33.

ON THE GENERAL THEORY OF ASSOCIATED ALGEBRAICAL FORMS.

[*Cambridge and Dublin Mathematical Journal*, VI. (1851), pp. 289—293.]

THE following brief exposition of the general theory of Associated Forms, as far as it has been as yet developed by the labours or genius of mathematicians, is intended as elucidatory and, to a certain extent, emendative of some of the statements in my paper* on Linear Transformations, in the preceding number of the *Journal*.

In the first place, let a linear equivalent of any given homogeneous function be understood to mean what the function becomes when linear functions of the variables are substituted in place of the variables themselves, subject to the condition of the modulus of transformation (that is, the value of the determinant formed by the coefficients of transformation) being unity.

Secondly, let two square arrays of terms (the determinants corresponding to each of which are unity) be said to be complementary when each term in the one square is equal to the value of what the determinant represented by the other square becomes when the corresponding term itself is taken unity, but all the other terms in the same line and column with it are taken zero. This relation between the two squares is well known to be reciprocal. Thus, for instance,

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}$$

will be said to be reciprocally complementary to one another when the two determinants which they represent are each unity, and when we have

[* p. 184, above.]

$$a = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \beta' & \gamma' \\ 0 & \beta'' & \gamma'' \end{vmatrix} \qquad \alpha = \begin{vmatrix} 1 & 0 & 0 \\ 0 & b' & c' \\ 0 & b'' & c'' \end{vmatrix}$$

$$b = \begin{vmatrix} 0 & 1 & 0 \\ \alpha' & 0 & \gamma' \\ \alpha'' & 0 & \gamma'' \end{vmatrix} \qquad \beta = \begin{vmatrix} 0 & 1 & 0 \\ a' & 0 & c' \\ a'' & 0 & c'' \end{vmatrix}$$

$$b' = \begin{vmatrix} \alpha & 0 & \gamma \\ 0 & 1 & 0 \\ \alpha'' & 0 & \gamma'' \end{vmatrix} \qquad \beta' = \begin{vmatrix} a & 0 & c \\ 0 & 1 & 0 \\ a'' & 0 & c'' \end{vmatrix}$$

&c. &c.

Accordingly, two transformations, say of $F(x, y, z)$ and $G(u, v, w)$ respectively, may be said to be concurrent when in F for x, y, z, we write

$$\begin{aligned} &ax + by + cz, \\ &a'x + b'y + c'z, \\ &a''x + b''y + c''z; \end{aligned}$$

and in G for u, v, w, we write

$$\begin{aligned} &au + bv + cw, \\ &a'u + b'v + c'w, \\ &a''u + b''v + c''w; \end{aligned}$$

but complementary when for u, v, w, we write

$$\begin{aligned} &\alpha u + \beta v + \gamma w, \\ &\alpha' u + \beta' v + \gamma' w, \\ &\alpha'' u + \beta'' v + \gamma'' w; \end{aligned}$$

a, b, c, &c., α, β, γ, &c. being related in the manner antecedently explained.

Two forms, each of the same number of variables, may be said to be associate forms when the coefficients of the one are functions of those of the other; and when it happens that the coefficients of the first are all explicit functions of those of the second, the latter may be termed the originant and the former the derivant.

If now all the linear equivalents of one or of two associated forms are similarly related to corresponding linear equivalents of the other, so that each may be derived from each by the same law, the forms so associated will be said to be concomitant each to the other. This concomitance may be of two kinds, and very probably, in the nature of things, only of the two kinds about to be described.

The first species of concomitance is defined by the corresponding equivalents of the two associated forms being deduced by precisely similar, or, as we have expressed it, concurrent transformations or substitutions, each from its given primitive. The second species of concomitance is defined by the corresponding equivalents being deduced not by similar but by contrary, that is, reciprocal or complementary substitutions. Concomitants of the first kind may be called covariants; concomitants of the second kind may be called contravariants. When of the two associated forms one is a constant, the distinction between co- and contra-variants disappears, and the constant may be termed an invariant of the form with which it is associated*. It follows readily from these definitions that a covariant of a covariant and a contravariant of a contravariant are each of them covariants; but a covariant of a contravariant and a contravariant of a covariant are each of them contravariants; and also that an invariant, whether of a covariant or of a contravariant, is an invariant of the original function†.

It will also readily be seen that as regards functions of two letters a contravariant becomes a covariant by the simple interchange of x, y with $-y$, x, respectively. Covariants are Mr Cayley's hyperdeterminants; contravariants include, but are not coincident with, M. Hermite's formes-adjointes, if we understand by the last-named term such forms as may be derived by the process described by M. Hermite in the third of his letters to M. Jacobi, "Sur différents objets de la Théorie des Nombres," (which process is an extension of that employed for determining the polar reciprocal of an algebraical locus‡). M. Hermite appears, however, elsewhere to have used

* Accordingly an invariant to a given form may be defined to be such a function of the coefficients of the form, as remains absolutely unaltered when instead of the given form any linear equivalent thereto is substituted. Of course if the determinant of the coefficients of the transformations correspondent to the respective equivalents be not taken unity as supposed in this definition, the effect will be merely to introduce as a multiplier some power of the determinant formed by the coefficients of transformation.

† It may likewise be shown that linear equivalents of covariants and contravariants are themselves related to one another as covariants and contravariants respectively, the transformations by which the equivalents are obtained being taken concurrent in the one case and contrary or reciprocal in the other; and of course any algebraic function of any number of covariants is a covariant and of contravariants a contravariant.

‡ This has been further generalized by me in the theorem§ given in the last number of this *Journal*, where I have shown in effect that any invariant in respect to $\xi, \eta \ldots \theta$ of

$$f(\xi, \eta \ldots \theta) + (x\xi + y\eta + \ldots + t\theta + \rho)\rho^{n-1},$$

(f being supposed to be of the degree n) is a contravariant of $f(x, y \ldots t)$. When this invariant is the determinant of f, it may be shown that we obtain M. Hermite's theorem. It is somewhat remarkable that contravariants should have been in use among mathematicians as well in geometry as the theory of numbers (although their character as such was not recognized) before covariants had ever made their appearance. Invariants of course first came up with the theory of the equation to the squares of the differences of the roots of equations, the last term in such equation being an invariant. I believe that I am correct in saying that covariants first made their appearance in one of Mr Boole's papers, in this *Journal*; but Hesse's brilliant application

[§ p. 186 above.]

the term forme-adjointe in a sense as wide as that in which I employ contravariants. For instance, he has given a most remarkable theorem, which admits of being stated as follows:

If we have a function of any number of letters, say of x, y, z, as

$$ax^m + mbx^{m-1}y + mcx^{m-1}z + \frac{m(m-1)}{2}dx^{m-2}y^2 + \&c.,$$

and if I be any invariant of this function, then will

$$\left(x^m \frac{d}{da} + x^{m-1}y\frac{d}{db} + x^{m-1}z\frac{d}{dc} + x^{m-2}y^2\frac{d}{dd}\ \&c.\right)^r I$$

be a "*forme-adjointe*" of the given function. It is perfectly true and admits of being very easily proved, as I shall show in your next number, that this is a contravariant of the given function*; but it is not (as far as I can see) a forme-adjointe in the sense in which the use of that word is restricted in the letter alluded to. If, however, we adopt as the *definition* of *formes-adjointes* generally, that property in regard to their transformées which M. Hermite has demonstrated of the particular class treated of by him in the letter alluded to, then his formes-adjointes become coincident with my contravariants. It will thus be seen that covariants and contravariants form two distinct and coextensive species of associated forms, which divide between them the wide and fertile empire of linear transformations so far as its provinces have been as yet laid open by the researches of analysts. In your next number I propose to enter much more largely into the subject generally. More particularly I shall describe the new method of Permutants, including the theory of Intermutants and Commutants (which latter are a species of the former, but embrace Determinants as a particular case), and their application to the theory of Invariants. I shall also exhibit the connexion between the theory of Invariants and that of Symmetrical Functions, and some remarkable theorems on Relative Invariants†.

Some of your readers may like to be informed that a Supplement to my last paper, under the title of "An Essay on Canonical Forms," has been since published‡; and that I have there given a much simpler method of solution of the problem of the reduction of quintic functions to their canonical form than in the original memoir, and extended the method successfully to the

of one from among the infinite variety of these forms to the discovery of the points of inflexion in a curve of the third order, in other words, to the Canonical Reduction of the Cubic Function of Three Letters, appears to have been the first occasion of their being turned to practical account.

* This is also true if I be taken any *covariant* instead of an invariant of the function.

† It will be readily apprehended that the definitions and conceptions above stated, respecting covariants and contravariants of two single functions, may be extended so as to comprehend systems of functions covariantive or contravariantive to one another.

‡ By Mr George Bell, University Bookseller, Fleet Street. [p. 203 below.]

reduction of all odd-degreed functions to their canonical form. I may take this occasion to state that the Lemma given in Note (B) of the Supplement, upon which this method of reduction is based, is an immediate deduction from the well-known theorem for the multiplication of Determinants.

There is a numerical error in "The Cubical Hyperdeterminant of the Twelfth Degree," worked out after the method of commutants by Mr Spottiswoode, given at the end of my paper in the May Number. The correct result will be stated in the next number of the *Journal*, where I hope also to be able to fix the number of distinct solutions of the problem of reducing a Sextic Function to its canonical form

$$u^6 + v^6 + w^6 + mu^2v^2w^2.$$

For odd-degreed functions there is never more than one solution possible, as shown in the Supplement referred to.

P.S. Since the above was sent to press, I have discovered an uniform mode of solution for the canonical reduction of functions, whether of odd or even degrees. The canonical form however, except for the fourth and eighth degrees, requires to be varied from that assumed in my previous paper. Thus, for the sixth degree the canonical form will be

$$au^6 + bv^6 + cw^6 + muvw\,(v - w)\,(w - u)\,(u - v),$$

where u, v, w are supposed to be connected by the identical equation $u + v + w = 0$. And there will be only *two* solutions—a remarkable and most unexpected discovery. For functions of the eighth degree there are five distinct solutions, and in general there is the strongest reason for believing (indeed it may be positively affirmed) that *when the canonical form has been rightly assumed* for a function of the even degree n, the number of solutions will be $\frac{1}{2}(n+2)$ when $\frac{1}{2}n$ is even, but $\frac{1}{4}(n+2)$ when $\frac{1}{2}n$ is odd. It turns out therefore that the theory for functions of the sixth degree is in some respects simpler than for those of the fourth. The investigation into canonical forms here referred to has led me to the discovery of a most unexpected theorem for finding all the invariants of a certain class, belonging to functions of two letters of an even degree.

34.

AN ESSAY ON CANONICAL FORMS, SUPPLEMENT TO A SKETCH OF A MEMOIR* ON ELIMINATION, TRANSFORMATION AND CANONICAL FORMS.

SINCE the above paper was in print I have succeeded in obtaining a canonical representation of the quadratic and cubic functions adjunctive to the general quintic (5th degreed) functions of two letters.

Let F the quintic function of x, y,

$$= u^5 + v^5 + w^5,$$

and

$$au + bv + cw = 0,$$

M being the modulus of the transformation, whereby transition is made from x, y to u, v. Then the quadratic adjunctive is

$$\frac{M^4}{c^4}\{a^4vw + b^4wu + c^4uv\};$$

and the cubic adjunctive is simply

$$\frac{1}{c^6}M^6(abc)^2\,uvw\dagger.$$

Hence we can, in accordance with what I ventured to predict in the preceding sketch, find u, v, w, by means of a simple and practical co-process. To wit, call

$$F = lx^5 + 5mx^4y + 10nx^3y^2 + 10px^2y^3 + 5qxy^4 + ry^5.$$

[* p. 184 above. See p. 201, note ‡.]

† The knowledge of the existence of these *lower* adjunctive forms is mainly a consequence of Mr Cayley's splendid discovery of hyperdeterminant constants. In fact, they are respectively the quadratic and cubic hyperdeterminants in respect to ξ and η of $\frac{1}{1.2.3.4.5}\left(\xi\frac{d}{dx} + \eta\frac{d}{dy}\right)^4 F$; x and y being treated as constants.

The fortunate proclaimer of a new outlying planet has been justly rewarded by the offer of a baronetcy and a national pension, which the writer of this wishes him long life and health to enjoy. In the meanwhile, what has been done in honour of the discoverer of a new and inexhaustible region of exquisite analysis?

Form the determinant

$$\begin{vmatrix} lx+my, & mx+ny, & nx+py \\ mx+ny, & nx+py, & px+qy \\ nx+py, & px+qy, & qx+ry \end{vmatrix}.$$

Let this cubic function, by solving it as a cubic equation, be made equal to

$$L(x+fy)(x+gy)(x+hy),$$

then

$$u=k(x+fy),\quad v=l(x+gy),\quad w=m(x+hy).$$

By means of the identity, $F=u^5+v^5+w^5$, l^5, m^5, n^5, are known by the solution of linear equations, and thus u, v, w, are determined by solving a cubic equation instead of one of the eighth degree, as in the method first given, and the process of canonising a quintic function is rendered *practically* possible.

For brevity sake let c represent unity. The constant determinant of the cubic adjunctive will be found to be

$$3M^{30}(abc)^{10}.$$

Calling, then, the cubic adjunctive of F, $C(F)$, we have the remarkable equation

$$uvw=\frac{C(F)}{\sqrt[3]{\{\frac{1}{3}\square C(F)\}}}.$$

It may also be shown that if we call the Hessian of F, $H(F)$, we shall have the following equally remarkable equation:

$$\square H(F)=\tfrac{1}{3}\square F\times\square C(F).$$

Again, calling the quadratic adjunctive of F, $Q(F)$, we shall easily find

$$\square Q(F)=M^{10}\begin{Bmatrix}(a^{\frac{5}{2}}+b^{\frac{5}{2}}+c^{\frac{5}{2}})\\(a^{\frac{5}{2}}+b^{\frac{5}{2}}-c^{\frac{5}{2}})\\(a^{\frac{5}{2}}-b^{\frac{5}{2}}+c^{\frac{5}{2}})\\(a^{\frac{5}{2}}-b^{\frac{5}{2}}-c^{\frac{5}{2}})\end{Bmatrix},$$

or, if we please,

$$=M^{10}\begin{Bmatrix}a^{10}+b^{10}+c^{10}\\-2a^5b^5-2a^5c^5-2b^5c^5\end{Bmatrix}.$$

When u, v, w are known, a, b, c, which are the resultants of v, w; w, u; u, v respectively are known. But their ratios, or, if we please to say so, the ratios of $a^5:b^5:c^5$, may be found independently and very elegantly as follows:—

Let $M^{10}\times$ product of the 4 forms of $a^{\frac{5}{2}}+1^{\frac{1}{2}}b^{\frac{5}{2}}+1^{\frac{1}{2}}c^{\frac{5}{2}}=A$,

$M^{20}\times$ product of the 16 forms of $a^{\frac{5}{4}}+1^{\frac{1}{4}}b^{\frac{5}{4}}+1^{\frac{1}{4}}c^{\frac{5}{4}}=B$,

$M^{30}\times a^{10}.b^{10}.c^{10}=C$.

A, B, C are known quantities, being respectively what we have called $\square Q(F)$, $\square(F)$*, $\frac{1}{3}\square C(F)$.

It may easily be shown that

$$B-A^2=128M^{20}a^5b^5c^5(a^5+b^5+c^5).$$

Hence M^5a^5, M^5b^5, M^5c^5 are the roots of ρ in the cubic equation

$$\rho^3+\frac{B-A^2}{2^7C^{\frac{1}{2}}}\rho^2+\tfrac{1}{4}\left\{\frac{(B-A^2)^2}{2^{14}C}-A\right\}\rho+C^{\frac{1}{2}}=0.$$

A, B, C, it will be observed, are independent and, as they may be termed, prime or radical adjunctive constants. Hitherto much mystery and uncertainty have attached to the theory of hyperdeterminants, from its having been tacitly assumed that they were always either of lower dimensions than the ordinary determinant, or else algebraical functions of such, and of the determinant. Whereas we now see that, whilst the determinant of a function in two letters of the fifth degree is of eight dimensions, one of its radical or primitive hyperdeterminants is of four, but the other of twelve dimensions. This is a most valuable consequence, and would seem to indicate that the number of radical hyperdeterminants to a function, over and above the common determinant, is always equal to the number of parameters entering into its canonical form. The importance of this ascertainment of an unsuspected third *radical* constant, adjunctive to a quintic function of two letters, in making to march the theory of hyperdeterminants, can hardly be over-estimated.

From the equation last given we are enabled to assign the conditions in order that two functions of the fifth degree may be capable of being linearly transformed either into the other. For if we call F and F' two such linearly equivalent quintic functions, they must be capable each of being thrown under the same form $u^5+v^5+(lu+mv)^5$, where l and m shall be the same for each. Consequently we must have the roots of ρ in the same ratio for F and F', which conditions may be expressed by means of the two equations

$$\frac{B-A^2}{C^{\frac{2}{3}}}=\frac{B'-A'^2}{C'^{\frac{2}{3}}},$$

$$\frac{(B-A^2)^2-2^{14}AC}{C^{\frac{4}{3}}}=\frac{(B'-A'^2)^2-2^{14}A'C'}{C'^{\frac{4}{3}}},$$

* More strictly speaking (and this correction should be supplied throughout in the "Sketch"), B is the negative determinant of $\frac{1}{5}F$. After finding, by the method of characteristics, or any special artifices, the algebraic part of the value of a resultant or determinant, a process frequently of some complexity remains over in assigning its numerical multiplier; this part of the operation being analogous to that which occurs in the Integral Calculus, of determining the constant to be added after the general form of an integral has been determined. In the "Sketch," a correction for the numerical multiplier remains also to be applied to the expressions given for the successive Hessian determinants.

A', B', C', of course representing the same functions of the coefficients of F' as A, B, C, respectively of F.

The two conditions required in their simplest form are accordingly

$$\frac{A}{C^{\frac{1}{2}}} = \frac{A'}{C'^{\frac{1}{2}}},$$

$$\frac{B}{C^{\frac{1}{3}}} = \frac{B'}{C'^{\frac{1}{3}}},$$

or

$$A^2 : B^3 : C :: A'^2 : B'^3 : C',$$

that is to say, *all quintic functions of two letters of which the determinant is to the subduplicate power of the radical hyperdeterminant of the twelfth order and to the sesquiduplicate power of the radical hyperdeterminant of the fourth order in given ratios, are mutually convertible.*

So for the quartic (that is, biquadratic) function of two letters, calling R and S the radical adjunctive constants of the second and third orders, the condition of convertibility between different forms of the same is, that $R^3 : S^2$ shall be a given ratio. And, in general, we may infer that the condition of convertibility between different functions of any degree is, that the several radical adjunctive constants of each raised respectively to such powers as will make them of like dimensions, shall be to one another in given ratios. Of course all cubic functions of two letters, according to this rule, are mutually convertible without any condition, they having but one radical adjunctive constant; and in fact all such functions, being representable as the sum of two cubes of new variables linearly related to those given, are necessarily convertible.

I have further succeeded in obtaining the canonical form of the *quadratic* adjunctive to *any odd degreed* function of two letters, which presents a wonderful analogy to the theory of relative determinants of *quadratic functions of any number* of letters, and constitutes an important step towards the construction of the theory of relative hyperdeterminants.

Let a function of two letters of the odd degree $m \, (= 2n - 1)$ be thrown under its canonical form,

$$u_1^m + u_2^m + \ldots + u_n^m,$$

and let there exist the $n - 2$ equations,

$$a_1 u_1 + a_2 u_2 + \ldots + a_n u_n = 0, \qquad (1)$$

$$b_1 u_1 + b_2 u_2 + \ldots + b_n u_n = 0, \qquad (2)$$

....

$$l_1 u_1 + l_2 u_2 + \ldots + l_n u_n = 0. \qquad (n-2)$$

Then, if M be the modulus of the transformation which converts u_1, u_2 into

x, y, and if, on making $\theta_1, \theta_2 \ldots \theta_n$ disjunctively equal to $1, 2 \ldots n$ we use (θ_{n-1}, θ_n) to denote in general the determinant

$$\begin{vmatrix} a_{\theta_1}, & a_{\theta_2} \ldots a_{\theta_{n-2}} \\ b_{\theta_1}, & b_{\theta_2} \ldots b_{\theta_{n-2}} \\ \cdots & \cdots \\ l_{\theta_1}, & l_{\theta_2} \ldots l_{\theta_{n-2}} \end{vmatrix},$$

the quadratic adjunctive of $\dfrac{1}{m(m-1)\ldots 2} F$ will be

$$\frac{M^{m-1}}{(1 . 2)^{m-1}} \Sigma \{(\theta_r, \theta_s)^{m-1} (u_r . u_s)\}^*.$$

N.B. By means of this formula, and of the theorem for finding relative determinants of quadratic functions, we can obtain the general canonical form for one set of the biquadratic adjunctive constants (hyperdeterminants of the fourth order in Mr Cayley's language) of any odd degreed function of two letters †.

Thus, for the fifth degree, preserving the notation of the "Sketch," we have the biquadratic adjunctive constant

$$= \begin{vmatrix} 0, & c^4, & b^4, & a \\ c^4, & 0, & a^4, & b \\ b^4, & a^4, & 0, & c \\ a, & b, & c, & 0 \end{vmatrix} \times \frac{M^{10}}{c^{10}}.$$

For the seventh degree, if we suppose the function to be equal to

$$u^7 + v^7 + w^7 + \theta^7,$$

and

$$au + bv + cw + d\theta = 0,$$

$$a'u + b'v + c'w + d'\theta = 0;$$

the biquadratic adjunctive constant will be $\dfrac{M^{14}}{(cd' - c'd)^{14}}$ multiplied by the determinant

$$\begin{vmatrix} 0, & (ab' - a'b)^6, & (ac' - a'c)^6, & (ad' - a'd)^6, & a, & a' \\ (ba' - b'a)^6, & 0, & (bc' - b'c)^6, & (bd' - b'd)^6, & b, & b' \\ (ca' - c'a)^6, & (cb' - c'b)^6, & 0, & (cd' - c'd)^6, & c, & c' \\ (da' - d'a)^6, & (db' - d'b)^6, & (dc' - d'c)^6, & 0, & d, & d' \\ a, & b, & c, & d, & 0, & 0 \\ a', & b', & c', & d', & 0, & 0 \end{vmatrix}.$$

* The condition $m = 2n - 1$ is only necessary in order that $\Sigma_n'(u^m)$ may be a canonical, because a possible and determinate, form for any given function of the mth degree. But the theorem in the text, so far as it serves to obtain the quadratic adjunctive of $\Sigma_n'(u^m)$, is true for *all* odd values of m, whether greater or less than $2n - 1$.

† See Note (A) of Appendix.

The determinants of the Hessian, the post-Hessian, and the præter-post-Hessian of F will be found (in the case of the quintic function) to be always multiples of powers of the determinant of the given function, and of its cubic adjunctive; and I believe that in general for a function of two letters of any degree the determinants of all the derived forms in the Hessian scale*, will be necessarily algebraical functions of any two of them.

I hope very shortly to accomplish the reduction of functions, as high as the seventh degree of two letters, to their canonical form, and also to present a complete theory of the failing or singular cases of canonical forms.

Since the above was in print I have discovered the following

General Theorem

for reducing a function of two letters of any odd degree to its canonical form.

Let the degree of the function be $(2n-1)$; then its canonical form is

$$u_1^{2n-1}+u_2^{2n-1}+\ldots+u_n^{2n-1},$$

with $(n-2)$ linear relations between $u_1, u_2, \ldots u_n$.

To find $u_1, u_2, \ldots u_n$, proceed as follows. Let the given function of the $(2n-1)$th degree be supposed to be

$$a_1x^{2n-1}+(2n-1)\,a_2x^{2n-2}y+(2n-1)\,\frac{2n-2}{2}\,a_3x^{2n-3}y^2+\ldots+a_{2n}y^{2n-1}.$$

Form the determinant

$$\begin{vmatrix} a_1x+a_2y, & a_2x+a_3y, & a_3x+a_4y \ldots & a_nx+a_{n+1}y \\ a_2x+a_3y, & a_3x+a_4y, & \ldots\ldots\ldots & a_{n+1}x+a_{n+2}y \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_nx+a_{n+1}y, & a_{n+1}x+a_{n+2}y, & \ldots\ldots\ldots & a_{2n-1}x+a_{2n}y \end{vmatrix}.$$

This determinant is a function of x and y of the nth degree, and by resolving an equation of the nth degree, may be decomposed into n factors, say

$$(l_1x+m_1y)\,(l_2x+m_2y)\ldots(l_nx+m_ny);$$

* I use the term Hessian (more properly speaking the Boolian) Scale, to denote the *determinants* in respect of ξ and η of $\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}+\&c.\right)^2F$.

Neither Hesse, however, nor any other writer up to the present time, had thought of constructing, and still less of turning to account, the functions (the first only excepted) which figure in this scale.

we shall then have

$$u_1=p_1(l_1x+m_1y),$$
$$u_2=p_2(l_2x+m_2y),$$
$$\ldots\ldots\ldots\ldots$$
$$u_n=p_n(l_nx+m_ny),$$

where the l's and m's are known, and the $(2n-1)$th powers of the p's may be found linearly, by means of the identical equation $\Sigma u^{2n-1}=F(x, y)$. Thus for example a function of the seventh degree of two letters may be reduced to its canonical form

$$(lx+my)^7+(l'x+m'y)^7+(l''x+m''y)^7+(l'''x+m'''y)^7,$$

by the resolution of a biquadratic equation. My demonstration of this extraordinary and unexpected consequence rests upon the following lemma*, itself a very beautiful and striking theorem (no doubt capable of much generalisation) in the theory of determinants. Form the rectangular matrix consisting of n rows and $(n+1)$ columns

$$\begin{matrix} T_1, & T_2, & T_3 & \ldots T_{n+1}, \\ T_2, & T_3, & T_4 & \ldots T_{n+2}, \\ T_3, & T_4, & T_5 & \ldots T_{n+3}, \\ \ldots & \ldots & \ldots & \ldots \\ T_n, & T_{n+1}, & T_{n+2} & \ldots T_{2n}, \end{matrix}$$

where

$$T_i=a_1^{r-i}b_1^{s+i}+a_2^{r-i}b_2^{s+i}+\ldots+a_{n-1}^{r-i}b_{n-1}^{s+i}.$$

Then all the $n+1$ determinants that can be formed by rejecting any *one* column at pleasure out of this matrix are identically zero.

In order the better to realise the proof, suppose

$$n=4, \text{ so that } 2n-1=7.$$

Let

$$F(x, y)=a_1x^7+7a_2x^6y+21a_3x^5y^2+35a_4x^4y^3+35a_5x^3y^4$$
$$+21a_6x^2y^5+7a_7xy^6+a_8y^7.$$

Suppose

$$t^7+u^7+v^7+w^7=F(x, y)=G(u, v),$$
$$at+bu=v,$$
$$a't+b'u=w.$$

Then, if M is the modulus of transition from x, y to u, v the hyper-

* See Note (B) of Appendix.

determinant, or, to adopt my new expression, the permutant P_4 (meaning thereby)

$$\begin{vmatrix} a_1x+a_2y, & a_2x+a_3y, & a_3x+a_4y, & a_4x+a_5y \\ a_2x+a_3y, & a_3x+a_4y, & a_4x+a_5y, & a_5x+a_6y \\ a_3x+a_4y, & a_4x+a_5y, & a_5x+a_6y, & a_6x+a_7y \\ a_4x+a_5y, & a_5x+a_6y, & a_6x+a_7y, & a_7x+a_8y \end{vmatrix},$$

which is a constant adjunctive in respect to ξ and η of $\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}\right)^6 F$, will, according to the principles laid down in the preceding "Sketch," be the product of a power of M multiplied by the corresponding adjunctive constant of $\left(\xi\frac{d}{du}+\eta\frac{d}{dv}\right)^6 G(u, v)$, and is therefore a multiple of the determinant

$$\begin{vmatrix} (1+A_1)t+A_2u, & A_2t+A_3u, & A_3t+A_4u, & A_4t+A_5u \\ A_2t+A_3u, & A_3t+A_4u, & A_4t+A_5u, & A_5t+A_6u \\ A_3t+A_4u, & A_4t+A_5u, & A_5t+A_6u, & A_6t+A_7u \\ A_4t+A_5u, & A_5t+A_6u, & A_6t+A_7u, & A_7t+(1+A_8)u \end{vmatrix},$$

where

$$A_1=a^7+a'^7, \quad A_2=a^6b+a'^6b', \quad A_3=a^5b^2+a'^5b'^2 \ldots A_8=b^7+b'^7.$$

In this determinant the coefficient of u^4 is

$$\begin{vmatrix} A_2, & A_3, & A_4, & A_5 \\ A_3, & A_4, & A_5, & A_6 \\ A_4, & A_5, & A_6, & A_7 \\ A_5, & A_6, & A_7, & 1+A_8 \end{vmatrix}$$

which is numerically equal to

$$A_5\begin{vmatrix} A_3, & A_4, & A_5 \\ A_4, & A_5, & A_6 \\ A_5, & A_6, & A_7 \end{vmatrix} - A_6\begin{vmatrix} A_2, & A_4, & A_5 \\ A_3, & A_5, & A_6 \\ A_4, & A_6, & A_7 \end{vmatrix}$$

$$+A_7\begin{vmatrix} A_2, & A_3, & A_5 \\ A_3, & A_4, & A_6 \\ A_4, & A_5, & A_7 \end{vmatrix} - (1+A_8)\begin{vmatrix} A_2, & A_3, & A_4 \\ A_3, & A_4, & A_5 \\ A_4, & A_5, & A_6 \end{vmatrix}$$

$=0$, because the second factors of the products are all zero by the lemma. Hence the permutant P_4 vanishes when $t=0$, and consequently it contains t as a factor, and in like manner it may be proved to contain u, v, w.

Hence t, u, v, w are the algebraical factors of P_4, and precisely the same proof applies to show in the case of a function in x and y, say F_{2n-1}, of any

odd degree $(2n-1)$ whatever, that the corresponding permutant P_n will contain the factors $u_1, u_2 \ldots u_n$ linear functions of x, y, such that

$$u_1^{2n-1}+u_2^{2n-1}+\ldots+u_n^{2n-1}=F_{2n-1}$$

as was to be shown.

Whenever P_n has equal roots, this will denote either (which is the more general case) that the usual canonical form fails and gives place to a singular form, (owing to some of the coefficients of transformation becoming infinite), or, which is the more special supposition, that the canonical form becomes catalectic by one or more of the linear roots* disappearing. Thus in the cubic function, if P_2 has equal roots, and consequently its determinant (which is coincident with that of the function itself) vanish, then the canonical form in general fails; so that, for example, ax^3+bx^2y cannot in general be exhibited as the sum of two cubes: if, however, certain further relations obtain between the coefficients of F, the canonical form reappears catalectically, the function becoming in fact representable as a single cube. So, again, for the quintic function (referring back to the notation above [page 205]), if P_3 have equal roots, that is if $C=0$, the canonical form fails, unless at the same time $B-A^2=0$, in which case the function becomes the sum of two fifth powers; but if furthermore $A=0$, then this catalectic form again gives place to a singular form, which, on the satisfaction of a further condition between the coefficients, again in its turn gives way before a (bicatalectic, that is) doubly catalectic form, namely, a single fifth power.

It is remarkable, that the form to which Mr Jerrard's method reduces the function of the fifth degree, expressed homogeneously as $ax^5+bxy^4+cy^5$, is a singular form, being incapable of being exhibited as the sum of three cubes; such, however, is not the case with the form $ax^5+bx^3y^2+cy^5$. It may further be remarked, that although the singly catalectic form of the quintic function is expressible by two conditions only, namely, $C=0$, $B-A^2=0$, it will be indicated by P_3 (which being a cubic function of x and y contains four terms) completely disappearing, so that apparently four conditions would appear to be required or implied. But of course these must be capable of being shown to be non-independent, and to be merely tantamount to the two independent ones, $C=0$, $B-A^2=0$. The theory of the catalectic forms of functions of the higher degrees of two variables presents many strong points of resemblance and of contrast to that of the catalectic forms of quadratic functions of several variables.

One important and immediate corollary from the General Theorem is, that the constants which enter into the linear functions appurtenant to the canonical form of any function of an odd degree form a single and unique system; or, in other words, the canonical forms for such functions are void of

* $u_1, u_2 \ldots u_n$ may be termed the linear roots of the form F_{2n-1}.

multiplicity, a result contrary to what might have been anticipated, and to what we know is the case for the canonical forms of functions of an even degree.

It may further be shown that if we have the $(n-2)$ equations

$$a_1u_1 + a_2u_2 + \ldots + a_nu_n = 0,$$
$$b_1u_1 + b_2u_2 + \ldots + b_nu_n = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$l_1u_1 + l_2u_2 + \ldots + l_nu_n = 0,$$

and call M the modulus of transformation in respect to u_1, u_2, and if we make

$$P_n = Ku_1u_2 \ldots u_n,$$

then

$$\begin{vmatrix} a_3, & a_4 \ldots a_n \\ b_3, & b_4 \ldots b_n \\ \ldots & \ldots \\ l_3, & l_4 \ldots l_n \end{vmatrix}^{\frac{1}{2}n(n-1)} K$$

is equal to the product of the $\frac{1}{2}n(n-1)$ factors of the form

$$\begin{vmatrix} a_{\theta_1}, & a_{\theta_2} \ldots a_{\theta_{n-2}} \\ b_{\theta_1}, & b_{\theta_2} \ldots b_{\theta_{n-2}} \\ \ldots & \ldots \\ l_{\theta_1}, & l_{\theta_2} \ldots l_{\theta_{n-2}} \end{vmatrix},$$

$\theta_1, \theta_2 \ldots \theta_{n-2}$ being any $(n-2)$ numbers out of the n numbers $1, 2, 3 \ldots n$.

It may hence be shown that

$$u_1u_2 \ldots u_n = \frac{P_n}{\left(\frac{1}{m} \underset{x,y}{\Box} P_n\right)^{\frac{1}{2n-1}}} \text{*},$$

m being a number which is a function of n, and which may be shown to be equal to $\underset{x,y}{\Box}(x^{n-1}y + xy^{n-1}) \div$ product of the squared differences of the roots of $l^{n-2} = 1$, that is

$$m = \left\{\frac{(n-1)^2 - 1}{-(n-2)}\right\}^{n-2} = (-n)^{n-2},$$

and thus

$$u_1u_2 \ldots u_n = \frac{P_n}{\sqrt[2n-1]{\{(-n)^{-(n-2)} . \underset{x,y}{\Box} P_n\}}}.$$

* $\underset{x,y}{\Box}$ means the determinant in respect to x and y.

As an example of the mode of finding $u_1, u_2 \ldots u_n$, let

$$F = 3x^5 + 20x^3y^2 + 10xy^4,$$

then

$$P_3 = \begin{vmatrix} 3x, & 2y, & 2x \\ 2y, & 2x, & 2y \\ 2x, & 2y, & 2x \end{vmatrix} = 4x^3 - 4y^2x.$$

Hence

$$u = fx, \quad v = g(x+y), \quad w = h(x-y).$$

To find f, g, h, we have $u^5 + v^5 + w^5 = F$, hence

$$f^5 + g^5 + h^5 = 3; \quad g^5 + h^5 = 2; \quad g^5 - h^5 = 1;$$

whence we have

$$F = x^5 + (x+y)^5 + (x-y)^5.$$

Again, we find

$$\Box(4x^3 - 4y^2x) = -4^4 \times 12,$$

$$\left(\frac{1}{(-3)} \Box P_3\right)^{\frac{1}{5}} = 4,$$

and accordingly

$$x(x+y)(x-y) = \frac{P_3}{\{(-3)^{-1} . \Box P_3\}^{\frac{1}{5}}},$$

according to the general formula above given.

As a second example let

$$F = 3x^7 + 42x^5y^2 + 70x^3y^4 + 14xy^6 + y^7;$$

then

$$P_4 = \begin{vmatrix} 3x, & 2y, & 2x, & 2y \\ 2y, & 2x, & 2y, & 2x \\ 2x, & 2y, & 2x, & 2y \\ 2y, & 2x, & 2y, & 2x+y \end{vmatrix} = 4(x^3y - xy^3) = 4xy(x-y)(x+y),$$

and accordingly we shall find

$$x^7 + y^7 + (x-y)^7 + (x+y)^7 = F.$$

Moreover

$$\Box(4x^3y - 4xy^3) = 4^9,$$

and

$$4^{4-2} \qquad = 4^2.$$

Thus

$$\frac{P_4}{tuvw} = \sqrt[7]{\frac{\Box P_4}{4^{4-2}}},$$

agreeable to the general formula.

As a corollary to our general proposition, it may be remarked, that if F_{2n-1} be a symmetrical function of x, y of the $(2n-1)$th degree, $P_n(F_{2n-1})$ will be also a symmetrical function of x and y, and may therefore be resolved into its factors by solving a *recurring* equation of the nth degree, which may, by well-known methods, be made to depend on the solution of an equation of the $\frac{1}{2}n$th or $\frac{1}{2}(n-1)$th degree, according as n is even or odd.

Hence the reduction of a function of two letters of the degree $4m \pm 1$ to its canonical form as the sum of powers may be made to depend on the solution of an equation of the mth degree; so that, for example, a symmetrical function of x, y, as high as the fifteenth or seventeenth degree, may be reduced by means of a biquadratic equation only.

In a short time I hope to present to the public a complete solution of the canonical forms of functions of two letters of even degrees, and possibly to exhibit some important applications of the principles of the method to the theory of numbers.

APPENDIX.

Note (A).

The permutants (meaning, in Mr Cayley's language, the hyperdeterminants) of $F_{2n+1}(x, y)$ of the fourth dimension in respect to the coefficients of F, may be all obtained by taking the quadratic permutant in respect to x and y of the quadratic permutant in respect of ξ and η of

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^{2l} F_{2n+1}(x, y),$$

l having any integer value from 1 to n.

In extension of a theorem in the foregoing Supplement, which applies only to the case of $l = n$, I am able to state the following more general theorem, in which the same notation is preserved as above [page 207]. The quadratic permutant in respect to ξ and η of

$$\frac{1}{(2n+1)\,2n \ldots (2n-2l+2)} \left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^{2l} F_{2n+1}(x, y),$$

is equal to

$$\frac{M^{2l}}{(1, 2)^{2l}} \Sigma \{(\theta_r, \theta_s)^{2l} (u_r . u_s)^{2n+1-2l}\}.$$

If now we proceed to form the quadratic permutant of the above sum in respect to x and y, we know *à priori*, by reason of Mr Cayley's invaluable researches, that we shall not get radically distinct results for all values, but only for certain periodically changing values of l.

I have not yet had leisure to seek for an explicit demonstration of this remarkable law, founded upon the above given canonical representation.

Note (B).

The lemma, upon which the general method for reducing odd degreed functions to their canonical form is founded, may be stated rather more simply and more generally as follows :—

The determinant

$$\begin{vmatrix} T_{r_1}, & T_{r_2} & \ldots\ T_{r_n} \\ T_{r_1+l_1}, & T_{r_2+l_1} & \ldots\ T_{r_n+l_1} \\ T_{r_1+l_2}, & T_{r_2+l_2} & \ldots\ T_{r_n+l_2} \\ \cdots & \cdots & \cdots \\ T_{r_1+l_{n-1}}, & T_{r_2+l_{n-1}} & \ldots\ T_{r_n+l_{n-1}} \end{vmatrix},$$

where T_θ denotes $A_1 a_1^\theta + A_2 a_2^\theta + \ldots + A_m a_m^\theta$ *provided that m is less than n*, is identically zero. In the theorem, as thus stated, there is no substantial loss of generality arising from the omission of the b's.

Thus stated the theorem and its extensions evidently repose upon the same or the like basis as the theory of partial fractions.

Note (C), *referring to the original "Sketch."*

The Boolo-Hessian scale of determinants furnishes a very pretty general theorem of geometrical reciprocity in connexion with the doctrine of successive polars. Let $F(x, y, z)$, a cubic homogeneous function of x, y, z equated to zero, express in general a curve of the third degree; then $\left(a\frac{d}{dx} + b\frac{d}{dy} + c\frac{d}{dz}\right) F$ will express its first polar in respect to the point a, b, c, that is, the conic which passes through the six points in which the tangents drawn from a, b, c to touch the given curve meet the same.

Again, if we take l, m, n the coordinates of any new point,

$$\left(l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right)\left(a\frac{d}{dx} + b\frac{d}{dy} + c\frac{d}{dz}\right) F$$

will express the polar, that is the chord of contact of the above conic, in respect to the last named point. If now we eliminate l, m, n between the three equations

$$\left(l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right)\left(a\frac{d}{dx} + b\frac{d}{dy} + c\frac{d}{dz}\right) F = 0,$$

$$\left(l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right)\left(a'\frac{d}{dx} + b'\frac{d}{dy} + c'\frac{d}{dz}\right) F = 0,$$

$$\left(l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right)\left(a''\frac{d}{dx} + b''\frac{d}{dy} + c''\frac{d}{dz}\right) F = 0,$$

it is easily seen that the resultant of the elimination is the square of the determinant

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix},$$

multiplied by the Hessian of the given function. And, moreover, that if we eliminate x, y, z we shall obtain precisely the same result with the letters l, m, n substituted for x, y, z. Hence it follows, that if we take the doubly infinite system of first polars to a given curve of the third degree, in respect to all the points lying in its plane, and then from any point in the *Hessian* to the given curve, draw pairs of tangents to each conic of the system so generated, then all the chords of contact will meet in one and the same point, which will itself be also a point situated upon the Hessian and conjugate to the former.

So, in general, for a function of any degree of any number of letters, viewed with relation to the doctrine of successive polars, the determinants of the Boolo-Hessian scale take one another up in pairs; namely the first takes up the last but one, the second the last but two, and so on; and consequently, if the degree of the function be odd, that function which (making abstraction of the constant determinant at the end) lies in the middle of the scale pairs with itself, and, in a sense analogous to that above exhibited for a function of the third degree, may be said to be always its own reciprocal.

P.S. I have just discovered the method of reducing functions of two letters of *even degrees* to their canonical form, which will shortly be published in a second Supplement.

At present I offer the annexed theorem (which strikingly contrasts with the law of uniqueness demonstrated of functions of an odd degree) as a foretaste of the enchanting developments with which I hope shortly to present my readers:—

If a given homogeneous function of x and y of the degree $2n$ be supposed to be thrown under its canonical form,

$$u_1^{2n} + u_2^{2n} + \ldots + u_n^{2n} + K(u_1 u_2 \ldots u_n)^2,$$

then will K^n have n^2-1 in general distinct values, to each of which will correspond a single distinct system of the linear functions of x and y,

$$1^{\frac{1}{n}}u_1, \quad 1^{\frac{1}{n}}u_2, \ldots 1^{\frac{1}{n}}u_n.$$

35.

EXPLANATION OF THE COINCIDENCE OF A THEOREM GIVEN BY MR SYLVESTER IN THE DECEMBER NUMBER OF THIS JOURNAL, WITH ONE STATED BY PROFESSOR DONKIN IN THE JUNE NUMBER OF THE SAME.

[*Philosophical Magazine*, (Fourth Series) I. (1851), pp. 44—46.]

I WISH to state, without loss of time, that in the theorem given by me* for the composition of two successive rotations about different axes, I have been anticipated by Prof. Donkin in the June Number of your *Journal*.

To my shame I must confess, that, although an occasional contributor to, I am not invariably a constant reader of your valuable miscellany, otherwise I should not have introduced the theorem in question without due acknowledgment of Professor Donkin's claims to whatever merit may attach to the priority of publication. The fact is, that I made out the theorem for myself nine years ago, and had some communication on the subject with Professor De Morgan, who was then writing the seventeenth chapter of his *Differential Calculus*. A recent conversation with this gentleman has brought back to my mind a vivid recollection of the course of that communication. I brought under Professor De Morgan's notice the analytical memoir of Sr Gabrio Pola on the subject in the Memoirs of the Italian Society of Modena, and satisfied myself of the existence of the single axis of displacement by compounding the two rotations in the manner given in my paper, which, for the case of two axes fixed in space, is the same as Professor Donkin's, and for two axes fixed in the rotating body is materially, although not formally the same.

It then occurred to me that a more simple demonstration ought to be deducible from the possibility of always *finding* the point on a sphere, by revolution about which, as a pole, one equal arc could actually be shown to be transportable into the place of another. But in proceeding to work out this idea I fell into a remarkable blunder, in which I have since been followed by more than one able friend to whom I have proposed the question. The

[* p. 158 above.]

blunder was of this kind:—Two arcs have to be drawn, bisecting at right angles the arcs joining the extremities of two equal arcs; the point of intersection of the two bisecting arcs *must* in all cases fall outside the quadrilateral formed by the equal and joining arcs. I supposed it to fall inside. There appears to be a fatal tendency to do so in all who take the subject in hand. In consequence of this error, the cause of which I did not at the moment perceive, I was driven to deny and admit in one breath the same proposition. Mr De Morgan sent me the correct proof after this method (the same as that given by him at page 489 of his *Calculus*), I am inclined to think after I had myself detected my error; but of this I cannot feel certain.

This is the method alluded to by me in the words "it is right to bear in mind, &c.," at the time of writing which all recollection of the same thing having been published by Mr De Morgan had vanished from my memory.

The proof of the triangle of rotations is so simple, that, as Professor Donkin states (in a letter which he has done me the favour of addressing me on the subject) was the case with himself, I thought it incredible that it should not have appeared in some elementary work, and I was therefore at no pains to publish it as my own; nor should I have written at all on the subject, had it not been for the surprise occasioned to my mind by falling in with Professor Stokes's article in the *Cambridge and Dublin Mathematical Journal*, to demonstrate the existence of an instantaneous axis, which proceeds in apparent unconsciousness of the so simply demonstrable law, that any number of rotations of any kind (and therefore those that take place in an instant of time) are representable by a single rotation about a single axis. I shall feel obliged by the early insertion of this explanation, more in justice to myself than to Professor Donkin, whose high and worthily earned reputation, not to speak of the disinterested love of truth for its own sake, apart from personal considerations, which animates the labours of the genuine votary of science, must make him indifferent to whatever credit might be supposed to result from the first authorship or publication of the very simple (however important) theorem in question.

36.

AN ENUMERATION OF THE CONTACTS OF LINES AND SURFACES OF THE SECOND ORDER.

[*Philosophical Magazine*, I. (1851), pp. 119—140.]

IT is well known that in general any two homogeneous quadratic functions of the same system of variables may be simultaneously transformed, so as to be expressed each of them as pure quadratic functions of a new system of variables equal in number and linearly connected with the original ones; a pure quadratic function meaning one in which only the squares of the variables are retained.

Every homogeneous quadratic function may be treated as the characteristic* of a locus of the second degree: if the function be of two letters, the locus is a binary system of points in a line wherein the distances of two fixed points from either point of the given system or given multiples of such distances correspond to the variables; if of three letters, the locus is a conic, the distances or given multiples of the distances of every point in which from three given lines in the plane of the conic are represented by the variables; if of four letters, the locus is a surface of the second order, the coordinates being the distances or multiples of the distances of any point therein from four planes drawn in the space in which the surface is contained, and so on for loci of four and higher dimensions.

I propose, however, in the present paper to restrict myself to the theory of the contacts of loci not transcending the limits of vulgar space, by which I mean the space cognizable through the senses†, and shall accordingly be

* According to the definition stated by me in a previous paper, the characteristic of a locus is the function which, equated to zero, constitutes the equation thereto.

† If the impressions of outward objects came only through the sight, and there were no sense of touch or resistance, would not space of three dimensions have been physically inconceivable? The geometry of three dimensions in ordinary parlance would then have been called transcendental. But in very truth the distinction is vain and futile. Geometry, to be properly understood, must be studied under a universal point of view; every (even the most elementary) proposition must be regarded as a fact, and but as a single specimen of an infinite series of homologous facts.

In this way only (discarding as but the transient outward form of a limited portion of an infinite system of ideas, all notion of extension as essential to the conception of geometry, however useful as a suggestive element) we may hope to see accomplished an organic and vital development of the science.

almost exclusively concerned in determining the singular cases of conjugate systems of quadratic forms of two, three, and four letters respectively.

In order that the reduction of any such system, say U and V, to a pure quadratic form may be possible (as it generally is), it is necessary that none of the roots of the complete determinant of $U+\lambda V$ shall be equal; if any relation of equality exist between these roots, the *general* reduction is *generally* no longer possible; under peculiar conditions, however, as will hereafter appear, in spite of the equality of certain of the roots, the irreducibility in its turn will cease, and the ordinary reduction be capable of being effected. It is easily seen, that to every relation of equality between the roots of the determinant of $U+\lambda V$ must correspond a particular species of contact between the loci which U and V characterize. But we should make a great mistake were we to suppose that every such relation of equality corresponded with but one species of contact; for instance, the characteristics of U and V of two conics are functions of three letters, and $\square(U+\lambda V)$ will be a cubic function of λ. Such a function may have two roots, or all its roots equal: this would seem to give but two species of contact, whereas we well know that there are no less than four species of contact possible between two conics. Accordingly we shall find, that, in order to determine the distinctive characters of each species of contact, we must look beyond the complete determinant, and examine into the relations (in themselves and to one another) of the several systems of minor determinants that can be formed from $U+\lambda V$.

By pursuing this method, we may assign *à priori* all the possible species of contact between any two loci of the second degree. How important this method is will be apparent from the fact, that not only have the distinctive characters of the various contacts possible between surfaces of the second order never been determined, but their number and the nature of certain of them have remained until this hour unknown and unsuspected.

The method which we shall pursue is an exhaustive one, and will conduct us by a natural order to a systematic arrangement of all the different modes and gradations of such contacts.

In a paper* in this *Magazine* for November 1850, I explained the decline of minor determinants, and stated a law, called the homaloidal law, concerning them.

If U and V be characteristics of the two loci whose contacts are to be considered, $U+\lambda V$ will be the function, the properties of whose complete determinant, and of the minor systems of determinants belonging to it, will serve to specify the nature of the contact.

It will be remembered, that, whatever be the number of variable letters in any quadratic function U, three of its first minor determinants being zero,

[* p. 150 above.]

makes all the first minors zero; six of its second minors being zero, makes all the second minors zero; and so on for the third, fourth, &c. minor systems according to the progression of the triangular numbers.

It is well known that whatever linear transformations be applied to a quadratic function W, the complete determinant thereof will remain unaltered, except by a multiplier depending upon the coefficients introduced into the equations of transformation; consequently the roots of λ in the equation obtained by making the determinant of $U+\lambda V$ zero remain unaffected by such transformation; and any relation or relations of equality among the roots of the equation $\square(U+\lambda V)=0$ is an immutable property of the system U, V, which is unaffected by linear transformations. Another and more general kind of immutable property (comprehending the above as a particular case), to which I shall have occasion to refer, is the following.

Suppose all the minors of any order of $U+\lambda V$ have a factor $\lambda+\epsilon$ in common; this factor will continue common to the same system of minors when U and V are simultaneously transformed. This is a very important proposition, and easily demonstrated; for if $\lambda+\epsilon$ be a common factor to all the rth minors of $U+\lambda V$, $(U-\epsilon V)$ will have its rth minors zero, and therefore, as explained by me in the paper above referred to, $U-\epsilon V$ will be degraded r orders below U or V. This is clearly a property independent of linear transformation, consequently $\lambda+\epsilon$ will remain a factor of the transformed rth minors.

In like manner it is demonstrable that any number of *distinct* factors $\lambda+\epsilon_1$, $\lambda+\epsilon_2$... common to the rth minors of one form of $U+\lambda V$, will remain common factors of any other linearly derived form of the same. It is consequently necessary that each rth minor of one form of any quadratic function W shall be a syzygetic* function of *all* the rth minors of any other form of the same; and consequently a function of λ of any degree, whether all its factors be or *be not distinct*, which is common to the rth minors of one form of $U+\lambda V$, will remain so to the rth minors of any other form of the same.

The law exhibiting the connexion of each rth minor of one form of W (any homogeneous quadratic function) with all the rth minors of any other form of W, will form the subject of a distinct communication.

Finally, to fully comprehend the annexed discussion, the following principle must be apprehended.

* If $A=pL+qM+rN+\&c.$, where p, q, r... do not any of them become infinite when L, M, N ... or any of them become zero, A may be termed a syzygetic function of L, M, N....

In the theorem above alluded to, it will be shown (as might be expected) that the syzygy in the case concerned is of the simplest kind, that is, that each rth minor of a quadratic function of any number of letters is a homogeneous linear function of all the rth minors of the same quadratic function linearly transformed.

If any factor K^e enter into all the rth minors of W, and if K^i be the highest power of K common to all the $(r+1)$th minors, then K^{2e-i} will be a common factor to all the $(r-1)$th minors.

Let r be taken unity; it is easily proved* that the complete determinant of *any* square matrix may be expressed by the difference between two products†, each of two first minor determinants divided by a certain second minor determinant. The proposition is therefore demonstrated for this case, and thereby in fact implicitly for every case, inasmuch as the first minors of every rth minor are $(r+1)$th minors of the original matrix. Hence it follows, that if any system of rth minor determinants have a common factor ϵ^i, the complete determinant must contain *at lowest* the factor $\epsilon^{(r+1)i}$, and any system of $(r-s)$th minor determinants thereunto will contain *at lowest* the factor $\epsilon^{(s+1)i}$.

I now proceed to apply these principles to the determination of the relative forms of conjugate quadratic functions representing geometrical loci of the second order. I shall begin with two binary systems of points in a right line.

The general characteristics U and V of two such systems may be thrown under the form

$$\left.\begin{array}{l} U = x^2 + y^2 \\ V = ax^2 + by^2 \end{array}\right\}.$$

When $\square(V + \lambda U) = 0$ has its two roots equal, these systems have a point in common. The above forms cease to be applicable, and convert into

$$\left.\begin{array}{l} U = xy \\ V = ax^2 + bxy \end{array}\right\}$$

where $x = 0$ represents the common point.

* This will appear in my promised paper on Determinants and Quadratic Functions.

† When the matrix is symmetrical about one of its diagonals (as it is in the case which we are concerned with), one of these products becomes a square. I may take this occasion of hinting, that the theory of quadratic functions merges in a larger theory of binary functions, consisting of the sum of the multiples of binary products formed by combining each of one set of quantities, $x, y, z \ldots$ with each of the same number of quantities of another set, as $x', y', z' \ldots$. For instance,

$$\begin{array}{l} axx' + bxy' + cxz' \\ + a'yx' + b'yy' + c'yz' \\ + a''zx' + b''zy' + c''zz' \end{array}$$

would be a binary function, and its determinant (no longer, as in a quadratic function, symmetrical about either diagonal) would correspond to the square matrix

$$\begin{array}{ccc} a & b & c \\ a' & b' & c' \\ a'' & b'' & c''. \end{array}$$

Almost all the properties of quadratic apply, with slight modifications, to binary functions.

Let U and V now represent two conics. When there is no contact, we have as the types of their characteristics

$$U = x^2 + y^2 + z^2,$$
$$V = ax^2 + by^2 + cz^2.$$

The three roots of $\square(V + \lambda U) = 0$ are

$$\lambda = -a, \quad \lambda = -b, \quad \lambda = -c,$$

showing that there are three distinct pairs of lines in which the intersections of U and V are contained, the equations to three pairs being respectively

$$(b-a)y^2 + (c-a)z^2 = 0,$$
$$(c-b)z^2 + (a-b)x^2 = 0,$$
$$(a-c)x^2 + (b-c)y^2 = 0;$$

the four points of the intersection being defined by the equations corresponding to the proportions

$$x : y : z :: \sqrt{(b-c)} : \sqrt{(c-a)} : \sqrt{(a-b)}.$$

Now let $\square(U + \lambda V)$ have two equal roots; the characteristics assume the form

$$U = x^2 + y^2 + xz,$$
$$V = ax^2 + by^2 + cxz^*.$$

Two of the pairs of lines become identical, that is, two of the four points of intersection coincide.

* We may if we please make $a = b$; for it may be shown that the equations, in their present forms, contain an arbitrariness of 10 degrees; namely, 9 on account of x, y, z being arbitrary linears of ζ, η, θ; 2 on account of the ratios $a : b : c$; together 11 reduced by one degree on account of x, y, z, changed into lx, ly, lz, leaving $U=0$, $V=0$ unaffected. Now the degrees of arbitrariness in two conics, subject to satisfy only one condition, is $2 \times 5 - 1$ or 9. Hence there is one degree of arbitrariness to spare. In fact, on making $a = b$, the axis z becomes the line joining the two points of intersection distinct from the point of contact; x remaining the tangent at the point of contact, and y, strange to say, still arbitrary, subject only to passing through the point of contact; if, however, y be made to pass through the point of contact, and either one of the distinct intersections, this form,

$$U = x^2 + y^2 + xz,$$
$$V = ax^2 + ay^2 + cxz,$$

becomes no longer tenable, but gives place to

$$U = y^2 + yx + xz,$$
$$V = ay^2 + ayx + cxz,$$

where x is the tangent at the point of contact, z the line joining the two intersections with one another, and x, $x + y$ respectively the lines joining either of them with the point of contact; if the multiplier of yx in V in the above be made b instead of a, x remains the tangent as before, y becomes any line through the point of contact, and z any line through one of the distinct intersections. A systematic view of similar modulations of form and the study of the laws of arbitrariness connected with them, as applicable to the general subject-matter of this paper, must be deferred to a subsequent occasion.

This may be termed "Simple Contact." The tangent at the point of contact is $x=0$; this equation making U and V each become of only one order.

The intersections are

$$x=0, \quad y=0, \qquad (1)$$

$$x=0, \quad y=0, \qquad (2)$$

$$\sqrt{(a-c)}\,x+\sqrt{(b-c)}\,y=0, \quad z=0, \qquad (3)$$

$$\sqrt{(a-c)}\,x-\sqrt{(b-c)}\,y=0, \quad z=0. \qquad (4)$$

These are obtained by making $V-aU=0$, which gives $x=0$ or $z=0$.

$x=0$ gives $y^2=0$, that is, $y=0$ twice over, and $z=0$ gives

$$(a-c)\,x^2+(b-c)\,y^2=0.$$

The number of conditions to be satisfied in this case is one only.

Next let $\square(U+\lambda V)$ have all its roots equal. This condition will be satisfied (still leaving U and V as general as they can remain consistent with these conditions) by making

$$U=x^2+yz+yx,$$

$$V=ax^2+ayz+byx.$$

Here only one *distinct* pair of lines can be drawn to contain the intersections, showing that three out of the four points come together.

This may be termed "Proximal Contact." The number of affirmative conditions to be satisfied is two, and the contact is therefore entitled of the second degree.

The tangent at the point of contact is $y=0$, and the four intersections become

$$x=0, \quad y=0,$$

$$x=0, \quad y=0,$$

$$x=0, \quad y=0,$$

$$x=0, \quad z=0.$$

These may be obtained from the equation $V-aU=0$, which gives $y=0$ or $z=0$; the former implying concurrently with itself $x^2=0$, and the latter $yz=0$.

Thus we obtain three systems,

$$x=0, \quad y=0,$$

and one

$$x=0, \quad z=0,$$

corresponding to three consecutive points and the single distinct one.

The determinant of $U+\lambda V$ being only of the third degree in λ, we have exhausted the singularities of the system U, V dependent on the form of the complete determinant of $U+\lambda V$.

Let now the first minors of $U+\lambda V$ have a factor in common; this will indicate that $U+\lambda V$ may be made to lose *two* orders by rightly assigning λ, in other words, that the intersections of U and V are contained upon a pair of *coincident* lines. Here it is remarkable that the original forms of U and V reappear, but with a special relation of equality between the coefficients: we shall have, in fact,

$$U=x^2+y^2+z^2,$$

$$V=ax^2+ay^2+bz^2.$$

This gives the law for double, or, as I prefer to call it, diploidal contact*. By virtue of the Homaloidal law, we know that if three first minors of $U+\lambda V$ be zero, all are zero; we have therefore to express that three quadratic functions of λ have a root in common. This implies the existence of two *affirmative* conditions; the contact of the two conics taken collectively may therefore be still entitled of the second degree, although the contact at each of the two points where it takes place is simple, or of the first degree.

These points are evidently defined by the equation

$$\{x+\sqrt{(-1)}\,y=0, \; z=0\},$$

$$\{x-\sqrt{(-1)}\,y=0, \; z=0\},$$

and the ordinary algebraical solution of the equations $U=0$, $V=0$ would naturally lead to the four systems

$$x+\sqrt{(-1)}\,y=0, \quad z=0,$$

$$x+\sqrt{(-1)}\,y=0, \quad z=0,$$

$$x-\sqrt{(-1)}\,y=0, \quad z=0,$$

$$x-\sqrt{(-1)}\,y=0, \quad z=0;$$

the two tangents at the point of contact are $x+\sqrt{(-1)}\,y=0$, $x-\sqrt{(-1)}\,y=0$, and the coincident pair of lines containing the intersections is $z^2=0$.

* See my remarks† on the conditions which express double contact in the *Cambridge Journal*, Nov. 1850. If n functions, being all zero, be the condition of a fact, but r independent syzygetic equations admit of being formed between these functions, the number of affirmative conditions required is not n, but $(n-r)$; because the fact may be expressed by affirming $(n-r)$ equations and denying certain others. Thus if $P=0$, $Q=0$, $R=0$, $S=0$ express a fact, and

$$PP'+QQ'+RR'+SS'=0,$$

$$PP''+QQ''+RR''+SS''=0,$$

the fact is expressible by affirming $P=0$, $Q=0$, and denying $R'S''-R''S'=0$, for then $P=0$, $Q=0$ will imply $R=0$, $S=0$; or, in like manner, by affirming any other two out of the four necessary equations, and denying the other equations. Observe, however, that all the required equations may coexist in the absence of such right of denial.

[† p. 129 above.]

It may at first view appear strange, that whilst no condition is required in order that U and V may be simultaneously metamorphosed into the forms of $x^2+y^2+z^2$, $ax^2+by^2+cz^2$, a, b and c being all unequal, for this metamorphosis to be possible when any two become equal, not one but two conditions must be satisfied. The reason of this is, that the coefficients of transformation, which, as well as a, b, c, are functions of the coefficients of the given quadratic functions, become infinite on constituting between the said coefficients such relations as are necessary for satisfying the equation $a=b$, or $a=c$, or $b=c$, except upon the assumption of some further particular relations between them over and above that implied in such equality.

In the ordinary case of diploidal contact, the first minors having a factor in common, this factor will enter twice into the complete determinant of $U+\lambda V$, but it *may* enter three times: this will indicate, that not only do the four intersections lie on a coincident pair of lines, but furthermore, that there is but one pair of lines of any kind on which they lie.

In the ordinary case of diploidal contact, it will be observed that this latter condition does not obtain; the four intersections lie on a coincident pair of lines; but they lie also on a crossing pair, namely, in the two tangents at the points of contact. In this higher species of diploidal contact, it is clear that the two points of contact, which are ordinarily distinct, come together, and that all four intersections coincide.

This I call *confluent* contact; the forms of U and V corresponding thereto will be

$$U=x^2+y^2+xz,$$
$$V=ay^2+axz;$$

the common tangent at the point of contact being $x=0$, and the four coincident points $x^2=0$, $y^2=0$.

The number of affirmative conditions to be satisfied being three, the contact is to be entitled of the third degree.

Observe, that it is of no use to descend below the first minors in this case; because the second minors, being linear functions of λ, could not have a factor in common, unless $V:U$ becomes a numerical ratio, which would imply that the conics coincided*.

Fortified by the successful application of our general principles to the preceding more familiar cases of contact, we are now in a condition to apply with greater confidence the same *à priori* method to the exhaustion and characterization of all the varied species of contact possible between surfaces

* No-contact and complete coincidence may be conceived as the two extreme cases in the scale of relative conjugate forms.

of the second order; a portion of the subject comparatively unexplored, and never before thought susceptible of reduction to a systematic arrangement.

When there is no contact, we may write

$$U=x^2+y^2+z^2+t^2,$$
$$V=ax^2+by^2+cz^2+dt^2,$$

and the intersection of the surfaces will lie in each of the four cones,

$$(a-d)x^2+(b-d)y^2+(c-d)z^2=0,$$
$$(a-b)x^2+(c-b)z^2+(d-b)t^2=0,$$
$$(a-c)x^2+(b-c)y^2+(d-c)t^2=0,$$
$$(b-a)y^2+(c-a)z^2+(d-a)t^2=0.$$

Whenever the surfaces are in contact, certain of these cones will coincide with certain others, so that their number will be always less than four. Also, as we shall find in such event, they may degenerate into pairs of intersecting or coincident planes.

Let us begin with considering the cases of contact for which the first minors (and consequently *à fortiori* the minors inferior to the first) have no factor in common.

Here $\square(V+\lambda U)$ is a biquadratic function.

If λ have all its roots unequal, we have U and V as above given.

If two roots are equal, the characteristics assume the form

$$\left.\begin{array}{l} U=x^2+y^2+z^2+xt \\ V=ax^2+by^2+cz^2+dxt \end{array}\right\}.$$

The touching plane is $x=0$; the point of contact is $x=0$, $y=0$, $z=0$; the curve of intersection is one of the fourth degree, with a double point at the point of contact.

There is but one condition to be satisfied, and the contact may be entitled "simple" and of the first degree.

Next let λ have three equal values, the equations become

$$U=x^2+yz+t^2+xy,$$
$$V=x^2+yz+at^2+bxy.$$

The tangent plane at the point of contact $y=0$, and the point itself $x=0$, $y=0$, $t=0$. The curve of intersection is a curve of the fourth order, with a cusp at the point of contact. The number of affirmative conditions to be satisfied is two; the contact is of the second degree, and may be termed "proximal" or cuspidal.

