, in the complete analytical expression for $\dfrac{n;}{a, b, c, \ldots l;}$. Such a term is called a wave. Reason for this name.

There are as many waves as distinct factors in $a, b, c, \ldots l$; every such factor as q giving rise to a term W_q.

If $a, b, c, \ldots l$ become the series of natural numbers $1, 2, 3, \ldots r$, the number of waves is r.

The value* of W_q for $\dfrac{n;}{a, b, c, \ldots l;}$ is the coefficient of $\dfrac{1}{t}$ in

$$\frac{1}{abc \ldots l} \Sigma \frac{(\rho e^t)^n}{[1 - (\rho e^t)^{-a}][1 - (\rho e^t)^{-b}] \ldots [1 - (\rho e^t)^{-l}]},$$

where ρ is a *primitive* root of $\rho^q = 1$, that is, a root not belonging to $\rho^{q/i} = 1$.

The only cases where the quantity under the sign of summation reduces to a single term is when $q = 1$, for which case $\rho = 1$, and when $q = 2$, for which case $\rho = -1$.

W_1 considered. It is non-periodic. It is the coefficient of t^n in the development of $\dfrac{\phi(t)}{(1-t)^i}$, when the Eulerian function

$$\frac{1}{(1-t^a)(1-t^b) \ldots (1-t^l)}$$

is supposed capable of being represented under the form

$$\frac{\phi t}{(1-t)^i} + \frac{\psi t}{\dfrac{1-t^a}{1-t}\,\dfrac{1-t^b}{1-t} \cdots \dfrac{1-t^l}{1-t}}.$$

It is also the *mean* of the m algebraical forms, m being the least common

[* Cf. p. 91 above.]

multiple of $a, b, c, \ldots l$, which represent $\frac{n\,;}{a, b, c, \ldots l\,;}$ when n is made to go through the m forms

$$\frac{km\,;}{a, b, c, \ldots l\,;},\ \frac{km+1\,;}{a, b, c, \ldots l\,;},\ \ldots\ \frac{km+(m-1)\,;}{a, b, c, \ldots l\,;}.$$

It is therefore the mean value of $\frac{n\,;}{a, b, c, \ldots l\,;}$.

W_2 is $(-)^n B$, where B is the coefficient of $\frac{1}{t}$ in

$$\frac{1}{a\,b\,c \ldots kl}\,\frac{1}{(1-e^{-at})(1-e^{-bt})\ldots(1+e^{-kt})(1+e^{-lt})},$$

$a, b, \ldots$ being the odd, and $\ldots k, l$ the even, integers among $a, b, c, \ldots k, l$.

If the first wave is A, $A+B$ will be the mean of $\frac{n\,;}{a, b, c, \ldots l\,;}$ for even values of n, and $A-B$ the mean of the same for odd values of n.

Observe that the degree in n of W_1 is one unit less than the number of the elements $a, b, c, \ldots l$; in the algebraical part of W_2 is one unit less than the number of even elements among $a, b, c, \ldots l$, and in general W_q is one unit less than the number of elements which contain q as a factor.

Provisional notation co_{-1}, co_r explained.

The equations

$$co_l\,\phi(t) = co_{l+\omega}\,[t^{\omega}\,\phi(t)],\quad co_{2t}\,\phi(t^2) = co_t\,\phi(T)$$

identically true.

Mode of developing

$$\frac{(\rho e^t)^n}{[1-(\rho e^t)^{-a}][1-(\rho e^t)^{-b}]\ldots \text{to } i \text{ terms}},$$

under the form $\rho^n . e^{nt-R}$; where

$$R = \Sigma \log\,[1-(\rho e^t)^{-a}].$$

This an essential part of the theorem.

The expression for $\frac{1}{1-ke^u}$ in terms of u being known, $\log(1-ke^u)$ is known by integration from the identity

$$\frac{d}{du}\log(1-ke^u) = \frac{-ke^u}{1-ke^u} = 1 - \frac{1}{1-ke^u},$$

so that in the first and second waves the only numerical constants to be determined are the numbers of Bernouilli.

Thus when $\rho = 1$, corresponding to W_1, R becomes

$$\Sigma\left\{\log(at) - \frac{at}{2} + \frac{B_1}{2^2}a^2t^2 - \frac{B_2}{2.3.4^2}a^4t^4 + \frac{B_3}{2.3.4.5.6^2}a^6t^6 - \&\text{c.}\right\}$$

$$= \log(a_1a_2\ldots a_i) + i\log t - \tfrac{1}{2}\Sigma a\,.\,t + \frac{B_1}{2^2}(\Sigma a^2)\,t^2 - \frac{B_2}{2.3.4^2}(\Sigma a^4)\,t^4 \pm \&\text{c.},$$

so that $nt - R$ becomes

$$-\log(a_1a_2\ldots a_i) - i\log t + \left\{(n+\tfrac{1}{2}\Sigma a)\,t - \frac{B_1}{2^2}(\Sigma a^2)\,t^2 \pm \&\text{c.}\right\},$$

and

$$e^{nt-R} = \frac{t^{-i}}{a_1a_2\ldots a_i}\,e^{\nu t-(B_1/2^2)s_2t^2+\&\text{c.}};$$

and, finally, $W_1 = \frac{1}{a_1a_2\ldots a_i}\,co_{i-1}\left\{e^{\nu t - \frac{1}{24}s_2t^2 + \frac{1}{2880}s_4t^4 - \frac{1}{181440}s_6t^6 \pm \&\text{c.}}\right\}$;

where

$$\nu = n + \tfrac{1}{2}\Sigma a.$$

In like manner, when $\rho = -1$, corresponding to W_2, if $a_1, a_2, \ldots a_e$ are the even, and $b_1, b_2, \ldots b_\omega$ the odd, elements,

$$R = \Sigma\log(1-e^{-at}) + \Sigma\log(1+e^{-bt}),$$

and we shall obtain

$$W_2 = \frac{(-)^n}{2^{\omega}a_1a_2\ldots a_e}\,co_{e-1}\left\{e^{\nu t - \frac{1}{24}(s_2+3\sigma_2) + \frac{1}{2880}(s_4+15\sigma_4) \pm \&\text{c.}}\right\},$$

where $\nu = n + \tfrac{1}{2}(a_1+a_2+\ldots+a_e+b_1+b_2+\ldots+b_\omega)$,

$$s_l = \Sigma a^l,\quad \sigma_l = \Sigma b^l.$$

Calculation of Mean Values for any given Number of Elements.

Example 1. To find the mean value of $\frac{n\,;}{a, b, c, d\,;}$.

This will be the coefficient of t^3 in

$$\frac{1}{abcd}\left\{e^{\nu t}\times e^{-\frac{1}{24}s_2t^2}\right\}$$

$$= \frac{1}{abcd}\,co_{+3}\left\{\begin{matrix}1+\nu t+\dfrac{\nu^2t^2}{1.2}+\dfrac{\nu^3t^3}{1.2.3}+\ldots\\ \times\\ 1-\dfrac{1}{24}s_2t^2\end{matrix}\right\}$$

$$= \frac{\nu}{abcd}\,co_{+1}\left\{\left(1+\frac{\nu^2t}{1.2.3}\right)\left(1-\frac{1}{24}s_2t_2\right)\right\}$$

$$= \frac{\nu}{abcd}\left\{\frac{\nu^2}{6}-\frac{s_2}{24}\right\}.$$

Example 2. To find the mean value of $\dfrac{n;}{a, b, c, d, e;}$.

This will be the coefficient of t^4 in

$$\frac{1}{abcde}\{e^{\nu t} \times e^{\frac{1}{24}s_2t^2} \times e^{\frac{1}{2880}s_4t^4}\}$$

$$= \frac{1}{abcde} co_{+2} \left\{ \begin{array}{l} 1 + \dfrac{\nu^2}{1 . 2} \quad t + \dfrac{\nu^4}{1 . 2 . 3 . 4} t^2 \\ \quad \times \\ 1 - \dfrac{1}{24} \quad s_2 t + \dfrac{1}{1152} s_2^2 t^2 \\ \quad \times \\ 1 + \dfrac{1}{2880} s_4 t^2 \end{array} \right\}$$

$$= \frac{1}{abcde}\left\{\frac{\nu^4}{24} - \frac{s_2}{24}\nu^2 + \left(\frac{s_2^2}{1152} + \frac{s_4}{2880}\right)\right\}.$$

Examples of Arithmetical Calculation of Simple Denumerants.

Example 1. To find the complete expression for $\dfrac{n;}{1, 2, 3;}$.

$$W_1 = \frac{1}{1 . 2 . 3} co_2 \{e^{(n+3)t} \times e^{-\frac{14}{24}t^2}\}$$

$$= \frac{1}{1 . 2 . 3} co_1 \left\{1 + \frac{(n+3)^2}{2} t\right\} \left\{1 - \frac{14}{24} t\right\} \text{ (since } 1 + 4 + 9 = 14)$$

$$= \frac{1}{12}\left\{(n+3)^2 - \frac{7}{12}\right\};$$

$$W_2 = co_{-1} \frac{(-)^n e^{nt}}{(1 + e^{-t})(1 + e^{-2t})(1 - e^{-3t})}$$

$$= \frac{(-)^n 1}{8} + \frac{1}{8}\left\{\frac{n;}{2;} - \frac{n-1;}{2;}\right\},$$

$$W_3 = co_{-1} \Sigma \frac{\rho^n . e^{nt}}{(1 - \rho^{-1}e^{-t})(1 - \rho^{-2}e^{-2t})(1 - e^{-3t})} \text{ (where } \rho^2 + \rho + 1 = 0)$$

$$= \frac{1}{3}\left\{\frac{\rho^n}{(1-\rho)(1-\rho^2)} + \frac{\rho'^n}{(1-\rho')(1-\rho'^2)}\right\}$$

$$= \frac{1}{9}(\rho^n + \rho'^n)$$

$$= \frac{1}{9}\left\{2\frac{n;}{3;} - \frac{n-1;}{3;} - \frac{n+1;}{3;}\right\}.$$

Thus the complete analytical value of $\dfrac{n;}{1, 2, 3;}$ is

$$\frac{(n+3)^2}{12} - \frac{7}{144} + \frac{1}{8}\left\{\frac{n;}{2;} - \frac{n-1;}{2;}\right\} + \frac{1}{9}\left\{2\frac{n;}{3;} - \frac{n-1;}{3;} - \frac{n+1;}{3;}\right\}.$$

Since $$\frac{7}{144} + \frac{1}{8} + \frac{2}{9} = \frac{57}{144} < \frac{1}{2},$$

the arithmetical value of $\dfrac{n;}{1, 2, 3;}$ is the nearest integer to $\dfrac{(n+3)^2}{12}$, as had been early observed under a different form of statement by Mr De Morgan.

Example 2. The same process applied to $\dfrac{n;}{1, 4, 7;}$ will give

$$W_1 = \frac{1}{2 \times 4 \times 7}\left\{\left(n + \frac{1+4+7}{2}\right)^2 - \frac{1+16+49}{24}\right\}$$

$$= \frac{1}{56}\left\{(n+6)^2 - \frac{11}{4}\right\},$$

$$W_2 = \frac{(-)^n}{2^2 . 4} = \frac{1}{16}\left\{\frac{n;}{2;} - \frac{n-1;}{2;}\right\},$$

$$W_4 = co_{-1} \Sigma \frac{i^n e^{nt}}{(1 - i^{-1}e^{-t})(1 - i^{-7}e^{-7t})(1 - e^{-4t})} \text{ (where } i^2 + 1 = 0)$$

$$= \frac{1}{4}\left\{\frac{i^n}{(1-i^3)(1-i)} + \frac{i'^n}{(1-i'^3)(1-i')}\right\}$$

$$= \frac{1}{8}\{i^n + i'^n\}$$

$$= \frac{1}{8}\left\{2\frac{n;}{4;} - 2\frac{n-2;}{4;}\right\}$$

$$= \frac{1}{4}\left\{\frac{n;}{4;} - \frac{n-2;}{4;}\right\}.$$

$$W_7 = co_{-1} \Sigma \frac{\theta^n e^{nt}}{(1 - \theta^6 e^{-t})(1 - \theta^3 e^{-nt})(1 - e^{-7t})}$$

$$\text{(where } \theta^6 + \theta^5 + \theta^4 + \theta^3 + \theta^2 + \theta + 1 = 0)$$

$$= \frac{1}{7}\Sigma \frac{\theta^n}{(1-\theta^3)(1-\theta^6)}$$

$$= \frac{1}{49}\Sigma \theta^n (1-\theta)(1-\theta^2)(1-\theta^4)(1-\theta^5)$$

$$= \frac{1}{49}\{\theta^n + \theta^{n+5} - 2\theta^{n+1} - 2\theta^{n+6}\}$$

$$= \frac{1}{7}\left\{\frac{n;}{7;} - 2\frac{n+1;}{7;} - 2\frac{n+4;}{7;} + \frac{n+5;}{7;} + 2\frac{n+6;}{7;}\right\}.$$

Thus the complete value of $\frac{n;}{1, 4, 7;}$, expressed in terms exclusively of

$$\nu = n + 6,$$

is the following:—

$$\frac{1}{56}\left\{\nu^2 - \frac{11}{4}\right\} + \frac{1}{16}\left\{\frac{\nu;}{2;} - \frac{\nu-1;}{2;}\right\} + \frac{1}{4}\left\{\frac{\nu-2;}{4;} - \frac{\nu;}{4;}\right\}$$
$$+ \frac{1}{7}\left\{2\frac{\nu;}{7;} + \left(\frac{\nu+1;}{7;} + \frac{\nu-1;}{7;}\right) - \left(2\frac{\nu+2;}{7;} + 2\frac{\nu-2;}{7;}\right)\right\}.$$

The limiting values of the sum of the second, third, and fourth waves for any value of n will be

$$\frac{1}{16} + \frac{1}{4} + \frac{2}{7} = \frac{67}{112}$$

on the positive side, and

$$+\frac{1}{16} - \frac{1}{4} - \frac{2}{7} = \frac{-53}{112}$$

on the negative side.

Hence the difference between the exact value and $\frac{\nu^2}{56}$ must lie between $\frac{123}{224}$ and $\frac{-117}{224}$.

So that in the greatest number of cases the *nearest* integer to $\frac{(n+6)^2}{56}$ gives the value of $\frac{n;}{1, 4, 7;}$; and the result can never be in error by more than a single unit.

An analogous approximate form of representation can be made for the number of modes of composing an integer with any number of elements mutually prime to each other.

Observe in the foregoing expression that the form $\frac{\nu+i;}{q;}$ is always paired with $\frac{\nu-i;}{q;}$: and $\frac{\nu;}{q;}$ $\left(\text{which in the actual case under consideration is } \frac{\nu;}{7;}\right)$ affords no exception, for this may be expressed as

$$\frac{1}{2}\left\{\frac{\nu+0;}{q;} + \frac{\nu-0;}{q;}\right\}.$$

Neither does $\frac{\nu-r;}{2r;}$ $\left(\text{in the actual case } \frac{\nu-1;}{2;}\right)$, for this may be expressed as

$$\frac{1}{2}\left\{\frac{\nu-r;}{2r;} + \frac{\nu+r;}{2r;}\right\}.$$

The sign of the pairing may be positive or negative according to a rule which the exhibition of the result worked out in the following examples will render clear.

Example 3. The denumerant $\frac{n;}{1, 3, 5;}$, expanded in a similar manner, gives rise to the following expression:—

$$\frac{1}{30}\left\{(\nu)^2 - \frac{35}{24}\right\} + \frac{2}{9}\left\{\frac{\nu+\frac{3}{2};}{3;} + \frac{\nu-\frac{3}{2};}{3;} - \frac{\nu+\frac{1}{2};}{3;} - \frac{\nu-\frac{1}{2};}{3;}\right\}$$
$$+ \frac{1}{5}\left\{\frac{\nu+\frac{1}{2};}{5;} + \frac{\nu-\frac{1}{2};}{5;} - \frac{\nu+\frac{3}{2};}{5;} - \frac{\nu-\frac{3}{2};}{5;}\right\}.$$

And, since

$$\frac{2}{9} + \frac{1}{5} + \frac{7}{144} = \frac{330}{720} < \frac{1}{2},$$

the arithmetical value of $\frac{n;}{1, 3, 5;}$ is always the *nearest integer* to $\frac{(2n+9)^2}{120}$.

This arithmetical mode of statement, how capable of extension to any set of elements following the natural order of the prime numbers, and to other cases.

Example 4. The denumerant $\frac{n;}{1, 2, 3, 4, 5, 6, 7;}$ in its expanded form is expressed by the following function of ν, which here represents $n + 14$, namely,

$$\frac{1}{725760}\left\{\frac{\nu^6}{5} - 35\nu^4 + \frac{13419}{10}\nu^2 - \frac{190325}{126}\right\}$$
$$+ \frac{1}{768}\left\{\frac{\nu^2}{2} - \frac{77}{6}\right\}\left(\frac{\nu;}{2;} - \frac{\nu-1;}{2;}\right)$$
$$+ \frac{1}{162}\left\{\frac{\nu+1;}{3;} - \frac{\nu-1;}{3;}\right\}\nu - \frac{5}{972}\left\{\left(\frac{\nu;}{3;} + \frac{\nu;}{3;}\right) - \left(\frac{\nu-1;}{3;} + \frac{\nu+1;}{3;}\right)\right\}$$
$$+ \frac{1}{64}\left\{\left(\frac{\nu+2;}{4;} + \frac{\nu-2;}{4;}\right) - \left(\frac{\nu;}{4,} + \frac{\nu;}{4;}\right)\right\}$$
$$+ \frac{1}{25}\left\{\left(\frac{\nu+1;}{5;} + \frac{\nu-1;}{5;}\right) - \left(\frac{\nu+2;}{5;} + \frac{\nu-2;}{5;}\right)\right\}$$
$$+ \frac{1}{36}\left\{\left(\frac{\nu+3;}{6;} + \frac{\nu-3;}{6;}\right) - \left(\frac{\nu;}{6;} + \frac{\nu;}{6;}\right) + \left(\frac{\nu-2;}{6;} + \frac{\nu+2;}{6;}\right) - \left(\frac{\nu-1;}{6;} + \frac{\nu+1;}{6;}\right)\right\}$$
$$+ \frac{1}{49}\left\{3\left(\frac{\nu;}{7;} + \frac{\nu;}{7;}\right) - \left(\frac{\nu+1;}{7;} + \frac{\nu-1;}{7;}\right) - \left(\frac{\nu+2;}{7;} + \frac{\nu-2;}{7;}\right) - \left(\frac{\nu+3;}{7;} + \frac{\nu-3;}{7;}\right)\right\}^{*}.$$

Observation. Provided that $\frac{-p;}{i;}$ shall be understood to signify the same thing as $\frac{p;}{i;}$, every wave in the above expansion remains entirely unaltered when ν becomes $-\nu$.

* The arithmetical value of $\frac{\nu-14;}{1, 2, 3, 4, 5, 6, 7;}$ is obviously the nearest integer to

$$\frac{1}{5760}\left\{\frac{\nu^6}{5} - 35\nu^4 + \frac{13419}{10}\nu^2\right\} + \frac{1}{1536}\left(\frac{\nu;}{2;} - \frac{\nu-1;}{2;}\right)\nu^2 + \frac{1}{162}\left\{\frac{\nu+1;}{3;} - \frac{\nu-1;}{3;}\right\}\nu.$$

A priori view of the form of such expansions of $\frac{n\,;}{a, b, c, \ldots l\,;}$.

First, the algebraical part is, as it ought to be, a *homogeneous* function of $n\,;\ a, b, c, \ldots l$.

Secondly, the change of ν into $-\nu$ either leaves the expansion absolutely unaltered, or unaltered save as to algebraical sign.

This depends on the theory of the *denumerantive functions* as distinguishable from denumerants. The latter discontinuous quantities, the former continuous.

Binary denumerants have in general several functions attached to them, namely, one less than the number of their denominatives*.

All generated forms have arithmetical and functional values.

Example. The form u_n generated by

$$\frac{1}{(1-x)^2} = u_0 + u_1 x + \ldots u_n x^n + \&c.$$

Property of the denumerantive function $\phi(n, a, b, c, \ldots l)$ to $\frac{n\,;}{a, b, c, \ldots l\,;}$; namely, that

$$\phi(n, a, b, c, \ldots l) = \pm\,\phi(n', a, b, c, \ldots l),$$

if

$$n + n' = -a - b - c \ldots - l,$$

the + sign being used when the number of elements is odd, and the negative sign when it is even.

This explains the pairing of the terms observed in $\frac{n\,;}{1, 2, 3, \ldots 7\,;}$. Great importance of this fact of pairing.

The number of modes of dividing n into seven parts is represented by the above formula, namely, with repetitions and zero values of parts allowed by making

$$\nu = n + 14,$$

with repetitions and zero values disallowed by making

$$\nu = n - 14,$$

with repetitions allowed, but zero values disallowed, by making

$$\nu = n + 7.$$

And so in general with the values $n + \frac{(r-1)\,r}{4}$; $n - \frac{(r-1)\,r}{4}$; $n - r$ respectively substituted for ν.

Mr Kirkman's representation of partitions to the modulus 7.

* This is when the form zero is not counted as a function. Zero occurs as a form once only in simple, but twice over in binary denumerants.

Example 5. Expansion of $\frac{n\,:}{1, 2, 3, 4, 5, 6\,:}$ as a function of n.

$$\frac{1}{17280}\left\{\frac{\nu^5}{5} - \frac{91}{6}\nu^3 + \frac{9191}{48}\right\} + \frac{1}{768}\left\{\frac{\nu+\frac{1}{2}\,:}{2\,:} - \frac{\nu-\frac{1}{2}\,:}{2\,:}\right\}\left(\nu^2 - \frac{161}{12}\right)$$
$$+\frac{1}{162}\left\{\left[\left(\frac{\nu+\frac{3}{2}\,:}{3\,:} + \frac{\nu-\frac{3}{2}\,:}{3\,:}\right) - \left(\frac{\nu+\frac{1}{2}\,:}{3\,:} + \frac{\nu-\frac{1}{2}\,:}{3\,:}\right)\right]\nu + \left(\frac{\nu+\frac{1}{2}\,:}{3\,:} - \frac{\nu-\frac{1}{2}\,:}{3\,:}\right)\right\}$$
$$+\frac{1}{32}\left\{\left(\frac{\nu+\frac{3}{2}\,:}{4\,:} - \frac{\nu-\frac{3}{2}\,:}{4\,:}\right) + \left(\frac{\nu+\frac{1}{2}\,:}{4\,:} - \frac{\nu-\frac{1}{2}\,:}{4\,:}\right)\right\}$$
$$+\frac{1}{25}\left\{\left(\frac{\nu-\frac{1}{2}\,:}{4\,:} - \frac{\nu+\frac{1}{2}\,:}{4\,:}\right) + \left(\frac{\nu-\frac{3}{2}\,:}{4\,:} - \frac{\nu+\frac{3}{2}\,:}{4\,:}\right)\right\}$$
$$+\frac{1}{18}\left\{\left(\frac{\nu+\frac{3}{2}\,:}{6\,:} - \frac{\nu-\frac{3}{2}\,:}{6\,:}\right) + \frac{1}{36}\left(\frac{\nu+\frac{5}{2}\,:}{6\,:} - \frac{\nu-\frac{5}{2}\,:}{6\,:}\right) + \left(\frac{\nu+\frac{1}{2}\,:}{6\,:} - \frac{\nu-\frac{1}{2}\,:}{6\,:}\right)\right\},$$

where

$$\nu = n + \frac{21}{2}.$$

Observe the substitution of the colon for the semicolon above and below the line in the fraction-form to distinguish a denumerantive function from a denumerantive proper. The arithmetical value of the foregoing is the nearest integer to the sum of its first, second, and third waves; and in the two latter it is only necessary to retain those terms which contain ν^2 and ν respectively.

On the expression for the number of waves when the denominatives of a denumerant or the elements of a partition are given.

On the blending of waves, and its advantages in some cases, as when the elements are all prime to each other without all being absolute primes.

Easy mode of deducing the fundamental theorem, by the application of a formula in the calculus of residues to the Eulerian. Its capital importance in the theory of partitions.

Close of the analytical portion of the course.

SEVENTH LECTURE*.

The representation of systems of linear equations by clusters of points recalled.

A single equation by a cluster of points in a line, of two simultaneous equations by a cluster of points in a plane, three simultaneous equations by a cluster of points in space—geometrical criterion between definite and indefinite systems.

The linear cluster which corresponds to a single equation is unique in form, not so the plane cluster which corresponds to two, or the solid cluster

* Delivered at King's College, London, July 11th, 1859.

which corresponds to three equations. One arbitrary parameter enters into the former, three into the latter.

All such clusters balance about their respective origins with the same weights at corresponding points.

Clusters so related might be termed homobaric.

Mechanical representation of the property of homobarism by a series of jointed parallelograms having two axes in common (see Fig. *A*). [Plate I. at the end of the volume.]

Criterion between definite and indefinite systems recalled.

"To determine the chance that three points thrown anywhere within a parallelogram may contain the centre."

Solution of this problem by theory of definite and indefinite binary equation-systems with three variables alluded to.

The chance is $\frac{5}{32}$ths against the points including the centre, whatever the form or dimensions of the parallelogram.