Next let $\square(U + \lambda V)$ have two pairs of equal roots, we shall find

$$U = x^2 + xy + zt,$$
$$V = ayz + bxy + czt.$$

The line $x = 0$, $z = 0$ will be common to both surfaces. The curve of intersection will therefore break up into a right line and a line of the third order.

The former will meet the latter in two points, which will be each of them points of contact. The contact is therefore diploidal; but as there is another species of diploidal contact to which we shall presently come, it will be expedient to characterize each of them by the nature of the intersections of the two surfaces; accordingly this may be termed unilinear-intersection contact, or more briefly, unilinear contact.

The number of affirmative conditions to be satisfied being two, it may be said to be collectively of the second degree, but (obviously?) the contact at each of the two points is of the nature of simple contact.

Lastly, let us suppose that all four roots of $U + \lambda V$ are equal; we shall find, as the most simple expressions of the most general forms of the two surfaces,

$$U = x^2 + xy + yz + zt,$$
$$V = axy + bz^2 + azt.$$

In this case the two points of intersection of the curve of the third degree, and the right line on which the surfaces intersect, come together, so that the right line becomes a tangent to the curve. The number of conditions to be satisfied is three: there is but one point of contact which may be considered as the union of two which have coalesced, and the species may be defined as confluent-unilinear contact.

If we throw the equations to the conoids having an unilinear contact into the form

$$x(x + y) + zt = 0,$$
$$xy + z(y + ct) = 0,$$

we obtain

$$(x + y)(y + ct) - yt = 0,$$

which last equation is no longer satisfied by $x = 0$, $z = 0$, these systems of roots having been made to disappear by the process of elimination.

The curve of the third degree, in which the two given conoids intersect, may thus be defined as their common intersection with the new conical surface defined by the third of the above equations.

More generally, it is apparent that the three conoids,

$$\left.\begin{array}{r} xu - yt = 0 \\ yv - zu = 0 \\ zt - xv = 0 \end{array}\right\},$$

in which x, y, z, t, u, v may any of them be considered as a homogeneous linear function of four others, intersect in the same line of the third degree. *Besides* which, the first and second intersect in the right line y, u; the second and third in z, v; the third and first in x, t; each of which lines it is evident is a chord of the common curve of intersection. For instance, $y = 0$, $u = 0$ may be satisfied concurrently with all the above three equations by satisfying the equation $zt - xv = 0$, which, as two linear relations exist originally between the six letters, and two more have been thrown in, becomes a quadratic equation between any two of the letters.

The only case of exception to this reasoning is, when $y = 0$, $u = 0$ can be satisfied concurrently with $z = 0$, $v = 0$, and with $x = 0$, $t = 0$; but in this case the surfaces all become cones; and as there is no longer a curve of the third degree, "Cadit quæstio." Even here, however, the intersection of any two of the surfaces becomes a conic, and two coincident generating lines on the two cones; so that if we take one of these and the conic to represent a degenerate form of a line of the third degree, the remaining straight line passes through a double point of such degenerate form, and the case passes into that of confluent-unilinear contact.

The two double points in the intersection of the two conoids

$$U = x(x + y) + zt = 0,$$
$$V = xy + z(y + ct) = 0,$$

by which I mean the points of intersection of the conic with the right line common to them, are found by making $x = 0$, $z = 0$, and substituting in the derived equation

$$(x + y)(y + ct) - ty = 0,$$

which gives $y = 0$, or $y + (c - 1)t = 0$; so that the two points required are

$$x = 0, \quad y = 0, \quad z = 0,$$
$$x = 0, \quad y = (1 - c)t, \quad z = 0.$$

It appears also that the entire intersection is contained in each of the two cones,

$$U - V, \text{ that is, } x^2 + z\{(1 - c)t - y\}$$

and

$$cU - V, \text{ that is, } cx^2 + y\{(c - 1)x - z\},$$

the respective vertices of which are at the points above determined.

The equations for confluent-unilinear contact,

$$x(x+y)+z(y+t)=0,$$
$$xy+z(cz+t)=0,$$

give

$$(x+y)(cz+t)-(y+t)y=0;$$

which, on making $x=0$, $z=0$, is satisfied by $y^2=0$; showing that the *confluence* takes place at the point

$$x=0, \quad y=0, \quad z=0.$$

The number of terms in the two equations for ordinary unilinear contact being six, and in those given for confluent unilinears seven, and the empirical rule in all other cases being that the terms tend to diminish and never increase in number as the degree of the contact (expressed by the number of conditions to be satisfied) rises, I am led to suspect that the conjugate system for the latter species of contact may admit of being reduced to some more simple form.

I must state here once for all, that all the *distinct* systems of (at least consecutive) conjugate forms that have been, and will be given, are *mutually* untransformable. This it is which distinguishes *singular* from *particular* forms.

A particular form is included in its primitive; but a singular form is one, which, while it responds to the same conditions as some other *more general* form, is incapable of being expressed as a particular case of the latter, on account of the *additional* condition or conditions which attach to it.

I pass now to the singularities which arise from the first minor determinants of $U+\lambda V$ having a factor in common, the second minors being supposed to be still without a common factor.

When this common factor is linear in respect to λ, let it be supposed to enter not more than twice (twice, we know, by the general principle enunciated at the commencement of this paper, it must enter) into the complete determinant.

Two of the cones containing the intersection of U and V then become coincident, and degenerate each into the same pair of crossing planes. This may be termed biplanar-contact. The characteristics of such contact are

$$U=x^2+y^2+z^2+t^2,$$
$$V=ax^2+ay^2+bz^2+ct^2;$$

the points of contact are two in number, being at the intersection of the two plane conics into which the curve of intersection breaks up. The two planes

in which these lie are given by the equation $(b-a)z^2+(c-a)t^2=0$; these intersect in the right line $z=0$, $t=0$, which meets both surfaces in the same two points,

$$z=0, \quad t=0, \quad x+\sqrt{(-1)}\,y=0,$$
$$z=0, \quad t=0, \quad x-\sqrt{(-1)}\,y=0,$$

the two common tangent planes at these points being

$$x+\sqrt{(-1)}\,y=0, \quad x-\sqrt{(-1)}\,y=0$$

respectively.

This, then, is another species of double contact between two conoids, and, as far as I know, the only kind hitherto recognized as such. The number of conditions to be satisfied remains two, as in the former species.

Next suppose that the common factor of the first minor enters three times into the complete determinant instead of *twice* only, as in the last case.

The corresponding characteristics will be found to be

$$U=x^2+zt+y^2+z^2,$$
$$V=ax^2+azt+by^2+cz^2.$$

The intersection of U, V still lies in two planes,

$$(b-a)y^2+(c-a)z^2=0;$$

but the intersection of these two planes,

$$y=0, \quad z=0,$$

meets the surfaces in the two coincident points,

$$y=0, \quad z=0, \quad x^2=0.$$

This, therefore, I call confluent-biplanar contact; the two conics constituting the complete intersection, instead of cutting, touch and at their point of contact the two conoids have a contact of a superior order. The conditions to be satisfied for this case are three in number.

Next suppose that the common factor of the first minors enters only twice into the complete determinant, but that the remaining two factors become equal.

Here the analytical characters of unilinear and biplanar contact are blended; in fact, the intersection consists of a conic and a pair of right lines meeting one another and the conic. The characteristics are

$$U=x^2+y^2+z^2+zt,$$
$$V=ax^2+ay^2+bz^2+czt.$$

The intersection is contained in the two planes

$$z = 0, \quad (b-a)z + (c-a)t = 0,$$

and consists of the two lines $z = 0$, $x^2 + y^2 = 0$, lying in the common tangent plane $z = 0$, and the conic

$$\left.\begin{array}{l}(b-a)z + (c-a)t = 0 \\ (a-c)x^2 + (a-c)y^2 + (b-c)z^2 = 0\end{array}\right\}.$$

There are *three* points of contact, namely, the point $x = 0$, $y = 0$, $z = 0$, where the two right lines cut, and $x^2 + y^2 = 0$, $t = 0$, $z = 0$, where these lines meet the conic. This, then, is a case of triple contact. I distinguish it by the name of bilinear-contact. The number of conditions is still three.

Now all else remaining as before, let the two pairs of equal roots in the complete determinant become identical, or, in other words, let the common factor of the first minors be contained four times in the complete determinant. The characteristics become

$$U = xz + xt + y^2 + z^2,$$
$$V = axz + bxt + by^2 + bz^2.$$

The intersection becomes the two right lines

$$x = 0, \quad y^2 + z^2 = 0,$$

and the conic

$$z = 0, \quad x^2 + y^2 = 0.$$

All these meet in the same point,

$$x = 0, \quad y = 0, \quad z = 0;$$

so that instead of contact in three points, the contact takes place about one only, in which the *three* may be conceived as merging. This I call confluent-bilinear contact. It requires the satisfaction of four conditions.

Next let us suppose that the two distinct factors are common to each of the first minors. This will imply the existence of four affirmative conditions.

The complete determinant will of necessity contain each of these factors twice, so that no additional singularity can enter through this determinant. The characteristics assume the form

$$U = x^2 + y^2 + z^2 + t^2,$$
$$V = ax^2 + ay^2 + bz^2 + bt^2.$$

The two surfaces will meet in four straight lines, forming a wry quadrilateral, whose equations are

$$x \pm \sqrt{(-1)}\, y = 0,$$
$$z \pm \sqrt{(-1)}\, t = 0.$$

These intersect each other in the four points

$$x = 0, \quad y = 0, \quad z^2 + t^2 = 0,$$
$$z = 0, \quad t = 0, \quad x^2 + y^2 = 0,$$

each of which will be a distinct point. This I term quadrilinear contact.

Now let the two factors common to each of the first minors become identical; so that a *squared* function, instead of an ordinary *quadratic* function of λ, is now their common measure.

The factor which enters twice into each of the first minors will enter four times into the complete determinant; the number of conditions to be satisfied is one more than in the preceding case, namely five, and the characteristics become

$$U = x^2 + y^2 + xz + yt,$$
$$V = ax^2 + by^2 + cxz + cyt.$$

Here arises a singularity of form in the intersections utterly unlike anything which has been remarked in the preceding cases. For it will not fail to have been observed, that the intersection in the nine preceding cases was always a line or system of lines of the fourth degree, so as to be cut by any plane in four points.

But in this case, the fact of the first minors having a factor in common, shows that the intersection is contained in two planes (which is of course to be viewed as a degenerate species of cone); and the fact of the complete determinant having all its roots equal, shows that there is but one system of a pair of planes in which the intersection is contained, and no more.

So that the two pairs of planes, into which the wry quadrilateral was divisible in the case immediately preceding, now become a single pair. This can only be explained by two of the opposite sides of the quadrilateral becoming indefinitely near to one another, but still not coinciding in the same planes; so that the actual *visible* or quasi-visible* intersection will be in three right lines, of which the middle one meets each of the two others.

This will further appear by proceeding regularly to solve the equations

$$U = 0, \quad V = 0.$$

$V - cU = 0$ gives $y = \pm kx$, where $k = \sqrt{\left(\frac{a-c}{c-b}\right)}$, and therefore $xz + kxt = 0$, or $xz - kxt = 0$; whence we see that the complete intersection is represented by the lines

$$(x = 0, \ y = 0); \quad (z + kt = 0, \ y - kx = 0),$$
$$(x = 0, \ y = 0); \quad (z - kt = 0, \ y + kx = 0),$$

* I use the term quasi-visible, because the intersection may become in part or whole imaginary.

showing that there are but three physically distinct lines, as already premised.

This, then, may be considered as derived from the preceding case of a wry quadrilateral intersection, by conceiving two opposite sides of the quadrilateral to come indefinitely near, but without coinciding.

Let these two lines be called P and P'; take any point in P and any two points in P' indefinitely near to one another and the point first taken, then this indefinitely small plane will be common to both surfaces, and consequently they ought to touch along *every point in the line P*. This is again confirmed by the forms given to U and V. For at any point where the coordinates are $0, 0, \zeta, \theta$ the equations to the tangent planes to the two surfaces respectively are

$$\zeta x + \theta y = 0,$$
$$c\zeta x + c\theta y = 0,$$

that is to say, are identical.

Whilst, therefore, certain grounds of geometrical, and still stronger grounds of analytical analogy, might seem to justify this species of contact taking the name of confluent quadrilinear, yet as, in fact, the intersection is trilinear, and as, moreover, the two indefinitely proximate lines must be considered, not as coincident, but as turned away from one another through an indefinitely small angle and out of the same plane, I prefer to take advantage of this striking property of contact *at every point* along a line (a property entirely distinct from any that we have yet considered), and confer upon the species of contact we have been considering the designation of unilinear-indefinite contact.

Where the line of indefinite contact meets the two other lines of the intersection, the contact is of course of a higher order; thus offering a parallel to what takes place in ordinary unilinear contact, in which there is no contact, except only *at two points* of the right line forming part of the complete intersection.

I believe that this kind of contact, which forms a natural family with two others about to be described, and which will close the list, has never before been imagined, and would at first sight have been rejected as impossible.

Having now exhausted the cases of the first class, in which the minors have no factor in common, and the two sections of the second class, in which the second minors have no common factor, but the first minors of $U + \lambda V$ a linear or quadratic function of λ in common, I descend to the third class, in which the second minors, which are quadratic functions of λ, are supposed to have a common factor.

This common factor must enter twice into each of the first minors by virtue of the law previously indicated, and cannot enter more than twice, as

otherwise the first minors of $U + \lambda V$ could only differ from one another by a numerical multiplier, which is obviously impossible, except when $U + \lambda V$ is of the form $(k + \lambda) U$, that is, when the two surfaces coincide.

Again, the common factor of the first minor must enter three times into the complete determinant; but there is no reason why it may not enter four times, and thus two cases arise. In the first, the characteristics take the form

$$U = x^2 + y^2 + z^2 + t^2,$$
$$V = ax^2 + ay^2 + az^2 + bt^2.$$

The second determinant having a factor in common, shows that the intersection U, V is contained in a pair of coincident planes; but the complete determinant, having two distinct factors, evidences that these plane intersections, viewed as indefinitely near but still distinct, lie in the same cone, which will be a cone enveloping both the surfaces U and V all along their mutual intersections. This is also seen easily from the forms of U and V; for we have $V - aU = (b - a) t^2$, which proves that the intersection lies in the coincident, or, to speak more strictly, consecutive planes $t^2 = 0$; and at any point $x = \xi$, $y = \eta$, $z = \zeta$, the tangent plane to each surface becomes

$$\xi x + \eta y + \zeta z = 0.$$

As there are six independent, that is, non-necessarily co-evanescent second minors, that the second minor systems shall all have a common factor, implies the satisfaction of five conditions. This species of contact I call curvilineo-indefinite; it is, I believe, the only kind of indefinite contact between two surfaces of the second order hitherto taken account of.

There is still, however, a higher species of contact, *videlicet*, when all the four roots of the complete determinant of $U + \lambda V$ are identical with the root common to each of its second minors. In this case the common enveloping cone becomes identical with the plane (considered as a coincident pair of planes) in which the surfaces intersect.

The characteristics take the form

$$U = x^2 + xy + zt,$$
$$V = \quad xy + zt.$$

The intersection is contained completely in the common tangent plane $x = 0$, and consists of the *two* right lines,

$$(x = 0, \quad z = 0),$$
$$(x = 0, \quad t = 0).$$

This, the highest and crowning species of contact, I call bilineo-indefinite. It is defined by six conditions.

At each point of the two lines of intersection of U and V there is contact, and a very peculiar species of contact at the intersection of these two lines themselves.

To form a distinct idea of this, let the physical visible or quasi-visible intersection of U, V take place along the two lines L, M; the *rational* intersection must be conceived as made up of the wry quadrilateral, L, M; L', M', in which L is indefinitely near to L', and M to M'. It follows, therefore, that there is contact at the four angles of the quadrilateral; but as there is nothing to fix the relative directions of the diagonal joining the intersection of L and M to that of L' and M', because there is nothing to restrict the position of the latter point, except that it shall lie upon either surface*, it appears that not only is there contact at the junction of the two lines constituting the complete intersection of the two surfaces, but that these surfaces continue to touch at consecutive points taken all round this first, and indefinitely near to it in any direction†.

Bilineo-indefinite (the highest) contact for two conoids is strictly analogous to confluence, the highest species of contact between conics. For this latter may be conceived as an intersection made up of two coincident pairs of coincident points; and the former, as an intersection made up of two coincident pairs of crossing right lines; and a pair of crossing lines is to a plane locus of the second degree what a coincident pair of points is to a rectilinear locus of the same degree.

In the subjoined table I have brought under one point of view the characters and algebraic forms which I call the *condensed forms* corresponding to each species of contact above detailed.

A. *Quadratic loci in a right line.*

Simple contact. One condition.	xy $x^2 + xy$

B. *Quadratic loci in a plane.*

1st Class.

Simple contact. One condition.	$x^2 + y^2 + xz$ $ax^2 + by^2 + cxz$
Proximal contact. Two conditions.	$x^2 + yx + yz$ $ax^2 + byx + ayz$

2nd Class.

Diploidal contact. Two conditions.	$x^2 + y^2 + z^2$ $ax^2 + ay^2 + bz^2$
Confluent contact. Three conditions.	$x^2 + y^2 + xz$ $y^2 + xz$

* This will be better seen by reference to the analogy presented by the case when the two conoids touch all along a curve. The rational intersection is made up of this curve and another indefinitely near it. The two curves, whatever be the position of their node, will lie in the same enveloping cone, so that the position of the node is indeterminate.

† As the two surfaces jut one close into the other at this point, it would perhaps be not improper to designate the contact at such point as umbilical.

C. *Quadratic loci in space.*

1st Class.

Simple contact. One condition.	$x^2 + y^2 + z^2 + xt$ $ax^2 + by^2 + cz^2 + dxt$
Proximal contact. Two conditions.	$x^2 + y^2 + xt + zt$ $ax^2 + by^2 + cxt + azt$
Unilinear contact. 1st species of diploidal. Two conditions.	$x^2 + xy + zt$ $ayz + bxy + czt$
Confluent-unilinear, or triple contact. Three conditions.	$x^2 + yz + xy + zt$ $az^2 + bxy + bzt$

2nd Class, 1st Section.

Biplanar contact. 2nd species of diploidal. Two conditions.	$x^2 + y^2 + z^2 + t^2$ $ax^2 + ay^2 + bz^2 + ct^2$
Confluent-biplanar contact. Three conditions.	$x^2 + zt + y^2 + z^2$ $ax^2 + azt + by^2 + cz^2$
Bilinear contact. Three conditions.	$x^2 + y^2 + z^2 + zt$ $ax^2 + ay^2 + bz^2 + czt$ or $xz + yt$ $axt + byz$
Confluent-bilinear contact. Four conditions.	$xz + xt + y^2 + z^2$ $axz + bxt + by^2 + bz^2$

2nd Class, 2nd Section.

Quadrilinear, or quadruple contact. Four conditions.	$x^2 + y^2 + z^2 + t^2$ $ax^2 + ay^2 + bz^2 + bt^2$ or $xy + zt$ $axy + bzt$
Unilineo-indefinite contact. Five conditions.	$x^2 + y^2 + xz + yt$ $ax^2 + by^2 + cxz + cyt$

3rd Class.

Curvilineo-indefinite contact. Five conditions.	$x^2 + y^2 + z^2 + t^2$ $ax^2 + ay^2 + az^2 + bt^2$
Bilineo-indefinite contact. Six conditions.	$x^2 + xy + zt$ $xy + zt$

Another (and, in a physical sense, more) natural mode of grouping the twelve species of conoidal contact, which, without observing the same lines of demarcation, leaves intact the sequence of the species, is into the three families. The first, or definite-continuous, for which the surfaces touch in a single point, and intersect in an unbroken curve, comprises simple and cuspidal contact.

The second definite-discontinuous, for which the surfaces touch in one, two, three or four points, but intersect in a curve more or less broken up into distinct parts, comprises all the species from the third to the ninth inclusive. The third natural family is that of indefinite contact, and comprises the three last species. It will of course be observed that there are five species of single contact, that is, contact at one point, namely, simple, cuspidal, and the three confluent species, two of double, one of treble, one of quadruple, and three of indefinite contact; the last being distinguishable *inter se*—lineo-indefinite as being special at two points, curvilineo-indefinite as having no speciality, and bilineo-indefinite as being special at one point only.

I might now proceed to discuss more particularly the nature of the contact taken, not collectively, but with reference to each single point where it exists. This, however, must be reserved for a future communication; as also, among other important and curious matter, the ascertainment of the singular forms of quadratic conjugate functions of five or more letters. At present I shall content myself with stating the following general proposition, which naturally suggests itself from a consideration of the cases already considered.

In a conjugate quadratic system of any number of letters, the lowest and also the highest degree of singularity will be always unique; the conditions to be satisfied in the former case being only one in number, and in the latter $\frac{1}{2}r(r-1)$, where r denotes the number of the letters. The first part of this proposition is self-apparent, the latter part may be inferred from the homaloidal law; for the $(r-2)$nd minors will be quadratic functions, and the highest degree of contact will correspond to those having a factor in common, which would involve the satisfaction of $\frac{1}{2}r(r-1)-1$ conditions only; but over and above this, that the complete determinant, instead of containing this common factor, as it needs must, $(r-1)$ times, shall contain it r times: this gives one condition more, making up the entire number to $\frac{1}{2}r(r-1)$.

The total number of different species of singularity for conjugate functions of a given number of letters, can only be expressed by aid of formulæ containing expressions for the number of various ways in which numbers admit of being broken up into a given number of parts.

The computation of this number in particular cases, upon the principle of the foregoing method, is attended with no difficulty.

We have seen that this number for two, three and four letters, is respectively one, four, twelve.

I have found that for five letters the number is twenty-four, for six letters fifty, for seven letters a hundred, and (subject to further examination) for eight letters one hundred and ninety-three. The series, therefore, as far as I have yet traced it, is 1, 4, 12, 24, 50, 100, 193. The last number must not be relied upon at present.

It will be observed, that the foregoing table for the contacts of surfaces of the second order contains no form corresponding to a complete intersection in two non-intersecting lines and an undegenerated conic. In fact, if two such lines form part of the intersection, *at least* one other right line intersecting them both, must go to make up the remaining part. This is easily verified; for it is readily seen that the most general representation of two conoids intersecting in two non-meeting lines will be

$$U = xy + zt,$$
$$V = axy + bzt + cxt + eyz,$$

where the two lines in question are

$$(x=0, \quad z=0),$$
$$(y=0, \quad t=0).$$

Now it will be found that the first minors of $V+\lambda U$ formed from the above equation will all contain the common factor $(a+\lambda)(b+\lambda)-ce$, showing that the contact is quadrilinear or linear-indefinite, that is bilinear, according as the roots of

$$\lambda^2+(a+b)\lambda+ab-ce=0$$

are distinct or equal; which explains how it is that only one species of bilinear contact (that is to say, the case corresponding to the two conoids agreeing in the two right lines in which each is cut by a common tangent plane) comes to find a place in the preceding enumeration.

It may not be uninteresting, under an euristic point of view, to state that the above theory, which, as well in what it accomplishes as in what it suggests (the author cannot but feel conscious), constitutes a substantial accession to analytical science, arose out of a theorem which occurred to him as likely to be true, in the act of reviewing for the press his paper "On Certain Additions" in the last November Number* of this *Magazine*, and which he had only then time to throw into a foot-note as a probable conjecture.

Wishing to subject it to an analytical test, he found it necessary to obtain the *condensed forms* which serve to characterize the confluent contact of

[* p. 148 above.]

conics. In this way he became aware of the great utility of these condensed forms, and of the desideratum to be supplied in obtaining a complete list of them applicable to all varieties of contact. The happy thought then occurred to him of inverting the process which he had applied in the treatment of the contacts of conics, in the November Number* of the *Cambridge and Dublin Mathematical Journal*; for whereas the nature of the contacts was there assumed and translated into the language of determinants, he soon discovered that it was the more easy and secure course to assume the relations of every possible *immutable* kind that could exist between the complete and minor determinants corresponding to the characteristics, by aid of these relations to construct the characteristics, and from the characteristics so obtained, determine the geometrical character of each resulting species of contact. Thus he has been able to effect the very results stated by himself as desiderata at the close of the paper in this *Magazine* above referred to.

Note.—It is proper to remark, that all the condensed forms given in this paper have actually been obtained by the author in the way above pointed out. The limits imposed by the objects to which the *Magazine* is devoted have restricted him from exhibiting the method at full; but any of his readers will be able without difficulty to make it out for himself.

The process consists in finding $U+\lambda V$ by means of solving for each case a problem of position (a kind of chess-board problem) on a square table, containing three places in length and breadth for conics, four places by four for surfaces, and so on (if need be) according to the number of variable letters involved. $U+\lambda V$ being thus determined in form, U and V become readily cognizable. It is right also to add, that some of the condensed forms here set forth have been incidentally noticed and employed by previous authors, as Plücker and Mr Cayley.

The conditions in each case to which the position-problem is subject are immediately deducible from the laws which the complete determinant, and the successive minor systems of determinants of $U+\lambda V$, are required to satisfy.

[* p. 119 above.]

37.

ON THE RELATION BETWEEN THE MINOR DETERMINANTS OF LINEARLY EQUIVALENT QUADRATIC FUNCTIONS.

[*Philosophical Magazine*, I. (1851), pp. 295—305.]

I SHOWED in the preliminary part of my paper on Contacts in the February Number of this *Magazine**, by *à priori* reasoning, that if a quadratic function (U) be linearly converted into another (V), any minor determinant of any order of V must be a syzygetic function of all the minor determinants of U of the same order.

The object of my present communication is to exhibit the syzygy in question, which, as I indicated, is linear; by which I mean that a determinant of the one function is equal to the sum of the pari-ordinal determinants of the other affected respectively with multipliers formed exclusively out of the coefficients of the equations of transformation. In order that a clear enunciation of the theorem in view may be possible, it is necessary to premise a new but simple, and, as experience has proved to me, a most powerful, because natural, method of notation applicable to all questions concerning determinants.

Every determinant is obtained by operating upon a square array of quantities, which, according to the ordinary method, might be denoted as follows:

$$\begin{array}{l} a_{1,1},\quad a_{1,2}\ldots a_{1,n},\\ a_{2,1},\quad a_{2,2}\ldots a_{2,n},\\ a_{3,1},\quad a_{3,2}\ldots a_{3,n},\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ a_{n,1},\quad a_{n,2}\ldots a_{n,n}. \end{array}$$

My method consists in expressing the same quantities biliterally as below:

$$\begin{array}{l} a_1\alpha_1,\quad a_1\alpha_2\ldots a_1\alpha_n,\\ a_2\alpha_1,\quad a_2\alpha_2\ldots a_2\alpha_n,\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ a_n\alpha_1,\quad a_n\alpha_2\ldots a_n\alpha_n, \end{array}$$

[* p. 221 above.]

where of course, whenever desirable, instead of $a_1, a_2 \dots a_n$, and $\alpha_1, \alpha_2 \dots \alpha_n$, we may write simply $a, b \dots l$, and $\alpha, \beta \dots \lambda$ respectively. Each quantity is now represented by two letters; the letters themselves, taken separately, being symbols neither of quantity nor of operation, but mere umbræ or ideal *elements* of quantitative symbols. We have now a means of representing the determinant above given in a compact form; for this purpose we need but to write one set of umbræ over the other as follows: $\begin{pmatrix} a_1, a_2 \dots a_n \\ \alpha_1, \alpha_2 \dots \alpha_n \end{pmatrix}$. If we now wish to obtain the algebraic value of this determinant, it is only necessary to take $\alpha_1, \alpha_2 \dots \alpha_n$ in all its $1, 2, 3 \dots n$ different positions, and we shall have

$$\begin{Bmatrix} a_1, & a_2 \dots a_n \\ \alpha_1, & \alpha_2 \dots \alpha_n \end{Bmatrix} = \Sigma \pm \{a_1\alpha_{\theta_1} \times a_2\alpha_{\theta_2} \times \dots \times a_n\alpha_{\theta_n}\},$$

in which expression $\theta_1, \theta_2 \dots \theta_n$ represents some order of the numbers $1, 2 \dots n$, and the positive or negative sign is to be taken according to the well-known dichotomous law. Thus, for example,

$$\begin{Bmatrix} a\,b\,c \\ \alpha\beta\gamma \end{Bmatrix} \text{ will represent } \left.\begin{matrix} a\alpha \times b\beta \times c\gamma \\ + a\beta \times b\gamma \times c\alpha \\ + a\gamma \times b\alpha \times c\beta \\ - a\beta \times b\alpha \times c\gamma \\ - a\alpha \times b\gamma \times c\beta \\ - a\gamma \times b\beta \times c\alpha \end{matrix}\right\}.$$

Although not necessary for our immediate object, it may not be inopportune to observe how readily this notation lends itself to a further natural extension of its application.

$\begin{Bmatrix} \overline{ab} & \overline{cd} \\ \alpha\beta & \gamma\delta \end{Bmatrix}$ will naturally denote

$$\frac{ab}{\alpha\beta} \times \frac{cd}{\gamma\delta} - \frac{ab}{\gamma\delta} \times \frac{cd}{\alpha\beta};$$

that is

$$\begin{Bmatrix} (a\alpha \times b\beta) \\ -(a\beta \times b\alpha) \end{Bmatrix} \times \begin{Bmatrix} (c\gamma \times d\delta) \\ -(c\delta \times d\gamma) \end{Bmatrix} - \begin{Bmatrix} (a\gamma \times b\delta) \\ -(a\delta \times b\gamma) \end{Bmatrix} \times \begin{Bmatrix} (c\alpha \times d\beta) \\ -(c\beta \times d\alpha) \end{Bmatrix}.$$

And in general the compound determinant

$$\begin{Bmatrix} \overline{a_1, b_1 \dots l_1}, & \overline{a_2, b_2 \dots l_2} \dots & \overline{a_r, b_r \dots l_r} \\ \alpha_1, \beta_1 \dots \lambda_1, & \alpha_2, \beta_2 \dots \lambda_2 & \alpha_r, \beta_r \dots \lambda_r \end{Bmatrix}$$

will denote

$$\Sigma \pm \begin{Bmatrix} a_1, & b_1 \dots l_1 \\ \alpha_{\theta_1}, & \beta_{\theta_1} \dots \lambda_{\theta_1} \end{Bmatrix} \times \begin{Bmatrix} a_2, & b_2 \dots l_2 \\ \alpha_{\theta_2}, & \beta_{\theta_2} \dots \lambda_{\theta_2} \end{Bmatrix} \times \dots \times \begin{Bmatrix} a_r, & b_r \dots l_r \\ \alpha_{\theta_r}, & \beta_{\theta_r} \dots \lambda_{\theta_r} \end{Bmatrix},$$

where, as before, we have the disjunctive equation

$$\theta_1, \theta_2 \dots \theta_r = 1, 2 \dots r.$$

As an example of the power of this notation, I will content myself with stating the following remarkable theorem in compound determinants, one of the most prolific in results of any with which I am acquainted, but which is derived from a more particular case of another vastly more general. The theorem is contained in the annexed equation

$$\begin{Bmatrix} \overline{a_1, a_2 \dots a_r, a_{r+1}}, & \overline{a_1, a_2 \dots a_r, a_{r+2}} \dots & \overline{a_1, a_2 \dots a_r, a_{r+s}} \\ \alpha_1, \alpha_2 \dots \alpha_r, \alpha_{r+1}, & \alpha_1, \alpha_2 \dots \alpha_r, \alpha_{r+2} & \alpha_1, \alpha_2 \dots \alpha_r, \alpha_{r+s} \end{Bmatrix}$$

$$= \begin{Bmatrix} a_1, & a_2 \dots a_r \\ \alpha_1, & \alpha_2 \dots \alpha_r \end{Bmatrix}^{s-1} \times \begin{Bmatrix} a_1, & a_2 \dots a_r, & a_{r+1}, & a_{r+2} \dots a_{r+s} \\ \alpha_1, & \alpha_2 \dots \alpha_r, & \alpha_{r+1}, & \alpha_{r+2} \dots \alpha_{r+s} \end{Bmatrix}. \quad (1)$$

It is obvious, that, without the aid of my system of umbral or biliteral notation, this important theorem could not be made the subject of statement without an enormous periphrasis, and could never have been made the object of distinct contemplation or proof.

To return to the more immediate object of this communication, suppose that we have any binary function of two sets of quantities, $x_1, x_2 \dots x_n$; $\xi_1, \xi_2 \dots \xi_n$, of which the general term will be of the form $c_{r,s} \times x_r\xi_s$; according to the principles of notation above laid down, nothing can be more natural than to represent $c_{r,s}$ by the biliteral group $a_r\alpha_s$; the function in question will then take the form

$$\Sigma a_r\alpha_s \,.\, x_r\xi_s;$$

the x's and ξ's denoting quantities, but the a's and α's mere umbræ. The function may then be thrown under the convenient symbolical form

$$(a_1x_1 + a_2x_2 + \dots + a_nx_n)$$
$$\times (\alpha_1\xi_1 + \alpha_2\xi_2 + \dots + \alpha_n\xi_n).$$

So if we confine ourselves to quadratic functions, for which $x_1, x_2 \dots x_n$; $\xi_1, \xi_2 \dots \xi_n$ become respectively identical, the general symbolical representation of any such will be

$$(a_1x_1 + a_2x_2 + \dots + a_nx_n)^2.$$

The complete determinant will be denoted by

$$\begin{Bmatrix} a_1, & a_2 \dots a_n \\ \alpha_1, & \alpha_2 \dots \alpha_n \end{Bmatrix},$$

and any minor determinant of the rth order by

$$\begin{Bmatrix} a_1, & a_2 \dots a_r \\ \alpha_{\theta_1}, & \alpha_{\theta_2} \dots \alpha_{\theta_r} \end{Bmatrix},$$

where $\theta_1, \theta_2 \ldots \theta_r$ are some certain r distinct numbers taken out of the series $1, 2, 3 \ldots r$. Suppose now that we have

$$U = (a_1x_1 + a_2x_2 + \ldots + a_nx_n)^2$$

linearly transformable into

$$V = (b_1y_1 + b_2y_2 + \ldots + b_ny_n)^2,$$

by means of the n equations

$$\left.\begin{array}{l} x_1 = a_1b_1 . y_1 + a_1b_2 . y_2 + \ldots + a_1b_n . y_n \\ x_2 = a_2b_1 . y_1 + a_2b_2 . y_2 + \ldots + a_2b_n . y_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n = a_nb_1 . y_1 + a_nb_2 . y_2 + \ldots + a_nb_n . y_n \end{array}\right\};$$

in which equations, be it observed, each coefficient a_rb_s is a single quantity, perfectly independent of the quantities denoted generally by a_ra_s, b_rb_s which enter into U and V. Our object is to be able to express the minor determinant

$$\begin{Bmatrix} b_{k_1}, & b_{k_2} \ldots b_{k_r} \\ b_{l_1}, & b_{l_2} \ldots b_{l_r} \end{Bmatrix},$$

in which the one group of distinct numbers, $k_1, k_2 \ldots k_r$ may either differ wholly from, or agree wholly or in part with the other group of distinct numbers $l_1, l_2 \ldots l_r$, under the form of

$$\Sigma \left\{ \begin{pmatrix} a_{\theta_1}, & a_{\theta_2} \ldots a_{\theta_r} \\ b_{\phi_1}, & b_{\phi_2} \ldots b_{\phi_r} \end{pmatrix} \times Q \right\}.$$

The particular value of Q corresponding to each double group, $\begin{pmatrix} \theta_1, & \theta_2 \ldots \theta_r \\ \phi_1, & \phi_2 \ldots \phi_r \end{pmatrix}$, may be denoted by $Q\begin{pmatrix} \theta_1, & \theta_2 \ldots \theta_r \\ \phi_1, & \phi_2 \ldots \phi_r \end{pmatrix}$; so that our problem consists in determining the value of $Q\begin{pmatrix} \theta_1, & \theta_2 \ldots \theta_r \\ \phi_1, & \phi_2 \ldots \phi_r \end{pmatrix}$ in the equation

$$\begin{Bmatrix} b_{k_1}, & b_{k_2} \ldots b_{k_r} \\ b_{l_1}, & b_{l_2} \ldots b_{l_r} \end{Bmatrix} = \Sigma \left\{ Q\begin{pmatrix} \theta_1, & \theta_2 \ldots \theta_r \\ \phi_1, & \phi_2 \ldots \phi_r \end{pmatrix} \times \begin{pmatrix} a_{\theta_1}, & a_{\theta_2} \ldots a_{\theta_r} \\ a_{\phi_1}, & a_{\phi_2} \ldots a_{\phi_r} \end{pmatrix} \right\}.$$

Accordingly I enunciate that

$$Q\begin{pmatrix} \theta_1, & \theta_2 \ldots \theta_r \\ \phi_1, & \phi_2 \ldots \phi_r \end{pmatrix} = \begin{Bmatrix} a_{k_1}, & a_{k_2} \ldots a_{k_r} \\ b_{\theta_1}, & b_{\theta_2} \ldots b_{\theta_r} \end{Bmatrix} \times \begin{Bmatrix} a_{l_1}, & a_{l_2} \ldots a_{l_r} \\ b_{\phi_1}, & b_{\phi_2} \ldots b_{\phi_r} \end{Bmatrix}$$

$$+ \begin{Bmatrix} a_{l_1}, & a_{l_2} \ldots a_{l_r} \\ b_{\theta_1}, & b_{\theta_2} \ldots b_{\theta_r} \end{Bmatrix} \times \begin{Bmatrix} a_{k_1}, & a_{k_2} \ldots a_{k_r} \\ b_{\phi_1}, & b_{\phi_2} \ldots b_{\phi_r} \end{Bmatrix}, \qquad (2)$$

subject to one sole exception in the case of $\theta_1, \theta_2 \ldots \theta_r$ being identical with $\phi_1, \phi_2, \ldots \phi_r$; namely, that for the terms (for such case) of the form

$Q\begin{pmatrix} \theta_1, & \theta_2 \ldots \theta_r \\ \theta_1, & \theta_2 \ldots \theta_r \end{pmatrix}$, the value to be taken is not that which the general formula would give, namely,

$$2\begin{Bmatrix} a_{k_1}, & a_{k_2} \ldots a_{k_r} \\ b_{\theta_1}, & b_{\theta_2} \ldots b_{\theta_r} \end{Bmatrix}^2,$$

but the half of this, that is simply the square of

$$\begin{Bmatrix} a_{k_1}, & a_{k_2} \ldots a_{k_r} \\ a_{\theta_1}, & a_{\theta_2} \ldots a_{\theta_r} \end{Bmatrix}.$$

The value of $Q\begin{pmatrix} \theta_1, & \theta_2 \ldots \theta_r \\ \phi_1, & \phi_2 \ldots \phi_r \end{pmatrix}$, it is obvious, contains only *quantities* of the form $a_r . b_s$, which are coefficients in the equations of transformation, but none of the form $a_r . a_s$ or $b_r . b_s$; showing that the syzygetic connexion between the minor determinants of U and V of the same order is linear, as has been already anticipatively announced.

The problem which I have treated above is only a particular case of a more general one, which may be stated as follows: given

$$U = (a_1x_1 + a_2x_2 + \ldots + a_nx_n)^2,$$

and supposing m linear equations to be instituted between $x_1, x_2 \ldots x_n$, so that U may be made a function of $(n-m)$ letters only, to express any minor determinant of the reduced form of U without performing the process of elimination between the given equations. Let the given equations be written under the form

$$\begin{array}{l} a_1a_{n+1}x_1 + a_2a_{n+1}x_2 + \ldots + a_na_{n+1}x_n = 0, \\ a_1a_{n+2}x_1 + a_2a_{n+2}x_2 + \ldots + a_na_{n+2}x_n = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_1a_{n+m}x_1 + a_2a_{n+m}x_2 + \ldots + a_na_{n+m}x_n = 0, \end{array}$$

and let it be convened (which takes nothing away from the generality of these equations) that $a_{n+r}a_{n+s}$ shall signify zero for all values of r and s *concurrently* greater than zero. Suppose that $x_1, x_2 \ldots x_m$, being eliminated, U becomes of the form

$$(b_{m+1}x_{m+1} + b_{m+2}x_{m+2} + \ldots + b_nx_n)^2;$$

and suppose that we wish to determine the value of the complete determinant of this last function; it will be found to be

$$\begin{pmatrix} b_{m+1}, & b_{m+2} \ldots b_n \\ b_{m+1}, & b_{m+2} \ldots b_n \end{pmatrix} = \begin{Bmatrix} a_1, & a_2 \ldots a_n, & a_{n+1} \ldots a_{n+m} \\ a_1, & a_2 \ldots a_n, & a_{n+1} \ldots a_{n+m} \end{Bmatrix} \div \begin{pmatrix} a_1, & a_2 & \ldots a_m \\ a_{n+1}, & a_{n+2} & \ldots a_{n+m} \end{pmatrix}^2,$$

the squared divisor being, as is obvious, a function only of the coefficients of the transforming equations, and depending for its value upon the particular

m quantities selected for elimination. The dividend, on the contrary, is independent of this selection, but involves the coefficients of the function combined with the coefficients of transformation. This is the symbolical representation of the theorem given by me in the postscript to my paper in the *Cambridge and Dublin Mathematical Journal* for November 1850*.

Suppose, now, more generally that we wish to find any minor determinant. The solution is given† by the equation

$$\begin{Bmatrix} b_{\theta_{m+1}}, & b_{\theta_{m+2}} \dots b_{\theta_{m+s}} \\ b_{\phi_{m+1}}, & b_{\phi_{m+2}} \dots b_{\phi_{m+s}} \end{Bmatrix}$$

(wherein the two groups $\theta_{m+1}, \theta_{m+2}, \dots \theta_{m+s}$; $\phi_{m+1}, \phi_{m+2} \dots \phi_{m+s}$ are each of them s differing, or wholly or in part agreeing individuals arbitrarily selected out of the $(n-m)$ numbers $m+1, m+2, \dots n$)

$$= \begin{Bmatrix} a_{\theta_1}, & a_{\theta_2} \dots a_{\theta_m}, & a_{\theta_{m+1}}, & a_{\theta_{m+2}} \dots a_{\theta_{m+s}} \\ a_{\phi_1}, & a_{\phi_2} \dots a_{\phi_m}, & a_{\phi_{m+1}}, & a_{\phi_{m+2}} \dots a_{\phi_{m+s}} \end{Bmatrix} \div \begin{Bmatrix} a_1, & a_2 & \dots a_m \\ a_{n+1}, & a_{n+2} & \dots a_{n+m} \end{Bmatrix}^2. \quad (3)$$

If we make $n = 2\gamma$ and $m = \gamma$, and $a_{\gamma+r} a_{\gamma+s} = 0$ for all positive values of either r or s, and $a_{\gamma-i} a_{n+e} = 0$ for all values of i and e differing from one another, and for equal values $a_{\gamma-e} a_{\gamma+e} = -1$, it will readily be seen that this last theorem reduces to the one first considered; and on careful inspection it will be found, that the solution given of the general question includes within it that presented for the particular case in question. Such inclusion, however, I ought in fairness to state is far from being obvious; and to demonstrate it exactly, and in general terms, requires the aid of methods which my readers would probably find to exceed their existing degree of knowledge or familiarity with the subject.

The theorem above enunciated was in part suggested in the course of a conversation with Mr Cayley (to whom I am indebted for my restoration to the enjoyment of mathematical life) on the subject of one of the preliminary theorems in my paper on Contacts in this *Magazine*.

It is wonderful that a theory so purely analytical should originate in a geometrical speculation. My friend M. Hermite has pointed out to me, that some faint indications of the same theory may be found in the *Recherches Arithmétiques* of Gauss. The notation which I have employed for determinants is very similar to that of Vandermonde, with which I have become acquainted since writing the above, in Mr Spottiswoode's valuable treatise *On the Elementary Theorems of Determinants*. Vandermonde was evidently on the right road. I do not hesitate to affirm, that the superiority of his and my notation over that in use in the ordinary methods is as great and almost as important to the progress of analysis, as the superiority of the notation of the differential calculus over that of the fluxional system. For what is the theory of determinants? It is an algebra upon algebra; a

[* p. 136 above.] [† see p. 251 below.]

calculus which enables us to combine and foretell the results of algebraical operations, in the same way as algebra itself enables us to dispense with the performance of the special operations of arithmetic. All analysis must ultimately clothe itself under this form*.

I have in previous papers defined a "Matrix" as a rectangular array of terms, out of which different systems of determinants may be engendered, as from the womb of a common parent; these cognate determinants being by no means isolated in their relations to one another, but subject to certain simple laws of mutual dependence and simultaneous deperition. The condensed representation of any such Matrix, according to my improved Vandermondian notation, will be

$$\begin{Bmatrix} a_1, & a_2 \dots a_n \\ \alpha_1, & \alpha_2 \dots \alpha_m \end{Bmatrix}.$$

To return to the theorems of the text. Theorem (2) admits of being presented in a more convenient form for the purposes of analytical operation, so as to become relieved from all cases of exception appertaining to particular terms.

The limitation to the generality of the expression for Q arises from our treating

$$\begin{Bmatrix} a_{\theta_1}, & a_{\theta_2} \dots a_{\theta_r} \\ a_{\phi_1}, & a_{\phi_2} \dots a_{\phi_r} \end{Bmatrix}$$

as *identical* with its *equal*,

$$\begin{Bmatrix} a_{\phi_1}, & a_{\phi_2} \dots a_{\phi_r} \\ a_{\theta_1}, & a_{\theta_2} \dots a_{\theta_r} \end{Bmatrix}.$$

If, however, we now convene to treat these two forms as distinct, so that in theorem (2)

$$\Sigma \left\{ Q \begin{pmatrix} \theta_1, & \theta_2 \dots \theta_r \\ \phi_1, & \phi_2 \dots \phi_r \end{pmatrix} \times \begin{pmatrix} a_{\theta_1}, & a_{\theta_2} \dots a_{\theta_r} \\ a_{\phi_1}, & a_{\phi_2} \dots a_{\phi_r} \end{pmatrix} \right\}$$

will contain $\left\{ \dfrac{n(n-1)\dots(n-r+1)}{1 \cdot 2 \dots r} \right\}^2$ terms, then we may write simply

$$Q \begin{pmatrix} \theta_1, & \theta_2 \dots \theta_r \\ \phi_1, & \phi_2 \dots \phi_r \end{pmatrix} = \begin{Bmatrix} a_{k_1}, & a_{k_2} \dots a_{k_r} \\ b_{\theta_1}, & b_{\theta_2} \dots b_{\theta_r} \end{Bmatrix} \times \begin{Bmatrix} a_{l_1}, & a_{l_2} \dots a_{l_r} \\ b_{\phi_1}, & b_{\phi_2} \dots b_{\phi_r} \end{Bmatrix},$$

* Perhaps the most remarkable indirect question to which the method of determinants has been hitherto applied is Hesse's problem of reducing a cubic function of 3 letters to another consisting only of 4 terms by linear substitutions—a problem which appears to set at defiance all the processes and artifices of common algebra. I have succeeded in applying a method founded upon this calculus to the linear reduction of a biquadratic function of two letters to Cayley's form $x^4 + mx^2y^2 + y^4$, and of a 5^c function of two letters to the new form $x^5 + y^5 + (ax + by)^5$. This last reduction is effected by means of the properties of a certain other function of the 8th degree connected with the given function of the 5th degree. See a paper on this subject in the forthcoming May Number of the *Cambridge and Dublin Mathematical Journal*. [p. 191 above.]

which equation is subject to no exception for the case of the θ's and ϕ's becoming identical. As regards this theorem, it will not fail to strike the reader that it ought to admit of verification; for that U may be derived from V in the same manner as V from U if we express $y_1, y_2 \dots y_n$ in terms of $x_1, x_2 \dots x_n$, by solving the system of equations (2), which there is no difficulty in doing. In fact, if we write

$$y_1 = \alpha_1\beta_1 x_1 + \alpha_1\beta_2 x_2 + \dots + \alpha_1\beta_n x_n,$$
$$y_2 = \alpha_2\beta_1 x_1 + \alpha_2\beta_2 x_2 + \dots + \alpha_2\beta_n x_n,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$y_n = \alpha_n\beta_1 x_1 + \alpha_n\beta_2 x_2 + \dots + \alpha_n\beta_n x_n,$$

we shall obtain

$$\alpha_r\beta_s = \left\{\begin{matrix} a_1, & a_2 \dots a_{r-1}, & a_{r+1}, & a_{r+2} \dots a_n \\ b_1, & b_2 \dots b_{s-1}, & b_{s+1}, & b_{s+2} \dots b_n \end{matrix}\right\} \div \left\{\begin{matrix} a_1, & a_2 \dots a_n \\ b_1, & b_2 \dots b_n \end{matrix}\right\}.$$

Accordingly we shall find

$$\left\{\begin{matrix} a_{m_1}, & a_{m_2} \dots a_{m_r} \\ a_{p_1}, & a_{p_2} \dots a_{p_r} \end{matrix}\right\} = \Sigma\left\{Q\begin{pmatrix} \psi_1, & \psi_2 \dots \psi_r \\ \omega_1, & \omega_2 \dots \omega_r \end{pmatrix} \times \begin{pmatrix} b_{\psi_1}, & b_{\psi_2} \dots b_{\psi_r} \\ b_{\omega_1}, & b_{\omega_2} \dots b_{\omega_r} \end{pmatrix}\right\},$$

and

$$Q\begin{pmatrix} \psi_1, & \psi_2 \dots \psi_r \\ \omega_1, & \omega_2 \dots \omega_r \end{pmatrix} = \begin{pmatrix} \alpha_{m_1}, & \alpha_{m_2} \dots \alpha_{m_r} \\ \beta_{\psi_1}, & \beta_{\psi_2} \dots \beta_{\psi_r} \end{pmatrix} \times \begin{pmatrix} \alpha_{p_1}, & \alpha_{p_2} \dots \alpha_{p_r} \\ \beta_{\omega_1}, & \beta_{\omega_2} \dots \beta_{\omega_r} \end{pmatrix};$$

substituting for the α's and β's their symbolical equivalents given above, and applying the theorem given below, we shall easily obtain

$$Q\begin{pmatrix} \psi_1, & \psi_2 \dots \psi_r \\ \omega_1, & \omega_2 \dots \omega_r \end{pmatrix} = \begin{pmatrix} a_{m_{r+1}}, & a_{m_{r+2}} \dots a_{m_n} \\ b_{\psi_{r+1}}, & b_{\psi_{r+2}} \dots b_{\psi_n} \end{pmatrix} \times \begin{pmatrix} \alpha_{p_{r+1}}, & \alpha_{p_{r+2}} \dots \alpha_{p_n} \\ \beta_{p_{r+1}}, & \beta_{p_{r+2}} \dots \beta_{p_n} \end{pmatrix}$$
$$\div \left\{\begin{matrix} a_1, & a_2 \dots a_n \\ b_1, & b_2 \dots b_n \end{matrix}\right\}^2.$$

If, now, in the expression

$$\left\{\begin{matrix} b_{k_1}, & b_{k_2} \dots b_{k_r} \\ b_{l_1}, & b_{l_2} \dots b_{l_r} \end{matrix}\right\} = \Sigma\left\{\begin{pmatrix} a_{k_1}, & a_{k_2} \dots a_{k_r} \\ b_{\theta_1}, & b_{\theta_2} \dots b_{\theta_r} \end{pmatrix}\begin{pmatrix} a_{l_1}, & a_{l_2} \dots a_{l_r} \\ b_{\phi_1}, & b_{\phi_2} \dots b_{\phi_r} \end{pmatrix}\begin{pmatrix} a_{\theta_1}, & a_{\theta_2} \dots a_{\theta_r} \\ a_{\phi_1}, & a_{\phi_2} \dots a_{\phi_r} \end{pmatrix}\right\},$$

we resubstitute for $\left\{\begin{matrix} a_{\theta_1}, & a_{\theta_2} \dots a_{\theta_r} \\ a_{\phi_1}, & a_{\phi_2} \dots a_{\phi_r} \end{matrix}\right\}$ its value in the form of

$$\Sigma\left\{\begin{pmatrix} b_{\omega_1}, & b_{\omega_2} \dots b_{\omega_r} \\ b_{\phi_1}, & b_{\phi_2} \dots b_{\phi_r} \end{pmatrix} Q\right\},$$

we shall obtain $\begin{pmatrix} b_{k_1}, & b_{k_2} \dots b_{k_r} \\ b_{l_1}, & b_{l_2} \dots b_{l_r} \end{pmatrix}$ under the form of

$$\Sigma\left\{R\begin{pmatrix} \omega_1, & \omega_2 \dots \omega_r \\ \psi_1, & \psi_2 \dots \psi_r \end{pmatrix} \times \begin{pmatrix} b_{\omega_1}, & b_{\omega_2} \dots b_{\omega_r} \\ b_{\phi_1}, & b_{\phi_2} \dots b_{\phi_r} \end{pmatrix}\right\};$$

and $R\begin{pmatrix} \omega_1, & \omega_2 \dots \omega_r \\ \psi_1, & \psi_2 \dots \psi_r \end{pmatrix}$ must $=0$, except for the case of $\omega_1, \omega_2 \dots \omega_r$; $\psi_1, \psi_2 \dots \psi_r$ being respectively identical with $k_1, k_2 \dots k_r$; $l_1, l_2 \dots l_r$, for which case $R\begin{pmatrix} k_1, & k_2 \dots k_r \\ l_1, & l_2 \dots l_r \end{pmatrix}$ must be unity. I have gone through this calculation and verified the result; in order to effect which, however, the following important generalization of theorem (1) must be apprehended.

Suppose two sets of umbræ,

$$a_1, \; a_2 \dots a_{m+n},$$
$$b_1, \; b_2 \dots b_{m+n},$$

and let r be any number less than m, and let any r-ary combination of the m numbers 1, 2, 3 ... m be expressed by ${}^q\theta_1, {}^q\theta_2 \dots {}^q\theta_m$, where q goes through all the values intermediate between 1 and μ, μ being

$$\frac{m(m-1)\dots(m-r+1)}{1.2\dots r};$$

then I say that the compound determinant,

$$\begin{vmatrix} a_{{}^1\theta_1}, & a_{{}^1\theta_2} \dots a_{{}^1\theta_m}, & a_{m+1}, & a_{m+2} \dots a_{m+n} \\ b_{{}^1\theta_1}, & b_{{}^1\theta_2} \dots b_{{}^1\theta_m}, & b_{m+1}, & b_{m+2} \dots b_{m+n} \end{vmatrix} \quad \begin{vmatrix} a_{{}^2\theta_1}, & a_{{}^2\theta_2} \dots a_{{}^2\theta_m}, & a_{m+1}, & a_{m+2} \dots a_{m+n} \\ b_{{}^2\theta_1}, & b_{{}^2\theta_2} \dots b_{{}^2\theta_m}, & b_{m+1}, & b_{m+2} \dots b_{m+n} \end{vmatrix}$$
$$\dots\dots \begin{vmatrix} a_{{}^\mu\theta_1}, & a_{{}^\mu\theta_2} \dots a_{{}^\mu\theta_m}, & a_{m+1}, & a_{m+2} \dots a_{m+n} \\ b_{{}^\mu\theta_1}, & b_{{}^\mu\theta_2} \dots b_{{}^\mu\theta_m}, & b_{m+1}, & b_{m+2} \dots b_{m+n} \end{vmatrix},$$

is equal to the following product,

$$\begin{vmatrix} a_{m+1}, & a_{m+2} \dots a_{m+n} \\ b_{m+1}, & b_{m+2} \dots b_{m+n} \end{vmatrix}^{\mu'}, \; \begin{vmatrix} a_1, & a_2 \dots a_{m+n} \\ b_1, & b_2 \dots b_{m+n} \end{vmatrix}^{\mu''} \qquad (4)$$

where

$$\mu'' = \frac{(m-1)(m-2)\dots(m-r+1)}{1.2\dots(r-1)},$$

and

$$\mu' = \frac{(m-1)(m-2)\dots(m-r)}{1.2\dots r};$$

when $r=1$, we have the case already given in theorem (2), and of course μ'' is to be taken unity.

This very general theorem is itself several degrees removed from my still unpublished Fundamental Theorem which is a theorem for the expansion of the products of determinants.

Obs. The analogy upon which the extension of the Vandermondian notation from simple to compound determinants is grounded, would be better apprehended if the biliteral symbols of simple quantities were written with the umbral elements disposed vertically, as $\frac{a}{b}$, instead of horizontally, as ab; which latter is the method for the purposes of typographical uniformity adopted in the text above. The other mode is, however, much to be preferred, and is what I propose hereafter to adhere to. For my two general umbræ, a, b, Vandermonde uses two *numbers*, one set a-cock upon the other, as 5^4. The objection to the use of numbers is apparent as soon as it becomes necessary to treat of the mutual relations of diverse systems of determinants, and his mode of writing the umbræ militates against the perception of the most valuable algebraical analogies. The one important point in which Vandermonde has anticipated me, consists in expressing a simple determinant by two horizontal rows of umbræ one over the other. But the idea upon which this depends is so simple and natural, that it was sure to reappear in any well-constructed system of notation.

38.

NOTE ON QUADRATIC FUNCTIONS AND HYPER-DETERMINANTS.

[*Philosophical Magazine*, I. (1851), p. 415.]

Permit me to correct an error of transcription in the MS. of my paper "On Linearly Equivalent Quadratic Functions" in the last number of the *Magazine*. The theorem [p. 246 above] marked (3), should read as follows:—

$$\begin{Bmatrix} b_{\theta_{m+1}}, & b_{\theta_{m+2}} \dots b_{\theta_{m+s}} \\ b_{\phi_{m+1}}, & b_{\phi_{m+2}} \dots b_{\phi_{m+s}} \end{Bmatrix}$$

$$= \begin{Bmatrix} a_1, & a_2 \dots a_m, & a_{\theta_{m+1}}, & a_{\theta_{m+2}} \dots a_{\theta_{m+s}}, & a_{n+1}, & a_{n+2} \dots a_{n+m} \\ a_1, & a_2 \dots a_m, & a_{\phi_{m+1}}, & a_{\phi_{m+2}} \dots a_{\phi_{m+s}}, & a_{n+1}, & a_{n+2} \dots a_{n+m} \end{Bmatrix}$$

$$\div \begin{Bmatrix} a_1, & a_2 & \dots a_m \\ a_{n+1}, & a_{n+2} & \dots a_{n+m} \end{Bmatrix}^2.$$

I may take this opportunity of mentioning, that by extending to algebraical functions generally a multiliteral system of umbral notation, analogous to the biliteral system explained in the paper above referred to as applicable to quadratic functions, I have succeeded in reducing to a mechanical method of compound permutation the process for the discovery of those memorable forms invented by Mr Cayley, and named by him hyper-determinants, which have attracted the notice and just admiration of analysts all over Europe, and which will remain a perpetual memorial, as long as the name of algebra survives, of the penetration and sagacity of their author.

39.

ON A CERTAIN FUNDAMENTAL THEOREM OF DETERMINANTS.

[*Philosophical Magazine*, II. (1851), pp. 142—145.]

THE subjoined theorem, which is one susceptible of great extension and generalization, appears to me, and indeed from use and acquaintance (it having been long in my possession) I know to be so important and fundamental, as to induce me to extract it from a mass of memoranda on the same subject; and as an act of duty to my fellow-labourers in the theory of determinants, more or less forestall time (the sure discoverer of truth) by placing it without further delay on record in the pages of this *Magazine*. Its developments and applications must be reserved for a more convenient occasion, when the interest in the New Algebra (for such, truly, it is the office of the theory of determinants to establish), and the number of its disciples in this country, shall have received their destined augmentation. In a recent letter to me, M. Hermite well alludes to the theory of determinants as "That vast theory, transcendental in point of difficulty, elementary in regard to its being the basis of researches in the higher arithmetic and in analytical geometry."

The theorem is as follows:—Suppose that there are two determinants of the ordinary kind, each expressed by a square array of terms made up of n lines and n columns, so that in each square there are n^2 terms. Now let n be broken up in any given manner into two parts p and q, so that $p+q=n$. Let, firstly, one of the two given squares be divided in a given *definite* manner into two parts, one containing p of the n given lines, and the other part q of the same; and secondly, let the other of the two given squares be divided *in every possible way* into two parts, consisting of q and p lines respectively, so that on tacking on the part containing q lines of the second square to the part containing p lines of the first square, and the part containing p lines of the second square to the part containing q of the first, we

get back a new couple of squares, each denoting a determinant different from the two given determinants; the number of such new couples will evidently be

$$\frac{n(n-1)\dots(n-p+1)}{1.2\dots p};$$

and my theorem is, that *the product of the given couple of determinants is equal to the sum of the products (affected with the proper algebraical sign) of each of the new couples formed as above described.* Analytically the theorem may be stated as follows.

Let $$\begin{Bmatrix} a_1, & a_2 \dots a_n \\ b_1, & b_2 \dots b_n \end{Bmatrix}, \quad \begin{Bmatrix} \alpha_1, & \alpha_2 \dots \alpha_n \\ \beta_1, & \beta_2 \dots \beta_n \end{Bmatrix},$$

according to the notation heretofore* employed by me in the preceding numbers of this *Magazine*, denote any two common determinants, each of the nth order, and let the numbers $\theta_1, \theta_2 \dots \theta_n$ be disjunctively equal to the numbers $1, 2 \dots n$ and $p+q=n$; then will

$$\begin{Bmatrix} a_1, & a_2 \dots a_n \\ b_1, & b_2 \dots b_n \end{Bmatrix} \times \begin{Bmatrix} \alpha_1, & \alpha_2 \dots \alpha_n \\ \beta_1, & \beta_2 \dots \beta_n \end{Bmatrix}$$

$$= \Sigma \pm \begin{Bmatrix} a_1, & a_2 & & \dots & & a_n \\ b_1, & b_2 \dots b_p, & \beta_{\theta_{p+1}}, & \beta_{\theta_{p+2}} \dots \beta_{\theta_n} \end{Bmatrix} \times \begin{Bmatrix} \alpha_1, & \alpha_2 & & \dots & & \alpha_n \\ \beta_{\theta_1}, & \beta_{\theta_2} \dots \beta_{\theta_p}, & b_{p+1}, & b_{p+2} \dots b_n \end{Bmatrix}.$$

The general term under the sign of summation may be represented by aid of the disjunctive equations

$$\phi_1, \phi_2 \dots \phi_n = 1, 2 \dots n,$$

$$\psi_1, \psi_2 \dots \psi_n = 1, 2 \dots n,$$

under the form of

$$(a_{\phi_1}.b_1 \times a_{\phi_2}.b_2 \times \dots \times a_{\phi_p}.b_p)(a_{\psi_{p+1}}.b_{p+1} \times a_{\psi_{p+2}}.b_{p+2} \times \dots \times a_{\psi_n}.b_n)$$
$$\times (\alpha_{\phi_{p+1}}.\beta_{\theta_{p+1}} \times \alpha_{\phi_{p+2}}.\beta_{\theta_{p+2}} \times \dots \times \alpha_{\phi_n}.\beta_{\theta_n})(\alpha_{\psi_1}.\beta_{\theta_1} \times \alpha_{\psi_2}.\beta_{\theta_2} \times \dots \times \alpha_{\psi_p}.\beta_{\theta_p}).$$

1st. When $\phi_1, \phi_2 \dots \phi_p = \psi_1, \psi_2 \dots \psi_p$, it will readily be seen, that for given values of $\phi_1, \phi_2 \dots \phi_p$, the product of the third and fourth factors becomes *substantially* identical with the general term of the determinant

$$\begin{Bmatrix} \alpha_1, & \alpha_2 \dots \alpha_n \\ \beta_1, & \beta_2 \dots \beta_n \end{Bmatrix},$$

and consequently, making the system $\phi_1, \phi_2 \dots \phi_p$ (or, which is the same thing, its equivalent $\psi_1, \psi_2 \dots \psi_p$) go through all its values, we get back for the sum of the terms corresponding to the equation

$$\phi_1, \phi_2 \dots \phi_p = \psi_1, \psi_2 \dots \psi_p,$$

[* p. 242 above.]

the product of the determinants

$$\begin{Bmatrix} a_1, & a_2 \ldots a_n \\ b_1, & b_2 \ldots b_n \end{Bmatrix} \text{ and } \begin{Bmatrix} \alpha_1, & \alpha_2 \ldots \alpha_n \\ \beta_1, & \beta_2 \ldots \beta_n \end{Bmatrix}.$$

2nd. When we have not the equality above supposed between the ϕ's and the ψ's, let

$$\phi_{p-h} = \psi_{p+k} \text{ and } \phi_{p+\eta} = \psi_{p-\zeta};$$

the corresponding term included under the Σ will contain the factor

$$\alpha_{\phi_{p+\eta}} . \beta_{\theta_{p+\eta}} \times \alpha_{\phi_{p-\zeta}} . \beta_{\theta_{p-\zeta}}.$$

Now leaving $\phi_1, \phi_2 \ldots \phi_p$, and $\psi_1, \psi_2 \ldots \psi_p$ unaltered, we may take a system of values $\theta_1', \theta_2' \ldots \theta_n'$, such that

$$\theta'_{p+\eta} = \theta_{p-\zeta},$$

and

$$\theta'_{p-\zeta} = \theta_{p+\eta}.$$

and for all other values of q except $p+\eta$, or $p-\zeta$, $\theta'_q = \theta_q$. The corresponding new value of the general term so formed by the substitution of the θ' for the θ series, will be identical with that of the term first spoken of, but will have the contrary algebraical sign, because the θ' arrangement of the figures $1, 2, 3 \ldots p$ is deducible by a single interchange from the θ arrangement of the same, the rule for the imposition of the algebraical sign plus or minus being understood to be, that the term in which

$$\beta_{\theta_{p+1}}, \beta_{\theta_{p+2}} \ldots \beta_{\theta_n}; \ \beta_{\theta_1}, \beta_{\theta_2} \ldots \beta_{\theta_p}$$

enter into the symbolical forms of the respective derived couples of determinants, has the same sign as, or the contrary sign to, that in which

$$\beta_{\theta'_{p+1}}, \beta_{\theta'_{p+2}} \ldots \beta_{\theta'_n}; \ \beta_{\theta'_1}, \beta_{\theta'_2} \ldots \beta_{\theta'_p}$$

so enter, according as an odd or an even number of interchanges is required to transform the arrangement

$$\theta_{p+1}, \theta_{p+2} \ldots \theta_n; \ \theta_1, \theta_2 \ldots \theta_p$$

into the arrangement

$$\theta'_{p+1}, \theta'_{p+2} \ldots \theta'_n; \ \theta'_1, \theta'_2 \ldots \theta'_p.$$

I have therefore shown that all the terms arising from the expansion of the products included under the sign of summation, for which the disjunctive identity $\phi_1, \phi_2 \ldots \phi_p = \psi_1, \psi_2 \ldots \psi_p$ does not exist, enter into the final sum in pairs, equal in quantity and differing in sign, which consequently mutually destroy, and that the terms for which the said identity does exist together make up the sum

$$\begin{Bmatrix} a_1, & a_2 \ldots a_n \\ b_1, & b_2 \ldots b_n \end{Bmatrix} \times \begin{Bmatrix} \alpha_1, & \alpha_2 \ldots \alpha_n \\ \beta_1, & \beta_2 \ldots \beta_n \end{Bmatrix};$$

which proves, upon first principles drawn direct from that notion of polar dichotomy of permutation systems which rests at the bottom of the whole theory of the subject, the fundamental, and, as I believe, perfectly new theorem, which it is the object of this communication to establish.

In applying the theorem thus analytically formulized, it is of course to be understood that, under the sign Σ, *permutations* within the separate parts of a given arrangement,

$$\theta_{p+1}, \theta_{p+2} \ldots \theta_n; \ \theta_1, \theta_2 \ldots \theta_p,$$

are inadmissible, the total number of terms so included being restricted to

$$\frac{n(n-1) \ldots (n-p+1)}{1 . 2 \ldots p}.$$

The theorem may be extended so as to become a theorem for the expansion of the product of any number of determinants, and adapted so as to take in that far more general class of functions known to Mr Cayley and myself under the new name of commutants, of which determinants present only a particular, and that the most limited instance.

40.

ON EXTENSIONS OF THE DIALYTIC METHOD OF ELIMINATION.

[*Philosophical Magazine*, II. (1851), pp. 221—230.]

THE theory about to be described is a natural extension of the method of elimination presented by me ten years ago (in June, 1841) in the pages of this *Magazine*, which I have been induced to review in consequence of the flattering interest recently expressed in the subject by my friend M. Terquem, and some other continental mathematicians, and because of the importance of the geometrical and other applications of which it admits, and of the inquiries to which it indirectly gives rise. We shall be concerned in the following discussion with systems of homogeneous rational integral functions of a peculiar form, to which for present purposes I propose to give the name of aggregative functions, consisting of ordinary homogeneous functions of the same variables but of different degrees, brought together into one sum made homogeneous by means of powers of new variables entering factorially.

Thus if $F, G, H \ldots L$ be any number of functions of any number of letters $x, y \ldots t$ of the degrees $m, m-\iota, m-\iota' \ldots m-(\iota)$ respectively,

$$F + G\lambda^{\iota} + H\mu^{\iota'} + \ldots + L\theta^{(\iota)}$$

will be an aggregative function of the variables entering into F, G, &c. and of $\lambda, \mu \ldots \theta$. I shall further call such a function binary, ternary, quaternary, and so forth, according to the number of variables contained in the functions F, G, H, &c. thus brought into coalition.

It will be convenient to recall the attention of the reader to the meaning of some of the terms employed by me in the paper above referred to.

If F be any homogeneous function of $x, y, z \ldots t$, the term augmentative of F denotes any function obtained from F of the form

$$x^{\alpha} y^{\beta} z^{\gamma} \ldots t^{\delta} \times F.$$

Again, if we have any number of such functions $F, G, H \ldots K$ of as many

variables $x, y, z \ldots t$, and we decompose $F, G, H \ldots K$ in any manner so as to obtain the equations

$$F = x^a P_1 + y^b P_2 + z^c P_3 + \&c. \ldots + t^d (P),$$
$$G = x^a Q_1 + y^b Q_2 + z^c Q_3 + \&c. \ldots + t^d (Q),$$
$$H = x^a R_1 + y^b R_2 + z^c R_3 + \&c. \ldots + t^d (R),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$K = x^a S_1 + y^b S_2 + z^c S_3 + \&c. \ldots + t^d (S),$$

and then form the determinant

$$\begin{vmatrix} P_1, & P_2, & P_3 \ldots (P) \\ Q_1, & Q_2, & Q_3 \ldots (Q) \\ R_1, & R_2, & R_3 \ldots (R) \\ \ldots & \ldots & \ldots \\ S_1, & S_2, & S_3 \ldots (S) \end{vmatrix},$$

this determinant, expressed as a function of $x, y, z \ldots t$, is what, in the paper referred to, I called a secondary derivee, but which for the future I shall cite by the more concise and expressive name of a *connective* of the system of functions $F, G, H \ldots K$ from which it is obtained. One prevailing principle regulates all the cases treated of in this and the antecedent memoir, namely that of forming linearly independent systems of augmentatives or connectives, or both, of the given system whose resultant is to be found, of the same degree one with the other, and equal in number (when this admits of being done) to the number of distinct terms in the functions thus formed. The resultant of these functions, treated as linear functions of the several combinations of powers of the variables in each term, will then be the resultant of the given system clear of all irrelevant factors. If the number of terms to be eliminated exceed the number of the functions, the elimination of course cannot be executed. If the contrary be the case, but the equality is restored by the rejection of a certain number of the equations, the resultant so obtained will vary according to the choice of the equations retained for the purpose of the elimination. The true resultant will not then coincide with any of the resultants so obtained, but will enter as a common factor into them all.

The following simple arithmetical principles will be found applicable and useful for quotation in the sequel:—

(*a*) The number of terms in a homogeneous function of p letters of the mth degree is

$$\frac{m(m+1)\ldots(m+p-1)}{1 \cdot 2 \ldots p}.$$

(*b*) The number of augmentatives of the $(m+n)$th degree belonging to a function of p letters of the mth degree is

$$\frac{(n+1)(n+2)\dots(n+p-1)}{1.2\dots p}.$$

(*c*) The number of solutions in integers (excluding zeros) of the equation $a_1+a_2+\dots+a_p=k$ is

$$\frac{(k-1)(k-2)\dots(k-p+1)}{1.2\dots(p-1)}.$$

To begin with the case of binary aggregatives. Let

$$\left.\begin{array}{l} F_m(x,y)+F_{m-\iota}(x,y)\lambda^\iota+F_{m-\iota'}(x,y)\mu^{\iota'}+\&c.\dots+F_{m-(\iota)}(x,y)\theta^{(\iota)} \\ G_n(x,y)+G_{n-\iota}(x,y)\lambda^\iota+G_{n-\iota'}(x,y)\mu^{\iota'}+\&c.\dots+G_{n-(\iota)}(x,y)\theta^{(\iota)} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ K_p(x,y)+K_{p-\iota}(x,y)\lambda^\iota+K_{p-\iota'}(x,y)\mu^{\iota'}+\&c.\dots+K_{p-(\iota)}(x,y)\theta^{(\iota)} \end{array}\right\} \quad (A)$$

be a system of functions (whose Resultant it is proposed to determine) equal in number to the variables $x, y, \lambda, \mu \dots \theta$, and similarly aggregative, that is having only the same powers of λ, μ, &c. entering into them, but of any degrees equal or unequal $m, n \dots p$. Let the number of the functions be r. Raise each of the given functions by augmentation to the degree s, where

$$s=(m+n+\dots+p)-(\iota+\iota'+\dots+(\iota))-1,$$

the number of augmentatives of the several functions will be

$$\begin{array}{c}(s+1)-m,\\ (s+1)-n,\\ \dots\dots\dots\\ (s+1)-p,\end{array}$$

and the total number will therefore be

$$r(s+1)-(m+n+\dots+p),$$

which $$=(r-1)(m+n+\dots+p)-r(\iota+\iota'+\dots+(\iota)).$$

Again, the number of terms to be eliminated will be the sum of the numbers of terms in functions respectively of the sth, $(s-\iota)$th, $(s-\iota')$th, ... $(s-(\iota))$th degrees, which are respectively

$$\begin{array}{c}s+1,\\ s+1-\iota,\\ s+1-\iota',\\ \dots\dots\dots\\ s+1-(\iota),\end{array}$$

and the number of these partial functions is $r-1$. Hence the number of terms to be eliminated is

$$(r-1)\{m+n+\&c.+p-(\iota+\iota'+\&c.+(\iota))\}-(\iota+\iota'+\&c.+(\iota))$$
$$=(r-1)(m+n+\&c.+p)-r(\iota+\iota'+\dots+(\iota)),$$

which is exactly equal to the number of the augmentative functions. Hence the Resultant* of the given functions can be found dialytically by linear elimination, and the exponent of its dimensions in respect to the coefficients of the given functions will be the number

$$(r-1)\Sigma m-r\Sigma\iota,$$

as above found.

The method above given may be replaced by another more compendious, and analogous to that known by the name of Bezout's abridged method for ordinary functions of two letters. As the method is precisely the same whatever the number of the functions employed may be, I shall for the sake of greater simplicity restrict the demonstration to the case of three functions, U, V, W, whose degrees (if unequal, written in ascending order of magnitude) are m, n, p respectively. Let

$$\begin{array}{l} U=F_m(x,y)+F_{m-\iota}(x,y)z^\iota,\\ V=G_n(x,y)+G_{n-\iota}(x,y)z^\iota,\\ W=H_p(x,y)+H_{p-\iota}(x,y)z^\iota.\end{array}$$

Let θ, ω be taken any two numbers which satisfy in integers greater than zero the equation $\theta+\omega=m+1$, and let

$$\begin{array}{l} F_m(x,y)=\phi_{m-\theta}.x^\theta+\phi_{m-\omega}.y^\omega,\\ G_n(x,y)=\gamma_{n-\theta}.x^\theta+\gamma_{n-\omega}.y^\omega,\\ H_p(x,y)=\eta_{p-\theta}.x^\theta+\eta_{p-\omega}.y^\omega,\end{array}$$

where the ϕ's, γ's, η's may be always considered rational integer functions of x and y; for every term in each of the functions F_m, G_n, H_p must either contain x^θ or y^ω, since, if not, its dimensions in x and y would not exceed

$$(\theta-1)+(\omega-1),$$

that is $m-1$, whereas each term is of m conjoined dimensions, at least, in x and y. Hence from the equations

$$\begin{array}{c}U=0,\\ V=0,\\ W=0,\end{array}$$

* The Resultant of a system of functions means in general the same thing as the left-hand side of the final equation (clear of extraneous factors) resulting from the elimination of the variables between the equations formed by equating the said functions severally to zero.

by eliminating x^{ω}, y^{θ} and z^{ι} we obtain the connective determinant

$$\begin{vmatrix} \phi_{m-\theta}, & \phi_{m-\omega}, & F_{m-\iota} \\ \gamma_{n-\theta}, & \gamma_{n-\omega}, & G_{n-\iota} \\ \eta_{p-\theta}, & \eta_{p-\omega}, & H_{p-\iota} \end{vmatrix},$$

which will be of the degree

$$m+n+p-(\theta+\omega+\iota),$$

that is of the degree $(n+p-\iota-1)$ in x and y; and the number of such connectives by principle (*c*) is p.

Again, by augmentation we can raise each of the functions U, V, W to the same degree as the connectives, and by principle (*b*) the number of such will be

$$n+p-m-\iota,$$
$$p-\iota,$$
$$n-\iota,$$

from U, V, W respectively, together making up the number

$$2n+2p-m-3\iota.$$

Hence in all we have $2n+2p-3\iota$ equations; and the number of terms to be eliminated will be, $n+p-\iota$ arising from F_m, G_n, H_p, and $n+p-2\iota$ from $F_{m-\iota}$, $G_{n-\iota}$, $H_{p-\iota}$; together making up the proper number $2n+2p-3\iota$.

Each connective contains ternary combinations of the coefficients, namely one of the coefficients belonging to that part of U, V, W which contains z^{ι}, and two coefficients from the other part: the dimensions of the resultant in respect of the coefficients of the former will hence be readily seen to be equal to the number of connectives + the number of terms in the augmentatives into which z^{ι} enters, that is, will equal $m+n+p-2\iota$; the total dimensions of the resultant in respect to all the coefficients of U, V, W will be

$$3m+(2n+2p-m-3\iota),$$

that is,

$$2m+2n+2p-3\iota;$$

and consequently, in respect to the coefficients of F_m; G_n; H_p, will be of

$$(2m+2n+2p-3\iota)-(m+n+p-2\iota),$$

that is, of $m+n+p-\iota$ dimensions. This result, which is of considerable importance, may be generalized as follows.

Returning to the general system (A), for which we have proved that the total dimensions of the resultant are

$$(r-1)(m+n+\ldots+p)-r(\iota+\iota'+\ldots+(\iota)),$$

let the coefficients of the column of partial functions

$$F_m,$$
$$G_n,$$
$$\vdots$$
$$K_p,$$

be called the first set; the coefficients of the column

$$F_{m-\iota},$$
$$G_{n-\iota},$$
$$\vdots$$
$$K_{p-\iota},$$

the second set, and so forth; then the dimensions in respect of the 1st, 2nd ... $(r-1)$th sets respectively are s, $s-\iota$, $s-\iota'$ ... $s-(\iota)$, where

$$s=m+n+\&c.+p-(\iota+\iota'+\&c.+(\iota)).$$

The important observation remains to be made, that all the above results remain good although any one or more of the indices of dimension of the partial functions in the system (A), as $m-\iota$, $m-\iota'$, $n-\iota$, &c., should become negative, provided that the terms in which such negative indices occur be taken zero, as will be apparent on reviewing the processes already indicated upon this supposition. If we take

$$m=n=\ldots=p, \text{ and } \iota=\iota'=\&c.=(\iota)=m-\epsilon,$$

the exponent of the total dimensions of the resultant becomes

$$(r-1)\,rm-r\,(r-2)\,(m-\epsilon)$$
$$=rm+r\,(r-2)\,\epsilon,$$

when $\epsilon=0$, this becomes mr, which is made up of $2m$ units of dimension belonging to the coefficients of the first column, and of m belonging to each of the $(r-2)$ remaining columns. Consequently, if we have

$$F_m(x,\ y)+\xi\lambda+\xi'\lambda'=0,$$
$$G_m(x,\ y)+\eta\lambda+\eta'\lambda'=0,$$
$$H_m(x,\ y)+\zeta\lambda+\zeta'\lambda'=0,$$
$$K_m(x,\ y)+\theta\lambda+\theta'\lambda'=0,$$

or any other number of equations similarly formed, the result of the elimination is always of m dimensions only in respect of ξ, η, ζ, θ, or of ξ', η', ζ', θ', and of $2m$ in respect of the coefficients in F, G, H, K.

I now proceed to state and to explain some seeming paradoxes connected with the degree of the resultant of such systems of defective functions as have been previously treated of in this memoir, as compared with the degree of the general resultant of a corresponding system of *complete* functions of the same number of variables.

In order to fix our ideas, let us take a system of only three equations of the form

$$\left.\begin{aligned} F_m(x,\ y) + F_{m-\iota}(x,\ y)\, z^{\iota} = 0 \\ G_n(x,\ y) + G_{n-\iota}(x,\ y)\, z^{\iota} = 0 \\ H_p(x,\ y) + H_{p-\iota}(x,\ y)\, z^{\iota} = 0 \end{aligned}\right\} . \qquad \text{(B)}$$

The resultant of this system found by the preceding method is in all of $2m + 2n + 2p - 3\iota$ dimensions. But in general, the resultant of three equations of the degrees m, n, p is of $mn + mp + np$ dimensions.

Now in order to reason firmly and validly upon the doctrine of elimination, nothing is so necessary as to have a clear and precise notion, never to be let go from the mind's grasp, of the proposition that every system of n homogeneous functions of n variables has a single and invariable Resultant. The meaning of this proposition is, that a function of the coefficients of the given functions can be found, such that, *whenever* it becomes zero, and *never except* when it becomes zero, the functions may be simultaneously made zero for some certain system of ratios between the variables. The function so found, which is sufficient and necessary to condition the possibility of the coexistence of the equality to zero of each of the given functions, is their resultant, and by analogy they may be termed its components. It follows that if R be a resultant of a given system of functions, any numerical multiple of any power of R or of any root of R when (upon certain relations being supposed to be instituted between the coefficients of its components) R breaks up into equal factors, will also be a resultant. This is just what happens in system (B) when $m = n = p = \iota$; the resultant found by the method in the text is of the degree $3m$; the general resultant of the system of three equations to which it belongs is of the degree $3m^2$; the fact being, that the latter resultant becomes a perfect mth power for the particular values of the coefficients which cause its components to take the form of the functions in system (B).

Suppose, however, that we have still $m = n = p$, but ι less than m, $6m - 3\iota$ will express the degree of the resultant of system (B); but this is no longer in general an aliquot part of $3m^2$, and consequently the resultant of system (B) that we have found is no longer capable in general of being a root of the general resultant. The truth is, that on this supposition the general resultant is zero; as it evidently should be, because the values $\frac{x}{z} = 0$, $\frac{y}{z} = 0$ satisfy the equations in system (B), except for the case of $m = \iota$; consequently the resultant furnished in the text, although found by the same process, is something of a different nature from an ordinary resultant; it

expresses, not that the system of equations (B) may be capable of coexisting, but that they may be capable of coexisting for values of $\frac{x}{z}$, $\frac{y}{z}$ other than 0 and 0. This is what I have elsewhere termed a sub-resultant. But there is yet a further case, to which neither of the above considerations will apply. This is when m, n, p are not equal, but $p - \iota = 0$.

On this supposition the degree of the resultant of (B) becomes $2m + 2n - p$, which in general will not be a factor of $mn + mp + np$; and in this case it will no longer be true that the values $\frac{x}{z} = 0$, $\frac{y}{z} = 0$ will satisfy the system (B), inasmuch as the last equation therein cannot so be satisfied. Now, calling the general resultant R and the particular resultant R', if R' should break up into factors so as to become equal to $(r')^a \times (s')^b \ldots (t')^c$, it might be the case that R should equal $(r')^{\alpha} . (s')^{\beta} \ldots (t')^{\gamma}$, and there would be nothing in this fact which would be inconsistent with the theory of the resultant as above set forth; but suppose that R' is indecomposable into factors, then it is evident that we must have $R = R' . R''$, and consequently that the existence of such a particular resultant as R' will argue the necessity of the existence of another resultant R''; in other words, the resultant so found cannot be in a strict sense the true and complete resultant for the particular case assumed, and yet the process employed appears to give the complete resultant, or at least it is difficult to see how the wanting factor escapes detection. To make this matter more clear, take a particular and a very simple case, where $m = 2$, $n = 2$, $p = \iota = 1$, so as to form the system of equations

$$\left.\begin{aligned} Ax^2 + Bxy + Cy^2 + (Dx + Ey)\, z &= 0 \\ A'x^2 + B'xy + C'y^2 + (D'x + E'y)\, z &= 0 \\ lx + my + nz \qquad &= 0 \end{aligned}\right\} . \qquad \text{(C)}$$

By virtue of my theorem, the degree of the resultant R' is

$$2(2 + 2 + 1) - 3 . 1 = 7,$$

but the resultant R of the system

$$\left.\begin{aligned} Ax^2 + Bxy + Cy^2 + (Dx + Ey)\, z + Fz^2 &= 0 \\ A'x^2 + B'xy + C'y^2 + (D'x + E'y)\, z + F'z^2 &= 0 \\ lx + my + nz \qquad &= 0 \end{aligned}\right\} , \qquad \text{(D)}$$

which becomes identical with the former when $F = 0$, $F' = 0$ is of

$$2 \times 2 + 2 \times 1 + 2 \times 1,$$

that is, of 8 dimensions. Hence it is evident that when $F = 0$, $F' = 0$, R must become $R' \times R''$.

It will be found in fact*, that on the supposition of $F=0, F'=0$, R becomes equal to $N \times R'$; and accordingly, besides the portion R' of the resultant of system (C), found by the method in the text, there is another portion N which has dropped through; but it may be asked, is N truly a relevant factor? were it not so, the theory of the resultant would be completely invalidated; but in truth *it is*; for $N=0$ will make the equations in system (C), *considered as a particular case* of system (D), capable of co-existing; the peculiarity, which at first sight prevents this from being obvious, consisting in the fact that the values of $\frac{x}{z}, \frac{y}{z}$ which satisfy the three equations when $N=0$ become *infinite*.

Thus, finally, we have arrived at a clear and complete view of the relation of the particular to the general resultant.

The general resultant may be zero, in which case the particular resultant is something altogether different from an ordinary resultant; or the particular resultant may be a root of the general resultant, or it may be more generally the product of powers of the simple factors, which enter into the composition of the general resultant; or lastly, it may be an incomplete resultant, the factors wanting to make it complete being such as when equated to zero, will enable the components of the resultant to coexist, but not for other than infinite values of certain of the ratios existing between the variables.

Without for the present further enlarging on the hitherto unexplored and highly interesting theory of Particular Resultants, I will content myself with stating one beautiful and general theorem relating to them; to wit, "if $F=0$, $G=0$, &c. be a given system of equations with the coefficients left general, and R be the resultant of F, G, &c., and if now the coefficients in F, G be so taken that R comes to contain as a factor or be coincident with R'^m, then will $R'=0$ indicate that (when the coefficients are so taken as above supposed) $F=0$, $G=0$, &c. will be capable of being satisfied, not, as in general, by one only, but by m distinct systems of values of the variables in F, G, &c., subject of course to the possibility, in special cases, of certain of the systems becoming multiple coincident systems."

I pass on now* to the more recondite and interesting theory of the resultant of Ternary Aggregative Functions, that is to say, functions of the form

$$F_m(x, y, z) + F_{m-\iota}(x, y, z)\, t^{\iota} + \&c. \ldots + F_{m-(\omega)}(x, y, z)\, t^{(\omega)},$$

which will be seen to admit of some remarkable applications to the theory of reciprocal polars.

[* See the Author's remarks below, p. 283.]

41.

ON A REMARKABLE DISCOVERY IN THE THEORY OF CANONICAL FORMS AND OF HYPERDETERMINANTS.

[*Philosophical Magazine*, II. (1851), pp. 391—410.]

In a recently printed continuation* of a paper which appeared in the *Cambridge and Dublin Mathematical Journal*, I published a complete solution of the following problem. A homogeneous function of x, y of the degree $2n+1$ being given, required to represent it as the sum of $n+1$ powers of linear functions of x, y. I shall prepare the way for the more remarkable investigations which form the proper object of this paper, by giving a new and more simple solution of this linear transformation.

Let the given function be

$$a_0 x^{2n+1} + (2n+1)\, a_1 x^{2n} y + \tfrac{1}{2}(2n+1)(2n)\, a_2 x^{2n-1} y^2 + \ldots + a_{2n+1} y^{2n+1},$$

and suppose that this is identical with

$$(p_1 x + q_1 y)^{2n+1} + (p_2 x + q_2 y)^{2n+1} + \&c. + (p_{n+1} x + q_{n+1} y)^{2n+1}.$$

The problem is evidently possible and definite, there being $2n+2$ equations to be satisfied, and $(2n+2)$ quantities p_1, q_1, &c. for satisfying the same.

In order to effect the solution, let

$$q_1 = p_1 \lambda_1,$$
$$q_2 = p_2 \lambda_2,$$
$$\&c. = \&c.$$
$$q_{n+1} = p_{n+1} \lambda_{n+1},$$

[* p. 203 above.]

we have then

$$
\begin{aligned}
&p_1^{2n+1} + p_2^{2n+1} + \ldots + p_{n+1}^{2n+1} = a_0,\\
&p_1^{2n+1}\lambda_1 + p_2^{2n+1}\lambda_2 + \ldots + p_{n+1}^{2n+1}\lambda_{n+1} = a_1,\\
&p_1^{2n+1}\lambda_1^2 + p_2^{2n+1}\lambda_2^2 + \ldots + p_{n+1}^{2n+1}\lambda^2_{n+1} = a_2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&p_1^{2n+1}\lambda_1^n + p_2^{2n+1}\lambda_2^n + \ldots + p_{n+1}^{2n+1}\lambda^n_{n+1} = a_n,\\
&p_1^{2n+1}\lambda_1^{n+1} + p_2^{2n+1}\lambda_2^{n+1} + \ldots + p_{n+1}^{2n+1}\lambda_{n+1}^{n+1} = a_{n+1},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&p_1^{2n+1}\lambda_1^{2n+1} + p_2^{2n+1}\lambda_2^{2n+1} + \ldots + p_{n+1}^{2n+1}\lambda_{n+1}^{2n+1} = a_{2n+1}.
\end{aligned}
$$

Eliminate $p_1, p_2 \ldots p_{n+1}$ between the 1st, 2nd, 3rd ... $(n+2)$th equations, and it is easily seen that we obtain

$$a_{n+1} - a_n \Sigma\lambda_1 + a_{n-1}\Sigma\lambda_1\lambda_2 \ldots \pm a_0\lambda_1\lambda_2 \ldots \lambda_{n+1} = 0.$$

Again, eliminating in like manner $p_1^{2n+1}\lambda_1, p_2^{2n+1}\lambda_2 \ldots p_{n+1}^{2n+1}\lambda_{n+1}$ between the 2nd, 3rd ... $(n+3)$th equations, we obtain

$$a_{n+2} - a_{n+1}\Sigma\lambda_1 + \ldots \mp a_1\lambda_1\lambda_2 \ldots \lambda_{n+1} = 0;$$

and proceeding in the same way until we come to the combination of the $(n+1)$th ... $(2n+2)$th equations, and writing

$$
\begin{aligned}
\Sigma\lambda_1 &= s_1,\\
\Sigma\lambda_1\lambda_2 &= s_2,\\
&\ldots\ldots\ldots\\
\lambda_1\lambda_2 \ldots \lambda_{n+1} &= s_{n+1},
\end{aligned}
$$

we find

$$
\begin{aligned}
&a_{n+1} - a_n\ s_1 + a_{n-1}\ s_2 \ldots \pm a_0 s_{n+1} = 0,\\
&a_{n+2} - a_{n+1}s_1 + a_n\ s_2 \ldots \mp a_1 s_{n+1} = 0,\\
&a_{n+3} - a_{n+2}s_1 + a_{n+1}\ s_2 \ldots \pm a_2 s_{n+1} = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&a_{2n+1} - a_{2n}\ s_1 + a_{2n-1}s_2 \ldots + a_n s_{n+1} = 0^*.
\end{aligned}
$$

Hence it is obvious that

$$(x + \lambda_1 y)(x + \lambda_2 y) \ldots (x + \lambda_{n+1} y)$$

is a constant multiple of the determinant

$$
\begin{vmatrix}
x^{n+1}, & -x^n y, & x^{n-1}y^2 & \ldots & \pm y^{n+1}\\
a_{n+1}, & a_n, & a_{n-1} & \ldots & a_0\\
a_{n+2}, & a_{n+1}, & a_n & \ldots & a_1\\
\ldots & \ldots & \ldots & \ldots & \ldots\\
a_{2n+1}, & a_{2n}, & a_{2n-1} & \ldots & a_n
\end{vmatrix}.
$$

* These equations in their simplified form arise from the ordinary result of elimination, in this case containing as a factor the product of the differences of the quantities $\lambda_1, \lambda_2, \ldots \lambda_{n+1}$.

Hence $\lambda_1, \lambda_2 \ldots \lambda_{n+1}$ are known, and consequently

$$p_1, p_2 \cdots p_{n+1}, q_1, q_2 \cdots q_{n+1}$$

are known, by the solution of an equation of the $(n+1)$th degree.

Thus suppose the given function to be

$$
\begin{aligned}
F &= ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + 10fy^5\\
&= (p_1x + q_1y)^5 + (p_2x + q_2y)^5 + (p_3x + q_3y)^5,
\end{aligned}
$$

we shall have, by an easy inference from what has preceded,

$$(p_1x + q_1y)(p_2x + q_2y)(p_3x + q_3y)$$

equal to a numerical multiple of the determinant

$$
\begin{vmatrix}
x^3, & -x^2y, & xy^2, & -y^3\\
d, & c, & b, & a\\
e, & d, & c, & b\\
f, & e, & d, & c
\end{vmatrix}.
$$

The solution of the problem given by me in the paper before alluded to presents itself under an *apparently* different and rather less simple form. Thus, in the case in question, we shall find according to that solution,

$$(p_1x + q_1y)(p_2x + q_2y)(p_3x + q_3y)$$

equal to a numerical multiple of the determinant

$$
\begin{vmatrix}
ax + by, & bx + cy, & cx + dy\\
bx + cy, & cx + dy, & dx + ey\\
cx + dy, & dx + ey, & ex + fy
\end{vmatrix}.
$$

The two determinants, however, are in fact identical, as is easily verified, for the coefficients of x^3 and y^3 are manifestly alike; and the coefficient of x^2y in the second form will be made up of the three determinants,

$$
\begin{vmatrix}
a, & b, & d\\
b, & c, & e\\
c, & d, & f
\end{vmatrix},
\quad
\begin{vmatrix}
a, & c, & c\\
b, & d, & d\\
c, & e, & e
\end{vmatrix},
\quad
\begin{vmatrix}
b, & b, & c\\
c, & c, & d\\
d, & d, & e
\end{vmatrix},
$$

of which the latter two vanish, and the first is identical with the coefficient of x^2y in the first solution. The same thing is obviously true in regard of the coefficients of xy^2 in the two forms, and a like method may be applied to show that in all cases the determinant above given is identical with the determinant of my former paper, namely

$$
\begin{vmatrix}
a_0x + a_1y, & a_1x + a_2y & \ldots & a_nx + a_{n+1}y\\
a_1x + a_2y, & a_2x + a_3y & \ldots & a_{n+1}x + a_{n+2}y\\
\ldots & \ldots & \ldots & \ldots\\
a_nx + a_{n+1}y, & a_{n+1}x + a_{n+2}y & \ldots & a_{2n}x + a_{2n+1}y
\end{vmatrix}.
$$

Thus, then, we see that for odd-degreed functions, the reduction to their canonical form of the sum of $(n+1)$ powers depends upon the solution of one single equation of the $(n+1)$th degree, and can never be effected in more than one way.

This new form of the resolving determinant affords a beautiful criterion for a function of x, y of the degree $2n+1$ being composed of n instead of, as in general, $(n+1)$ powers. In order that this may be the case, it is obvious that two conditions must be satisfied; but I pointed out in my supplemental paper on canonical forms, that all the coefficients of the resolving determinant must vanish, which appears to give far too many conditions. Thus, suppose we have

$$ax^7 + 7bx^6y + 21cx^5y^2 + 35dx^4y^3 + 35ex^3y^4 + 21fx^2y^5 + 7gxy^6 + hy^7.$$

The conditions of catalecticism, that is, of its being expressible under the form of the sum of three (instead of, as in general, four) seventh powers, requires that all the coefficients of the different powers of x and y must vanish in the determinant

$$\begin{vmatrix} y^4, & -y^3x, & y^2x^2, & -yx^3, & x^4 \\ a, & b, & c, & d, & e \\ b, & c, & d, & e, & f \\ c, & d, & e, & f, & g \\ d, & e, & f, & g, & h \end{vmatrix};$$

in other words, we must have five determinants,

$$\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}, \quad \begin{vmatrix} a, & c, & d, & e \\ b, & d, & e, & f \\ c, & e, & f, & g \\ d, & f, & g, & h \end{vmatrix}, \quad \begin{vmatrix} a, & b, & c, & e \\ b, & c, & d, & f \\ c, & d, & e, & g \\ d, & e, & f, & h \end{vmatrix},$$

$$\begin{vmatrix} a, & b, & d, & e \\ b, & c, & e, & f \\ c, & d, & f, & g \\ d, & e, & g, & h \end{vmatrix}, \quad \begin{vmatrix} b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \\ e, & f, & g, & h \end{vmatrix},$$

all separately zero. But by my homaloidal law*, all these five equations amount only to $(5-4)(5-3)$, that is, to 2. I may notice here, that a theorem substantially identical with this law, and another absolutely identical with the theorem of compound determinants given by me in this *Magazine*, and afterwards generalized in a paper also published† in this *Magazine*, entitled

[* p. 150 above.] [† p. 241 above.]

"On the Relations between the Minor Determinants of Linearly Equivalent Quadratic Forms," have been subsequently published as original in a recent number of M. Liouville's journal.

The general condition of mere singularity, as distinguished from catalecticism, that is, of the function of the degree $2n+1$, being incapable of being expressed as the sum of $n+1$ powers, is that the resolving resultant shall have two equal roots; in other words, that its determinant shall be zero.

Mr Cayley has pointed out to me a very elegant mode of identifying the two forms of the resolving resultant, which I have much pleasure in subjoining. Take as the example a function of the fifth degree, we have by the multiplication of determinants,

$$\begin{vmatrix} y^3, & -y^2x, & yx^2, & -x^3 \\ a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \end{vmatrix} \times \begin{vmatrix} 1, & 0, & 0, & 0 \\ x, & y, & 0, & 0 \\ 0, & x, & y, & 0 \\ 0, & 0, & x, & y \end{vmatrix}$$

$$= \begin{vmatrix} y^3, & a, & b, & c \\ 0, & ax+by, & bx+cy, & cx+dy \\ 0, & bx+cy, & cx+dy, & dx+ey \\ 0, & cx+dy, & dx+ey, & ex+fy \end{vmatrix},$$

which dividing out each side of the equation by y^3, immediately gives the identity required, and the method is obviously general.

Turn we now to consider the mode of reducing a biquadratic function of two letters to its canonical form, *videlicet*

$$(fx+gy)^4 + (hx+ky)^4 + 6m(fx+gy)^2(hx+ky)^2.$$

Let the given function be written

$$ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4.$$

Let $\quad g = f\lambda_1, \quad k = h\lambda_2, \quad mf^2h^2 = \mu, \quad \lambda_1 + \lambda_2 = s_1, \quad \lambda_1\lambda_2 = s_2,$

then we have

$$f^4 + h^4 + 6\mu = a,$$

$$4f^4\lambda_1 + 4h^4\lambda_2 + 6\mu(2s_1) = 4b,$$

$$6f^4\lambda_1^2 + 6h^4\lambda_2^2 + 6\mu(s_1^2 + 2s_2) = 6c,$$

$$4f^4\lambda_1^3 + 4h^4\lambda_2^3 + 6\mu(2s_1s_2) = 4d,$$

$$f^4\lambda_1^4 + h^4\lambda_2^4 + 6\mu s_2^2 = e.$$

Eliminating f and h between the first, second and third; the second, third and fourth; and the third, fourth and fifth equations successively, we obtain

$$as_2 - bs_1 + c - \mu(8s_2 - 2s_1^2) = 0,$$
$$bs_2 - cs_1 + d - \mu(4s_1s_2 - s_1^3) = 0,$$
$$cs_2 - ds_1 + e - \mu(8s_2^2 - 2s_1^2s_2) = 0.$$

Let now
$$(2s_1^2 - 8s_2)\mu = \nu,$$
and we shall have
$$as_2 - bs_1 + (c + \nu) = 0,$$
$$bs_2 - \left(c - \frac{\nu}{2}\right)s_1 + d = 0,$$
$$(c + \nu)s_2 - ds_1 + e = 0.$$

Hence ν will be found from the cubic equation

$$\begin{vmatrix} a, & b, & c + \nu \\ 2b, & 2c - \nu, & 2d \\ c + \nu, & d, & e \end{vmatrix} = 0,$$

that is,
$$\nu^3 - \nu(ae - 4bd + 3c^2) + \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix} = 0,$$

in which equation it will not fail to be noticed that the coefficient of ν^2 is zero, and the remaining coefficients are the two well-known hyperdeterminants, or, as I propose henceforth to call them, the two Invariants of the form

$$ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4;$$

be it also further remarked that

$$\nu = 8\left(\tfrac{1}{4}s_1^2 - s_2\right)\mu,$$

in which equation the coefficient of 8μ is the Determinant or Invariant of

$$x^2 + s_1xy + s_2y^2.$$

When ν is thus found, s_1, s_2, and μ, being given by the equations in terms of ν, are known, and by the solution of a quadratic λ_1, λ_2 become known in terms of s_1, s_2, and f, h in terms of λ_1, λ_2, μ, and the problem is completely determined. The most symmetrical mode of stating this method of solution is to suppose the given function thrown under the form

$$(fx + gy)^4 + (f_1x + g_1y)^4 + 6\epsilon(fx + gy)^2(f_1x + g_1y)^2.$$

Then writing
$$(fx + gy)(f_1x + g_1y) = Lx^2 + Mxy + Ny^2,$$

$-\nu$, the quantity to be found by the solution of the cubic last given, becomes

$$8\epsilon\left(LN - \frac{M^2}{4}\right).$$

I shall now proceed to apply the same method to the reduction of the function

$$a_0x^8 + 8a_1x^7y + 28a_2x^6y^2 + 56a_3x^5y^3 + 70a_4x^4y^4 + 56a_5x^3y^5$$
$$+ 28a_6x^2y^6 + 8a_7xy^7 + a_8y^8,$$

under the form of

$$(p_1x + q_1y)^8 + (p_2x + q_2y)^8 + (p_3x + q_3y)^8 + (p_4x + q_4y)^8$$
$$+ 70\epsilon(p_1x + q_1y)^2(p_2x + q_2y)^2(p_3x + q_3y)^2(p_4x + q_4y)^2.$$

It will be convenient to begin, as in the last case, by taking

$$q_1 = p_1\lambda_1, \quad q_2 = p_2\lambda_2, \quad q_3 = p_3\lambda_3, \quad q_4 = p_4\lambda_4,$$
$$\epsilon p_1^2p_2^2p_3^2p_4^2 = m,$$

and

$$(x + \lambda_1y)(x + \lambda_2y)(x + \lambda_3y)(x + \lambda_4y) = x^4 + s_1x^3y + s_2x^2y^2 + s_3xy^3 + s_4y^4 = U,$$

we shall then have nine equations for determining the nine unknown quantities of the general form

$$p_1^8\lambda_1^\iota + p_2^8\lambda_2^\iota + p_3^8\lambda_3^\iota + p_4^8\lambda_4^\iota + M_\iota m = a_\iota,$$

where ι has all values from 0 to 8 inclusive, and where

$$M_\iota = 70 . \frac{1 . 2 \ldots \iota . 1 . 2 \ldots (8 - \iota)}{1 . 2 \ldots 8}$$

multiplied into the coefficient of $y^\iota x^{8-\iota}$ in U^2.

Taking these nine equations in consecutive fives, beginning with the first, second, third, fourth, fifth, and ending with the fifth, sixth, seventh, eighth, ninth, we obtain the five equations following:—

$$a_0s_4 - a_1s_3 + a_2s_2 - a_3s_1 + a_4s_0 - mN_1 = 0,$$
$$a_1s_4 - a_2s_3 + a_3s_2 - a_4s_1 + a_5s_0 - mN_2 = 0,$$
$$a_2s_4 - a_3s_3 + a_4s_2 - a_5s_1 + a_6s_0 - mN_3 = 0,$$
$$a_3s_4 - a_4s_3 + a_5s_2 - a_6s_1 + a_7s_0 - mN_4 = 0,$$
$$a_4s_4 - a_5s_3 + a_6s_2 - a_7s_1 + a_8s_0 - mN_5 = 0,$$

where

$$N_1 = M_0s_4 - M_1s_3 + M_2s_2 - M_3s_1 + M_4,$$
$$N_2 = M_1s_4 - M_2s_3 + M_3s_2 - M_4s_1 + M_5,$$
$$N_3 = M_2s_4 - M_3s_3 + M_4s_2 - M_5s_1 + M_6,$$
$$N_4 = M_3s_4 - M_4s_3 + M_5s_2 - M_6s_1 + M_7,$$
$$N_5 = M_4s_4 - M_5s_3 + M_6s_2 - M_7s_1 + M_8.$$

Developing now U^2, we have

$$M_0 = 70, \quad M_1 = \frac{35}{2} s_1, \quad M_2 = 5s_2 + \frac{5}{2} s_1^2, \quad M_3 = \frac{5}{2} s_3 + \frac{5}{2} s_1 s_2,$$

$$M_4 = 2s_4 + 2s_1 s_3 + s_2^2, \quad M_5 = \frac{5}{2} s_1 s_4 + \frac{5}{2} s_2 s_3, \quad M_6 = 5s_2 s_4 + \frac{5}{2} s_3^2,$$

$$M_7 = \frac{35}{2} s_3 s_4, \quad M_8 = 70 s_4^2.$$

Hence

$$N_1 = 72s_4 - 18s_1 s_3 + 6s_2^2,$$

$$N_2 = 18s_1 s_4 - \frac{9}{2} s_1^2 s_3 + \frac{3}{2} s_1 s_2^2,$$

$$N_3 = 12s_2 s_4 - 3s_1 s_2 s_3 + s_2^3,$$

$$N_4 = 18s_3 s_4 - \frac{9}{2} s_1 s_3^2 + \frac{3}{2} s_2^2 s_3,$$

$$N_5 = 72s_4^2 - 18s_1 s_3 s_4 + 6s_2^2 s_4.$$

Hence we have

$$N_1 = 72I, \quad N_2 = 72I \frac{s_1}{4}, \quad N_3 = 72I \frac{s_2}{6}, \quad N_4 = 72I \frac{s_3}{4}, \quad N_5 = 72I s_4,$$

where it will be observed that I is the quadratic invariant of U.

Making now

$$72mI = \nu,$$

we shall have the five following equations:—

$$a_0 s_4 - a_1 s_3 + a_2 s_2 - a_3 s_1 + (a_4 - \nu) = 0,$$

$$a_1 s_4 - a_2 s_3 + a_3 s_2 - \left(a_4 + \frac{\nu}{4}\right) s_1 + a_5 = 0,$$

$$a_2 s_4 - a_3 s_3 + \left(a_4 - \frac{\nu}{6}\right) s_2 - a_5 s_1 + a_6 = 0,$$

$$a_3 s_4 - \left(a_4 + \frac{\nu}{4}\right) s_3 + a_5 s_2 - a_6 s_1 + a_7 = 0,$$

$$(a_4 - \nu) s_4 + a_5 s_3 - a_6 s_2 - a_7 s_1 + a_8 = 0;$$

so that the problem reduces itself to finding ν, which is found from the equation of the fifth degree:—

$$\begin{vmatrix} a_0, & a_1, & a_2, & a_3, & a_4 - \nu \\ a_1, & a_2, & a_3, & a_4 + \frac{\nu}{4}, & a_5 \\ a_2, & a_3, & a_4 - \frac{\nu}{6}, & a_5, & a_6 \\ a_3, & a_4 + \frac{\nu}{4}, & a_5, & a_6, & a_7 \\ a_4 - \nu, & a_5, & a_6, & a_7, & a_8 \end{vmatrix} = 0,$$

ν, it will be observed, being 72 times the quadratic invariant of

$$(p_1 x + q_1 y)(p_2 x + q_2 y)(p_3 x + q_3 y)(p_4 x + q_4 y),$$

the function being supposed to be thrown under the form of

$$\Sigma (p_1 x + q_1 y)^8 + 70\epsilon (p_1 x + q_1 y)^2 (p_2 x + q_2 y)^2 (p_3 x + q_3 y)^2 (p_4 x + q_4 y)^2.$$

It is obvious that in the equation for finding ν, all the coefficients being functions of the invariable quantities p_1, q_1, &c., and ϵ, must be themselves invariants of the given function; so that the determinant last given will present under one point of view four out of the six invariants belonging to a function of the eighth degree, and these four will be of the degrees 2, 3, 4, 5 respectively*.

I shall now proceed to generalize this remarkable law, and to demonstrate the existence and mode of finding $2n$ consecutively-degreed independent invariants of any homogeneous function of the degree $4n$, and of $n+1$ consecutively-even-degreed independent invariants of any homogeneous function of the degree $4n+2$; a result, whether we look to the fact of such invariants existing, or to the simplicity of the formula for obtaining them, equally unexpected and important, and tending to clear up some of the most obscure, and at the same time interesting points in this great theory of algebraical transformations.

In the first place, let me recall to my readers in the simplest form what is meant by an invariant† of a homogeneous function, say of two variables x and y. If the coefficients of the function $f(x, y)$ be called $a, b, c \ldots l$, and if when for x we put $lx + my$, and for y, $nx + py$, where $lp - mn = 1$, the coefficients of the corresponding terms become $a', b' \ldots l'$; and if

$$I(a, b \ldots l) = I(a', b' \ldots l'),$$

then I is defined to be an invariant of f.

Let now $f(x, y)$ be a homogeneous function in x, y of the 2ιth degree, and write

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^{\iota} f(x, y) + \lambda (\eta x - \xi y)^{\iota} = P,$$

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^{\iota} f(lx + my,\ nx + py) + \lambda (\eta x - \xi y)^{\iota} = P',$$

where ξ and η are independent of x, y, and $lp - mn = 1$.

Let

$$x' = lx + my,$$

$$y' = nx + py,$$

then

$$\xi \frac{d}{dx} + \eta \frac{d}{dy} = \xi \frac{dx'}{dx} \frac{d}{dx'} + \xi \frac{dy'}{dx} \frac{d}{dy'} + \eta \frac{dx'}{dy} \frac{d}{dx'} + \eta \frac{dy'}{dy} \frac{d}{dy'},$$

* The reasoning in this paragraph seems of doubtful conclusiveness. It may be accepted, however, as a fact of observation confirmed and generalized by the subsequent theorem, that the coefficients are invariants.

† *Olim*, Hyperdeterminant, Constant derivative.

and if we now write

$$l\xi + m\eta = \xi',$$
$$n\xi + p\eta = \eta',$$

we find
$$\xi \frac{d}{dx} + \eta \frac{d}{dy} = \xi' \frac{d}{dx'} + \eta' \frac{d}{dy'}.$$

Again, from the equations between x', y', x, y, we find

$$x = \frac{px' - my'}{pl - mn} = px' - my',$$
$$y = \frac{ly' - nx'}{pl - mn} = ly' - nx';$$

therefore
$$\eta x - \xi y = (p\eta + n\xi)\, x' - (m\eta + l\xi)\, y' = \eta' x' - \xi' y'.$$

Hence
$$P' = \left(\xi' \frac{d}{dx'} + \eta' \frac{d}{dy'}\right)^{\iota} f(x', y') + \lambda\, (\eta' x' - \xi' y')^{\iota}.$$

Again,
$$\frac{d}{d\xi} = l \frac{d}{d\xi'} + n \frac{d}{d\eta'},$$
$$\frac{d}{d\eta} = m \frac{d}{d\xi'} + p \frac{d}{d\eta'}.$$

Hence

$$\left(\frac{d}{d\xi}\right)^{\iota} P' = l^{\iota} \left(\frac{d}{d\xi'}\right)^{\iota} P' + \iota l^{\iota-1} n \left(\frac{d}{d\xi'}\right)^{\iota-1} \frac{d}{d\eta'} P' + \&c. + n^{\iota} \left(\frac{d}{d\eta'}\right)^{\iota} P',$$

$$\left(\frac{d}{d\xi}\right)^{\iota-1} \frac{d}{d\eta} P' = l^{\iota-1} m \left(\frac{d}{d\xi'}\right)^{\iota} P' + \{l^{\iota-1} p + (\iota - 1)\, l^{\iota-2} mn\} \left(\frac{d}{d\xi'}\right)^{\iota-1} \frac{d}{d\eta'} P' + \&c.$$
$$+ n^{\iota-1} p \left(\frac{d}{d\eta'}\right)^{\iota} P',$$

....

$$\left(\frac{d}{d\eta}\right)^{\iota} P' = m^{\iota} \left(\frac{d}{d\xi'}\right)^{\iota} P' + \iota m^{\iota-1} p \left(\frac{d}{d\xi'}\right)^{\iota-1} \frac{d}{d\eta'} P + \&c. + p^{\iota} \left(\frac{d}{d\eta'}\right)^{\iota} P'.$$

But P' being of ι dimensions in ξ' and η', and also in x and y, each of the equations above written will be of ι dimensions in x and y, and of no dimensions in ξ', η'; in fact, the successive terms of the right-hand members of the above $\iota + 1$ equations will be multiples of the $(\iota + 1)$ quantities

$$(x')^{\iota},\ (x')^{\iota-1} y',\ (x')^{\iota-2} y'^{\iota} \dots (y')^{\iota}.$$

Consequently a linear resultant may be taken of

$$\left(\frac{d}{d\xi}\right)^{\iota} P',\ \left(\frac{d}{d\xi}\right)^{\iota-1} \frac{d}{d\eta} P' \dots \left(\frac{d}{d\eta}\right)^{\iota} P',$$

treating x'^{ι}, $x'^{\iota-1} y' \dots y'^{\iota}$ as independent, and as quantities to be eliminated; and this, according to a well-known principle of elimination, will prove

the linear resultant of the foregoing equations to be equal to the linear resultant of

$$\left(\frac{d}{d\xi'}\right)^{\iota} P',\ \left(\frac{d}{d\xi'}\right)^{\iota-1} \frac{d}{d\eta'} P' \dots \left(\frac{d}{d\eta'}\right)^{\iota} P',$$

multiplied by the determinant

$$\left|\begin{matrix} l^{\iota}, & \iota n l^{\iota-1}, & \tfrac{1}{2}\iota(\iota-1)\, n^{\iota} l^{\iota-2}, & \dots, & n^{\iota} \\ l^{\iota-1} m, & l^{\iota-1} p + (\iota-1)\, mnl^{\iota-2}, & & \dots, & n^{\iota-1} p \\ \dots & \dots & \dots & \dots & \dots \\ m^{\iota}, & \iota m^{\iota-1} p, & \tfrac{1}{2}\iota(\iota-1)\, m^{\iota} p^{\iota-2}, & \dots, & p^{\iota} \end{matrix}\right|.$$

This last written determinant may be shown from the method of its formation to be equal to $(lp - mn)^{\frac{\iota(\iota+1)}{2}}$, that is, to unity, because $lp - mn = 1$. Again, since

$$x'^{\iota} = l^{\iota} x^{\iota} + \iota l^{\iota-1} m x^{\iota-1} y + \&c. + m^{\iota} y^{\iota},$$
$$x'^{\iota-1} y' = l^{\iota-1} n x^{\iota} + (l^{\iota-1} n + (\iota - 1)\, l^{\iota-2} mn)\, x^{\iota-1} y + \&c. + m^{\iota-1} p y^{\iota},$$

....

$$y'^{\iota} = n^{\iota} x^{\iota} + \iota n^{\iota-1} p x^{\iota-1} y + \dots + p^{\iota} y^{\iota},$$

the resultant of $\left(\frac{d}{d\xi}\right)^{\iota} P' \dots \left(\frac{d}{d\eta}\right)^{\iota} P'$, obtained by treating x^{ι}, $x^{\iota-1} y \dots y^{\iota}$ as the eliminables, will be equal to the resultant of the same functions when x'^{ι}, $x'^{\iota-1} y' \dots y'^{\iota}$ are taken as the eliminables* multiplied by a power of the determinant

$$\left|\begin{matrix} l^{\iota}, & \dots, & m^{\iota} \\ l^{\iota-1} n, & \dots, & m^{\iota-1} p \\ \dots & \dots & \dots \\ n^{\iota}, & \dots, & p^{\iota} \end{matrix}\right|,$$

which determinant, like the last, is unity. Thus, then, we have succeeded in showing that the resultant obtained by eliminating x^{ι}, $x^{\iota-1} y \dots y^{\iota}$ between

$$\left(\frac{d}{d\xi}\right)^{\iota} P',\ \left(\frac{d}{d\xi}\right)^{\iota-1} \frac{d}{d\eta} P' \dots \left(\frac{d}{d\eta}\right)^{\iota} P'$$

is equal to the resultant obtained by eliminating $(x')^{\iota}$, $x'^{\iota-1} y' \dots y'^{\iota}$ between

$$\left(\frac{d}{d\xi'}\right)^{\iota} P',\ \left(\frac{d}{d\xi'}\right)^{\iota-1} \frac{d}{d\eta'} P' \dots \left(\frac{d}{d\eta'}\right)^{\iota} P';$$

* For the statement of the general principle of the change of the variables of elimination, see my paper in the March Number, 1851, of the *Camb. and Dub. Math. Jour.* [p. 186 above].

or, which is evidently the same thing, the resultant obtained by eliminating $x^\iota, x^{\iota-1}y \ldots y^\iota$ between

$$\left(\frac{d}{d\xi}\right)^\iota P, \quad \left(\frac{d}{d\xi}\right)^{\iota-1}\frac{d}{d\eta}P \ldots \left(\frac{d}{d\eta}\right)^\iota P;$$

that is to say, this last resultant remains absolutely unaltered in value when for x, y we write respectively

$$lx + my,$$
$$nx + py,$$

provided that $lp - mn = 1$.

Hence by definition this resultant is an invariant $f(x, y)$, and λ being arbitrary, all the separate coefficients of the powers of λ in this resultant must also be invariants. I proceed to express this resultant in terms of λ and the coefficients of (x, y). Let $\varpi = 1.2.3 \ldots \iota$ and

$$\frac{1}{\varpi}\left(\frac{d}{d\xi}\right)^\iota P = \left(\frac{d}{dx}\right)^\iota f + \lambda(-y)^\iota \qquad = E_1,$$

$$\frac{1}{\varpi}\left(\frac{d}{d\xi}\right)^{\iota-1}\frac{d}{d\eta}P = \left(\frac{d}{dx}\right)^{\iota-1}\frac{d}{dy}f + \lambda(-y)^{\iota-1}x \qquad = E_2;$$

$$\frac{1}{\varpi}\left(\frac{d}{d\xi}\right)^{\iota-2}\left(\frac{d}{d\eta}\right)^2 P = \left(\frac{d}{dx}\right)^{\iota-2}\left(\frac{d}{dy}\right)^2 f + \lambda(-y)^{\iota-2}x^2 = E_3,$$

....

$$\frac{1}{\varpi}\left(\frac{d}{d\eta}\right)^\iota P = \left(\frac{d}{dy}\right)^\iota f + \lambda x^\iota \qquad = E_{\iota+1};$$

and

$$f(x, y) = a_0 x^{2\iota} + 2\iota a_1 x^{2\iota-1}y + \tfrac{1}{2}(2\iota)(2\iota-1)a_2 x^{2\iota-2}y^2 + \&c. + a_{2\iota}y^{2\iota}.$$

We find, writing $\sigma\lambda$ for λ, where $\sigma = 2\iota(2\iota-1)\ldots(\iota+1)$,

$$\frac{1}{\sigma}E_1 = a_0 x^\iota + \iota a_1 x^{\iota-1}y + \tfrac{1}{2}\iota(\iota-1)a_2 x^{\iota-2}y^2 \ldots$$
$$+ \tfrac{1}{2}\iota(\iota-1)a_{\iota-2}x^2y^{\iota-2} + \iota a_{\iota-1}xy^{\iota-1} + a_\iota y^\iota + \lambda(-y)^\iota,$$

$$\frac{1}{\sigma}E_2 = a_1 x^\iota + \iota a_2 x^{\iota-1}y + \tfrac{1}{2}\iota(\iota-1)a_3 x^{\iota-2}y^2 \ldots$$
$$+ \tfrac{1}{2}\iota(\iota-1)a_{\iota-1}x^2y^{\iota-2} + \iota a_\iota xy^{\iota-1} + a_{\iota+1}y^\iota + \lambda(-y)^{\iota-1}x,$$

$$\frac{1}{\sigma}E_3 = a_2 x^\iota + \iota a_3 x^{\iota-1}y \ldots$$
$$+ \tfrac{1}{2}\iota(\iota-1)a_\iota x^2y^{\iota-2} + \iota a_{\iota+1}xy^{\iota-1} + a_{\iota+2}y^\iota + \lambda(-y)^{\iota-2}x^2,$$

....

$$\frac{1}{\sigma}E_{\iota+1} = a_\iota x^\iota + \&c. + \lambda x^\iota;$$

accordingly, by eliminating

$$x^\iota, \quad \iota x^{\iota-1}y, \quad \tfrac{1}{2}\iota(\iota-1)x^{\iota-2}y^2 \ldots y^\iota,$$

we obtain as the required resultant*,

$$\begin{vmatrix} a_\iota \pm \lambda, & a_{\iota-1}, & a_{\iota-2}, & \ldots & a_0 \\ a_{\iota+1}, & a_\iota \mp \frac{\lambda}{\iota}, & a_{\iota-1}, & \ldots & a_1 \\ a_{\iota+2}, & a_{\iota+1}, & a_\iota \pm \frac{\lambda}{\frac{1}{2}\iota(\iota-1)}, & \ldots & a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{2\iota}, & a_{2\iota-1}, & \ldots\ldots, & \ldots & a_\iota + \lambda \end{vmatrix}.$$

Inasmuch as all the coefficients of λ in this expression are invariants of $f(x, y)$, and there are no invariants of the first order, it is clear that the coefficient of λ^ι must be always zero, which is easily verified.

Again, if ι is odd, the determinant remains unaltered if we write $-\lambda$ for λ; hence when $f(x, y)$ is of the degree $4\epsilon + 2$, all the coefficients of the odd powers of λ disappear. Thus, then, our theorem at once demonstrates that a function of x, y of the degree 4ϵ has 2ϵ invariants of all degrees from 2 up to $2\epsilon+1$ inclusive, and that a function of x, y of the degree $4\epsilon+2$ has $\epsilon+1$ invariants whose degrees correspond to all the even numbers in the series from 2 to $2\epsilon+2$.

But in order that the proposition, as above stated, may be understood in its full import and value, it is necessary to show that these invariants are independent of one another, which is usually a most troublesome and difficult task in inquiries of this description, but which the peculiar form of our grand determinant enables us to accomplish with extraordinary facility. In order to make the spirit of the demonstration more apparent, take the case of a function of the twelfth degree, whose coefficients, divided by the successive binomial numbers 1, 12, $\frac{12.11}{2}$, &c. may be called

$$a, b, c, d, e, f, g, h, i, j, k, l, m.$$

* Mr Cayley has made the valuable observation, that λ (given by equating to zero the above determinant) may be defined by means of the equation

$$\left(\frac{d}{dx}\frac{d}{d\eta} - \frac{d}{dy}\frac{d}{d\xi}\right)^\iota \{f(x, y) \times \phi(\xi, \eta)\} = \lambda\phi(x, y),$$

ϕ being itself a certain rational integral form of a function of the ιth degree, the ratio of whose coefficients would be given by virtue of the above equations as functions of λ and the coefficients of $f(x, y)$.

Our grand determinant then takes the form

$$\begin{vmatrix} g+\lambda, & f, & e, & d, & c, & b, & a \\ h, & g-\frac{\lambda}{6}, & f, & e, & d, & c, & b \\ i, & h, & g+\frac{\lambda}{15}, & f, & e, & d, & c \\ j, & i, & h, & g-\frac{\lambda}{20}, & f, & e, & d \\ k, & j, & i, & h, & g+\frac{\lambda}{15}, & f, & e \\ l, & k, & j, & i, & h, & g-\frac{\lambda}{6}, & f \\ m, & l, & k, & j, & i, & h, & g+\lambda \end{vmatrix}.$$

Here it will be observed that

a and m appear only		1 time.
b and l	...	2 times.
c and k	...	3 ...
d and j	...	4 ...
e and i	...	5 ...
f and h	...	6 ...
g	...	7 ...

Let now the coefficients be called

$$H_2,\ H_3,\ H_4,\ H_5,\ H_6,\ H_7,$$

H_2 and H_3 manifestly are independent.

Again, if possible, let $H_4 = pH_2^2$, then a and m would appear twice in H_4, contrary to the rule.

Hence H_4 is independent of H_2, H_3.

For a similar reason H_5 cannot depend on H_2, H_3.

Again, if possible, let

$$H_6 = pH_2^3 + qH_2H_4 + rH_3^2,$$

H_2^3 will contain b^6l^6, which by the rule cannot appear in H_2H_4 or in H_3^2.

Hence $p = 0$.

Also H_4 will contain $b^2l^2 \times$ the coefficient of λ^2 in

$$\left(g+\frac{\lambda}{15}\right)\left(g-\frac{\lambda}{20}\right)\left(g+\frac{\lambda}{15}\right),$$

which is not zero. And H_2 also contains bl; hence H_2H_4 will contain b^3l^3. But H_3 will evidently not contain b^3 or l^3, or b^2l or bl^2, nor can H_6 contain b^3l^3; hence $q=0$. Finally, H_3^2 will contain c^6 and k^6, but H_6 can only contain as to these letters the combination c^3k^3; hence $r=0$.

Consequently H_6 does not depend on H_2, H_4, H_3. As regards H_2, H_3, H_4, H_5, H_6 not vanishing, this may be made at once apparent by making all the letters but g vanish; the H's then become identical with the coefficients of

$$(g+\lambda)^2\left(g-\frac{\lambda}{6}\right)^2\left(g+\frac{\lambda}{15}\right)^2\left(g-\frac{\lambda}{20}\right),$$

none of which are zero except that of λ^6. The same or a similar demonstration may be extended to H_7 and easily generalized; hence, then, this most unexpected and surprising law is fully made out*.

To return to the subject of canonical forms, I have not found the method so signally successful in its application to the 4th and 8th degrees, conduct to the solution of other degrees, such as the 6th, 12th, or 16th, of all of which I have made trial; possibly another canonical form must be substituted to meet the exigency of these cases†; and it may be remarked in general, that if we have a function of the $(2n)$th degree, the canonical form assumed may be taken,

$$\Sigma\,(p_1x+q_1y)^{2n}+V;$$

where V, in lieu of being the squared product of

$$(p_1x+q_1y),\,(p_2x+q_2y),\,\ldots,\,(p_nx+q_ny),$$

* This demonstration, however, does not extend to show that the coefficients of the powers of λ may not possibly be dependents, that is, explicit functions of one another combined with other invariants not included among their number, or of these latter alone. For example, in the case of the 12th degree, we know by Mr Cayley's law that there must be two invariants of the 4th order. Our determinant gives only one of these. Call the other one K_4; by the above reasoning it is not disproved but that we may have

$$H_6 = pH_2^3 + qH_2H_4 + rH_3^2 + sH_2K_4.$$

I believe, however, that the H's may be demonstrated without much difficulty to be primitive or fundamental invariants. The law of Mr Cayley here adverted to admits of being stated in the following terms:—The number of independent invariants of the 4th order belonging to a function of x, y of the nth degree is equal to the number of solutions in integers (not less than zero) of the equation $2x+3y=n-3$. *Vide* his memorable paper (in which several numerical errors occur against which the reader should be cautioned) "On Linear Transformations," vol. I. *Cambridge and Dublin Mathematical Journal*, new series. There is no great difficulty in showing, by aid of the doctrine of symmetrical functions, that there can never be *more* than one quadratic or one cubic invariant, and in what cases there is one or the other, or each, to any given function of two variables. The general law, however, for the number of invariants of any order other than 2, 3, 4 remains to be made out, and is a great desideratum in the theory of linear transformations.

† See the Postscript [p. 283] for a verification of this conjecture.

may be any hyperdeterminant, or (as I shall in future call such functions) covariant of this product, understanding $P(x, y)$ to be a covariant of $f(x, y)$ when $P(lx+my, nx+py)$ stands in precisely the same relation to $f(lx+my, nx+py)$ as $P(x, y)$ to $f(x, y)$, provided only that $lp-mn=1$. For the relation and distinction between covariants and contravariants, see a short article of mine* in the *Cambridge and Dublin Mathematical Journal* for this month. In endeavouring to apply the method of the text to the Sextic Function

$$ax^6+6bx^5y+15cx^4y^2+20dx^3y^3+15ex^2y^4+6fxy^5+gy^6,$$

thrown under the form

$$\Sigma\,(px+qy)^6+20\epsilon U^2,$$

where

$$U=(p_1x+q_1y)(p_2x+q_2y)(p_3x+q_3y)=s_0x^3+s_1x^2y+s_2xy^2+s_3y^3,$$

I obtain the following equations:

$$as_3-bs_2+cs_1-ds_0=\epsilon\,(162s_0^2s_3-54s_0s_1s_2+12s_1^3),$$
$$bs_3-cs_2+ds_1-es_0=\epsilon\,(54s_0s_1s_3+6s_1^2s_2-36s_0s_2^2),$$
$$cs_3-ds_2+es_1-fs_0=\epsilon\,(-54s_0s_2s_3-6s_1s_2^2+36s_3s_1^2),$$
$$ds_3-es_2+fs_1-gs_0=\epsilon\,(-162s_0s_3^2+54s_1s_2s_3+12s_2^3).$$

In these equations, if we call the quantities multiplied by ϵ respectively L, M, N, P, we shall find

$$s_3L-\frac{1}{3}s_2M-\frac{1}{3}s_1N+s_0P=0,$$

and

$$s_3L-s_2M-s_1N+s_0P=I;$$

where I denotes the determinant, or, as I shall in future call such function (in order to avoid the obscurity and confusion arising from employing the same word in two different senses), the Discriminant†, which is the biquadratic (and of course sole) invariant of the cubic function

$$s_0x^3+s_1x^2y+s_2xy^2+s_3y^3.$$

The reduction of the function of the fourth degree to its canonical form may be effected very easily by means of the properties of the invariants of

[* p. 200 above.]