Hence the chance is $\frac{5}{32}$ths against three points capable of being taken anywhere in an indefinite plane including a *given* fourth point in the same plane.

But it would be incorrect to infer from this that the chance of some three out of four points (capable of being taken anywhere in an indefinite plane), including the fourth, is $\frac{5}{8}$; it will be much less than this.

Explanation of this seeming paradox. Geometrical and analytical modes of treating this second question alluded to.

Experimental method of verification. New game of odd and even.

The natural order of the variables in a single homonymous equation recalled.

Unless definite there is no natural order.

The importance of obtaining such natural orders to the theory of compound partition, namely, in applying the process of eduction. Example of natural and disturbed order.

General *analytical* condition of normal sequence.

Difficulty of seeing any natural order among the rays of a solid cluster; it will presently appear that such orders do exist, but that instead of one natural order there are several; the number depending (1) upon the number of points in the cluster, (2) upon the mode in which the rays are grouped, subject to the observation that the distinct modes of grouping in the view of this theory are always limited in number, and determinable *a priori*.

Passage by perspective from the grouping of rays in a plane to the grouping of points in a line; from the grouping of rays *in solido* to the grouping of points in a plane; and from the grouping of rays in plu-space to the grouping of points *in solido*.

Observe that in studying the character of an equation-system no attention need in the first instance be paid to the primary, because the process of eduction concerns the variables only, and not the constant terms in the system; by the act of taking the perspective, the origin no longer appears—thus, nothing is left but a perspective cluster or group of points, as many in number as the variables of the system.

Theorem. The number of classes of definite binary systems of linear equations for any given number of variables is *one*, because there is but one species of arrangement of a given number of points in a line. The number of classes of definite ternary systems of equations with r variables is the number of distinct modes of grouping together r points in a plane; the number of classes of definite quaternary systems of equations with r variables is the number of distinct modes of grouping together r points in space.

Observe the fact of space being made subservient through the method of perspective to systems of linear equations greater in number than the so-called dimensions of space.

In determining the natural order in a binary system, the perspective group may be substituted for the cluster, provided the line of projection cuts all the rays on the same side of the origin, so that a line through the origin parallel to the line of projection falls *outside* the cluster; but, if this condition is not observed, the order will be disturbed. (See Fig. *B*.) [Plate I. at the end of the volume.]

In like manner, for a ternary system, the plane of projection must be supposed to be drawn parallel to a plane through the origin external to the solid cluster.

Observation on Plane and Solid Groups, considered as representing definite Ternary and Quaternary Equation-systems respectively. If we suppose a ternary system of which one of the equations is of the form

$$x + y + z + \dots + u - 1 = 0,$$

the others being

$$ax + by + cz + \dots + lu - m = 0,$$

$$\alpha x + \beta y + \gamma z + \dots + \lambda u - \mu = 0,$$

such a system may evidently be represented by a group of points in a plane whose coordinates are

$$(a, \alpha)\,(b, \beta)\,(c, \gamma) \dots (l, \lambda);\ (m, \mu),$$

and

$$x, y, z, \dots u;\ -1,$$

will be the weights to be placed at these points respectively in order to *balance* each other.

Moreover, if we start with any *definite* ternary system, we may substitute for one of the equations in it a homonymous equation reducible to the form of

$$x' + y' + z' + \ldots + u' - 1 = 0,$$

on taking $x', y', z', \ldots u',$

all homonymous multiples of

$$x, y, z, \ldots u.$$

Consequently, the *form* of the plexus of principal derivatives which depends essentially only on the relations of algebraical signs in the coefficients of these derivatives will be the same whether the system be considered as involving explicitly $x, y, z, \ldots u$, or $x', y', z', \ldots u'$, and, consequently, every definite ternary system of equations whatever may be represented in its essential properties of form by a plane group of points, in lieu of a solid cluster of rays. And in like manner, without going from hypersolidity to the solid, we see that any definite quaternary system may be represented by a group of points *in solido*.

The property indicated of the self-balancing group *in plano* being substitutable for the solid cluster with its centre of gravity at the origin may be deduced easily from this more general theorem, that if two groups of weighted points are in perspective about a given point G, and the weights at the corresponding points in the two groups are in the inverse ratios of their distances from G, if one of them has its centre of gravity at G, the other also will have its centre of gravity there. Hence, if one of these groups be considered as a derivative from the first, and all the points of the derivative group be brought to lie within the same plane, it must become self-balancing, since otherwise a plane group of statical points would have its centre of gravity outside the plane.

Notice that the geometrical construction for determining whether a system of equations is definite or indefinite would fail for a quaternary system, but the analytical method operative through the principal plexus continues to hold good.

Ternary Systems, and Plane Groups.

Imagine a sphere to be drawn with the origin of a solid cluster as its centre; the general arrangement of the points on the sphere will correspond to the arrangement of the points on the perspective plane, and, when convenient to do so, the one may be substituted for the other.

Illustration by examples with four and five points.

Recall eduction and the condition of its giving rise to definite systems, namely, that the systems deduced by the successive deformations of the given system shall remain definite, that is, external to the centre. If a group of

points on a sphere be contained within the boundary of a hemisphere, the centre will be external to such group; but, if the bounding contour of the group formed by arcs of great circles cover more than half the sphere, the centre will be contained within the group. The effect upon the spherical perspective of a cluster representing a ternary system of equations due to the change of a variable x into $-x$ is to make the point corresponding to the coefficients of x pass to the opposite end of the diameter passing through it; such a change may be termed a *reversal* of the point, and the point so obtained the opposite of the original point. The problem of normal orders for ternary systems may therefore be stated geometrically as follows:—

A given number of points being contained within a hemisphere, to discover what orders of sequence of these points will possess this property that on the first, second, third, &c., to the last of them, one after the other undergoing *reversal*, the transformed group shall never occupy more than half the surface of the sphere.

From this it follows that, if

$$xyz \ldots tuv$$

be a normal sequence, $vut \ldots xyz$

will be so likewise, for, if we denote the opposite points to

$$xyz \ldots uv \text{ by } x'y'z' \ldots u'v',$$

respectively, it is clear that, if the groups

$$\left.\begin{array}{l} x'yz \ \ldots uv \\ x'y'z \ldots uv \\ x'y'z' \ldots uv \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ x'y'z' \ldots u'v \end{array}\right\}$$

are respectively contained, each within their own hemispheres, the groups

$$\left.\begin{array}{l} v'u \ \ldots zyx \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ v'u' \ \ldots zyx \\ v'u' \ldots z'yx \\ v'u' \ldots z'y'x \end{array}\right\}$$

will each also be contained in hemispheres opposite to the former, taken in reverse order.

The *contour* of a spherical group defined.

What is meant by a peripheral and what by an internal point to a group on a sphere.

Again, to obtain the law of normal sequences, we have the following propositions:—

(1) Any sequence $x, y, z, t, \ldots u, v, w$ of points in a sphere will be a normal order of sequence, provided the following condition is satisfied, namely, that, on joining those points with each other in the order of their succession by arcs of great circles, the broken line or spherical zigzag so formed shall be capable at every one of its angular points of being divided into two parts by a great circle which does not cut the line at any other point; for evidently in such case, if we draw a great circle through a point u, which does not cut any of the sides, or any other angle except u, the points x, y, z, t being all reversed, will lie together with v, w in a hemisphere bounded by the great circle so drawn.

(2) A normal sequence of points in a group cannot be bounded at either extremity by an interior point of the group. For, on joining the opposite of such exterior point with the closed figure surrounding that point, we evidently obtain a figure clasping the hemisphere bounded by the great circle perpendicular to the diameter through these points, and stretching into the hemisphere beyond. On the other hand, a normal order may always be commenced from any peripheral point in the group at pleasure, for, if u, v, y, z, x, t be any group contained within a hemisphere H, and u' a point in the contour, it is apparent that u, v, y, z, x, t, u' will also be contained within the same hemisphere H, so that, in fact, one way of characterizing a normal sequence would be as a sequence in which each point in turn becomes a peripheral point, alike when all the points preceding as when all the points following it are *reversed*.

(3) No arc joining y, z, two consecutive points in a normal sequence $x, y, z, t, u, v, w, \omega$, can cut uv any other such arc, for it is clear that, if yz crosses uv, x will be contained within the triangle $y'uv$ (y' meaning the opposite point to y), and, consequently, z will not be external to $tuvw\omega x'y'$, as it must be if the given order is normal.

(4) It follows also, as an immediate corollary from 2, that no point t can be contained within the contour of xyz or within that of $uvw\omega$.

Hence (5) it follows from (3) and (4) that the two spherical areas bounded respectively by the contours of xyz, $uvw\omega$ have no part whatever in common, and, consequently, may be separated by a great circle drawn through t.

Hence, combining the conclusions of (1) and (5), we arrive at the theorem that the sole necessary and sufficient condition for determining $x, y, z, t, u, v, w, \omega$ to be in normal sequence upon a sphere is that through any point as t a great circle can be drawn upon the sphere not cutting this line in any other point; and, consequently, the sole necessary and sufficient condition

for a number of points in a plane group being in normal sequence is that the zigzag formed by drawing straight lines from any one point to the next in the sequence shall be capable of being cut in twain at any of its angles by a right line.

General definition of a diatomic line continuous or discontinuous in a *plane* or in *space* (see plate) [at the end of the volume].

The condition of normal sequence may be extended from plane to solid groups, that is, from ternary to quaternary equation-systems, the sole necessary and sufficient condition for determining a normal order of points, as well *in solido* as *in plano*, being that the zigzag following the succession of the points in the order shall be a *diatomic* line.

In order to depress a ternary system so as to make its denumerant depend upon binary denumerants, we must be able to form orders of normal sequence among its variables.

Every such order will or may furnish two distinct forms of solution, provided the requisite conditions of relative primeness and asyzygeticism are satisfied.

The easiest way of determining such normal orders is by means of diatomic lines drawn from point to point of the representative plane group.

Every ternary system may be identified by means of its principal plexus, as will presently be shown with some specific form of group, corresponding in number to the number of the variables in the system. It becomes necessary, therefore, to facilitate the solution of the problem of denumeration of ternary systems, to classify and register the distinct forms of arrangement of plane groups (and in like manner, in order to make denumerants of the fourth order depend upon those of the third, we must begin with classifying and registering the various dispositions of which a given number of points is susceptible in space).

Plane Groups.

For three points only one species of arrangement is possible, and all the orders are normal orders.

For four points two distinct arrangements only are possible, namely, of four points external to one another, or three points with a fourth point in the interior.

Morph defined—its geometrical and analytical meaning.

Exclusion of syzygetic cases.

The morph corresponding to the one case (see plate) [at the end of the volume] will be the following:—

$$xy: \quad yz: \quad zt: \quad tx:$$
$$x:z \quad y:t.$$

The morph corresponding to the other case (see plate) will be

$$\begin{array}{lll} zt: & xt: & yt: \\ y:z & z:x & x:y. \end{array}$$

Accordingly, the plexus of principal derivatives to every definite ternary system with four variables, omitting the constant terms, will be of one or the other of the two following forms:—

$$\left.\begin{array}{ll} ax+by=? & lx-mz=? \\ cy+dz=? & ny-pt=? \\ ez+ft=? & \\ gx+ht=? & \end{array}\right\}\text{ 1st form,}$$

$$\left.\begin{array}{ll} az+bt=? & \alpha y-\beta z=? \\ cx+dt=? & \gamma z-\delta x=? \\ ey+ft=? & \epsilon x-\phi y=? \end{array}\right\}\text{ 2nd form.}$$

The constants are of course not independent of each other, nor are the *signs* of the quantities left unexpressed, but symbolized by ?, independent. How to assign the laws of the dependence is not material to our immediate object.

By means of these two forms, we can by examination of the plexus of principal derivatives to a ternary system of linear equations with four variables, ascertain whether it corresponds to the form of group in Figs. 2, 3, or to that in Fig. 4.

The normal orders belonging to these groups respectively are shown in the plate [at the end of the volume].

If we proceed now to the groupings of five points, we shall find that the only essentially distinct arrangements are the following:—

Five points all external to each other.

Four points with one inside.

Three points with two inside.

No differences of any other kind than those above denoted will give rise to any difference in the character of the morph resulting. The three morphs corresponding to these three arrangements are the following:—

$$\left.\begin{array}{lllll} ztu: & tux: & uxy: & xyz: & yzt: \\ zt:x & tu:y & ux:z & xy:t & yz:u \end{array}\right\}\text{ 1st form,}$$

$$\left.\begin{array}{llll} ztu: & xtu: & xyu: & yzu: \\ & zu:x & tu:y & \\ yz:t & xt:z & xy:t & xy:z \end{array}\right\}\text{ 2nd form,}$$

$$\left.\begin{array}{lll} ztu: & xtu: & ytu: \\ uz:y & uz:x & yu:x \\ yt:z & xt:z & xt:y \\ & xz.:y & \end{array}\right\}\text{ 3rd form.}$$

And by simply observing whether the principal plexus has three, four, or five homonymous derivatives, the perspective representation of any definite ternary system of linear equations with five variables can be identified with one or the other of the Figures 5–7, 8–11, or 12–15. The diatomic zigzags expressing the normal orders to these figures are given in the plate.

So far we have found that the number of morphs has followed the progression 1; 2; 3; arising from the fact that there are no essential differences of position of a point or a couple of points within a triangle or of one point within a quadrilateral. Very different is the case for a figure of six points, corresponding to a definite ternary system, with six variables. The following cases arise:—

(1) The six points may be at the angles of a hexagon.

(2) Five of the six points may be at the angles of a pentagon, and the interior point may occupy any one of three essentially distinct regions, each such position giving rise to a distinct species of morph confined to the inner.

These three distinct kinds of position defined.

(3) Four of the points may be external, and the pair of points within occupy six distinct kinds of position.

These six different relative positions described.

Finally, three of the six points may be external, so that the figure may be viewed as a triangle within a triangle, and there will be six distinct relative positions of triangles so related.

These six different relative positions described.

There are thus in all sixteen different classes of arrangement of six points in a plane, and therefore sixteen different classes of ternary systems with six variables.

The normal orders for each of these sixteen cases have been completely worked out by Captain Noble, R.A., and the numbers for each figure attached to them in the plate (16–31).

On the arrangement of plane groups in natural families.

Classes belong to the same natural family which are capable of appearing simultaneously in the same course of the same process of eduction.

Any two classes of systems belong to the same family which may be obtained from one or other by altering the signs concurrently of one or more of such of the coefficient groups as may be so altered without the system ceasing to be definite. If in the morph of any class we make any letter pass from right to left of the colon, and *vice versa* throughout, and after such change the morph still contains characters of the form $xyz \ldots u$: corresponding to homonymous principal derivatives, the morph so obtained will belong to the same family with that from which it is derived. Thus from one morph all others of the same family may be derived by simple inspection and transposition.

Conversion defined.

Scales of derivation in general are divaricative, but for principal family are lineal.

Example four-, five-, and six-point systems.

How to determine *a priori* what letters in a given morph are convertible.

Examples in six-point systems.

The number of families for six-point arrangements is four.

The two classes of four-point systems, and the three classes of five-point systems, belong respectively to a single family. Proof.

Numerical and natural modes of classifying groups contrasted.

The principal class and principal family of any r-point group defined.

The tactical rule which serves to define any normal order in the principal class. Example in five-point system. Example in six-point system.

In four- and five-point systems all the classes belong to the principal family, there being no other.

The numerical system of arrangement in families gives rise to a new question in the partition of numbers.

Thus a seven-point system, and an eight-point system arranged after the numerical system, consist respectively of families which may be typified as follows:—

7 6, 1 5, 2 4, 3 3, 4 3, 3, 1

8 6, 2 5, 3 4, 4 4, 3, 1 3, 5 3, 4, 1 3, 3, 2

Theorem. All classes of the same family may be derived from one another by perspective projection.

Conversion balls and their use.

Theorem. Normal orders are orders of perspective sequence (see plate).

Application of perspective regions to finding normal orders by exhaustive method.

The position of the eye must be external to the group.

The entire plane outside the group may be divided into as many distinct perspective regions as there are normal orders (see plate).

A ship tacking along a diatomic zigzag is continually making angular way in reference to a point taken anywhere in some determinate region.

Normal orders of points in space are also orders of double perspective sequence, a line of view and planes of light being substituted for the point of view and rays of light.

Four-, five-, and six-point systems in space, like three-, four-, and five-point systems in planes, are reducible respectively to one, two, and three classes.

The classes of quaternary point-systems like those of ternary, and by the same method, may be arranged in natural families.

Reasons for believing a higher or more complex colligation of classes possible for quaternary systems.

Although the geometry of dispositions does not explicitly recognise distinctions grounded on magnitude, still the relations which it contemplates must admit of quantitative discrimination.

The *cæsura* in the *eduction* process following any normal order of the variables; how determined geometrically for ternary or quaternary systems by the principle of *denudation*; conformity of this with the rule for binary systems.

How the neutral region in a normal order which does not exist for binary arises for ternary and higher systems (see plate).

Example. The distances from each other of four points in a plane being given (six quantities connected by one equation) it must be possible to form one or more rational functions of these quantities of which the values as positive or negative must serve to discriminate between the two kinds of disposition in which four points may be grouped.

General character of the new geometry of disposition.

"It is the theory of permutation of space."—*Cayley*.

27.

THÉORIE DES NOMBRES.

(*Extrait d'une Lettre adressée à* M. Hermite *par* M. Sylvester.)

[*Comptes Rendus de l'Académie des Sciences*, L. (1860), p. 367.]

...En désignant par $(n\,;\,a,\,b)$ le nombre des solutions entières et positives de l'équation

$$ax + by = r$$

pour la série des valeurs $r = 0, 1, 2, \ldots, n$, j'ai obtenu ces deux théorèmes :

1°. Soit $n + 1 = kab + n'$, on aura

$$(n\,;\,a,\,b) = k\,\frac{kab + a + b + 2n' - 1}{2} + (n'\,;\,a,\,b).$$

Cette relation permet déjà de remplacer n par son résidu minimum suivant le module ab dans $(n\,;\,a,\,b)$.

2°. Soit ν un nombre entier inférieur à ab; on pourra déterminer les entiers positifs a' et b' de manière à avoir

$$ab' - ba' = 1,$$

a' étant moindre que a, et b' moindre que b. Cela posé, si l'on désigne par $E(x)$ l'entier compris dans une quantité quelconque x, et qu'on pose

$$E\left(\frac{b'\nu}{b}\right) = \nu',$$

on aura
$$(\nu\,;\,a,\,b) = (\nu'\,;\,a',\,b') - \mathfrak{R},$$

ou
$$\mathfrak{R} = \left[\nu' - E\left(\frac{a'\nu}{a}\right)\right] E\left(\frac{a\nu' - \nu a' + 1}{a'}\right).$$

Par ce second théorème on peut diminuer les deux coefficients a et b, en les remplaçant par a' et b'; donc en le joignant au précédent et appliquant successivement les deux propositions, on voit qu'on pourra exprimer $(n\,;\,a,\,b)$ par une série contenant au plus autant de termes qu'il a de fonctions convergentes vers $\frac{a}{b}$.

28.

THÉORIE DES NOMBRES.

(*Extrait d'une Lettre adressée à* M. Hermite *par* M. Sylvester.)

[*Comptes Rendus de l'Académie des Sciences*, L. (1860), p. 489.]

La somme $\sum\limits_{i=1}^{i=\frac{q-1}{2}} E\left(i\frac{p}{q}\right)$ où $E(x)$ désigne, suivant l'usage, l'entier contenu dans la quantité x, et qui joue un si grand rôle dans la théorie des résidus quadratiques, peut se calculer complétement par la méthode suivante, plus simple et plus facile que celle d'Eisentein pour déterminer seulement si la somme est paire ou impaire. Je développe $\frac{p}{q}$ sous la forme d'une fraction continue avec ces conditions, que le nombre des quotients soit impair et que chaque quotient de rang impair après le premier soit pair, ce qu'on réalisera en faisant le premier quotient congru à p suivant le module 2. Soit donc ainsi :

$$\frac{p}{q} = a_0 + \frac{\epsilon_1}{a_1 +}\,\frac{\epsilon_2}{a_2 +}\,\cdots + \frac{\epsilon_{2\omega}}{a_{2\omega}}.$$

Les quotients $a_2, a_4, \ldots, a_{2\omega}$ étant pairs, a_1, a_3, etc., quelconques, et ϵ_1, ϵ_2, etc., égaux à ± 1, on aura, en faisant $\lambda_i = \epsilon_1, \epsilon_2, \ldots \epsilon_i$,

$$\sum_{i=1}^{i=\frac{q-1}{2}} E\left(i\frac{p}{q}\right) = \frac{1}{8}\left[(p-2)\,q - \Sigma\,\{2\lambda_{2i-1} + (a_{2i} - 2)\,\lambda_{2i}\}\right].$$

A cette proposition je joindrai la suivante qui en est une généralisation.

Soit k un diviseur quelconque de $q - 1$; et supposons que dans le développement en fraction continue

$$\frac{p}{q} = a_0 + \frac{\epsilon_1}{a_1 +}\,\frac{\epsilon_2}{a_2 +}\,\cdots + \frac{\epsilon_{2\omega}}{a_{2\omega}},$$

tous les quotients à partir de a_1 soient multiples de k, le premier a_0 étant congru à p module k; alors on aura

$$E\left(\frac{p}{q}\right) + E\left(2\,\frac{p}{q}\right) + \ldots + E\left(\frac{q-1}{k}\cdot\frac{p}{q}\right)$$
$$= \frac{(q-2)\,p + k\,(p-q) - \Sigma\,[k\lambda_{2i-1} + \{(k-1)\,a_{2i} - k\}\,\lambda_{2i}]}{2k^2}.$$

Les conditions énoncées seront toujours d'ailleurs possibles, si l'on a $k < 5$.

29.

NOTE SUR CERTAINES SÉRIES QUI SE PRÉSENTENT DANS LA THÉORIE DES NOMBRES.

[*Comptes Rendus de l'Académie des Sciences*, L. (1860), p. 650.]

Soit $E\left(\frac{p}{q}\right)$ le plus grand nombre entier contenu dans la fraction $\frac{p}{q}$, et faisons

$$F(p, q, k, l) = E\left(\frac{p}{q}\right) + E\left(2\frac{p}{q}\right) + \ldots + E\left(l\frac{q-1}{k}\frac{p}{q}\right),$$

en supposant $q-1$ divisible par k. Il existe entre trois fonctions quelconques F, qui ont les mêmes valeurs de p, q, k, mais où la quantité l varie en restant moindre que k, l'équation algébrique suivante :

$$\frac{k-l'-l''}{(l-l')(l-l'')} F(p, q, k, l) + \frac{k-l''-l}{(l'-l'')(l'-l)} F(p, q, k, l') + \frac{k-l-l'}{(l''-l)(l''-l')} F(p, q, k, l'') = \frac{(p-1)(q-1)}{2k}.$$

Quand $l'+l''=k$, cette relation devient

$$F(p, q, k, l) - F(p, q, k, k-l) = (2l-k)\frac{(p-1)(q-1)}{2k},$$

ce qu'on peut vérifier par un procédé tout élémentaire. Il existe aussi entre les fonctions F, où k et l restent les mêmes, p et q étant changés entre eux, l'équation

$$F(p, q, k, l) + F(q, p, k, l) = \frac{l^2(p-1)(q-1)}{2k^2}.$$

Pour le cas de $l=1$, ce théorème a été déjà donné par Eisenstein, qui a exprimé alors la fonction F par une série trigonométrique finie. Mais quel que soit l, je suis parvenu à exprimer d'une manière analogue cette fonction, et dans le même ordre d'idées, c'est également par une série trigonométrique que j'ai été amené à représenter les valeurs de p' et q', moindres que p et q, satisfaisant à l'équation

$$p'q - q'p = 1,$$

valeurs qu'on obtient habituellement par le procédé du plus grand commun diviseur.

30.

SUR LA FONCTION $E(x)$.

[*Comptes Rendus de l'Académie des Sciences*, L. (1860), pp. 732—734.]

Soient p et q deux quantités positives quelconques, λ une quantité moindre que la plus petite valeur a qui rende en même temps ap et aq entiers, de sorte que, p et q étant incommensurables, λ est arbitraire ; mais si l'on suppose que ces quantités aient un plus grand commun diviseur k, on aura

$$\lambda < \frac{1}{k}.$$

Cela étant, je dis qu'on aura l'égalité suivante :

$$\sum_{\omega=0}^{\omega=E(\lambda q)} E\left(\omega\frac{p}{q}\right) + \sum_{\omega=0}^{\omega=E(\lambda p)} E\left(\omega\frac{q}{p}\right) = E(\lambda p)\,E(\lambda q).$$

Supposons-la satisfaite, en effet, pour toutes les valeurs de λ inférieures à une certaine limite, et faisons croître λ par degrés insensibles à partir de cette limite. Aucun des membres de l'équation ne changera de valeur qu'autant que λp ou λq deviendront des nombres entiers, ce qui, par hypothèse, n'arrivera jamais en même temps. Supposons que λp le premier devienne entier : à ce moment la seconde somme du premier membre s'augmente de $E\left(\lambda p \cdot \frac{q}{p}\right)$, c'est-à-dire de $E(\lambda q)$, la première ne changeant pas. Quant au second membre de l'équation, il est évident que $E(\lambda q)$ ne change pas, mais $E(\lambda p)$ est augmenté d'une unité, donc le second membre comme le premier s'accroît de $E(\lambda p)$. Donc le théorème subsiste pour la première valeur de λ qui fait varier les deux membres de l'équation, par conséquent, pour la seconde, la troisième, etc., et enfin pour toutes les valeurs inférieures à la plus petite quantité qui rend en même temps λp et λq entiers. Donc, étant vrai pour $\lambda = 0$, le théorème a lieu également pour toutes les valeurs de λ moindres que la limite supposée.