† "Discriminant," because it affords the *discrimen* or test for ascertaining whether or not equal factors enter into a function of two variables, or more generally of the existence or otherwise of multiple points in the locus represented or characterized by any algebraical function, the most obvious and first observed species of singularity in such function or locus. Progress in these researches is impossible without the aid of clear expression; and the first condition of a good nomenclature is that different things shall be called by different names. The innovations in mathematical language here and elsewhere (not without high sanction) introduced by the author, have been never adopted except under actual experience of the embarrassment arising from the want of them, and will require no vindication to those who have reached that point where the necessity of some such additions becomes felt.

the canonical form, as I have shown in the *Cambridge and Dublin Mathematical Journal*. Accordingly I have endeavoured to ascertain whether the reduction of the sixth degree might not be effected by a similar method.

If we start with the form $ax^6+by^6+cz^6+90mx^2y^2z^2$, where $x+y+z=0$, which is only another mode of representing the canonical form previously given, we shall find that there are four independent invariants, of the second, fourth, sixth and tenth degrees. Calling these H_2, H_4, H_6, H_{10}, and writing s_1, s_2, s_3 for $a+b+c$, $ab+ac+bc$, abc it will be found, after performing some extremely elaborate computations, that

$$H_2=s_2-270m^2,$$
$$H_4=6ms_3+45m^2s_2+216m^3s_1+891m^4,$$
$$H_6=4s_3^2+120s_2s_3m-[684s_2^2+432s_1s_3]\,m^2$$
$$+(13\,.\,27\,.\,64s_3-64\,.\,81s_1s_2)\,m^3+8\,.\,81\,.\,169s_2m^4$$
$$+7\,.\,128\,.\,729s_1m^5+16\,.\,729\,.\,239m^6.$$

H_{10} is too enormously long to attempt to compute; but we can easily prove its independent existence by making $m=0$, in which case the (determinant, or, to use the new term proposed, the) discriminant of $ax^6+by^6+cz^6$ becomes the product of the twenty-five forms of the expression

$$(ab)^{\frac{1}{5}}+(ac)^{\frac{1}{5}}\,.\,1^{\frac{1}{5}}+(bc)^{\frac{1}{5}}\,.\,1^{\frac{1}{5}}\,*.$$

Now in general the value of such a product for $\alpha^{\frac{1}{5}}+\beta^{\frac{1}{5}}.\,1^{\frac{1}{5}}+\gamma^{\frac{1}{5}}.\,1^{\frac{1}{5}}$ is obviously of the form

$$(\alpha+\beta+\gamma)^5+\alpha\beta\gamma\,\{f(\alpha+\beta+\gamma)^2+g\,(\alpha\beta+\alpha\gamma+\beta\gamma)\};$$

for when $\alpha=0$ or $\beta=0$ or $\gamma=0$, the product must become respectively $(\beta+\gamma)^5$, $(\gamma+\alpha)^5$ and $(\alpha+\beta)^5$. Moreover, without caring to calculate f, g†, it is enough for our present purpose to satisfy ourselves that g cannot be zero, as then the product would have a factor $(\alpha+\beta+\gamma)^2$. Hence, then, on putting

* Such a product in the language of the most modern continental analysis is, I believe, termed a Norm. If we suppose the general function of x, y of the 4th degree thrown under the form $Au^4+Bv^4+Cw^4$, where $u+v+w=0$, and the general function of x, y, z of the 3rd degree thrown under the form $Au^3+Bv^3+Cw^3+D\theta^3$, where $u+v+w+\theta=0$, the theory of norms will afford an instantaneous and, so to speak, intuitive demonstration of the respective related theorems, and the discriminant (*aliter* determinant) of each such function is decomposable into the sum of a square and a cube. Each of these forms is indeterminate, in either case there being but two relations fixed between the coefficients A, B, C; A, B, C, D; and we may easily establish the following singular species of algebraical porism. In the first case

$$(ABC)^2:(AB+AC+BC)^3,$$

and in the second case

$$(ABCD)^3:(\Sigma A^2B^2C^2-2ABCD\Sigma AB)^2$$

are *invariable ratios*.

† $f=-625$, $g=3125$.

$\alpha = bc$, $\beta = ac$, $\gamma = ab$, we see that the discriminant, when m is 0, will be of the form

$$s_2^5 + fs_2^2 s_3^2 + gs_3^3 s_1.$$

But when m is 0, H_4 vanishes, and there is no term s_1 or s_3 in H_2. Hence evidently the discriminant H_{10} just found cannot be dependent on H_2, H_4, or H_6; nor is it possible to make

$$H_{10} + pH_2^5 + qH_2^2 H_6,$$

that is,

$$(p+1)\,s_2^5 + fs_2^2 s_3^2 + gs_3^3 s_1$$

a perfect square on account of g not vanishing; so there is no H_8 upon which H_{10} can depend. Hence, admitting, as there seems every reason to do, that the number of invariants of a function of x, y of the degree m is $m-2$, we find that the four invariants in the case of the first degree are respectively of the second, fourth, sixth, and tenth dimensions, a determination in itself, as a step to the completion of the theory of invariants, of no minor importance.

But it seems hopeless by means of these forms to arrive at the desired canonical reduction. The forms, however, of H_2, H_4, H_6 are very remarkable as not rising above the first, first and second degrees respectively in s_1, s_2, s_3. Also H_4 vanishes when $m=0$ and H_4 has been obtained by putting

$$ax^6 + by^6 + cz^6 + 90mx^2y^2z^2$$

under the form of

$$Ax^6 + 6Bx^5y + 15Cx^4y^2 + 20Dx^3y^3 + 15Ex^2y^4 + 6Fxy^5 + Gy^6,$$

and taking the determinant

$$\begin{vmatrix} A & B & C & D \\ B & C & D & E \\ C & D & E & F \\ D & E & F & G \end{vmatrix}.$$

Consequently *in general* the vanishing of the above-written determinant will express the condition that a function of the sixth degree may be decomposable into three sixth powers. This also is true more generally. If $F(x, y)$ be a function of $2i$ dimensions, the vanishing of the resultant in respect to x^i, $x^{i-1}y \ldots y^i$ (taken dialytically) of

$$\left(\frac{d}{dx}\right)^i F,\ \left(\frac{d}{dx}\right)^{i-1}\frac{d}{dy}F \ldots \left(\frac{d}{dy}\right)^i F$$

will indicate that F admits of being decomposed into i powers of linear functions of x, y*.

In consequence of the greater interest, at least to the author, of the preceding investigations, I have delayed the insertion of the promised continuation of my paper on extensions of the dialytic method, which will

* Such a function so decomposable may be termed meio-catalectic. Meio-catalecticism for even-degreed functions is the analogue of singularity for odd-degreed functions.

appear in a subsequent Number. I take this opportunity of correcting a trifling slip of the pen which occurs towards the end* of the paper alluded to. The values of $\frac{x}{z}$ and $\frac{y}{z}$ become zero, and not infinite, when $N=0$; and the antepenultimate paragraph should end with the words "an incomplete resultant." The theorem also, in the last paragraph but one, should be stated more distinctly as subject to an important exception as follows.

Whenever the resultant of a system of equations $F=0$, $G=0$, &c. contains a factor R'^m, this will indicate that, on making $R'=0$, the given system of equations will admit of being satisfied by m algebraically distinct systems of values of the variables, except in those cases where there is a singularity in the forms of F, G, &c., taken either separately, or in partial combination with one another. An example will serve to make the meaning of the exception apparent. Let F, G, H denote three quadratic equations in x and y, so that $F=0$, $G=0$, $H=0$ may be conceived as representing three conic sections. Let R be the resultant of F, G, H, and suppose the relations of the coefficients in F, G, H to be such that $R = R'^2$; then $R'=0$ will imply the existence of one or the other of the three following conditions: namely, either that the three conics have a chord in common, which is the most general inference; or, which is less general, that two of the conics touch one another; or, which is the most special case of all, that one of the conics is a pair of right lines.

So, again, if we have two equations in x, and their resultant contains F^2, this may arise either from one of the functions containing a square factor, or from their being susceptible, on instituting one further condition, namely of $F=0$, of having a quadratic factor in common between them.

P.S. The conjecture made in the preceding pages has been since confirmed by the discovery of a modification in the canonical form applicable to functions of the sixth degree, which simplifies the theory in a remarkable manner. Assume $f(x, y)$, a function of the sixth degree, as equal to

$$au^6 + bv^6 + cw^6 \pm muvw\,(u-v)(v-w)(w-u),$$

where u, v, w, linear functions of x and y, satisfy the equation

$$u + v + w = 0;$$

then will the product of uvw be capable of being determined by means of the solution of a quadratic equation, of the square root of whose roots the coefficients of uvw will be known linear functions. Thus by an affected quadratic, a pure quadratic, and a cubic equation, the values of u, v, w may be completely ascertained. The discussion of this theory, and of a general inverse method for assigning the true (in the sense of the most manageable) Canonical Form for functions of any even degree, will form the subject of a subsequent communication.

[* p. 264 above.]

42.

ON THE PRINCIPLES OF THE CALCULUS OF FORMS.

[*Cambridge and Dublin Mathematical Journal*, VII. (1852), pp. 52—97.]

PART I. GENERATION OF FORMS*.

SECTION I. *On Simple Concomitance.*

THE primary object of the Calculus of Forms is the determination of the properties of Rational Integral Homogeneous Functions or systems of functions: this is effected by means of transformation; but to effect such transformation experience has shown that forms or form-systems must be contemplated not merely as they are in themselves, but with reference to the ensemble of forms capable of being derived from them, and which constitute as it were an unseen atmosphere around them. The first part of this essay will therefore be devoted to the theory of the external relations of forms or form-systems; the second part to the analysis of forms: that is to say, the first part will treat of the Generation and affinities, and the second part of the Reduction and equivalences of forms.

In its most crude and absolute, or, so to speak, archetypal condition a Rational Integral Homogeneous Function may be regarded as a linear function of several distinct and perfectly independent classes of variables.

* It may be well at the outset to give notice to my readers of the exact meaning to be attached to the following terms:

1. The linear-transformations are supposed to be always taken such that the modulus, that is, the determinant of the coefficients of transformation, is unity; or, as it may be phrased, the transformations are uni-modular.

2. The word Determinant is restricted in all cases to signify the alternate function formed in the usual manner from a group of quantities arranged in square order.

3. The word *Discriminant* (typified by the prefix-symbol □) is used to denote the determinant (usually but most perplexingly so called) of a homogeneous function of variables.

4. The resultant of two or more homogeneous functions of as many variables is the left-hand side of the final equation (in its *complete form* and free from extraneous factors) which results from eliminating the variables between the equations obtained by making each of the functions zero.

The first step towards the limitation of this very general but necessary conception consists in imagining the total number of classes to become segregated into groups, and certain correspondences to obtain between the variables of a class in any group with some the variables in each other class of the same group. The investigations in this and the subsequent section will be confined exclusively to the theory of functions where the several classes of variables, if more than one, all belong to a single group, so that the variables in one class have each their respective correspondents in the remaining classes. Such a group may again be conceived to become subdivided into sets each of the same number of variables, and the corresponding variables in the different sets to become absolutely identical. This leads to the conception of a homogeneous function of related classes of variables of various degrees of exponency in respect to the several classes. The relation of the different classes, if containing the same number of variables (in which case the relation may be termed Simple) will be understood to be defined by their being simultaneously subject to similar or contrary operations of linear substitution; so that, for example, if x, y, z; ξ, η, ζ are two such classes, when x, y, z are replaced by $ax+by+cz$, $a'x+b'y+c'z$, $a''x+b''y+c''z$, respectively, ξ, η, ζ will be, according to the *species* of the relation, subject to be at the same time replaced either by $a\xi+b\eta+c\zeta$, $a'\xi+b'\eta+c'\zeta$, $a''\xi+b''\eta+c''\zeta$, or otherwise by $\alpha\xi+\beta\eta+\gamma\zeta$, $\alpha'\xi+\beta'\eta+\gamma'\zeta$, $\alpha''\xi+\beta''\eta+\gamma''\zeta$, where

$$\alpha = \begin{vmatrix} 1 & 0 & 0 \\ 0 & b' & c' \\ 0 & b'' & c'' \end{vmatrix} \qquad \beta = \begin{vmatrix} 0 & 1 & 0 \\ a' & 0 & c' \\ a'' & 0 & c'' \end{vmatrix} \qquad \gamma = \begin{vmatrix} 0 & 0 & 1 \\ a' & b' & 0 \\ a'' & b'' & 0 \end{vmatrix}$$

&c. &c. &c.*

On the former supposition the related classes $x, y, z, \xi, \eta, \zeta$ will be said to be cogredient, and on the latter supposition contragredient †. If now we have one or more functions of classes of variables so related ‡, such function or system of functions may have associated with it a concomitant, also made up of distinct but related classes of variables, such classes being capable of being either greater or fewer in number than the classes of the given function or system of functions.

In the primitive function or system, as also in the concomitant, the related classes may be all of the same species, or some of one and the others of the contrary species. Even if we limit ourselves to the conception of a

* See my paper in the previous number of this *Journal* [p. 199 above.]

† The germ of the notion of contragredience will be found in the immortal *Arithmetic* of the great and venerable Gauss.

‡ The relation here spoken of will be observed to be of a *dynamical* character, not referring to the systems as they are in themselves, but to the movements to which they are simultaneously subject.

primitive function or system of functions with only one class of variables, its concomitant may be composed of various classes of variables, in respect to some of which it will be covariant with, and in respect to the others contravariant to, the primitive function or system*. This is an immense and most important extension of the conception of a concomitant given in my preceding paper in this *Journal*, and will be shown to have the effect of reducing the whole existing theory under subjection to certain simple abstract and universal laws of operation.

The relation of concomitance is purely of form. A being a given form, B is its concomitant, when A' being derived from A by simultaneous substitutions impressed upon the class of variables or upon each of the classes (if there be more than one) in A, and B' from B by corresponding (coincident or contrary) substitutions impressed upon the class or classes of variables in B, B' is capable of being derived from A' after the same law as B from A; or, as it may be otherwise expressed, "functions are concomitant when their correlated linear derivatives are homogeneous in point of form †."

This definition implies that one at least of the forms must be the most general possible of its kind: in a secondary but very important sense, however, functions obtained by impressing particular values or relations upon the quantities entering into the primitive and its associate form, will still be called concomitant. Thus $x^3 - y^3$ will be termed a concomitant to $x^3 + y^3$, not that we can affirm that $(ax + by)^3 - (cx + dy)^3$:

that is $\quad (a^3 - c^3)\,x^3 + 3\,(a^2b - c^2d)\,x^2y + 3\,(ab^2 - cd^2)\,xy^2 + (b^3 - d^3)\,y^3$,

treated as a function of x and y, can be derived from $(ax + by)^3 + (cx + dy)^3$,

that is $\quad (a^3 + c^3)\,x^3 + 3\,(a^2b + c^2d)\,x^2y + 3\,(ab^2 + cd^2)\,xy^2 + (b^3 + d^3)\,y^3$,

when $ad - bc = 1$ by the same law as $(x^3 - y^3)$ from $(x^3 + y^3)$, for the elements for forming such comparison are wanting, but because $x^3 + y^3$ and $x^3 - y^3$ are the correspondent particular values respectively assumed by

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$

and its concomitant

$$(ad^2 + 2c^3 - 3bcd)\,x^3 - (6b^2d - 3c^2b - 3acd)\,x^2y$$
$$+ (6ac^2 - 3cb^2 - 3cba)\,xy^2 - (a^2d + 2b^3 - 3bca)\,y^3,$$

when $\qquad a = 1, \quad b = 0, \quad c = 0, \quad d = 1.$

With the aid of this extended signification of the term concomitant (whether it be a covariant or contravariant) we can in all cases speak (as otherwise we in general could not) of the concomitant of a concomitant. The relation

between systems of variables has been stated to be Simple (whether they be cogredient or contragredient) when each variable in one system corresponds with some one in each other. Compound relation arises as follows:—Suppose $x, y;\ \xi, \eta$ two independent systems of two variables each, and that the system of four variables u, v, w, t is subject to linear variations imitating, in the way of cogredience or contragredience, those to which $x\xi, x\eta, y\xi, y\eta$ are subject; then u, v, w, t may be said to be cogredient or contragredient to the continued systems $x, y;\ \xi, \eta$. If $x, y;\ \xi, \eta$ be themselves cogredient, then a system of only three variables u, v, w, may be cogredient or contragredient in respect to $x\xi,\ x\eta + y\xi,\ y\eta$, and if $x, y;\ \xi, \eta$ be coincident, u, v, w may be similarly related to x^2, xy, y^2. The illustration may easily be generalized, and it will be seen in the sequel that its conception of compound-relation between systems of a differing number of variables will greatly extend the power and application of the methods about to be developed. Without having recourse to a formal definition, it is obvious that the notion of a concomitant conveyed in my former paper in this *Journal* lends itself without difficulty to the most general supposition which can be made of functions between which any number of systems of related variables are distributed, whatever such relation be, whether simple or compound, and whether of cogredience or of contragredience. The proposition stated in my last paper relative to a concomitant of the concomitant of a function being a concomitant of the original still applies to concomitants in the wider sense in which we now understand that term, and the species of each system of variables in the second concomitant with respect to the species or either species (if there be systems of both kinds in the primitive) will be determined upon the general principle which determines the effect of concurrence and contrariety being made to operate each upon itself or one in either order upon the other.

The highest law and the most powerful in its applications which I have yet discovered in the theory of concomitants may be expressed by affirming that when several related classes of variables are present in any concomitant, a new concomitant, *derived from the former by treating one or any number of these classes as independent of the remaining classes*, will still be a concomitant of the primitive. I shall quote this hereafter as the Law of Succession. This law, to which I have been led up inductively, requires an extended examination and a rigorous proof. It is the keystone of the subject, and any one who should suppose that it is a self-evident proposition (as from the simplicity of the enunciation it might be supposed to be) will commit no slight error.

If $\phi(x, y \dots z)$ be any homogeneous form of function of $x, y, \dots z$, every homogeneous sum in the expansion by Taylor's theorem of

$$\phi\,(u + u',\ v + v' \dots w + w'),$$

* And of course the concomitant may be an invariant to its originant in respect of one or more systems of variables entering into the former.

† Or, more generally, it may be said that concomitance consists in the persistence of morphological affinity.

which in fact, on making $u'=x$, $v'=y \ldots w'=z$, becomes identical (to a numerical factor *près*) with $\left(u\frac{d}{dx}+v\frac{d}{dy}+w\frac{d}{dz}\right)^i\phi$, is what I have elsewhere termed an Emanant, and by a partial method I had demonstrated that every invariant of such an emanant in respect to $u, v \ldots w$, in which $x, y \ldots z$ are treated as constants, or *vice versâ*, would give a covariant of ϕ. The reason of this is now apparent. For it may easily be shown* that every emanant is in fact itself a covariant of the function to which it belongs with respect to each of the related classes of variables which enter into it, or is as it may be termed a double covariant. The law of Succession shows therefore that a concomitant to an emanant from which one of the classes has disappeared will be a covariant of the primitive in respect to the remaining class.

In applying the law of Succession, great use can be made of a function of two classes of letters which may be termed a Universal Mixed Concomitant; this is $x\xi+y\eta+\ldots+z\zeta$, which has the property of remaining unaltered when any linear substitution (for which the modulus is unity) is impressed upon $x, y \ldots z$, and the contrary one upon $\xi, \eta \ldots \zeta$†.

If $f(x, y)$ be any function of x, y, of the degree m, $f+\lambda(x\xi+y\eta)^m$ will

* To demonstrate this it is only necessary to observe that if $u, v, \ldots w, u', v', \ldots w'$ be cogredient with themselves and with $x, y, \ldots z$,

$$\phi(u+\lambda u', v+\lambda v', \ldots w+\lambda w')$$

will evidently be a concomitant of $\phi(x, y, \ldots z)$; and, λ being arbitrary, the coefficients of the different powers of λ must be separately concomitants of $\phi(x, y, \ldots z)$, but these coefficients are the emanants of ϕ. Q. E. D.

† Thus, if

$$\begin{aligned} x&=ax'+by'+cz', & \xi&=(gn-hm)\xi'+(hl-fn)\eta'+(fm-gl)\zeta',\\ y&=fx'+gy'+hz', & \eta&=(-nb+mc)\xi'+(-lc+na)\eta'+(-ma+lb)\zeta',\\ z&=lx'+my'+nz', & \zeta&=(bh-cg)\xi'+(cf-ah)\eta'+(ag-bf)\zeta', \end{aligned}$$

then
$$x\xi+y\eta+z\zeta=\begin{pmatrix} a & b & c\\ f & g & h\\ l & m & n\end{pmatrix}\times(x'\xi'+y'\eta'+z'\zeta')$$
$$=x'\xi'+y'\eta'+z'\zeta'.$$

When the coefficients of transformation correspond to the direction-cosines between one system of rectangular axes and another, the reciprocal system is identical with the direct system; so that $x, y, z; \xi, \eta, \zeta$, on this particular supposition, may be regarded indifferently as contragredient or as cogredient; accordingly they may be made identical, and then $x^2+y^2+z^2$ remains invariable, which is the well-known characteristic of orthogonal transformation. It may be observed here that there exists a special theory of concomitance limited to such species of linear transformations, which may be termed Conditional Concomitance, and I have found in several cases that the invariants of conditional concomitants turn out to be absolute invariants of the primitive. Much more important is the remark that there exists a theory of universal concomitants for an indefinite number instead of merely two systems of variables, as used in the text. In the sequel it will be seen that the application of this universal concomitant (like the touch of an enchanter's wand) serves to transmute covariants into contravariants, and back again, and causes single invariants to germinate and fructify into complete connected systems of forms.

be a mixed concomitant of f, it being evident that every function of concomitants of a function is itself a concomitant of the same.

Suppose now
$$f=ax^m+mbx^{m-1}y+\tfrac{1}{2}m(m-1)cx^{m-2}y^2+\&c.,$$
the concomitant becomes
$$(a+\lambda\xi^m)x^m+m(b+\lambda\xi^{m-1}\eta)x^{m-1}y+\tfrac{1}{2}m(m-1)(c+\lambda\xi^{m-2}\eta^2)+\&c.$$
Consequently if P be any concomitant of f, P' obtained from P by writing $a+\lambda\xi^m$, $b+\lambda\xi^{m-1}\eta$, &c. for a, b, &c., will still be a concomitant of f; and by Taylor's theorem P' evidently equals

$$P+\left(\xi^m\frac{d}{da}+\xi^{m-1}\eta\frac{d}{db}+\&c.\right)P$$
$$+\frac{1}{1.2}\left(\xi^m\frac{d}{da}+\xi^{m-1}\eta\frac{d}{db}+\&c.\right)^2P$$
$$+\&c.$$

If we take P an invariant of f, we have M. Hermite's theorem* for $f(x, y)$, and precisely the same demonstration applies to the general case of $f(x, y \ldots z)$. P' is, by virtue of the general rule, a contravariant of f in respect to $\xi, \eta \ldots \zeta$: if P be taken a function containing one single system, and is also a contravariant to f in respect to that system, P' will be a double contravariant; and if we make the two systems in P' identical, we have the extension of M. Hermite's theorem alluded to by me in one of the notes† to my last paper, wherein I have stated that "I may be taken any *covariant* of the function": as regards the purpose of that statement, the word covariant was used in error for contravariant.

The preceding method may be viewed as a particular application of the general principle, that if $U_1, U_2 \ldots U_m$ be any m functions (whether concomitants any of them of the others or not), then any concomitant of $\lambda_1U_1+\lambda_2U_2+\ldots+\lambda_mU_m$ being expressed as a function of $\lambda_1, \lambda_2 \ldots \lambda_m$, every coefficient in such expression will be a concomitant of the system $U_1, U_2 \ldots U_m$. Thus, for example, if U and V be two quadratic functions of n variables $x, y \ldots z$, the *discriminant* $\square(\lambda U+\mu V)$ will contain $n+1$ terms, of which the coefficients of the first and last will be $\square U$ and $\square V$; and every one of the $(n+1)$ coefficients will be a concomitant (of course an invariant) of U and V. These $(n+1)$ invariants will in fact constitute the fundamental scale of invariants to the system U and V, and every other invariant of U

* This theorem was first stated to me by Mr Cayley, who, I understand, derived it from M. Eisenstein, under the form of a theorem of covariants, which of course it becomes on interchanging x, y with $-y, x$. But as a theorem of covariants it could not be extended to functions of more than two variables. M. Hermite appears to have discovered this theorem, under its more eligible form, subsequently to, but independently of, M. Eisenstein.

[† p. 201 above, note *.]

and V will be an explicit rational function of the $(n+1)$ terms of the scale. In connexion with this principle may be stated another relative to any system of homogeneous functions of a greater number of variables of the same class, namely, that if any set of the variables one less in number than the number of the functions be selected at will, and any invariant of a given kind be taken of the resultant of the functions in respect to the variables selected, all such invariants so formed will have an integral factor in common, and this common factor will be an invariant of the given system of functions.

It will be convenient to speak hereafter of systems for which the march of the linear substitutions is coincident as cogredient, and those for which the march is contrary as contragredient systems.

Suppose m cogredient classes of m variables, the determinant formed by writing the $m \times m$ quantities in square order will evidently be a universal covariant. Thus, take the two systems x, y; ξ, η. $x\eta - y\xi$ is a universal covariant, and evidently therefore F, which I use to denote

$$\phi(x, y) \times \phi(\xi, \eta) + \lambda(x\eta - y\xi)^m,$$

will be a covariant to $\phi(x, y)$. Let $\phi(x, y)$ be of m dimensions; any invariant of F will be an invariant of ϕ; thus, let the two systems x, y; ξ, η be treated as perfectly independent, and take the discriminant of F (viewed as a function of x, y; ξ, η), that is the resultant of the four functions $\frac{dF}{dx}, \frac{dF}{dy}, \frac{dF}{d\xi}, \frac{dF}{d\eta}$; this resultant will be an invariant of ϕ; and λ being arbitrary, all the coefficients of its different powers will be invariants of ϕ. We thus fall upon another theorem of M. Hermite, namely that if $\lambda = \frac{\phi(x, y) \times \phi(\xi, \eta)}{(x\xi - y\eta)^m}$, the coefficients of the equation which will give the minimum values of λ are invariants of ϕ. So more generally, any invariant of $f(x, y, \xi, \eta) - \lambda(x\xi - y\eta)^m$, f being of the degree m in x, y and in ξ, η, will be an invariant of f; and among other invariants may be taken the discriminant obtained by treating x, ξ, y, η as absolutely unrelated.

If f be a function of various classes each containing n covariables, and if not less than n of these classes be covariable classes, and after selecting at will any n of such systems, as $x_1, y_1 \ldots z_1$; $x_2, y_2 \ldots z_2$; ...... $x_n, y_n \ldots z_n$, the symbolical determinant

$$\left|\begin{matrix} \frac{d}{dx_1}, & \frac{d}{dy_1} & \cdots & \frac{d}{dz_1} \\ \frac{d}{dx_2}, & \frac{d}{dy_2} & \cdots & \frac{d}{dz_2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{d}{dx_n}, & \frac{d}{dy_n} & \cdots & \frac{d}{dz_n} \end{matrix}\right|$$

be expanded and written equal to D, then $D^{\iota} f$ will be a concomitant of f; and, more generally, by selecting different combinations of the covariable systems n and n together in every way possible, and forming corresponding symbols of operation $E, F \ldots H$, we shall have $D^{\iota} . E^{\iota'} \ldots H^{(\iota)} . f$, for all values of $\iota, \iota' \ldots (\iota)$, a covariant of f in respect to the classes so combined. This explains and contains the whole pith and marrow of Mr Cayley's simple but admirable method of obtaining covariants and invariants (or, as termed by their author, hyperdeterminants) to a function ϕ_1 of a single system $x_1, y_1 \ldots z_1$; he forms similar functions $\phi_2 \ldots \phi_\mu$ of $x_2, y_2 \ldots z_2$; ... $x_\mu, y_\mu \ldots z_\mu$, and uses the product $\phi_1 \times \phi_2 \times \ldots \times \phi_\mu$ as a function f of μ systems: the multiple covariant obtained by operating thereupon becomes a simple covariant on identifying the different classes of covariables introduced in the procedure.

SECTION II. *On Complex Concomitance.*

We have hitherto been engaged in considering only a particular case of concomitance, the true idea of which relates not to an individual associated form (as such), but to a complex of forms capable of degenerating into an individual form. Such a complex may be called a Plexus. A plexus of forms is concomitant to a given form or combination of forms under the following circumstances.

If (O) be the originant, meaning thereby the primitive form or system of forms, and P the concomitant plexus made up of the μ forms $P_1, P_2 \ldots P_\mu$, and if, when by duly related linear substitutions, O becomes O', the plexus P becomes P', made up of the forms $P'_1, P'_2 \ldots P'_\mu$, and if the plexus $'P$ formed from O' after the same law as P from O be made up of the forms $'P_1, 'P_2 \ldots 'P_\mu$, then will each form in either of the plexuses $'P, P'$ be a linear function of all the forms in the other plexus, and the connecting constants in every such linear function will be functions of the coefficients of the substitution whereby O and P have become transformed into O' and P'.

A function forming part of a concomitant plexus may be termed a concomitantive. Concomitantives therefore usually have a joint relation to a common plexus and a concomitant is only another name for an unique concomitantive. Every plexus contains a definite number of concomitantives; in place of any one of these may be substituted an arbitrary linear function of all the rest, but the total number of independent forms *sufficient and necessary* to make the complete plexus respond to the requirements of the definition will remain constant.

If now we combine together the whole number of functions contained in one or more plexuses concomitant to any given originant, all of the same degree relative to any given selected system or systems of variables, and if the number of the concomitantives so combined be exactly equal to the number of terms in each, arranged as a function of the selected class or classes of variables, then the dialytic resultant (obtained by treating each combination of the selected variables as an independent variable, and forming a determinant in the usual manner), will be a concomitant to the given originant. This, which is only the partial expansion of some much higher law, may be termed the "Law of Synthesis."

Let f be any function of a single class of variables $x_1, x_2 \dots x_n$. Let χ represent any product of these variables or of their several powers of any given degree r; the number of different values of χ will be μ, where

$$\mu = \frac{n(n+1)\dots(n+r-1)}{1.2\dots r},$$

and $\chi_1 f, \chi_2 f \dots \chi_\mu f$ will form a covariantive plexus to f.

Again, let ϑ represent any product of the degree r of the symbols

$$\frac{d}{dx_1}, \frac{d}{dx_2} \dots \frac{d}{dx_n};$$

$\vartheta_1 f, \vartheta_2 f \dots \vartheta_\mu f$ will also form a covariant plexus to f.

The coefficients of connexion between the forms of either plexus depend in an analogous manner upon the coefficients of the substitution supposed to be impressed upon the variables, with the sole difference that every coefficient taken from the line r and column s of the determinant of substitution which appears in any coefficient of connexion of the one plexus is replaced by the coefficient taken from the line s and the column r in the corresponding coefficient of connexion for the other plexus.

Let $f(x, y)$ be any function of x, y of the degree $2m$; then

$$\left(\frac{d}{dx}\right)^m, \left(\frac{d}{dx}\right)^{m-1} \frac{d}{dy}, \dots\dots \left(\frac{d}{dy}\right)^m$$

will form a covariantive plexus; thus, suppose

$$f(x, y) = a_1 x^{2m} + 2m a_2 x^{2m-1} y + \dots + a_{2m+1} y^{2m};$$

omitting numerical factors, the plexus will be composed of the $(m+1)$ lines following:

$$\begin{array}{l} a_1 x^m + m a_2 x^{m-1} y + \dots + a_{m+1} y^m, \\ a_2 x^m + m a_3 x^{m-1} y + \dots + a_{m+2} y^m, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m+1} x^m + m a_{m+2} x^{m-1} y + \dots + a_{2m+1} y^m, \end{array}$$

and consequently, by the law of synthesis, the determinant

$$\begin{vmatrix} a_1, & a_2 \dots\dots a_{m+1} \\ a_2, & a_3 \dots\dots a_{m+2} \\ \dots\dots & \dots\dots\dots\dots \\ a_{m+1}, & a_{m+2} \dots a_{2m+1} \end{vmatrix}$$

is an invariant of f.

When this determinant is zero, I have proved in my paper* on Canonical Forms, in the *Philosophical Magazine* for November last, that f is resoluble into the sum of m powers of linear functions of x and y. I shall hereafter refer to a determinant formed in this manner from the coefficients of f as its catalecticant. Mr Cayley was, I believe, the first to observe that all catalecticants† are invariants.

Again, more generally, let $f(x, y, \xi, \eta)$ be a function of the mth degree of x, y, and of a like degree in respect of ξ, η, which are supposed to be cogredient with x and y; then

$$f(x, y, \xi, \eta) + \lambda(x\eta - y\xi)^m$$

(say F) will be a concomitant of f; and therefore if we take the system

$$\left(\frac{d}{dx}\right)^m F, \left(\frac{d}{dx}\right)^{m-1} \frac{d}{dy} F \dots\dots \left(\frac{d}{dy}\right)^m F,$$

which will be functions of ξ and η alone, and take their resultant, this resultant will be an invariant of f. As a particular case of this theorem, let

$$f = \left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^m \phi,$$

where ϕ is supposed to be a function of x and y only and of $2m$ dimensions, f is a concomitant of ϕ, and therefore the invariant of f, obtained in the manner just explained, will be an invariant of ϕ. Thus then we have an instantaneous demonstration of the theorem given‡ by me in the paper of the *Philosophical Magazine* before named, namely, if

$$\phi(x, y) = a_1 x^{2m} + 2m a_2 x^{2m-1} y + \dots + a_{2m+1} y^{2m},$$

say, in order to fix the ideas, $= ax^6 + 6bx^5 y + 15cx^4 y^2 + \dots + gy^6$; then the determinant

$$\begin{vmatrix} a, & b, & c, & d+\lambda \\ b, & c, & d-\frac{1}{3}\lambda, & e \\ c, & d+\frac{1}{3}\lambda, & e, & f \\ d-\lambda, & e, & f, & g \end{vmatrix},$$

(and the analogously formed determinant for the general case) will be an invariant of ϕ. The general determinant so formed is peculiarly interesting, because it furnishes when equated to zero the one sole equation necessary to be solved in order to be able to effect the reduction of $\phi(x, y)$ to its canonical form, and gives the means, irrespective of any other view of the theory of invariants, of determining completely and absolutely the condition

[* See p. 282 above.]

† But the catalecticant of the biquadratic function of x, y was first brought into notice as an invariant by Mr Boole; and the discriminant of the quadratic function of x, y is identical with its catalecticant, as also with its Hessian. Meicatalecticizant would more completely express the meaning of that which, for the sake of brevity, I denominate the catalecticant.

[‡ p. 277 above.]

of the possibility of two given functions of the same degree of x, y being linearly transformable one into the other. This theorem will be obtained in a more general manner in the following section. I only pause now to make the very important observation, that not only is the determinant an invariant, but every minor system* of determinants that can be formed from it (there are of course m such systems) is an invariantive plexus to the given function ϕ.

The form under which this theorem presents itself suggests a theorem vastly more general and of peculiar interest, as showing a connexion between the theory of functions of a certain degree and of a certain number of variables with other functions of a lower degree but of a greater number of variables. Here again, under a different aspect, is reproduced the great principle of dialysis, which, originally discovered in the theory of elimination, in one shape or another pervades the whole theory of concomitance and invariants.

Let ϕ represent any function of the degree pq (of any number, or, to fix the ideas, say of three variables x, y, z); let the general term of ϕ be represented by

$$\frac{pq\,(pq-1)\ldots 1}{(1.2\ldots\alpha)(1.2\ldots\beta)(1.2\ldots\gamma)}(\alpha,\beta,\gamma)\,x^\alpha y^\beta z^\gamma,$$

where $\alpha+\beta+\gamma=pq$, and (α, β, γ) represents a portion of the coefficient of $x^\alpha y^\beta z^\gamma$.

Let

$$\frac{1.2\ldots p}{(1.2\ldots r)(1.2\ldots s)(1.2\ldots t)}\,x^r y^s z^t=\theta_{r,s,t},$$

where $r+s+t=p$, so that there are as many θ's as there are modes of

* These minor systems mean as follows:—the system of rth minors comprises all the distinct determinants that can be got by striking out from the square array (which I call the Matrix) from which the complete determinant is formed, any r lines and any r columns selected at will. The last, or mth minor, is of course a system consisting of the coefficients of $\phi(x, y)$, and it is evident that if $\phi(x, y \ldots z)$ be any function of any number of variables $x, y \ldots z$, the coefficients will form an invariantive plexus to ϕ.

The following remark as to the changes undergone by the coefficients of ϕ when the variables undergo any substitution, is not without interest and importance for the theory.

Let x become $fx+f'y+\ldots+(f)z$,

y .... $gx+g'y+\ldots+(g)z$,

....

z .... $hx+h'y+\ldots+(h)z$.

Then the coefficient of the highest power of x becomes

$$\phi(f, g\ldots h),$$

and the coefficient of the term containing $y^r \ldots z^s$ becomes

$$\left(f'\frac{d}{df}+g'\frac{d}{dg}+\ldots+h'\frac{d}{dh}\right)^r\times\&c.\times\left\{(f)\frac{d}{df}+(g)\frac{d}{dg}+\ldots+(h)\frac{d}{dh}\right\}^s\phi(f, g\ldots h).$$

subdividing p into three integral parts (zeros being admissible); that is $\frac{1}{6}(p+1)(p+2)(p+3)$. Then any product such as $x^\alpha y^\beta z^\gamma$ may be divided in a variety of ways into the product of q of these θ's, and it may be shown that the entire quantity

$$\frac{pq\,(pq-1)\ldots 1}{(1.2\ldots\alpha)(1.2\ldots\beta)(1.2\ldots\gamma)}(x^\alpha y^\beta z^\gamma)$$

$$=\Sigma\left\{\frac{1.2\ldots q}{(1.2\ldots m_1)(1.2\ldots m_2)\ldots(1.2\ldots m_r)}(\theta_{\mu_1}{}^{m_1}\theta_{\mu_2}{}^{m_2}\ldots\theta_{\mu_r}{}^{m_r})\right\},$$

where $m_1+m_2+\ldots+m_r=q$. Consequently ϕ may be represented under the form of a function of the degree q of $\frac{1}{6}(p+1)(p+2)(p+3)$ (say ι) variables $\theta_1, \theta_2\ldots\theta_\iota$, and its general term will be of the form

$$\frac{1.2\ldots q}{(1.2\ldots m_1)(1.2\ldots m_2)\ldots(1.2\ldots m_r)}(\alpha,\beta,\gamma)\{\theta^{m_1}\theta^{m_2}\ldots\theta^{m_r}\},$$

where α, β, γ are the indices respectively of x, y, z, when the last factor is expressed as a function of these variables*. Now if ϑ be used to denote this new representation of ϕ when $\theta_1, \theta_2\ldots\theta_\iota$ are treated as absolutely independent variables, and if we attach to it any universal concomitant, as $(x\xi+y\eta+z\zeta)^p$ admitting of being written under the form $\omega(\theta_1, \theta_2\ldots\theta_\iota)$, wherein the coefficients will be functions of ξ, η, ζ; then any invariant to ϑ and ω, treated as two systems of ι variables, will be a concomitant to ϕ, the original function in x, y, z†. ϑ and ω may be termed respectively, for facility of reference, the Particular and Absolute functions. Thus, for example, we take ϕ a function of x, y of the degree $4n$, say

$$a_1x^{4n}+4na_2x^{4n-1}y+\&c.+a_{4n+1}y^{4n},$$

and make $p=2n$, $q=2$, so that ϑ becomes a quadratic function of $(2n+1)$ variables obtained by making $x^{2n}=\theta_1$, $x^{2n-1}y=\theta_2\ldots y^{2n}=\theta_{2n+1}$‡, and the concomitant ω, formed from $(\xi x+\eta y)^{2n}$, becomes

$$\theta_1\xi^{2n}+2n\theta_2\xi^{2n-1}\eta+\ldots+\theta_{2n+1}\eta^{2n};$$

then if we take R the quadratic invariant of ω, that is

$$R=\theta_1\theta_{2n+1}-2n\theta_2\theta_{2n}\ \&c.\pm\frac{1.2.3\ldots(2n)}{(1.2\ldots n)^2}\tfrac{1}{2}(\theta_{n+1})^2,$$

* See Note (1) in Appendix. [p. 322 below.]

† In fact ϑ is a concomitant to ϕ, and ω to a power of the universal concomitant; the θ's forming a system of variables cogredient with the compound system $x^{r_1}y^{s_1}z^{t_1}$, $x^{r_2}y^{s_2}z^{t_2}$, &c.: and it must be well observed that the same substitutions which render ϑ and ω respectively identical with ϕ and a power of the universal concomitant, would render an infinite number of other functions also coincident with the same; but none of these other functions would be concomitants. Herein we see the importance of the definition and conception of compound relation; the θ system being compound by relation with the x, y, z system, after the manner of cogredience.

‡ A slight variation upon the method as above explained for the general case has been here introduced inadvertently by writing $x^{2n-1}y=\theta_2$, &c., in lieu of $2nx^{2n-1}y=\theta_2$, &c., which, as it does not in any degree affect the reasoning, I have not deemed it worth while to alter.

it will readily be seen that the determinant of $\vartheta + \lambda R$, treated as a quadratic function of $(2n+1)$ variables, will give an invariant of ϕ, and this will be the same as that obtained by the particular method above given. Thus, suppose

$$\phi(x, y) = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4.$$

Let $$x^2 = \theta_1,\ \ 2xy = \theta_2,\ \ y^2 = \theta_3,$$

$$\vartheta = a\theta_1^2 + 2b\theta_1\theta_2 + c\theta_2^2 + 2c\theta_1\theta_3 + 2d\theta_2\theta_3 + e\theta_3^2,$$

$$\omega = (x\xi + y\eta)^2 = x^2\theta_1 + xy\theta_2 + y^2\theta_3,$$

$$R = \theta_1\theta_3 - \frac{\theta_2^2}{4}.$$

Then Λ the discriminant of $\vartheta + 2\lambda R$ in respect to $\theta_1, \theta_2, \theta_3$

$$= \begin{vmatrix} a, & b, & c+\lambda \\ b, & c-\frac{1}{2}\lambda, & d \\ c+\lambda, & d, & e \end{vmatrix},$$

and I may remark that the relations between the several transformees of the invariantive plexuses formed by the minor determinant systems of Λ (in this, and in general for the case of an evenly-even index) may be found by treating $\vartheta + 2\lambda R$ as a quadratic function of the variables (in this case $\theta_1, \theta_2, \theta_3$) and applying the rule given by me in the *Philosophical Magazine* in my* paper "On the relation between the Minor Determinants of linearly-equivalent Quadratic Forms."† This second method, however, is not immediately applicable to the case of indices oddly even, that is of the form $4n+2$, to which the first method applies, equally as to the case $4n$; for if we make $2n+1 = p$ and $q = 2$, ω being of an odd degree, has no quadratic invariant; it has however a quadratic covariant, which will be of the second degree in respect to $\theta_1, \theta_2 \ldots \theta_{p+1}$ as well as in respect to ξ, η; and if we call this R and take the discriminant of $\vartheta + \lambda R$ in respect to the variables $\theta_1, \theta_2 \ldots \theta_{p+1}$, we shall obtain, as I am indebted to a remark of my valued friend M. Hermite for bringing under my notice, a very beautiful and interesting function of λ, of which all the coefficients will be contravariants of ϕ. Thus, let

$$\phi = ax^6 + 6bx^5y + 15cx^4y^2 + 20dx^3y^3 + 15ex^2y^4 + 6fxy^5 + gy^6,$$

[* p. 241 above.]

† Moreover, upon the supposition made in the text, the particular and absolute functions ϑ and ω may be treated in all respects as if they were functions characterizing quadratic loci, and any singularity in their relation will correspond to and denote a singularity in the given function ϕ to which ϑ refers. Thus, for instance, if ϕ be a function of x, y of the eighth degree, ϑ and ω will be quadratic functions of five letters each. Quadratic loci have no other singularity of relation than what corresponds to different species of contact. The number of contacts between loci, characterized by 5 letters, is 24 (see my paper ‡ in the *Philosophical Magazine*, "On the contacts of lines and surfaces of the second order"). Consequently this mode of representing ϑ and ω will give rise to the discovery and specification of 24 different kinds of singularity in ϕ, and the analytical characteristics of each of them. But there of course may, and in fact will, exist other singularities in ϕ besides those which have their correspondencies in the relations of these quadratic concomitants. [‡ p. 237 above.]

make $$x^3 = \theta_1,\ \ 3x^2y = \theta_2,\ \ 3xy^2 = \theta_3,\ \ y^3 = \theta_4,$$

so that

$$\vartheta = a\theta_1^2 + 2b\theta_1\theta_2 + c\theta_2^2 + 2c\theta_1\theta_3 + 2d\theta_2\theta_3 + 2d\theta_1\theta_4 + g\theta_4^2 + 2f\theta_3\theta_4 + e\theta_3^2 + 2e\theta_4\theta_2,$$

$$\omega = (x\xi + y\eta)^3 = \theta_1\xi^3 + \theta_2\xi^2\eta + \theta_3\xi\eta^2 + \theta_4\eta^3,$$

$$R = \begin{vmatrix} 3\theta_1\xi + \theta_2\eta, & \theta_2\xi + \theta_3\eta \\ \theta_2\xi + \theta_3\eta, & \theta_3\xi + 3\theta_4\eta \end{vmatrix},$$

$$-R = \xi^2\theta_2^2 + \eta^2\theta_3^2 + \xi\eta\theta_2\theta_3 - 9\xi\eta\theta_1\theta_4 - 3\xi^2\theta_1\theta_3 - 3\eta^2\theta_2\theta_4.$$

Consequently the discriminant in respect to $\theta_1, \theta_2, \theta_3, \theta_4$ of $\vartheta - 2\lambda R$ becomes

$$\begin{vmatrix} a, & b, & c-3\lambda\xi^2, & d-9\lambda\xi\eta \\ b, & c+2\lambda\xi^2, & d+\lambda\xi\eta, & e-3\lambda\eta^2 \\ c-3\lambda\xi^2, & d+\lambda\xi\eta, & e+2\lambda\eta^2, & f \\ d-9\lambda\xi\eta, & e-3\lambda\eta^2, & f, & g \end{vmatrix}.$$

If this determinant be expanded as a function of λ, all the coefficients of the various powers of λ will be contravariants to the given function ϕ. The term involving λ^4 is zero. Let ξ become $-y$ and η become x, then the remaining terms (abstraction made of the powers of λ) become covariants of ϕ. The first term (the coefficient of λ^3) becomes ϕ itself; the last term is the catalecticant, and thus we see, in general, that for functions of x and y of an oddly-even degree, a whole series of covariants may be interpolated between the function and its catalecticant, the dimensions in respect of the coefficients of ϕ in arriving at each step increasing by 1 unit and the degree in respect of the variables diminishing by 2 units. This is consequently a much simpler and more available scale than one with which I have been long previously acquainted, and which applies alike to functions of any even degree.

Thus, let $\phi(x, y)$ be of $2k$ dimensions; form all the even emanants of ϕ, which will be all of the form $\left(\xi\frac{d}{dx} + \eta\frac{d}{dy}\right)^{2\iota}\phi$, and take their respective catalecticants in respect to ξ and η. We shall in this way obtain a regular scale of covariants interpolated between the Hessian of ϕ (corresponding to $\iota = 1$) and the catalecticant of ϕ (corresponding to $\iota = k$). If ϕ be of the degree $2k+1$, we shall have an analogous scale interpolated between the Hessian of ϕ and its canonizant; the latter term denoting the function which is the product of the $k+1$ linear functions of x and y, the sum of whose $(2k+1)$th powers is identically equal to ϕ*.

By means of the Theory of the Plexus we may obtain various representa-

* See Note (2) in Appendix. [p. 322 below.]

tions of the same invariant; thus, for example, if we take F a function of x, y of the fifth degree and form its Hessian H, that is

$$\begin{vmatrix} \frac{d^2F}{dx^2}, & \frac{d^2F}{dxdy} \\ \frac{d^2F}{dydx}, & \frac{d^2F}{dy^2} \end{vmatrix},$$

this will be a function of the sixth degree in x, y, and of the two orders in the coefficients. If we combine the two plexuses

$$\frac{dF}{dx}, \frac{dF}{dy}; \quad \frac{d^2H}{dx^2}, \frac{d^2H}{dxdy}, \frac{d^2H}{dy^2},$$

we shall have five equations between which x^4, x^3y, x^2y^2, xy^3, y^4 may be eliminated dialytically; the resultant will be of the $2+3.2$, that is the eighth order in the coefficients, and of the form $\square F - I_4^2$, where $\square F$ and I_4 are respectively the determinant and quintic invariant of F, each affected with a proper numerical multiplier (the "$B - A^2$" of my supplemental* essay on canonical forms) which, as Mr Cayley has remarked, may also be represented by the resultant of P; $\frac{dQ}{dx}$; $\frac{dQ}{dy}$ where P and Q are respectively the quadratic and cubic invariants in respect to ξ and η of $\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^4 F$.

It will be well at this point to recapitulate in brief a method of elimination applicable to certain systems of functions published by me many years since in the *Philosophical Magazine*, and to compare this method with that afforded by the theory of the plexus for finding an invariant for each of the very same systems, possessing all the external characters, formed in a precisely similar manner to, and not impossibly identical *with*, the resultant of every such system. I shall devote my first moments of leisure to the ascertainment of this last most important point, as to the identity or otherwise of the plexus-invariant with the resultant. Take the case of three functions of x, y, z (say ϕ, ψ, ω) each of the same degree n; to fix the ideas, suppose $n=3$: there are two purely algebraical processes (modifications of the same method and leading to identical results) by which the resultant of ϕ, ψ, ω may be found. I shall call these processes the first and second respectively.

First process: Write

$$\phi = x^2P + yQ + zR,$$
$$\psi = x^2P' + yQ' + zR',$$
$$\omega = x^2P'' + yQ'' + zR'',$$

decompositions which may be effected in an infinite variety of manners, so that P, Q, R shall be integer functions of x, y, z; take the linear resultant of ϕ, ψ, ω, in respect to x^2, y, z, which call $H_{2,1,1}$; this will evidently be

[* p. 205 above.]

of $9-4$, that is, of 5 dimensions. Form analogously the functions $H_{1,2,1}$, $H_{1,1,2}$; $H_{2,1,1}$, $H_{1,2,1}$, $H_{1,1,2}$ constitute an auxiliary system of functions which vanish when ϕ, ψ, ω vanish together; combine this auxiliary system with the augmentative system

$$x^2\phi, \; y^2\phi, \; z^2\phi, \; xy\phi, \; yz\phi, \; zx\phi,$$
$$x^2\omega, \; y^2\omega, \; z^2\omega, \; xy\omega, \; yz\omega, \; zx\omega,$$
$$x^2\psi, \; y^2\psi, \; z^2\psi, \; xy\psi, \; yz\psi, \; zx\psi.$$

We shall thus have in all $3+3\times 6$, that is, 21 functions into which the 21 terms x^5, x^4y, x^4z, &c. enter linearly: the linear resultant of these 21 functions is the resultant of ϕ, ψ, ω, clear of all extraneousness.

Second process: Write

$$\phi = x^3P + yQ + zR,$$
$$\psi = x^3P' + yQ' + zR',$$
$$\omega = x^3P'' + yQ'' + zR'',$$

and, as before, take the linear resultant $H_{3,1,1}$, which will however be of $9-5$, that is, of only 4 dimensions.

Again, take

$$\phi = x^2L + y^2M + zN,$$
$$\psi = x^2L' + y^2M' + zN',$$
$$\omega = x^2L'' + y^2M'' + zN'',$$

and form the determinant $H_{2,2,1}$; we shall thus have the auxiliary system

$$H_{3,1,1}, \; H_{1,3,1}, \; H_{1,1,3}, \; H_{2,2,1}, \; H_{2,1,2}, \; H_{1,2,2}.$$

Let this be combined with the augmentative system

$$x\omega, \; y\omega, \; z\omega; \; x\phi, \; y\phi, \; z\phi; \; x\psi, \; y\psi, \; z\psi.$$

Between these $6+9$, that is, 15 functions, the 15 terms x^4, x^3y, x^3z, &c. may be linearly eliminated, and the resultant thus obtained will be precisely the same as that got by the preceding process.

Here we have 6 auxiliaries and 6 augmentatives; the auxiliaries are of three dimensions in respect to the coefficients of ϕ, ψ, ω; the augmentatives of one dimension only; in the former process there were 3 auxiliaries and 18 augmentatives, $6\times 3+9 = 27 = 3\times 3+18$.

Now let this method be compared with the following:

First process: Take the 18 augmentatives $x^2\phi$, $x^2\omega$, $x^2\psi$, &c. as in the first process of the algebraical method above explained; but in place of the 3 auxiliaries therein given, take another system of 9 as follows:

Write the determinant

$$\begin{vmatrix} \frac{d\phi}{dx}, & \frac{d\phi}{dy}, & \frac{d\phi}{dz} \\ \frac{d\psi}{dx}, & \frac{d\psi}{dy}, & \frac{d\psi}{dz} \\ \frac{d\omega}{dx}, & \frac{d\omega}{dy}, & \frac{d\omega}{dz} \end{vmatrix} = R;$$

$\frac{dR}{dx}, \frac{dR}{dy}, \frac{dR}{dz}$ form a concomitantive plexus; the 18 augmentatives form another; the linear resultant of these two plexuses will be an invariant of ϕ, ψ, ω, and of precisely the same dimensions as the resultant last found; if they are not identical it will be indeed a matter of exceeding wonder, and even more interesting than if they should be proved so to be.

Second process: Combine the augmentative plexus

$$x\omega,\ y\omega,\ z\omega;\quad x\phi,\ y\phi,\ z\phi;\quad x\psi,\ y\psi,\ z\psi,$$

with the differential plexus

$$\frac{d^2R}{dx^2}, \frac{d^2R}{dxdy}, \frac{d^2R}{dy^2}, \frac{d^2R}{dydz}, \frac{d^2R}{dz^2}, \frac{d^2R}{dzdx},$$

we thus obtain a linear resultant in a manner precisely similar to that afforded by the second process of our algebraical method.

In general, if ϕ, ψ, ω be of the degrees n, n, n, as there are two algebraical varieties of the linear method for finding the resultant, so are there two varieties of the concomitantive method for finding the resembling invariant. In both methods the augmentatives are identical; the only difference being in the auxiliary system.

In the first process the augmentative system will be got by operating upon each of the functions ϕ, ψ, ω, with the multipliers $x^{n-1}, y^{n-1}, z^{n-1}$, and the other homogeneous products of x, y, z; the auxiliary system by operating upon R with the symbolical multipliers $\left(\frac{d}{dx}\right)^{n-2}, \left(\frac{d}{dy}\right)^{n-2}, \left(\frac{d}{dz}\right)^{n-2}$, and the other homogeneous products of $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ of the degree $n-2$.

In the second process the augmentative system is formed by the aid of the multipliers $x^{n-2}, y^{n-2}, z^{n-2}$, &c., and the auxiliary system by aid of

$$\left(\frac{d}{dx}\right)^{n-1}, \left(\frac{d}{dy}\right)^{n-1}, \left(\frac{d}{dz}\right)^{n-1}, \text{\&c.}$$

For the particular case of $n=2$ the first process of the concomitantive method is merely an application under its most symmetrical form of the first

process of the general algebraical method. The second process of the concomitantive method for this same case (at least when ϕ, ψ, ω are the partial differential coefficients of the same function of the third degree) has been shown by Dr Hesse to give the resultant, so that for this case, at all events, we know that each concomitantive auxiliary must be a linear function of the augmentatives and the algebraical auxiliaries.

Again, if we go to the system where ϕ, ψ, ω are of the respective degrees $n, n, n+1$. In the algebraical method (for applying which there are no longer two, but one only process), the augmentative system is obtained by multiplying ϕ by the homogeneous products of $x^{n-1}, x^{n-1}y, x^{n-1}z$, &c., ψ by the like products, and ω by the homogeneous products $x^{n-2}, x^{n-2}y$, &c. The auxiliary system is made up of functions of the general form

$$H_{p,q,r} \text{ where } p+q+r=n+2,$$

$H_{p,q,r}$ being the determinant obtained by writing

$$\phi = Lx^p + My^q + Nz^r,$$
$$\psi = L'x^p + M'y^q + N'z^r,$$
$$\omega = L''x^p + M''y^q + N''z^r.$$

And in like manner for the case of ϕ, ψ, ω, being of the respective degrees $n, n, n-1$, the augmentative system is obtained by affecting ϕ, ψ each with multipliers $x^{n-2}, x^{n-2}y$, &c., and ω with the multipliers $x^{n-1}, x^{n-1}y$, &c.

The number of functions (for either case) in the augmentative and auxiliary plexuses thus obtained will be found to be exactly equal to the number of terms in each such function, as shown by me in the paper alluded to. Let this be compared with the transcendental method (I use this word at this point in preference to concomitantive, because in fact the algebraical and differential auxiliary systems are both alike concomitantive plexuses to ϕ). For the case of $n, n, n+1$, the Jacobian determinant R of ϕ, ψ, ω will be of the degree $3n-2$, and the system $\left(\frac{d}{dx}\right)^{n-1} R, \left(\frac{d}{dx}\right)^{n-2}\left(\frac{d}{dy}\right) R$, &c. combined with the augmentative systems

$$x^{n-2}\omega,\quad x^{n-3}y\omega,\quad \text{\&c.}$$
$$x^{n-1}\phi,\quad x^{n-2}y\phi,\quad \text{\&c.}$$
$$x^{n-1}\psi,\quad x^{n-2}y\psi,\quad \text{\&c.}$$

will give an invariant resembling (at least in generation and form) if not identical with the resultant of ϕ, ψ, ω. For the case of ϕ, ψ, ω being of the degrees $n, n, n-1$, the Jacobian R is of the degree $3n-4$ and

$$\left(\frac{d}{dx}\right)^{n-2} R,\quad \left(\frac{d}{dx}\right)^{n-3} \frac{d}{dy} R, \text{\&c.}$$

is the system which, combined with the augmentative systems

$$x^{n-2}\phi,\quad x^{n-3}y\phi,\ \&c.$$
$$x^{n-2}\psi,\quad x^{n-3}y\psi,\ \&c.$$
$$x^{n-1}\omega,\quad x^{n-2}y\omega,\ \&c.$$

will produce the resembling invariant.

Finally, for the last and more special case which the algebraical method applies to, namely of ϕ, ψ, ω, θ, four quadratic functions of x, y, z, t, there can be here little doubt (upon the first impression) that in place of the algebraically obtained plexus

$$H_{2,1,1,1},\quad H_{1,2,1,1},\quad H_{1,1,2,1},\quad H_{1,1,1,2},$$

may be substituted the differential plexus

$$\frac{dR}{dx},\ \frac{dR}{dy},\ \frac{dR}{dz},\ \frac{dR}{dt},$$

which, combined with the augmentatives

$$x\phi,\ x\psi,\ x\omega,\ x\theta;\quad y\phi,\ y\psi,\ y\omega,\ y\theta;\quad z\phi,\ z\psi,\ z\omega,\ z\theta;\quad t\phi,\ t\psi,\ t\omega,\ t\theta,$$

will render possible the dialytic elimination of the 20 homogeneous products

$$x^3,\ x^2y,\ x^2z,\ x^2t,\ xyz,\ y^3,\ \&c.\ \&c.^*$$

Upon precisely the same principles may be verified instantaneously the method given by Hesse (without demonstration) for finding the polar reciprocal of lines of the third and fourth orders, at least to the extent of seeing that the functions obtained by his methods are contravariants (of the right degree and order) of the function from which they are derived. The polar reciprocal to a *surface* of the third degree may be obtained in the same manner.

Let $\phi(x, y, z, t)$ be the characteristic of such a surface. If we form a differential plexus of the first emanant of ϕ taken together with the concomitant $w = x\xi + y\eta + z\zeta + t\theta$, by operating with

$$\frac{d}{dx},\ \frac{d}{dy},\ \frac{d}{dz},\ \frac{d}{dt}\ \text{upon}\ \left(\xi'\frac{d}{dx} + \eta'\frac{d}{dy} + \zeta'\frac{d}{dz} + \theta'\frac{d}{dt}\right)(\phi + \lambda w),$$

and combining this plexus with $x\xi' + y\eta' + z\zeta' + t\theta'$, the resultant taken in respect to ξ', η', ζ', θ' (say R) will (according to the law of synthesis) be a

* Subsequent reflection induces me to reject as very improbable the (at first view likely) conjecture of the identity of the resultant with the invariant which simulates its form, except in the proved cases of three quadratic functions and the strongly resembling case of four quadratic functions last adverted to in the text above. Did this identity obtain, analogy would indicate that the catalecticant of the Hessian of two homogeneous functions of the same degree in x, should be identical with their resultant, which is easily demonstrated to be false, except when the functions are of the third degree.

contravariant to the system $\phi + \lambda w$ and w, and therefore to ϕ, because w is itself a concomitant to ϕ. R is of the third degree in x, y, z, t, as also in the coefficients of ϕ. If we form a differential plexus of $R + \mu w$ analogous to that formed above with $\phi + \lambda w$, and combine these two plexuses with the augmentative system xw, yw, zw, tw, there will be $4+4+4$, that is, 12 functions containing the 12 terms x^2, y^2, z^2, t^2, xy, xz, xt, yz, yt, zt, λ, μ, and the dialytic resultant, which will be found to be a contravariant of the twelfth degree in ξ, η, ζ, θ, and of the twelfth order in respect of the coefficients of ϕ, will be (there can be little doubt) the polar reciprocal to the characteristic ϕ.

A few remarks upon the analytical character of a polar reciprocal may be not out of place here. If ϕ be any homogeneous function of the degree m of any number (n) of variables $(x, y \dots z)$, the object of the theory of polar reciprocals is to discover what is the relation between $\xi, \eta \dots \zeta$ expressed in the simplest terms such that, when this equation is satisfied, $\xi x + \eta y + \dots + \zeta z = 0$ will be tangential to $\phi = 0$. In order for this to take effect it is necessary that when any one of the variables z is expressed in terms of the others $\dots y, x$, and this value established in ϕ, the discriminant of ϕ, so transformed, should be zero. Consequently the characteristic of the polar reciprocal to ϕ is that rational integral function which is common to all the discriminants obtained by expressing ϕ (by aid of the equation $\xi x + \eta y + \dots + \zeta z$) as a function of any $(n-1)$ of the variables. Let I_x be any invariant whatever of the order r of ϕ_x (meaning by this last symbol what ϕ becomes when x is eliminated), and $I_y \dots I_z$ the corresponding invariants when $y \dots z$ respectively are eliminated; I_x will evidently be of the form $\frac{E_x}{(\xi)^{mr}}$, the numerator being an integer of r dimensions in the coefficients of ϕ and of mr dimensions in respect of $\xi, \eta \dots \zeta$; and by the fundamental definition of invariants it may easily be shown that

$$I_x : I_y : \dots : I_z :: \frac{1}{\xi^{\frac{mr}{n-1}}} : \frac{1}{\eta^{\frac{mr}{n-1}}} : \dots : \frac{1}{\zeta^{\frac{mr}{n-1}}}{}^*,$$

and therefore

$$\frac{E_x}{\xi^p} = \frac{E_y}{\eta^p} = \dots = \frac{E_z}{\zeta^p},\quad \text{where } p = \frac{m(n-2)r}{n-1}.$$

Consequently all these quotients must be essentially integer, and any one of them will be of the order r in respect of the coefficients of ϕ and of the

* We see indirectly from this, that for a function of $(n-1)$, say γ, variables of the degree m, an invariant of the order r must be subject to the condition that $\frac{mr}{\gamma}$ = an integer. This is easily shown upon independent grounds; when $\gamma = 2$, $\frac{mr}{\gamma}$ must be not merely an integer but an *even integer*, and doubtless some analogous law applies to the general case.

degree $\frac{mr}{n-1}$ in respect of $\xi, \eta \dots \zeta$. Consequently the polar characteristic of ϕ, which is the common factor of the *discriminants* of $I_x, I_y \dots I_z$ (for which species of invariant r evidently is equal to $(n-1)(m-1)^{n-2}$, the function being in fact the discriminant of a function of the mth degree of $(n-1)$ variables), will be of the order $(n-1)(m-1)^{n-2}$ in respect of the coefficients of ϕ and of the degree $m(m-1)^{n-2}$ in respect of the contragredients $\xi, \eta \dots \zeta$.

As to what relates to the reciprocity which exists between ϕ and its polar reciprocal ψ, this is included in a much higher theory of elimination, one proposition of which may be enunciated somewhat to the effect following, namely that if ϕ be a homogeneous function of $x, y \dots z$, and ω of $x, y \dots z$, $u, v \dots w$, and if, by aid of the equations

$$\phi = 0,$$

$$\frac{d\phi}{dx} + \lambda \frac{d\omega}{dx} = 0,$$

$$\frac{d\phi}{dy} + \lambda \frac{d\omega}{dy} = 0,$$

$$\dots\dots\dots\dots\dots\dots$$

$$\frac{d\phi}{dz} + \lambda \frac{d\omega}{dz} = 0,$$

$x, y \dots z$ be eliminated and the resultant be called ψ, then the effect of performing a similar operation upon ψ, ω, with respect to $u, v \dots w$, as that just above indicated for the system ϕ, ω, with respect to $x, y \dots z$, will be to give a resultant, one factor of which will be the primitive function ϕ over again. There is some reason for supposing that polar reciprocals, which are scarcely ever (if ever, except indeed for quadratic functions) the simplest contravariants to a given function, may be expressed algebraically by means of the simpler contravariants, in the same way as discriminants admit (in many, if not in all cases, with the same exception as above) of being represented as algebraical functions of invariants of a lower order or simpler form.