Si l'on supprime la restriction admise à l'égard de λ, j'observe que toutes les fois que, cette quantité croissant d'une manière continue, λp et λq deviennent entiers en même temps, l'expression

$$\sum_{\omega=0}^{\omega=E(\lambda q)} E\left(\omega\frac{p}{q}\right) + \sum_{\omega=0}^{\omega=E(\lambda p)} E\left(\omega\frac{q}{p}\right),$$

recevra un accroissement

$$E(\lambda p) + E(\lambda q) = \lambda p + \lambda q,$$

tandis que $E(\lambda p) + E(\lambda q)$ ne recevra que l'accroissement

$$\lambda p \lambda q - (\lambda p - 1)(\lambda q - 1) = \lambda p + \lambda q - 1.$$

Par conséquent, on aura pour toutes les valeurs de λ, l'égalité suivante:

$$\sum_{\omega=0}^{\omega=E(\lambda q)} E\left(\omega \frac{p}{q}\right) + \sum_{\omega=0}^{\omega=E(\lambda p)} E\left(\omega \frac{q}{p}\right) = E(\lambda p)\, E(\lambda q) + L,$$

où L désigne combien de fois px et qx deviennent entiers lorsque x croît de zéro à λ, ou, si l'on veut, le nombre des solutions positives moindres que λ de l'équation

$$(p+q)\, x = E(px) + E(qx).$$

Supposons maintenant p et q entiers, et $\lambda = \frac{k'}{k}$, k et k' étant aussi entiers avec la condition $k' < k$. En désignant par e et f les résidus minima positifs de p et q suivant le module k, les quantités $k'e$ et $k'f$ soient toutes deux moindres que k et le théorème se présente sous la forme suivante:

$$\sum_{\omega=0}^{\omega=\frac{k'(q-f)}{k}} E\left(\omega \frac{p}{q}\right) + \sum_{\omega=0}^{\omega=\frac{k'(p-e)}{k}} E\left(\omega \frac{q}{p}\right) = \left(\frac{k'}{k}\right)^2 (p-e)(q-f).$$

Lorsque $e=1$, $f=1$, les inégalités $k'e<k$, $k'f<k$ ont séparément lieu et on obtient l'équation

$$\sum_{\omega=0}^{\omega=\frac{k'(q-1)}{k}} E\left(\omega \frac{p}{q}\right) + \sum_{\omega=0}^{\omega=\frac{k'(p-1)}{k}} E\left(\omega \frac{q}{p}\right) = \frac{k'^2 (p-1)(q-1)}{k^2},$$

qui donne le théorème d'Eisenstein en posant $k'=1$. On voit aussi qu'on aura toujours si e et f sont les résidus minima de p et q par rapport au module k,

$$\sum_{\omega=0}^{\omega=\frac{q-f}{k}} E\left(\omega \frac{p}{q}\right) + \sum_{\omega=0}^{\omega=\frac{p-e}{k}} \Sigma\left(\omega \frac{q}{p}\right) = \frac{(p-e)(q-f)}{k^2}.$$

Il m'a paru qu'une démonstration tellement simple, on peut presque dire intuitive, de la proposition fondamentale de la théorie des résidus quadratiques, par l'emploi d'une variable continue, ne serait pas sans intérêt pour les géomètres.

31.

ON PONCELET'S APPROXIMATE LINEAR VALUATION OF SURD FORMS.

[*Philosophical Magazine*, XX. (1860), pp. 203—222.]

M. PONCELET'S method of approximately representing surd forms, and more particularly the square roots of homogeneous quadratic functions, by linear functions of the variables, is given in *Crelle's Journal*, Vol. XIII. 1834, pp. 277—291, under the title "Sur la Valeur approchée des radicaux." By this method, as applied to two variables, the resultant of two forces in a plane may be approximately expressed as a linear function of its two components, a case fully considered by M. Poncelet; and tables have been worked out applicable to this case, which appear to have been found of great utility in some important problems of mechanical and practical engineering. But the illustrious author of this beautiful method has left his theory imperfect in respect of its application to three variables.

To supply this slight but not unimportant omission, and to indicate how this more general case admits of being treated, more especially with reference to the approximate representation of the resultant of three forces in space as a linear function of its three components, is the object of this communication. At the close of the memoir referred to, M. Poncelet uses these words:—"Il serait inutile de pousser plus loin cet examen (referring to a discussion of the form $\sqrt{(a^2-b^2)}$), attendu que dans les applications de la mécanique aux machines les radicaux de la forme $\sqrt{(a^2-b^2)}$ sont rarement à considérer. Nous en dirons autant de ceux de la forme $\sqrt{(a^2+b^2+c^2)}$, qui représentent la résultante de trois forces rectangulaires entre elles et situées dans l'espace. D'ailleurs, si l'on connait les limites entre lesquelles demeurent compris les rapports des composantes a, b, c, ou de leurs résultantes partielles $\sqrt{(a^2+b^2)}$, &c., on pourra toujours ramener ce cas au premier de ceux que nous avons examinés," meaning to the case of $\sqrt{(a^2+b^2)}$. Now, in the first place, it is not clear how this reduction can be effected in general, or indeed in the vast majority of cases that might be proposed. For instance, if we have given

$a < \sqrt{(b^2+c^2)}$, $a > b$, $a > c$, I do not see how after, according to M. Poncelet's process, $\sqrt{(a^2+b^2+c^2)}$ is put under the form $\alpha a + \beta\sqrt{(b^2+c^2)}$ by aid of the limit $a < \sqrt{(b^2+c^2)}$, any use can be made of the other limits $a > b$, $a > c$ in further reducing this to the ultimate form $\alpha a + \alpha'\beta b + \beta'\beta c$. Or if we take the still simpler case, where a, b, c are left unlimited, in whatever way we attempt to proceed we shall obtain different approximations, according to the order in which we effect the successive reductions.

Furthermore, in those few exceptional cases where the process indicated by M. Poncelet leads to the use of all the limits given, the form arrived at is not and never can be the true *best* form, defined as such, according to M. Poncelet's own principles, as that which within the given limits has its *maximum* proportional error the least possible. Thus M. Poncelet indicates as the linear form for $\sqrt{(a^2+b^2+c^2)}$, when the given limits are $a^2 > b^2+c^2$, $b^2 > c^2$, $\cdot 96046a + \cdot 38201b + \cdot 15827c$, with a maximum error textually quoted from his memoir, $\cdot 0507$. It will be seen hereafter that the true best linear form gives a maximum error about one-tenth less than this. But it would be quite easy to give examples in which the maximum error by Poncelet's process should exceed in an indefinite proportion the necessary maximum error. This, for instance, would be the case if we imposed the limitations

$$x^2+y^2 > \lambda z^2, \quad y^2+z^2 > \lambda x^2, \quad z^2+x^2 > \lambda y^2,$$

on taking λ inferior but indefinitely near to 2.

The geometrical method of demonstration given by M. Poncelet for the case of two variables, labours under the inconvenience of *beginning* with a figure of three dimensions, and consequently does not admit of being carried beyond that case, although the results for three variables geometrically stated, when the conditions of the question are set under an appropriate form, are precisely analogous to that obtained by M. Poncelet for two variables; for whilst his construction is begun in space, his result subsides to a representation *in plano*. But between these two cases there is a very marked distinction; which is, that whilst for a surd radical with two variables every change in the limits proposed gives rise to a change in the corresponding linear form, such is never the case with a surd form with three or more variables, unless the limits be expressed by a *single linear* inequality between the variables which enter into the surd form, and the surd form itself. Thus, for instance, if $\sqrt{(x^2+y^2+z^2)}$ is to be represented linearly within the limits $z > x$, $z > y$ (for greater conciseness I throughout suppose the variables to be positive), the linear representation will be precisely the same as for the single limit $z > \sqrt{(x^2+y^2)}$, or, which is the same thing, $z - \sqrt{\tfrac{1}{2}}\,\sqrt{(x^2+y^2+z^2)} > 0$; and accordingly for the problem with three variables there is usually a preliminary question to be solved, namely, to find the single inequality of the

kind proposed which involves the satisfaction of the given limits, and is capable of being substituted for them without increasing the maximum proportional error. This preliminary question may be reduced, as will be seen, to an elementary geometrical form, and is strictly tantamount to the problem following:—Imagine a pincushion with a number of pins stuck into it, to find the least ring which can be made to take them all in,—a problem proposed by myself some four or five years ago with reference to points in a plane, in the *Quarterly Mathematical Journal*, and of which Professor Peirce of Cambridge University, U.S., has favoured me with a complete solution, which is equally applicable to the sphere, the case with which we shall be principally concerned in what follows.

I shall begin, then, with supposing R to be an integer homogeneous quadratic function of x, y, z, where x, y, z, R are subject to the linear inequality $Ax + By + Cz - \sqrt{R} > 0$. The geometrical solution, as such, will be seen to be equally applicable to the case of two, and the analytical representation to which it leads to any number of variables.

The problem to be solved is to find a linear form $Lx + My + Nz$ such that the greatest value of $\dfrac{Lx+My+Nz}{\sqrt{R}} - 1$ shall have the least possible arithmetical magnitude, without regard to *sign* as positive or negative, for all values of x, y, z satisfying the proposed inequality.

It is clear that, as the entire question is one of ratios, we may subject x, y, z to the condition expressed by $R = 1$ without affecting the result; in other words, we may consider x, y, z as the coordinates of a point limited to lie on the segment of the surface $R = 1$ cut off by the plane $Ax + By + Cz = 1$. Suppose, then, that $Lx + My + Nz$ is the linear form sought. The proportional error is $Lx + My + Nz - 1$; so that if we draw the plane

$$Lx + My + Nz - 1 = 0,$$

the error is expressible geometrically (paying no attention to sign) as the quotient of the perpendicular upon this plane from any point x, y, z in the segment, namely, $\dfrac{Lx+My+Nz-1}{\sqrt{(L^2+M^2+N^2)}}$, divided by the perpendicular from the origin to the same plane, namely, $\dfrac{1}{\sqrt{(L^2+M^2+N^2)}}$. Hence, then, the geometrical question to be resolved is simply to draw a plane for which the greatest value of this quotient, restricted to points within the segment, shall be the least possible. From this it is immediately seen to follow, that the portion of the surface cut off by the plane $Lx + My + Nz - 1 = 0$ must be a portion of the segment cut off by the given plane $Ax + By + Cz - 1 = 0$. And its actual position may be determined by means of a principle generally known, but which, as it will occupy but a few words, it may be well to deduce from first principles.

Suppose there are $(r+1)$ quantities, each containing the same system of r parameters; for greater brevity, say three quantities, p, q, r, each functions of the same two parameters λ, μ: let us call the greatest of the quantities p, q, r, corresponding to assigned values of λ, μ, the *dominant*; so that, according as we change λ, μ, the name of the dominant is liable to change; and that we wish to find M the minimum value of the dominant upon the supposition that the variations of p, q, r in respect to λ or μ are never simultaneously zero, and may be made positive or negative at will; then M will be found from the equations $M=p=q=r$. For if we had $M=p$ and $p>q, p>r$, by varying at will λ or μ we could make δp negative; and consequently since by hypothesis p differs sensibly from q and r, the dominant of $p+\delta p$, $q+\delta q$, $r+\delta r$ would necessarily be less than that of p, q, r, and thus M would not be the minimum dominant.

In like manner, if $M=p=q$, $p>r$, we could by means of the equations

$$\frac{dp}{d\lambda}\delta\lambda+\frac{dp}{d\mu}\delta\mu=-\epsilon,$$

$$\frac{dq}{d\lambda}\delta\lambda+\frac{dq}{d\mu}\delta\mu=-\eta,$$

so determine $\delta\lambda, \delta\mu$ as to diminish simultaneously p and q; and thus the dominant of $p-\epsilon$, $q-\eta$, $r+\delta r$ would, as before, be less than that of p, q, r. The same reasoning applies to any number $(r+1)$ functions of r variables. And if the number of functions should exceed $r+1$, it would still serve to show that when the dominant is a minimum, $(r+1)$ out of the whole number of the functions must all alike represent that dominant. Thus leaving for a moment in our original problem the case of three variables, and going down to that of only two variables, in which case we have to deal with a curve of the second order in lieu of a surface, and are to suppose that a segment of such curve is cut off by a right line A, and are required to draw another right line B such that the maximum square of the quotient of a perpendicular upon B from any point in the segment by the perpendicular from the centre upon B is to be a minimum, we evidently have to solve the same problem as if we had to find the least value of the dominant of three quantities involving two parameters, two being the number of constants required to fix the line B; those three quantities being the squares of the fractions whose numerators are the three perpendiculars from the extremities of A, and from the vertex of the arc cut off by B upon B, and their denominators the perpendicular upon B from the origin; accordingly the line B must be so chosen as to make the three perpendiculars in the numerators, without reference to sign, all equal, so that B is parallel to A, and bisects the sagitta of the segment cut off by A, that is, the longest perpendicular from any point in the segment upon A.

In the case of R being, as originally supposed, a function of x, y, z, we may take an indefinite number of points in the section of the surface $R=1$ made by the plane $Ax+By+Cz-1=0$, and the summit of the segment made by the plane to be determined $Lx+My+Nz=1$, and may show by the same reasoning as above (there being now three parameters) that four of these perpendiculars must be equal *inter se*, which proves, to begin with, that at all events the two planes must be parallel; and then the reasoning applied to two functions of one parameter will further show that this plane must bisect the sagitta of the segment cut off by the *given plane* $Ax+By+Cz-1=0$*. And we have now a geometrical solution of the question, which it is important to observe is in general, but, as will be presently seen, not universally applicable to the case when the limiting relations of x, y, z are defined by means of the position of a variable point limited to lie within a triangular area upon the surface $R=1$, whose sides are determined by the traces upon that surface of three planes drawn through the origin; the plane drawn through the angular points of this triangle will then take the place of the plane $Ax+By+Cz-1=0$ in the preceding investigation.

The next thing to be done is to obtain the quantities L, M, N in terms of A, B, C, and the coefficients of R, which is an easy matter to accomplish. Let

$$R=ax^2+by^2+cz^2+2fyz+2gzx+2hxy=\phi(x, y, z),$$

and call ξ, η, ζ the coordinates at the summit of the segment; the equation to the tangent plane at that point, which is of the form $Ax+By+Cz=0$, will be identical with

$$(a\xi+h\eta+g\zeta)X+(h\xi+b\eta+f\zeta)Y+(g\xi+f\eta+c\zeta)Z=1.$$

Hence

$$a\xi+h\eta+g\zeta=\frac{A}{\sigma},$$

$$h\xi+b\eta+f\zeta=\frac{B}{\sigma},$$

$$g\xi+f\eta+c\zeta=\frac{C}{\sigma},$$

and

$$\frac{A}{\sigma}\xi+\frac{B}{\sigma}\eta+\frac{C}{\sigma}\zeta=1;$$

* The absolute liberty of the plane sought for $(Lx+My+Nz=1)$ to take up all positions in space, and the absence of singular points in the segment cut off by the plane $Ax+By+Cz=1$, suffice to show that the conditions of variation necessary for the legitimate application of the theorem employed above are satisfied. If the minimum dominant is not at one of the *points of equality* given by the theorem, it must lie either at some minimum, or at all events at some singular point of one of the functions of the system to which the dominant belongs, or else at some point corresponding to the contour, so to say, if there be one, of the space within which the parameters are contained. In the case before us, the parameters, however chosen, to fix the position of the plane are perfectly independent, so that there is no limiting contour; and it is obvious that the functions representing the distances concerned from this variable plane have no

and therefore $$\frac{1}{\sigma^2}\frac{P\phi(A, B, C)}{\Delta\phi(A, B, C)} = 1,$$

where $\Delta\phi$ is the *discriminant*, and $P\phi$ the *polar reciprocal* of $\phi(A, B, C)$. Hence

$$\sigma = \sqrt{\frac{P}{\Delta}}\,*.$$

and the perpendicular upon the tangent plane is

$$\frac{1}{\sqrt{(A^2+B^2+C^2)}}\sqrt{\frac{P}{\Delta}}.$$

Consequently the mean between this and the perpendicular upon the given plane is

$$\frac{1}{\sqrt{(A^2+B^2+C^2)}}\,\frac{\sqrt{P}+\sqrt{\Delta}}{2\sqrt{\Delta}};$$

and therefore the equation to the plane required is

$$Ax + By + Cz = \frac{\sqrt{P}+\sqrt{\Delta}}{2\sqrt{\Delta}},$$

so that $$L = \frac{2\sqrt{\Delta}}{\sqrt{P}+\sqrt{\Delta}}A, \quad M = \frac{2\sqrt{\Delta}}{\sqrt{P}+\sqrt{\Delta}}B, \quad N = \frac{2\sqrt{\Delta}}{\sqrt{P}+\sqrt{\Delta}}C,$$

$Lx + My + Nz$ being the approximate representation of $\sqrt{\{\phi(x, y, z)\}}$, and the maximum error being evidently

$$\frac{\sqrt{P}-\sqrt{\Delta}}{\sqrt{P}+\sqrt{\Delta}}.$$

These results are perfectly general, and apply to a quadratic radical of an integer homogeneous quadratic function of any number of variables; thus for $\sqrt{\{\phi(x, y, z, t)\}}$ the linear representative form is

$$\frac{2\sqrt{\Delta}.A}{\sqrt{P}+\sqrt{\Delta}}x + \frac{2\sqrt{\Delta}.B}{\sqrt{P}+\sqrt{\Delta}}y + \frac{2\sqrt{\Delta}.C}{\sqrt{P}+\sqrt{\Delta}}z + \frac{2\sqrt{\Delta}.D}{\sqrt{P}+\sqrt{\Delta}}t,$$

maxima or minima values. I do not (nor ought I to) pretend to have presented the theoretical principles involved in the limitation of the general *law of equality* with all the logical rigour and precision of which the subject might admit, as this would be beside my present object, which is not to call in question the grounds of admitted truth applicable to the question in hand, but to advance it one step further in the direction of practical application.

* We see from the above, that if $Ax+By=1$, or $Ax+By+Cz=1$ be the equation to the chordal line or plane of a segment of a line or surface of the second degree, the ratio of the perpendiculars to such line or plane from the centre of the line or surface and the vertex of the segment respectively, or, which is the same thing, of a ray to any point in the segment to the portion of this ray produced, intercepted between the line or surface and the tangent at the vertex, is expressed by $\sqrt{\Delta} : \sqrt{P}$. It may at first sight appear strange that P should be of the *form* of a *contravariant* (in lieu of a covariant); but it must be remembered that the axes to which the line or surface and its chord are referred are supposed to be orthogonal, and for orthogonal substitutions, contravariants and covariants are indistinguishable.

and the greatest proportional error is still

$$\frac{\sqrt{P}-\sqrt{\Delta}}{\sqrt{P}+\sqrt{\Delta}};$$

D signifying the discriminant, and P the polar reciprocal of $\phi(A, B, C, D)$.

For the sphere, the perpendicular upon any tangent plane being 1, the linear form ought to be that obtained from the equation $Ax + By + Cz = K$, where

$$\frac{K}{\sqrt{(A^2+B^2+C^2)}} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{(A^2+B^2+C^2)}}\right),$$

or $$K = \tfrac{1}{2}\{\sqrt{(A^2+B^2+C^2)}+1\},$$

that is to say, the approximation is

$$\frac{2A}{1+\sqrt{(A^2+B^2+C^2)}}x + \&c.,$$

the maximum error being

$$\frac{\sqrt{(A^2+B^2+C^2)}-1}{\sqrt{(A^2+B^2+C^2)}+1},$$

which is easily seen to agree with the general formulæ above given.

When, as is usually the case in applying these results, the plane $Ax + By + Cz - 1 = 0$ is not directly given, but is to be found as the plane passing through three given points whose coordinates are a, b, c; a', b', c'; a'', b'', c'' respectively, we may use the equations

$$A = \frac{F}{Q}, \quad B = \frac{G}{Q}, \quad C = \frac{H}{Q},$$

where
$$F = (b'c'' - b''c') + (b''c - bc'') + (bc' - b'c),$$
$$G = (c'a'' - c''a') + (c''a - ca'') + (ca' - c'a),$$
$$H = (a'b'' - a''b') + (a''b - ab'') + (ab' - a'b),$$
$$Q = \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}.$$

But it may also sometimes be needful in practice, as will presently appear, to determine the plane with immediate reference to only two points upon the surface.

Application to the surd form which represents the resultant of three forces at right angles to each other.

Here $R = \sqrt{(x^2+y^2+z^2)}$, and $R=1$ represents a sphere. Two cases will be shown to arise. The first, the more frequent one, is that already alluded to, where a limiting plane has to be drawn through three given points. For this case, using F, G, H in the sense in which they have immediately above been employed, the linear representation of $\sqrt{(x^2+y^2+z^2)}$ becomes

$$\frac{2F}{Q+N}x + \frac{2G}{Q+N}y + \frac{2H}{Q+N}z,$$

with a maximum proportional error

$$\frac{N-Q}{N+Q},$$

N representing

$$\sqrt{(F^2+G^2+H^2)}.$$

The second case is where the limiting plane has to be drawn through two points upon the sphere so as to cut it in a circle, of which the line joining the two points is a diameter.

In this case, calling the coordinates of the two points respectively α, β, γ; α', β', γ', and writing $\alpha\alpha' + \beta\beta' + \gamma\gamma' = m$, it is easily seen that the perpendicular upon the limiting plane is $\sqrt{\frac{1+m}{2}}$, and consequently the perpendicular upon the plane

$$Lx + My + Nz = 1 \text{ is } \frac{1}{2}\left\{1 + \frac{\sqrt{(1+m)}}{2}\right\}.$$

Also this plane being parallel to the limiting plane, is perpendicular to the line joining the origin to the point

$$x : y : z :: \frac{\alpha+\alpha'}{2} : \frac{\beta+\beta'}{2} : \frac{\gamma+\gamma'}{2},$$

and therefore

$$L = \frac{\alpha+\alpha'}{\rho}, \quad M = \frac{\beta+\beta'}{\rho}, \quad N = \frac{\gamma+\gamma'}{\rho},$$

and

$$\frac{\rho}{\sqrt{\{(\alpha+\alpha')^2+(\beta+\beta')^2+(\gamma+\gamma')^2\}}} = \frac{1}{2}\left\{1 + \sqrt{\frac{1+m}{2}}\right\};$$

that is to say,

$$\rho = \sqrt{\{2(1+m)\}} \cdot \frac{1}{2}\left(1 + \sqrt{\frac{1+m}{2}}\right)$$
$$= \tfrac{1}{2}\left[\sqrt{\{2(1+m)\}} + (1+m)\right];$$

so that the linear form required is

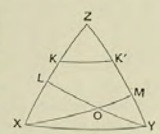
$$\left[\sqrt{\{2(1+m)\}} + 1 + m\right]\left\{\frac{\alpha+\alpha'}{2}x + \frac{\beta+\beta'}{2}y + \frac{\gamma+\gamma'}{2}z\right\},$$

with a maximum proportional error

$$\frac{\sqrt{2} - \sqrt{(1+m)}}{\sqrt{2} + \sqrt{(1+m)}}.$$

(m is of course identical with the cosine of the angle between the radii joining the two given points.)

The conditions of inequality which obtain between x, y, z may be, and usually will be, such as correspond to the limitation of the point (x, y, z) to an area contained within a triangle or polygon upon the surface of the sphere. Thus take X, Y, Z each a quadrant apart from the other, the points where the surface of the sphere $x^2+y^2+z^2=1$ is pierced by the axes. If no limitation is placed upon the values of x, y, z further than the one throughout supposed of their remaining always positive, the limiting area will be XYZ. If we suppose

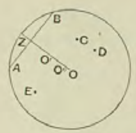


$$z > k\sqrt{(x^2+y^2)},$$

we may take $\tan XK = k$, and drawing the small circle KK', ZKK' will be the limiting area; if, again, $z < k\sqrt{(x^2+y^2)}$, $KK'YX$ will be the limiting area; if, again, $z < k\sqrt{(x^2+y^2)}$, $z > lx$, $z > my$ be the limiting conditions, taking $\tan LX = l$, $\tan MY = m$, and drawing LY, XM to intersect in O, $KK'MOL$ will be the corresponding area, and so in general. Even so simple a set of conditions as $z > x$, $z > y$ it is seen will give rise to a quadrilateral area, limited in the figure by $ZLOM$, when $ZL = ZM = 45°$. Thus, then, we approach the preliminary question to which allusion has been already made, which is to determine the *least circle* that will cut off from a given sphere a segment containing a given system of points lying upon it. The solution is precisely the same, substituting arcs of great circles for right lines, as the problem of drawing upon a plane the least circle containing a set of points given in the plane.