I close this section with the remark that every complete and unambiguous system of functions of the constants in a given form or set of forms *characteristic** of any singularity absolute or relative in such form or forms must

* I repeat here that a function or system of functions which severally equated to zero express unequivocally and completely the existence of any position or negation, is termed the characteristic of such position or negation. Thus for example the resultant of a group of equations is the characteristic of the possibility of their coexistence. The discriminant of a function of two variables is the characteristic of its possession of two equal factors; the catalecticant is the characteristic of its decomposability into the sum of a defined number of powers of linear functions of the variables, &c.

constitute an invariantive plexus or set of invariantive plexuses. The system unambiguously characteristic of a singularity of an order n will (except when $n=1$) almost universally consist of far more than n functions, subject of course to the existence of syzygetic* relations between any $(n+1)$ of such functions. The existence of multiple roots of a function of two variables is a specific, but by no means a peculiar case of singularity, and requires, for its complete and systematic elucidation, to be treated in connexion with the general theory of the subject.

SECTION III. *On Commutants.*

The simplest species of commutant is the well-known common determinant.

If we combine each of the n letters $a, b \dots l$ with each of the other n, $\alpha, \beta \dots \lambda$, we obtain n^2 combinations which may be used to denote the terms of a determinant of n lines and columns, as thus:

$$a\alpha, \quad a\beta \dots a\lambda,$$

$$b\alpha, \quad b\beta \dots b\lambda,$$

$$\dots\dots\dots\dots\dots\dots$$

$$l\alpha, \quad l\beta \dots l\lambda.$$

It must be well understood that the single letters of either set are mere umbræ, or shadows of quantities, and only acquire a real signification when one letter of one set is combined with one of the other set. Instead of the inconvenient form above written, we may denote the determinant more simply by the matrix

$$a, \quad b, \quad c \dots l,$$

$$\alpha, \quad \beta, \quad \gamma \dots \lambda;$$

and to find the expanded value of such a matrix the rule is evidently to take one of the lines in all its 1, 2, 3 ... n different forms, arising from the permutations of the letters (or umbræ) which it contains; and then form the product of the n quantities formed by the combination of the respective pairs of letters in the same vertical column, affecting such product with the sign of + or − according to the rule, that all products corresponding to arrangements of the terms subject to the permutation derivable from one another by an even number of interchanges are of the same, and by an odd number of interchanges of a contrary sign. If both lines are permuted and a similar rule applied, with the additional circumstance that the sign of the products

* Rational integer functions which admit of being multiplied severally by other rational integer functions such that the sum of the products is identically zero, are said to be "syzygetically related."

is made to depend on the product of the algebraical signs due to the respective arrangements in the two lines of umbræ, it is evident that the result will be the same as when only one line is put into motion, save and except that a numerical factor $1.2.3 \ldots n$ will affect each term. If the two sets of umbræ $a, b, c \ldots l$; $\alpha, \beta, \gamma \ldots \lambda$ be taken identical, and if it be convened that the order of the combination of any two letters shall not affect the value of the quantity thereby denoted, $\begin{matrix} a, b, c \ldots l \\ a, b, c \ldots l \end{matrix}$ will denote a symmetrical determinant.

If instead of two lines of umbræ, three or more be taken, the same principle of solution will continue to be applicable. Thus, if there be a matrix of any even number r of lines each of n umbræ,

$$\begin{matrix} a_1, & b_1 \ldots l_1, \\ a_2, & b_2 \ldots l_2, \\ \ldots & \ldots \\ a_r, & b_r \ldots l_r, \end{matrix}$$

the first may be supposed to remain stationary, and the remaining $(r-1)$ lines each be taken in $1, 2 \ldots n$ different orders; every order in each line will be accompanied by its appropriate sign + or −; and each different grouping in each line will give rise to a particular grouping of the letters read off in columns. The value of the commutant expressed by the above matrix will therefore consist of the sum of $(1.2 \ldots n)^{r-1}$ terms, each term being the product of n quantities respectively symbolized by a group of r letters and affected with the sign + or − according as the number of negative signs in the total of the arrangements of the lines (from the columnar reading off of which each such term is derived) is even or odd.

For example, the value of

$$\begin{matrix} a, & b, \\ c, & d, \\ e, & f, \\ g, & h, \end{matrix}$$

will be found by taking the $(1.2)^3$ arrangements, as below,

$$\begin{matrix} a, b, & a, b, & a, b, & a, b, & a, b, & a, b, & a, b, & a, b, \\ c, d, & d, c, & c, d, & d, c, & c, d, & d, c, & c, d, & d, c, \\ e, f, & e, f, & f, e, & f, e, & e, f, & e, f, & f, e, & f, e, \\ g, h, & g, h, & g, h, & g, h, & h, g, & h, g, & h, g, & h, g. \end{matrix}$$

The signs of c, d; e, f; g, h being supposed +, those of d, c; f, e and h, g will be each −. Consequently the sum of the terms will be expressed by

$$aceg \times bdfh - adeg \times bcfh - acfg \times bdeh + adfg \times bceh$$
$$- aceh \times bdfg + adeh \times bcfg + acfh \times bdeg - adfh \times bceg.$$

Commutants thus formed may be termed total commutants, because the entire of each line is made to pass through all its possible forms of arrangement. In total commutants it is necessary that the number of lines r be even; for if taken odd, on making all the r lines to change, instead of obtaining $1.2 \ldots n$ lines, the result obtained when all but one are made to change, it will be found that the latter will be repeated $\frac{1}{2}(1.2 \ldots n)$ times with the sign +, and $\frac{1}{2}(1.2 \ldots n)$ times with the sign −, so that the algebraical sum of the terms will be zero. Moreover the commutants of the species above described, besides being total, are simple, inasmuch as all the umbræ to be termed consist of single letters.

My first proposition in the application of the theory of commutants to that of forms is as follows:

Let ϕ be a function homogeneous and linear in respect to an even number r of any systems whatever of variables, as

$$x_1, y_1 \ldots t_1; \quad x_2, y_2 \ldots t_2; \quad x_r, y_r \ldots t_r.$$

Form the commutant

$$\begin{matrix} \frac{d}{dx_1}, & \frac{d}{dy_1} \cdots & \frac{d}{dt_1}, \\ \frac{d}{dx_2}, & \frac{d}{dy_2} \cdots & \frac{d}{dt_2}, \\ \ldots & \ldots & \ldots \\ \frac{d}{dx_r}, & \frac{d}{dy_r} \cdots & \frac{d}{dt_r}. \end{matrix}$$

Let the general term of this commutant, expanded, be called

$$F_{\theta_1} \times F_{\theta_2} \times \ldots \times F_{\theta_r},$$

then is

$$\Sigma F_{\theta_1} . \phi \times F_{\theta_2} . \phi \times \ldots \times F_{\theta_r} . \phi$$

a covariant or invariant*, as the case may be, of ϕ.

Be it observed that the march of the substitution for the different sets of variables in the above proposition is supposed to be perfectly independent. All the systems but one may undergo linear transformation, or they may all undergo distinct and disconnected transformations at the same time, and the proposition still continue applicable. It will however evidently be no less applicable should the march of substitution for any of the systems become cogredient or contragredient to that of any other systems.

If we suppose ϕ to be a function of an even degree r of a single system of n variables $x, y \ldots t$, so that the r systems x_1, y_1, &c., x_2, y_2, &c. ... x_r, y_r, &c. become identical, we can at once infer from the above scheme the existence and mode of forming an invariant to ϕ of the order n. This last appears

[* See below, p. 324.]

for the case $n=2$, and ought, for all other values of n, to have been known* to the author of the immortal discovery of invariants, termed by him hyperdeterminants, in the sense which, according to the nomenclature here adopted, would be conveyed by the term hyperdiscriminants.

Before proceeding to discuss the theory of compound total commutants, or enlarging upon that of partial commutants, I shall make an interesting application of the preceding general proposition to the discovery of Aronhold's S and T, the two invariants respectively of the fourth and sixth orders appertaining to a homogeneous cubic function (say F) of three variables x, y, z. These may be termed respectively H_4 and H_6. As to H_6 a theoretically possible but eminently prolix and ungraceful method immediately presents itself, namely to take $F^2 = G$, and after forming the commutant with six lines,

$$\begin{matrix} \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \end{matrix}$$

to operate with the 6^3 ternary products of which this is made up upon G: the result being an invariant of G, will be so to F, and being of the third degree in respect to the coefficients of G, will be of the sixth in respect to those of F. It will evidently therefore be H_6, or at least a numerical multiple of H_6, the form of which, inasmuch as the only other invariant is H_4, we know in form to be unique. But the general theorem affords another and probably the

* That this was not known explicitly to and should have escaped the penetration of the sagacious author of the theory, and those who had studied his papers, must be attributed to the imperfection of the notation heretofore employed for denoting the coefficients of a homogeneous polynomial function. The umbral method of denoting such a function ϕ of the degree r under the form of $(ax+by+\ldots+cz)^r$, which is equivalent to, but a more compendious and independent mode of mentally conceiving and handling the representation

$$\left(x\frac{d}{dx}+y\frac{d}{dy}+\ldots\ldots+z\frac{d}{dz}\right)\phi,$$

exhibits the true internal constitution of such functions, and necessarily leads to the discovery of their essential properties and attributes.

most practically compendious* solution as regards H_6, of which the question admits.

Let G† represent the mixed concomitant to F formed by the bordered determinant

$$\begin{vmatrix} \frac{d^2F}{dx^2}, & \frac{d^2F}{dxdy}, & \frac{d^2F}{dxdz}, & \xi \\ \frac{d^2F}{dydx}, & \frac{d^2F}{dy^2}, & \frac{d^2F}{dydz}, & \eta \\ \frac{d^2F}{dzdx}, & \frac{d^2F}{dzdy}, & \frac{d^2F}{dz^2}, & \zeta \\ \xi, & \eta, & \zeta, & 0 \end{vmatrix}.$$

G is a function of the second order as to x, y, z, and of the like order in respect to ξ, η, ζ, which two systems will be respectively cogredient and contragredient in respect to the x, y, z system in F. In other words, which is all we need to look to, G is a concomitant of F, and so also will be

$$G+\lambda(x\xi+y\eta+z\zeta)^2,$$

which may be termed H. Form now the commutant

$$\begin{matrix} \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz}, \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta}, \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta}, \end{matrix}$$

this being applied to H will give an invariant (the fact that the march of the substitutions for the systems x, y, z; ξ, η, ζ is contrary, being completely immaterial to the applicability of the general theorem above given);

* Having since this was printed been favoured with a view of some of the proof-sheets of Mr Salmon's most valuable Second Part of his *System of Analytical Geometry* (about to appear, and which is calculated, in my opinion, to awaken a higher idea of and excite a new taste for geometrical researches in this country), I find that I am mistaken in this point; the less symmetrical method operated with by Mr Salmon being decidedly the shortest for practically obtaining S and T in the general case. Symmetry, like the grace of an eastern robe, has not unfrequently to be purchased at the expense of some sacrifice of freedom and rapidity of action.

† G is the mixed concomitant to the given cubic function, which is halfway (so to speak) between it and its polar reciprocal. In fact, when the operation is repeated upon G, which was executed upon the given function to obtain G (that is, when we *border* the Hessian of G in respect to x, y, z, vertically and horizontally with the column and line ξ, η, ζ) the determinant thereby represented becomes the polar reciprocal to the given function.

the commutant so formed will be a cubic function of λ, in which the coefficient of λ^3 is a numerical quantity, that of λ^2 is zero, that of λ is H_4 and the constant term is H_6.

Thus for example let $F = x^3 + y^3 + z^3 + 6mxyz$, then

$$G = \begin{vmatrix} x, & mz, & my, & \xi \\ mz, & y, & mx, & \eta \\ my, & mx, & z, & \zeta \\ \xi, & \eta, & \zeta, & 0 \end{vmatrix}$$

and therefore

$$H = \Sigma \{(\lambda - m^2) x^2\xi^2 + (\lambda + m^2)\, 2yz\eta\zeta + yz\xi^2 - 2mx^2\eta\zeta\},$$

the Σ implying the sum of similar terms with reference to the interchanges between x, ξ; y, η; z, ζ.

In developing the commutant above, the first line may be kept in a fixed position; for the sake of brevity, (x), (y), (z); (ξ), (η), (ζ) may be written in the place of

$$\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}; \quad \frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta},$$

and it will readily be seen that the only effective arrangements will be as underwritten:

$$\begin{array}{ccc} (x)(y)(z) & (x)(y)(z) & (x)(y)(z) \\ (x)(y)(z) & (x)(y)(z) & (x)(y)(z) \\ (\xi)(\eta)(\zeta) & (\eta)(\zeta)(\xi) & (\zeta)(\xi)(\eta) \\ (\xi)(\eta)(\zeta) & (\zeta)(\xi)(\eta) & (\eta)(\zeta)(\xi) \end{array}$$

$$\begin{array}{cccccc} (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) \\ (x)(z)(y) & (x)(z)(y) & (\zeta)(\eta)(\xi) & (z)(y)(x) & (y)(x)(z) & (y)(x)(z) \\ (\xi)(\eta)(\zeta) & (\xi)(\zeta)(\eta) & (\xi)(\eta)(\zeta) & (\zeta)(\eta)(\xi) & (\xi)(\eta)(\zeta) & (\eta)(\xi)(\zeta) \\ (\xi)(\zeta)(\eta) & (\xi)(\eta)(\zeta) & (\zeta)(\eta)(\xi) & (\xi)(\eta)(\zeta) & (\eta)(\xi)(\zeta) & (\xi)(\eta)(\zeta) \end{array}$$

$$\begin{array}{cccccc} (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) \\ (x)(z)(y) & (x)(z)(y) & (z)(x)(y) & (z)(y)(x) & (y)(x)(z) & (y)(x)(z) \\ (\eta)(\zeta)(\xi) & (\zeta)(\eta)(\xi) & (\xi)(\zeta)(\eta) & (\zeta)(\xi)(\eta) & (\eta)(\zeta)(\xi) & (\xi)(\zeta)(\eta) \\ (\zeta)(\eta)(\xi) & (\eta)(\zeta)(\xi) & (\zeta)(\xi)(\eta) & (\xi)(\zeta)(\eta) & (\xi)(\zeta)(\eta) & (\eta)(\zeta)(\xi) \end{array}$$

$$\begin{array}{cccccc} (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) & (x)(y)(z) \\ (y)(z)(x) & (z)(x)(y) & (y)(z)(x) & (y)(z)(x) & (z)(x)(y) & (z)(x)(y) \\ (\zeta)(\xi)(\eta) & (\eta)(\zeta)(\xi) & (\xi)(\eta)(\zeta) & (\eta)(\zeta)(\xi) & (\xi)(\eta)(\zeta) & (\zeta)(\xi)(\eta) \\ (\xi)(\zeta)(\eta) & (\eta)(\zeta)(\xi) & (\eta)(\zeta)(\xi) & (\xi)(\eta)(\zeta) & (\zeta)(\xi)(\eta) & (\xi)(\eta)(\zeta) \end{array}$$

The signs of the four lines in each of these arrangements are two alike, and two contrary to the signs of the correspondent lines in the first arrangement; hence the effective sign is the same for all, and the result, after rejecting from each term the common factor -16, is seen, from inspection, to be

$$4(\lambda - m^2)^3 - 8m^3 + 6(\lambda - m^2)(\lambda + m^2)^2 - 12m(\lambda + m^2) + 2(\lambda + m^2)^3 + 1,$$

which is equal to

$$12\lambda^3 + 0 \, . \, \lambda^2 - 12(m - m^4)\lambda + 1 - 20m^3 - 8m^6;$$

here the coefficients $m - m^4$ and $1 - 20m^3 - 8m^6$ are the two invariants (Aronhold's S and T) for the canonical form operated upon; and it will be observed that

$$(1 - 20m^3 - 8m^6)^2 + 64(m - m^4)^3 = (1 + 8m^3)^3,$$

which is easily proved to be the discriminant of

$$x^3 + y^3 + z^3 + 6mxyz.$$

It may however be observed, that this is not the discriminant of the function in λ just found, as reasons of analogy* might have suggested it probably would be: in order that this might be the case, the coefficient of λ^3 should be 4 instead of 12, and of λ, $m - m^4$ instead of $m^4 - m$. There is ground for supposing that another function of λ may be found by a different method, in which this relation will take effect.

The theorem above given for simple total commutants admits of an interesting application to the general case of a function F of the nth degree, in respect to each of two independent systems of two variables x, y; ξ, η. Let F be symbolically represented by $(ax + by)^n (\alpha\xi + \beta\eta)^n$, so that $a^n\alpha^n$ represents the coefficient of $x^n\xi^n$, $na^{n-1}b\alpha^n$ of $x^{n-1}y\xi^2$, &c. &c.; then the commutant

$$\begin{array}{cr} a, \; b, & (1) \\ a, \; b, & (2) \\ \cdots & \\ a, \; b, & (n) \\ \alpha, \; \beta, & (1) \\ \alpha, \; \beta, & (2) \\ \cdots & \\ \alpha, \; \beta, & (n) \end{array}$$

will represent a quadratic invariant of F, which will contain $(n+1)^2$ coefficients. By expanding this commutant we obtain a general expression for the invariant under a very interesting form.

* The biquadratic function of x, y having only one parameter, and therefore two invariants, its theory possesses striking analogies to the theory of the cubic function of three letters. The function in λ which gives these invariants for the first-named function, according to the method given in the first section, has the same discriminant as the function itself.

I now proceed to give the general theorem for compound total commutants as applicable to the discovery of invariants.

Let there be a function of m disconnected classes of systems of variables; let the systems in the same class be supposed all distinct but cogredient with one another. The function is supposed to be linear in respect to each system in each class, and the number of systems is the same for all the classes, and the number of variables the same in each system. This function may then be represented symbolically under the form

$$({}^1a_1 . {}^1x_1 + {}^1b_1 . {}^1y_1 + \dots + {}^1l_1 . {}^1t_1)({}^1a_2 . {}^1x_2 + {}^1b_2 . {}^1y_2 + \dots + {}^1l_2 . {}^1t_2)$$

$$\dots({}^1a_n . {}^1x_n + {}^1b_n . {}^1y_n + \dots {}^1l_n . {}^1t_n)$$

$$\times({}^2a_1 . {}^2x_1 + {}^2b_1 . {}^2y_1 + \dots + {}^2l_1 . {}^2t_1)({}^2a_2 . {}^2x_2 + {}^2b_2 . {}^2y_2 + \dots + {}^2l_2 . {}^2t_2)$$

$$\dots({}^2a_n . {}^2x_n + {}^2b_n . {}^2y_n + \dots {}^2l_n . {}^2t_n)$$

$$\times \text{ \&c.}$$

$$\times({}^pa_1 . {}^px_1 + {}^pb_1 . {}^py_1 + \dots + {}^pl_1 . {}^pt_1)({}^pa_2 . {}^px_2 + {}^pb_2 . {}^py_2 + \dots + {}^pl_2 . {}^pt_2)$$

$$\dots({}^pa_n . {}^px_n + {}^pb_n . {}^py_n + \dots {}^pl_n . {}^pt_n).$$

In this expression the $x, y \dots t$'s are all real, but the $a, b \dots l$'s all umbral; in fact, ${}^fa_g, {}^fb_g$, &c. may be understood to denote $\frac{d}{d^fx_g}, \frac{d}{d^fy_g}$, &c.

The n systems of variables in each of the sets above written are supposed to be cogredient *inter se*.

Take the symbolical product of the first set, first making *for the moment*

$${}^1x_1 = {}^1x_2 = \dots {}^1x_n = x, \text{ \&c. \&c.}, \quad {}^1t_1 = {}^1t_2 = \dots {}^1t_n = t;$$

and let the coefficients of the several terms

$$x^n, x^{n-1}y \dots \text{\&c.},$$

be called $\qquad {}^1A_1, {}^1A_2 \dots {}^1A_\mu,$

where μ is the number of terms contained in a homogeneous function of the nth degree of the m variables $x, y \dots t$. In like manner proceed with each of the lines, and then write down the commutant

$${}^1A_1, {}^1A_2 \dots {}^1A_\mu,$$

$${}^2A_1, {}^2A_2 \dots {}^2A_\mu,$$

$$\dots\dots\dots\dots\dots\dots$$

$${}^pA_1, {}^pA_2 \dots {}^pA_\mu.$$

This commutant is an invariant of F: it will of course be remembered that, unless p is even, the commutant vanishes.

Thus, for example, take two sets of two systems of two variables: in all four systems,

$$x, y; \; \xi, \eta \; : p, q; \; \phi, \psi,$$

each couple of systems on either side of the colon (:) being cogredient *inter se*: and let F be symbolically represented by

$$(ax + by)(\alpha\xi + \beta\eta)(lp + mq)(\lambda\phi + \mu\psi);$$

then the invariant given by the theorem will be the commutant

$$a\alpha; \; a\beta + \alpha b; \; b\beta,$$

$$l\lambda; \; l\mu + \lambda m; \; m\mu.$$

The six positions of this are as below written (the first three being positive and the second three negative)

$$a\alpha; a\beta + \alpha b; b\beta, \quad a\alpha; a\beta + \alpha b; b\beta, \quad a\alpha; a\beta + \alpha b; b\beta,$$

$$l\lambda; l\mu + \lambda m; m\mu, \quad l\mu + \lambda m; m\mu; l\lambda, \quad m\mu; l\lambda; l\mu + \lambda m,$$

$$a\alpha; a\beta + \alpha b; b\beta, \quad a\alpha; a\beta + \alpha b; b\beta, \quad a\alpha; a\beta + \alpha b; b\beta,$$

$$l\mu + \lambda m; l\lambda; m\mu, \quad l\lambda; m\mu; l\mu + \lambda m, \quad m\mu; l\mu + \lambda m; l\lambda.$$

If we write F under its explicit form,

$$A x\xi p\phi + B x\xi p\psi + C x\xi q\phi + D x\xi q\psi$$

$$+ A' x\eta p\phi + B' x\eta p\psi + C' x\eta q\phi + D' x\eta q\psi$$

$$+ A'' y\xi p\phi + B'' y\xi p\psi + C'' y\xi q\phi + D' y\xi q\psi$$

$$+ A''' y\eta p\phi + B''' y\eta p\psi + C''' y\eta q\phi + D''' y\eta q\psi,$$

we have identically the relations following,

$$a\alpha l\lambda = A, \quad a\alpha l\mu = B, \quad a\alpha m\lambda = C, \quad a\alpha m\mu = D,$$

$$a\beta l\lambda = A', \quad a\beta l\mu = B', \quad a\beta m\lambda = C', \quad a\beta m\mu = D',$$

$$b\alpha l\lambda = A'', \quad b\alpha l\mu = B'', \quad b\alpha m\lambda = C'', \quad b\alpha m\mu = D'',$$

$$b\beta l\lambda = A''', \quad b\beta l\mu = B''', \quad b\beta m\lambda = C''', \quad b\beta m\mu = D''',$$

and the commutant expanded becomes

$$A(B' + C'' + C' + B'')D''' + (B + C)(D' + D'')A''' + D(A' + A'')(B''' + C''')$$

$$-(B + C)(A' + A'')D''' - A(D' + D'')(B''' + C''') - D(B' + C'' + C' + B'')A'''.$$

In the foregoing the x's in the several lines were *for the moment* taken identical, in order the more easily to explain the law of formation of the quantities A. But suppose that they become actually identical for the same line. F then becomes a function of the nth degree in respect to each of p systems of variables, and may be represented symbolically under the form

$$({}^1a^1x + {}^1b^1y + \ldots + {}^1l^1t)^n \times ({}^2a^2x + {}^2b^2y + \ldots + {}^2l^2t)^n$$
$$\ldots \times ({}^pa^px + {}^pb^py + \ldots + {}^pl^pt)^n.$$

We may still further limit the generality of the theorem by supposing

$${}^1x = {}^2x = \ldots {}^px = x,$$
$${}^1y = {}^2y = \ldots {}^py = y,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$${}^1t = {}^2t = \ldots {}^pt = t;$$

F then becomes $\quad (ax + by + \ldots + lt)^{np}.$

Accordingly, as many different factors as can be found contained an even number of times in the exponent of the function, so many invariants can be formed immediately from a function of any number of variables m by the method of total commutation.

If one of these factors be called n, the commutant corresponding thereto will be of the order

$$\frac{(n+1)(n+2)\ldots(n+m-1)}{1.2\ldots(m-1)}$$

in respect to the coefficients. Thus take $m = 2$, so that

$$F = (ax + by)^{np}.$$

The general form of such a commutant will be found by taking A_1, $A_2 \ldots A_{n+1}$ the coefficients of the several combinations of x, y in $(ax + by)^n$, from which the numerical coefficients n, $\frac{1}{2}n(n-1)$, &c. may be rejected, as only introducing a numerical factor into the result; the commutant will therefore be expressed by means of the form

$$a^n;\ a^{n-1}b;\ a^{n-2}b^2 \ldots;\ b^n, \qquad (1)$$
$$a^n;\ a^{n-1}b;\ a^{n-2}b^2 \ldots;\ b^n, \qquad (2)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a^n;\ a^{n-1}b;\ a^{n-2}b^2 \ldots;\ b^n. \qquad (p)$$

If $p = 2$, the compound commutant

$$a^n;\ a^{n-1}b;\ \ldots;\ b^n,$$
$$a^n;\ a^{n-1}b;\ \ldots;\ b^n,$$

will easily be seen to be only another form for the catalecticant of $(ax + by)^{2n}$. Thus, let $n = 2$,

$$(ax + by)^4 = Ax^4 + 4Bx^3y + 6Cx^2y^2 + 4Dxy^3 + Ey^4;$$

so that $\quad a^4 = A,\ a^3b = B,\ a^2b^2 = C,\ ab^3 = D,\ b^4 = E.$

The commutant (which is of the form of the matrix to an ordinary determinant, with the exception that the umbræ enter compoundly instead of simply into the several terms separated by the marks of punctuation), will be

$$a^2;\ ab;\ b^2,$$
$$a^2;\ ab;\ b^2:$$

this, written in the six forms

$$\left.\begin{matrix} a^2; & ab; & b^2 \\ a^2; & ab; & b^2 \end{matrix}\right\} \quad \left.\begin{matrix} a^2; & ab; & b^2 \\ a^2; & b^2; & ab \end{matrix}\right\} \quad \left.\begin{matrix} a^2; & ab; & b^2 \\ ab; & a^2; & b^2 \end{matrix}\right\}$$
$$\left.\begin{matrix} a^2; & ab; & b^2 \\ b^2; & ab; & a^2 \end{matrix}\right\} \quad \left.\begin{matrix} a^2; & ab; & b^2 \\ ab; & b^2; & a^2 \end{matrix}\right\} \quad \left.\begin{matrix} a^2; & ab; & b^2 \\ b^2; & a^2; & ab \end{matrix}\right\}$$

gives the expression

$$a^4 \times a^2b^2 \times b^4 - a^4 \times (ab^3)^2 - b^4 \times (a^3b)^2 - (a^2b^2)^3 + 2a^3b \times ab^3 \times a^2b^2;$$

that is $\quad ACE - AD^2 - EB^2 - C^3 + 2BCD.$

One important observation may here be made of a fact which otherwise might easily escape attention, which is, that commutants, where the same terms simple or compound are found in all or several of the lines, in general give rise to products, some of them equal and with the same sign, and others equal but with the *contrary* sign.

This last phenomenon does not manifest itself in commutants appertaining to functions of two variables of the two particular and different species which first and most naturally present themselves, namely where there are only two lines or only two columns*—I believe that it displays itself in every other case of commutantives to functions of two variables. Thus it is that algebraical expressions derived from given functions disguise their symmetry; to make which come to light it becomes necessary to add terms of contrary sign to such expressions. As an example, the reader is invited to develope the cubic invariant of a function of x and y, symbolically expressed by $(ax + by)^8$, where

$$a^8 = A,\quad a^7b = B \ldots ab^7 = H,\quad b^8 = I,$$

* These commutants give respectively the quadrinvariant and the catalecticant, the former of which alone was formerly recognised by Mr Cayley as a commutant.

by means of the commutant

$$\begin{array}{lll} a^2, & ab, & b^2, \\ a^2, & ab, & b^2, \\ a^2, & ab, & b^2, \\ a^2, & ab, & b^2.\text{*} \end{array}$$

Suppose F to be the general even-degreed function of two variables of the degree $2np$.

Let $$H = \left(\xi \frac{d}{dy} - \eta \frac{d}{dx}\right)^{np} F + \lambda (x\xi + y\eta)^{np},$$

and express H umbrally under the form

$$(ax + by)^{np} (\alpha\xi + \beta\eta)^{np}.$$

* [See p. 346 below.] The number of terms resulting from the independent permutation of each of the 3 linear lines is 6^3, that is 216; but the actual result is (using small letters instead of large) $P - Q$, where

$$P = aei + 3ag^2 + 12beh + 3c^2i + 24cf^2 + 24d^2g + 15e^3,$$
$$Q = 4afh + 4bid + 8bgf + 22ceg + 8chd + 36def,$$

so that the effective number of permutations is only 164. The difference between this and 216 divided by 216 may be termed the Index of Demolition, which we see in this case is $\frac{52}{216}$ or $\frac{13}{54}$; that is, somewhat less than $\frac{1}{4}$. For the cubic invariant of the function of the fourth degree this index is zero, all the permutations being effective. If we take the cubic invariant of the function $ax^{12} + 12bx^{11}y + 66cx^{10}y^2 + \&c. + my^{12}$ under the form $P - Q$, we shall find

$$P = 6ahl + 10ajj + 6bfm + 54bhk + 54cfl + 155cii + 10ddm + 430dgj$$
$$+ 155eek + 520ehh + 520ffl + 280ggg,$$
$$Q = agm + 15aik + 30bgl + 50bij + 15cem + 4cgk + 150chj + 30del + 210dfk$$
$$+ 250dhi + 230efj + 555egl + 660fgh.$$

The number of terms in P and Q is of course the same, and will be found to be 2200 for each; so that out of the 6^5, that is 7776 permutations of the 5 lower rows, only 4400 are effective, and the index of demolition becomes $\frac{3376}{7776}$, that is $\frac{211}{486}$, or rather greater than $\frac{5}{12}$. The Index of Demolition thus goes on constantly increasing as the degree of the function rises; probably (?) it converges either towards $\frac{1}{2}$ or else towards unity. In arranging the terms it will be found most convenient to adopt, as I have done above, the dictionary method of sequence. The computations are greatly facilitated by the circumstance of the effect of any arrangement of each of the 5 lower lines not being altered when these *lines* are permuted with one another; this gives rise to the subdivision of the 7776 permutations into groups as follows: 6 of 120 identical terms, 60 of 60, 36 of 20, 60 of 30, 24 of 20, 30 of 10, 30 of 5, and 6 of 1. So that the total number of permutational arrangements to be constructed is only 252. Other methods of abridging the labour will readily suggest themselves to the practical computer. The total number of the *groups* of terms is of course always known *à priori*, and, for instance, in the case before us, must be equal to the number of ways in which $\frac{1}{2}(12 \times 3)$, that is the number 18, can be divided into 3 parts, none of which is to exceed the number 12, that is 25; for the cubic invariant of the function of the eighth degree of two variables it is the number of ways in which 12 can be divided into 3 parts, of which none shall exceed 8, and so forth, zeros being always understood to be admissible; and of course in general for an invariant of the order r to a function of the degree n of i variables, the number of distinct terms is in general the number of ways in which $\frac{nr}{i}$ can be divided into r parts, of which none shall exceed n, subject however always to the possibility in particular cases of a diminution in consequence of some of the groups assuming zero for their coefficient.

The commutant

$$\begin{array}{lll} a^n, & a^{n-1}b \ldots b^n, & (1) \\ a^n, & a^{n-1}b \ldots b^n, & (2) \\ \ldots\ldots & & \\ a^n, & a^{n-1}b \ldots b^n, & (p) \\ \alpha^n, & \alpha^{n-1}\beta \ldots \beta^n, & (1) \\ \alpha^n, & \alpha^{n-1}\beta \ldots \beta^n, & (2) \\ \ldots\ldots & & \\ \alpha^n, & \alpha^{n-1}\beta \ldots \beta^n, & (p) \end{array}$$

will be a function of λ, and all the several coefficients will be invariants of F*.

When $p = 1$ we obtain the Λ given in the preceding section, and originally published by me in the *Philosophical Magazine* for the month of November, 1851. The Λ obtained on this supposition has for its coefficients a series of independent invariants, commencing with the catalecticant and closing with the quadratic invariant. When p has any other value, we observe a similar series commencing with a commutantive invariant of a lower order than the catalecticant, but always closing with the quadratic invariant. Thus, for example, when $2np = 8$, we may obtain by the preceding theorem three different quadratic functions; one giving the invariants of the orders 5, 4, 3, 2, the second those of the orders 3, 2, the third the invariant of the order 2.

In this case the invariants of the same order given by the different Λ's are the same to numerical factors *près*. Whether this is always necessarily the case is a point reserved for further examination.

The commutants applied in the preceding theorems have been called by me total commutants, because the total of each line of umbræ is permuted in every possible manner. If the lines be divided into segments, and the permutation be local for each segment instead of extending itself over the whole line, we then arrive at the notion of partial commutants, to which I have also (in concert with Mr Cayley) given the distinctive name of Intermutants. In order to find the invariants of functions of odd degrees, the theory of total commutants requires the process of commutation to be applied, not immediately to the coefficients of the proposed function, but to some derived concomitant form. I became early sensible of this imperfection, and stated to the friend above named, to whom I had previously

* By substituting the symbols $\frac{d}{dx}, \frac{d}{dy}$, &c. in place of the *umbræ* a, b, &c., the theorem is easily stated for covariants generally. But in applying the commutantive method to obtain covariants, or rather in the statement of the results flowing from each application, it is never necessary to go beyond the case of invariants, because the commutantive covariants of any given homogeneous function are always identical with commutantive invariants of emanants of the same function.

imparted my general method of total commutation, my conviction of the existence of a qualified or restricted method of permutation, whereby the invariants of the cubic function, for instance, of two and of three letters would admit, without the aid of a derived form, of being represented. Many months ago, when I was engaged in this important research, and had made some considerable steps towards the representation of the invariant, that is, the discriminant of the cubic function of x and y, under the form of a single permutant, I was surprised by a note from the friend above alluded to, announcing that he had succeeded in fixing the form of the permutant of which I was at that moment in search. It is with no intention of complaining of this interference on the part of one to whose example and conversation I feel so deeply indebted, (and the undisputed author of the theory of Invariants,) that I may be permitted to say that, independent of the intervention of this communication, I must inevitably have succeeded in shaping my method so as to furnish the form in question; and that with greater certainty, after my theory of commutants had furnished me with the precedent of permutable forms giving rise to terms identical in value but affected with contrary signs. As I have understood that Mr Cayley is likely to develop this part of the subject in the present number of the *Journal*, it will be the less necessary for me to enter at any length into the theory of partial commutants on the present occasion.

The method of partial commutation is a simple but most important corollary from that of total commutation hereinbefore explained. To fix the ideas, conceive a class of p cogredient systems, and that there are qr such classes perfectly independent. Proceed to divide these qr classes in any manner whatever into r sets, each containing q classes; and form the symbol of the total commutant corresponding to each such set. Now let these commutants be placed side by side against one another, and transpose the terms in each compound line thus formed once for all, but in any arbitrary manner. Then permute in every possible way all those symbols in each line, *inter se*, which belong to the same class, and operate with the symbols thus produced by reading off the vertical columns and attending to the rule of the + and − signs, as in the case of a total commutant; the result will be a commutant of the form operated upon. For instance, let $p=1$, $q=3$, $r=2$, and let the number of variables in each system be 2. Form the commutant operators

$$\begin{array}{cc|cc} \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{d\xi}, & \frac{d}{d\eta}, \\ \frac{d}{dp}, & \frac{d}{dt}, & \frac{d}{d\phi}, & \frac{d}{d\theta}, \\ \frac{d}{dr}, & \frac{d}{ds}, & \frac{d}{d\rho}, & \frac{d}{d\sigma}. \end{array}$$

Interchange *in any manner* but *once for all* the symbols in each line, as thus:

$$\begin{array}{cccc} \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{d\xi}, & \frac{d}{d\eta}, \\ \frac{d}{d\phi}, & \frac{d}{dp}, & \frac{d}{dt}, & \frac{d}{d\theta}, \\ \frac{d}{ds}, & \frac{d}{d\rho}, & \frac{d}{dr}, & \frac{d}{d\sigma}. \end{array}$$

Now permute, *inter se*, the variables of each system, as

$$\frac{d}{dx}, \frac{d}{dy}; \quad \frac{d}{dp}, \frac{d}{dt}, \text{ \&c.};$$

the total number of the operative forms resulting will be $(1.2)^6$, and the sum of the $(1.2)^6$ quantities, half positive and half negative, formed after the type of

$$\Sigma \left\{ \begin{array}{c} \frac{d}{dx}\frac{d}{d\phi}\frac{d}{ds}U \times \frac{d}{dy}\frac{d}{dp}\frac{d}{d\rho}U \\ \times \frac{d}{d\xi}\frac{d}{d\theta}\frac{d}{dr}U \times \frac{d}{d\eta}\frac{d}{dt}\frac{d}{d\sigma}U \end{array} \right\},$$

U being supposed to be a function homogeneous in

$$x, y;\ \xi, \eta;\ p, t;\ \phi, \theta;\ r, s;\ \rho, \sigma,$$

will be a covariant of U.

The proof of the truth of this proposition is contained in what is shown in the Notes of the Appendix for total commutants, it being only necessary to make the systems which are independent vary consecutively, and then apply the inference to the supposition of their varying simultaneously.

It may be extended to the more general supposition of classes of an unequal number of cogredient systems of unequal numbers of variables in each, the only condition apparently required being that the number of distinct terms shall be the same in each line of the final commutantive operator. The important remark to be made is, that in applying this theorem there is nothing to prevent any of the systems being made *identical*; or, in other words, a given function of one system of variables may be regarded as a function of as many different, although coincident, sets as we may choose to suppose. Thus, suppose

$$U = Ax^2 + 2Bxy + Cy^2,$$

we may take the partial commutant formed of the two total commutant operators

$$\begin{array}{cc} \frac{d}{dx}, & \frac{d}{dy}, \\ \frac{d}{dx}, & \frac{d}{dy}, \end{array}$$

combined with itself. If we write them in the same order,

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\acute{d}}{dx}, \frac{\acute{d}}{dy},$$

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\acute{d}}{dx}, \frac{\dot{d}}{dy},$$

(where I use the dots and dashes to distinguish those in the same line which are considered as belonging to the same class, and therefore as permutable, *inter se*), we shall evidently obtain $4\{AC - B^2\}^2$; if we commence with a permutation, so as to have the form of operation

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\acute{d}}{dx}, \frac{\acute{d}}{dy},$$

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\acute{d}}{dx}, \frac{\dot{d}}{dy},$$

it will be found that we obtain $2\{AC - B^2\}^2$.

Again, suppose that we have

$$U = Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3.$$

If we write

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\acute{d}}{dx}, \frac{\acute{d}}{dy},$$

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\acute{d}}{dx}, \frac{\acute{d}}{dy},$$

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\acute{d}}{dx}, \frac{\acute{d}}{dy},$$

the value of the commutant would come out zero; but if we make a permutation, and write

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\acute{d}}{dx}, \frac{\acute{d}}{dy},$$

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\dot{d}}{dx}, \frac{\acute{d}}{dy},$$

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\acute{d}}{dx}, \frac{\dot{d}}{dy},$$

the operation indicated by the above performed upon U, will give a multiple of the discriminant of U.

In like manner we may represent Aronhold's Sextic Invariant of the form $(x, y, z)^3$ by means of the partial commutant

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\dot{d}}{dz}, \frac{\dot{d}}{dx}, \frac{\acute{d}}{dy}, \frac{\dot{d}}{dz},$$

$$\frac{\acute{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\dot{d}}{dz}, \frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\acute{d}}{dz},$$

$$\frac{\acute{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\dot{d}}{dz}, \frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\acute{d}}{dz}.$$

If we make

$$V = \left(\xi' \frac{d}{d\xi} + \eta' \frac{d}{d\eta} + \zeta' \frac{d}{d\zeta}\right) \left(\xi \frac{d}{dx} + \eta \frac{d}{dy} + \zeta \frac{d}{dz}\right)^2 (x, y, z)^3,$$

and use H to signify the determinant

$$\begin{vmatrix} x, & y, & z \\ \xi, & \eta, & \zeta \\ \xi', & \eta', & \zeta' \end{vmatrix},$$

which is evidently an universal triple covariant, and make

$$W = V + \lambda H,$$

and apply to W the partial commutantive symbol

$$\frac{\dot{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\dot{d}}{dz}, \frac{\acute{d}}{dx}, \frac{\dot{d}}{dy}, \frac{\dot{d}}{dz},$$

$$\frac{\dot{d}}{d\xi}, \frac{\dot{d}}{d\eta}, \frac{\dot{d}}{d\zeta}, \frac{\dot{d}}{d\xi}, \frac{\dot{d}}{d\eta}, \frac{\dot{d}}{d\zeta},$$

$$\frac{\dot{d}}{d\xi'}, \frac{\dot{d}}{d\eta'}, \frac{\dot{d}}{d\zeta'}, \frac{\dot{d}}{d\xi'}, \frac{\dot{d}}{d\eta'}, \frac{\dot{d}}{d\zeta'},$$

we shall obtain a function of λ of which all the odd powers and the second power will disappear, and such that the coefficients of λ^4 and the constant term will be Aronhold's S and T, and the discriminant of the entire function in respect to λ^2 (if not for the distribution assigned to the dots and dashes in the foregoing, at least for some other distribution) may not improbably be the discriminant of the given function $(x, y, z)^3$.

NOTES IN APPENDIX.

(1) [p. 295 above.] More generally, in as many ways as the number n can be divided into parts, in so many ways can a given function of one set of variables be as it were *unraveiled* so as to furnish concomitant forms.

For instance, the form $ax^3 + 3bx^2y + 3cxy^2 + dy^3$ has for a concomitant

$$aux + buy + bvx + cvy + cwx + dwy,$$

where u, v, w are cogredient with $x^2, 2xy, y^2$; and also

$$auu'x + buu'y + buv'x + bvu'x + cvv'x + cvu'y + cuv'y + dvv'y,$$

where u, v; u', v' are cogredient with each other and with x and y; and the proposition in the text may be best derived from this more general theorem by dividing the index into equal parts, forming as many systems as there are such parts, and then identifying the systems so formed.

(2) [p. 297 above.] The following additional example will illustrate the power of this method.

Let $\phi = (x, y, z)^4$ be the general function of the fourth degree. Form by *unravelment* the concomitant form $(u, v, w, p, q, r)^2$ (say P) where u, v, w, p, q, r are cogredient with $x^2, y^2, z^2, 2zy, 2xz, 2yx$.

Again, the universal concomitant $(x\xi + y\eta + z\zeta)^2$ will have for its concomitant

$$u\xi^2 + v\eta^2 + w\zeta^2 + p\eta\zeta + q\zeta\xi + r\xi\eta,$$

where ξ, η, ζ are contragredient to x, y, z. Now take the reciprocal polar of this last form with respect to ξ, η, ζ; that is,

$$\Sigma\,(vw - \tfrac{1}{4}p^2)\,x_1^2 + 2\Sigma\,(\tfrac{1}{4}qr - \tfrac{1}{2}pu)\,y_1z_1 \text{ (say } G),$$

where x_1, y_1, z_1, being contragredient to ξ, η, ζ, will be cogredient with x, y, z. $P + \lambda G$ is a quadratic function of the six variables u, v, w, p, q, r, and its discriminant will give a function of λ of the sixth degree, all of whose even coefficients will be covariants of ϕ. If we replace x_1, y_1, z_1 by x, y, z, these even coefficients will be respectively (understanding that *order* refers to the dimensions *quoad* the coefficients of ϕ and *degree* to the dimensions *quoad* x, y, z) as follows:

Of order	6	degree	0,
„	5	„	2,
„	4	„	4,
„	3	„	6,
„	2	„	8,
„	1	„	10,
„	0	„	12.

The two last coefficients must evidently be identically zero. It is *possible* that some of the others may be so too: as regards the one of the third order and sixth degree, this is of the same form as, and may be identical with, the Hessian of ϕ; as regards the one of the fourth order and fourth degree, this may be ϕ itself multiplied by the cubic invariant (which the theory of Section III. proves to exist) of ϕ. But the covariants of the fifth order and second degree, and of the second order and eighth degree, if they are not identically zero, and if the latter is not ϕ^2 (which a trial or two of some very simple cases will easily establish one way or the other) are probably irreducible forms. The existence of a correlated conic section to a curve of the fourth order, if established, would be particularly interesting, and its geometrical meaning would well deserve being elicited.

(3) [p. 303 above.] If any form (f) of the degree n be written symbolically,

$$(a_1x_1 + a_2x_2 + \ldots + a_\iota x_\iota)^n,$$

where $x_1, x_2 \ldots x_\iota$ are real but $a_1, a_2 \ldots a_\iota$ umbral, and if I_r be any invariant of the order r in respect of the real coefficients of (f), it is easily seen by reason of I_r remaining unaltered when $x_1, x_2 \ldots x_\iota$ become respectively $f_1x_1, f_2x_2 \ldots f_\iota x_\iota$, provided that $f_1, f_2 \ldots f_\iota = 1$, that each term in I_r expressed by means of the umbræ, must contain an equal number of times $a_1, a_2 \ldots a_\iota$, so that each such term will contain $\frac{nr}{\iota}$ of each of them, of course differently subdivided and grouped; hence we have the universal condition that $\frac{nr}{\iota}$ must be an integer; but this is less stringent than the actual condition, which is that $\frac{nr}{\iota}$ must be an integer of a certain form; for instance, as before observed, when $\iota = 2$, $\frac{nr}{\iota}$ must be an *even* integer.

(4) [p. 307 above.] To prove the theorem given in the text for total simple commutants it is only necessary to bear in mind that whenever two columns in any total commutant become identical, the commutant vanishes. To fix the ideas, take the commutant formed of lines similar to $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, written

under one another; let there be r such lines, the total number of terms will be $(1.2.3)^r$: the $1.2.3$ positions of the line written above will correspond to $(1.2.3)^{r-1}$ several groupings of the remaining lines. Now when x, y, z undergo a unimodular linear substitution, $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ will undergo a related substitution not coincident with that of x, y, z, but still unimodular; let x, y, z change, all the other systems remaining fixed, and suppose $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ to become respectively

$$f\frac{d}{dx}+g\frac{d}{dy}+h\frac{d}{dz},$$

$$f'\frac{d}{dx}+g'\frac{d}{dy}+h'\frac{d}{dz},$$

$$f''\frac{d}{dx}+g''\frac{d}{dy}+h''\frac{d}{dz},$$

then each of the $(1.2.3)^{r-1}$ groups of the terms arising from the permutation of $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ will subdivide into 27 groups, of which we may reject those in which any of the terms $\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)$ occurs twice or three times; accordingly there will be left only the six effective orders of permutations,

$$\left(f\frac{d}{dx}, g'\frac{d}{dy}, h''\frac{d}{dz}\right); \left(f\frac{d}{dx}, h'\frac{d}{dz}, g''\frac{d}{dy}\right); \text{ \&c.}$$

consequently each of the $(1.2.3)^{r-1}$ groups gives rise to 6 times 6 products whose sum will be $\begin{vmatrix} f'', & g'', & h'' \\ f', & g', & h' \\ f, & g, & h \end{vmatrix}$ × the sum of the 6 products corresponding to the permutations of $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$; and therefore, the transformations being unimodular, the sum of the products corresponding to the entire $(1.2.3)^r$ permutations remains constant when x, y, z change. In like manner, all the systems may change one after the other, and consequently all of them at the same time without affecting the value of the commutant: and in like manner for the general case. Q.E.D.

(5) [p. 312 above.] The truth of the proposition relative to compound commutants and the mode of the demonstration will be apparent from the subjoined example.

Let the function be supposed to be

$$(ax+by)(a'x'+b'y')(\alpha\xi+\beta\eta)(\alpha'\xi'+\beta'\eta'),$$

where x, y; x', y' are cogredient and ξ, η; ξ', η' cogredient; the a, b, α, β, &c. are of course mere umbræ. Now take the compound commutant

$$\begin{array}{lll} aa', & ab'+a'b, & bb', \\ \alpha\alpha', & \alpha\beta'+\alpha'\beta, & \beta\beta'. \end{array}$$

Let x, y; x', y' undergo a linear substitution, and, accordingly,

$$\begin{array}{lll} \text{let } a & \text{become} & fa+gb, \\ a' & \text{,,} & fa'+gb', \\ b & \text{,,} & ha+kb, \\ b' & \text{,,} & ha'+kb', \end{array}$$

f, g, h, k being of course actual and not umbral; then the above commutant will be easily seen to decompose into 6 others, which will be equal to the original commutant multiplied by the determinant

$$\begin{vmatrix} f^2, & 2fg, & g^2 \\ fh, & fk+gh, & gk \\ h^2, & 2hk, & k^2 \end{vmatrix},$$

which is equal to $(fk-gh)^3$, that is $=1$.

And so in general, which shows, as in the preceding note, that all the classes of cogredient systems may be transformed successively one after the other, and therefore simultaneously, without altering the value of the commutant.

(6) In the last May Number* of the *Journal*, Mr Boole, to whose modest labours the subject is perhaps at least as much indebted as to any one other writer, has given a theorem†, (14) p. 94, the excellent idea contained in which there is no difficulty in shaping so as to render it generalizable by aid of the theory of contravariants. It may be regarded in some sort a pendant or reciprocal to the Eisenstein-Hermite theorem, presented by me under a wider aspect in the First Section of this paper.

[* *Camb. and Dub. Math. Journ.* Vol. VI. (1851), pp. 87—106.]

† Mr Boole applied his theorem to obtain the cubic invariant of $(x, y)^4$, say $\phi(x, y)$, by operating upon its Hessian with $\phi\left(\frac{d}{dy}, -\frac{d}{dx}\right)$. More generally, when $\phi(x, y)=(x, y)^{2n}$, the catalecticant of the antepenultimate emanant of ϕ is also of the degree $2n$; and this, when operated upon by $\phi\left(\frac{d}{dy}, -\frac{d}{dx}\right)$, will give an invariant of the order $n+1$, which is probably identical with the catalecticant of ϕ itself. There exists a most interesting transformation of the catalecticant of any emanant of a function of any degree in x, y, whether even or odd, under the form of a determinant some of the lines of which contain combinations only of x and y, without any of the coefficients, and all the rest the coefficients only of the given function without x or y. The Hessian being the catalecticant of the second emanant is of course included within this statement.

Let $\phi(x, y \ldots z)$ have any contravariant $\theta(x, y \ldots z)$; then will

$$\phi\left(\frac{d}{dx}, \frac{d}{dy} \cdots\cdots \frac{d}{dz}\right) . \theta(x, y \ldots z)$$

be a contravariant of ϕ. For orthogonal transformations the terms contravariant and covariant coincide, and the theorem for this case appears to have been known to Mr Boole, see (15), same page. More generally, if ψ and θ be any two concomitants of ϕ, the algebraical product $\psi\theta$ will also be a concomitant of ϕ, provided that the systems of variables in ψ and θ have all distinct names, or that those which bear the same names are cogredient with one another. If this proviso does not hold good, the product in question will evidently be no longer a concomitant of ϕ. Let however Ψ denote what ψ becomes, and ϑ what θ becomes, when in place of the variables $x, y \ldots z$ of every two contragredient synonymous systems in ψ and θ we write $\frac{d}{dx}, \frac{d}{dy} \cdots\cdots \frac{d}{dz}$, then will $\vartheta\psi$ and $\Psi\theta$ be each of them concomitants of ϕ, the synonymous systems becoming cogredient with ψ in the one case and with θ in the other.

(7) There is one principle of *paramount* importance which has not been touched upon in the preceding pages, which I am very far from supposing to exhaust the fundamental conceptions of the subject, (indeed, not to name other points of enquiry, I have reason to suppose that the idea of contragredience itself admits of indefinite extension through the medium of the reciprocal properties of commutants; the particular kind of contragredience hereinbefore considered having reference to the reciprocal properties of ordinary determinants only).

The principle now in question consists in introducing the idea of *continuous* or *infinitesimal* variation into the theory. To fix the ideas, suppose C to be a function of the coefficients of $\phi(x, y, z)$, such that it remains unaltered when x, y, z become respectively fx, gy, hz, provided that $fgh = 1$. Next, suppose that C does not alter when x becomes $x + ey + \epsilon z$, when e and ϵ are indefinitely small: it is easily and obviously demonstrable that if this be true for e and ϵ indefinitely small, it must be true for *all* values of e and ϵ. Again, suppose that C alters neither when x receives such an infinitesimal increment, y and z remaining constant, nor when y nor z separately receive corresponding increments, z, x and x, y in the respective cases remaining constant; it then follows from what has been stated above that this remains true for finite increments to x or y or z separately; and hence it may easily be shown that C will remain constant for any *concurrent* linear transformations of x, y, z, when the modulus is unity. This all-important principle enables us at once to fix the form of the symmetrical functions of the roots of $\phi\left(\frac{x}{y}, 1\right)$ which represent invariants of $\phi(x, y)$ when the coefficient of the

highest power of x is made unity. It also *instantaneously* gives the necessary and *sufficient* conditions to which an invariant of any given order of any homogeneous function whatever is subject, and thereby reduces the problem of discovering invariants to a definite form. But as these conditions coincide with those which have been stated to me as derived from other considerations by the gentleman whose labours in this department are *concomitant* with my own, I feel myself bound to abstain from pressing my conclusions until he has given his results to the press.

(8) By aid of the general principle enunciated in Note (6) above, we can easily obtain Aronhold's S and T. Let U be the given cubic function of x, y, z, and let $G(x, y, z; \xi, \eta, \zeta)$ be the polar reciprocal in respect to ξ, η, ζ of $\left(\xi\frac{d}{dx} + \eta\frac{d}{dy} + \zeta\frac{d}{dz}\right)^2 U$, then $G(\xi, \eta, \zeta; x, y, z)$ as well as the former G will be a concomitant to U, but the homonymous systems of variables in the two G's will be contragredient; and, accordingly,

$$G\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}; \frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}\right) . G(\xi, \eta, \zeta; x, y, z)$$

will be a concomitant to U; this concomitant is readily seen to be an invariant of the fourth order; that is, Aronhold's S. Again, from S, by means of the Eisenstein-Hermite theorem, we may derive a form $K(x, y, z)$ of the third degree in x, y, z, and whose coefficients will be of three dimensions; and, accordingly, if the Hessian of U be called $H(U)$,

$$K\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right) . H(U)$$

will be a Sextic Invariant of U, that is, Aronhold's T.

43.

ON THE PRINCIPLES OF THE CALCULUS OF FORMS.

[*Cambridge and Dublin Mathematical Journal*, VII. (1852), pp. 179—217.]

PART I. SECTION IV. *Reciprocity, also Properties and Analogies of certain Invariants, &c.*

IT will hereafter be found extremely convenient to represent all systems of variables cogredient with the original system in the primitive form by letters of the Roman, and all contragredient systems by letters of the Greek alphabet; the rules for concomitance may then be applied without paying any regard to the distinction between the direction of the march of the substitutions, the variables at the close of each operation as it were telling their own tale in respect of being cogredients or contragredients. This distinction has not (as it should have) been uniformly observed in the preceding sections; as, for instance, in the notation for emanants which have been derived by the application of the symbol $\left(\xi\frac{d}{dx}+\eta\frac{d}{dy}+\&c.\right)^2$, instead of the more appropriate one $\left(x'\frac{d}{dx}+y'\frac{d}{dy}+\&c.\right)^2$.

The observations in this section will refer exclusively to points of doctrine which have been started in the preceding sections in such order as they more readily happen to present themselves. And, first, as to some important applications of the reciprocity method referred to in Notes (6) and (8) of the Appendix [pp. 325, 327 above].

The practical application of this method will be found greatly facilitated by the rule that x, y, z, &c. may always in any combination of concomitants be replaced respectively by $\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}$, &c., and *vice versâ*. I shall apply this prolific principle of reciprocity to elucidate some of the properties and relations of Aronhold's S and T, and certain other kindred forms. This S and T are the quartinvariant and sextinvariant respectively of a cubic of three variables. I give the names of s and t to the quadrinvariant and cubinvariant of the quartic function of two variables. Furthermore, whoever will consider attentively the remarks made in Section II. of the foregoing relative to reciprocal polars, will apprehend without any difficulty that to every invariant of a function of any degree of any number of variables will

correspond a contravariant of a function of the same degree of variables one more in number, and that between such invariants, whatever relations exist expressed independently of all other quantities, precisely the same relations must exist between the corresponding contravariants. Thus, then, to s and t the two invariants of $(x, y)^4$ will correspond two contravariants σ and τ of $(x, y, z)^4$, and to S and T the two invariants of $(x, y, z)^3$ will correspond Σ and ϑ two contravariants of $(x, y, z, t)^3$. Calling r the resultant of $(x, y)^4$, R the resultant of $(x, y, z)^3$, ρ the polar reciprocal, or, more briefly, the reciprocant of $(x, y, z)^4$, and (R) the reciprocant of $(x, y, z, t)^3$, we have the following equations (presuming that all the quantities are previously affected with the proper numerical multipliers), namely

$$r = s^3 + t^2, \qquad \rho = \sigma^3 + \tau^2,$$

$$R = S^3 + T^2, \quad (R) = \Sigma^3 + \vartheta^2.$$

I propose in this First Annotation to point out the remarkable analogies which exist between the modes of generating the four pairs of quantities s, t, &c., the functions severally corresponding to which I shall call u, ω, U, Ω. The Hessian corresponding to any of these functions will be denoted by an H prefixed, and when we have to consider, not the pure Hessian, but the matrix formed from it by adding a vertical and horizontal border of variables, the same in number but contragredient to the variable of the function (as, for instance, the Hessian of u bordered with ξ, η horizontally and vertically, or of U with ξ, η, ζ), then I shall denote the result by the ruled symbol $\overline{H}$, and if there be occasion to add two borders, as $\xi, \eta, \zeta;\ \xi', \eta', \zeta'$, both repeated in the horizontal and vertical directions, the result will be typified by the doubly ruled $\overline{\overline{H}}$.

Now, in the first place, as observed by me in Note (8) of the Appendix in the last number; if we call the coefficients of U (10 in number) a, b, c, d, &c., we have

$$S = \overline{H}\left\{\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta};\ \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right\}\overline{H}\{x, y, z;\ \xi, \eta, \zeta\},$$

also

$$T = \frac{dS}{da}\frac{d^3H}{dx^3} + \frac{dS}{db}\frac{d^3H}{d^2x\,dy} + \frac{dS}{dc}\frac{d^3H}{d^2x\,dz} + \&c.$$

I will now add the further important relation

$$S^2 = \frac{dT}{da}\frac{d^3H}{dx^3} + \frac{dT}{db}\frac{d^3H}{d^2x\,dy} + \frac{dT}{dc}\frac{d^3H}{d^2x\,dz} + \&c.^*,$$

* It will be found hereafter convenient to designate contravariants formed in this manner from invariants as *Evects* of such invariants or contravariants, and according to the number of times that such process of derivation is applied, 1st, 2nd, 3rd, &c. evects. Such evects form a peculiar class, and when considered generally, without reference to the base to which they refer, they may be termed evectants. Evectants will be again distinguishable according as their base is an invariant simply or a contravariant. Perhaps the terms pure and affected evectants may serve to mark this distinction.

so that it will be observed if all the derivatives of S are zero, T is zero, and *vice versâ*.

Precisely in the same way, using h and $\bar{\text{h}}$ to denote respectively the Hessian of u and the same bordered with ξ, η, we have

$$s = \bar{\text{h}}\left(\frac{d}{d\xi}, \frac{d}{d\eta}; \frac{d}{dx}, \frac{d}{dy}\right)\bar{\text{h}}\,(x, y; \xi, \eta),$$

$$t = \frac{ds}{da}\frac{d^4h}{dx^4} + \frac{ds}{db}\frac{d^4h}{dx^3dy} + \frac{ds}{dc}\frac{d^4h}{dx^2dy^2} + \&c.$$

$$s^2 = \frac{dt}{da}\frac{d^4h}{dx^4} + \frac{dt}{db}\frac{d^4h}{dx^3dy} + \frac{dt}{dc}\frac{d^4h}{dx^2dy^2} + \&c.$$

Again, taking $(\bar{\bar{H}})$ the second bordered Hessian of Ω; that is, Ω bordered as well horizontally as vertically with the double lines and columns ξ, η, ζ, θ; $\xi', \eta', \zeta', \theta'$,

$$\Sigma = (\bar{\bar{H}})\left(\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}, \frac{d}{d\theta}; \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \frac{d}{dt}; \xi', \eta', \zeta', \theta'\right)$$
$$\times (\bar{\bar{H}})(x, y, z, t; \xi, \eta, \zeta, \theta; \xi', \eta', \zeta', \theta').$$

$$\vartheta = \frac{d\Sigma}{da}\frac{d^3\bar{H}}{dx^3} + \frac{d\Sigma}{db}\frac{d^3\bar{H}}{dx^2dy} + \frac{d\Sigma}{dc}\frac{d^3\bar{H}}{dx^2dz} + \frac{d\Sigma}{dd}\frac{d^3\bar{H}}{dx^2dt} + \&c.,$$

$$\Sigma^2 = \frac{d\vartheta}{da}\frac{d^3\bar{H}}{dx^3} + \frac{d\vartheta}{db}\frac{d^3\bar{H}}{dx^2dy} + \&c.$$

In like manner again

$$\sigma = (\bar{h})\left\{\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}; \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}; \xi', \eta', \zeta'\right\}$$
$$\times \bar{\bar{h}}\{x, y, z; \xi, \eta, \zeta; \xi', \eta', \zeta'\},$$

$$\tau = \frac{d\sigma}{da}\frac{d^4(\bar{h})}{dx^4} + \&c.,$$

$$\sigma^2 = \frac{d\tau}{da}\frac{d^4\bar{h}}{dx^4} + \&c.,$$

σ and τ are the same quantities as are calculated by Mr Salmon, in his inestimable work *On Higher Plane Curves*, but are there expressed under the names of S and T, with the sole difference that in place of x, y, z, used by Mr Salmon, the contragredient variables ξ', η', ζ' are used in the expressions above. Mr Salmon has also pointed out to me that σ may be obtained by operating with

$$\left(\xi^4\frac{d}{da} + \xi^3\eta\frac{d}{db} + \xi^3\zeta\frac{d}{dc} + \&c.\right)$$

directly upon I a cubic invariant of the function u, or $(x, y, z)^4$. This I is no other than the simple commutant obtained by operating upon u with the commutantive symbol formed by taking four times over the line $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, agreeable to the remark made in the third section that

every function of an even degree of n variables possesses an invariant of the nth order in extension of Mr Cayley's observation that every such function of two variables possesses a quadrinvariant, that is an invariant of the second order.

I need hardly remark that σ is of 2 dimensions in the coefficients and of 4 in the contragredient variables, τ of 3 in the coefficients and of 5 in the contragredients, Σ of 4 in the constants and 4 in the contragredients, ϑ of 6 in the constants and 6 in the contragredients, or that the single-bordered Hessians of u and U and the double-bordered Hessians of ω and Ω are each of them quadratic in respect of the x &c. as well as of the ξ &c. systems.

If the right numerical factors be attributed to S, T, Aronhold has shown that

$$H\{H(U)\} + T.H(U) + S^2U = 0,$$

and in my paper in the last May Number*, I gave the equation

$$h\{h(u)\} + s.h(u) + tu = 0.$$

I think it highly probable that it will be found that the analogous equations obtain, namely

$$\bar{H}\{\bar{H}(\Omega)\} + \vartheta.\bar{H}(\Omega) + \Sigma^2\Omega = 0,$$
$$\bar{h}\{\bar{h}(\omega)\} + \sigma.\bar{h}(\omega) + \tau\omega = 0.$$

These remarkable equations, if verified (of which I can scarcely doubt), will be most powerful aids to the dissection of the forms ω, Ω, and thereby to the detection of the fundamental properties of curves of the fourth and surfaces of the third degree, of which at present so little is known. It will have been observed that in the preceding developments the contravariants of ω and Ω were derived in precisely the same way from ω and Ω as the corresponding invariants of u and U from u and U, with the sole difference that the Hessian used in the two latter cases is replaced by a single-bordered Hessian in the two former cases, and a single-bordered Hessian in the two latter by a double-bordered Hessian in the two former. The analogies are not even yet stated exhaustively; for it will be remembered (as shown in the third section), that T and S can be derived directly and concurrently by means of operating with the commutantive symbol

$$\left.\begin{matrix} \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz} \\ \frac{d}{dx}, & \frac{d}{dy}, & \frac{d}{dz} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \end{matrix}\right\} \text{upon } \bar{H}(U) + \lambda(x\xi + y\eta + z\zeta)^2,$$

[* p. 192 above.]

which gives a result of the form $m(\lambda^3 + S\lambda + T)$, m being a number; and I conjecture that if

$$\frac{d}{dx},\ \frac{d}{dy},\ \frac{d}{dz},\ \frac{d}{dt},$$
$$\frac{d}{dx},\ \frac{d}{dy},\ \frac{d}{dz},\ \frac{d}{dt},$$
$$\frac{d}{d\xi},\ \frac{d}{d\eta},\ \frac{d}{d\zeta},\ \frac{d}{d\theta},$$
$$\frac{d}{d\xi},\ \frac{d}{d\eta},\ \frac{d}{d\zeta},\ \frac{d}{d\theta},$$

be made to operate upon

$$\overline{\overline{H}}\Omega + \lambda(x\xi + y\eta + z\zeta + t\theta)^2,$$

and the result be put under the form

$$m(\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D),$$

that A will be zero, B and C will be respectively Σ and ϑ, and perhaps D (a contravariant, if it effectively exist, of 8 dimensions in the coefficients of Ω, and of a like number in the contragredients ξ', η', ζ', θ'), also zero. But of the evanescence of D I do not speak with any degree of assurance.