We may, in the first place, obviously reject all those points that are contained within the contour formed by arcs joining the remaining points, so that the case of points lying at the angles of a convex polygon alone remains to be studied. Now if we confine our attention even to the simplest case of a system of three points, we shall see at once that two cases arise. If a circle be drawn through them, and these three points do not lie in the same semicircle, no smaller circle than this can be drawn to contain the three; but if they do lie in the same semicircle, it is obvious that a circle described upon the line joining the outer two as a diameter will be smaller than the circle passing through all three, and will contain them all. It was this simple but striking fact in the geometry of situation which led me to propose the question for any number of points in the *Quarterly Mathematical Journal*; and as Prof. Peirce's exhaustive method of solution has not appeared in print, I may take this occasion of presenting it.

Let A, Z, B, C, D, E be the given points. Let AZB be a circle whose centre is drawn through A, Z, B, chosen so as to include all the others; then if A, Z, B are not contained in the same semicircle, AZB is the circle required. But if AZB be less than a semicircle, as in the figure, we may first reject the consideration of all the points contained between the arc AB and its chord. We must then find O', O'', &c., the centres of the circles passing through A, B, C; A, B, D, &c.: these will all lie in the same straight line $O'O''O$. Selecting the one nearest to O, say O'', we describe the corresponding circle, in which AC will now take the place of AB in the former circle. If the points A, B, C are not contained in less than a semicircle, that is, if ABC is an acute-angled or right-angled triangle, ABC is the circle required; but if they do lie within the same semicircle so that ABC forms an obtuse angle, B will now have to be rejected, and we must find a new centre as before, and so on continually. By this process we must inevitably at last exhaust all the given points; and the final circle so obtained will be the circle sought, unless the three points through which it has been drawn are distributed over the same semicircle, in which case the circle required is that described upon the chord joining the two extreme points as its diameter. The solution will evidently be *unique*, and (as already hinted at) merely require the construction upon the sphere either of a circle passing through a certain set of three out of all the given points, or else passing through only two of them, so as to be perpendicular to the radius bisecting their joining line.



If we imagine an india-rubber band (similar, we may suppose, in form to a "parlour quoit" but more elastic) having the faculty of maintaining its figure always circular, or which is more simple in the case before us, capable of maintaining itself in the same plane, and imagine this sufficiently stretched over the surface of the sphere to contain all the given points (represented by very minute pins' heads given upon it), this band will by its contraction upon the surface of the sphere, however originally placed, imitate the steps of Prof. Peirce's method of solution; and after (it may be) passing through and quitting successive sets of three points, come to a position of *geometrical equilibrium*, either when its circumference contains a triad of the

given points lying at the angles of an acute-angled triangle, or a duad at the extremities of one of its diameters*.

The following observation, which constitutes a veritable theorem, and is presupposed in Prof. Peirce's solution, is very important:—"Any circle being found which, either passing through three of the given points such that no two of their joining lines form an obtuse angle, or which described upon the line joining two of the given points as a diameter, includes all the rest, is the minimum circle which contains all the points of the given cluster; so that one, and only one, circle exists satisfying the above *alternative* condition."

It may be instructive to proceed to the application of the method now fully explained to some of the more salient cases of inequality, it being understood that these cases are given to afford some general notion of the precision of the method, and by no means as specimens of such as it would be applied to in practice, for which the limits I shall suppose would be far too wide to furnish any useful result.

Example 1. x, y, z unlimited. Here the values of F, G, H, Q are the minor determinants of the matrix,

$$\begin{vmatrix} 1, & 0, & 0, & \bar{1} \\ 0, & 1, & 0, & \bar{1} \\ 0, & 0, & 1, & \bar{1} \end{vmatrix}$$

$F=G=H=1$, $Q=1$, and the linear approximation to $\sqrt{(x^2+y^2+z^2)}$ becomes $\dfrac{2}{\sqrt{3}+1}x+$ &c., or $(\sqrt{3}-1)x+(\sqrt{3}-1)y+(\sqrt{3}-1)z$, or say

$$\cdot 73025x + \cdot 73025y + \cdot 73025z,$$

* The annexed is a more complete and, I think, a correct account of what would happen to the band under the supposed conditions. It will begin to move parallel to its own plane, and continue so to do until it comes in contact with one of the physical points (call it A) upon the surface of the sphere. Supposing that the position of equilibrium is not then attained by the band passing at the same moment through *one* other point at the opposite extremity of a diameter to A, or through *two* other of the given points forming a non-obtuse-angled triangle with A, it will begin to revolve (always contracting the while) about a tangent at A to its intersection with the sphere as an axis, until it meets a second of the given points, say B. If the line AB is a diameter of the band, *cadit quæstio*, the problem is solved. If not, the band will go on further contracting, revolving meanwhile round AB as an axis until either AB *becomes* a diameter in virtue of the contraction of the band's dimensions (and so the problem is solved), or else before this can take place the band is arrested at a third point C, either forming a non-obtuse-angled triangle with AB and so solving the problem, or else an obtuse-angled triangle with AB and lying exterior to the arc AB on one side of it or the other; on the latter supposition the line joining C with the extremity of AB nearest to it, will (it appears to me) form a new axis of rotation for the band, which will quit the further extremity of the old axis, and thus the motion will continue with an intermitting change of axes, until at last the band either finds out for itself an axis which in the course of the contraction becomes a diameter, or else brings the band into contact with a third point forming a non-obtuse-angled triangle with such axis, in either of which cases the minimum periphery is attained, the contraction comes to an end, and the problem is solved.

with a maximum proportional error

$$\frac{\sqrt{3}-1}{\sqrt{3}+1}, \text{ or } 2-\sqrt{3} = {\cdot}26895.$$

The corresponding error for $\sqrt{(x^2+y^2)}$ under the form ${\cdot}8284x + {\cdot}8284y$ is ${\cdot}17160$, or about two-thirds of the one in question*.

Example 2. $z > \sqrt{(y^2+x^2)}$. Here the determining matrix is

$$\begin{vmatrix} 0, & 0, & 1, & \bar{1} \\ 0, & \sqrt{\tfrac{1}{2}}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \sqrt{\tfrac{1}{2}}, & 0, & \sqrt{\tfrac{1}{2}}, & \bar{1} \end{vmatrix}$$

$$F = G = \sqrt{\tfrac{1}{2}} - \tfrac{1}{2} = {\cdot}207107$$

$$H = \tfrac{1}{2}$$

$$Q = \tfrac{1}{2}$$

$$N^2 = F^2 + G^2 + H^2 = 1 - \sqrt{\tfrac{1}{2}} = {\cdot}292893$$

$$N = {\cdot}541196$$

$$N + Q = 1{\cdot}041196 \qquad N - Q = {\cdot}041196.$$

Thus the linear approximation becomes

$${\cdot}397825x + {\cdot}397825y + {\cdot}960430z,$$

with a maximum error ${\cdot}039493$.

Example 3. $z > \sqrt{(y^2+x^2)}, y > x$. This is M. Poncelet's example (*Crelle*, Vol. XIII. p. 291). His a, b, c correspond respectively with my z, y, x; there are some misprints in line 6 of this page (in M. Poncelet's Memoir) which may perplex the reader; it is intended to stand thus:

$$\delta \sqrt{(a^2+b^2+c^2)} + \beta\delta' \sqrt{(b^2+c^2)} = \sqrt{(a^2+b^2+c^2)} \cdot \left(\delta + \beta\delta' \sqrt{\frac{b^2+c^2}{a^2+b^2+c^2}}\right).$$

Here the determining matrix corresponds to the area ZKN (the coordinates of N being found from the equations $z^2 = x^2+y^2$, $y = x$, $z^2+x^2+y^2 = 1$), and the matrix will be as subjoined.

* It would have been more exact to have treated this as a case of a circle to be drawn through four points, namely, Z the middle points of ZX, ZY and the middle or lowest point (in reference to Z) of the small circle drawn through these two, and having Z for its pole. But it is easily seen that the small circle drawn through the three former will contain the one last named, for the tangent of its circular radius will be $\sqrt{2} \times \tan\frac{45^\circ}{2}$, and consequently its summit will be further from Z than from the point in question. A similar remark applies to the subsequent and some other examples.

$$\begin{vmatrix} 0, & 0, & 1, & \bar{1} \\ 0, & \sqrt{\tfrac{1}{2}}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \tfrac{1}{2}, & \tfrac{1}{2}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \end{vmatrix}.$$

$$F = \sqrt{\tfrac{1}{2}} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}} - \tfrac{1}{2} - \tfrac{1}{2} = 3\sqrt{\tfrac{1}{8}} - 1 = {\cdot}060660$$

$$G = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{\tfrac{1}{2}} = \tfrac{1}{2} - \sqrt{\tfrac{1}{8}} = {\cdot}146447$$

$$H = \tfrac{1}{2}\sqrt{\tfrac{1}{2}} = {\cdot}353553$$

$$Q = \tfrac{1}{2}\sqrt{\tfrac{1}{2}} = {\cdot}353553$$

$$\begin{aligned} N^2 = F^2 + G^2 + H^2 &= \tfrac{17}{8} + \tfrac{3}{8} + \tfrac{1}{8} - 7\sqrt{\tfrac{1}{8}} \\ &= \tfrac{21}{8} - \tfrac{1}{2}\sqrt{24{\cdot}5} \\ &= 2{\cdot}625 - 2{\cdot}474874 \\ &= {\cdot}150126 \end{aligned}$$

$$N = {\cdot}387461, \qquad N + Q = {\cdot}741014, \qquad N - Q = {\cdot}033908.$$

The maximum error therefore is $\frac{33908}{741014} = {\cdot}0457$, or about one-tenth less than that given by M. Poncelet's form.

$$\frac{2F}{N+Q} = \frac{6066}{37051} = {\cdot}1637,$$

$$\frac{2G}{N+Q} = \frac{14645}{37051} = {\cdot}3953,$$

$$\frac{2H}{N+Q} = \frac{35355}{37051} = {\cdot}9542.$$

The last of these quantities is less, the first two greater, than the corresponding coefficients in M. Poncelet's form.

Examples 4 *and* 5. The inequality system, $\sqrt{(x^2+y^2)} > z > y > x$, is represented by the triangle KNQ, and the corresponding determining matrix will be

$$\begin{vmatrix} 0, & \sqrt{\tfrac{1}{2}}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \tfrac{1}{2}, & \tfrac{1}{2}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \sqrt{\tfrac{1}{3}}, & \sqrt{\tfrac{1}{3}}, & \sqrt{\tfrac{1}{3}}, & \bar{1} \end{vmatrix}.$$

So, too, the inequality system, $\sqrt{(x^2+y^2)} < z < y > x$, has for its locus the triangle ZKN, its determining matrix

$$\begin{vmatrix} 0, & \sqrt{\tfrac{1}{2}}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \tfrac{1}{2}, & \tfrac{1}{2}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ 0, & 0, & 1, & \bar{1} \end{vmatrix}.$$

It would be superfluous to go on multiplying numerical examples, that may be left to those who feel the want of the Tables which this method affords. If the limiting conditions were supposed to be $z > y$, $z > x$, this would correspond to the quadrilateral $ZK'OK$ in the last figure: it may easily be ascertained that a circle passing through $K'ZK$ would contain O, and would have its centre between N and Z. Hence by the application of Peirce's law, we know that the minimum circle in this case is that which can be drawn through $K'ZK$, and consequently the linear form and maximum error will be precisely the same as for the simpler case already considered, $z > \sqrt{(x^2+y^2)}$. On the other hand, if the conditions imposed were simply $z < x$, $z < y$ (conditions, be it remembered, far wider than ever would be admitted in practice), the limiting figure becomes XOY; and since $MO < MX$ or MY, the centre of the circle through XOY would fall under XY, so that the limiting circle in this case would be that having M for its pole; the linear substitutive form would not contain z, but would be the same as if z did not appear, namely $\cdot 96046x + \cdot 960467y$, with $\cdot 03954$ as the maximum proportional error. The same remark would apply to the system of conditions $z < \lambda x$, $z < \lambda y$ for any value of λ not inferior to $\sqrt{\tfrac{1}{2}}$.

The conditions $z > x$, $z > y$, $z < \sqrt{(x^2+y^2)}$ would correspond to the limiting area $KK'O$, which would give rise to the determining matrix,

$$\begin{vmatrix} 0, & \sqrt{\tfrac{1}{2}}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \sqrt{\tfrac{1}{2}}, & 0, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \sqrt{\tfrac{1}{3}}, & \sqrt{\tfrac{1}{3}}, & \sqrt{\tfrac{1}{3}}, & \bar{1} \end{vmatrix}.$$

The condition $z < \sqrt{(x^2+y^2)}$ would correspond to a limiting area, $KK'XY$. If KY be bisected in G, and $K'X$ in G', and $G'YGX$ intersect in H, it is obvious that a small circle may be described with H as its pole passing through all four points X, Y, K, K', which will be the minimum circle of limitation. To assign the determining matrix, we may take any three of these four points, as, for example, Y, X, K, which will give

$$\begin{vmatrix} 0, & 1, & 0, & \bar{1} \\ 1, & 0, & 0, & \bar{1} \\ \sqrt{\tfrac{1}{2}}, & 0, & \sqrt{\tfrac{1}{2}}, & \bar{1} \end{vmatrix}.$$

This gives

$$Q = \sqrt{\tfrac{1}{2}} = \cdot 70711,$$
$$F = \sqrt{\tfrac{1}{2}},\ G = \sqrt{\tfrac{1}{2}},\ H = 1 - \sqrt{\tfrac{1}{2}} = \cdot 29289,$$
$$N^2 = \tfrac{5}{2} - \sqrt{2} = 1\cdot 085786,$$
$$N = 1\cdot 04200,$$
$$N + Q = 1\cdot 74911, \quad N - Q = \cdot 33489.$$

The linear approximation is accordingly

$$\cdot 8090x + \cdot 8090y + \cdot 3351z,$$

with a maximum proportional error $\cdot 1914$.

Finally, for $z > y$, $y > x$ the limiting triangle will be ZKO, the determining matrix

$$\begin{vmatrix} 0, & 0, & 1, & \bar{1} \\ 0, & \sqrt{\tfrac{1}{2}}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \sqrt{\tfrac{1}{3}}, & \sqrt{\tfrac{1}{3}}, & \sqrt{\tfrac{1}{3}}, & \bar{1} \end{vmatrix}.$$

$$F = \sqrt{\tfrac{1}{2}} - \sqrt{\tfrac{1}{3}} = \cdot 1297, \quad G = \sqrt{\tfrac{1}{3}}\,\{1 - \sqrt{\tfrac{1}{2}}\} = \cdot 1692,$$
$$H = \sqrt{\tfrac{1}{6}} = \cdot 4082,$$
$$N^2 = \tfrac{3}{2} - \sqrt{\tfrac{2}{3}} - \sqrt{\tfrac{2}{9}} = \cdot 21207,$$
$$N = \cdot 4605, \qquad N + Q = \cdot 8687,$$
$$Q = \sqrt{\tfrac{1}{6}} = \cdot 4082, \qquad N - Q = \cdot 0523.$$

The linear approximation is $\cdot 2986x + \cdot 3895y + \cdot 9397z$, with a maximum error $\cdot 06$ (more precisely $\cdot 0602$). This is a trifle beyond half as much again as the maximum error of the best linear approximation to $\sqrt{(x^2+y^2)}$, subject to the limitation $x > y$, which (see Poncelet's Memoir, p. 280) is a little under $\cdot 04$.

Poncelet has shown that for $\sqrt{(x^2+y^2)}$, when x, y are the coordinates of a point limited within a sector whose bounding radii make angles ϕ and ψ with the axis of X, the approximate linear form is

$$\frac{\cos\tfrac{1}{2}(\phi+\psi)}{\cos^2\dfrac{\phi-\psi}{4}}x + \frac{\sin\tfrac{1}{2}(\phi+\psi)}{\cos^2\dfrac{\phi-\psi}{4}}y,$$

with a maximum error $\tan^2\dfrac{\phi-\psi}{4}$.

In like manner it follows immediately from the method given in the text, that if the summit of the limiting segment make angles λ, μ, ν with the axes of X, Y, Z, and its spherical radius be ρ, the approximate expression for $\sqrt{(x^2+y^2+z^2)}$ is

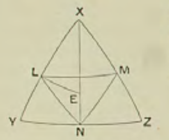
$$\frac{\cos\lambda}{\cos^2\dfrac{\rho}{2}}x + \frac{\cos\mu}{\cos^2\dfrac{\rho}{2}}y + \frac{\cos\nu}{\cos^2\dfrac{\rho}{2}}z,$$

with a maximum error $\tan^2\dfrac{\rho}{2}$, which expressions are the precise analogues of the former, as will immediately appear from the consideration that the summit of the spherical segment corresponds with the centre of the circular arc.

As an example of the use of these formulæ, suppose the given limits to be

$$x < \sqrt{(y^2+z^2)}, \quad y < \sqrt{(z^2+x^2)}, \quad z < \sqrt{(x^2+y^2)}.$$

If we bisect the quadrants XY, YZ, ZX in L, N, M respectively, the variable point will be limited to lie in LMN, and the base of the corresponding segment will be the circle passing through LMN whose summit will be at E, the point where the perpendicular to XY at L and the arc bisecting the angle X meet.



Here then we have

$$\rho = LE, \quad \lambda = \mu = \nu = XE,$$

$$\tan\rho = \cos 45^\circ = \sqrt{\tfrac{1}{2}}, \qquad \cot\lambda = \sqrt{\tfrac{1}{2}},$$

$$\cos\rho = \sqrt{\tfrac{2}{3}}, \quad \cos^2\frac{\rho}{2} = \tfrac{1}{2}\{1 + \sqrt{\tfrac{2}{3}}\}, \quad \cos\lambda = \sqrt{\tfrac{1}{3}},$$

$$\tan^2\frac{\rho}{2} = \frac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2}}.$$

Hence the linear approximation is

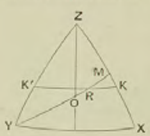
$$\frac{2}{\sqrt{3} + \sqrt{2}}(x + y + z) = 2(\sqrt{3} - \sqrt{2})\{x + y + z\}$$
$$= \cdot 6356744\,(x + y + z),$$

with a maximum proportional error $5 - \sqrt{24} = \cdot 10102$.

More generally, if we assume the system of conditions

$$\sqrt{(x^2 + y^2)} > cz, \quad \sqrt{(y^2 + z^2)} > cx, \quad \sqrt{(z^2 + x^2)} > cy,$$

c being any number intermediate between 1 and $\sqrt{2}$, if in the figure annexed, we take $\tan ZK = \tan ZK' = c$, and join KK' by a small circle intersecting YM, which bisects ZX, in R, O remaining still the summit of XZY, it is easy to perceive that the limiting area will be included within the triangular space cut out between KK' and the two other analogous small circles; λ, μ, ν will remain the same as before, and OR will represent ρ. Accordingly we have from the quadrantal triangle ZYR,



$$\cos ZR = \sin RY \cos RYZ,$$

that is
$$\sin RY = \sqrt{\frac{2}{c^2 + 1}};$$

therefore
$$RY = \tan^{-1}\sqrt{\frac{2}{c^2 - 1}},$$

$$\tan\rho = \tan RO = \tan(RY - OY) = \frac{\sqrt{\left(\frac{2}{c^2 - 1}\right)} - \sqrt{2}}{1 - \frac{2}{\sqrt{(c^2 - 1)}}}$$
$$= \sqrt{2}\left\{\frac{1 - \sqrt{(c^2 - 1)}}{\sqrt{(c^2 - 1)} - 2}\right\}.$$

When $c = \sqrt{2}$, this vanishes; and when $c > \sqrt{2}$, the conditions become incompatible.

The equations $\tan\phi = \sqrt{\frac{2}{c^2 - 1}}$, or $\cos 2\phi = \frac{c^2 - 3}{c^2 + 1}$, and

$$\rho = \phi - \tan^{-1}\sqrt{2} = \phi - 54^\circ\, 44',$$

are well adapted for logarithmic computation. Suppose

$$c = \tfrac{4}{3}, \quad \cos 2\phi = -\tfrac{11}{25} = -\cdot 44, \quad 2\phi = 180^\circ - 63^\circ\, 54' = 116^\circ\, 6',$$
$$\phi = 58^\circ\, 3', \quad \rho = 3^\circ\, 19',$$

giving a maximum error $\tan(1^\circ\, 39'\, 30'')^2 = \cdot 0008375$. The linear form corresponding to this is

$$\frac{2\sqrt{\frac{1}{3}}}{1 + \cos\rho}\{x + y + z\} = \cdot 5778x + \cdot 5778y + \cdot 5778z.$$

If $c < 1$, the formula changes; the limiting area, from a triangle, becoming a hexagon through all the angles of which a circle will admit of being drawn, which circle will give the limiting segment. ρ becomes the third side of a spherical triangle of which the other two sides are $\tan^{-1}\sqrt{2}$ and $\tan^{-1} c$ respectively, and the included angle 45°; so that

$$\cos\rho = \sqrt{\frac{1}{3(1 + c^2)}} + \sqrt{\frac{c}{3(1 + c^2)}} = (1 + \sqrt{c})\sqrt{\frac{1}{3(1 + c^2)}},$$

and the maximum error, that is $\tan^2\frac{\rho}{2}$, becomes

$$\frac{\sqrt{\{3(1 + c^2)\}} - 1 - \sqrt{c}}{\sqrt{\{3(1 + c^2)\}} + 1 + \sqrt{c}}.$$

The only real difficulty in extending M. Poncelet's method in the manner pursued in the above unpretending study, consisted in forming a clear preconception of the mode in which any given system of limits require for the purpose in view to be regarded, namely, as enveloped, so to say, in a single condition (no wider than absolutely necessary) expressed by a linear equation between the given surd function and the variables which enter into it.

I may in conclusion just observe that if the relative values of the variables be limited, not by a system of conditions giving rise to a polygonal area of limitation, but by a condition expressed by the positivity of a single homogeneous function of the variables of any degree, the variable point will then be limited by the intersection of the sphere with a cone, and we should have to solve a preliminary geometrical problem of circumscribing a spherical curve by the least possible circle,—a question which I have neither leisure nor inclination to discuss, but to which I believe Mr Cayley has paid some attention.

Before taking final leave of my readers and the subject, I devote a word to the *inverse case* of *Three Rectangular Forces.* This is the case where the resultant and two of the rectangular components are given, and it is the third component which is to be expressed linearly in terms of them. In this case an approximate expression is to be found for $\sqrt{(z^2 - y^2 - x^2)}$, and the geometrical locus which replaces the sphere becomes an equilateral hyperboloid of revolution of two sheets.

If the variable point be supposed to be limited to a segment of one sheet of the hyperboloid cut off by the plane $Ax + By + Cz = 1$, the discriminant of $z^2 - y^2 - x^2$ being 1, and its polar reciprocal of the same form as itself, the approximate linear form of the surd becomes

$$\frac{2Cz}{\sqrt{(C^2 - B^2 - A^2)} + 1} + \frac{2By}{\sqrt{(C^2 - B^2 - A^2)} + 1} + \frac{2Ax}{\sqrt{(C^2 - B^2 - A^2)} + 1},$$

with a maximum proportional error

$$\frac{1 - \sqrt{(C^2 - B^2 - A^2)}}{1 + \sqrt{(C^2 - B^2 - A^2)}}.$$

To *envelope,* however, any given arbitrary system of inequalities between the coordinates x, y, z on the hyperboloid within a single condition,

$$Ax + By + Cz - 1 > 0,$$

becomes a geometrical problem of somewhat greater difficulty than the corresponding one for the sphere, and I do not propose to enter upon the discussion of it here.

I shall content myself, as M. Poncelet has done in the corresponding case *in plano,* with exhibiting a single numerical application of the method.

Suppose the given limits to be defined by the equations

$$z^2 > \tfrac{3}{2}(y^2 + x^2), \quad y > x.$$

Here it is obvious that the *enveloping condition* will be expressible by means of the equation to a plane drawn through three points on the hyperboloid, the coordinates of one of which are found by writing

$$y = 0, \quad x = 0;$$

of a second by writing

$$z^2 - \tfrac{3}{2}y = 0, \quad x = 0;$$

and of the third by writing

$$z^2 - \tfrac{3}{2}(y^2 + x^2) = 0, \quad y - x = 0;$$

and for all three

$$z^2 - y^2 - x^2 = 1.$$

Hence we obtain the matrix

$$\begin{vmatrix} 1, & 0, & 0, & \bar{1} \\ \sqrt{3}, & \sqrt{2}, & 0, & \bar{1} \\ \sqrt{3}, & 1, & 1, & \bar{1} \end{vmatrix}.$$

And if we call the minors obtained by leaving out the first, second, third, fourth columns respectively H, G, F, Q, the linear form becomes

$$\frac{2Hz}{\sqrt{(H^2 - G^2 - F^2)} + Q} + \frac{2Gy}{\sqrt{(H^2 - G^2 - F^2)} + Q} + \frac{2Fx}{\sqrt{(H^2 - G^2 - F^2)} + Q},$$

with a maximum error

$$\frac{Q - \sqrt{(H^2 - G^2 - F^2)}}{Q + \sqrt{(H^2 - G^2 - F^2)}}.$$

And since

$$Q = \sqrt{2}, \quad H = \sqrt{2}, \quad -G = \sqrt{3} - 1, \quad -F = (\sqrt{2} - 1)(\sqrt{3} - 1),$$

we have

$$\sqrt{(H^2 - G^2 - F^2)} = 1{\cdot}1714 \text{ and } Q = 1{\cdot}4142,$$

so that the representative form becomes $1{\cdot}093z - {\cdot}566y - {\cdot}089x$, with a maximum relative error of about ·094.

32.

MEDITATION ON THE IDEA OF PONCELET'S THEOREM.

[*Philosophical Magazine*, XX. (1860), pp. 307—316.]

HITHERTO Poncelet's theorem has been regarded as a method *sui generis* and complete in itself; but in truth it is but the first germ or rudiment of a vast and prolific algebraical theory; and not only so, but the principle which it contains admits of applications of the utmost value in various dynamical and analytical questions, which it is surprising should have been allowed to lay so long dormant. For the present, however, I mean to confine myself to a very brief indication of one direction in which the theorem admits of being generalized. And first I will make a remark upon so simple a matter as the extraction of the square root, which seems to have escaped observation, and at all events is so far from being generally known, that two of the highest authorities for mathematical erudition in this country whom I have consulted on the subject provisionally accept it as new.

Let r be an approximate value of $\sqrt{N}$; then by that mode of application of Newton's method of approximation to the equation $x^2 = N$ which is equivalent to the use of continued fractions, we may easily establish the following theorem, namely, that

$$\frac{r^2+N}{2r},\quad \frac{r^3+3rN}{3r^2+N},\quad \frac{r^4+6r^2N+N^2}{4r^3+4rN},\quad \frac{r^5+10r^3N+5rN^2}{5r^4+10r^2N+N^2},\ \ldots\text{*}$$

will be successive approximations to $\sqrt{N}$, whose limits of error can be

* In other words, if r be the first approximation to $\sqrt{N}$, the ith approximation will be

$$\frac{(r+\sqrt{N})^i+(r-\sqrt{N})^i}{(r+\sqrt{N})^i-(r-\sqrt{N})^i}\sqrt{N},$$

so that the relative error becomes

$$\frac{2(r-\sqrt{N})^i}{(r+\sqrt{N})^i-(r-\sqrt{N})^i},$$

in which form the theorem is self-subsistent, and needs no proof. But the fact remains interesting, that the application of Newton's method of approximation to the equation $x^2=N$ will be found to lead to the form above written at the ith step of the process conducted after the continued-fraction fashion.

assigned when a limit to the error of the first approximation r is given. The coefficients of the qth approximation, it will be observed, are for the numerator the alternate binomial coefficients

$$1,\ q\frac{q-1}{2},\ q\frac{q-1}{2}\frac{q-2}{3}\frac{q-3}{4},\ \&c.;$$

and for the denominator the intermediate ones,

$$q,\ q\frac{q-1}{2}\frac{q-2}{3},\ \&c.$$

Mr Cayley has reminded me that the third approximation, $\frac{r^3+3rN}{3r^2+N}$, is a special case of a formula for *any* root of N given in the books; and to Mr De Morgan I am indebted for a hint which has led me to notice that all these forms may be deduced from the Newtonian method of approximation*.

If we call the ith approximation $\phi(i, r)$, we shall find that the functional equation $\phi\{j, \phi(i, r)\} = \phi(ij, r)$ will be satisfied; which is not so mere a truism as might at first sight be supposed, as any one may satisfy himself by studying the analogous theory for cubic or higher roots, a part of the subject to which I may hereafter return.

Now as to the limits of accuracy afforded by the successive approximations. Let e be a known limit to the relative error of the first approximation r, by which I mean that $\left(\frac{\sqrt{N}-r}{\sqrt{N}}\right)^2 < e^2$. For greater simplicity, I take separately the cases where r is too great and r is too small.

1. Let $\sqrt{N} < r < (1+\epsilon)\sqrt{N}$; then the errors will be throughout in excess; and we may assign as a limit of error to the ith approximation a quantity, say ϵ_i, which is a known function of ϵ, namely, $\frac{2}{(2\epsilon^{-1}+1)^i-1}$, which it may be noticed is less than $\frac{\epsilon^i}{2^{i-1}}$.

2. Let $\sqrt{N} > r > (1-\eta)\sqrt{N}$; then the errors will be alternately in defect and excess, and to the ith approximation we may assign a limit of error η_i, where $\eta_i = \frac{2}{(2\eta^{-1}-1)^i-(-1)^i}$ †.

* The expansion (after Newton) of $\sqrt{N}$ introduces the binomial coefficients—a curious fact! What are the analogous integers which the continued-fraction process applied to $\sqrt[3]{N}$ will produce?

† If we write

$$\epsilon_i = \theta(\epsilon, i) \text{ and } \eta_i = \vartheta(\eta, i),$$

then if i be any *odd* number,

$$\theta\{\theta(\epsilon, i), j\} = \theta(\epsilon, ij),$$
$$\vartheta\{\vartheta(\eta, i), j\} = \vartheta(\eta, ij);$$

and if i be any *even* number,

$$\vartheta\{\theta(\epsilon, i), j\} = \theta(\epsilon, ij),$$
$$\theta\{\vartheta(\eta, i), j\} = \vartheta(\eta, ij).$$

We may now apply these results to Poncelet's linear approximate representation of $\sqrt{(a+bx+cx^2)}$. Suppose $f+gx$ is the first approximation, as found by Poncelet's method, with a maximum relative error e, then

$$\frac{(f+gx)^2+(a+bx+cx^2)}{2(f+gx)}$$

will be a much closer approximation, with a relative error never exceeding $\frac{e^2}{2+e}$ in excess, nor $\frac{e^2}{2-e}$ in defect. So a still nearer approximation will be $\frac{(f+gx)^3+3(f+gx)(a+bx+cx^2)}{3(f+gx)^2+a+bx+cx^2}$, with a relative error never exceeding $\frac{e^3}{4+6e+3e^2}$ in excess, nor $\frac{e^3}{4-6e+3e^2}$ in defect, and so on. The marvellous facility which these formulæ afford for the calculation of elliptic and ultra-elliptic functions, and not merely for their computation as by a method of quadratures, but (which is of far greater importance) their quasi-representation under circular and logarithmic forms, with assignable limits of proportional error, will be illustrated in a future communication. As regards the idea of substituting rational for irrational functions, I have only to-day learned from Mr Cayley that I am anticipated in this by Mr Merrifield*,

Or more simply, if the error in excess be treated as positive, and in defect as negative, and δ be the first and δ_i the ith limit of error, we shall have

$$\delta_i=\frac{2\delta^i}{(2+\delta)^i-\delta^i};$$

and calling $\delta_i=\theta(i, \delta)$,

$$\theta\{j, \theta(i, \delta)\}=\theta(ij, \delta).$$

Thus, then, if we call $\frac{N+x^2}{2x}=\psi x$, $\psi^q x$ will correspond to the (2^q)th order of approximation, and the absolute value of the error will be less than

$$\frac{2\delta^{2^q}}{(2+\delta)^{2^q}-\delta^{2^q}}.$$

By way of example, suppose we take 6 as our first approximation to $\sqrt{31}$, then

$$\delta<\frac{\frac{1}{2}}{5\frac{1}{2}}<\frac{1}{11};$$

and if we make $\psi x=\frac{31+x^2}{2x}$, we shall have

$$\psi^4 6 : \sqrt{31} :: 1+\omega : 1,$$

where

$$\omega<\frac{2}{23^{16}-1},$$

which serves to exemplify the prodigious rapidity of the approximation in this method of extracting the square roots of numbers.

* I quite concur with Mr Merrifield, and in fact before being made acquainted with the existence of his paper, had emitted the same opinion (among others to Dr Borchardt of Berlin), that the substitutive method, consisting in the employment of rational functions in place of the radical, affords by far the most expeditious means for the calculation of elliptic functions of all orders, especially the third, and supersedes the necessity for the construction of special

in a paper very recently read before the Royal Society, but not yet printed in the *Transactions**.

auxiliary tables. I believe, however, that my substitutions, founded on Poncelet's views, are in general the best that can be employed for the purpose. In addition to other advantages they possess this, which deserves notice—that as we know *à priori* a superior limit to the proportional error, the arithmetical values of the integrals to which they are applied may be brought out correct to any required place of decimals, without its being necessary to calculate and compare a superior and inferior limit to the integral, either one of these being sufficient in my method to indicate its own reliable degree of precision.

* In general it is obvious, if ϕx between the limits a and b retain always the same sign, and ψx within these limits be sometimes greater and sometimes less than ϕx, but the difference between them be always less than $\epsilon\phi x$, then $\int_b^a dx\,\psi x$ will differ from $\int_b^a dx\,\phi x$ by *considerably* less than $\epsilon\int_b^a dx\,\phi x$. Paradoxical, however, as it may at first sight appear, there are extreme cases where this difference tends to a ratio of equality with $\epsilon\int_b^a dx\,\phi x$. The complete elliptic function of the first order may be made to furnish an example of this. Let

$$\phi x=\frac{1}{\sqrt{\{(1-x^2)(1-c^2x^2)\}}}=\frac{\sqrt{\{(1-x^2)+b^2x^2\}}}{(1-c^2x^2)\sqrt{(1-x^2)}}$$

(so that $b^2=1-c^2$), and let

$$\psi x=\frac{f\sqrt{(1-x^2)}+gbx}{(1-c^2x^2)\sqrt{(1-x^2)}};$$

if we make

$$f=\frac{2}{1+\sqrt{2}},\quad g=\frac{2}{1+\sqrt{2}},\quad \epsilon=\frac{\sqrt{2}-1}{\sqrt{2}+1},$$

it follows from Poncelet's theorem, that for all values of x intermediate between 0 and 1, ψx will differ from ϕx by less than $\epsilon\phi x$.

Now it will easily be found by ordinary integration that

$$\int_0^1 dx\,\psi x=\frac{f}{2c}\log\frac{1+c}{1-c}+\frac{g}{c}\tan^{-1}\frac{b}{c}.$$

Hence $\int_0^1 dx\,\phi x$ must be always less than

$$\frac{f}{2(1-\epsilon)c}\log\frac{1+c}{1-c}+\frac{g}{(1-\epsilon)c}\tan^{-1}\frac{b}{c},$$

that is,

$$<\frac{1}{2c}\log\frac{1+c}{1-c}+\frac{1}{c}\tan^{-1}\frac{b}{c},$$

when c becomes indefinitely near to unity; that is, when b becomes indefinitely small, this approaches indefinitely near to $\log\frac{2}{b}+\frac{\pi}{2}$. But we know, by a theorem of Legendre, that the approximate value for the integral in such case is $\log\frac{4}{b}$; so that the superior limiting value of $\int_0^1 dx\,\phi x$, found by the application of Poncelet's method, approaches in this instance indefinitely near to the value itself. The explanation of this is easy. As c approximates to unity, the only important values of x in the integral

$$\int_0^1\frac{dx}{\sqrt{\{(1-x^2)(1-c^2x^2)\}}},$$

are those which lie in the *immediate* vicinity of 1; and for all such values the relative error is at a *negative maximum*.

The method, however, of Mr Merrifield in working out this conception is, I believe, entirely different from that here indicated: how the many mathematicians of a practical stamp, English and foreign, who have worked with

It is not a little remarkable that so rude an application of Poncelet's method should serve to indicate almost with the force of rigorous demonstration the approximate formula

$$F(c)=\log\frac{1}{b}+\text{constant},$$

when c approaches indefinitely near to unity, the constant left undetermined being known to be less than $\log 2+\frac{\pi}{2}$.

Nay, the demonstration may be made absolutely rigid if we set about to find an inferior limit. To this end make

$$\psi x=\frac{1}{\{f\sqrt{(1-x^2)}+gbx\}\sqrt{(1-x^2)}},$$

we shall find without difficulty

$$\int_0^1 \psi dx=\frac{1}{f\gamma}\log\frac{1+\gamma-b}{\gamma-1+b}\frac{\gamma+b}{\gamma-b}, \text{ where } \gamma=\sqrt{(1+b^2)},$$

and consequently we shall obtain as an *inferior* limit to $F(c)$ the expression

$$\frac{1}{\gamma}\log\frac{1+\gamma-b}{\gamma-1+b}\frac{\gamma+b}{\gamma-b},$$

which approaches indefinitely near to $\log\frac{2}{b}$ as c approaches indefinitely near to unity. It is thus seen that Legendre's $F(c)$, when c is indefinitely near to 1, lies between $\log\frac{2}{b}$ and $\log\frac{2}{b}+\frac{\pi}{2}$; the arithmetical mean between these limits is $\log\frac{2}{b}+\frac{\pi}{4}$, that is, $\log\frac{1}{b}+1{\cdot}4785$, differing by only ·0923 from the true value $\log\frac{1}{b}+\log 4$. Of course, when the form of $F(c)$ in the case supposed is known, namely, $\log\frac{1}{b}+C$, there is no difficulty in determining C (as may be seen in Verhulst's *Traité des Fonctions Elliptiques*); but the process above given of throwing the general value of $F(c)$ between limits, is, I believe, by far the easiest and most natural method of obtaining this form. The limits themselves, it should be noticed, have virtually been found by the method, simple to *naïveté*, of writing $\sqrt{(1-c^2x^2)}=\sqrt{(p^2+q^2)}$, where $p=\sqrt{(1-x^2)}$ and $q=bx$, and then substituting for $\frac{1}{\sqrt{(p^2+q^2)}}$, $\frac{1}{p+q}$ as an inferior, and $\frac{p+q}{p^2+q^2}$ as a superior limit in the quantity to be integrated. *Closer* and *calculable* limits *ad libitum* to the integral may be arrived at by substituting for $\frac{1}{\sqrt{(p^2+q^2)}}$ one or the other of the two following rational functions of p, q, according as we wish to obtain an inferior or superior limit to the integral, namely,

$$\frac{\{p+q+\sqrt{(p^2+q^2)}\}^i-\{p+q-\sqrt{(p^2+q^2)}\}^i}{\{p+q+\sqrt{(p^2+q^2)}\}^i+\{p+q-\sqrt{(p^2+q^2)}\}^i}\frac{1}{\sqrt{(p^2+q^2)}},$$

or

$$\frac{\{p+q+\sqrt{(p^2+q^2)}\}^i+\{p+q-\sqrt{(p^2+q^2)}\}^i}{\{p+q+\sqrt{(p^2+q^2)}\}^i-\{p+q-\sqrt{(p^2+q^2)}\}^i}\frac{1}{\sqrt{(p^2+q^2)}},$$

in which formulæ the greater i is taken the closer will be the approximation. I am not aware that any of these limits to $F(c)$ (even the simplest of which, namely, those given above, may have some value for computational purposes, and have fallen thus very incidentally in my way) have ever before been noticed.

It is not unworthy of notice that the second superior limit to $\frac{1}{\sqrt{(p^2+q^2)}}$, namely, $\frac{p^2+pq+q^2}{(p+q)(p^2+q^2)}$, is an arithmetic mean between the first superior and first inferior limits, and

Poncelet's method during the last quarter of a century, should have managed to overlook so obvious and important an extension of the principle and its applications, I find hard to realize; and my wonder is even greater that I should not have been anticipated twenty years ago, than that I should have been anticipated so recently. But the algebraical theory to which this extension points the way is replete with interest of a far higher order than its applications to practice; for plainly the derived approximate fractions, however sufficient for the purposes of computation, are not, nor ever can be the best and closest of their respective kinds*. To fix the ideas, let us

consequently our second superior limit to the integral when b is indefinitely small becomes $\log\frac{2}{b}+\frac{\pi}{4}$, which brings the constant much nearer to its true value than did the use of the first limit; and as this approximation will evidently not stop at the second step of the process, we may safely infer that the integral derived from either formula when $i=\infty$ (for all values of b, whether finite or indefinitely small), not merely bears to $F(c)$ a ratio differing infinitely little from that of equality, but is absolutely equal to, and may for all analytical purposes be employed to represent $F(c)$.

I have been at the trouble of calculating the inferior limit afforded by the second approximation, and find that for b indefinitely small it is $\log\frac{2}{b}+\frac{\pi}{3\sqrt{3}}$, that is, $\log\frac{1}{b}+1{\cdot}2977$; the superior limit has been shown to be $\log\frac{1}{b}+1{\cdot}4785$, the mean is therefore $\log\frac{1}{b}+1{\cdot}3881$, differing by only ·0018 from the true value! As the constant continues for all values of i to be a multiple of π, the ith approximations *à suprà* and *ab infrà*, which are always effectible, will give (on making $i=\infty$) two new expansions for π, one infinitesimally in excess, the other infinitesimally in defect of its true value expressed as a multiple of log 2, which it might well repay the trouble of some young analyst to develope.

* That the fractional forms derived from the linear substitutive form are not the best of their respective kinds, appears immediately, so far as the derivatives of the odd order (subsequent to the first) are concerned, from the consideration that the limits of error in excess and in defect will be actually attained for values of x lying within the prescribed limits; but these errors, ϵ_i and η_i (when $\epsilon=\eta$, which is true by hypothesis), are never equal, the former (the extreme error in defect) being always the greater of the two; but if any such derivative were the best of its kind, the absolute values of the extreme errors of excess and defect ought to be *equal* to each other. But more generally, if possible, let the ith derivative to $L(x)$ (where $L(x)$ represents the radical linear approximant $fx+b$ to $\sqrt{(a+bx+cx^2)}$, say $Q(x)$), namely,

$$\frac{(Lx+Qx)^i+(Lx-Qx)^i}{(Lx+Qx)^i-(Lx-Qx)^i}Q(x),$$

be supposed the best of its kind: then the relative error is $\frac{2(Lx-Qx)^i}{(Lx+Qx)^i-(Lx-Qx)^i}$, and the maximum value of this must be equal (to the sign *près*) to the value which it has when we give to x either of its extreme connecting values. Now obviously the above is a maximum only when $\frac{Lx+Qx}{Lx-Qx}$ is a minimum, and therefore when $\frac{Lx}{Qx}$ is a maximum; but by hypothesis, the value of x, say m, which makes this a maximum, gives to $\frac{Lx}{Qx}-1$ the same value with the opposite sign to that which it would have in writing for x either of its limiting values, say k or k'.

Thus we have two equations for determining $\frac{Lk}{Qk}$, $\frac{Lm}{Qm}$, namely,

$$\frac{Lm}{Qm}-1=1-\frac{Lk}{Qk},$$

confine ourselves to the *second* Ponceletic approximation to $\sqrt{(a+bx+cx^2)}$, namely, that which has the form $\frac{\lambda+\mu x+\nu x^2}{1+qx}$, where λ, μ, ν are to be determined. The problem to be solved is the following.

Let $$\lambda+\mu x+\nu x^2 = V,$$
$$(1+qx)\sqrt{(a+bx+cx^2)} = U;$$

it is required to assign the four constants λ, μ, ν, q, so that the maximum value of $\left(\frac{V}{U}-1\right)^2$ for values of x intermediate between a and b shall be the least possible. Some little way, but only a little way, into the solution of this problem we can look in advance. In the first place, if we seek for the maximum values of $\left(\frac{V}{U}-1\right)^2$, we obtain the rational equation

$$\sqrt{(a+bx+cx^2)}\left(U\frac{dV}{dx}-V\frac{dU}{dx}\right)=0,$$

which will easily be seen to be a *cubic* (not a *biquadratic*) equation in x. Call $\left(\frac{V}{U}-1\right)=\phi(x)$; then the three roots of this equation being named x_1, x_2, x_3, the law of equality explained in my preceding paper would seem to show* that we must be able to satisfy the following equations,

$$(\phi x_1)^2 = (\phi x_2)^2 = (\phi x_3)^2 = (\phi a)^2 = (\phi b)^2,$$

which amount to four independent equations, the precise number of constants λ, μ, ν, q to be determined. So in like manner the ith rational *approximant* will contain $2i$ disposable constants; the differentiation of the quantity analogous to $\frac{V}{U}$ will give rise to an equation of the $(2i-1)$th degree; and

and $$\left(\frac{Lm}{Qm}+1\right)^i-\left(\frac{Lm}{Qm}-1\right)^i=(-)^{i-1}\left\{\left(\frac{Lk}{Qk}+1\right)^i-\left(\frac{Lk}{Qk}-1\right)^i\right\}.$$

Thus, suppose $i=2$, we should obtain from the second equation $\frac{Lm}{Qm}=-\frac{Lk}{Qk}$, which is inconsistent with the first; so if $i=3$, we should obtain $\left(\frac{Lm}{Qm}\right)^2=\left(\frac{Lk}{Qk}\right)^2$, and therefore, on account of the first equation, $\frac{L(m)}{Q(m)}=1$; and so in like manner for any value of i, we should derive one or more *numerical* values for $\frac{Lm}{Qm}$, which is absurd, since this quantity is a function of k, k', the two connecting values of x.

* Is it not, however, somewhat uncertain whether the equalities

$$(\phi x_1)^2=(\phi x_2)^2=(\phi x_3)^2$$

must all, in all cases (that is to say, for all given values of the limits) subsist? since the law of equality will not apply to such values of x as lie without the prescribed limits, and *non constat à priori* that the roots of the cubic do all lie within these limits. The subject at the very threshold is beset with doubts and difficulties of a peculiar kind, which we can hardly hope to overcome without calling in geometrical imagination to our aid.

there will be $2i-1+2$, that is, $2i+1$ functions of these $2i$ quantities to be equated, which furnish precisely the required number of equations to make the problem definite. It is, however, apparent that in solving these equations we shall find a *multiplicity* of systems, by which I mean a *definite* number of systems of values of the disposable constants which will equally well satisfy the equations. For instance, in the theory of the second approximation, the equalities

$$(\phi x_1)^2=(\phi x_2)^2=(\phi x_3)^2$$

will be satisfied by supposing $x_1=x_2=x_3$*. But it is by no means evident *à priori* that this system of equalities will correspond to the absolute minimum of which we are in quest: nay, though even we had $\phi x_1=\phi x_2=\phi x_3$, those equations do not necessarily imply $x_1=x_2=x_3$. Of the multiplicity of solutions referred to, one only gives the true minimum; but to assign *à priori* the distinguishing marks of this truest and best, *hic labor, hoc opus est*. It will be delightful to find, if it turn out to be true, that for the best form, $\frac{P}{Q}$ representing $\sqrt{X}$ (P being a rational function of the ith degree, and Q of the $(i-1)$th in x), the rational quantity

$$XQ\frac{dP}{dx}-\sqrt{X}\,P\frac{d}{dx}(Q\sqrt{X})$$

must be a perfect $(2i-1)$th power of a linear function of x; but in the present state of my ignorance I dare not do more than affirm that there is a bare probability in favour of this being true: whoever shall first succeed in discovering the true form of the expression will have established a remarkable theorem. Here for the moment I break off, contented with having pointed to a theory as yet, if the expression may be allowed, sleeping in its cradle, but destined, I am persuaded, at no distant day to set in motion as large a mass of algebraical thought as has been set in motion by the never-to-be-forgotten Hessian discussion of the flexures of the cubic curve,—the turning-point between the old algebra and the new.

Henceforward Poncelet's theorem figures no longer as a detached method, a mere stroke of art in aid of the computer, but becomes integrally attached to the grand and progressive body of doctrine of the modern algebra.

* If this is so, we shall have for determining the four constants the following equations:

$$x_1=x_2=x_3,\quad \phi a=\phi b=-\phi x_1.$$

But more probable than this seems the conjecture, that, supposing x_1, x_2, x_3 to be arranged in the order of their relative magnitudes, the determining equations might be

$$x_1=x_3,\quad \phi a=\phi b=\phi x_2=-\phi x_1.$$

Or is it possible that the *character* of the solution may be discontinuous, and may depend upon the magnitudes, relative or absolute, of the given limits a and b? Probably Dr Tchebitcheff would be able better than any other living analyst to answer these queries. But what an endless vista of future research does the prosecution of the Ponceletic method open out to us!

33.

NOTES TO THE MEDITATION ON PONCELET'S THEOREM, INCLUDING A VALUATION OF THE TWO NEW DEFINITE INTEGRALS

$$\int_0^{\frac{\pi}{2}} \frac{\log \cos \phi \, d\phi}{\sqrt{\{1 - b^2 (\cos \phi)^2\}}}, \quad \int_0^{\frac{\pi}{2}} \frac{\log [1 + \sqrt{\{1 - b^2 (\cos \phi)^2\}}] \, d\phi}{\sqrt{\{1 - b^2 (\cos \phi)^2\}}}.$$

[*Philosophical Magazine*, XX. (1860), pp. 525—533.]

NOTE A.

THE method given in the October Number of the *Magazine* for approximately representing a quadratic surd by a rational fraction is equally applicable to a surd of any degree. To fix the ideas, suppose we wish to approximate in this manner to $\sqrt[3]{R}$.