Mr Salmon has made an excellent observation to the effect that if we call (σ) what σ becomes when ξ', η', ζ' are replaced by $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$, $(\sigma)h(\omega)$ will represent a covariant to ω of $3 + 2$, that is, 5 dimensions in the coefficients, and of $6 - 4$, that is, of 2 dimensions in x, y, z, $h(\omega)$ being of 3 and 6 dimensions in these respectively, and σ of 2 and 4 dimensions respectively in the same. Now these resulting dimensions 5 and 2 precisely agree with the form especially noticed by me in Note* (2) of the Appendix, where it was derived as one of a group by the method of unravelment. There can be little doubt that these two conics each of them indissolubly connected with every curve of the fourth degree are identical. The form $(\sigma)h(\omega)$ enables us to prove readily (thanks to Mr Salmon's calculation of σ, given in his *Higher Plane Curves*, under the name of S) that this is a *bonâ fide* existent conic.

For if we take a particular case of ω, say

$$\omega = a_1x^4 + b_2y^4 + c_3z^4 + 6dy^2z^2,$$

we find

$$h(\omega) = \begin{vmatrix} a_1x^2, & 0, & 0 \\ 0, & b_2y^2 + dz^2, & dyz \\ 0, & dyz, & c_3z^2 + dy^2 \end{vmatrix}$$
$$= a_1(b_2c_3 + d^2)x^2y^2z^2 + a_1b_2dx^2y^4 + a_1c_3dx^2z^4,$$

[* p. 323 above.]

and σ becomes

$$a_1d\eta'^2\zeta'^2,$$

and consequently (σ) is

$$a_1d\left(\frac{d}{dy}\right)^2\left(\frac{d}{dz}\right)^2,$$

and therefore

$$(\sigma)h(\omega) = 4a_1^2d(b_2c_3 + d^2)x^2,$$

the conic here reducing to a pair of coincident straight lines. This example demonstrates that the conic is in general actually existent.

As I have said so much upon S and T it may not be irrelevant to state in this place how I obtained the conditions for U, the characteristic of the curve of the third degree becoming the characteristic of a conic and a straight line, that is breaking up into a linear and a quadratic factor, which Mr Salmon has inserted in the notes to his work above referred to. When U is of this form it may obviously by linear transformations be expressed by $ax^3 + 6dxyz$, but when starting with the general form,

$$a_1x^3 + b_2y^3 + c_3z^3 + \text{\&c.} + 6Dxyz,$$

we form two contravariants from S and T, to wit

$$\left(\xi^3\frac{d}{da_1} + \eta^3\frac{d}{db_2} + \zeta^3\frac{d}{dc_3} + \text{\&c.} + \xi\eta\zeta\frac{d}{dD}\right)S, \text{ say } S',$$
$$\left(\xi^3\frac{d}{da_1} + \eta^3\frac{d}{db_2} + \zeta^3\frac{d}{dc_3} + \text{\&c.} + \xi\eta\zeta\frac{d}{dD}\right)T, \text{ say } T',$$

and then make $a_1 = a$, $D = d$, and all the other coefficients zero, it will easily be seen on examining the forms of S and T, given by Mr Salmon, that (S) and (T) (the evectants of S and T) become respectively

$$4d^3\xi\eta\zeta,\quad 31d^5\xi\eta\zeta;$$

we have therefore $(T) + \lambda(S) = 0$: and (T) and (S), although contravariantive to their primitive U, are covariantive with one another, so that $(T) + \lambda(S) = 0$ is a persistent relation unaffected by linear transformations; it follows therefore that when U is of, or reducible to, the form supposed,

$$\frac{dS}{da_1} : \frac{dS}{db_2} : \frac{dS}{dc_3} : \text{\&c.} : \frac{dS}{dD}$$
$$= \frac{dT}{da_1} : \frac{dT}{db_2} : \frac{dT}{dc_3} : \text{\&c.} : \frac{dT}{dD},$$

which is the criterion given in the note referred to*.

I am also able to obtain these equations more directly by another method founded upon a New View of the Theory of Elimination, an account of which,

* Mr Salmon has remarked that the two evectants (S) and (T) intersect in the nine cuspidal points of the polar reciprocal to the curve.

however, I must reserve for another occasion, but which, I may mention, serves to fix not merely the conditions, as in the ordinary restricted theory, that a given set of equations may be simultaneously satisfiable by some one system of values of the variables, but the *conditions* that such set of equations may be simultaneously satisfiable by any given number of distinct systems of variables.

Mr Salmon has remarked to me to the effect that if in τ we write $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, in place of the contragredients, and call τ so altered (τ), then $(\tau)h(\omega)$ will be an invariant of 6 dimensions in the coefficients of ω. This sextinvariant I have little doubt is identical with that obtained by operating upon ω with the commutantive symbol

$$\left\{\begin{matrix} \left(\frac{d}{dx}\right)^2, & \frac{d}{dx}\frac{d}{dy}, & \left(\frac{d}{dy}\right)^2, & \frac{d}{dy}\frac{d}{dz}, & \left(\frac{d}{dz}\right)^2, & \frac{d}{dz}\frac{d}{dx} \\ \left(\frac{d}{dx}\right)^2, & \frac{d}{dx}\frac{d}{dy}, & \left(\frac{d}{dy}\right)^2, & \frac{d}{dy}\frac{d}{dz}, & \left(\frac{d}{dz}\right)^2, & \frac{d}{dz}\frac{d}{dx} \end{matrix}\right\}.$$

This, like every other commutant of 2 lines only, is of course capable of being expressed under the form of an ordinary determinant, and the remark is not without interest, as showing how the proposition known with respect to quadratic functions of any number of variables, namely of every such having an invariantive determinant, lends itself to the general case of functions of any even degree of any number of variables which also have always an invariantive determinant attached to them, of which the terms are simple coefficients of such functions. The only peculiarity (if it be one) of quadratic functions in this respect being that they have each but one invariant of such form and no other. In the case before us, if we write

$$\omega = a_1x^4 + b_2y^4 + c_3z^4 + 4a_2x^3y + 4a_3x^3z + 4b_1y^3x + 4b_3y^3z + 4c_1z^3x + 4c_2z^3y$$
$$+ 6dy^2z^2 + 6ez^2x^2 + 6fx^2y^2 + 12lx^2yz + 12mxy^2z + 12nxyz^2,$$

the sextinvariant in question becomes representable under the form of the determinant

$$\begin{vmatrix} a_1, & a_2, & f, & l, & e, & a_3 \\ a_2, & f, & b_1, & m, & n, & l \\ f, & b_1, & b_2, & b_3, & d, & m \\ l, & m, & b_3, & d, & c_2, & n \\ e, & n, & d, & c_2, & c_3, & c_1 \\ a_3, & l, & m, & n, & c_1, & e \end{vmatrix}\,*.$$

* This determinant is identical with the determinant formed by taking the second differential coefficients of the function and arranging in the usual manner the coefficients of the several powers and combinations of powers of the variables treated as if they were independent quantities.

Before quitting the subject of S and T the two invariants of the cubic function of 3 variables, or, as it may be termed, of the cubic curve, it may not be amiss to give the complete table which I have formed corresponding to all the singular cases which can befall such curve, which will be seen below to be eight in number; it is of the highest importance to push forward the advanced posts of geometry, and for this purpose to obtain the same kind of absolute power and authority over, and clear and absolute knowledge of, the properties and affections of cubic forms as have been already attained for forms of the second degree.

Let $$U = ax^3 + 4bx^2y + 4cx^2z + \&c.$$

(1) When U has one double point $S^3 + T^2 = 0$.

(2) When U has two double points, that is becomes a conic and right line

$$\frac{dS}{da}\frac{dT}{db} - \frac{dS}{db}\frac{dT}{da} = 0, \ \&c. \ \&c.$$

(3) When U has a cusp $S = 0$, $T = 0$.

(4) When U has two coincident double points, that is, is a conic and a tangent line thereto, which comprises the two preceding cases in one,

$$\frac{dT}{da} = 0, \quad \frac{dT}{db} = 0, \ \&c.$$

and also *therefore* $$S = 0.$$

(5) When U becomes three right lines forming a triangle

$$\frac{d^2S}{da\,db}\frac{d^2T}{dc\,de} - \frac{d^2T}{da\,db}\frac{d^2S}{dc\,de} = 0, \ \&c.$$

where a, b, c, e each represent any of the coefficients arbitrarily chosen, whether distinct or identical.

Another, and lower in degree system of equations, may be substituted for the above, obtained by affirming the equality of the ratios between the coefficients of U and the corresponding coefficients of its Hessian.

(6) When U represents a pencil of three rays meeting in a point

$$\frac{dS}{da} = 0, \quad \frac{dS}{db} = 0, \ \&c.$$

and also *therefore* $$T = 0.$$

Also in place of this system may be substituted the system obtained by taking all the coefficients of the Hessian zero.

(7) When U becomes a line, and two other coincident lines,

$$\frac{dS}{da}=0, \quad \frac{dS}{db}=0, \text{ \&c.}$$

and also

$$\frac{d^2T}{da^2}=0, \quad \frac{d^2T}{da\,db}=0, \text{ \&c.}$$

I have not ascertained whether this second system necessarily implies the first; I rather think that it does not. In the preceding case also it would be interesting to show the direct algebraical connexion between the system formed by the coefficients of the Hessian and the system consisting of the first derivatives of S.

(8) When U becomes a perfect cube representing three coincident right lines

$$\frac{d^2S}{da^2}=0, \quad \frac{d^2S}{da\,db}=0, \text{ \&c.}$$

and

$$\frac{d^2T}{da^2}=0, \quad \frac{d^2T}{da\,db}=0, \text{ \&c.}$$

The first of these systems of equations necessarily implies the equations $\frac{dT}{da}=0$, $\frac{dT}{db}=0$, &c., as is obvious from the equation

$$T=\frac{dS}{da}\frac{d^3H}{dx^3}+\frac{dS}{db}\frac{d^3H}{dx^2dy}+\text{ \&c.}$$

but not necessarily the second and lower system $\frac{d^2T}{da^2}=0$, &c. above written.

So if we take

$$u=ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4$$

when 2 roots are equal

$$s^3+t^2=0,$$

when 2 pairs of roots are equal

$$\frac{ds}{da}\frac{dt}{db}-\frac{ds}{db}\frac{dt}{da}=0, \text{ \&c.,}$$

when 3 roots are equal $\quad s=0, \quad t=0,$

and when all 4 roots are equal

$$\frac{dt}{da}=0, \quad \frac{dt}{db}=0, \text{ \&c.}$$

Before closing this Section I may make a remark, in reference to the sextic invariant of ω, which admits of being extended to all commutants formed by operating upon the function with a commutantive symbol obtained by writing over one another lines consisting of powers of $\frac{d}{dx}$, $\frac{d}{dy}$, &c. and

their combinations (to which, in the Third Section, I gave the name of *compound* commutants, a qualification which, for reasons that will hereafter be adduced, I think it advisable to withdraw). The remark I have to make is this, namely that the invariant obtained by operating upon ω with

$$\left.\begin{matrix} \left(\frac{d}{dx}\right)^2, & \frac{d}{dx}\frac{d}{dy}, & \left(\frac{d}{dy}\right)^2, & \frac{d}{dy}\frac{d}{dz}, & \left(\frac{d}{dz}\right)^2, & \frac{d}{dz}\frac{d}{dx} \\ \left(\frac{d}{dx}\right)^2, & \frac{d}{dx}\frac{d}{dy}, & \left(\frac{d}{dy}\right)^2, & \frac{d}{dy}\frac{d}{dz}, & \left(\frac{d}{dz}\right)^2, & \frac{d}{dz}\frac{d}{dx} \end{matrix}\right\},$$

is precisely the same as may be obtained by operating with

$$\left.\begin{matrix} \frac{d}{du}, & \frac{d}{dv}, & \frac{d}{dw}, & \frac{d}{dp}, & \frac{d}{dq}, & \frac{d}{dr} \\ \frac{d}{du}, & \frac{d}{dv}, & \frac{d}{dw}, & \frac{d}{dp}, & \frac{d}{dq}, & \frac{d}{dr} \end{matrix}\right\}$$

upon the concomitant quadratic function to ω obtained by the method of unravelment, as in Note (2) of the Appendix [p. 322 above]; and so, in general, every commutant obtained by operating upon a function of any number of variables of the degree $2mp$ with a symbol consisting of $2p$ lines in which the mth powers of $\frac{d}{dx}$, $\frac{d}{dy}$, &c. and their mth combinations occur, will be identical with the commutant obtained by operating with a symbol also of $2p$ lines, in which only the simple powers occur of $\frac{d}{du}$, $\frac{d}{dv}$, &c. (where u, v, &c. are cogredient with x^p, $x^{p-1}y$, &c.), upon a function of u, v, &c., formed by the method of unravelment from the given function.

Finally, before quitting the subject of reciprocity, I may state, it follows from the general statement made at the commencement of this Section, that inasmuch as

$$(x\xi+y\eta+z\zeta+\text{\&c.})^2$$

is a universal concomitant form, so also must

$$\left(\frac{d}{d\xi}\frac{d}{dx}+\frac{d}{d\eta}\frac{d}{dy}+\frac{d}{d\zeta}\frac{d}{dz}+\text{\&c.}\right)^2$$

be a universal concomitant symbol of operation; accordingly it is certain that any concomitant in which x, y, z, &c., ξ, η, ζ, &c. enter, operated upon with this symbol, will remain a concomitant: in several cases which I have examined, the effect of this operation will be to produce an evanescent form, but I see no ground for supposing that this is other than an accidental, or at all events for supposing that it is a necessary and universal consequence of the operation. It may also be observed that in the case of as many cogredient sets of variables as variables in each set, as for instance 3 sets of 3 variables each, the determinant which may be formed by arranging them in regular order, as

$$\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix},$$

is evidently a universal concomitant, and moreover an equivocal concomitant, possessing the property of remaining a concomitant when the variables are respectively but simultaneously exchanged for their contragredients ξ, η, ζ; ξ', η', ζ'; ξ'', η'', ζ''; which shows also that in place of the variables may be written the differential operators

$$\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}; \frac{d}{dx'}, \frac{d}{dy'}, \frac{d}{dz'}; \frac{d}{dx''}, \frac{d}{dy''}, \frac{d}{dz''}:$$

a remark which leads us to see the exact place in the general theory occupied by Mr Cayley's method of generating covariants given in the concluding paragraph of the First Section [p. 290 above]. I may likewise add, that inasmuch as $(x'\xi + y'\eta + z'\zeta + \&c.)^r$ is a universal concomitant,

$$\left(x' \frac{d}{dx} + y' \frac{d}{dy} + \&c.\right)^r$$

will be so too, by virtue of the general law of interchange, which conducts immediately to the theory of emanation, showing that this last symbol, operating upon any function, furnishes covariants thereunto for any integer value of z.

One additional interesting remark presents itself to be made concerning U, the cubic function of x, y, z, which is, that calling as before T its sextic invariant, and $a, 3b, 3c, d$, &c. the coefficients, the formula

$$\left(\xi^3 \frac{d}{da} + \xi^2\eta \frac{d}{db} + \xi^2\zeta \frac{d}{dc} + \xi\eta\zeta \frac{d}{dd} + \&c.\right)^2 T$$

will give the polar reciprocal, or, as it has been agreed to term it, the reciprocant of U. I believe the remark of the probability of this being the case originated with myself, but Mr Cayley first verified it by actual calculation, using for that purpose the value of T, given by Mr Salmon in his work *On the Higher Plane Curves*, already frequently alluded to, which is an indispensable manual equally for the objects of the higher special geometry as for the new or universal algebra, being in fact a common ground where the two sciences meet and render mutual aid.

Mr Salmon also observed, that the first evect of T, namely

$$\left(\xi^3 \frac{d}{da} + \xi^2\eta \frac{d}{db} + \&c.\right) T,$$

was identical in form with what may be termed the first devect of the polar reciprocal, that is, the result of operating upon the polar reciprocal with what U becomes when $\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}$, are substituted in the stead of x, y, z. And inasmuch as, by Euler's law,

$$\left\{a\left(\frac{d}{d\xi}\right)^3 + 3b\left(\frac{d}{d\xi}\right)^2 \frac{d}{d\eta} + \&c.\right\} \times \left\{\xi^3 \frac{d}{da} + \xi^2\eta \frac{d}{db} + \&c.\right\} T$$
$$= 6\left\{a \frac{d}{da} + b \frac{d}{db} + \&c.\right\} T = 36T,$$

it follows that T is the second *devect* of the polar reciprocal, or at least identical with it in point of form. But, since the preceding matter was printed, I have discovered in the course of a most instructive and suggestive correspondence with Mr Salmon, the principle upon which these and similar identifications depend, thereby dispensing with the necessity for the excessively tedious labour of verification which, even in the simple example before us, would be found to extend over several pages of work.

The theory in which this principle is involved will be given, along with other very important matter, in the next number of the *Journal*.

Supplementary Observations on the Method of Reciprocity.

It has been observed, that ξ, η, &c. may always be inserted in place of $\frac{d}{dx}, \frac{d}{dy}$, &c., and *vice versâ*, in a concomitant form, without destroying its concomitance. Accordingly, instead of the evector symbol

$$\xi^3 \frac{d}{da} + \xi^2\eta \frac{d}{db} + \&c.,$$

we may employ

$$\left(\frac{d}{dx}\right)^3 \frac{d}{da} + \left(\frac{d}{dx}\right)^2 \frac{d}{dy} \frac{d}{db} + \&c.;$$

and operating with this upon any concomitant, the result will be a concomitant. Hence we see, for example, that if we take the concomitant SH formed by the product of the invariant S and the covariant H,

$$\left\{\left(\frac{d}{dx}\right)^3 \frac{d}{da} + \left(\frac{d}{dx}\right)^2 \frac{d}{dy} \frac{d}{db} + \&c.\right\} SH$$

will be a covariant; in fact this will be found to be T, the difference between this and the expression before given for T, namely

$$\left(\frac{d}{dx}\right)^3 H \frac{dS}{da} + \left(\frac{d}{dx}\right)^2 \frac{d}{dy} H \frac{dS}{db} + \&c.,$$

being

$$S \times \left\{\frac{d}{da}\left(\frac{d}{dx}\right)^3 H + \frac{d}{db}\left(\frac{d}{dx}\right)^2 \frac{d}{dy} H + \&c.\right\},$$

which is zero, there being no invariant to $(x, y, z)^3$ of the 3rd degree in a, b, c, &c., as the factor multiplied by S would be were it not evanescent. The same observation may be extended to analogous equations given previously.

I have chiefly, however, made the above observation with a view to making more clear the enunciation of the theorem which I am now about to state, the most important perhaps in its application of any yet brought to light on the subject, but the consequences of which, as I have but quite recently discovered it, must be reserved for a future number of the *Journal*.

Let any function of any number of variables be supposed to have for its coefficients the letters a, b, &c. affected with the ordinary binomial or multinomial coefficients; and let another function be taken identical with the former in all respects, except in the circumstance that all their numerical multipliers are suppressed. Let this function or form be termed the respondent to the primitive: furthermore, by the inverse of any form understand what that form becomes when, in place of x, y, z, &c., ξ, η, ζ, &c.,

$$\frac{d}{dx},\ \frac{d}{dy},\ \frac{d}{dz},\ \text{\&c.,}\ \frac{d}{d\xi},\ \frac{d}{d\eta},\ \frac{d}{d\zeta},\ \text{\&c.,}$$

are respectively substituted (and so for all the systems of the variables), and likewise at the same time similar substitutions are made of $\frac{d}{da}, \frac{d}{db}, \frac{d}{dc}$, &c., in place of a, b, c, &c.; then we have this grand and simple law—*The inverse of any concomitant to a respondent is a concomitant to its primitive.* When the inverse of any concomitant to the respondent is made to operate upon the same concomitant of the primitive, it will be found that the result is a power of the universal concomitant. If the concomitant to the respondent be an invariant thereof, the rule indicates that on merely replacing in the respondent a, b, c, &c. by $\frac{d}{da}, \frac{d}{db}, \frac{d}{dc}$, &c., the result operating on any invariant or other concomitant of the primitive, leaves it still an invariant or other concomitant. For instance, if we take the function

$$ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + fy^5,$$

which has three invariants L, M, N, of the degrees 4, 8, 12, respectively: and if we call λ, μ, ν what L, M, N become when, in place of a, b, c, d, e, f respectively, we write

$$\frac{d}{da},\ \frac{1}{5}\frac{d}{db},\ \frac{1}{10}\frac{d}{dc},\ \frac{1}{10}\frac{d}{dd},\ \frac{1}{5}\frac{d}{de},\ \frac{d}{df},$$

we shall find that

$$\lambda M = L,\quad \mu N = L,$$

and

$$\lambda N = \text{a linear function of } M \text{ and } L^2.$$

Again, if in the case of any function of x, y, z, &c., we take, instead of any other concomitant to the respondent, the respondent itself, its inverse gives the symbol of operation

$$\left(\frac{d}{da}\right)\left(\frac{d}{dx}\right)^3 + \frac{d}{db}\left(\frac{d}{dx}\right)^2\left(\frac{d}{dy}\right) + \text{\&c.,}$$

just previously treated of. If again, in the case of a function of x, y, say

$$ax^n + nbx^{n-1}y + \ldots + nb'xy^{n-1} + a'y^n,$$

we take the inverse of the polar reciprocal of the respondent, we get the operator

$$\frac{d}{da}\left(\frac{d}{d\eta}\right)^n - \frac{d}{db}\left(\frac{d}{d\eta}\right)^{n-1}\frac{d}{d\xi} + \text{\&c.;}$$

and replacing $\frac{d}{d\eta}, \frac{d}{d\xi}$ by y, x, we find that

$$y^n\frac{d}{da} - y^{n-1}x\frac{d}{db} + \text{\&c.,}$$

operating on any concomitant, leaves it still a concomitant, which is M. Eisenstein's theorem before adverted to, only generalized by the introduction of any concomitant in lieu of the discriminant.

This extraordinary theorem of respondence will be found on reflection to favour the notion of treating the coefficients of a general function as themselves a system of variables, in a manner contragredient to the terms to which they are affixed.

Finally, there is yet another mode of applying the principle of reciprocity, which must be carefully distinguished from any previously stated in these pages.

I have said that in place of the quantitative symbols of one alphabet, as x, y, z, &c., we may always substitute the operation symbols $\frac{d}{d\xi}, \frac{d}{d\eta}, \frac{d}{d\zeta}$, &c. of the opposite alphabet. But now I say, in place of the quantitative symbols x, y, z, &c. occurring in the concomitant to any form f, may be substituted the quantities (observe, no longer operative symbols but quantities) $\frac{dF}{d\xi}, \frac{dF}{d\eta}, \frac{dF}{d\zeta}$, &c., F being itself any concomitant to f. Thus, for instance, taking F identical with f, we see that $f\left(\frac{df}{d\xi}, \frac{df}{d\eta}, \frac{df}{d\zeta}, \text{\&c.}\right)$ is concomitant to f: or again, if f be a function of x, y only, say $f(x, y)$, taking F the polar reciprocal of f, that is $f(-\eta, \xi)$, we see that $f\left(-\frac{df}{dy}, \frac{df}{dx}\right)$ will be a concomitant to f: this concomitant, by the way it may be observed, will always contain f as a factor, because when $f=0$, $x\frac{df}{dx}+y\frac{df}{dy}=0$. Possibly it may be true that, when f is a function of any number of variables x, y, z, &c., and $F(\xi, \eta, \zeta, \text{\&c.})$ its polar reciprocal,

$$f\left(\frac{dF(x, y, z, \text{\&c.})}{dx}, \frac{dF(x, y, z, \text{\&c.})}{dy}, \text{\&c.}\right),$$

which is a concomitant to f, contains f as a factor; but I have not had time to see how this is. It is rather singular that Mr Cayley and Professor Borchardt of Berlin have both independently made to me the observation that, when $f(x, y)$ is taken a cubic function of x and y, $f\left(\frac{df}{dy}, \frac{-df}{dx}\right)$ is equal to the product of f by the first evectant of the discriminant of f. The general consideration of the consequences of this new and important application of the idea of reciprocity must be reserved for a future section.

SECTION V. *Applications and Extension of the Theory of the Plexus.*

If
$$\phi = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4,$$

we can obtain, by operating catalectically with x', y' upon

$$\left(x'\frac{d}{dx}+y'\frac{d}{dy}\right)^2\phi, \quad \left(x'\frac{d}{dx}+y'\frac{d}{dy}\right)^4\phi,$$

the two concomitants

$$\begin{vmatrix} ax^2+2bxy+cy^2, & bx^2+2cxy+dy^2 \\ bx^2+2cxy+dy^2, & cx^2+2dxy+ey^2 \end{vmatrix}, \qquad (1)$$

$$\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}, \qquad (2)$$

the one in fact being the Hessian, the other the catalecticant of ϕ itself. Again, if

$$\phi = ax^5 + 5bx^4y + 10cx^3y^2 + \ldots + fy^5,$$

by operating catalectically with x', y' upon the second and fourth emanants, as in the last case, we obtain the two covariants

$$\begin{vmatrix} ax^3+3bx^2y+3cxy^2+dy^3, & bx^3+3cx^2y+3dxy^2+ey^3 \\ bx^3+3cx^2y+3dxy^2+ey^3, & cx^3+3dx^2y+3exy^2+fy^3 \end{vmatrix}, \qquad (1)$$

$$\begin{vmatrix} ax+by, & bx+cy, & cx+dy \\ bx+cy, & cx+dy, & dx+ey \\ cx+dy, & dx+ey, & ex+fy \end{vmatrix}, \qquad (2)$$

which are in fact the Hessian and canonizant respectively of ϕ. So in general, for a function of x, y of the degree 2ι or $2\iota+1$, we can obtain ι covariantive forms, the first being the Hessian, and the last the catalecticant on the first supposition and the canonizant on the second: calling the index of the function for either case n, the forms appearing in this scale will be of the degree $(r+1)$ in the constants, and of the degree $(r+1)(n-2r)$ in x and y.

It has previously* been intimated that all these determinants admit of a remarkable transformation.

This transformation may be expressed more elegantly by dealing not directly with the covariant forms as above given, but with their polar reciprocants obtained immediately by writing ξ for $-y$ and η for x.

(1) Suppose $\phi = ax^3 + 2bx^2y + 3cxy^2 + dy^3$;

$$\begin{vmatrix} a, & 2b, & c \\ b, & 2c, & d \\ \xi^2, & 2\xi\eta, & \eta^2 \end{vmatrix}$$

will be found to be the reciprocant of its Hessian.

(2) Let $\phi = ax^4 + 4bx^3y + \ldots + ey^4$;

the reciprocant of its Hessian will be found to be

$$\begin{vmatrix} a, & 3b, & 3c, & d \\ b, & 3c, & 3d, & e \\ \xi^2, & 2\xi\eta, & \eta^2, & \\ & \xi^2, & 2\xi\eta, & \eta^2 \end{vmatrix}.$$

(3) Let $\phi = ax^5 + 5bx^4y + \ldots + fy^5$;

the reciprocant of its Hessian will be

$$\begin{vmatrix} a, & 4b, & 6c, & 4d, & e \\ b, & 4c, & 6d, & 4e, & f \\ \xi^2, & 2\xi\eta, & \eta^2, & & \\ & \xi^2, & 2\xi\eta, & \eta^2, & \\ & & \xi^2, & 2\xi\eta, & \eta^2 \end{vmatrix};$$

[* p. 325 above, note †].

and the reciprocant of its canonizant is

$$\begin{vmatrix} a, & 3b, & 3c, & d \\ b, & 3c, & 3d, & e \\ c, & 3d, & 3e, & f \\ \xi^3, & 3\xi^2\eta, & 3\xi\eta^2, & \eta^3 \end{vmatrix}.$$

The numerical coefficients in this and in the first case are inserted for the sake of uniformity, but it will of course be readily observed that when there is but one line of ξ and η, that the numerical coefficients being the same for each column may be rejected without affecting the form of the result.

So again, if

$$\phi = ax^6 + 6bx^5y + \ldots + gy^6,$$

the reciprocant of the Hessian is

$$\begin{vmatrix} a, & 5b, & 10c, & 10d, & 5e, & f \\ b, & 5c, & 10d, & 10e, & 5f, & g \\ \xi^2, & 2\xi\eta, & \eta^2, & & & \\ & \xi^2, & 2\xi\eta, & \eta^2, & & \\ & & \xi^2, & 2\xi\eta, & \eta^2, & \\ & & & \xi^2, & 2\xi\eta, & \eta^2 \end{vmatrix},$$

and the reciprocant of the second form in the scale, which comes between the Hessian and the catalecticant, is

$$\begin{vmatrix} a, & b, & c, & d, & e \\ b, & c, & d, & e, & f \\ c, & d, & e, & f, & g \\ \xi^3, & \xi^2\eta, & \xi\eta^2, & \eta^3, & \\ & \xi^3, & \xi^2\eta, & \xi\eta^2, & \eta^3 \end{vmatrix};$$

and so in general. The rule of formation is sufficiently plain not to need formulating in general terms. It is easy to see that all these forms are concomitants to the function from which they are formed; for example, take

$$\phi = ax^6 + 6bx^5y + \ldots + gy^6;$$

then

$$\left(\frac{d}{dx}\right)^2\phi, \quad \frac{d}{dx}\frac{d}{dy}\phi, \quad \left(\frac{d}{dy}\right)^2\phi$$

form a plexus.

So likewise if we take $\psi = (x\xi + y\eta)^4$,

$$\frac{d\psi}{d\xi}, \quad \frac{d\psi}{d\eta}$$

form a plexus. But ψ and ϕ are concomitantive, ψ being a universal concomitant. Hence we may combine together these two plexuses, that is

$$\left.\begin{matrix} ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4 \\ bx^4 + 4cx^3y + 6dx^2y^2 + 4exy^3 + fy^4 \\ cx^4 + 4dx^3y + 6ex^2y^2 + 4fxy^3 + gy^4 \end{matrix}\right\},$$

$$\left.\begin{matrix} \xi^3x^4 + 3\xi^2\eta x^3y + 3\xi\eta^2x^2y^2 + \eta^3xy^3 \\ \xi^3x^3y + 3\xi^2\eta x^2y^2 + 3\xi\eta^2xy^3 + \eta^3y^4 \end{matrix}\right\},$$

and, by the principle of the plexus, x^4, x^3y, x^2y^2, xy^3, y^4 may be eliminated dialytically, and the resultant will be the determinant last given, which is therefore a contravariant to ϕ.

The manner in which I was led to notice this singular transformation is somewhat remarkable.

In the supplemental part of my essay *On Canonical Forms* [p. 203 above], my method of solution of the problem of throwing the quintic function of two variables under the form $u^5 + v^5 + w^5$, led me to see that u, v, w are the three factors of

$$\begin{vmatrix} ax + by, & bx + cy, & cx + dy \\ bx + cy, & cx + dy, & dx + ey \\ cx + dy, & dx + ey, & ex + fy \end{vmatrix};$$

the more simple mode of the solution of the same problem, given by me in the *Philosophical Magazine* for the month of November last [p. 266 above], led to

$$\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ y^3, & -xy^2, & x^2y, & -x^3 \end{vmatrix},$$

as the product of the same three factors; whence the identity of the two forms becomes manifest. In the paper last named I gave two proofs, one my own, the other Mr Cayley's, of a like kind of identity for the canonizant of any odd-degreed function of x, y in general. The proof of the identity of the corresponding forms in the much more general proposition above indicated [p. 325 above, footnote †] must be reserved until more pressing and important matters are disposed of. In the footnote referred to I ought to have added, in order to make the sense more clear, that the degree of the catalecticant there referred to in respect of the coefficients would be n.

I regret to think that there are many other typographical errors in the earlier sections; the most unfortunate of these is in the note at page [316], in the values of P and Q belonging to the cubic commutant dodecadic function of x and y, the corrected values of which will be given in my next communication. I ought also to observe, in correction of the remark made in the footnote to page [302], that it follows as a consequence of a recent paper by Dr Hesse in *Crelle's Journal*, that the method given by me in the text applied (according to what I have there termed the 1st process for obtaining an invariant resembling the resultant) to a system of three cubic equations (in which application only the 1st powers of $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ enter) produces for that case also, as well as for the cases specified in the note, not a counterfeit resemblance of, but the actual resultant itself.

Returning to the theory of the plexus of which I am about to enunciate a most important extension, I beg to refer my readers to the last paragraph, p. [291], in the last number of the *Journal*, where I have shown how to form, under certain conditions, a determinant by combining together various concomitants and eliminating dialytically one set of the variables, which determinant will be concomitantive to the concomitants out of which it is formed, and of course also therefore to their common original.

Now the extension of this theorem, to which I wish to call attention, is this, that not only such determinant as a whole is a concomitant to such original, but every minor system of determinants that can be formed out of it will form a concomitantive plexus complete within itself to the same original. But, much more generally, it should be observed that there is no occasion to begin with a square determinant; it is sufficient to have a rectangular array of terms formed by taking the several terms of one plexus or of several plexuses combined, provided that they are of the same degree in respect to the variables (or to the selected system of variables, if there be several systems), and forming out of such rectangular array any minor system of determinants at will. Every such system will be a concomitantive plexus. The simple illustrations which follow will make my meaning clear.

Suppose

$$\phi = ax^6 + 6bx^5y + 15cx^4y^2 + 21dx^3y^3 + 15ex^2y^4 + 6fxy^5 + gy^6.$$

I have previously remarked, in the foregoing sections, that a, b, c, d, e, f, g, the coefficients form an invariantive plexus to ϕ; so also we know that the catalecticant

$$\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}$$

is an invariant to ϕ. But we are now able to couple together these facts and see the law which is contained between them; for if we take

$$\left(\frac{d}{dx}\right)^{\iota}\phi,\ \left(\frac{d}{dx}\right)^{\iota-1}\frac{d}{dy}\phi \dots \left(\frac{d}{dy}\right)^{\iota}\phi,$$

ι being any number, as for instance, if we take $\iota = 3$, we shall have as a plexus

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$
$$bx^3 + 3cx^2y + 3dxy^2 + ey^3,$$
$$cx^3 + 3dx^2y + 3exy^2 + fy^3,$$
$$dx^3 + 3ex^2y + 3fxy^2 + gy^3;$$

accordingly not only is the determinant

$$\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}$$

an invariant, but also the system obtained by striking out any one line and one column, being what I term the first minors, will be an invariantive plexus, so too will the system of second minors

$$ac - b^2,\ bd - c^2,\ ce - d^2,\ ad - bc,\ ae - bd,\ be - cd,\ \&c.$$

form an invariantive plexus, as well as the last minors, that is, the simple terms a, b, c, d, e, f, g. Again, we might have taken the plexus

$$\left(\frac{d}{dx}\right)^2\phi,\ \frac{d}{dx}\frac{d}{dy}\phi,\ \left(\frac{d}{dy}\right)^2\phi,$$

which would give the array

$$\begin{matrix} a, & b, & c, & d, & e \\ b, & c, & d, & e, & f \\ c, & d, & e, & f, & g; \end{matrix}$$

but the minor systems of determinants herein comprised will be found to be identical with those last considered, with the exception that the highest system, containing a single determinant only, will now be wanting. So in general it will easily be seen that a similar method in general, when ϕ is of 2ι dimensions, will lead to $\iota + 1$ invariantive plexuses comprising the given coefficients grouped together at one extremity of the scale, and the catalecticant alone at the other; and if ϕ is of $2\iota + 1$ dimensions, there will still be $\iota + 1$ such plexuses, commencing with the coefficients as one group and ending with a system of combinations of the $(\iota + 1)$th degree in regard to the coefficients, which system accordingly takes the place of the catalecticant of the former case, which for this case is non-existent.

As a profitable example of the application of this law of synthesis, in its present extended form, let it be required to determine the conditions that a function of x, y of the fifth degree may have three equal roots. In general, let $\phi = ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + fy^5$, then ϕ has a quadratic and cubic covariant of which I have written at large in my supplemental essay above referred to, being in fact the s and t (that is the quadrinvariant and cubinvariant) in respect to x', y' (x, y being treated as constants) of

$$\left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^4 \phi.$$

Let these covariants respectively be called

$$Ax^2 + 2Bxy + Cy^2 = u,$$

$$\alpha x^3 + 3\beta x^2y + 3\gamma xy^2 + \delta y^3 = v;$$

then

$$\left.\begin{matrix} Ax + By \\ Bx + Cy \end{matrix}\right\}$$

forms a plexus, and

$$\left.\begin{matrix} \alpha x^2 + 2\beta xy + \gamma y^2 \\ \beta x^2 + 2\gamma xy + \delta y^2 \end{matrix}\right\}$$

will form another.

Now when $a = 0$, $b = 0$, $c = 0$, ϕ will have three equal roots, and

$$\left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^4 \phi$$

becomes

$$6dy \,.\, x'^2y'^2 + 4(dx + ey)\,x'y'^3 + (ex + fy)\,y'^4,$$

of which the quadrinvariant in respect to x', y' is easily seen to be d^2y^2 and the cubinvariant d^3y^3. Accordingly the grouping

$$\begin{matrix} A, & B \\ B, & C \end{matrix} \text{ becomes } \left.\begin{matrix} 0, & 0 \\ 0, & d^2 \end{matrix}\right\},$$

and the grouping

$$\begin{matrix} \alpha, & \beta, & \gamma \\ \beta, & \gamma, & \delta \end{matrix} \text{ becomes } \left.\begin{matrix} 0, & 0, & 0 \\ 0, & 0, & d^3 \end{matrix}\right\}.$$

Accordingly, we see that the determinant $\begin{vmatrix} A, & B \\ B, & C \end{vmatrix}$ and all the first minors of $\begin{vmatrix} \alpha, & \beta, & \gamma \\ \beta, & \gamma, & \delta \end{vmatrix}$, that is $\alpha\gamma - \beta^2$, $\beta\delta - \gamma^2$, $\alpha\delta - \beta\gamma$, become zero; but the former single quantity $\begin{vmatrix} A, & B \\ B, & C \end{vmatrix}$ being an invariant, and this last system being an invariantive plexus, all the quantities so affirmed to be zero will remain zero, notwithstanding any linear transformations to which ϕ may be subjected; thus then we obtain an immediate proof of the theorem that

when a function of x and y of the fifth degree contains three equal roots the determinant of its quadratic covariant, which in fact is its sole quartinvariant, and the first minors of its cubinvariant will be all separately zero. This theorem may be made still more stringent; for by combining

$$Ax^2 + 2Bxy + Cy^2,$$

$$\alpha x^2 + 2\beta xy + \gamma y^2,$$

$$\beta x^2 + 2\gamma xy + \delta y^2,$$

it becomes manifest that in the case supposed all the first minor determinants of

$$\begin{vmatrix} A, & B, & C \\ \alpha, & \beta, & \gamma \\ \beta, & \gamma, & \delta \end{vmatrix}$$

will be zero, showing in addition to the theorem last enunciated that also

$$A : B : C :: \alpha : \beta : \gamma :: \beta : \gamma : \delta.$$

It is curious and instructive to remark that this last set of equations, stringent as *they appear*, and far more than enough to express a duplex condition, are not sufficient to imply unequivocally the existence of three equal roots, unless we have also $AC - B^2 = 0$; for suppose ϕ to take the form $ax^5 + fy^5$ (b, c, d, e all vanishing); then it will easily be seen that

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 0,$$

$$A = 0; \quad B = af, \quad C = 0.^*$$

* If we take L, M, N a system of fundamental invariants to ϕ, of which all the other invariants of ϕ are rational integer functions, then $L = \begin{vmatrix} A, & B \\ B, & C \end{vmatrix}$ and the simplest forms for M and N are

$$M = \begin{vmatrix} A, & B, & C \\ \alpha, & \beta, & \gamma \\ \beta, & \gamma, & \delta \end{vmatrix} \text{ and } N = \begin{vmatrix} \alpha, & 2\beta, & \gamma & \\ & \alpha, & 2\beta, & \gamma \\ \beta, & 2\gamma, & \delta, & \\ & \beta, & 2\gamma, & \delta \end{vmatrix},$$

where L and N are the discriminants of the quadratic and cubic covariants of ϕ respectively, and a linear function of M, L^2 is the discriminant of ϕ itself (L, M, N being of 4, 8, and 12 dimensions respectively in the coefficients of ϕ).

For many purposes of the calculus of forms it is desirable to have the command of cases for which any two out of these three invariants may be made to vanish without the third vanishing; and it will be found that when ϕ is of the form $y^2(cx^3 + fy^3)$, $L = 0$, $M = 0$; when ϕ is of the form $y(bx^4 + fy^4)$, $N = 0$, $L = 0$; and when ϕ is of the form $ax^5 + ey^5$, $M = 0$, $N = 0$; and of course when ϕ is of the form $y^3(dx^2 + fy^2)$, $L = 0$, $M = 0$, $N = 0$; it being obviously true in general, as remarked by Mr Cayley, that when not less than half the roots of a function of two variables are equal, all its invariants must vanish together.

Consequently we shall still have all the first minors of

$$\begin{vmatrix} A, & B, & C \\ \alpha, & \beta, & \gamma \\ \beta, & \gamma, & \delta \end{vmatrix}$$

zero, although there is not even so much as a pair of equal roots in ϕ; $AC - B^2$ however, it will be observed, is not zero in this supposition.

The theory of Hessians, simple or bordered, may be regarded as one among the infinite diversity of applications of the principle of the plexus. Let U, V, W, &c. be any number of concomitants having the common system of variables $x, y \ldots z$. Let χ represent

$$x'\frac{d}{dx} + y'\frac{d}{dy} + \ldots + z'\frac{d}{dz},$$

and take

$$\chi^2 U + \lambda\chi V + \&c. + \mu\chi W = S;$$

then

$$\frac{dS}{dx'}, \frac{dS}{dy'} \cdots \frac{dS}{dz'}$$

forms a plexus; and this, combined with χV, &c. ... χW, enables us to eliminate dialytically x', y', z', $\lambda \ldots \mu$. The result is a Hessian of U, bordered with

$$\frac{dV}{dx}, \frac{dV}{dy} \cdots \frac{dV}{dz}$$

horizontally and vertically, and also with

$$\frac{dW}{dx}, \frac{dW}{dy} \cdots \frac{dW}{dz},$$

&c. &c.

similarly dispersed; which Hessian, so bordered, is thus seen to be a concomitant to $U, V \ldots W$. The Hessian, as ordinarily bordered with $\xi, \eta \ldots \zeta$, is derived by taking for V the universal concomitant

$$x\xi + y\eta + \ldots + z\zeta,$$

and for W (if there be a double border)

$$x\xi' + y\eta' + \ldots + z\zeta',$$

and so forth.

If V be taken identical with U, the resulting form, consisting of U bordered with $\frac{dU}{dx}, \frac{dU}{dy} \cdots \frac{dU}{dz}$, has been shown* in my paper "On certain general Properties of Homogeneous Functions," in this *Journal*, to be equal to the product of the simple Hessian of U and of U itself multiplied by a

[* p. 173 above.]

numerical factor. The theory of the bordered Hessian may be profitably extended by taking

$$S = \chi^{2r} U + \lambda\chi^r V + \ldots + \mu\chi^r W,$$

and combining with $\chi^r V \ldots \chi^r W$ the plexus obtained by operating upon S with the rth powers and products of $\frac{d}{dx}, \frac{d}{dy} \cdots \frac{d}{dz}$, and eliminating dialytically the rth powers and products of $x', y' \ldots z'$. Thus if

$$U = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4 \text{ and } V = (x\xi + y\eta)^2,$$

we obtain, by taking $S = \chi^4 U + \lambda\chi^2 V$, and proceeding as indicated in the preceding,

$$\begin{vmatrix} a, & b, & c, & \xi^2 \\ b, & c, & d, & \xi\eta \\ c, & d, & e, & \eta^2 \\ \xi^2, & \xi\eta, & \eta^2, & \end{vmatrix}$$

as a concomitant to U. So again, if

$$U = ax^5 + 5bx^4y + \ldots + fy^5,$$

we find

$$\begin{vmatrix} ax + by, & bx + cy, & cx + dy, & \xi^2 \\ bx + cy, & cx + dy, & dx + ey, & \xi\eta \\ cx + dy, & dx + ey, & ex + fy, & \eta^2 \\ \xi^2, & \xi\eta, & \eta^2, & \end{vmatrix}$$

a concomitant to U.

These extensions of the ordinary theory of Hessians will be found to be of considerable practical importance in the treatment of forms, for which reason they are here introduced.

Section VI. *On the Partial Differential Equations to Concomitants, Orthogonal and Plagiogonal Invariants, &c.*

In the 7th note of the Appendix to the three preceding sections* I alluded to the partial differential equations by which every invariant may be defined.

This method may also be extended to concomitants generally. M. Aronhold, as I collect from private information, was the first to think of the application of this method to the subject; but it was Mr Cayley who communicated to me the equations which define the invariants of functions of

[* p. 326 above.]

two variables*. The method by which I obtain these equations and prove their sufficiency is my own, but I believe has been adopted by Mr Cayley in a memoir about to appear in *Crelle's Journal.* I have also recently been informed of a paper about to appear in *Liouville's Journal* from the pen of M. Eisenstein, where it appears the same idea and mode of treatment have been made use of. Mr Cayley's communication to me was made in the early part of December last, and my method (the result of a remark made long before) of obtaining these and the more general equations, and of demonstrating their sufficiency, imparted a few weeks subsequently—I believe between January and February of the present year.

The method which I employ, in fact, springs from the very conception of what an invariant means, and does but throw this conception into a concise analytical form.

Suppose, to fix the ideas,

$$\phi = ax^n + nbx^{n-1}y + \tfrac{1}{2}n(n-1)cx^{n-2}y^2 + \dots + ly^n,$$

and let $I(a, b, c \dots l)$ be any invariant to ϕ.

Now suppose x to become $x + ey$, but y to remain unchanged; the modulus of the transformation, $\begin{vmatrix} 1, e \\ 0, 1 \end{vmatrix}$, being unity, I cannot alter in consequence of this substitution; but the effect of this substitution is to convert ϕ into the form

$$\alpha x^n + n\beta x^{n-1}y + \tfrac{1}{2}n(n-1)\gamma x^{n-2}y^2 + \dots + \lambda y^n,$$

where $\qquad \alpha = a, \ \beta = b + ae, \ \gamma = c + 2be + ae^2, \ \text{\&c. \&c.}$

$$\lambda = l + \dots + nbe^{n-1} + ae^n.$$

Consequently, if we make

$$\Delta b = ae, \quad \Delta c = 2be + ae^2, \ \text{\&c. \&c.},$$

we have by Taylor's theorem, observing that $\Delta a = 0$,

$$\Delta I = \left(\Delta b \frac{d}{db} + \Delta c \frac{d}{dc} + \text{\&c.}\right) I + \frac{1}{1.2}\left(\Delta b \frac{d}{db} + \Delta c \frac{d}{dc} + \text{\&c.}\right)^2 I$$
$$+ \frac{1}{1.2.3}\left(\Delta b \frac{d}{db} + \text{\&c.}\right)^3 I + \text{\&c.} = 0;$$

* It is extremely desirable to know whether M. Aronhold's equations are the same in form as those here subjoined. It is difficult to imagine what else they can be in substance. Should these pages meet the eye of that distinguished mathematician he will confer a great obligation on the author and be rendering a service to the theory by communicating with him on the subject: and I take this opportunity of adding that I shall feel grateful for the communication of any ideas or suggestions relating to this new Calculus from any quarter and in any of the ordinary mediums of language—French, Italian, Latin or German, provided that it be in the Latin character.

and this being true for all the values of e, every separate coefficient of e in ΔI must be zero: hence we obtain n different equations by equating to zero the coefficients of $e, e^2 \dots e^n$ respectively. The first of these equations will be

$$\left(a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \text{\&c.}\right)\phi = 0,$$

and it is obvious that this will imply all the rest; for, when e is taken indefinitely small, $I(a, b, c \dots)$ does not alter (when this equation is satisfied) by changing $a, b, c \dots$ into $a', b', c' \dots$; consequently $I(a', b', c', \text{\&c.})$ will not alter, when in place of a', b', c' we write a'', b'', c'', &c., obtained from a', b', c', &c., by the same law as a', b', c', &c., from a, b, c, &c.

Thus we may go on giving an indefinite number of increments, ey to x, without changing the value of I. Consequently, if the equation above written be satisfied, *à priori* all the rest must be so too. But there is not any difficulty in showing the same thing by a direct method*.

For we have

$$\left(a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \text{\&c.}\right) I = 0,$$

an identical equation. Hence

$$\left(a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \text{\&c.}\right)\left\{\left(a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \text{\&c.}\right) I\right\} = 0;$$

hence

$$\left\{\left(a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \text{\&c.}\right)\left(a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \text{\&c.}\right)\right\} I$$
$$+ \left\{a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \text{\&c.}\right\}^2 I = 0,$$

that is

$$\left\{2\left(a\frac{d}{dc} + 3b\frac{d}{dd} + 6c\frac{d}{de} + \text{\&c.}\right) + \left(a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \text{\&c.}\right)^2\right\} I = 0;$$

repeating the application of the symbolic operator

$$\left(a\frac{d}{db} + 2b\frac{d}{dc} + \text{\&c.}\right),$$

* The method above given has the advantage however of being immediately applicable to every species of concomitant, and we learn from it that concomitance, whether absolute or conditional, is sufficiently determined when affirmed to exist for *infinitesimal* variations; it cannot exist for infinitesimal variations without, by *necessary implication*, existing for finite variations also; a most important consideration this in conducing to a true idea of the nature of invariance and the other kinds of concomitance, and in cutting off all superfluous matter from the statement of the conditions by which they are defined.

we obtain

$$\left.\begin{array}{l} 1.2.3\left\{a\dfrac{d}{dd}+4b\dfrac{d}{de}+10c\dfrac{d}{df}+\&c.\right\} \\ +1.2\left\{a\dfrac{d}{db}+2b\dfrac{d}{dc}+\&c.\right\}\left\{a\dfrac{d}{dc}+3b\dfrac{d}{dd}+\&c.\right\} \\ +\left(a\dfrac{d}{db}+2b\dfrac{d}{dc}+3c\dfrac{d}{dd}+\&c.\right)^3 \end{array}\right\} I=0,$$

and so on; the numerical multipliers of the terms of the several series within the parentheses forming the regular succession of figurate numbers

1, 2, 3, &c.

1, 3, 6, &c.

1, 4, 10, &c.

It is easy to see that these equations correspond to the results of making the coefficients of the successive powers of e equal to zero.

I may remark, that the first instance as far as I know on record of this, (as some may regard it rather bold) but in point of fact perfectly safe and legitimate method of differentiating conjointly operator and operand, occurs in a paper by myself in this *Journal*, Feb. 1851, "On certain General Properties of Homogeneous Functions" [p. 165 above]; where I have applied it in operating with

$$\left\{(x_1-a_1e)\frac{d}{da_1}+(x_2-a_2e)\frac{d}{da_2}+\&c.\right\}$$

upon

$$\left\{(x_1-a_1e)\frac{d}{da_1}+(x_2-a_2e)\frac{d}{da_2}+\&c.\right\}^r\omega,$$

which, as I have there noticed, gives the result

$$\left\{(x_1-a_1e)\frac{d}{da_1}+\&c.\right\}^{r+1}\omega$$

$$-re\left\{(x_1-a_1e)\frac{d}{da_1}+\&c.\right\}^r\omega.$$

The equation $\left(a\dfrac{d}{db}+2b\dfrac{d}{dc}+\&c.\right)I=0$ is evidently not enough to define I as an invariant; it merely serves to show that I does not alter when in place of x we write $x+ey$, but this is true for any function of the differences of the roots of the form multiplied by a suitable power of a, namely that power which is just sufficient to cause the product to become integer. But if we now, for convenience, write

$$\phi=ax^n+nbx^{n-1}y+\tfrac{1}{2}n(n-1)cx^{n-2}y^2+\dots$$
$$+\tfrac{1}{2}n(n-1)c'x^2y^{n-2}+nb'xy^{n-1}+a'y^n,$$

and form the similar equation from the other side, namely

$$\left(a'\frac{d}{db'}+2b'\frac{d}{dc'}+3c'\frac{d}{dd'}+\&c.\right)I=0,$$

these two equations together will suffice to define any invariant, as I shall proceed to show—these are the two equations alluded to brought under my notice by Mr Cayley. If they coexist, it follows from the method by which I have deduced them that x may be changed into $x+ey$, or y into $y+fx$, without I being altered, e and f having any values whatever: and it is obvious that these substitutions may be performed, not merely alternatively but successively, because the equations between the coefficients are identical equations, and depend only on the form of I.

Let now x become $x+ey$, and then y become $y+fx$; the result of these substitutions is to convert

x into $x+efx+ey$,

and y into $fx+y$.

Finally, let x become $x+gy$; then x is converted into $(1+ef)(x+gy)+ey$, and y into $y+f(x+gy)$,

that is x becomes $(1+ef)x+(eg+efg)y$,

and y becomes $fx+(1+fg)y$.

The modulus of substitution it is evident, *à priori*, always remains unity, and nothing would be gained by pushing the substitutions any further, as it is clear that we may satisfy the equations

$$1+ef=p,\quad e+g+efg=q,$$
$$f=p',\qquad 1+fg=q',$$

for all values of p, q, p', q', which satisfy the equation

$$pq'-p'q=1,$$

and for none other except such values; hence I remains unaltered for any unit-modular linear transformation of x, y, and is therefore an invariant by definition.

If ϕ be taken a function of three variables, x, y, z, and be thrown under the form

$$az^n+(a_1x+b_1y)z^{n-1}+(a_2x^2+2b_2xy+c_2y^2)z^{n-2}+\&c.,$$

and I be any invariant of ϕ, by supposing x to become $x+ey$, and giving b_1, b_2, c_2, &c., the corresponding variations, and taking e indefinitely small, we obtain

$$\left\{a_1\frac{d}{db_1}+\left(a_2\frac{d}{db_2}+2b_2\frac{d}{dc_2}\right)+\left(a_3\frac{d}{db_3}+2b_3\frac{d}{dc_3}+3c_3\frac{d}{dd_3}\right)+\&c.\right\}I=0,$$

$$\left\{b_1\frac{d}{da_1}+\left(c_2\frac{d}{db_2}+2b_2\frac{d}{da_2}\right)+\&c.\ \&c.\right\}I=0:$$

and in like manner, by arranging ϕ according to the powers of y and of x, we obtain two other pairs of equations: it is clear, however, that three equations (it would seem any three out of the six) would suffice and imply the other three. The method of demonstration will be the same as in the instance of two variables: First, it can be shown by the method of successive accretions, that I remaining invariable when x receives an indefinitely small increment ϵy, or y an indefinitely small increment ϵz, or z an indefinitely small increment ϵx, it will also remain invariable when these increments are taken of any finite magnitude. Secondly, by eight successive transformations, admissible by virtue of the preceding conclusion, x, y, z may be changed into any linear functions of x, y, z, consistent with the modulus of transformation being unity. And in general for a function of m variables, m partial differential equations similarly constructed (but not however arbitrarily selected) will be necessary and sufficient to determine any invariant: and it is clear that all the general properties of invariants must be contained in and be capable of being educed out of such equations.

The same method enables us also to establish the partial differential equations for any covariant, or indeed any concomitant whatever.

Thus let

$$\phi = ax^n + nbx^{n-1}y + \tfrac{1}{2}n(n-1)cx^{n-2}y^2 + \ldots + nb'xy^{n-1} + a'y^n = 0,$$

and let $K(a, b, c, \&c.;\ x, y, x', y', \&c.;\ \xi, \eta, \&c.)$ represent any concomitant, x, y; x', y' being cogredient, and ξ, η, &c. contragredient systems; when x, y become $x + ey$, y, any such system x', y' becomes $x' + ey', y'$; and any such system as ξ, η becomes $\xi, \eta - e\xi$; and taking e indefinitely small, the second coefficients a, b, c, &c. become $a, b + ae, c + 2be$, &c. as before; hence the equation to the concomitant becomes

$$\left\{a\frac{d}{db} + 2b\frac{d}{dc} + \ldots - y\frac{d}{dx} - y'\frac{d}{dx'} + \ldots + \xi\frac{d}{d\eta} - \&c.\right\} = 0^{*};$$

and in like manner, by changing y into $y + ex$, results the corresponding equation

$$\left\{a'\frac{d}{db'} + 2b'\frac{d}{dc'} + \ldots - x\frac{d}{dy} - x'\frac{d}{dy'} + \ldots + \eta\frac{d}{d\xi} - \&c.\right\} K = 0.$$

These two equations define in a perfectly general manner every concomitant (with any given number of cogredient and contragredient systems) to the form ϕ; and the due number of pairs of similarly constituted equations will serve to define the concomitant to a function of any given number of variables†.

* For we have

$$K(a, b+ae, c+2be, \&c.;\ x, y, \&c.;\ \xi, \eta, \&c.)$$
$$= K(a, b, c, \&c.;\ x, x+ey, \&c.;\ \xi, \eta-e\xi, \&c.;\ \&c.).$$

† *Vide* Note (10) [p. 361 below].

In like manner we may proceed to form the equations corresponding to what may be termed *conditional* concomitants, whether *orthogonal* or *plagiogonal*. The concomitants previously considered may be termed absolute, the linear transformations admissible being independent of any but the one general relation, imposed merely for the purpose of convenience, namely of their modulus being made unity. An orthogonal concomitant is a form which remains invariable, not for arbitrary unit-modular, but for orthogonal transformation, that is for linear substitutions of $x, y \ldots z$, which leave unchanged $x^2 + y^2 + \ldots + z^2$: in like manner, a plagiogonal concomitant may be defined of a form which remains invariable for all linear substitutions of $x, y \ldots z$, which leave unaltered any given quadratic function of $x, y \ldots z$. Thus, let it be required to express the condition of $Q(a, b, c \ldots x, y;\ \xi, \eta)$, being an orthogonal concomitant to the form

$$ax^n + nbx^{n-1}y + \ldots + nb'xy^{n-1} + a'y^n.$$

Let x become $x + ey$, e being indefinitely small, then y must become $y - ex$, and the variations of $a, b \ldots b', a'$ will be the sum of the variations produced by taking separately $x + ey$ for x and $y - ex$ for y. Hence the one sole condition for Q being of the required form becomes

$$\left\{\begin{matrix} \left(a\dfrac{d}{db} + 2b\dfrac{d}{dc} + \ldots - y\dfrac{d}{dx} + \xi\dfrac{d}{d\eta}\right) \\ -\left(a'\dfrac{d}{db'} + 2b'\dfrac{d}{dc'} + \ldots - x\dfrac{d}{dy} + \eta\dfrac{d}{d\xi}\right) \end{matrix}\right\} Q = 0,$$

or, as it may be written, $\theta Q - \omega Q = 0$, where $\theta Q = 0$, $\omega Q = 0$ are the two equations expressing the conditions of Q, being an unconditional or absolute concomitant; and so in general if ϕ be a function of m variables, we may obtain $\tfrac{1}{2}m(m-1)$ equations of the form $L - M = 0$ for the concomitant, of which however $(m-1)$ only will be independent.

Supposing, again, the substitutions to which x, y are subject to be conditioned by $lx^2 + 2mxy + ny^2$ remaining unalterable, or which is a more convenient and only in appearance less general supposition by $x^2 + 2mxy + y^2$ remaining unalterable, the general type of an infinitesimal system of substitutions will be rendered by supposing x, y to become $(1 + me)x + ey$, $-ex + (1 - me)y$, respectively, for then $x^2 + 2mxy + y^2$ becomes

$$(1 - m^2e^2)x^2 + \{2m + (2m - 2m^2)e^2\}xy + (1 - m^2e^2)y^2,$$

which differs from $x^2 + 2mxy + y^2$ only by quantities of the second order of smallness which may be neglected, and ξ and η will therefore become $(1 - me)\xi - e\eta$, $-ex + (1 + me)y$, respectively: then, as to the coefficients of ϕ, in addition to the variations which they undergo when m is zero, there will be the variations consequent upon x assuming the increment mex, and y

the increment $-mey$: but by making x become $x+mex$, a, b, c, &c., b', a' assume respectively the variations

$$n\,.\,mea,\ (n-1)\,meb,\ \ldots\ meb',\ 0, \text{ respectively};$$

and by making y become $y-mey$, the corresponding variations become

$$0,\ -meb,\ \ldots -(n-1)\,meb',\ -n\,.\,mea', \text{ respectively.}$$

Hence the equation becomes

$$\theta Q - \omega Q + m\,(\lambda Q - \mu Q) = 0,$$

where θ and ω have the same signification as before, and where λ denotes

$$na\frac{d}{da} + (n-1)\,b\frac{d}{db} + \ldots + b'\frac{d}{db'} + x\frac{d}{dx} - \xi\frac{d}{d\xi},$$

and μ denotes

$$b\frac{d}{db} + 2c\frac{d}{dc} + \ldots\ldots\ldots + na'\frac{d}{da'} - y\frac{d}{dy} + \eta\frac{d}{d\eta}.$$

If there be several systems of x, y or of ξ, η, or of both, the only difference in the equation of condition will consist in putting

$$\Sigma\left(y\frac{d}{dx}\right),\ \ \Sigma\left(x\frac{d}{dy}\right),\ \ \Sigma\left(x\frac{d}{dx}\right),\ \ \Sigma\left(y\frac{d}{dy}\right),$$

$$\Sigma\left(\eta\frac{d}{d\xi}\right),\ \ \Sigma\left(\xi\frac{d}{d\eta}\right),\ \ \Sigma\left(\xi\frac{d}{d\xi}\right),\ \ \Sigma\left(\eta\frac{d}{d\eta}\right),$$

instead of the single quantities included within the sign of definite summation.

Fearing to encroach too much on the limited space of the *Journal*, I must conclude for the present with showing how to integrate the general equation to the orthogonal invariant of ϕ, the general function of x, y.

Beginning with $\phi = ax^2 + 2bxy + cy^2$, the equation becomes

$$\left\{-2b\frac{d}{da} + (a-c)\frac{d}{db} + 2b\frac{d}{dc} + y\frac{d}{dx} - x\frac{d}{dy}\right\}Q = 0.$$

Write now

$$da = -2bd\theta, \qquad dx = yd\theta,$$
$$db = (a-c)\,d\theta, \quad dy = -xd\theta,$$
$$dc = +2bd\theta;$$

we have then

$$\lambda da + \mu db + \nu dc = d\theta\,\{\mu a + 2(\nu-\lambda)\,b - \mu c\}.$$

Let $\qquad \mu = \kappa\lambda, \quad 2(\nu-\lambda) = \kappa\mu, \quad -\mu = \kappa\nu;$

then $\qquad d\log(\lambda a + \mu b + \nu c) = \kappa d\theta;$

or $\qquad \lambda a + \mu b + \nu c = be^{\kappa\theta}.$

To find κ we have the determinant

$$\begin{vmatrix} \kappa, & -1, & 0 \\ 2, & \kappa, & -2 \\ 0, & 1, & \kappa \end{vmatrix} = 0,$$

that is, $\qquad \kappa^3 + 4\kappa = 0,$

and calling the three roots of this equation κ_1, κ_2, κ_3, we have

$$\kappa_1 = 0, \quad \kappa_2 = 2\iota, \quad \kappa_3 = -2\iota;$$

accordingly we may put

$$\kappa = 0, \qquad \lambda = 1, \quad \mu = 0, \qquad \nu = 1,$$

or $\qquad \kappa = 2\iota, \quad \lambda = 1, \quad \mu = 2\iota, \quad \nu = -1,$

or $\qquad \kappa = -2\iota, \quad \lambda = 1, \quad \mu = -2\iota, \quad \nu = -1.$

Again, $\qquad pdx + qdy = (py - qx)\,d\theta;$

and putting $-q = ep$, $p = eq$, so that $px + qy = Ee^{e\theta}$,

$$e^2 = -1, \quad e_1 = \iota, \quad e_2 = -\iota;$$

and we may put

$$e = \iota, \qquad p = 1, \quad q = -\iota,$$

or $\qquad e = -\iota, \quad p = 1, \quad q = +\iota.$

Consequently the complete integral of the given partial differential equation is found by writing

$$a + c = l, \qquad\qquad x - \iota y = Ee^{\iota\theta},$$
$$a + 2\iota b - c = l'e^{2\iota\theta}, \quad x + \iota y = E'e^{-\iota\theta},$$
$$a - 2\iota b - c = l''e^{-2\iota\theta}.$$

By means of these five equations, after eliminating θ, we may obtain four independent equations between a, b, c; x, y. Suppose

$$Q_1 = 0, \quad Q_2 = 0, \quad Q_3 = 0, \quad Q_4 = 0;$$

then $Q = F(Q_1, Q_2, Q_3, Q_4)$ is the complete integral required.