If we assume P as the first approximation, and make

$$L = P + \sqrt[3]{R}, \quad M = P + \rho \sqrt[3]{R}, \quad N = P + \rho^2 \sqrt[3]{R},$$

where $\rho^3 = 1$, and write

$$F_1 = L^i + M^i + N^i,$$
$$F_2 = L^i + \rho^2 M^i + \rho N^i,$$
$$F_3 = L^i + \rho M^i + \rho^2 N^i,$$
$$U_1 = \frac{F_1}{F_2} R^{\frac{1}{3}}, \quad U_2 = \frac{F_2}{F_3} R^{\frac{1}{3}}, \quad U_3 = \frac{F_3}{F_1} R^{\frac{1}{3}},$$
$$V_1 = \frac{F_2}{F_1} R^{\frac{1}{3}}, \quad V_2 = \frac{F_3}{F_2} R^{\frac{1}{3}}, \quad V_3 = \frac{F_1}{F_3} R^{\frac{1}{3}},$$

we may easily establish the following propositions, which indeed are almost self-evident:—

(1) Each U and V is a rational fraction.

(2) When $i = \infty$, each $U = R^{\frac{1}{3}}$, each $V = R^{\frac{1}{3}}$.

(3) For all finite values of i, $R^{\frac{1}{3}}$ is intermediate between the least and greatest U, and $R^{\frac{1}{3}}$ between the least and greatest V.

So in general if k is any prime number, we may form $(k-1)$ cycles, each cycle containing k fractions possessing precisely analogous properties as regards representing approximately and limiting the successive powers of $R^{\frac{1}{k}}$. By means of these formulæ [the theory of which might be extended to algebraic quantities of every order (in Abel's sense of the word)], we obtain a complete command over the integration of surd quantities in general as they may appear in any physical problem, being thereby enabled to represent the integrals, not merely arithmetically, but analytically (which is of much higher importance) by logarithmic and circular functions to any degree of accuracy that may be required, and with known assignable numerical limits of error.

NOTE B.

This note relates to the concluding paragraph of the long note at page 313 in the October Number of the *Magazine* [203, above]. I find that the ith inferior limit to $F(c) - \log \frac{2}{b}$, when c differs indefinitely little from unity given by the method therein explained, is

$$\log \frac{2}{b} + \frac{2}{i} \sum_{k=E\frac{i}{2}}^{k=1} \frac{\cos \frac{2k-1}{2k} \pi}{\sqrt{\left\{1 + \left(\sin \frac{2k-1}{2i} \pi\right)^2\right\}}} \cos^{-1} \left(\sin \frac{2k-1}{2i} \pi\right)^2,$$

and that the superior limit is

$$\log \frac{2}{b} + \frac{\pi}{2i} + \frac{2}{i} \sum_{k=E\frac{i}{2}}^{k=1} \frac{\cos \frac{k\pi}{i}}{\sqrt{\left\{1 + \left(\sin \frac{k\pi}{i}\right)^2\right\}}} \cos^{-1} \left(\sin \frac{k\pi}{i}\right)^2.$$

When $i = \infty$ these limits of course come together, and the finite sums resolve themselves into the definite integral

$$\frac{2}{\pi} \int_0^{\frac{\pi}{2}} d\tau \frac{\cos \tau}{\sqrt{\{1 + (\sin \tau)^2\}}} \cos^{-1} (\sin \tau)^2,$$

of which, therefore, the value must be log 2. Hence, writing $(\sin \tau)^2 = \cos 2\theta$, we obtain

$$\int_0^{\frac{\pi}{4}} dt \frac{\theta \sin \theta}{\sqrt{(\cos 2\theta)}} = \frac{\log 2}{\sqrt{2}} \frac{\pi}{4}.$$

NOTE C.

It may be shown that any of the expressions for $N^{\frac{1}{k}}$ derived from making $i = \infty$ in the general formulæ given in Note A, are in fact tantamount to its representation as a definite integral of a very simple kind. I shall not go into the proof of this here; it may be sufficient to indicate that it depends upon the fact that the equation of infinite degree $(\phi x)^i + (\psi x)^i + (\vartheta x)^i + \&c.$, may be resolved into sets of factors of a known form. In the question before us, the function to be so resolved is the denominator of any one of the quantities analogous to U or V in Note A; and ϕx, ψx, ϑx ... become linear functions of x with imaginary coefficients. Its resolution into factors is rendered possible by the circumstance that only two of the quantities ϕ, ψ, ϑ ... can bear a finite ratio to each other for any given value of x, and consequently all the roots of the equation

$$(\phi x)^i + (\psi x)^i + (\vartheta x)^i \ldots = 0\,*$$

are contained *among* the roots of several binary equations

$$(\phi x)^i = (\psi x)^i, \quad (\phi x)^i = (\vartheta x)^i, \ \&c.:$$

which are the roots of any one of these equations (as for example of the first) that belong to the *given* equation will be determined by the condition that they must make the norms of all the other functions (for example of ϑx) indefinitely small as compared with the norms of those two which appear in it (for example ϕx, ψx). In this manner, if the total number of the functions is k, supposing ϕ, ψ, ϑ ... to be all linear functions of x, each binomial equation out of its entire stock of i roots will contribute $\dfrac{i}{k^{\frac{k-1}{2}}}$ roots available towards the solution of the given equation. Mr Cayley has remarked to me the analogy between this determination and Newton's method of finding the form of the several parabolic equations $y = cx^\lambda$ which represent the branches of a given algebraical curve at its origin. In the equation to the given curve cx^λ is to be substituted for y; the terms will then all become powers of x (an infinitesimal) whose indices will be linear functions of λ; every pair of them in turn is equated to zero, and of all the values of λ thus obtained only those will be preserved which cause the two equated linear functions of λ belonging to any given pair of terms to be less than all the others, and consequently the terms themselves (whose indices the linear functions are) infinitely greater than all the other terms.

Linear functions of a variable figure in both investigations, namely, in Newton's as indices of the same infinitesimal quantity, in mine as quantities whose infinite index is the same†; but the logic and mode of procedure (utterly unlike as are the questions in their origin and subject matter) is the same in either case.

* My friend, M. Jordan, of the École des Mines (author of a remarkable thesis on *groups*), has developed some interesting geometrical consequences arising out of the study of this equation, which I hope he may be induced to publish.

† In a word, Newton's equation is an exponential one made up of nothings, mine an algebraical one made up of infinities.

NOTE D.

The remark contained in the preceding note, as to the effect of representing $N^{\frac{1}{k}}$ by an infinite rational fraction being identical with that of expressing it as a definite integral, combined with a consideration of the cause of the success of the particular method referred to in Note B, has led me to the investigation following, of the value of the complete elliptic function of the first species. As usual denoting it by $F(c)$, we have

$$F(c) = \int_0^{\frac{\pi}{2}} d\theta \frac{1}{\sqrt{\{1 - c^2(\sin\theta)^2\}}}$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta \int_0^{\infty} dx \frac{\cos\theta}{1 - c^2(\sin\theta)^2 + (\cos\theta)^2 x^2}$$

$$= \frac{2}{\pi}\int_0^{\infty} dx I,$$

where

$$I = \int_0^{\frac{\pi}{2}} d\theta \frac{\cos\theta}{(1+x^2) - (c^2+x^2)(\sin\theta)^2}$$

$$= \frac{1}{\sqrt{\{(1+x^2)(c^2+x^2)\}}}\left[\log\{\sqrt{(1+x^2)} + \sqrt{(c^2+x^2)}\} - \log\sqrt{(1-c^2)}\right].$$

Let $x = \tan\phi$, $b = \sqrt{(1-c^2)}$; then

$$F(c) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\{1 - b^2(\cos\phi)^2\}}}\{\log[\sec\phi + \sqrt{\{(\sec\phi)^2 - b^2\}}] - \log b\}$$

$$= \frac{2}{\pi}\log\frac{1}{b}F(b) + \frac{2}{\pi}R,$$

where

$$R = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\{1 - b^2(\cos\phi)^2\}}}\log[\sec\phi + \sqrt{\{(\sec\phi)^2 - b^2\}}]$$

$$= \int_{\frac{\pi}{2}}^{0} d\phi \left\{\frac{\log(\cos\phi)}{\sqrt{\{1 - b^2(\cos\phi)^2\}}}\right\} + \int_0^{\frac{\pi}{2}} d\phi \frac{\log[1 + \sqrt{\{1 - b^2(\cos\phi)^2\}}]}{\sqrt{\{1 - b^2(\cos\phi)^2\}}}.$$

It will presently appear that these two definite integrals are equal to one another!

Let $$V_{2r} = \int_{\frac{\pi}{2}}^{0} (\cos\phi)^{2r}\log(\cos\phi)\,d\phi.$$

Then we may easily establish the formula of reduction,

$$V_{2r} = \frac{2r-1}{2r}V_{2r-2} - \frac{1.3.5..(2r-3)}{2.4.6..(2r-2)2r}\frac{\pi}{2};$$

and since (as is well known) $V_0 = \frac{\pi}{2}\log 2$, we have

$$V_2 = \frac{1}{2}\frac{\pi}{2}\left(\log 2 - \frac{1}{1.2}\right),$$

$$V_4 = \frac{1.3}{2.4}\frac{\pi}{2}\left(\log 2 - \frac{1}{1.2} - \frac{1}{3.4}\right),$$

$$V_6 = \frac{1.3.5}{2.4.6}\frac{\pi}{2}\left(\log 2 - \frac{1}{1.2} - \frac{1}{3.4} - \frac{1}{5.6}\right),$$

$$\&c. = \&c.$$

Hence, by expanding the denominator in a series proceeding according to powers of $(\cos\phi)^2$, it is readily seen that the first integral becomes

$$\frac{\pi}{2}\left\{\log 2 + \left(\frac{1}{2}\right)^2\left(\log 2 - \frac{1}{1.2}\right)b^2 + \left(\frac{1.3}{2.4}\right)^2\left(\log 2 - \frac{1}{1.2} - \frac{1}{3.4}\right)b^4 + \&c.\right\}.$$

To find the second integral, we must obtain the general term in the expansion in a series of powers of t of

$$\frac{\log\{1+\sqrt{(1-t^2)}\}}{\sqrt{(1-t^2)}}$$

(where t stands for $b\cos\phi$), that is, of

$$\frac{1}{\sqrt{(1-t^2)}}\int dt\left(\frac{1}{t} - \frac{1}{t\sqrt{(1-t^2)}}\right),$$

say of $\phi t = \frac{1}{\sqrt{(1-t^2)}}\psi t$. Now

$$\left(\frac{d}{dt}\right)^2\{(1-t^2)\phi t + \int dt\,(t\phi t)\} = \left(\frac{d}{dt}\right)^2\left(\sqrt{(1-t^2)}\,\psi t + \int dt\,\frac{t\psi t}{\sqrt{(1-t^2)}}\right)$$

$$= \psi t\left\{\left(\frac{d}{dt}\right)^2\sqrt{(1-t^2)} + \frac{d}{dt}\frac{t}{\sqrt{(1-t^2)}}\right\} + \psi' t\left(\frac{-2t}{\sqrt{(1-t^2)}} + \frac{t}{\sqrt{(1-t^2)}}\right) + \sqrt{(1-t^2)}\,\psi'' t$$

$$= -\frac{t}{\sqrt{(1-t^2)}}\psi' t + \sqrt{(1-t^2)}\,\psi'' t$$

$$= -\frac{t}{\sqrt{(1-t^2)}}\left\{\frac{1}{t} - \frac{1}{t\sqrt{(1-t^2)}}\right\} - \sqrt{(1-t^2)}\left\{\frac{1}{t^2} + \frac{2t^2-1}{t^2(1-t^2)^{\frac{3}{2}}}\right\}$$

$$= -\frac{1}{\sqrt{(1-t^2)}} - \frac{\sqrt{(1-t^2)}}{t^2} + \frac{1}{1-t^2} - \frac{2t^2-1}{t^2(1-t^2)}$$

$$= \frac{-1}{t^2\sqrt{(1-t^2)}} + \frac{2}{t^2}$$

$$= \frac{1}{t^2} - \frac{1}{2} - \frac{1.3}{2.4}t^2 - \frac{1.3.5}{2.4.6}t^4, \ \&c.$$

Hence, writing

$$\frac{\log\{1+\sqrt{(1-t^2)}\}}{\sqrt{(1-t^2)}} = \log 2 + K_2 t^2 + \ldots + K_{2i-2}t^{2i-2} + K_{2i}t^{2i} + \&c.,$$

and equating the coefficients of t^{2i}, we obtain

$$2i(2i-1)(K_{2i} - K_{2i-2}) + (2i-1)K_{2i-2} = -\frac{1.3.5\ldots(2i-1)}{2.4.6\ldots 2i},$$

that is,
$$K_{2i} = \frac{2i-1}{2i}K_{2i-2} - \frac{1.3.5\ldots 2i-3}{2.4.6\ldots(2i-2)(2i)^2}.$$

Thus $K_0 = \log 2$, $K_2 = \frac{1}{2}\left(\log 2 - \frac{1}{1.2}\right)$, $K_4 = \frac{1.3}{2.4}\left(\log 2 - \frac{1}{1.2} - \frac{1}{3.4}\right)$,

$$\&c. = \&c.,$$

and consequently
$$\int_0^{\frac{\pi}{2}} d\phi\,\frac{\log[1+\sqrt{\{1-b^2(\cos\phi)^2\}}]}{\sqrt{\{1-b^2(\cos\phi)^2\}}}$$

$$= \frac{\pi}{2}\left\{\log 2 + \left(\frac{1}{2}\right)^2\left(\log 2 - \frac{1}{1.2}\right)b^2 + \left(\frac{1.3}{2.4}\right)^2\left(\log 2 - \frac{1}{1.2} - \frac{1}{3.4}\right)b^4 + \&c.\right\}.$$

Thus, then, we obtain the following remarkable equalities *:

$$\frac{\pi}{2}F(c) = \log\frac{1}{b}F(b) + 2\int_{\frac{\pi}{2}}^{0} d\phi\,\frac{\log(\cos\phi)}{\sqrt{\{1-b^2(\cos\phi)^2\}}}$$

$$= \log\frac{1}{b}F(b) + 2\int_0^{\frac{\pi}{2}} d\phi\,\frac{\log[1+\sqrt{\{1-b^2(\cos\phi)^2\}}]}{\sqrt{\{1-b^2(\cos\phi)^2\}}},$$

or
$$\int_0^{\frac{\pi}{2}} d\phi\,\frac{\log[1+\sqrt{\{1-b^2(\cos\phi)^2\}}]}{\sqrt{\{1-b^2(\cos\phi)^2\}}} = \int_{\frac{\pi}{2}}^{0} d\phi\,\frac{\log(\cos\phi)}{\sqrt{\{1-b^2(\cos\phi)^2\}}}$$

$$\left[= \int_0^{\frac{\pi}{2}} d\phi\,F(b,\phi)\cot\phi\right] = \frac{\pi}{4}F(c) + \frac{1}{2}\log b\,F(b).$$

When b is indefinitely small, it is obvious from either of these equations that

$$F(c) = -\frac{2}{\pi}\log b\,\frac{\pi}{2} + 2\log 2 = \log\frac{4}{b},$$

Legendre's well-known formula previously referred to.

The equality of the first two definite integrals in the *sorites* above given, is, as we have seen, a consequence of the equality

$$\frac{\log\{1+\sqrt{(1-t^2)}\}}{\sqrt{(1-t^2)}} = \frac{2}{\pi}\int_{\frac{\pi}{2}}^{0} d\theta\,\{\log\cos\theta + \log\cos\theta\,(\cos\theta)^2 t^2$$

$$+ \log\cos\theta\,(\cos\theta)^4 t^4 + \&c.\}.$$

[* The reader may be glad to have the references: *Schlömilch Zeitschrift*, II. (1857), pp. 49, 414; *Tortolini Annali*, III. (1860), p. 254. See also below, p. 298.]

Hence we have

$$\int_{\frac{\pi}{2}}^{0} \frac{\log\cos\theta}{1-b^2(\cos\theta)^2}\,d\theta = \frac{\pi}{2}\,\frac{\log\{1+\sqrt{(1-b^2)}\}}{\sqrt{(1-b^2)}}\,^*.$$

The extreme facility and brevity with which the method in the text gives the value of $F(c)$ for b indefinitely small is worthy of notice, as in the usual text-books it is obtained by a very indirect and circuitous process. We may obtain in like manner the value of

$$\int_0^{\frac{\pi}{2}} d\theta\,\frac{1}{(1-e\sin\theta)}\cdot\frac{1}{\sqrt{\{1-c^2(\sin\theta)^2\}}}$$

on the same supposition as to c, whether $1-e$ vanishes with $1-c$ or remains finite when $c=1$. On the latter supposition, the definite integral in question has for its value

$$\frac{1}{1-e}\log\frac{2}{b}+\frac{1}{1-e^2}\log\frac{2}{(1+e)^e}.$$

When $e=1$, this becomes infinite; when $e=-1$, the second term becomes $\frac{1}{4}+\frac{1}{2}\log 2$, and the entire integral is $\frac{1}{2}\log\frac{4}{b}+\frac{1}{4}$; when $e=0$, it is $\log\frac{4}{b}$. Subtracting the half of the latter integral from the former, we shall obtain

$$\int_0^{\frac{\pi}{2}} d\theta\,\frac{(1-\sin\theta)^2}{(\cos\theta)^3}=\frac{1}{2},$$

which is easily verified.

By taking successively $e=\sqrt{(-n)}$, $e=-\sqrt{(-n)}$, and adding together the halves of the two integrals corresponding to these suppositions, we obtain the *ultimate* value of the complete elliptic integral of the third kind, namely,

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+n(\sin\theta)^2}\cdot\frac{1}{\sqrt{\{1-c^2(\sin\theta)^2\}}},$$

from the general formula above given, always of course subject to the condition that c is supposed indefinitely near to 1†.

* From this it will readily be seen that when n is any integer we may obtain

$$\int_0^{\frac{\pi}{2}} \frac{\log\cos\theta}{\{1-b^2(\cos\theta)^2\}^n}$$

by processes of differentiation in a form involving only algebraical and logarithmic quantities, and so, from what precedes, when n is any half-integer, in terms of such quantities and of complete elliptic functions.

† It seems to be expected of every pilgrim up the slopes of the mathematical Parnassus, that he will at some point or other of his journey sit down and invent a definite integral or two towards the increase of the common stock. The author of these notes has been somewhat late in acquitting himself of this debt of honour, but ventures to hope that the principal results contained in the text above may be thought not unworthy of a place in some future edition of that noble and sumptuous monument of Dutch learning, industry, and fine taste, the invaluable collection of definite integrals by M. Bierens de Haan.

34.

ON THE PRESSURE OF EARTH ON REVETMENT WALLS.

[*Philosophical Magazine*, xx. (1860), pp. 489—499.]

Part I.

Critique of the Hypothesis of Parallel "Planes of Rupture."

The ensuing investigation deals with the pressure of *Mathematical* earth. By mathematical earth, I mean earth treated according to the idea of Coulomb, namely, as a *continuous** mass separable by planes in all directions, but whose separating surfaces exert upon one another forces consisting of two parts, one of the nature of ordinary friction, the other of so-called cohesion. Of the latter, for greater simplicity, I shall commence with taking no account, so that the matter with which we have to deal becomes, so to say, "a frictional fluid." If we isolate in idea any element of this fluid—suppose, to fix the ideas, a molecule bounded by plane faces, this molecule will be kept at rest by its own weight, the pressures on the several faces, and the forces of friction acting along these faces: these last-named forces are limited not to exceed the product of the corresponding pressures by a certain coefficient, termed the coefficient of friction.

In order to render the inquiry before us quite definite, let us begin with supposing two vertical side walls and a back of solid immoveable masonry, between which the earth is piled up in a determinate form, fronted by a pier of given specific gravity, whose *minimum* thickness is to be determined by the condition that it may just suffice to prevent the pier from being either forced forward or turned round over its further edge. The earth is thus of course supposed to have only one free face, being entirely supported at the sides and the back by the masonry just spoken of. The problem then that we have to solve is evidently the following:—"Of all the possible states of equilibrium of the earth consistent with the assigned conditions, to determine that one which shall make the greater of two quantities to be

* The only *essential* quality of our mathematical earth which differentiates it from actual vulgar earth is this of *continuity*.

named the least possible,"—one of these quantities being the thickness of the wall determined by the condition that its friction with the ground shall be just equal to the sum of the horizontal pressures on the wall, the other by the condition that its moment about the edge most remote from the earth shall be just equal to the sum of the moments of the entire thrust upon the wall at each several element thereof in respect to the same edge.

Whenever Coulomb's method leads to a right solution of the problem of revetments, the thrusts on the several elements of the wall will be all parallel; and it may easily be seen that, in solving the problem for this case, we are solving the problem of making the statical sum of the thrusts a minimum; and the result will be the same, whether the pier can only be pushed bodily on its base, or can only turn over an edge, or can do both one and the other. But it must obviously be erroneous to assume as a universal principle, that in the state bordering upon motion, or what is going still further, in a state antecedent to this, the statical sum of the pressures will be a minimum; and if Mr Moseley's "principle of least resistance," quoted by Professor Rankine, means this, I have no scruple in proclaiming my entire dissent from such an assumption. I do not here enter at all into the question of determining pressure, except in the state of equilibrium bordering upon motion; and in that state common sense points out that it is not the pressure or sum of pressures, but the effect of such pressure or pressures in inducing motion in a certain possible manner, or in any one out of a choice of possible manners, that governs the determination of the minimum. This principle of least resistance is one of the shoals upon which Mr Rankine's investigation appears to me to have split.

Be it observed that the only physical assumption which I propose is this, that if *equilibrium can be preserved consistently with the imposed conditions, equilibrium will be preserved*. Without such a supposition the question would be incapable of treatment without further laws regulating the interior forces than we suppose given. The legitimacy of such an assumption cannot, I think, be seriously called into question, and once made, the problem of determining the wall's thickness becomes a purely mathematical question; one undoubtedly of great difficulty, but perfectly determinate, and falling under the dominion of the Calculus of Variations, as will easily be recognized from the circumstance that the integration of the general equations of equilibrium, if it could be performed, would necessarily contain arbitrary functions, whose form would have to be assigned so as to make a certain quantity or the greatest of a set of quantities a minimum; but the peculiar manner in which the internal forces are defined as subject to satisfy not an equation or system of equations, but a law of inequality, must render it a task exceeding the present powers, at all events, of the writer of this paper, to arrive at a result by the direct application of the Calculus referred to. In order to pave the

way to the discussion of the more general inquiry, I shall commence with examining whether under any and what circumstances the forced solution of Coulomb and his followers, founded upon the notion of what have been (it seems to me incautiously) termed planes of fracture or rupture (but which really mean no more than planes for which friction at each point thereof is acting with its utmost energy, that is, if we please so to say, planes of greatest frictional energy*), is the true solution; that is to say, I shall investigate under what conditions the surfaces of "rupture" or "of greatest energy of friction" are or can be planes; and I shall easily be able to ascertain these conditions, and to prove that when they are satisfied (but not otherwise) the results of the received theory are exact.

Professor Rankine, in the light in which he appears in a paper published in the *Transactions* of the Royal Society, is not to be ranked among those whom I have called the followers of Coulomb. He is entitled to the merit of

* It is obvious that the notion of the planes in question being the planes in which the earth would begin with crumbling, if the equilibrium were disturbed by the wall giving way (for such is the idea intended to be conveyed by their being called planes of rupture), is quite irrelevant to the determination of their position, and to the solution of the question of the thrust in the wall. But such a notion in itself is objectionable, as assuming a physical fact for which there is no just ground. The idea, or rather I may say the metaphysical process, which unconsciously has swayed Coulomb and his followers to give them this name, appears to me to be the following. "Since it is only along these planes that friction is acting at its full energy, and since, when motion ensues, friction must be acting at its full energy, therefore a change must have taken place in the friction of any other plane before motion can take place along it, which change does not take place along the planes in question. Now every change must operate in time, therefore the motion must have begun along the planes of greatest friction before it can have taken place along any other." But it is a most dangerous proceeding, and fraught with errors familiar to mathematicians, to attempt to reason from the conditions of equilibrium to those of incipient motion; and that dynamical considerations, and not statical, must decide the incipient directions of the motion in the case before us, will be obvious when we reflect that the friction might be supposed to become *nil*, and then we should be treating of a perfect fluid, in which case the planes of rupture disappear, but none the less would motion take place in determinate directions on any wall of the reservoir containing the fluid giving way. A notable example of the important distinction between rest and equilibrium is afforded by the question (which, I am informed, originated in Caius College, Cambridge) of finding the tension of a rope by which a bucket full of water, with a cork tied to its bottom, is fastened to a fixed point, at the moment when the fastening is cut or gives way. At that moment the vertical pressure in the bottom of the bucket, supposing the specific gravity of the cork to be one-fourth that of water, if it could be estimated on statical principles, that is with reference to the elevation of the surface of the fluid [and some non-mathematical physicists might easily suppose it could be so estimated, since motion has not *yet taken place*, but is only *imminent*], would be the weight of the bucket together with that of the water, together with four times that of the cork, and so it would appear as if the tension would be increased by the cutting of the string, whereas, in fact, precisely the contrary effect will take place; for since downward momentum must result from the impending motion of the cork upwards and the water downwards, part of the weight of the water and cork is spent as downward moving force, and consequently only a portion remains to act as vertical pressure upon the bucket, just as an air-cushion will press with less force than its weight on the seat which bears it, when, in consequence of the air being let out, part of the weight is being expended in lowering the top of the cushion.

having perceived that the received hypothesis rested on no solid foundation, and of having been the first (publicly at least) to assert that the equations of internal equilibrium must be resorted to for the satisfactory discussion of the question; but, notwithstanding the sincere esteem in which I hold the great abilities of this gentleman, I have been compelled to come to the conclusion, and trust to be able to satisfy himself, that the use he has made of these equations is illusory, and that his results bear upon their very face a demonstrable character of error.

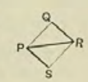
Under the supposed *data*, it is, if not obvious, at all events assumed by all writers on the subject, that the equilibrium of every vertical section of the earth, parallel to the side walls, may be determined *per se*, and that we may treat the question as one regarding space of only two dimensions. I shall therefore, with a view to clearness, treat of the equilibrium of any one such section; the molecules, whose equilibrium is to be considered, will be spoken of as bounded by lines instead of planes, and so we shall speak of lines instead of planes of "rupture," and we may thus conform our language to the relations of the figure actually represented upon the paper.

For the benefit of those to whom the conditions of molecular equilibrium are new, it may be well to indicate briefly how they may be obtained, still keeping within our prescribed framework of two-dimensioned space (although the reader will not experience the slightest difficulty in extending them to space of three dimensions)*. Through any point in the interior of the plane-mass at rest, imagine a small rectilinear element to be drawn. The entire molecular force exerted on this actual element might be termed the thrust; but by the thrust I shall understand *the unit of thrust*, corresponding to the well-known conception of *unit of pressure* for the particular case of a fluid mass. This thrust (or unit of thrust) may be imagined separated into two parts, one perpendicular to the element, which may be

* I have purposely begun with the beginning, because I wish to give perfect precision to the terms Thrust, Pressure, and Stress, as I shall use them. Some recent authors on mechanics have wished to distinguish force measured statically from force measured by acceleration, by giving to the former the name of pressure. But surely unnecessary confusion is introduced into mechanical language when we are thereby reduced to speak of the pressure of friction, and ought to enunciate the cardinal law of friction by stating that the *pressure of friction* bears to the *pressure of pressure* a certain limiting relation. I acknowledge an objection scarcely less valid (except that it has antiquity to plead in excuse) to the use of the term accelerating force; as we may be thereby reduced to speak of the accelerating force of a retarding influence, as friction, or of an influence which does not necessarily either accelerate or retard, as in the case of a centripetal pull upon a body moving uniformly in a circle. I think this difficulty in language may be met to some extent by giving to force, usually called accelerative, the designation of alternative, and to force measured by weight or momentum that of quantitative force. There is no magic in names, however well selected, but there may be a great deal of mischief arising out of a confused and uncertain nomenclature.

termed the *pressure*, the other parallel to it, which may be termed *face-force* [for the case we shall have more especially to consider, the face-force receives the name of *friction*, and is limited to be less than the *pressure* multiplied by the so-called coefficient of friction]. As the element acted upon turns round, the thrust changes in magnitude and direction, and to the totality of the thrusts going forth in all directions from a given point we may give the name of *stress**. We shall now be able to obtain two sorts of conditions—one giving the necessary law connecting the various *thrusts* of the same *stress*, the other expressing the law of the variation of the pressure and facial force (together constituting the thrust) upon an element given in direction in passing from one stress to another; we may call these respectively the equations of distribution and the equations of variation.

Let $PQRS$ be any infinitely small molecule bounded by lines at right angles to one another. Since this is kept at rest by its own weight, by the lines of pressures perpendicular to QR and PS, and the other pair perpendicular to PQ and RS, and by the facial forces acting along PQ, QR, RS, SP respectively, if we call f the face-force [that is, the unit of face-force] on PS, it is obvious that the corresponding quantity for QR will differ from it by an infinitely small quantity; in like manner f' may be taken as the face-force on QR, PS respectively. Hence the couples whose moments $(f \,.\, QR) \times QP$ and $(f' \,.\, PQ) \times QR$ respectively must be equal and opposite, or in other words, f being understood to act to or from Q, according as f' acts to or from Q, we must have $f = f'$. To fix the ideas, conceive the face-forces to tend towards Q. Let us now consider the equilibrium of the triangular molecule PQR. Call the pressure on $PQ\,(R)$, the pressure on $RQ\,(P)$, the face-force on PQ or $QR\,(Q)$. In comparison with the thrusts on the faces of our triangular molecule, gravity or other impressed forces may be neglected as giving rise to quantities of an inferior order of smallness.



Let QP, QR be regarded as two fixed rectangular axes, and let $QPR = \theta$. Let the pressure and face-force on PR (always understanding thereby the units of such forces) be called N and F respectively (F, to fix the ideas, being taken to act from P to R). Then resolving the forces perpendicular to PR, we obtain

$$N \,.\, PR = R \,.\, PQ \,.\, \cos QPR + P \,.\, QR \cos QRP$$
$$+ Q \,.\, PQ \sin QPR + Q \,.\, QR \sin QRP,$$

or

$$N = R (\cos \theta)^2 + 2Q \cos \theta \sin \theta + P (\sin \theta)^2;$$

* Thus, stress stands in somewhat the same relation to its component thrusts, as a radiant point to the luminous rays which it emits.

and resolving parallel to PR, we have

$$F \times PR = R \,.\, PQ \sin QPR - P \,.\, QR \sin QRP$$
$$+ Q \,.\, PQ \cos QPR - Q \,.\, QR \cos QRP,$$

or

$$F = (R - P) \sin\theta \cos\theta.$$

Imagine now QPR to be represented by a single point O. R, P are respectively the pressures, and N the face-force ("units of pressures and of face-force") on elements drawn in the orthogonal directions OX, OY; N the pressure, and F the face-force on an element drawn in the direction OP, making an angle θ with OX. Obviously, therefore, if we draw in all directions from O lines whose lengths are as the inverse square roots of the pressure-part of the thrust acting on those lines, calling the length of line corresponding to θ, r, we have

$$\frac{1}{r^2} = R (\cos\theta)^2 + 2Q \sin\theta \cos\theta + P (\sin\theta)^2,$$

R, Q, P being constant quantities.

Consequently the locus of the extremities of these lines is a conic; and taking new axes of coordinates in the directions of the principal axes of this conic, and understanding by R and P the pressures perpendicular to those axes respectively, the equations obtained assume the form

$$N = R (\cos\theta)^2 + P (\sin\theta)^2, \qquad (1)$$
$$F = (R - P) \sin\theta \,.\, \cos\theta; \qquad (2)$$

showing that in elements in the directions of the principal axes the face-forces vanish, and the thrusts become purely pressures, that is, forces perpendicular to the surfaces upon which they act. R and P are of course essentially positive, as otherwise the molecules would be subject to a force of separation instead of compression, and consequently the conic in question is an ellipse. The total value of the thrust $= \sqrt{(N^2 + F^2)}$

$$= \sqrt{\{R^2 (\cos\theta)^2 + P^2 (\sin\theta)^2\}}. \qquad (3)$$

R and P will evidently be in the directions in which, for a given point, the entire thrust, as well as the pressure-part of it, is the least and greatest. These directions may be said to be those of "principal thrust." If we start from any point and proceed from that point always in the direction of a line of principal thrust so as to form a continuous curve, two such curves cutting each other at right angles will intersect every point of the mass at rest, of which, in the case of mathematical earth, I may state, by way of anticipation, that only one can cut the free surface when that surface is supposed to form part of a horizontal plane.

These lines may also be termed the principal lines of pressure, or simply the lines of pressure; and this name may be considered indifferently to have

reference either to the fact that the thrust *in the direction of the tangent* at any point in any such curve is the thrust acting upon the normal, or to the fact that the thrust *upon the tangent* at any point is in the direction of the normal; as either one of such conditions implies the other.

The cosine of the angle between the pressure and the thrust will be

$$\frac{R (\cos\theta)^2 + P (\sin\theta)^2}{\sqrt{\{R^2 (\cos\theta)^2 + P^2 (\sin\theta)^2\}}};$$

which, calling the principal semiaxes of the ellipse referred to a and b respectively, and the rectangular coordinates of any point therein x and y, becomes

$$\frac{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}}{\sqrt{\left(\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4}\right)}} = \frac{1}{\sqrt{\left(\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4}\right)}}$$

which is equal to the perpendicular from the centre on the tangent divided by the radius vector, showing that the direction of the thrust on any radius of the ellipse in question is in the direction of the conjugate diameter, whereby it is seen that the line of thrust, and the line thrust upon, stand in a reciprocal relation to each other.

I may add the cursory remark as regards the value of the total thrusts in the case more immediately before us, that [as is apparent from the equation (3)] they will be represented in relative magnitude by the radius vector drawn in the direction of the line thrust upon, to meet, not the ellipse of pressures just described, but another ellipse whose major and minor axes are to one another in the duplicate ratio of the other two.

If we wish, however, to present the above results in a form more immediately translateable into the actual case of nature, I mean that of space with three dimensions, it becomes expedient to use a different ellipse, or rather the same ellipse in another position, to represent the stress at any point.

In the equations above found, connecting N and F with P, Q, R, θ is the angle made with a fixed axis, not by the line of pressure R, but by the element on which this pressure is exerted. Let ϕ be the angle made by the pressure itself, so that $\phi = \theta + \frac{\pi}{2}$, then we have

$$N = P (\cos\phi)^2 - 2Q \sin\phi \cos\phi + R (\sin\phi)^2$$
$$F = (P - R) \sin\phi \cos\phi.$$

And the same process as has been already employed will serve to show that we may construct an ellipse such that the inverse square of the radius vector in every direction may represent the magnitude of the *pressure* in that direction (that is the magnitude of the normal part of the thrust upon the element perpendicular to that direction), and in this ellipse the radius vector and perpendicular to the tangent at each point will represent the corresponding directions of pressure and thrust, which obviously will coincide for the directions of greatest and least pressure.

If, now, we go out into space of three dimensions, it will readily be anticipated, and may easily be proved, that an ellipsoid whose radii vectores represent the relative magnitudes of the inverse square roots of the pressures takes the place of the ellipse, the thrusts and pressures correspond respectively (in direction) to the normal and radius vector at each point, and in three directions, at right angles to each other, these latter come together.

It is desirable that the reader should bear in mind that the ellipse of which I have spoken is in fact only a principal section of this ellipsoid. The assumption which (following in the track of my predecessors) I shall make, that the greatest energy of friction exerted at any point will be exerted in some direction in a vertical plane parallel to the revetment wall, will be seen from what follows a little further on, to imply that every such plane contains the radius vector which makes the greatest angle with the normal, and consequently the section of the ellipsoid of stress with which we are dealing will be the plane of greatest and least thrust, or greatest and least pressure. By way of aid to the imagination in seizing this subtle conception of stress (a real conquest in physical ideology due to the last quarter of the present century, although its first germ may be recognized in the much earlier molecular view of the circumambient pressures round about each internal point of a perfect fluid), I have gone thus briefly into the generation of the ellipse and ellipsoid above described; but I shall have very little occasion, except for occasional facility of reference, to have resort to them, as the equations (1) and (2) will suffice for my purpose in the present inquiry.

These are the equations which govern the distribution of stress; and it may be convenient to confer upon the ellipse whose radii vectores are in length inversely as the square roots of the pressures acting upon them, the name of the ellipse of *pressures*, in order to obviate any possibility of the position of this ellipse being confounded with that of the one which would, I believe, more ordinarily go by the name of the ellipse of *stress*. Every point in the mass is the centre of such an ellipse; and those ellipses, if properly drawn, will represent completely, and on the same scale, the magnitude and distribution of the pressures round about any point. It is almost needless to add that for a perfect fluid these ellipses would become circles.

Let us now proceed to establish the law of the variation of the stresses, or, to speak more accurately, of the thrusts acting on planes drawn in

any given directions, on passing from one point of the mass to another. Returning to our little rectangular element $PQRS$, and considering the lines PQ, PS to be given in direction, so that we may consider $PQ = dx$ and $PS = dy$, and calling the units of pressure on PQ and RQ L and N, the unit of face-force M, the impressed forces of acceleration X in the direction of x, and Y in the direction of y, and the unit of mass ρ, by simple estimation of the forces in the directions of x and y respectively we obviously obtain, due attention being paid to the mode of fixing the positive directions of X and Y,

$$\frac{dL}{dy} + \frac{dM}{dx} = \rho Y,$$

$$\frac{dN}{dx} + \frac{dM}{dy} = \rho X.$$

If, as in the case with which we shall have to deal, the sole impressed force is that of gravity, and if we treat the weight of a unit of the mass as unity, and make the axis of x horizontal and that of y vertical, the equations become

$$\frac{dL}{dy} + \frac{dM}{dx} = 1,$$

$$\frac{dN}{dx} + \frac{dM}{dy} = 0.$$

These, being the equations which control the law of the variation of the thrusts estimated in given directions in passing from one stress to another, I call the equations of variation of stress.

I now proceed to the application of the principles above set forth to the treatment of the particular question in hand.

Let μ be the coefficient of friction of the earth upon itself, and $\mu = \tan\lambda$, so that λ is the angle of repose; by this is to be understood that the thrust on any element can never make, with the perpendicular to that element, an angle greater than λ. Now the general law of the distribution of stress proves that the actual angle between the perpendicular to the element and its thrust will in two directions be zero. Hence at any given point it will pass through all gradations, from zero up to a certain limit. Here presents itself the question, Is that limit λ, or can it be λ for every point in the mass? As we have no right to assume *à priori* that this limiting angle in that state of equilibrium which we wish to determine must be equal to λ throughout the mass, and obviously it will not be so for actual cases of equilibrium which arise, we want a name to distinguish the maximum ratio which friction bears to pressure in any specified stress from the absolute maximum which this ratio is capable of attaining. We may name the former the coefficient of frictional energy; and for every point where this is equal to the absolute coefficient of friction, we may say the friction of the stress is at its maximum energy. Let (μ) be the coefficient of frictional energy for any given stress, and $(\lambda) = \tan^{-1}(\mu)$ the corresponding angle of repose. [We may also, if we please, term (μ) and (λ) the relative coefficient and relative angle of repose respectively, that is, relative to any assigned stress.] Let the ratio between the maximum and minimum thrust of any stress be called γ^2: a simple relation connects γ and (λ)*.

For calling, as before, L the pressure, and M the face-force (now the friction), we have by equation (1),

$$L = P(\cos\theta)^2 + R(\sin\theta)^2,$$

$$M = (P - R)\sin\theta\cos\theta,$$

$$R = P\gamma^2,$$

$$\tan(\lambda) = (\mu) = \text{maximum value of } \frac{M}{L}.$$

To find this maximum, we have

$$\delta\{\cot\theta + \gamma^2\tan\theta\} = 0.$$

Hence $$\gamma\tan\theta = 1,$$

and therefore $$\cot(\lambda) = \frac{2\gamma}{1-\gamma^2},$$

therefore $$(1-\gamma^2) - 2\gamma\tan\lambda = 0,$$

or $$\gamma = \sec(\lambda) - \tan(\lambda) = \frac{1-\sin(\lambda)}{\cos(\lambda)} = \tan\left(45^\circ - \frac{(\lambda)}{2}\right).$$

This equation expresses the universal relation between the form of the ellipse of pressures for any stress and the relative angle of repose for such stress.

The problem we have just solved may be presented advantageously, in order to make the impression of it more vivid (as it is of cardinal importance), under a geometrical point of view. Taking any radius vector of the ellipse of pressures, the angle between it and its conjugate radius is 90° at any vertex; at some point therefore it will be at a minimum, and this minimum will be the complement of the relative angle of repose.

From the preceding investigation, it will easily be seen that, to find the ray-directions which give this minimum, we have only to construct a rectangle circumscribing the ellipse, and either of its two diagonals will be in the direction required, and the angle between either such ray and the principal axes *plus* or *minus* half the angle between it and the normal (which angle is the relative angle of repose) will be half a right angle†.

* This relation and its importance are well known to Professor Rankine.

† In fact the diameters which coincide with the directions of these diagonals are conjugate diameters, equally inclined to the principal axes; and these, as I suppose must be well known, are the conjugate diameters whose inclination to each other is a minimum.

35.

ON THE EQUATION

$$P(m) + E\left(\frac{m}{m-1}\right)P(m-1) + E\left(\frac{m}{m-2}\right)P(m-2) + \ldots + E(m) = m\frac{m+1}{2}.$$

[*Quarterly Journal of Mathematics*, III. (1860), pp. 186—190.]

$P(m)$ I use to denote the number of integers less than m and prime to it except when $m = 1$, in which case $P(m) = 1$. $E\left(\frac{m}{r}\right)$ I use to denote the integer part of $\frac{m}{r}$, or the whole of $\frac{m}{r}$ if $\frac{m}{r}$ is an integer.

Then evidently if we use $\frac{m}{r;}$ to denote *unity* when m contains r and *zero* in all other cases

$$E\left(\frac{m}{r}\right) - E\left(\frac{m-1}{r}\right) = \frac{m}{r;}.$$

Again, it is well known that the factors of any binomial function, as for instance $x^{12} - 1$, are made up of the prime factors of all the binomial factors of $x^{12} - 1$ as $x^2 - 1$, $x^3 - 1$, $x^4 - 1$, $x^6 - 1$, $x^{12} - 1$, and consequently that

$$m = \frac{m}{1;}P(1) + \frac{m}{2;}P(2) + \frac{m}{3;}P(3) + \ldots + \frac{m}{m;}P(m),$$

which equation may also be easily proved independently (*vide* note at end).

Let now

$$E\left(\frac{m}{m}\right)P(m) + E\left(\frac{m}{m-1}\right)P(m-1) + E\left(\frac{m}{m-2}\right)P(m-2) + \ldots + E\left(\frac{m}{1}\right)P(1) = u_m.$$

Then $$E\left(\frac{m-1}{m-1}\right)P(m-1) + E\left(\frac{m-1}{m-2}\right)P(m-2) + \ldots + E\left(\frac{m-1}{1}\right)P(1) = u_{m-1}.$$

Hence $u_m - u_{m-1} = \frac{m}{m;} P(m) + \frac{m}{m-1;} P(m-1) + \ldots + \frac{m}{1;} P(1) = m.$

Hence $$u_m = m\,\frac{m+1}{2} + C,$$

and since $u_1 = 1$ we must make $C = 0$, and

$$u_m = m\,\frac{m+1}{2},$$

as was to be shown.

NOTE.—Proof of the equation

$$P(m) + \frac{m}{(m-1);} P(m-1) + \frac{m}{(m-2);} P(m-2) + \ldots + 1 = m.$$

Let $a, b, c \ldots$ be the prime factors of m, so that

$$m = a^{\alpha} . b^{\beta} . c^{\gamma} \ldots,$$

and, for example, suppose

$$m = a^{\alpha} . b^{\beta} . c^{\gamma}.$$

Then the numbers contained in m may be divided into groups as follows: one group in which a, b, c all appear, another in which only two of the letters a, b, c appear, a third in which only one of them appears, and finally *unity* in which none of them appears.

The sum of the numbers of integers prime to m and less than it for the factors in the first group

$$= (a^{\alpha-1} + a^{\alpha-2} + \ldots + 1)(b^{\beta-1} + b^{\beta-2} + \ldots + 1)(c^{\gamma-1} + c^{\gamma-2} + \ldots + 1) \times (a-1)(b-1)(c-1)$$

$$= (a^{\alpha} - 1)(b^{\beta} - 1)(c^{\gamma} - 1).$$

In like manner the sum of the numbers of such integers for the factors in the second group

$$= (a^{\alpha} - 1)(b^{\beta} - 1) + (a^{\alpha} - 1)(c^{\gamma} - 1) + (b^{\beta} - 1)(c^{\gamma} - 1),$$

for the third group

$$= (a^{\alpha} - 1) + (b^{\beta} - 1) + (c^{\gamma} - 1),$$

and for *unity*

$$= 1.$$

Hence the total sum of such factors

$$= a^{\alpha} . b^{\beta} . c^{\gamma}$$
$$= m,$$

as was to be shown, and so in the like manner whatever may be the number of prime constituents $a, b, c \ldots$ in m.

Q. E. D.

P.S. 1. By successive integration the theorem first established may be generalized, and preserving the same notations as before, it emerges into the following proposition: [cf. the form below]

$$\Sigma_{\infty}^{0}{}_{i}\, P(i^r) \times \left[\frac{\left(E\frac{m}{i}\right)\left(E\frac{m}{i}+1\right)\ldots\left\{E\frac{m}{i}+(r-1)\right\}}{1.2\ldots r}\right]$$
$$= \Sigma_{m}^{1}(m^r).$$

Thus let $$r = 2.$$

Then $$\Sigma_{\infty}^{0}{}_{i}\, P(i^2)\left\{\frac{\left(E\frac{m}{i}\right)\left(E\frac{m}{i}+1\right)}{2}\right\}$$
$$= \Sigma m^2 = \frac{m(m+1)(2m+1)}{2.3},$$

or observing that

$$P(i^q) = i^{q-1} . P(i),$$

$$\Sigma_{\infty}^{0}{}_{i}\, iP(i)\left\{E\left(\frac{m}{i}\right)E\left(\frac{m}{i}+1\right)\right\}$$
$$= \frac{m(m+1)(2m+1)}{3}.$$

Example, let $$m = 5,$$

$$5P(5) = 20,$$
$$4P(4) = 8, \quad E\left(\tfrac{5}{4}\right) = 1,$$
$$3P(3) = 6, \quad E\left(\tfrac{5}{3}\right) = 1,$$
$$2P(2) = 2, \quad E\left(\tfrac{5}{2}\right) = 2,$$
$$E\left(\tfrac{5}{1}\right) = 5,$$
$$20 \times 2 + 8 \times 2 + 6 \times 2 + 2 \times 6 + 5 \times 6$$
$$= 110,$$
$$\frac{5 \times 6 \times 11}{3} = 110.$$

Or we may use the theorem under the form following:

$$\Sigma_{n}^{1}{}_{i}\left[P(i^r) \times S\left\{E\left(\frac{n}{i}\right)\right\}^{r-1}\right] = S(n^r),$$

where it is to be observed that

$$Sq^r \text{ means } 1^r + 2^r + \ldots + q^r.$$

Example, let $r=3,$

then $$S\left(E\frac{n}{i}\right)^2=\frac{\left(E\frac{n}{i}\right)\left(E\frac{n}{i}+1\right)\left(2E\frac{n}{i}+1\right)}{2\,.\,3},$$

$$Sn^3=\left\{n\left(\frac{n+1}{2}\right)\right\}^2,$$

accordingly $$\Sigma_n^1\left\{P(i^3)\times\frac{\left(E\frac{n}{i}\right)\left(E\frac{n}{i}+1\right)\left(2E\frac{n}{i}+1\right)}{6}\right\}$$

$$=\left(n\frac{n+1}{2}\right)^2.$$

Thus let $n=4,$

then $$E(\tfrac{4}{4})=1,\ \frac{1\,.\,2\,.\,3}{2\,.\,3}=1,\ \ P(4^3)=16\times 2=32,$$

$$E(\tfrac{4}{3})=1,\ \frac{1\,.\,2\,.\,3}{2\,.\,3}=1,\ \ P(3^3)=\ 9\times 2=18,$$

$$E(\tfrac{4}{2})=2,\ \frac{2\,.\,3\,.\,5}{2\,.\,3}=5,\ \ P(2^3)=\ 4\times 1=\ 4,$$

$$E(\tfrac{4}{1})=4,\ \frac{4\,.\,5\,.\,9}{2\,.\,3}=30,\ P(1^3)=\qquad 1,$$

$$32+18+20+30=100,$$

$$\left(\frac{4\,.\,5}{2}\right)^2=100.$$

P.S. 2. The fundamental theorem in its simplest terms is as follows: If $i_1, i_2 \ldots i_r$ be any arbitrary positive integers

$$n^r=(\Sigma)^r\left[P\{(i_1)^{r-1}\}\,P\{(i_2)^{r-2}\}\ldots P(i_{r-1})\times\frac{n}{i_1 i_2\ldots i_r}\right];$$

the $(\Sigma)^r$ meaning merely the sign of summation r times repeated.

Example, let $r=2,\quad n=4,$

4 is divisible by $1\times 1,\ 2\times 1,\ 4\times 1,$

$1\times 2,\ 2\times 2,$

$1\times 4,$

$$P(1)=1,\ P(2)=1,\ P(4)=2,$$

$$1\times 1\times 1+1\times 1\times 1+1\times 1\times 2$$
$$+2\times 1\times 1+2\times 1\times 1$$
$$+4\times 2\times 1$$
$$=4+4+8=16=4^2.$$

It is obvious that this theorem must be capable of being reduced to an algebraical identity by writing $n=a^\alpha.b^\beta.c^\gamma\ldots$ as I have shown in the note above for the case $r=1$.

The proof is left to the ingenuity of the reader.

36.

SUR UNE PROPRIÉTÉ DES NOMBRES PREMIERS QUI SE RATTACHE AU THÉORÈME DE FERMAT.

[*Comptes Rendus de l'Académie des Sciences*, LII. (1861), pp. 161—163.]