Pursuing precisely the same method for the general case, it will be found that, calling the degree of the given function n when n is even, the equation in κ to be solved will be

$$\kappa(\kappa^2+4)(\kappa^2+9)\ldots(\kappa^2+n^2) = 0;$$

and when n is odd (say $2m+1$), the equation in κ to solve will be

$$(\kappa+1)(\kappa^2+9)\ldots(\kappa^2+n^2) = 0;$$

and performing the necessary reductions, and calling the roots of the equation, arranged in order of magnitude, $\kappa_1\iota, \kappa_2\iota \dots \kappa_n\iota$, respectively, it will be found that the equations containing the integral become

$$\left.\begin{array}{l} L_1 = l_1' e^{\kappa_1\iota\theta} \\ L_2 = l_2 e^{\kappa_2\iota\theta} \\ L_3 = l_3 e^{\kappa_3\iota\theta} \\ \dots\dots\dots\dots \\ L_{n+1} = l_{n+1} e^{\kappa_{n+1}\iota\theta} \end{array}\right\} \quad \left.\begin{array}{l} x - \iota y = E e^{\iota\theta} \\ x + \iota y = E' e^{-\iota\theta} \end{array}\right\},$$

where $l_1, l_2 \dots l_{n+1}$; E, E' are arbitrary constants, and where $L_1, L_2 \dots L_{n+1}$ are the values assumed by the 1st, 2nd ... $(n+1)$th coefficients of the given function ϕ, or

$$ax^n + nbx^{n-1}y + \dots + nb'xy^{n-1} + a'y^n,$$

when it is transformed by writing $x + \iota y$ in place of x, and $y + \iota x$ in place of y. ι is of course employed in the foregoing according to the usual notation to represent $\sqrt{(-1)}$. The same method applies to the general theory of plagiogonal concomitants, where the linear substitutions are supposed such as to leave $lx^2 + 2mxy + ny^2$ unaltered in form, and the equations in θ which contain the integral present themselves under a similar aspect. But a more full discussion of these interesting integrals must be reserved until the ensuing number of the *Journal*.

NOTES IN APPENDIX.

(9) The scale of covariants to a function of (x, y) obtained by the method of unravelment [on p. 297 above], may be otherwise deduced in a form more closely analogous to that of the corresponding theorems for the corresponding invariantive scale [on p. 295 above], by a method which has the advantage of exhibiting the scale equally well for the case of functions of the degree $4\iota + 2$ or $4\iota + 4$, the only difference being that in the latter case the coefficients of the odd powers of λ will be found all to vanish, so that the degrees of the covariants will rise by steps of 4 instead of by steps of 2, just conversely to what happens in the invariantive scale; whereas in the invariantive scale alluded to the forms containing odd powers of λ vanish when the degree of the function is of the form $4\iota + 2$, but do not vanish when it is of the form 4ι. This method in the form here subjoined is a slight modification of one suggested to me by my friend Mr Cayley.

Let F be the given function of x, y of the degree $2n$; take the systems x', y'; x_1, y_1 cogredient with one another and with x, y. Then form the concomitant

$$K = \left(x'\frac{d}{dx} + y'\frac{d}{dy}\right)^n F + \lambda\,(x'y - y'x)^{n-1}(x'y_1 - y'x_1)(xy_1 - yx_1).$$

Then (by what may be termed the Divellent method, which has been previously applied by me in the *Philosophical Magazine* for Nov. 1851) calling $\theta_0, \theta_1, \theta_2 \dots \theta_n$, the coefficients of

$$x'^n,\ x'^{n-1},\ y',\ \dots\ y'^n \text{ in } K,$$

we shall have

$$\begin{array}{l} \theta_0 = A_0x^n + B_0x^{n-1}y + \dots + L_0y^n, \\ \theta_1 = A_1x^n + B_1x^{n-1}y + \dots + L_1y^n, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \theta_n = A_nx^n + B_nx^{n-1}y + \dots + L_ny^n, \end{array}$$

the coefficients being functions of the coefficients of f and of quadratic combinations of x_1, y_1, affected with the multiplier λ; and the determinant

$$\begin{vmatrix} A_0, & B_0 \dots L_0 \\ A_1, & B_1 \dots L_1 \\ \dots & \dots \\ A_n, & B_n \dots L_n \end{vmatrix}$$

will give a function of λ in which the coefficients of the several powers of λ will be all zero or covariants of F.

The actual form of this determinant is not here given for want of space and time, but will be exhibited hereafter. Precisely an analogous method applies to obtain the scale to $(x, y, z)^4$ given in Note (2) [p. 322 above]. Calling $F = (x, y, z)^4$, let the systems x', y', z'; x_1, y_1, z_1, be taken cogredient with one another and with x, y, z. Then, using R to express the determinant

$$\begin{vmatrix} x', & y', & z' \\ x, & y, & z \\ x_1, & y_1, & z_1 \end{vmatrix},$$

and making

$$K = \left(x'\frac{d}{dx} + y'\frac{d}{dy} + z'\frac{d}{dz}\right)^2 F + \lambda R,$$

and proceeding as above by the divellent method, we obtain the scale required.

(10) [p. 356 above.] It is obvious that these defining equations ought to give the means of discovering and verifying all the properties of concomitants; but it is very difficult to see how in the present state of analysis many of the general theorems that have been stated, readily admit of being deduced from them.

The comparatively simple but eminently important theory of the evector symbol does however admit of a very pretty verification by aid of these equations. Thus, suppose θ any concomitant; suppose a contravariant to a function F of x, y, say

$$ax^n + nbx^{n-1}y + \dots + nb'xy^{n-1} + a'y^n.$$

Then θ must satisfy the two equations

$$\left(L+\xi\frac{d}{d\eta}\right)\theta=0,\quad \left(L'+\eta\frac{d}{d\xi}\right)\theta=0,$$

where
$$L=a\frac{d}{db}+2b\frac{d}{dc}+\ldots+nb'\frac{d}{da'},$$
$$L'=a'\frac{d}{db'}+2b'\frac{d}{dc'}+\ldots+nb\frac{d}{da}.$$

Now let $\phi=\chi(\theta)$ where

$$\chi=\xi^n\frac{d}{da}+\xi^{n-1}\eta\frac{d}{db}+\xi^{n-2}\eta^2\frac{d}{dc}+\ldots+\eta^n\frac{d}{da'};$$

then
$$L(\chi\theta)=\chi(L\theta)-(\chi L)\theta$$
$$=\chi(L\theta)-\left(\xi^n\frac{d}{db}+2\xi^{n-1}\eta\frac{d}{dc}+\ldots+n\xi\eta^{n-1}\frac{d}{da'}\right)\theta,$$
$$\xi\frac{d}{d\eta}(\chi\theta)=\chi\left(\xi\frac{d}{d\eta}\theta\right)+\left(\xi\frac{d}{d\eta}\chi\right)\theta$$
$$=\chi\left(\xi\frac{d}{d\eta}\theta\right)+\left(\xi^n\frac{d}{db}+2\xi^{n-1}\eta\frac{d}{dc}+\ldots+n\xi\eta^{n-1}\frac{d}{da'}\right)\theta.$$

Hence
$$\left(L+\xi\frac{d}{d\eta}\right)\chi(\theta)=\chi\left\{\left(L+\xi\frac{d}{d\eta}\right)\theta\right\}=\chi(0)=0.$$

Similarly
$$\left(L'+\eta\frac{d}{d\xi}\right)\chi(\theta)=0.$$

Hence if θ is an integral of the two conditioning equations, so also is $\chi(\theta)$. In like manner, if θ be a covariant or any other kind of concomitant of F, it may be proved that its evectant $\chi(\theta)$ is the same.

(11) [p. 331 above.] Very much akin with the supposed equations is the following most remarkable equation, which can be proved to exist. Let ϕ be a function of x and y of the 5th degree. Let P and Q be the quadratic and cubic covariants of ϕ. P is of two dimensions in the coefficients and also in the variables, and Q of three dimensions in both; they are in fact the s and t (in respect to x' and y') of $\left(x'\frac{d}{dx}+y'\frac{d}{dy}\right)^4\phi$. Then, giving P and Q proper numerical factors, it will be found that

$$H_2\phi+PH\phi+Q\phi=0.$$

I believe that a similar equation connects any function of x and y above the 3rd degree with its first and second Hessians. The proof will be given in a subsequent Section, where also I shall give a complete proof, which occurred to me immediately after sending the preceding note to the press, of the complete Theory of the Respondent by means of the general equations of concomitance.

P.S. Since the preceding was in type, I have ascertained the existence and sufficiency of a general method for forming the polar reciprocal and probably also the discriminant to functions of any degree of three variables by an explicit process of permutation and differentiation. In particular I am enabled to give the actual rule for constructing the polar reciprocal and the discriminant curves of the 4th and 5th degrees. So far as regards the polar reciprocal of curves of the 4th degree M. Hesse has already given a method of obtaining it, but mine is entirely unlike to this, and rests upon certain extremely simple and universal principles of the calculus of forms. The only thing necessary to be done in order to carry on the process to curves of the 6th or higher degrees, is to ascertain the relation of the discriminants of functions of two variables of those respective degrees to such of the fundamental invariants as are of an inferior order to the discriminant.

The theory applies equally well to surfaces and to functions of any number of variables, and may, I believe, without any serious difficulty be extended so as to reduce to an explicit process the general problem of effecting the elimination between functions of any degree and of any number of variables. The method above adverted to will appear in a subsequent Section.

[*Continued pp.* 402 *and* 411 *below.*]

44.

SUR UNE PROPRIÉTÉ NOUVELLE DE L'ÉQUATION QUI SERT A DÉTERMINER LES INÉGALITÉS SÉCULAIRES DES PLANÈTES.

[*Nouvelles Annales de Mathématiques*, XI. (1852), pp. 438—440.]

[Extract.]

6. *Soit le déterminant carré symétrique*

$$\begin{vmatrix} a_{1,1}, & a_{1,2} \dots a_{1,n} \\ a_{2,1}, & a_{2,2} \dots a_{2,n} \\ \dots & \dots \\ a_{n,1}, & a_{n,2} \dots a_{n,n} \end{vmatrix}, \qquad \text{(M)}$$

dans lequel on a, d'après la définition,

$$a_{l,c} = a_{c,l}.$$

Élevant le déterminant à la puissance p, on obtient le déterminant

$$\begin{vmatrix} A_{1,1}, & A_{1,2} \dots A_{1,n} \\ A_{2,1}, & A_{2,2} \dots A_{2,n} \\ \dots & \dots \\ A_{n,1}, & A_{n,2} \dots A_{n,n} \end{vmatrix}; \qquad \text{(N)}$$

et ce déterminant est symétrique aussi par rapport à la diagonale $A_{1,1}, A_{2,2} \dots A_{n,n}$.

Retranchant de chaque terme de la diagonale symétrique de (M) *la même quantité* λ, *on obtient le déterminant*

$$\begin{vmatrix} a_{1,1} - \lambda, & a_{1,2} \dots a_{1,n} \\ a_{2,1}, & a_{2,2} - \lambda \dots a_{2,n} \\ \dots & \dots \\ a_{n,1}, & a_{n,2} \dots a_{n,n} - \lambda \end{vmatrix}. \qquad \text{(P)}$$

Développant ce déterminant et ordonnant par rapport à λ, *on obtient une expression qui, étant égalée à zéro, donne l'équation*

$$\lambda^n - f\lambda^{n-1} + g\lambda^{n-2} + \dots (-1)^n t = 0, \qquad (1)$$

équation qui a n racines réelles (voir t. X. p. 259).

Retranchant de chaque terme de la diagonale symétrique du déterminant (N) *la quantité* μ, *et opérant comme ci-dessus, on parvient à l'équation*

$$\mu^n - F\mu^{n-1} + G\mu^{n-2} + \dots (-1)^n T = 0, \qquad (2)$$

équation qui a aussi n racines réelles. Les racines de cette équation sont les racines de l'équation (1), *élevées chacune à la puissance p.*

Démonstration. Représentons par

$$\rho_1, \quad \rho_2, \quad \rho_3 \dots \rho_p,$$

les p racines de l'équation $\rho^p - 1 = 0$. Écrivons le déterminant

$$\begin{vmatrix} a_{1,1} - \rho_q\lambda, & a_{1,2} \dots a_{1,n} \\ a_{2,1}, & a_{2,2} - \rho_q\lambda \dots a_{2,n} \\ \dots & \dots \\ a_{n,1} \dots & a_{n,n} - \rho_q\lambda \end{vmatrix},$$

et faisons q égal successivement à tous les nombres de la suite $1, 2, 3 \dots p$, on aura p déterminants; le produit de tous ces déterminants reste évidemment le même dans quelque ordre qu'on prenne ces déterminants, et, d'après les propriétés connues des racines de l'unité, tous les termes en ρ qui ne seront pas élevés à une puissance p disparaîtront, et λ accompagnant toujours ρ, il ne reste donc que des λ^p, et le déterminant-produit sera

$$\begin{vmatrix} A_{1,1} - \lambda^p, & A_{1,2}, & A_{1,3} \dots A_{1,n} \\ A_{2,1}, & \dots, & A_{2,2} - \lambda^p \dots A_{2,n} \\ \dots & \dots & \dots \\ A_{n,1}, & A_{n,2}, & \dots\dots A_{n,n} - \lambda^p \end{vmatrix}; \qquad \text{(Q)}$$

où, faisant abstraction de λ, on a le déterminant (N). Ainsi

$$\mu = \lambda^p.$$

C. Q. F. D.

7. *Application.*

$$n = 2, \text{ et } p = 2;$$

déterminant

$$\begin{vmatrix} a, & b \\ b, & c \end{vmatrix}, \qquad \text{(M)}$$

élevant ce déterminant au carré, on a

$$\begin{vmatrix} a^2 + b^2, & ab + bc \\ ab + bc, & b^2 + c^2 \end{vmatrix}; \qquad \text{(N)}$$

déterminant

$$\begin{vmatrix} a-\lambda, & b \\ b, & c-\lambda \end{vmatrix}, \qquad (P)$$

$$\lambda^2 - (a+c)\lambda + ac - b^2 = 0; \qquad (1)$$

déterminant

$$\begin{vmatrix} a^2+b^2-\mu, & ab+bc \\ ab+bc, & b^2+c^2-\mu \end{vmatrix},$$

$$\mu^2 - (a^2+c^2+2b^2)\mu + (ac-b^2)^2 = 0, \text{ où } \mu = \lambda^2. \qquad (2)$$

Faisons

$$n = 2, \quad p = 3,$$

(M) ne change pas, et l'on a

$$\begin{vmatrix} a^3+2ab^2+b^2c, & a^2b+abc+b^3+bc^2 \\ a^2b+abc+b^3+b^2c, & ab^2+2b^2c+c^3 \end{vmatrix}; \qquad (N)$$

le déterminant (P) et l'équation (1) restent les mêmes; mais l'équation (2) devient

$$\mu^2 - (a^3+c^3+3ab^2+3cb^2)\mu + (ac-b^2)^3 = 0,$$

où

$$\mu = \lambda^3,$$

car, λ_1 et λ_2 étant les deux racines de l'équation (1), on a

$$\lambda_1^3 + \lambda_2^3 = a^3 + c^3 + 3ab^2 + 3cb^2, \quad \lambda_1^3\lambda_2^3 = (ac-b^2)^3.$$

8. M. Sylvester fait observer que son théorème est un cas particulier d'un théorème plus général, démontré par M. Borchardt, pour des déterminants quelconques, et qui devient le théorème démontré ci-dessus, lorsque le déterminant est symétrique (*Journal de Mathématiques*, t. XII. p. 63, 1847).

45.

ON A REMARKABLE THEOREM IN THE THEORY OF EQUAL ROOTS AND MULTIPLE POINTS.

[*Philosophical Magazine*, III. (1852), pp. 375—378.]

IN order that the theorem which I propose to state may be the more easily understood, and with the least ambiguity expressed, I shall commence with the case of a homogeneous function of two variables only, x and y.

Let

$$\phi = ax^n + nbx^{n-1}y + \tfrac{1}{2}n(n-1)cx^{n-2}y^2 + \ldots + nb'xy^{n-1} + a'y^n,$$

and let the result of operating with the symbol

$$x^n\frac{d}{da} + x^{n-1}y\frac{d}{db} + \ldots + y^{n-1}x\frac{d}{db'} + y^n\frac{d}{da'},$$

on any function of $a, b, c \ldots b', a'$ be called the Evectant of such function, and the result of repeating this process r times the rth Evectant.

Understand by the multiplicity of the equation the number of equalities between the roots that exist; so that a pair of equal roots will signify a multiplicity 1, two pairs of equal roots, or three equal roots a multiplicity 2; a pair of equal roots and a set of three equal roots, a multiplicity 1 + 2 or 3, and so on. Now suppose the total multiplicity of ϕ to be m: the first part of the proposition consists in the assertion that the 1st, 2nd, 3rd ... $(m-1)$th Evectants of the discriminant of ϕ, that is of the result of eliminating x and y between $\frac{d\phi}{dx}, \frac{d\phi}{dy}$ (as well as the discriminant itself), will all vanish in whatever way the multiplicity is distributed; the second part of the proposition about to be stated requires that the mode should be taken into account of the manner in which the multiplicity (m) is made up. Suppose, then, that there are r groups of roots, for one of which the multiplicity is m_1, for the second m_2, &c., and for the rth m_r, so that $m_1 + m_2 + \ldots + m_r = m$. Then, I say, that the mth evectant of the determinant of ϕ is of the form

$$(a_1x + b_1y)^{m_1n} (a_2x + b_2y)^{m_2n} \ldots (a_rx + b_ry)^{m_rn},$$

where $a_1 : b_1,\ a_2 : b_2 \ldots a_r : b_r$ are the ratios of $x : y$ corresponding to the several sets of equal roots.

This latter part of the theorem for the case of $m = 1$ was discovered inductively by Mr Cayley, by considering the cases when ϕ is a cubic, or a biquadratic function. I extended the theory to functions of any number of variables, and supplied a demonstration, that is for the case of one pair of equal roots. Mr Salmon showed that my demonstration could be applied to the case of two pairs of equal roots, or two double points, &c., and very nearly at the same time I made the like extension to the case of three equal roots, cusps, &c., and almost immediately after I obtained a demonstration for the theorem in its most general form. This demonstration reposes upon a very refined principle, which I had previously discovered but have not yet published, in the Theory of Elimination.

I have here anticipated a little in speaking of the theorem as applicable to curves and other loci.

Suppose $\phi(x, y, z) = 0$ to be the equation to a curve expressed homogeneously.

Let

$$\begin{aligned}\phi(x, y, z) = {} & ax^n + (na'x^{n-1}y + nb'x^{n-1}z)\\ & + \tfrac{1}{2}n(n-1)a''x^{n-2}y^2 + n(n-1)b''x^{n-2}yz + \tfrac{1}{2}n(n-1)c''x^{n-2}z^2,\\ & + \&c. \qquad \&c.,\end{aligned}$$

and understand by the evectant of any quantity the result of operating upon it with the symbol

$$x^n \frac{d}{da} + x^{n-1}y \frac{d}{da'} + x^{n-1}z \frac{d}{db'} + x^{n-2}y^2 \frac{d}{da''} + \&c.$$

Suppose, now, the curve to have double points, the $(r-1)$th evectant (and of course all the inferior evectants) of the discriminant of ϕ (meaning thereby the result of eliminating x, y, z between $\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz}$) will all vanish, and the rth evectant will be of the form

$$(a_1x + b_1y + c_1z)^n \times (a_2x + b_2y + c_2z)^n \ldots \times (a_rx + b_ry + c_rz)^n,$$

where $a_1 : b_1 : c_1,\ a_2 : b_2 : c_2 \ldots a_r : b_r : c_r$ are the ratios of the coordinates at the respective double points. If there be cusps the multiplicity of each

such will be 2; and calling the total multiplicity m, to every cusp will correspond a factor of the $2n$th power in the mth evectant; and so on in general for various degrees of multiplicity at the singular points respectively. The like theorem extends to conical and other singular points of surfaces; so that there exists a method, when a locus is given having any degree of multiplicity, of at once detecting the amount and distribution of this multiplicity, and the positions of the one or more singular points. In conclusion I may state, that precisely analogous results (*mutatis mutandis*) obtain, when, in place of a single function having multiplicity, we take the more general supposition of any number of homogeneous functions being subject to the condition of pluri-simultaneity, that is being capable of being made to vanish by each of several different systems of values for the ratios between the variables. Multiplicity in a single function is, in fact, nothing more nor less than pluri-simultaneity existing between the functions derived from it by differentiating with respect to each of the given variables successively. But as I purpose to give these theorems and their demonstration, which I have already imparted to my mathematical correspondents, in a paper destined for reading before the Royal Society, I need not further enlarge upon them on the present occasion.

P.S. In the above statement I have spoken only of cusps of curves which are the precise and unambiguous analogues of three coincident points in point-systems, in order to avoid the necessity of entering into any disquisition as to the species of singularity in curves or other loci corresponding to higher degrees of multiplicity in point-systems, a subject which has not hitherto been completely made out. I may here also add a remark, which gives a still higher interest to the theory, which is (to confine ourselves, for the sake of brevity, to functions of two variables), that if any root of $x : y$, say $a : b$, occur $1 + \mu$ times, the total multiplicity of the equation being supposed m, and its degree n, then taking ι any integer number not exceeding μ, the $(m + \iota)$th evectant of the discriminant will contain the factor $(ax + by)^{(\mu-\iota)n}$. So that, for instance, if there be but a single group of equal roots, and they be $1 + \mu$ in number, every evectant up to the $(\mu - 1)$th inclusive will vanish, and from the μth to the $(2\mu - \iota)$th will contain a power of $(ax + by)^n$.

46.

OBSERVATIONS ON A NEW THEORY OF MULTIPLICITY.

[*Philosophical Magazine*, III. (1852), pp. 460—467.]

IN the Postscript to my paper in the last number of the *Magazine*, I mis-stated, or to speak more correctly, I understated the law of Evection applicable to functions having any given amount of distributive multiplicity. The law may be stated more perfectly, and at the same time more concisely, as follows. Every point represented by the coordinates $\alpha_1, \beta_1 \ldots \gamma_1$, for which the multiplicity is m_1, will give rise in *every evectant* * of the discriminant of the function to a factor $(\alpha_1 x + \beta_1 y + \ldots + \gamma_1 z)^{m_1 n}$, n being supposed to be the degree of the function. Hence if there be r such points, for which the several multiplicities are $m_1, m_2 \ldots m_r$, every evectant must contain $(m_1 + m_2 + \ldots + m_r)\, n$ linear factors; and as the ιth evectant is of the degree ιn, it follows that all the evectants below the $(m_1 + m_2 + \ldots + m_r)$th evectant must vanish completely, and this Evectant itself be contained as a factor in all above it†. When a function of only two variables is in question, there is no difficulty in understanding what property of the function it is which is indicated by the allegation of the existence of multiplicities $m_1, m_2 \ldots m_r$;

* Frequent use being made in what follows of the word Evectant, I repeat that the evectant of any expression connected with the coefficients of a given function (supposed to be expressed in the more usual manner with letters for the coefficients affected with the proper binomial or polynomial numerical multipliers) means the result of operating upon such expressions with a symbol formed from the given function by suppressing all the binomial or polynomial numerical parts of the coefficients to be suppressed, and writing in place of the literal parts of the coefficients a, b, c, &c. the symbols of differentiation $\frac{d}{da}, \frac{d}{db}, \frac{d}{dc}$, &c.; in all that follows it is the successive evectants of the discriminant alone which come under consideration. I need hardly repeat, that the discriminant of a function is the result of the process of elimination (clear from extraneous factors) performed between the partial differential quotients of the function in respect to the several variables which it contains, or to speak more accurately, is the characteristic of their coevanescibility.

† The constitution of the quotients obtained by dividing all the other evectants of the discriminant by the first non-evanescent one, presents many remarkable features which remain yet to be fully studied out, and promise a wide extension of the existing theory.

as already remarked, this simply means that there are r distinct groups of equal roots, such groups containing $1 + m_1, 1 + m_2 \ldots 1 + m_r$ roots respectively. So for curves and higher loci, the total distributive multiplicity is the sum of the multiplicities at the several multiple points. But the true theory of the higher degrees of multiplicity separately considered at any point remains yet to be elaborated, and will be found to involve the consideration of the theory of elimination from a point of view under which it has never hitherto been contemplated.

Confining our attention for the present to curves, we have a clear notion of the multiplicity 1: this is what exists at an ordinary double point. As well known, its analytical character may be expressed by saying that the function of x, y, z, which characterizes the curve, is capable, when proper linear transformations are made, of being expanded under the form of a series descending according to the powers of z, such that the constant coefficient of the highest power of z, and the linear function of x, y, which is the coefficient of the next descending power of z, may both disappear. Again, when the multiplicity is 2, the third coefficient, which is a quadratic function of x and y, will become a perfect square. This is the case of a cusp, which, as I have said, is the precise analogue to that of three equal roots for a function of two variables. Before proceeding to consider what it is which constitutes a multiplicity 3 for a curve, it will be well to pause for a moment to fix the geometrical characters of the ordinary double point and the cusp.

If we agree to understand by a first polar to a curve the curve of one degree lower which passes through all the points in which the curve is met by tangents drawn from an arbitrary point taken anywhere in its own plane, we readily perceive that at an ordinary double point all the infinite number of first polars which can be drawn to the curve will intersect one another at the double point. Again, at a cusp all these polars will not only all intersect, they will moreover all touch one another at the cusp. Now we may proceed to inquire as to the meaning of a multiplicity of the third degree, which, strange to say, I believe has never yet been distinctly assigned by geometricians.

This is not the case of a so-called triple point, that is a point where three branches of the curve intersect. Supposing $x = 0$, $y = 0$, to represent such a point, the characteristic of the curve must be reducible to the form

$$(gx^3 + hx^2y + kxy^2 + ly^3)\, z^{n-3} + \&c.,$$

which, as is well known, involves the existence of four conditions. This, however, would not in itself be at all conclusive against the multiplicity at a triple point being only of the third degree; for it can readily be shown that there may exist singular points of any degree of *singularity* (as measured by the number of conditions necessary to be satisfied in order that such singularity may come into existence), but for which the multiplicity may be as low as we please; as, for instance, if at a double point (which is not a cusp) there be a point of inflexion on one branch or on both, or a point of undulation, or any other singularity whatever, still provided there be no cusps, the multiplicity will stick at the first degree and never exceed it; for only the discriminant itself will vanish on these suppositions, but no evectant of the discriminant. The reason, on the contrary, why a so-called triple point must be said to have a multiplicity of the degree 4, and not merely of the degree 3, springs from the fact that the 1st, 2nd and 3rd evectants of the discriminant all vanish at such a point.

It is clear, then, that there ought to exist a species of multiplicity for which the 1st and 2nd evectants vanish, but not the 3rd. In fact, as at a double point the first polars all merely intersect, but at a cusp have all a contact with one another of the first degree, so we ought to expect that there should exist a species of multiple point such that all the first polars should have with each other a contact of the second degree (or if we like so to say, the same curvature) at that point. When the curve has a triple point, all its first polars will have that point upon them as a double point; and it is not at the first glance, easy *à priori* to say what is the nature of the contact between two curves which intersect at a point which is a double point to each of them: we know upon settled analytical principles, that when one curve having a double point is crossed there by another curve not having a double point, that the two must be said to have with one another, a contact of the 1st degree; and we now learn from our theory of evection, that if each have a double point at the meeting-point, the degree of the contact must from principles of analogy be considered to be of the 3rd degree*. Now, then, we come to the question of deciding definitely what is a multiple point for which the degree of multiplicity is 3. It is, adopting either test, whether of first polar contact or of evection, a cusp situated or having its *nidus*, so to say, at a point of inflexion. In other words, $x=0$, $y=0$ will be a point whose multiplicity is intermediate between that of the cusp and that of a so-called triple point, when the characteristic of the curve admits of being written under the form

$$z^{n-2}x^2 + z^{n-3}(gx^3 + hx^2y + ixy^2) + z^{n-4} \text{ \&c.};$$

or in other words, when over and above the vanishing of the constant and linear coefficients, and the quadratic coefficient being a perfect square, as in the case of an ordinary cusp, this square has a factor in common with the next (the cubic) coefficient; or again, in other words, a curve has a point

* This may easily be verified by direct analytical means; as also the more general proposition, that two curves meeting at a point where there are m branches of the one and n branches of the other, must be considered to have mn coincident points in common, that is, if we like so to express it, to have a contact of the degree $mn-1$.

for which the multiplicity is 3 when its characteristic function admits of being expanded according to the powers of one of the variables, in such a manner that the first coefficient and the second (the linear) coefficient vanish, and that the discriminant of the third and the resultant of the third and fourth are both at the same time zero. This being the case, it may be shown that the first polars will all have with each other a contact of the second degree; and moreover, that all the evectants of the discriminant will have as a common factor a linear function of the variables, raised to a power whose index is three times that of the characteristic function. As, then, there is but one kind of ordinary double point, and but one kind of point with multiplicity 2, so there is one, and only one, kind of point with a multiplicity 3. A cusp is a peculiar double point; a flex-cusp (as for the moment I call the point last above discussed) is a peculiar cusp. This law of unambiguity, however, appears to stop at the third degree. A so-called triple point (which ought in fact to be called a *quintuple* point) is a point for which the multiplicity, as shown above, is of the fourth degree; but it is not the only point of that degree of multiplicity. Without assuming to have exhausted every possible supposition upon which such a degree of multiplicity may be brought into existence, it will be sufficient to take as an example a curve whose characteristic is capable of assuming the form

$$z^{n-2}x^2 + z^{n-3}(gx^3 + hx^2y) + z^{n-4}(kx^4 + lx^3y + mx^2y^2 + nxy^3) + z^{n-5} \text{ \&c.}$$

It may readily be demonstrated that the first polars of this curve have all with one another at the point x, y a contact of a degree exceeding the 2nd, that is of at least the 3rd degree (and, I believe, in general not higher). Now the point x, y is evidently not a triple-branched point, but a cusp with three additional degrees of singularity; so that we have evidence of the existence of a point whose degree of singularity is 5, and whose multiplicity is at least 4, but which is in no sense a modified triple point. It is probably true (but to demonstrate this requires a further advance to be made than has yet been realized in the theory of the constitution of discriminants) that a cusp may be so modified by the *nidus* at which it is posited, as, without ever passing into a triple point, to be capable of furnishing any amount of multiplicity whatever, curiously in this contrasting with an ordinary double point, no amount whatever of extraordinary singularity imparted to which, or so to speak, to its *nidus*, can ever heighten its multiplicity so as to make it surpass the first degree without first converting it into a cusp. I may illustrate the nature of a flex-cusp by what happens to a curve of the third degree. When it breaks up into a conic and a right line, there are two ordinary double points; for the existence of these double points, as for the existence of a cusp, two conditions are required. When, however, the right line and conic touch one another (a *casus omissus* this in the works of the special geometers), the characters of the cusp and the point of inflexion are combined at the point of contact; the multiplicity is of the third degree, and the singularity also of a degree not exceeding this; three conditions only being necessary to be satisfied in order that a given cubic may degenerate into such a form; and it will be found that the discriminant and the first and second evectants thereof vanish for this case, and that the third evectant of the discriminant will be a perfect 9th power; whereas in order that the cubic may have a so-called triple point, that is may degenerate into a trident of diverging rays, four conditions must be satisfied, and it will be found that when this is the case, the first, second, and third evectants of the discriminant will all vanish, and the fourth will be a perfect 12th power of a linear function of the variables. I may mention, by the way, at this place, that the law of a discriminant and the successive evectants up to the mth inclusive, all vanishing, may be expressed otherwise (not in *identical*, but in *equivalent* or *equipollent* terms), by saying that the discriminant and all its derivatives of a degree not exceeding the mth will all vanish—understanding by a derivative of the discriminant any function obtained from the discriminant by differentiating it any specified number of times with respect to the constants of the function to which it belongs, the same constants being repeated or not indifferently*. And very surprising it must be allowed to be, stated as a bare analytical fact, that $(m+1)$ conditions imposed upon the coefficients of a function of any number of variables and of any degree should suffice to make the inordinately greater number of functions which swarm among the derivatives of the mth and inferior degrees of the discriminant each and all simultaneously vanish.

Without pushing these observations too far for the patience of the general reader, it may be remarked by way of setting foot with our new theory upon the almost unvisited region of the singularities of surfaces, that by the light of analogy we may proceed with a safe and firm step as far as multiplicity of the third degree inclusive.

The function characteristic of the surface being supposed to be expressed in terms of the four variables x, y, z, t, and expanded according to descending powers of t, then when x, y, z is an ordinary double point of the first degree of multiplicity, the constant and the linear coefficient disappear; when the point has a multiplicity 2, the discriminant of the quadratic coefficient will be zero, that is this coefficient will be expressible by means of due linear transformations under the form of x^2+y^2; and when the multiplicity is to be of the degree 3, the cubic coefficient will, at the same time that the quadratic coefficient is put under the form x^2+y^2, itself (for the same system of x and y) assume the form of a cubic function of x, y, z, in which the highest power of z, that is z^3, will not appear; or in other words (restoring to x, y, z their

* Or, to speak more simply, the discriminant and its successive *differentials* up to the mth exclusive must all vanish simultaneously.

generality), not only will the first derivatives of the quadratic function be nullifiable simultaneously with each other, but likewise at the same time with the cubic function itself. These three cases will be for surfaces, the analogues so far, but only so far as regards the degree of the multiplicity, to the double point, cusp, and flex-cusp of curves*. The analogue to the so-called triple point of the curves will be a point whose degree of singularity, depending upon the vanishing of the six constants in the third coefficient (which is a quadratic function of x, y, z) at the same time as the three constants in the linear factor, would seem to be but 6 more than for a double point, that is in all $1+6$ or 7, but whose multiplicity, as inferred from the nature of the contact of its first polars, which will be of the 7th order, would appear to be 8 (a seeming incongruity which I am not at present in a condition to explain)†; so that there will apparently be 4 steps of multiplicity to interpolate between this case and the case analogous (*sub modo*) to the flex-cusp, last considered. Whether these intervening degrees correspond to singularities of an unambiguous kind, no one is at present in a condition to offer an opinion. I will conclude with a remark, the result of my experience in this kind of inquiry as far as I have yet gone in it, namely that it would be most erroneous to regard it as a branch of isolated and merely curious or fantastic speculation. Every singularity in a locus corresponds to the imposition of certain conditions upon the form of its characteristic; by aid of the theory of evection we are able to connect the existence of these conditions with certain consequences happening to the form of the discriminant, and thereby it becomes possible, upon known principles of analysis, to infer particulars relating to the constitution of the discriminant itself in its absolutely general form, very much upon the same principle as when the values of a function for particular values of its variable or variables are known, the general form of the function thereby itself, to some corresponding extent, becomes known. Thus, for instance, I have by the theory of evection in its most simple application, been led to a representation of the discriminant

* At an ordinary conical point of a surface for which the multiplicity is 1, every section of the surface is a curve with a double point. When the multiplicity is 2, the cone of contact becomes a pair of planes, through the intersection of which any other plane that can be drawn cuts the surface in a section having an ordinary cusp of multiplicity 2, but which themselves cut the surface in sections, having so-called triple points, so that for these two principal sections (which is rather surprising) the multiplicity suddenly jumps up from 2 to 4. All other things remaining unaltered when the multiplicity of the conical point is 3, the cusp belonging to any section of the surface drawn through any intersection of the two tangent planes passes from an ordinary cusp to a flex-cusp.

† So, too, at a so-called quadruple point in a curve, the degree of the contact of the 1st polars is 8, and therefore the multiplicity of the curve at such point is 9; but the number of constants which vanish for this case (namely all those of the cubic coefficient in x, y) over and above what vanish for the case of a so-called triple point is only 4, which is a unit less than the difference between the measures of the multiplicities at the respective points; and this difference continues to increase as we pass on to so-called quintuple and higher multiple points in the curves.

of a function of two variables under a form very different and very much more complete and fecund in consequences than has ever been supposed, or than I had myself previously imagined, to be possible.

According to the opinion expressed by an analyst of the French school, of pre-eminent force and sagacity, it is through this theory of multiplicity, here for the first time indicated, that we may hope to be able to bridge over for the purposes of the highest transcendental analysis, the immense chasm which at present separates our knowledge of the intimate constitution of functions of two from that of three, or any greater number of variables.

It is, as I take pleasure in repeating, to a hint from Mr Cayley*, who habitually discourses pearls and rubies, that I am indebted for the precious and pregnant observation on the form assumed by the first discriminantal evectant of a binary function with a pair of equal roots, out of which, combined with some antecedent reflections of my own, this new theory of multiplicity has taken its rise. The idea of the process of evection, and the discovery of its fundamental property of generating what, in my calculus of forms (*Cambridge and Dublin Mathematical Journal*), I have called *contravariants*, is due to my friend M. Hermite. The polar reciprocals of curves and other loci are contravariants and, as I have recently succeeded in showing, for curves at least, evectants, but of course not discriminantal evectants; and I am already able to give the actual explicit rule for the formation of the polar reciprocal of curves as high as the 5th degree, which with a little labour and consideration can be carried on to the 6th, and in fact to curves of any degree n when once we are acquainted with any mode of determining all such independent invariants of a function of two variables as are of dimensions not exceeding $2(n-1)$ in respect of the coefficients.

By the special geometers (by whom I mean those who, unvisited by a higher inspiration, continue to regard and to cultivate geometry as the science of mere sensible space) this problem has only been accomplished, and that but recently, for curves whose degrees do not exceed the 4th. Mr Salmon has made the happy and brilliant (and by the calculus of forms instantaneously demonstrable) discovery, communicated to me in the course of a most instructive and suggestive correspondence, that *a certain readily ascertainable evectant of every discriminant of any function whatever is an exact power of its polar reciprocal**.

* Mr Cayley's theorem stood thus:—If

$$ax^n + nbx^{n-1}y + \dots + nb'xy^{n-1} + a'y^n$$

have two equal roots, and ϖ be its discriminant, then will

$$\left\{y^n \frac{d}{da} - y^{n-1}x \frac{d}{db} \text{ \&c.} \pm x^n \frac{d}{da'}\right\} \varpi$$

be a perfect nth power. It will easily be seen that this theorem is convertible into a theorem of evection by interchanging in the result x and y with y and $-x$.

I believe that it may be shown, that, with the sole exception of odd-degreed functions of two variables, the *polar reciprocal itself* (as distinguished from a power thereof) of every function is an evectant, not (of course) of the discriminant, but of some determinable inferior invariant.

P.S. The terms pluri-simultaneous and pluri-simultaneity, used or suggested by me in my last paper in the *Magazine*, may be advantageously replaced by the more euphonious and regularly formed words consimultaneous, consimultaneity. Multiplicity and all its attributes and consequences are included as particular cases in the general conception and theory of consimultaneity, that is of consimultaneous equations, or, which is the same thing, of consimulevanescent functions.

* Namely, for a function of degree n, and variability (that is, having a number of variables) p, the $(n-1)^{p-1}$th evect of the discriminant is the $(n-1)$th power of the polar reciprocal.

47.

A DEMONSTRATION OF THE THEOREM THAT EVERY HOMOGENEOUS QUADRATIC POLYNOMIAL IS REDUCIBLE BY REAL ORTHOGONAL SUBSTITUTIONS TO THE FORM OF A SUM OF POSITIVE AND NEGATIVE SQUARES.

[*Philosophical Magazine*, IV. (1852), pp. 138—142.]

It is well known that the reduction of any quadratic polynomial

$$(1, 1)\, x^2 + 2\,(1, 2)\, xy + (2, 2)\, y^2 + \dots + (n, n)\, t^2$$

to the form $a_1\zeta^2 + a_2\eta^2 + \dots + a_n\theta^2$, where $\zeta, \eta \dots \theta$ are linear functions of $x, y \dots t$, such that $x^2 + y^2 + \dots + t^2$ remains identical with $\zeta^2 + \eta^2 + \dots + \theta^2$ (which identity is the characteristic test of orthogonal transformation), depends upon the solution of the equation

$$\begin{vmatrix} (1, 1) + \lambda, & (1, 2) \dots\dots\dots & (1, n) \\ (2, 1), & (2, 2) + \lambda \dots & (2, n) \\ \dots\dots & \dots\dots & \dots\dots \\ (n, 1), & (n, 2) \dots\dots\dots & (n, n) + \lambda \end{vmatrix} = 0.$$

The roots of this equation give $a_1, a_2 \dots a_n$; and if they are real, it is easily shown that the connexions between $x, y \dots t$; $\zeta, \eta \dots \theta$, are also real. M. Cauchy has somewhere given a proof of the theorem*, that the roots of λ in the above equation must necessarily always be real; but the annexed demonstration is, I believe, new; and being very simple, and reposing upon a theorem of interest in itself, and capable no doubt of many other applications, will, I think, be interesting to the mathematical readers of this *Magazine*.

* Jacobi and M. Borchardt have also given demonstrations; that of the latter consists in showing that Sturm's functions for ascertaining the total number of real roots expressed by my formulæ (many years ago given in this *Magazine*) are all, in the case of $f(\lambda)$, representable as the sums of squares, and are therefore essentially positive.

Let

$$f(\lambda) = \begin{vmatrix} (1, 1) + \lambda, & (1, 2) \dots\dots\dots\dots & & (1, n) \\ (2, 1), & (2, 2) + \lambda \dots\dots & & (2, n) \\ (3, 1), & (3, 2), & (3, 3) + \lambda \dots & (3, n) \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ (n, 1), & (n, 2) \dots\dots\dots\dots & & (n, n) + \lambda \end{vmatrix},$$

it is easily proved that $f(\lambda) \times f(-\lambda)$

$$= \begin{vmatrix} [1, 1] - \lambda^2, & [1, 2] \dots\dots & [1, n] \\ [2, 1], & [2, 2] - \lambda^2 \dots & [2, n] \\ \dots\dots & \dots\dots & \dots\dots \\ [n, 1], & [n, 2] \dots\dots & [n, n] - \lambda^2 \end{vmatrix},$$

where $[\iota, \epsilon] = (\iota, 1) \times (1, \epsilon) + (\iota, 2) \times (2, \epsilon) + \dots + (\iota, n) \times (n, \epsilon)$.

If, now, for all values of r and s, $(r, s) = (s, r)$, that is, if $f(0)$ becomes the complete determinant to a symmetrical matrix, then every term $[r, s]$ in the derived matrix becomes a sum of squares, and is essentially positive, and $(-1)^n f(\lambda) \times f(-\lambda)$ assumes the form

$$(\lambda^2)^n - F(\lambda^2)^{n-1} + G(\lambda^2)^{n-2} + \dots \pm L,$$

where $F, G, \dots L$ will evidently be all positive; for it may be shown that F will be the sum of the squares of the separate terms, that is, of the last minor determinants of the given matrix, G the sum of the squares of the last but one minors, and so on, L being the square of the complete determinant. For instance, if

$$f(\lambda) = \begin{vmatrix} a + \lambda, & \gamma, & \beta \\ \gamma, & b + \lambda, & \alpha \\ \beta, & \alpha, & c + \lambda \end{vmatrix}$$

$$-f(\lambda) \times f(-\lambda) = \lambda^6 - F\lambda^4 + G\lambda^2 - H,$$

where

$$F = a^2 + b^2 + c^2 + 2\alpha^2 + 2\beta^2 + 2\gamma^2,$$

$$G = (ab - \gamma^2)^2 + (bc - \alpha^2)^2 + (ac - \beta^2)^2$$
$$+ 2\,(a\alpha - \beta\gamma)^2 + 2\,(b\beta - \gamma\alpha)^2 + 2\,(c\gamma - \alpha\beta)^2,$$

$$H = \begin{vmatrix} a, & \gamma, & \beta \\ \gamma, & b, & \alpha \\ \beta, & \alpha, & c \end{vmatrix}^2.$$

Hence it follows immediately that $f(\lambda) = 0$ cannot have imaginary roots; for, if possible, let $\lambda = p + q\sqrt{(-1)}$, and write

$$a + p = a', \quad b + p = b', \quad c + p = c', \quad \lambda + p = \lambda',$$

$f(\lambda)$ becomes
$$\begin{vmatrix} a'+\lambda', & \gamma, & \beta \\ \gamma, & b'+\lambda', & \alpha \\ \beta, & \alpha, & c'+\lambda' \end{vmatrix},$$
or say $\phi(\lambda')$, and the equation $\phi(\lambda')\times\phi(-\lambda')=0$ will be of the form
$$\lambda'^6 - F'\lambda'^4 + G'\lambda'^2 - H' = 0,$$
where F', G', H' are all essentially positive. Hence, by Descartes' rule, no value of λ'^2 can be negative, that is, $(\lambda-p)^2$ cannot be of the form $-q^2$; that is to say, it is impossible for any of the roots of $f(\lambda)=0$ to be imaginary, or, as was to be demonstrated, all the roots are real.

I may take this occasion to remark, that by whatever linear substitutions, orthogonal or otherwise, a given polynomial be reduced to the form $\Sigma A_1\zeta_1^2$, the number of positive and negative coefficients is invariable: this is easily proved. If now we proceed to reduce the form (expressed under the umbral notation) $(a_1x_1 + a_2x_2 + \dots + a_nx_n)^2$ to the form
$$A_1\zeta_1^2 + A_2\zeta_2^2 + \dots + A_{n-1}\zeta^2_{n-1} + A_n\zeta_n^2,$$
by first driving out the mixed terms in which x_1 enters, then those in which x_2 enters, and so forth until eventually only x_n of the original variables is left, it may readily be shown that
$$A_1 = \begin{pmatrix} a_1 \\ a_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_1a_2 \\ a_1a_2 \end{pmatrix} \div \begin{pmatrix} a_1 \\ a_1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} a_1a_2a_3 \\ a_1a_2a_3 \end{pmatrix} \div \begin{pmatrix} a_1a_2 \\ a_1a_2 \end{pmatrix} \dots\dots$$
$$\dots A_n = \begin{pmatrix} a_1a_2\dots a_n \\ a_1a_2\dots a_n \end{pmatrix} \div \begin{pmatrix} a_1a_2\dots a_{n-1} \\ a_1a_2\dots a_{n-1} \end{pmatrix}.$$
It follows, therefore, that in whatever order we arrange the umbræ $a_1a_2\dots a_n$, the number of variations and of continuations of sign in the series
$$1, \begin{pmatrix} a_1 \\ a_1 \end{pmatrix}, \begin{pmatrix} a_1a_2 \\ a_1a_2 \end{pmatrix} \cdots \begin{pmatrix} a_1a_2\dots a_n \\ a_1a_2\dots a_n \end{pmatrix},$$
will be invariable, and in fact will be the same as the number of positive and negative roots in the generating function in λ above treated of, that is, since all the roots are real, will be the same as the number of variations and continuations in the series formed by the coefficients of the several powers of λ, that is
$$1, \Sigma\begin{pmatrix} a_1 \\ a_1 \end{pmatrix}, \Sigma\begin{pmatrix} a_1a_2 \\ a_1a_2 \end{pmatrix} \cdots \begin{pmatrix} a_1a_2\dots a_n \\ a_1a_2\dots a_n \end{pmatrix}.$$

The first part of this theorem admits of an easy direct demonstration; for by my theory of compound determinants, given in this *Magazine**, we know that
$$\begin{array}{cc} \overline{a_1a_2\dots a_{r-1}a_r}| & \overline{a_1a_2\dots a_{r-1}a_{r+1}}| \\ a_1a_2\dots a_{r-1}a_r & a_1a_2\dots a_{r-1}a_{r+1} \end{array}$$
$$= \begin{pmatrix} a_1a_2\dots a_{r-1} \\ a_1a_2\dots a_{r-1} \end{pmatrix} \times \begin{pmatrix} a_1a_2\dots a_{r-1}a_ra_{r+1} \\ a_1a_2\dots a_{r-1}a_ra_{r+1} \end{pmatrix}.$$

[* Cf. pp. 241, 252 above.]

The first member of this equation is equivalent to
$$\begin{pmatrix} a_1a_2\dots a_{r-1}a_r \\ a_1a_2\dots a_{r-1}a_r \end{pmatrix} \times \begin{pmatrix} a_1a_2\dots a_{r-1}a_{r+1} \\ a_1a_2\dots a_{r-1}a_{r+1} \end{pmatrix} - \begin{pmatrix} a_1a_2\dots a_{r-1}a_r \\ a_1a_2\dots a_{r-1}a_{r+1} \end{pmatrix}^2.$$
Hence it follows, that if the two factors on the right-hand side of the equation have the same sign,
$$\begin{pmatrix} a_1a_2\dots a_{r-1}a_r \\ a_1a_2\dots a_{r-1}a_r \end{pmatrix} \text{ and } \begin{pmatrix} a_1a_2\dots a_{r-1}a_{r+1} \\ a_1a_2\dots a_{r-1}a_{r+1} \end{pmatrix}$$
have also the same sign *inter se*, and consequently the two triads
$$\begin{bmatrix} a_1a_2\dots a_{r-1} \\ a_1a_2\dots a_{r-1} \end{bmatrix}, \begin{bmatrix} a_1a_2\dots a_{r-1}a_r \\ a_1a_2\dots a_{r-1}a_r \end{bmatrix}, \begin{bmatrix} a_1a_2\dots a_{r-1}a_ra_{r+1} \\ a_1a_2\dots a_{r-1}a_ra_{r+1} \end{bmatrix},$$
and
$$\begin{bmatrix} a_1a_2\dots a_{r-1} \\ a_1a_2\dots a_{r-1} \end{bmatrix}, \begin{bmatrix} a_1a_2\dots a_{r-1}a_{r+1} \\ a_1a_2\dots a_{r-1}a_{r+1} \end{bmatrix}, \begin{bmatrix} a_1a_2\dots a_{r-1}a_{r+1}a_r \\ a_1a_2\dots a_{r-1}a_{r+1}a_r \end{bmatrix},$$
will in all cases present the same number of changes and continuations, which proves that the contiguous umbræ, a_r, a_{r+1}, may be interchanged without affecting the number of variations and continuations in the entire series; but, as is well known, any one order of elements is always convertible into any other order by means of successive interchanges of contiguous elements, which demonstrates that, in whatever order the elements $a_1, a_2\dots a_n$ be arranged, the number of continuations and variations in
$$1, \begin{pmatrix} a_1 \\ a_1 \end{pmatrix}, \begin{pmatrix} a_1a_2 \\ a_1a_2 \end{pmatrix} \cdots \begin{pmatrix} a_1a_2\dots a_n \\ a_1a_2\dots a_n \end{pmatrix},$$
is invariable. But that the same thing is true (as we know it to be), for the relation between any one of these unsymmetrical series and the symmetrical series (resulting from the method of orthogonal transformation)
$$1, \Sigma\begin{pmatrix} a_1 \\ a_1 \end{pmatrix}, \Sigma\begin{pmatrix} a_1a_2 \\ a_1a_2 \end{pmatrix}, \dots \begin{pmatrix} a_1a_2\dots a_n \\ a_1a_2\dots a_n \end{pmatrix},$$
is by no means so easily demonstrable in the general case by a direct method, and the attention of algebraists is invited to supply such direct method of demonstration. My knowledge of the fact of this equivalence is, as I have stated, deduced from that remarkable but simple law to which I have adverted, which affirms the invariability of the number of the positive and negative signs between all linearly equivalent functions of the form $\Sigma \pm c_rx^r$ (subject, of course, to the condition that the equivalence is expressible by means of equations into which only real quantities enter); a law to which my view of the physical meaning of quantity of matter inclines me, upon the ground of analogy, to give the name of the Law of Inertia for Quadratic Forms, as expressing the fact of the existence of an invariable number inseparably attached to such forms.

48.

ON STAUDT'S THEOREMS CONCERNING THE CONTENTS OF POLYGONS AND POLYHEDRONS, WITH A NOTE ON A NEW AND RESEMBLING CLASS OF THEOREMS.

[*Philosophical Magazine*, IV. (1852), pp. 335—345.]

THE beautiful and important geometrical theorems of Staudt are, I believe, little, if at all, known to English mathematicians. They originally appeared in *Crelle's Journal* for the year 1843, and have been recently reproduced in M. Terquem's *Nouvelles Annales* for the August Number of the present year.

These theorems may be summed up, in a word, as intended to show the possibility and method of expressing the product of any two polygons or any two polyhedrons as entire functions of the squares of the distances of the angular points of the two figures from one another. The well-known expression for the square of the area of a triangle in terms of the sides (in which, when expanded, only even powers of the lengths of the sides appear), is but a particular case of Staudt's theorem for polygons, for it may be considered as the case of two equal and similar triangles whose angular points coincide. So in like manner, as observed by Staudt, a similar expression in terms of its sides may be found for the square of a pyramid. This expression had, however, been previously given (although, by a strangle negligence, not named for what it was) by Mr Cayley in the *Cambridge Mathematical Journal* for the year 1841*, in his paper on the relations between the mutual distances to one another of four points in a plane and five points in space; the singularly ingenious (and as singularly undisclosed) principle of that paper consisting in obtaining an expression for the volume of a pyramid in terms of its sides, and equating this, or rather its square, to zero as the conditions of the four angular points lying in the same plane.

* Query, Is not this expression for the volume of a pyramid in terms of its sides to be found in some previous writer? It can hardly have escaped inquiry.

The analogous condition for five points in space is virtually deduced by going out into rational space of four dimensions, and equating to zero the expression obtained for the volume of a plupyramid; meaning thereby the figure which stands in the same relation to space of four as a pyramid to space of three dimensions. Mr Cayley's method, if it had been pursued a step further, would have led him to a complete anticipation of the principal part of Staudt's discovery. The method here given is not substantially different from Mr Cayley's, but is made to rest upon a more general principle of transformation than that which he has employed. As to Staudt's own method, it is as clumsy and circuitous as his results are simple and beautiful. Geometry, trigonometry and statics, are laid under contribution to demonstrate relations which will be seen to flow as immediate and obvious consequences from the most elementary principles in the algorithm of determinants. Perhaps, however, M. Staudt's method is as good as could be found in the absence of the application of the method of determinants, the powers of which, even so recently as ten years ago, were not so well understood or so freely applied as at the present day.

The following new but simple theorem, of which I shall have occasion to make use, will be found to be a very useful addition to the ordinary method for the multiplication of determinants. "If the determinants represented by two square matrices are to be multiplied together, *any number of columns may be cut off* from the one matrix, and a corresponding number of columns from the other. Each of the lines in either one of the matrices so reduced in width as aforesaid being then multiplied by each line of the other, and the results of the multiplication arranged as a square matrix and bordered with the two respective sets of columns cut off arranged symmetrically (the one set parallel to the new columns, the other set parallel to the new lines), the complete determinant represented by the new matrix so bordered (abstraction made of the algebraical sign) will be the product of the two original determinants."

Thus $\begin{pmatrix} ab \\ cd \end{pmatrix} \times \begin{pmatrix} \alpha\beta \\ \gamma\delta \end{pmatrix}$ may be put under any one of the three following forms:—

$$\begin{vmatrix} a\alpha + b\beta, & a\gamma + b\delta \\ c\alpha + d\beta, & c\gamma + d\delta \end{vmatrix}$$

or

$$\begin{vmatrix} a\alpha, & a\gamma, & b \\ c\alpha, & c\gamma, & d \\ \beta, & \delta, & 0 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 2, & 2, & a, & b \\ 2, & 2, & c, & d \\ \alpha, & \beta, & 0, & 0 \\ \gamma, & \delta, & 0, & 0 \end{vmatrix}^*.$$

* Any quantities might be substituted instead of 2 in the places occupied by the figure in the above determinant, as such terms do not influence the result; this figure is probably, however, the proper quantity arising from the application of the rule, because (as all who have calculated with determinants are aware) the value of the determinant represented by a matrix of *no places* is not zero but unity.

And in general for two matrices of n^2 terms each, this rule of multiplication will give $(n+1)$ distinct forms representing their products.

Thus, as a further example,

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} \times \begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix},$$

besides the first and last forms, will be representable by the two intermediate forms

$$-\begin{vmatrix} a\alpha+b\beta, & a\alpha'+b\beta', & a\alpha''+b\beta'', & c \\ a'\alpha+b'\beta, & a'\alpha'+b'\beta', & a'\alpha''+b'\beta'', & c' \\ a''\alpha+b''\beta, & a''\alpha'+b''\beta', & a''\alpha''+b''\beta'', & c'' \\ \gamma, & \gamma', & \gamma'', & 0 \end{vmatrix}$$

and

$$+\begin{vmatrix} a\alpha, & a\alpha', & a\alpha'', & b, & c \\ a'\alpha, & a'\alpha', & a'\alpha'', & b', & c' \\ a''\alpha, & a''\alpha', & a''\alpha'', & b'', & c'' \\ \beta, & \beta', & \beta'', & 0, & 0 \\ \gamma, & \gamma', & \gamma'', & 0, & 0 \end{vmatrix}.$$

To arrive, for instance, at the latter of these two forms, we have only to write the two given matrices under the respective forms

$$\begin{vmatrix} a, & b, & c, & 0, & 0 \\ a', & b', & c', & 0, & 0 \\ a'', & b'', & c'', & 0, & 0 \\ 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0, & 1 \end{vmatrix}, \quad \begin{vmatrix} \alpha, & 0, & 0, & \beta, & \gamma \\ \alpha', & 0, & 0, & \beta', & \gamma' \\ \alpha'', & 0, & 0, & \beta'', & \gamma'' \\ 0, & 1, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0 \end{vmatrix}$$

and then apply the ordinary rule of multiplication. So, again, to arrive at the first of the above written two forms, we must write the two given matrices under the respective forms

$$\begin{vmatrix} a, & b, & c, & 0 \\ a', & b', & c', & 0 \\ a'', & b'', & c'', & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix} \text{ and } -\begin{vmatrix} \alpha, & \beta, & 0, & \gamma \\ \alpha', & \beta', & 0, & \gamma' \\ \alpha'', & \beta'', & 0, & \gamma'' \\ 0, & 0, & 1, & 0 \end{vmatrix}$$

and proceed as before.

This rule is interesting as exhibiting, as above shown, a complete scale whereby we may descend from the ordinary mode of representing the product of two determinants to the form, also known, where the two original deter-

minants are made to occupy opposite quadrants of a square whose places in one of the remaining quadrants are left vacant, and shows us that under one aspect at least this latter form may be regarded as a matrix *bordered* by the two given matrices.

A second but obvious theorem requiring preliminary notice is the following, namely that the value of the determinant to the matrix

$$\begin{matrix} a_{1,1}, & a_{1,2} \ldots a_{1,n}, & 1, \\ a_{2,1}, & a_{2,2} \ldots a_{2,n}, & 1, \\ \cdots & \cdots\cdots & \cdots \\ a_{n,1}, & a_{n,2} \ldots a_{n,n}, & 1, \\ 1, & 1, \ldots 1, & 0, \end{matrix}$$

is the same as the value of the determinant to the matrix

$$\begin{matrix} A_{1,1}, & A_{1,2} \ldots A_{1,n}, & 1, \\ A_{2,1}, & A_{2,2} \ldots A_{2,n}, & 1, \\ \cdots & \cdots\cdots & \cdots \\ A_{n,1}, & A_{n,2} \ldots A_{n,n}, & 1, \\ 1, & 1, \ldots 1, & 0, \end{matrix}$$

where in general

$$A_{r,s} = a_{r,s} + h_r + k_s,$$

$h_1, h_2 \ldots h_n$ and $k_1, k_2 \ldots k_n$ being any two perfectly arbitrary series of quantities. This simple transformation is of course derived by adding to the respective columns in the first matrix the last column (consisting of units) multiplied respectively by $h_1, h_2 \ldots h_n, 0$; and to the respective lines, the last line (consisting of units) multiplied respectively by $k_1, k_2 \ldots k_n, 0$.

Suppose, now, that we have two tetrahedrons whose volumes are represented respectively by one-sixth of the respective determinants

$$\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}, \quad \begin{vmatrix} \xi_1, & \eta_1, & \zeta_1, & 1 \\ \xi_2, & \eta_2, & \zeta_2, & 1 \\ \xi_3, & \eta_3, & \zeta_3, & 1 \\ \xi_4, & \eta_4, & \zeta_4, & 1 \end{vmatrix},$$

x_r, y_r, z_r representing the orthogonal coordinates of the point r in one tetrahedron, and ξ_r, η_r, ζ_r the same for the point r in the other.

By the first theorem their product may be represented (striking off the last column only from each matrix) by the matrix

$$\begin{vmatrix} \Sigma x_1\xi_1, & \Sigma x_1\xi_2, & \Sigma x_1\xi_3, & \Sigma x_1\xi_4, & 1 \\ \Sigma x_2\xi_1, & \Sigma x_2\xi_2, & \Sigma x_2\xi_3, & \Sigma x_2\xi_4, & 1 \\ \Sigma x_3\xi_1, & \Sigma x_3\xi_2, & \Sigma x_3\xi_3, & \Sigma x_3\xi_4, & 1 \\ \Sigma x_4\xi_1, & \Sigma x_4\xi_2, & \Sigma x_4\xi_3, & \Sigma x_4\xi_4, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix},$$

where, in general, any such term as $\Sigma x_r \xi_s$ represents

$$x_r \xi_s + y_r \eta_s + z_r \zeta_s.$$

Again, by virtue of the second theorem, adding

$$-\tfrac{1}{2}\Sigma x_1^2, \quad -\tfrac{1}{2}\Sigma x_2^2, \quad -\tfrac{1}{2}\Sigma x_3^2, \quad -\tfrac{1}{2}\Sigma x_4^2$$

to the respective lines, and

$$-\tfrac{1}{2}\Sigma \xi_1^2, \quad -\tfrac{1}{2}\Sigma \xi_2^2, \quad -\tfrac{1}{2}\Sigma \xi_3^2, \quad -\tfrac{1}{2}\Sigma \xi_4^2$$

to the respective columns, the above matrix becomes (after a change of signs not affecting the result) the $-\frac{1}{8}$th of

$$\begin{vmatrix} \Sigma(x_1-\xi_1)^2, & \Sigma(x_1-\xi_2)^2, & \Sigma(x_1-\xi_3)^2, & \Sigma(x_1-\xi_4)^2, & 1 \\ \Sigma(x_2-\xi_1)^2, & \Sigma(x_2-\xi_2)^2, & \Sigma(x_2-\xi_3)^2, & \Sigma(x_2-\xi_4)^2, & 1 \\ \Sigma(x_3-\xi_1)^2, & \Sigma(x_3-\xi_2)^2, & \Sigma(x_3-\xi_3)^2, & \Sigma(x_3-\xi_4)^2, & 1 \\ \Sigma(x_4-\xi_1)^2, & \Sigma(x_4-\xi_2)^2, & \Sigma(x_4-\xi_3)^2, & \Sigma(x_4-\xi_4)^2, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix},$$

or calling the angular points of the one tetrahedron a, b, c, d, and of the other p, q, r, s, 8×36, that is 288 times, their product is representable by $-1 \times$ the determinant

$$\begin{vmatrix} (ap)^2, & (aq)^2, & (ar)^2, & (as)^2, & 1 \\ (bp)^2, & (bq)^2, & (br)^2, & (bs)^2, & 1 \\ (cp)^2, & (cq)^2, & (cr)^2, & (cs)^2, & 1 \\ (dp)^2, & (dq)^2, & (dr)^2, & (ds)^2, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix},$$

and of course if p, q, r, s coincide respectively with a, b, c, d, 576 times the square of the tetrahedron $abcd$ will be represented under Mr Cayley's form,

$$\begin{vmatrix} 0, & (ab)^2, & (ac)^2, & (ad)^2, & 1 \\ (ba)^2, & 0, & (bc)^2, & (bd)^2, & 1 \\ (ca)^2, & (cb)^2, & 0, & (cd)^2, & 1 \\ (da)^2, & (db)^2, & (dc)^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix} *,$$

four out of the sixteen distances vanishing, and the remaining twelve reducing to six pairs of equal distances. The demonstration of Staudt's

* The corresponding quantity to the above determinant for the case of the triangle (hereafter given) is identical with the Norm to the sum of the sides. I have succeeded in finding the Factor (of ten dimensions in respect of the edges), which, multiplied by the above Determinant itself, expresses the Norm to the sum of the Faces, that is, the superficial area of the Tetrahedron.

theorem for triangles is obtained in precisely the same way by throwing the product of the two determinants

$$\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \xi_1, & \eta_1, & 1 \\ \xi_2, & \eta_2, & 1 \\ \xi_3, & \eta_3, & 1 \end{vmatrix}$$

under the form of $-\frac{1}{4}$th of

$$\begin{vmatrix} \Sigma(x_1-\xi_1)^2, & \Sigma(x_1-\xi_2)^2, & \Sigma(x_1-\xi_3)^2, & 1 \\ \Sigma(x_2-\xi_1)^2, & \Sigma(x_2-\xi_2)^2, & \Sigma(x_2-\xi_3)^2, & 1 \\ \Sigma(x_3-\xi_1)^2, & \Sigma(x_3-\xi_2)^2, & \Sigma(x_3-\xi_3)^2, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}.$$

When the two triangles coincide, calling their angular points a, b, c the above written determinant becomes

$$\begin{vmatrix} 0, & (ab)^2, & (ac)^2, & 1 \\ (ba)^2, & 0, & (bc)^2, & 1 \\ (ca)^2, & (cb)^2, & 0, & 1 \\ 1, & 1, & 1, & \end{vmatrix}.$$

or

$$(ab)^4 + (ac)^4 + (bc)^4 - 2(ab)^2.(ac)^2 - 2(ab)^2.(bc)^2 - 2(ac)^2.(bc)^2,$$

the negative of which is the well-known form expressing the square of four times the area of the triangle abc.