En étudiant les propriétés arithmétiques des nombres de Bernoulli et des autres nombres qui leur sont analogues, je suis tombé tout récemment sur une représentation du résidu par rapport au module p^2 de la même fonction exponentielle r^{p-1} dont le théorème de Fermat enseigne que le résidu par rapport à p est l'unité. Nommons le nombre entier $\frac{r^{p-1}-1}{p}$ le quotient de Fermat, dont p sera dit le module et r la base. En supposant que la base est un nombre premier, je trouve qu'on peut exprimer son résidu par rapport au module au moyen d'une série de fractions dont les dénominateurs seront tous les nombres inférieurs au module p, et les numérateurs des nombres périodiques qui ne dépendent que de la base r.

En effet, si le module est un nombre premier impair, les fractions qui expriment ce résidu auront pour dénominateurs successifs $p-1,\ p-2,\ p-3, \ldots, 2, 1,$ et pour numérateurs* le cycle toujours répété $1, 2, 3, \ldots, r-1, r,$ sauf à entendre que le cycle des numérateurs commence avec le terme qui est congru à $\frac{1}{p}$ par rapport à r. Par exemple, soit $r=5$, nous aurons d'après cette règle

$$\frac{5^{p-1}-1}{p}\equiv\frac{1}{p-1}+\frac{2}{p-2}+\frac{3}{p-3}+\frac{4}{p-4}+\frac{5}{p-5}+\frac{1}{p-6}+\frac{2}{p-7}+\ldots,$$

quand p est de la forme $10k+1$, mais* [à cause de $2\times 3\equiv 1 \pmod 5$]

$$\equiv\frac{3}{p-1}+\frac{4}{p-2}+\frac{5}{p-3}+\frac{1}{p-4}+\frac{2}{p-5}+\ldots,$$

quand p est de la forme $10k+2$. Il est bon de remarquer que la somme des réciproques des dénominateurs étant congrue à zéro pour le module p, on peut augmenter ou diminuer simultanément (à volonté) tous les termes

[* See correction below, p. 241.]

du cycle d'un même nombre quelconque, et conséquemment pour le cycle 1, 2, 3, ..., r, on peut substituer un cycle plus symétrique dans lequel le terme au milieu sera zéro. Ainsi on trouve en prenant $r=3$ (suivant le module p)

$$\frac{3^{p-1}-1}{p} \equiv -\frac{1}{p-1}+\frac{1}{p-3}-\frac{1}{p-4}+\frac{1}{p-6}-\frac{1}{p-7}\cdots,$$

ou

$$\equiv -\frac{1}{p-2}+\frac{1}{p-3}-\frac{1}{p-5}+\frac{1}{p-6}\cdots,$$

selon que p est de la forme $6n+1$ ou $6n-1$ respectivement.

Par exemple, faisons $p=7$, alors

$$-\frac{1}{6}+\frac{1}{4}-\frac{1}{3}+\frac{1}{1} \equiv -6+2-5+1 \equiv 6 \equiv \frac{3^6-1}{7}$$

c'est-à-dire $\equiv 104 \pmod 7$.

Prenons encore $p=11$, alors

$$\frac{1}{9}-\frac{1}{8}+\frac{1}{6}-\frac{1}{5}+\frac{1}{3}-\frac{1}{2} \equiv 5-7+2-9+4-6 \equiv 0 \equiv \frac{3^{10}-1}{11}$$

c'est-à-dire $\equiv 22 \times (3^5+1) \pmod{11}$.

Reste à donner la série pour le cas où la base du quotient de Fermat est le nombre 2 [cf. p. 235 below]. Par ce cas on trouve

$$\frac{2^{p-1}-1}{p} \equiv \frac{2}{p-3}+\frac{2}{p-4}+\frac{2}{p-7}+\frac{2}{p-8}+\frac{2}{p-11}+\ldots,$$

ou*

$$\equiv \frac{2}{p-2}+\frac{2}{p-3}+\frac{2}{p-6}+\frac{2}{p-7}+\frac{2}{p-10}+\ldots,$$

selon que p est de la forme $4k+1$ ou $4k-1$ respectivement. On peut énoncer des théorèmes plus généraux en substituant pour p et $p-1$ un nombre quelconque et un indicateur maximum respectivement. Pour le moment je me borne à faire une remarque sur la constitution arithmétique des nombres de Bernoulli et des nombres analogues qui entrent dans le développement des sécantes, dont l'étude m'a conduit à la loi donnée plus haut. Quant aux nombres de Bernoulli, on sait déjà par le théorème publié presque simultanément par MM. Clausen et Staudt, que le dénominateur de B_n est un produit de puissances simples de nombres premiers, étant composé du produit de tous les nombres premiers qui, diminués par l'unité, sont diviseurs de $2n$. Mais on paraît ne pas avoir fait la remarque importante que le numérateur de B_n contiendra tous les facteurs de n qui ne sont pas puissances des facteurs du dénominateur, de telle sorte que, si n contient

[* The sign of every term in the second expression should be changed. Stern, *Crelle*, Bd c. (1887), p. 188.]

p^i, mais ne contient pas $p-1$, le numérateur de B_n contiendra p^i; comme corollaire, on peut remarquer que, p étant un nombre premier quelconque, le numérateur de B_p contiendra toujours p. Quant aux nombres de la série pour la sécante qu'on peut nommer les nombres d'Euler qui le premier en a fait le calcul, et qui sont tous, comme on sait, des nombres entiers et positifs, et que je propose de dénoter par le symbole E, voici une propriété dont ils jouissent.

Désignons par p un nombre premier tel que $p-1$ ou plus généralement $(p-1)p^i$ soit un facteur de $2n$; alors, dans le cas où p est de la forme $4k+1$, p^{i+1} sera un facteur de E_n, mais dans le cas où p est de la forme $4k-1$, p^{i+1} sera un facteur de $2(-1)^{n-1}+E_n$. On comprend que E_n exprime le coefficient de $\dfrac{x^{2n}}{1.2\ldots 2n}$ dans le développement de sécante de x. Par parenthèse il sera bon de remarquer qu'en combinant les deux règles pour B_n et E_n on voit que le dénominateur de leur produit ne peut les contenir comme facteurs aucuns nombres premiers de la forme $4k+1$.

Euler a fait le calcul des E jusqu'à E_9, mais a donné une valeur erronée de cette dernière qui a été corrigée par M. Rothe, dans le *Journal de Crelle*, dans un Mémoire communiqué par M. Ohm*. Selon ma règle E_9+2 doit contenir les trois facteurs 3, 7, 19, ce qui s'accorde avec la valeur donnée par Rothe, mais non pas avec celle d'Euler. C'est à propos de ma nouvelle théorie des partitions des nombres que je me suis intéressé spécialement aux nombres de Bernoulli et d'Euler, qui tous les deux font une partie des développements qu'elle exige; en effet, on a besoin dans cette théorie de toutes les espèces de nombres dont la fonction génératrice est $\Sigma \dfrac{\rho^g}{e^t-\rho}$ (g étant un entier quelconque donné, et ρ une racine de l'unité d'un degré quelconque). Selon le degré de l'équation dont ρ est une racine primitive, on peut les nommer des nombres bernoulliens (ou si l'on veut sous-bernoulliens) d'un tel ou tel ordre. Jusqu'à présent on parait n'avoir tenu compte que des nombres bernoulliens du premier et du second ordre (qui sont liés entre eux par le facteur exponentiel si bien connu) et de ceux du quatrième ordre auquel appartiennent en effet les nombres dits d'Euler. Mais ces nombres pour tous les ordres possèdent des propriétés arithmétiques très-dignes d'être étudiées; j'espère pouvoir y revenir et avoir l'honneur d'en faire le sujet d'une nouvelle communication à l'Académie.

[* Bd xx.]

37.

ADDITION À LA NOTE INSÉRÉE DANS LE PRÉCÉDENT COMPTE RENDU.

[*Comptes Rendus de l'Académie des Sciences*, LII. (1861), pp. 212—214.]

DANS la Note que j'ai eu l'honneur de présenter lundi dernier à l'Académie et qui a été insérée au *Compte rendu*, j'ai fait connaître [p. 231 above] le résidu du nombre E_n par rapport au module p^{i+1}, pour le cas où n contient le facteur $(p-1)p^i$, p étant un nombre premier impair. Il restait à exprimer ce même résidu dans le cas de $p=2$, c'est-à-dire dans le cas où n contient le facteur 2^i. Je trouve alors que E_n est congru à 1 suivant le module 2^{i+1}.

Mais j'ai obtenu en même temps un autre théorème très-général et très-utile pour ce genre de calculs; voici en quoi il consiste. Si n et n' sont des nombres entiers différents de zéro, et que $2n$ et $2n'$ soient congrus suivant le module $(p-1)p^i$, on aura

1° $$(-)^n E_n \equiv (-)^{n'} E_{n'} \pmod{p^{i+1}},$$

lorsque p sera un nombre premier impair,

2° $$E_n = E_{n'} \pmod{2^i},$$

lorsque l'on aura $p=2$, c'est-à-dire lorsque n et n' seront congrus par rapport à 2^{i-1} *.

Si l'on se rappelle que $E_1=1$ et que l'on combine la dernière partie de ce théorème avec celui qui se trouve énoncé plus haut, on arrive immédiatement à cette conséquence remarquable, que: Tout nombre d'Euler est de la forme $4k+1$. Cette loi si simple paraît avoir échappé à l'illustre inventeur de ces nombres puisque la valeur qu'il a donnée pour E_9 est de la forme $4k-1$. En se reportant aux théorèmes que j'ai obtenus, on ne peut guère commettre

* Un théorème tout à fait analogue doit avoir lieu pour les nombres de Bernoulli du 2^me ordre, c'est-à-dire pour les nombres qui multiplient $\frac{x^{2n-1}}{1.2\ldots 2n}$ dans le développement de $\frac{1}{e^x+1}$ en série.

d'erreurs, sans les reconnaître, dans le calcul des nombres E. Par exemple, en partant des quatre valeurs

$$E_1=1,\quad E_2=5,\quad E_3=61,\quad E_4=1385,$$

on peut affirmer à priori que E_9 appartient à toutes les formes linéaires

$$5k+1,\quad 11k+1,\quad 13k+9,\quad 16k+1,\quad 17k+1;$$

en outre, à cause de la forme du double 18 de l'indice 9, lequel contient les facteurs 6, 2×3, 18, on sait que E_9 appartient encore aux formes linéaires

$$7k-2,\quad 9k-2,\quad 19k-2.$$

La valeur 24048 79661 671 obtenue par Euler ne satisfait à aucune de ces huit conditions; celles-ci, au contraire, sont toutes vérifiées par la valeur 24048 79675 441 donnée par M. Rothe. Ainsi on peut non-seulement affirmer que la première valeur est erronée, mais encore on a tout lieu de croire à l'exactitude de la seconde, quoique M. Rothe ne l'ait pas justifiée en présentant les détails de ses calculs.

Je remarquerai, en terminant, que le théorème énoncé plus haut offre le moyen de reconnaître si un nombre premier donné p peut figurer comme facteur dans quelqu'un des termes de la suite indéfinie E, ou dans quelqu'un des termes de la suite $E\pm\alpha$, tirée de la première en augmentant ou en diminuant ses termes d'un même nombre donné α. Car si cette circonstance se présente, à l'égard de l'une de ces suites, p sera nécessairement facteur d'un au moins des $\frac{p-1}{2}$ premiers termes de cette suite, et même de l'un des $\frac{p-3}{2}$ premiers termes dans le cas de $\alpha=0$. On reconnaît l'exactitude de ce dernier point en se rappelant que tous les nombres premiers $4k+1$ étant facteurs des nombres d'Euler, il n'y a lieu de considérer que les diviseurs $4k-1$; or, d'après le théorème de la Note précédente, $E_{\frac{p-1}{2}}$ ne peut être divisible par p quand ce nombre est de la forme $4k-1$. Par exemple, l'inspection des quatre premiers nombres d'Euler 1, 5, 61, 5×277, suffit pour démontrer qu'aucun des nombres de la suite indéfinie E n'est divisible par 3, 7 ou 11.

Des considérations analogues s'appliquent sans difficulté aux diviseurs p^i.

38.

NOTE RELATIVE AUX COMMUNICATIONS FAITES DANS LES SÉANCES DES 28 JANVIER ET 4 FÉVRIER 1861.

(*Extrait d'une Lettre adressée à* M. SERRET *par* M. SYLVESTER.)

[*Comptes Rendus de l'Académie des Sciences*, LII. (1861), pp. 307, 308.]

DANS la Note que j'ai eu l'honneur de présenter récemment à l'Académie et qui a été insérée au *Compte rendu* de la séance du 4 février dernier, j'ai fait connaître un théorème qui lie entre elles deux congruences, dont l'une se rapporte aux *indices* des nombres d'Euler et l'autre à ces nombres eux-mêmes; et, en même temps, j'ai avancé* qu'un théorème analogue doit avoir lieu pour les nombres de Bernoulli. Voici en quoi consiste ce théorème:

Soient p un nombre premier, n et n' deux nombres entiers dont les doubles $2n$, $2n'$ ne contiennent aucun des facteurs $p, p-1$, et soient congrus suivant le module $(p-1)p^i$ (i étant un entier quelconque positif ou nul); les nombres de Bernoulli B_n et $B_{n'}$ seront liés entre eux par la congruence

$$(-)^n \frac{B_n}{n} \equiv (-)^{n'} \frac{B_{n'}}{n'} \pmod{p^{i+1}}.$$

On doit remarquer que, d'après les conditions de l'énoncé, p ne peut être égal ni à 2, ni à 3.

Pour donner un exemple de ce théorème, prenons $n = 7$, $n' = 17$; les nombres $2n$ et $2n'$ seront congrus par rapport à $11 - 1$ et aussi par rapport à $(5-1)\,5$; d'ailleurs

$$\frac{B_7}{7} = \frac{1}{6}, \quad \frac{B_{17}}{17} = \frac{2\ 577\ 687\ 858\ 367}{17 \times 6};$$

par conséquent, on aura

$$\frac{B_7}{7} - \frac{B_{17}}{17} = -\frac{2\ 577\ 687\ 858\ 350}{102} \equiv 0 \pmod{11 \times 25},$$

ce que l'on peut vérifier immédiatement.

* Voir à la page [232] de ce volume.

Je profite de cette occasion pour présenter une remarque importante au sujet de la formule par laquelle j'ai exprimé [p. 230 above] le résidu de $\frac{r^{p-1}-1}{p}$ suivant le module p, dans le cas où l'on a $r = 2$. Cette formule peut être remplacée avec avantage par la suivante:

$$\frac{2^{p-1}-1}{p} \equiv -\frac{\frac{1}{2}}{p-1} + \frac{\frac{1}{2}}{p-2} - \frac{\frac{1}{2}}{p-3} + \dots \pmod{p},$$

qui est tout à fait semblable aux formules relatives au cas où r est un nombre premier impair, et qui n'exige pas, comme celle que j'avais trouvée d'abord, que l'on distingue les formes $4k+1$ et $4k-1$ du module premier p.

Pour ce qui concerne le cas où la base r du quotient de Fermat $\frac{r^{p-1}-1}{p}$ est un nombre composé, il n'y a aucune difficulté à exprimer le résidu de ce quotient suivant le module p, par des suites de fractions dont les dénominateurs sont les nombres inférieurs à p, et dont les numérateurs constituent des cycles exactement comme dans le cas où r est un nombre premier. Pour obtenir, en effet, les suites dont je viens de parler, il suffit de faire usage de la congruence évidente

$$\frac{(abc \dots k)^{p-1}-1}{p} \equiv \frac{(a^{p-1}-1)+(b^{p-1}-1)+(c^{p-1}-1)+\dots+(k^{p-1}-1)}{p} \pmod{p},$$

dans laquelle $a, b, c, \dots, k$, désignent des entiers quelconques égaux ou inégaux. Au moyen de cette congruence, on ramène immédiatement, par de simples additions, le cas où r est un nombre composé au cas où cette base est un nombre premier.

39.

SUR L'INVOLUTION DES LIGNES DROITES DANS L'ESPACE CONSIDÉRÉES COMME DES AXES DE ROTATION.

[*Comptes Rendus de l'Académie des Sciences*, LII. (1861), pp. 741—745.]

Note présentée par M. CHASLES.

ON sait qu'on peut représenter un déplacement infiniment petit quelconque d'un corps rigide au moyen des rotations du corps autour de six axes. En effet, la méthode usuelle de représenter ce déplacement au moyen de trois mouvements de rotation et de trois de translation rentre, comme un cas particulier, dans la méthode dont je parle, en prenant trois axes sur les six à une distance infiniment éloignée du corps. Cependant il n'est pas vrai que la disposition des six axes soit arbitraire dans un sens absolu. Car si les six axes sont choisis de telle façon qu'on peut trouver des forces qui, agissant dans leurs directions sur un corps rigide, feront équilibre entre elles, les rotations autour de ces axes ne restent plus indépendantes, c'est-à-dire une rotation autour d'un de ces axes peut être décomposée dans ses rotations autour des autres, et conséquemment les six axes n'équivaudront en réalité qu'à cinq tout au plus. Dans ce cas, on peut dire que les six axes forment un système en *involution*; et l'objet de cette Note est de préciser les caractères géométriques par lesquels on peut reconnaître une pareille involution et, de plus, de fournir les moyens de construire un tel système, et, en supposant cinq des axes donnés, de trouver le lieu le plus général du sixième.

L'auteur traite d'abord les cas où les droites données sont en nombre inférieur à cinq, et où il s'agit d'en déterminer une de plus qui fasse avec les droites données un système de droites pouvant représenter les directions d'un système de forces (ou de rotations, ce qui revient au même) se faisant équilibre. Il a occasion de citer la Statique de M. Mœbius (*Lehrbuch der Statik*; Lepzig, 1837), et surtout un Mémoire dans lequel ce savant géomètre a traité ces mêmes questions (*Ueber die Zusammensetzung unendlich kleiner Drehungen*; voir *Journal de Crelle*, 1, XVIII, p. 189—212).

Il continue ainsi :

Je passe à la question (objet principal de cette Note) de l'involution du nombre maximum de six lignes. Je suppose que ces lignes soient données, à l'exception d'une seule dont il s'agit de déterminer le lieu géométrique. Je combine les cinq lignes données quatre à quatre; et, quand cela peut se faire, je mène deux transversales rencontrant les quatre droites de chaque combinaison. L'on aura ainsi, en général, cinq paires de transversales.

Dans ces circonstances, je suis à même d'énoncer la proposition géométrique remarquable qui suit: En choisissant arbitrairement un point dans l'espace, et en menant par ce point une transversale à chacune des paires de transversales nommées plus haut, toutes ces transversales ainsi menées (en général au nombre de cinq) se trouveront dans le même plan; et corrélativement, en coupant les paires de transversales par un plan quelconque, les droites (généralement cinq en nombre) qui joignent les deux points d'intersection de la même paire, se croiseront toutes dans le même point. Je nomme un plan et un point ainsi déterminés réciproquement, *pôle* et *plan polaire*.

Je prends arbitrairement une droite qui coupe une paire quelconque de transversales, et je choisis à volonté deux points O et O' sur cette ligne; je trouve les plans polaires respectifs de O et O' (ce qu'il est toujours possible de faire, parce qu'il y a deux paires de transversales au moins, outre la paire coupée par la ligne OO'), disons P et P'. Dans le plan P, je prends à volonté deux points E et F, et par E et F je mène deux lignes qui coupent respectivement les deux lignes d'une quelconque des paires de transversales dont j'ai parlé et qui rencontrent le plan P' en E' et F'; je construis deux *faisceaux homographiques* situés dans P et P', pour lesquels les rayons OO', OE', OF' correspondent respectivement à $O'O$, $O'E$, $O'F$, et je dis que toute droite qui coupe deux rayons correspondants quelconques de ces deux faisceaux sera en involution avec les cinq lignes données, et *vice versâ*, chaque ligne en involution avec les cinq lignes données coupera deux rayons correspondants de ces deux faisceaux.

Jusqu'ici j'ai supposé que la ligne commune aux deux faisceaux a été choisie dans une direction qui traverse les deux droites d'une des paires de transversales connues. Cette restriction peut maintenant être abandonnée, car on pourra choisir pour la ligne des centres des faisceaux une droite quelconque qui coupe deux rayons correspondants; c'est-à-dire une sixième ligne quelconque qui se trouve en involution avec cinq lignes données, pourra servir de rayon commun à deux faisceaux plans homographiques ainsi disposés que chaque ligne coupant deux rayons correspondants dans deux faisceaux sera elle-même en involution avec les cinq lignes données.

J'ajoute, comme étant compris virtuellement dans ce qui précède, que le lieu de toutes les lignes qui sont en involution avec les lignes données et passent par un point donné, est le plan polaire de ce point (selon la définition expliquée ci-dessus du pôle et du plan polaire). M. Mœbius avait déjà démontré que ce lieu doit être un plan; mais il avait omis de donner le moyen de la construire.

On peut aussi remarquer que chacune des cinq lignes données passe par deux rayons correspondants dans chaque couple de faisceaux construit selon la méthode fournie plus haut; la même chose aura lieu pour chaque ligne droite qui se trouve dans l'hyperboloïde dont trois quelconques des lignes données sont des génératrices; et j'ajoute que six lignes quelconques, chacune desquelles passe par deux rayons correspondants dans un couple de faisceaux, seront en involution entre elles.

On peut donner le nom d'*axes conjugués* à chaque paire de lignes dont toutes les transversales sont en involution avec un système donné de cinq droites. Ces systèmes d'axes possèdent entre eux des propriétés remarquables dont, pour le moment, je veux seulement indiquer la suivante: *On peut toujours mener un hyperboloïde par deux paires quelconques d'axes conjugués.*

Voici les propriétés métriques les plus frappantes des couples de faisceaux homographiques dont il est question. Les deux droites* perpendiculaires à la ligne des centres dans les deux plans de l'homographie seront des rayons correspondants; en conséquence, si l'on fait tourner l'un des faisceaux autour de la ligne des centres jusqu'à ce qu'il se trouve dans le même plan avec l'autre faisceau, les rayons correspondants s'entrecouperont dans une ligne droite perpendiculaire à la ligne des centres*, et je trouve que le point où cette perpendiculaire coupe la ligne des centres sera le *pôle* du plan qui, passant par cette ligne, divise en deux parties égales l'angle dièdre formé par les deux plans homographiques. Nommons ce point le *pivot* de la ligne des centres: j'aurai tout à l'heure l'occasion d'y revenir.

Considérons l'ensemble de tous les axes conjugués, c'est-à-dire de toutes les paires de rayons correspondants de tous les couples de faisceaux appartenant à un système donné de cinq lignes, je dis qu'on peut appliquer dans les directions de ces deux axes deux forces dont le rapport de grandeur sera absolument constant pour le système donné, de façon qu'elles seront statiquement équivalentes à deux forces de grandeurs convenablement choisies dans les directions de deux autres axes conjugués quelconques. En considérant une ligne quelconque coupant ces deux axes comme la ligne des centres d'un couple homographique contenant ces deux axes pour rayons correspondants, les deux forces qui doivent agir dans leur direction pour balancer les deux forces fixes auront des *moments* égaux par rapport au *pivot* de cette ligne. Par conséquent, si l'on connaît le pivot d'une seule ligne de centres qui rencontre deux axes conjugués fixes porteurs des lignes en involution avec un système de cinq lignes données, on peut construire tous les couples de

[* See the correction below, p. 244.]

faisceaux homographiques dont les lignes et centres rencontrent ces mêmes axes. Car non-seulement les plans d'homographie de chaque couple seront connus, mais le rapport anharmonique de ses deux faisceaux le sera de même, et cela parce que la position des pivots devient déterminée. On peut ajouter que, puisque tous les pivots appartenant aux mêmes axes conjugués doivent être très-éloignés de ces deux axes par des distances perpendiculaires qui sont dans un rapport constant entre elles, le lieu géométrique qui les contient tous sera une surface du second degré et évidemment un hyperboloïde.

Puisque tous les axes conjugués appartenant à un système de cinq droites données peuvent être considérés comme les directions de deux forces qui équivalent statiquement à deux forces données en grandeur et en position, on voit par ce qui a été dit plus haut que l'ensemble infini de toutes les paires de forces équivalentes entre elles possède cette propriété remarquable, déjà donnée par M. Mœbius (*Journal de Crelle*, t. x. p. 317), que les transversales tirées du même point quelconque dans l'espace de manière à rencontrer les directions des forces dans chaque paire, seront situées dans le même plan, qu'on peut nommer le plan polaire au point donné. C'est une polarité réciproque tout aussi nettement définie que la polarité plus ordinaire qui se rattache à une surface donnée du second degré. On voit que la polarité dont il est ici question peut être considérée comme se rattachant à deux paires de lignes droites qui sont les génératrices du même hyperboloïde.

Dans une communication subséquente, j'ajouterai brièvement les caractères algébriques de tous les cas d'involution, et je ferai connaître un déterminant (composé de déterminants obtenus par la combinaison des coefficients des équations de six ou d'un moindre nombre de lignes droites, mises sous leurs formes les plus générales) au moyen duquel on peut s'assurer si ces droites sont en involution ou non, et, de plus, distinguer entre les diverses espèces d'involution, et même reconnaître d'autres dispositions singulières de ces lignes qui constituent une espèce d'involution imparfaite. Toute cette théorie découle, selon ma méthode de la traiter, des notions les plus élémentaires de la statique des corps rigides.