There is another and more general theorem of Staudt for two triangles not in the same plane, which may be obtained with equal facility. In fact, if we start from the determinant

$$\begin{vmatrix} (a\alpha)^2, & (a\beta)^2, & (a\gamma)^2, & 1 \\ (b\alpha)^2, & (b\beta)^2, & (b\gamma)^2, & 1 \\ (c\alpha)^2, & (c\beta)^2, & (c\gamma)^2, & 1 \\ 1, & 1, & 1, & \end{vmatrix},$$

and add to each column respectively the last column multiplied by $e\xi_1^2, e\xi_2^2, e\xi_3^2$ respectively, we arrive at the form

$$\begin{vmatrix} (a\alpha)^2 + e\xi_1^2, & (a\beta)^2 + e\xi_2^2, & (a\gamma)^2 + e\xi_3^2, & 1 \\ (b\alpha)^2 + e\xi_1^2, & (b\beta)^2 + e\xi_2^2, & (b\gamma)^2 + e\xi_3^2, & 1 \\ (c\alpha)^2 + e\xi_1^2, & (c\beta)^2 + e\xi_2^2, & (c\gamma)^2 + e\xi_3^2, & 1 \\ 1, & 1, & 1, & \end{vmatrix},$$

and considering $\xi_1, \eta_1; \xi_2, \eta_2; \xi_3, \eta_3$ as the coordinates of α, β, γ, the projections upon the plane of abc of a triangle ABC, whose plane intersects the former plane in the axis of y, and makes with that plane an angle whose tangent is e, it is easily seen that this determinant is term for term identical with the determinant

$$\begin{vmatrix} (aA)^2, & (aB)^2, & (aC)^2, & 1 \\ (bA)^2, & (bB)^2, & (bC)^2, & 1 \\ (cA)^2, & (cB)^2, & (cC)^2, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix},$$

which therefore expresses -16 times the product of the triangles abc and $\alpha\beta\gamma$, that is $abc \times ABC \times$ cosine of the angle between the two. A similar method, if we ascend from sensible to rational geometry, may be given for expressing in terms of the distances the product of *any* two pyramids (in a hyperspace) by the cosine of the angle included between the two infinite spaces * in which they respectively lie. To pass from the cases which have been considered of two triangles to two polygons, or of two tetrahedrons to two polyhedrons, generally presents no difficulty; and for Professor Staudt's method of doing so, which is simple and ingenious, and does not admit of material improvement, the reader is referred to the memoir in Crelle's *Journal* or Terquem's *Annales* already adverted to. It is, however, to be remarked (and this does not appear to be sufficiently noticed in the memoirs referred to), that whilst the expression for the product of any two polygons in terms of the distances given by Staudt's theorem is unique, that for the product of two polyhedrons given by the same is not so, but will admit of as many varieties of representation as there are units in the product of the numbers respectively expressing the number of ways in which each polygonal face of each polyhedron admits of being mapped out into triangles. I cannot help conjecturing (and it is to be wished that Professor Staudt or some other geometrician would consider this point) that in every case there exists, linearly derivable from Staudt's optional formulæ (but not coincident with any one of them), some unique and best, because most symmetrical, formula for expressing the product of two polyhedrons in terms of the distances of the angular points of the one from those of the other. In conclusion I may observe, that there is a theorem for distances measured on a given straight line, which, although not mentioned by Staudt, belongs to precisely the same class as his theorems for areas in a plane and volumes in space; namely a theorem which expresses twice the rectangle of any two such distances under the form of an aggregate of four squares, two taken positively and two

* In rational or universal geometry, that which is commonly termed infinite space (as if it were something absolute and unique, and to which, by the conditions of our being, the representative power of the understanding is limited), is regarded as a single homaloid related to a plane, precisely in the same way as a plane is to a right line. Universal geometry brings home to the mind with an irresistible force of conviction the truth of the Kantian doctrine of locality.

negatively; that is to say, if A, B, C, D be any four points on a right line $2AB \times CD = AD^2 + BC^2 - AC^2 - BD^2$. I know not whether this theorem be new, but it is one which evidently must be of considerable utility to the practical geometer.

Note on the above.

The fundamental theorem in determinants, published by me in the *Philosophical Magazine* in the course of last year*, leads immediately to a class of theorems strongly resembling, and doubtless intimately connected with, those of Staudt.

Thus for triangles we have by this fundamental theorem

$$\begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ 1, & 1, & 1 \end{vmatrix} \times \begin{vmatrix} \xi_1, & \xi_2, & \xi_3 \\ \eta_1, & \eta_2, & \eta_3 \\ 1, & 1, & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x_1, & \xi_1, & \xi_2 \\ y_1, & \eta_1, & \eta_2 \\ 1, & 1, & 1 \end{vmatrix} \times \begin{vmatrix} \xi_3, & x_2, & x_3 \\ \eta_3, & y_2, & y_3 \\ 1, & 1, & 1 \end{vmatrix} + \begin{vmatrix} x_1, & \xi_2, & \xi_3 \\ y_1, & \eta_2, & \eta_3 \\ 1, & 1, & 1 \end{vmatrix} \times \begin{vmatrix} \xi_1, & x_2, & x_3 \\ \eta_1, & y_2, & y_3 \\ 1, & 1, & 1 \end{vmatrix}$$

$$+ \begin{vmatrix} x_1, & \xi_3, & \xi_1 \\ y_1, & \eta_3, & \eta_1 \\ 1, & 1, & 1 \end{vmatrix} \times \begin{vmatrix} \xi_2, & x_2, & x_3 \\ \eta_2, & y_2, & y_3 \\ 1, & 1, & 1 \end{vmatrix}$$

and consequently, if ABC, DEF be any two triangles,

$$ABC \times DEF = ADE \times FBC + AEF \times DBC + AFD \times BCE.$$

This may be considered a theorem relating to two ternary systems of points in a plane. The analogous and similarly obtainable theorem for two binary systems of points in the same right line is

$$AB \times CD = AC \times DB - AD \times CB.$$

As in applying this last theorem to obtain correct numerical results we must give the same algebraical sign to any two lengths denoted by the two *arrangements* XY, ZT, according as the direction from X to Y is the same as that from Z to T, or contrary to it, so in the theorem for the products of triangles, the areas denoted by any two ternary arrangements XYZ, TUV must be taken with the like or the contrary sign, according as the direction of the rotation XYZ is consentient with or contrary to that of TUV; so that three of the six possible arrangements of XYZ may be used indifferently for one another, but the other three would imply a change of sign. If we

[* See pp. 249, 253 above.]

analyse what we mean by fixing the direction of the rotation of XYZ, and reduce this form of speech to its simplest terms, we easily see that it amounts to ascertaining on which side of B, C lies, that is whether to its right or left, to a spectator stationed at A on a given side of the plane ABC.

Let us now pass to the corresponding theorems for two tetrahedrons put respectively under the forms

$$\begin{vmatrix} x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \\ z_1, & z_2, & z_3, & z_4 \\ 1, & 1, & 1, & 1 \end{vmatrix}, \qquad \begin{vmatrix} \xi_1, & \xi_2, & \xi_3, & \xi_4 \\ \eta_1, & \eta_2, & \eta_3, & \eta_4 \\ \zeta_1, & \zeta_2, & \zeta_3, & \zeta_4 \\ 1, & 1, & 1, & 1 \end{vmatrix}.$$

We may represent this product in either of two ways by the application of our fundamental theorem, namely as

$$\begin{vmatrix} x_1, & \xi_1, & \xi_2, & \xi_3 \\ y_1, & \eta_1, & \eta_2, & \eta_3 \\ z_1, & \zeta_1, & \zeta_2, & \zeta_3 \\ 1, & 1, & 1, & 1 \end{vmatrix} \times \begin{vmatrix} \xi_4, & x_2, & x_3, & x_4 \\ \eta_4, & y_2, & y_3, & y_4 \\ \zeta_4, & z_2, & z_3, & z_4 \\ 1, & 1, & 1, & 1 \end{vmatrix} + \&c.$$

or as

$$\begin{vmatrix} x_1, & x_2, & \xi_1, & \xi_2 \\ y_1, & y_2, & \eta_1, & \eta_2 \\ z_1, & z_2, & \zeta_1, & \zeta_2 \\ 1, & 1, & 1, & 1 \end{vmatrix} \times \begin{vmatrix} \xi_3, & \xi_4, & x_3, & x_4 \\ \eta_3, & \eta_4, & y_3, & y_4 \\ \zeta_3, & \zeta_4, & z_3, & z_4 \\ 1, & 1, & 1, & 1 \end{vmatrix} + \&c.$$

there being four products to be added together in the first expression and six in the latter; and the rule, if we wish that all the products may be additive, being that on removing the sign of multiplication the determinant to the square matrix formed by the Greek letters *in situ* shall always preserve the same sign. Hence we derive two geometrical formulæ concerning the products of polyhedrons, namely

$$(1)\quad ABCD \times EFGH = ABCE \times FGHD - ABCF \times GHED$$
$$+ ABCG \times HEFD - ABCH \times FGED.$$

$$(2)\quad ABCD \times EFGH = ABEF \times GHCD + ABGH \times EFCD$$
$$+ ABEG \times HFCD + ABHF \times EGCD$$
$$+ ABEH \times FGCD + ABFG \times EHCD.$$

These formulæ give rise to an exceedingly interesting observation. In order that they shall be numerically true, we must have a rule for fixing the sign to be given to the solid content represented by any reading off of the four points of a tetrahedron, that is we must have a rule for determining

the sign of solid contents of figures situated anywhere in space analogous to that which, as applied to linear distances reckoned on a given right line, is the true foundation of the language of trigonometry, and the condition precedent for the possibility of any system of analytical geometry such as exists, and which, not altogether without surprise, I have observed in the pages of this *Magazine* one of the learned contributors has thought it necessary to vindicate the propriety of importing into his theory of quaternions.

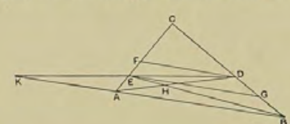
Various rules may be given for fixing the sign of a tetrahedron denoted by a given order of four letters. One is the following: the content of $ABCD$ is to be taken positive or negative, according as to a spectator at A the rotation of BCD is positive or negative. Another, again, is to consider AB and CD as representing, say two electrical currents, and to suppose a spectator so placed that the current AB shall pass through the longitudinal axis of his body from the head towards the feet, and looking towards the other current CD; the sign of the solid content of the tetrahedron (and, indeed, also the effect, in a general sense, of the action of the two currents upon one another) will depend upon the circumstance of this latter current appearing to flow from the right to the left, or contrariwise in respect of the spectator. Last and simplest mode of all, the sign of the solid content of $ABCD$ will depend upon the *nature* (in respect to its being a right-handed or left-handed-screw) of any regular screw-line (whether the common helix or one in which the increase or decrease of the inclination is always in the same direction) terminating at B and C, and so taken that BA shall be the direction of the tangent produced at B, and CD the direction of the tangent produced at C. Inasmuch as of the twenty-four permutations of a quaternary arrangement a defined twelve have one sign, and the other twelve the contrary sign, these various definitions of the direction, or, as it may be termed, polarity, of a tetrahedron corresponding to a given reading, whether as taken each in itself or compared one with another, give rise to, or rather imply a considerable number of interesting theorems included in our intuitions of space, and probably belonging to the, in my belief, inexhaustible class of primary and indemonstrable truths of the understanding.

49.

ON A SIMPLE GEOMETRICAL PROBLEM ILLUSTRATING A CONJECTURED PRINCIPLE IN THE THEORY OF GEOMETRICAL METHOD.

[*Philosophical Magazine*, IV. (1852), pp. 366—369.]

THE following theorem deserves attention as illustrating a principle of geometrical method which will be presently adverted to. It is curious, also, from the fact of its solution being by no means so obvious and self-evident as one would expect from the extreme simplicity of its enunciation. It appeared, and for the first time, it is believed, at the University of Cambridge about a twelvemonth back, where it excited considerable attention among some of the mathematicians of the place. The proposition, as originally presented, was merely to prove that if ABC be a triangle, and if AD and BE drawn bisecting the angles at A and B and meeting the opposite sides in D and E be equal, then the triangle must be isosceles. It is particularly



noticeable that all the geometrical demonstrations yet given of this theorem are indirect. Thus the first and simplest (communicated to me by a promising young geometrician, Mr B. L. Smith of Jesus College, Cambridge), was the following:—Assume one of the angles at DAB to be greater than the corresponding angle EBA; it can easily be shown that, upon this supposition, D will be higher up from AB than E; so that if DF and EG be drawn parallel to AB, DF will be above EG; it is then easily shown that $DF = AF$, $EG = BG$, and consequently DF and AF are each respectively less than EG

and BG; and also DFA, which is the supplement of twice DAB, will be less than EGB, which is the supplement of twice FBA; from which it is readily inferred, by an easy corollary to a proposition of Euclid, that DA will be less than FB, whereas it should be equal to it; so that neither of the half angles at the base can be greater than the other, and the triangle is proved to be isosceles. Another and independent demonstration by the writer of this article is less simple, but has the advantage of lending itself at once to a considerable generalization of the theorem as proposed. Assuming, as above, that DAB is greater than EBA, it is easily seen that DE produced will cut BA at K on the side of it: also if AD and BE intersect in H, it is readily demonstrable, by a suitably constructed apparatus of similar triangles, that

$$AH : BH :: CE : CD.$$

But as HBA is less than HAB, AH is less than BH, and therefore CE is less than CD, and therefore CED is greater than CDE; that is to say, CAB less K is greater than CBA plus K, and therefore DAB less K is greater than EBA, that is ADE is greater than ABE, and therefore the perpendicular from A upon DE is greater than that from E on AB, which is easily proved to be absurd. Hence, as before, the triangle is proved to be isosceles. This proof, it is obvious, remains good for *all cases* in which EB and DA, drawn on either side of the base, divide the angles at the base proportionally, provided that these lines remain equal, and make positive or negative angles with the base not *less* than one-half of the respective corresponding angles which the sides of the triangle are supposed to make with it. The analytical solution of the question, as might be expected, extends the result still further. To obtain this, let

$$BAC = n \,.\, BAD, \qquad ABC = n \,.\, ABE,$$

n for the present being any numerical quantity, positive or negative; calling $BAC = 2n\alpha$, $ABC = 2n\beta$, we readily obtain, by comparison of the equal dividing lines with the base of the triangle,

$$\frac{\sin(2n\alpha + 2\beta)}{\sin 2n\alpha} = \frac{\sin(2n\beta + 2\alpha)}{\sin 2n\beta},$$

or

$$\frac{\sin(2n\alpha + 2\beta)}{\sin(2n\beta + 2\alpha)} = \frac{\sin 2n\alpha}{\sin 2n\beta};$$

and by an obvious reduction,

$$\frac{\tan(n-1)(\alpha-\beta)}{\tan n(\alpha-\beta)} = \frac{\tan(n+1)(\alpha+\beta)}{\tan n(\alpha+\beta)}.$$

When this equation is put under an integer form, it is of course satisfied by making $\alpha = \beta$; on any other supposition than $\alpha = \beta$ it evidently cannot be satisfied by admissible values of the angles for any value of n between

$+1$ and $+\infty$; for on that supposition, since $(\alpha-\beta)$ and $(\alpha+\beta)$ are each less than $\frac{180}{2n}$, the first side of the equation will be necessarily a proper fraction and positive; but the second side, either a positive improper fraction if $(n+1)(\alpha+\beta)$ be less, and a negative proper or a negative improper fraction if $(n+1)(\alpha+\beta)$ be greater than a right angle.

If n be negative, let it equal $-\nu$, then

$$\frac{\tan(\nu+1)(\alpha-\beta)}{\tan\nu(\alpha-\beta)}=\frac{\tan(\nu-1)(\alpha+\beta)}{\tan\nu(\alpha+\beta)};$$

and for the same reason as before, if ν lies between ∞ and 1, this equation cannot be satisfied. Hence the theorem is proved to be true for all values of n, except between $+1$ and -1. For these values it ceases to be true; in fact, for such values for any given values of $(\alpha-\beta)$ there will be always, as it may be easily proved, one or more values of $(\alpha+\beta)$; thus if $n=\frac{1}{2}$, the equation becomes

$$\frac{\tan 3\left(\frac{\alpha+\beta}{2}\right)}{\tan\frac{\alpha+\beta}{2}}=-1;$$

and if $n=-\frac{1}{2}$,

$$\frac{\tan 3\left(\frac{\alpha-\beta}{2}\right)}{\tan\frac{\alpha-\beta}{2}}=-1,$$

showing that $\alpha+\beta=90$ and $\alpha-\beta=\pm 90$ in these respective cases will afford a solution over and above the solution $\alpha=\beta$, which is easily verified geometrically*. It would be an interesting inquiry (for those who have leisure for such investigations) to determine for any given value of n between $+1$ and -1 the superior and inferior limits to the number of admissible values of $\alpha+\beta$ corresponding to any given value of $\alpha-\beta$†.

My reader will now be prepared to see why it is that all the geometrical demonstrations given of this theorem, even in the simplest case of all, namely when $n=2$, are indirect, I believe I may venture to say *necessarily* indirect. It is because the truth of the theorem depends on the necessary non-existence of real roots (between prescribed limits) of the analytical equation expressing the conditions of the question; and I believe that it may be safely taken as an axiom in geometrical method, that whenever this is the case no other

* In the first of these cases, if the base of the triangle is supposed given, the locus of the vertex is a right line and a circle; in the second case, a right line and an equilateral hyperbola.

† When $\pm n$ lies between $\frac{1}{2\iota-1}$ and $\frac{1}{2\iota+1}$ (ι being any positive integer), it is easily seen that the superior limit must be at least as great as ι.

form of proof than that of the *reductio ad absurdum* is possible in the nature of things. If this principle is erroneous, it must admit of an easy refutation in particular instances.

As an example, I throw out (not a challenge, but) an invitation to discover a direct proof, if such exist, of the following geometrical theorem, as simple a one as it is perhaps possible to imagine:—"To prove that if from the middle of a circular arc two chords be drawn, and the remoter segments of these chords cut off by the line joining the end of the arc be equal, the nearer segments will also be equal." The analytical proof depends upon the fact of the equation $x^2+ax=b^2$ (where a is the given length of each segment, and b the length of the chord of half the given arc) having only one admissible root; and if the principle assumed or presumed to be true be valid, no other form of pure geometrical demonstration than the *reductio ad absurdum* should be applicable in this case. For the converse case, where the nearer segments are given equal, the reducing equation is $a(a+x)=b^2$, indicating nothing to the contrary of the possibility of there being a direct solution, which accordingly is easily shown to exist. The indirect form of demonstration, it may be mentioned, is sometimes liable to be introduced in a manner to escape notice. As, for instance, if it should be taken for granted in the course of an argument, that one triangle upon the same base and the same side of it as another triangle, and having the same vertical angle, must have its vertex lying on the same arc; this would seem to be *immediately* true by virtue of the well-known theorem, that angles in the same circular segment are equal, but in reality can only be *inferred* from it indirectly by showing the impossibility of its lying outside or inside the arc in question. To go one step further, I believe it to be the case, that granted to be true all those fundamental propositions in geometry which are presupposed in the principles upon which the language of analytical geometry is constructed, then that the *reductio ad absurdum* not only is of necessity to be employed, but moreover in propositions of an affirmative character never need be employed, except when as above explained the analytical demonstration is founded on the impossibility or inadmissibility of certain roots due to the degree of the equation implied in the conditions of the question. If this surmise turn out to be correct, we are furnished with *a universal criterion for determining when the use of the indirect method of geometrical proof should be considered valid and admissible and when not**.

* If report may be believed, intellects capable of extending the bounds of the planetary system and lighting up new regions of the universe with the torch of analysis, have been baffled by the difficulties of the elementary problem stated at the outset of this paper, in consequence, it is to be presumed, of seeking a form of geometrical demonstration of which the question from its nature does not admit. If this be so, no better evidence could be desired to evince the importance of such a criterion as that suggested in the text.

50.

ON THE EXPRESSIONS FOR THE QUOTIENTS WHICH APPEAR IN THE APPLICATION OF STURM'S METHOD TO THE DISCOVERY OF THE REAL ROOTS OF AN EQUATION.

[*Hull British Association Report* (1853), Part II., pp. 1—3.]

MANY years ago I published expressions for the residues which appear in the application of the process of common measure to fx and $f'x$, and which constitute Sturm's auxiliary functions. These expressions are complete functions of the factors of fx and of differences of the roots of fx, and are therefore in effect functions of the factors exclusively, since the difference between any two roots may be expressed as the difference between two corresponding factors. Having found that in the practical applications of Sturm's theorem the quotients may be employed with advantage to replace the use of the residues, I have been led to consider their constitution; and having succeeded in expressing these quotients (which are of course linear functions of x) under a similar form to that of the residues, that is, as complete functions of the factors and differences of the roots of fx, I have pleasure in submitting the result to the notice of the Mathematical Section of the British Association.

Let $h_1, h_2, h_3 \ldots h_n$ be the n roots of fx.

Let $\zeta(a, b, c \ldots l)$ in general denote the squared product of the differences of $a, b, c \ldots l$.

Let Z_i denote in general $\Sigma\zeta(h_{\theta_1} h_{\theta_2} \ldots h_{\theta_i})$, where $\theta_1, \theta_2 \ldots \theta_i$ indicate any combination of i out of the n quantities $a, b, c, \ldots l$, with the convention that $Z_0 = 1$, $Z_1 = n$; and let (i) denote $\frac{1}{2}\{1 + (-1)^i\}$, being zero when i is odd, and unity when i is even; then I find that the ith quotient Q_i may be written under the form

$$Q_i = {}_iP_1^2(x - h_1) + {}_iP_2^2(x - h_2) + \ldots + {}_iP_n^2(x - h_n),$$

where in general

$${}_iP_e = \frac{Z_{i-1}}{Z_i}\,\frac{Z^2_{i-3}}{Z^2_{i-2}}\,\frac{Z^2_{i-5}}{Z^2_{i-4}} \cdots \frac{Z^2_{(i)}}{Z^2_{(i)+1}}$$

$$\times\, \Sigma\,\{\zeta(h_{\theta_1} h_{\theta_2} \ldots h_{\theta_{i-1}}) \times (h_e - h_{\theta_1})(h_e - h_{\theta_2}) \ldots (h_e - h_{\theta_{i-1}})\}.$$

If we suppose $\frac{f'x}{fx}$, by means of the common measure process, to be expanded under the form of an improper continued fraction, the successive quotients will be the values of $Q_1, Q_2 \ldots Q_n$ above found, that is

$$\frac{f'x}{fx} = \frac{1}{Q_1 -}\,\frac{1}{Q_2 -}\,\frac{1}{Q_3 -} \cdots \frac{1}{Q_n};$$

the successive convergents of this fraction will be

$$\frac{1}{Q_1},\ \frac{Q_2}{Q_1 Q_2 - 1},\ \frac{Q_2 Q_3 - 1}{Q_1 Q_2 Q_3 - Q_1 - Q_3},\ \ldots,\ \frac{f'x}{fx}.$$

The numerators and denominators of these convergents will consequently also be functions of the factors exclusively. They are the quantities the sum of the products of which multiplied respectively by fx and $f'x$ produce (to constant factors *près*) the residues. The denominators are expressible very simply in terms of the factors and the differences of the roots; and their values under such forms were published by me about the same time as the values of the residues in the *Philosophical Magazine*; the expression for the numerators is much more complicated, but is given in my paper, "The Syzygetic Relations," &c., in the *Philosophical Transactions*. [p. 429 below.]

By comparing the expression for any quotient with the expressions for the two residues from which it may be derived, we obtain the following remarkable identity: $Z_{i-1} \times Z_i$, that is

$$\Sigma\zeta(h_1 h_2 \ldots h_{i-1}) \times \Sigma\zeta(h_1 h_2 \ldots h_i) = {}_iP_1^2 + {}_iP_2^2 + {}_iP_3^2 + \ldots + {}_iP^2_n.$$

When the roots are all real, we have thus the product of one sum of squares by the product of another sum of squares (the number in each sum depending upon the arbitrary quantity i), brought under the form of a sum of a constant number n of squares, which in itself is an interesting theorem.

The expression above given for Q_i leads to a remarkable relation between the quotients and convergents to $\frac{f'x}{fx}$.

Let it be supposed, as before, that

$$\frac{f'x}{fx} = \frac{1}{Q_1 x -}\,\frac{1}{Q_2 x -}\,\frac{1}{Q_3 x -} \cdots \frac{1}{Q_n x},$$

and let the successive convergents to this continued fraction be

$$\frac{N_1(x)}{D_1(x)},\ \frac{N_2(x)}{D_2(x)},\ \frac{N_3(x)}{D_3(x)},\ \cdots \frac{N_n(x)}{D_n(x)},$$

where the numerators and denominators are not supposed to undergo any reductions, but are retained in their crude forms as deduced from the law

$$N_i = Q_i N_{i-1} - N_{i-2},$$

$$D_i = Q_i D_{i-1} - D_{i-2}.$$

$N_1(x)$ being 1, and $D_1(x)$ being $Q_1(x)$; then it may be deduced from the published results above adverted to that

$$D_i(x) = \frac{Z^2_{i-1} Z^2_{i-3} \dots Z^2_{(i)}}{Z_i^2 Z^2_{i-2} \dots Z^2_{(i)+1}} \{\zeta(h_{\theta_1} h_{\theta_2} \dots h_{\theta_i})(x - h_{\theta_1})(x - h_{\theta_2}) \dots (x - h_{\theta_i})\}.$$

Hence $$\Sigma\{\zeta(h_{\theta_1} h_{\theta_2} \dots h_{\theta_{i-1}}) \times (h_e - h_{\theta_1})(h_e - h_{\theta_2}) \dots (h_e - h_{\theta_{i-1}})\}$$
$$= \frac{Z^2_{i-1} Z^2_{i-3} \dots Z^2_{(i-1)+1}}{Z^2_{i-2} Z^2_{i-4} \dots Z^2_{(i-1)}} D_{i-1}(h_e);$$

and we have therefore

$${}_iP_e = \frac{Z^2_{i-1}}{Z_i} \frac{Z^4_{i-3}}{Z^4_{i-4}} \frac{Z^4_{i-5}}{Z^4_{i-6}} \cdots \frac{Z^4_{(i)}}{Z^4_{(i)+1}} D_{i-1}(h_e),$$

and consequently

$$Q_i = \frac{Z^2_{i-1}}{Z_i^2} \frac{Z^8_{i-3}}{Z^8_{i-4}} \cdots \frac{Z^8_{(i)}}{Z^8_{(i)+1}} \Sigma\{(D_{i-1}(h_e))^2 (x - h_e)\},$$

which is the general equation connecting the form of each quotient with that of the denominator to the immediately preceding unreduced convergent in the expansion of $\frac{f'x}{fx}$ under the form of an improper continued fraction.

If instead of the denominator of the unreduced convergents, the denominators of the convergents reduced to their simplest forms be employed, the powers of Z in the constant factor will undergo a diminution. The essential part of this theorem admits of being stated in general terms as follows:—

"If the quotient of an algebraical function of x by its first differential coefficient be expressed under the form of a continued fraction whose successive partial quotients are linear functions of x, any one of these quotients may be found (to a constant factor *près*) by taking the sum of the products formed by multiplying each factor $(x - h)$ of the given function by the square of what the denominator of the immediately antecedent convergent fraction becomes after substituting in it for x the root corresponding to such factor."

P.S. Since the above was read before the British Association, the theory has been extended by the author to comprise the general case of the expansion of any two algebraical functions under the form of a continued fraction, and has been incorporated into the paper in the *Philosophical Transactions* above referred to.

51.

ON A THEOREM CONCERNING THE COMBINATION OF DETERMINANTS.

[*Cambridge and Dublin Mathematical Journal*, VIII. (1853), pp. 60—62.]

Let ${}^1A$ represent the line of terms ${}^1a_1, {}^1a_2, \dots {}^1a_m$,

${}^1B$ „ „ „ ${}^1b_1, {}^1b_2, \dots {}^1b_m$.

Let ${}^1A \times {}^1B$ represent $\Sigma({}^1a_r \times {}^1b_r)$, where of course there are m terms within the symbol of summation.

Again, let ${}^2A$ represent the line ${}^2a_1, {}^2a_2, \dots {}^2a_m$,

${}^2B$ „ „ ${}^2b_1, {}^2b_2, \dots {}^2b_m$,

and let $\begin{vmatrix} {}^1A \\ {}^2A \end{vmatrix} \times \begin{vmatrix} {}^1B \\ {}^2B \end{vmatrix}$ represent $\Sigma \begin{vmatrix} {}^1a_r, & {}^1a_s \\ {}^2a_r, & {}^2a_s \end{vmatrix} \times \begin{vmatrix} {}^1b_r, & {}^1b_s \\ {}^2b_r, & {}^2b_s \end{vmatrix}$,

$\begin{vmatrix} {}^1a_r, & {}^1a_s \\ {}^2a_r, & {}^2a_s \end{vmatrix}$ denoting the determinant $({}^1a_r . {}^2a_s - {}^1a_s . {}^2a_r)$,

$\begin{vmatrix} {}^1b_r, & {}^1b_s \\ {}^2b_r, & {}^2b_s \end{vmatrix}$ „ „ „ $({}^1b_r . {}^2b_s - {}^1b_s . {}^2b_r)$,

there being of course $\frac{1}{2}m(m-1)$ terms comprised within the sign of summation; and so, in general, let

$$\begin{vmatrix} {}^1A \\ {}^2A \\ {}^3A \\ \vdots \\ {}^nA \end{vmatrix} \times \begin{vmatrix} {}^1B \\ {}^2B \\ {}^3B \\ \vdots \\ {}^nB \end{vmatrix}, \text{ n being less than m,}$$

(and where in general ${}^rA$ denotes ${}^ra_1, {}^ra_2, \ldots {}^ra_m$ and ${}^rB$ denotes ${}^rb_1, {}^rb_2, \ldots {}^rb_m$) represent

$$\Sigma \begin{vmatrix} {}^1a_{h_1}, & {}^1a_{h_2}, & \ldots & {}^1a_{h_n} \\ {}^2a_{h_1}, & {}^2a_{h_2}, & \ldots & {}^2a_{h_n} \\ \ldots & \ldots & \ldots & \ldots \\ {}^na_{h_1}, & {}^na_{h_2}, & \ldots & {}^na_{h_n} \end{vmatrix} \times \begin{vmatrix} {}^1b_{h_1}, & {}^1b_{h_2}, & \ldots & {}^1b_{h_n} \\ {}^2b_{h_1}, & {}^2b_{h_2}, & \ldots & {}^2b_{h_n} \\ \ldots & \ldots & \ldots & \ldots \\ {}^nb_{h_1}, & {}^nb_{h_2}, & \ldots & {}^nb_{h_n} \end{vmatrix}.$$

Now let r be any integer less than m, and let

$$\mu = \frac{m(m-1)\ldots(m-r+1)}{1.2\ldots r},$$

and, supposing $\theta_1, \theta_2, \ldots \theta_r$ to be r numbers of the set $1, 2, \ldots m$, let $G_1, G_2, \ldots G_\mu$ denote the μ rectangular matrices of the forms

$$\begin{vmatrix} {}^{\theta_1}A \\ {}^{\theta_2}A \\ \ldots \\ {}^{\theta_r}A \end{vmatrix} \text{ respectively,}$$

and let $H_1, H_2, \ldots H_\mu$ denote the μ rectangular matrices of the forms

$$\begin{vmatrix} {}^{\theta_1}B \\ {}^{\theta_2}B \\ \ldots \\ {}^{\theta_r}B \end{vmatrix} \text{ respectively.}$$

Now form the determinant

$$\begin{vmatrix} G_1 \times H_1, & G_1 \times H_2, & \ldots & G_1 \times H_\mu \\ G_2 \times H_1, & G_2 \times H_2, & \ldots & G_2 \times H_\mu \\ \ldots & \ldots & \ldots & \ldots \\ G_\mu \times H_1, & G_\mu \times H_2, & \ldots & G_\mu \times H_\mu \end{vmatrix},$$

then, if we give r the successive values $1, 2, 3 \ldots m$ (in which last case the determinant in question reduces to a single term), the values of the determinant above written will be severally in the proportions of

$$K, K^m, K^{\frac{1}{2}m(m-1)}, \ldots K^m, K;$$

that is to say, the logarithms of these several determinants will be as the coefficients of the binomial expansion $(1+x)^m$.

When we make $r = m$, and equate the determinant corresponding to this value of r with that formed by making $r = 1$, the theorem becomes identical with a theorem previously given by M. Cauchy, for the Product of Rectangular Matrices.

It would be tedious to set forth the demonstration of the general theorem in detail. Suffice it here to say that it is a direct corollary from the formula marked (4) in my paper in the *Philosophical Magazine* for April 1851, entitled "On the Relations between the Minor Determinants of Linearly Equivalent Quadratic Functions*," when that formula is particularized by making

$$\begin{Bmatrix} a_{m+1}, & a_{m+2}, & \ldots & a_{m+n} \\ b_{m+1}, & b_{m+2}, & \ldots & b_{m+n} \end{Bmatrix}$$

represent a determinant all whose terms are zeros except those which lie in one of the diagonals, these latter being all units, which comes, in fact, to defining that

$$\begin{vmatrix} a_{m+e} \\ b_{m+e} \end{vmatrix} = 1, \text{ and } \begin{vmatrix} a_{m+e} \\ b_{m+\epsilon} \end{vmatrix} = 0.$$

The important theorem here referred to is made almost unintelligible by an unfortunate misprint of ${}^q\theta_m, {}^1\theta_m, {}^2\theta_m, {}^\mu\theta_m$, in place of ${}^q\theta_r, {}^1\theta_r, {}^2\theta_r, {}^\mu\theta_r$. I may here take notice of another and still more inexplicable blunder in the same paper, formula (3)†, in the latter part of the equation belonging to which

$$\begin{Bmatrix} a_{\theta_1}, & a_{\theta_2}, & \ldots & a_{\theta_m}, & a_{\theta_{m+1}}, & a_{\theta_{m+2}}, & \ldots & a_{\theta_{m+e}} \\ a_{\phi_1}, & a_{\phi_2}, & \ldots & a_{\phi_m}, & a_{\phi_{m+1}}, & a_{\phi_{m+2}}, & \ldots & a_{\phi_{m+e}} \end{Bmatrix}$$

is written in lieu of

$$\begin{Bmatrix} a_1, & a_2, & \ldots & a_m, & a_{\theta_{m+1}}, & a_{\theta_{m+2}}, & \ldots & a_{\theta_{m+e}} & a_{n+1} & a_{n+2} & \ldots & a_{n+m} \\ a_1, & a_2, & \ldots & a_m, & a_{\phi_{m+1}}, & a_{\phi_{m+2}}, & \ldots & a_{\phi_{m+e}} & a_{n+1} & a_{n+2} & \ldots & a_{n+m} \end{Bmatrix}.$$

[* p. 249 above.] [† See pp. 246, 251 above.]

52.

NOTE ON THE CALCULUS OF FORMS.

[See pp. 363 and 411.]

[*Cambridge and Dublin Mathematical Journal*, VIII. (1853), pp. 62—64.]

ACCIDENTAL causes have prevented me from composing the additional sections on the Calculus of Forms, which I had destined for the present Number of this *Journal*. In the meanwhile the subject has not remained stationary. Among the principal recent advances may be mentioned the following.

1. The discovery of Combinants; that is to say, of concomitants to systems of functions remaining invariable, not only when combinations of the variables are substituted for the variables, but also when combinations of the functions are substituted for the functions; and as a remarkable first-fruit of this new theory of double invariability, the representation of the Resultant of any three quadratic functions under the form of the square of a certain combinantive sextic invariant added to another combinant which is itself a biquadratic function of 10 cubic invariants. When the three quadratic functions are derived from the same cubic function, this expression merges in M. Aronhold's for the discriminant of the cubic. The theory of combinants naturally leads to the theory of invariability for non-linear substitutions, and I have already made a successful advance in this new direction.

2. The unexpected and surprising discovery of a quadratic covariant to any homogeneous function in x, y of the nth degree, containing $(n-1)$ variables cogredient with $x^{n-2}, x^{n-3}y \ldots y^{n-2}$ and possessing the property of indicating *the number of real and imaginary roots* in the given function. This covariant, on substituting for the $(n-1)$ variables the combinations of the powers of x, y with which they are cogredient, becomes the Hessian of the given function*.

* This covariant furnishes, if we please, functions symmetrical in respect to the two ends of an equation for determining the number of its real and imaginary roots. The ordinary Sturmian functions, it is well known, have not this symmetry. As another example of the successful application of the new methods to subjects which have been long before the mathematical world and supposed to be exhausted, I may notice that I obtain without an effort, by their aid, a much more simple, practical, and complete solution of the question of the simultaneous transformation of two quadratic functions, or the orthogonal transformation of one such function, than any previously given, even by the great masters Cauchy and Jacobi, who have treated this question.

3. The demonstration due to M. Hermite of a law of reciprocity connecting the degree or degrees of any function or system of functions with the order or orders of the invariants belonging to the system. The theorem itself was first propounded by me about a twelvemonth back, and communicated to Messrs Cayley, Polignac, and Hermite, as serving to connect together certain phenomena which had presented themselves to me in the theory: unfortunately it appeared to contradict another law too hastily assumed by myself and others as probably true, and I consequently laid aside the consideration of this great law of reciprocality. To M. Hermite, therefore, belongs the honour of reviving and establishing,—to myself whatever lower degree of credit may attach to suggesting and originating,—this theorem of numerical reciprocity, destined probably to become the corner-stone of the first part of our new calculus; that part, I mean, which relates to the generation and affinities of forms*.

4. I may notice that the Calculus of Forms may now with correctness be termed the Calculus of Invariants, by virtue of the important observation that every concomitant of a given form or system of forms may be regarded as an invariant of the given system and of an absolute form or system of absolute forms combined with the given form or system. As regards that particular branch of the theory of invariants which relates to resultants, or, in other words, to the doctrine of elimination, I may here state the theorem alluded to in a preceding Number of the *Journal*, to wit that if R be the resultant of a system of n homogeneous functions of n variables, written out in their complete and most general form (so that by definition $R=0$ is the condition that the equations got by making the n given functions zero, shall be simultaneously satisfiable by one system of ratios), then the condition that these equations may be satisfied by ι distinct systems of ratios between the n variables is $\delta^{\iota} R = 0$, the variation δ being taken in respect to every constant entering into each of the n equations.

* This theorem of numerical reciprocity promises to play as great a part in the Theory of Forms as Legendre's celebrated theorem of reciprocity in that of Numbers. Another demonstration of it, which leaves nothing to be desired for beauty and simplicity, has been since discovered by Mr Cayley, which ultimately rests upon that simple law (essentially although not on the face of it a law of reciprocity) given by Euler, which affirms that the number of modes in which a number admits of being partitioned is the same whether the condition imposed upon the mode of partitionment be that no part shall exceed a given number, or that the number of parts constituting any one partition shall not exceed the same number.

53.

ON THE RELATION BETWEEN THE VOLUME OF A TETRAHEDRON AND THE PRODUCT OF THE *SIXTEEN* ALGEBRAICAL VALUES OF ITS SUPERFICIES.

[*Cambridge and Dublin Mathematical Journal*, VIII. (1853), pp. 171—178.]

THE area of a triangle is related (as is well known) in a very simple manner to the eight algebraical values of its perimeter: If we call the values of the squared sides of the triangle a, b, c, there will be nothing to distinguish the algebraical affections of sign of the simple lengths so as to entitle one to a preference over the other. The area of the triangle can only vanish by reason of the three vertices coming into a straight line; hence, according to the general doctrine of characteristics, we must have the Norm of $\sqrt{a}+\sqrt{b}+\sqrt{c}$, containing as a factor some root or power of the expressions for the area of the triangle. The Norm in question being representable as $-N^2$ where N is the Norm of $a^{\frac{1}{2}} \pm b^{\frac{1}{2}} \pm c^{\frac{1}{2}}$, which is of four dimensions in the elements a, b, c, and undecomposable into rational factors, we infer that to a numerical factor *près* the square of the area must be identical with the Norm N, and thus, by a logical *coup-de-main*, completely supersede all occasion for the ordinary geometrical demonstration given of this proposition, which in its turn, with certain superadded definitions, would admit of being adopted as the basis of an absolutely pure system of Analytical Trigonometry that should borrow nothing from the methods and results of sensuous or practical geometry. But into this speculation it is not my present purpose to enter: what I propose to do is to extend a similar mode of reasoning to space of three dimensions, and to point out a general theorem in determinants which is involved as a consequence in the generalization of the result of the inquiry when pushed forward into the regions of what may be termed Absolute or Universal Rational Space.

Let F, G, H, K be the four squared areas of the faces of a tetrahedron, and V the volume; then, since V only becomes zero in the case of the four vertices coming into the same plane, which is characterised by the equation

$$\sqrt{F}+\sqrt{G}+\sqrt{H}+\sqrt{K}=0$$

subsisting, we infer that N the Norm of

$$\sqrt{F} \pm \sqrt{G} \pm \sqrt{H} \pm \sqrt{K}$$

must contain a power of V as a rational factor. V^2 is rational and of three dimensions in the squared edges; the Norm above spoken of is of eight dimensions in the same. Consequently there is a rational factor, say Q, remaining, which is of five dimensions in the squared edges, and this factor I now proceed to determine, the other factor V^2 being, as is well known, a numerical product of the determinant

$$\begin{vmatrix} 0, & ab^2, & ac^2, & ad^2, & 1 \\ ba^2, & 0, & bc^2, & bd^2, & 1 \\ ca^2, & cb^2, & 0, & cd^2, & 1 \\ da^2, & db^2, & dc^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix}$$

a, b, c, d being the four angular points of the tetrahedron. See *London and Edinburgh Philosophical Magazine*, 1852. [p. 386 above.]

The quantity Q possesses an interest of a geometrical character; for if we call the radii of the eight spheres which can be inscribed in a tetrahedron $r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8$, we evidently have $r_1r_2r_3r_4r_5r_6r_7r_8 \times N=(3V)^8$. Hence (R), the product of the eight radii in question, $=\frac{3^8V^8}{N}=\frac{3^8V^6}{Q}$.

Consequently Q is the quantity which characterises the fact of one or more of *the radii of the inscribed spheres becoming infinite.* For the triangle there exists no corresponding property; this we know *à priori*, and can explain also analytically from the fact that if we call P the product of the radii of the four inscribable circles, ν the Norm of the perimeter, and A the area, we have

$$P\nu = 2^4A^4,$$

and

$$\nu=\frac{2^4A^4}{P}=A^2,$$

which contains no denominator capable of becoming zero, so that as long as the sides remain finite the curvature of the inscribed circles is incapable of vanishing.

To determine N as a function of the edges, and then to discover by actual division the value of $\frac{N}{V^2}$, would be the direct but an excessively tedious and almost impracticably difficult process. I have ever felt a preference for the *à priori* method of discovering forms whose properties are known, and never yet have met with an instance where analysis has denied to gentle solicitation conclusions which she would be loth to grant to the application of force. The case before us offers no exception to the truth of this remark. Q is a function of five dimensions in terms of the squared edges: let us begin by finding the value of that part of Q in which at most a certain set of four of these edges make their appearance, and to find which consequently the other two edges may be supposed zero without affecting the result. We may make two distinct hypotheses concerning these two edges; we may suppose that they are opposite, that is non-intersecting edges, or that they are contiguous, that is intersecting edges.

To meet the first hypothesis suppose $ab = 0$, $ce = 0$.

For convenience sake, use F, G, H, K to denote 16 times the square of each area, instead of the simple square of the areas. Call

$$16\,(abc)^2 = K, \quad 16\,(abd)^2 = H, \quad 16\,(acd)^2 = G, \quad 16\,(bcd)^2 = F.$$

Then

$$-K = (ab)^4 + (ac)^4 + (bc)^4 - 2\,(ab)^2\,(ac)^2 - 2\,(ab)^2\,(bc)^2 - 2\,(ac)^2\,(bc)^2$$
$$= ac^4 + bc^4 - 2\,(ac)^2\,(bc)^2.$$

Similarly,

$$-H = ad^4 + bd^4 - 2\,(ad)^2\,(bd)^2,$$
$$-G = ca^4 + da^4 - 2ca^2 da^2,$$
$$-F = cb^4 + db^4 - 2cb^2 db^2.$$

Hence one value of $\sqrt{F} + \sqrt{G} + \sqrt{H} + \sqrt{K}$ will be

$$\sqrt{(-1)}\,\{(ac^2 - bc^2) + (bd^2 - ad^2) + (da^2 - ac^2) + (bc^2 - bd^2)\} = 0.$$

Hence, on this first supposition, the Norm vanishes. But V^2 does not vanish when $ab = 0$, $cd = 0$, for it becomes, *saving* a numerical factor,

$$\begin{vmatrix} 0, & 0, & ac^2, & ad^2, & 1 \\ 0, & 0, & bc^2, & bd^2, & 1 \\ ca^2, & cb^2, & 0, & 0, & 1 \\ da^2, & db^2, & 0, & 0, & 1 \\ 1, & 1, & 1, & 1, & \end{vmatrix},$$

that is

$$(ac^2 . bd^2 - ad^2 . bc^2)\,(cb^2 + ad^2 - ca^2 - bd^2)$$
$$+ (bc^2 - ac^2)\,(ca^2 . db^2 - cb^2 . da^2)$$
$$+ (ad^2 - bd^2)\,(ca^2 . db^2 - cb^2 . da^2)$$
$$= 2\,(ac^2 . bd^2 - ad^2 . bc^2)\,(ad^2 + bc^2 - ac^2 - bd^2);$$

and consequently, since N vanishes but V^2 does not vanish, Q vanishes, showing that there is no term in Q but what contains one at least of any

two opposite edges as a factor; or, in other words, there is no term in Q of which the product of the square of the product of all three sides of some one or other of the four faces does not form a constituent part.

Next, let us suppose $ab = 0$, $ac = 0$, then

$$K^2 = 16abc^2 = -bc^4,$$
$$H^2 = 16abd^2 = -(ad^2 - bd^2)^2,$$
$$G^2 = 16acd^2 = -(ad^2 - cd^2)^2,$$
$$F^2 = 16bcd^2 = -bc^4 - bd^4 - cd^4 + 2bc^2 . bd^2 + 2bc^2 . cd^2 + 2bd^2 . cd^2.$$

Four of the factors of N will be therefore

$$\{\iota\,(bc^2 + cd^2 - bd^2) \pm F\}, \quad \{\iota\,(bc^2 - cd^2 + bd^2) \pm F\},$$

ι denoting $\sqrt{(-1)}$, and the product of these four factors will be

$$\{(bc^2 + cd^2 - bd^2)^2 + F^2\} \times \{(bc^2 - cd^2 + bd^2)^2 + F^2\},$$

which is equal to

$$16bc^4 . bd^2 . cd^2;$$

and similarly, the remaining part of the Norm will be

$$\{(2ad^2 - bd^2 - cd^2 + bc^2)^2 + F^2\} \times \{(2ad^2 - bd^2 - cd^2 - bc^2)^2 + F^2\},$$

that is

$$\{4ad^4 - 4ad^2\,(bd^2 + cd^2 + bc^2) + 4bc^2 . bd^2 + 4bd^2 . cd^2 + 4cd^2 . bc^2\}$$
$$\times \{4ad^4 - 4ad^2\,(bd^2 + cd^2 - bc^2) + 4bd^2 . cd^2\}.$$

Again, since $ac^2 = 0$ and $bc^2 = 0$, V^2 becomes

$$\begin{vmatrix} 0, & 0, & 0, & ad^2, & 1 \\ 0, & 0, & bc^2, & bd^2, & 1 \\ 0, & cb^2, & 0, & cd^2, & 1 \\ da^2, & db^2, & dc^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix},$$

which is evidently equal to

$$2bc^2 \begin{vmatrix} 0, & 0, & ad^2, & 1 \\ 0, & cb^2, & cd^2, & 1 \\ da^2, & db^2, & 0, & 1 \\ 1, & 1, & 1, & \end{vmatrix} - bc^4 \begin{vmatrix} 0, & ad^2, & 1 \\ da^2, & 0, & 1 \\ 1, & 0, & \end{vmatrix},$$

$$= 2bc^2\,\{2bc^2 ad^2 + ad^4 - ad^2 bd^2 - cd^2 ad^2 + bd^2 cd^2\} - 2bc^4 ad^2$$
$$= 2bc^2\,\{ad^4 - ad^2\,(bd^2 + cd^2 - bc^2) + bd^2 . cd^2\}.$$

Hence, paying no attention to any mere numerical factor, we have found that when $ac=0$ and $bc=0$, Q or $\frac{N}{V^2}$ becomes

$$bc^2 . bd^2 . cd^2 \{ad^4 - ad^2 (bd^2 + cd^2 + bc^2) + bc^2 . bd^2 + bd^2 . cd^2 + cd^2 . bc^2\}.$$

Hence, with the exception of the terms in which five out of the six edges enter, the complete value of Q will be

$$\Sigma (bc^2 . bd^2 . cd^2) \{ad^4 - ad^2 (bd^2 + cd^2 + bc^2) + bc^2 . bd^2 + bd^2 . cd^2 + cd^2 . bc^2\},$$

or more fully expressed, and still abstracting from terms containing five edges,

$$= \Sigma bc^2 . bd^2 . cd^2 \{(ab^4 + ac^4 + ad^4) - (ab^2 + ac^2 + bc^2)(bd^2 + bc^2 + cd^2)$$
$$+ bc^2 . bd^2 + bd^2 . cd^2 + cd^2 . bc^2\}.$$

It remains only to determine the value of the numerical coefficient affecting each of the six terms of the form

$$ab^2 . ac^2 . ad^2 . bc^2 . bd^2.$$

To find this, let

$$ab^2 = ac^2 = ad^2 = bc^2 = bd^2 = cd^2 = 1;$$

then evidently, since all the squared areas are equal, several of the factors of N will become zero, but V^2 evidently does not become zero for a regular tetrahedron; hence Q becomes zero: and if we call the numerical factor sought for λ, we must have (observing that the Σ includes four parts corresponding to each of the four faces)

$$4\{3 - 9 + 3\} + 6\lambda = 0,$$

therefore $$-12 + 6\lambda = 0, \text{ or } \lambda = 2.$$

Hence the complete value of Q is

$$\Sigma ab^2 . bc^2 . ca^2 \{(da^4 + db^4 + dc^4) - (da^2 + db^2 + dc^2)(ab^2 + bc^2 + ca^2)$$
$$+ ab^2 . bc^2 + bc^2 . ca^2 + ca^2 . ab^2\}$$
$$+ 2\Sigma (ab^2 . bc^2 . cd^2 . da^2 . ac^2);$$

or, which is the same quantity somewhat differently and more simply arranged,

$$Q = \Sigma (ab^2 . bc^2 . ca^2) \{(da^4 + db^4 + dc^4 + da^2 . db^2 + db^2 . dc^2 + dc^2 . da^2)$$
$$+ (ab^2 . bc^2 + bc^2 . ca^2 + ca^2 . ab^2) - (da^2 + db^2 + dc^2)(ab^2 + bc^2 + ca^2)\},$$

and this quantity equated to zero expresses the conditions of a radius of an

inscribed sphere becoming infinite. The direct method would have involved, as the first step, the formation of the Norm of a numerator consisting of

$$\sqrt{F} \pm \sqrt{G} \pm \sqrt{H} \pm \sqrt{K},$$

the value of which is

$$\Sigma F^4 - 4\Sigma F^3 G + 6\Sigma F^2 G^2 + 4\Sigma F^2 GH - 40 FGHK,$$

and contains $4+6+12$, that is 22 positive terms, and 12, that is 13 negative terms, together 35 terms, each of which might be an aggregate of 6^4 or 1296 quantities, and thus involve in all the consideration of 45360 separate parts, for each of the quantities F, G, H, K being a quadratic function of three of the squared edges, will contain six terms. It is not uninteresting to notice that in addition to the case already mentioned of two opposite edges being each zero, as $ab=0$, $cd=0$, Q will also vanish for the case of $ab=cd$, $bc=ad$; that is for the case of two intersecting edges being each equal in length to the edges respectively opposite to them. This is evident from the fact that on the hypothesis supposed the face $acb=acd$ and the face $bdc=bda$; hence $N=0$, and therefore, V not vanishing, $\frac{N}{V^2}$, that is Q, will vanish.

We may moreover remark that since $ab=0$ and $cd=0$ does not make V vanish, the perpendicular distance of ab from cd, which, multiplied by $ab \times cd$, gives six times the volumes, must on this supposition become infinite. When three edges lying in the same plane all vanish simultaneously, Q vanishes, since one edge at least in every face of the pyramid vanishes, and V also vanishes, as is evident from the expression for V^2, when $ab=0$, $ac=0$, $bc=0$, becoming a multiple of

$$\begin{vmatrix} 0, & 0, & 0, & ad^2, & 1 \\ 0, & 0, & 0, & bd^2, & 1 \\ 0, & 0, & 0, & cd^2, & 1 \\ ad^2, & bd^2, & cd^2, & 0, & 0 \\ 1, & 1, & 1, & 0, & 0 \end{vmatrix},$$

which is evidently zero.

It appeared to me not unlikely, from the situation and look of Q (the characteristic of one of the inscribed spheres becoming infinite), that it might admit of being represented as a determinant, but I have not succeeded in throwing it under that form. I have a strong suspicion that if we take Q' a function corresponding to a tetrahedron $a'b'c'd'$, in the same way as Q corresponds to $abcd$, QQ', and not improbably $\sqrt{(QQ')}$, will be found to be (as we know from Staudt's Theorem of $\sqrt{(V^2 . V'^2)}$) a rational integral function of the squares of the distances of the points a, b, c, d from the points a', b', c', d'.

That N should divide out by V^2 is in itself an analytical theorem relating to 6 arbitrary quantities $ab^2, ac^2, ad^2, bc^2, bd^2, cd^2$, which evidently admits of extension to any triangular number 10, 15, &c. of arbitrary quantities. Thus we may affirm, *à priori*, that the norm of

$$\sqrt{L} \pm \sqrt{M} \pm \sqrt{N} \pm \sqrt{P} \pm \sqrt{Q},$$

where (for the sake of symmetry, retaining double letters, as AB, AC, &c., to denote *simple* quantities)

$$Q = \begin{vmatrix} 0, & AB, & AC, & AD, & 1 \\ AB, & 0, & BC, & BD, & 1 \\ AC, & BC, & 0, & CD, & 1 \\ AD, & BD, & CD, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix}, \quad P = \begin{vmatrix} 0, & AB, & AC, & AE, & 1 \\ AB, & 0, & BC, & BE, & 1 \\ AC, & BC, & 0, & CE, & 1 \\ AE, & BE, & CE, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix},$$

$$N = \&c., \quad M = \&c., \quad L = \&c.,$$

will contain as a factor the determinant

$$\begin{vmatrix} 0, & AB, & AC, & AD, & AE, & 1 \\ AB, & 0, & BC, & BD, & BE, & 1 \\ AC, & BC, & 0, & CD, & CE, & 1 \\ AD, & BD, & CD, & 0, & DE, & 1 \\ AE, & BE, & CE, & DE, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix},$$

and a similar theorem may evidently be extended to the case of any $\frac{n(n+1)}{2}$ arbitrary quantities whatever.

54.

ON THE CALCULUS OF FORMS, OTHERWISE THE THEORY OF INVARIANTS.

[*Continued from p.* 363 *above*.]

[*Cambridge and Dublin Mathematical Journal*, VIII. (1853), pp. 256—269.]

SECTION VII. *On Combinants.*

REASONS of convenience have induced me to depart from the plan to which I originally intended to adhere in the development of this theory, and I shall hereafter, from time to time, continue to add sections on such parts of the subject as may chance to be most present to my mind or most urgent upon my attention, without waiting for the exact place which they ought to occupy in a more formal treatise, and without having regard to the separation of the subject into the two several divisions stated at the outset of the first section. The present section will be devoted to a brief and partial exposition of the theory of Combinants*, with a view to the application of this theory to the solution of the problem of throwing the resultant of three general homogeneous quadratic functions under its most simple form, being analogous to that given by Aronhold in the particular case where the three functions are derived from the same cubic, and becoming identical therewith when the coefficients are accommodated to this particular supposition†. I shall confine myself for the present to combinants relating to systems of functions, all of the same degree.

If $\phi_1, \phi_2, \ldots \phi_r$, be homogeneous functions of any number of variables, any invariant or other concomitant of the system which remains unchanged, not only for linear substitutions impressed upon the variables contained within the functions, but also for linear combinations impressed upon the functions themselves, is what I term a Combinant. A Combinant is thus an invariant or other concomitant of a system in its corporate capacity (quâ *system*), being in fact

* Discovered by the Author of this paper in the winter of 1852.

† A similar method will subsequently be applied to the representation of the resultant of two cubic equations as a function of Combinants bearing relations to the quadratic and cubic invariants of a quartic function of x and y, precisely analogous to those which the Combinants that enter into the solution above alluded to bear to the Aronholdian invariants of a cubic function.

common to the whole family of forms designated by $\lambda_1\phi_1 + \lambda_2\phi_2 + \ldots + \lambda_r\phi_r$, where $\lambda_1, \lambda_2, \ldots \lambda_r$, are arbitrary constants. If the coefficients of $\phi_1, \phi_2, \ldots \phi_r$, be supposed to be written out in r lines (the coefficients of corresponding terms occupying the same place in each line), so as to form a rectangular matrix, any combinantive invariant will be a function of the determinants corresponding to the several squares of r^2 terms each that can be formed out of such matrix, or, as they may be termed, the *full* determinants belonging to such rectangular matrix. If we call any such combinant K, then, over and above the ordinary partial differential equations which belong to it in its character of an invariant, it will be necessary and sufficient, in order to establish its combinantive character, that K shall be subject to satisfy $(r-1)$ pairs of equations of the form

$$\left(a'\frac{d}{da} + b'\frac{d}{db} + c'\frac{d}{dc}\cdots\right)K = 0,$$

$$\left(a\frac{d}{da'} + b\frac{d}{db'} + c\frac{d}{dc'}\cdots\right)K = 0,$$

where $a, b, c \ldots$; $a', b', c' \ldots$, are respectively lines in the matrix above referred to.

So any combinantive concomitant will be a function of the full determinants of the matrix formed by the coefficients of the given system of forms and of the variables, and will be subject to satisfy the additional differential equations just above written.

It will readily be understood furthermore, that an invariant or other concomitant may be combinantive in respect to a certain number of forms of a system, and not in respect of other forms therein; or more generally, may be combinantive in respect of each, separately considered, of a series of groups into which a given system may be considered to be subdivided, without being so in respect of the several groups taken collectively.

In the fourth section of my memoir [p. 429 below] on a "Theory of the Conjugate Properties of two rational integral Algebraical Functions," recently presented to the Royal Society of London, the case actually arises of an invariant of a system of three functions, which is combinantive in respect only to two of them.

For greater simplicity, let the attention for the present be kept fixed upon combinants which are such in respect of a single group of functions, all of the same degree in the variables. (It will of course have been perceived that when the system is made up of several groups, there would be nothing gained by limiting the groups to be all of the same degree *inter se*; it is sufficient that all of the same group be of the same degree *per se*.)

All such combinants will admit of an obvious and immediate classification. Let us suppose that a combinant is proposed which is in its lowest terms, that is to say, incapable of being expressed as a rational integral algebraical function of combinants of an inferior order. Such a combinant may, notwithstanding this, admit of being decomposed into non-combinantive invariants of inferior dimensions to its own, and in such event will be termed a *complex* combinant; or it may be indecomposable after this method, in which event it will be termed a *simple* combinant. It will presently be shown, that the resultant of a system of three quadratic functions is made up of a complex combinant of twelve dimensions, and of the square of a simple combinant of six dimensions, expressible as a biquadratic function of ten non-combinantive invariants, each of three dimensions in the coefficients. There is an obvious mode of generating complex combinants; according to which they admit of being viewed as invariants of invariants. Supposing $\phi_1, \phi_2, \ldots \phi_r$, to be the functions of the given system, $\lambda_1\phi_1 + \lambda_2\phi_2 + \ldots + \lambda_r\phi_r$ may conveniently be termed the conjunctive of the system: if now one or more invariants or other concomitants be taken of this conjunctive, there results a derivative function or system of functions of the quantities $\lambda_1, \lambda_2, \ldots \lambda_r$, in which every term affecting any power or combination of powers of the λ series is necessarily an invariant or concomitant of the given system. If now an invariant or other concomitant be taken of the new system in respect to $\lambda_1, \lambda_2, \ldots \lambda_r$, (the original variables (supposing them to enter) being treated as constants), this secondarily derived invariant will be itself an Invariant, or at all events a Concomitant in respect of the original system, and being unaffected by linear substitutions impressed upon the λ system, is by definition a combinant of such system. A similar method will obviously apply if the original system be made up of various groups; each group will give rise to a conjunctive, and one or more concomitants being taken of this system of conjunctives and treated as in the case first supposed, (the only difference being, that there will on the present supposition be several *unrelated* systems instead of a single system of new variables, that is, several λ systems instead of one only) the result, when all the λ systems have been *invariantized out* (that is, made to disappear by any process for forming invariants), will be a combinant in respect to each of the groups, severally considered, of the given system of functions.

Here let it be permitted to me to make a momentary digression, in order to be enabled to avoid for the future the inconvenience of using the phrase "invariant or other concomitant," and so to be enabled at one and the same time to simplify the language and to give a more complete unity to the matter of the theory, by showing how every concomitant may in fact be viewed as a simple invariant, so that the calculus of forms may hereafter admit of being cited, as I propose to cite it, under the name of the Theory of Invariants.

Thus, to begin with the case of *simple* contragredience and cogredience, if $\xi, \eta, \zeta \ldots$ are contragredient to $x, y, z \ldots$, any form containing $\xi, \eta, \zeta \ldots$, which is concomitantive to a given form or system of forms S, which contains $x, y, z \ldots$, may be regarded as concomitantive to the system S', made up of S and the superadded *absolute* form $\xi x + \eta y + \zeta z + \ldots$, say ϑ; where $\xi, \eta, \zeta \ldots$ are treated no longer as variables, but as *constants*. In like manner every system of variables contragredient to $x, y, z \ldots$, or to any other system of variables in S, will give rise to a superadded form analogous to ϑ, the totality of which may be termed S_1; and thus the various systems $\xi, \eta, \zeta \ldots$ will no longer exist as variables in the derived form, but purely as constants. Again, if S contain any system of variables ϕ, ψ, ϑ, &c., contragredient to x, y, z, &c., the system of variables u, v, w, &c., cogredient with x, y, z, &c., may be considered as constants belonging to the superadded form $\phi u + \psi v + \vartheta w \ldots$; but if S do not contain any system contragredient to x, y, z, &c., then u, v, w, &c. may be treated as constants belonging to the superadded system of forms $xv - yu$, $yw - zv$, $zu - xw$, &c.; and so in general any concomitant containing any sets of variables in simple relation, whether of cogredience or contragredience, with any of the sets in the given system S, may in all cases be treated as an *invariant* of the system S', made up of S and a certain superadded system S_1, all the forms contained in which are absolute, by which I mean, that they contain no literal coefficient. The same conclusion may be extended to the case of concomitants containing sets of variables in *compound* relation with the sets in the given system of forms S. Thus, suppose $u_1, u_2, \ldots u_n$, to be in compound relation of cogredience with $x^{n-1}, x^{n-2}y, x^{n-3}y^2, \ldots y^{n-1}$; $u_1, u_2, \ldots u_n$, may be regarded as constants belonging to the superadded form

$$u_1 y^{n-1} - (n-1)\, u_2 y^{n-2} x + \tfrac{1}{2}(n-1)(n-2)\, u_3 y^{n-3} x^2 \mp \ldots \pm u_n x^{n-1},$$

say Ω. And thus universally we are enabled to affirm, that a concomitant of whatever nature to a given system of forms, may be reduced to the form of an invariant of a system made up of the given system and a certain other superadded system of absolute forms: without, therefore, abandoning the use of the terms concomitant, cogredience, contragredience, &c., which for many purposes are highly convenient and save much circumlocution, we may regard every concomitant as a disguised invariant, and under the name of the Theory of Invariants comprise the totality of the theory of Concomitance. I have already had occasion to make use of the superadded form Ω in discussing the theory of the Bezoutiant (a quadratic form concomitant to two functions of the same degree in x, y, which plays a most important part in the theory of the relations of their real roots), in the memoir for the Royal Society previously adverted to.

I now return to the question of applying the theory of combinants to the decomposition of the resultant of three general quadratic functions of

x, y, z. It will of course be apparent that every resultant of any system of n functions of the same degree of a single set of n variables is a combinantive invariant of the system. This is an immediate and simple corollary to the theorem given by me in this *Journal*, in May, 1851. Accordingly, in proceeding to analyse the composition of the resultant of three quadratic functions, I may, besides impressing linear combinations upon the variables, impress linear combinations upon the functions themselves, in any way most conducive to simplicity and facility of expression and calculation; and whatever relations shall be proved to exist between the resultant and other combinants for such specific representation, must be universal, and hold good for the functions in their most general form.

(1) The system, by means of linear substitutions impressed upon the variables which enter into the functions, may be made to assume the form

$$x^2 + y^2 + z^2,$$
$$ax^2 + by^2 + cz^2,$$
$$lx^2 + my^2 + nz^2 + 2pyz + 2qzx + 2rxy.$$

(2) By means of linear combinations of the functions themselves the system may evidently be made to take the form

$$(c-a)\, x^2 + (c-b)\, y^2,$$
$$(a-b)\, y^2 + (a-c)\, z^2,$$
$$ky^2 + 2pyz + 2qzx + 2rxy;$$

and finally, by taking suitable multipliers of x, y, z in lieu of x, y, z, it may be made to become

$$\rho\,(x^2 - y^2),$$
$$\sigma\,(y^2 - z^2),$$
$$y^2 + 2fyz + 2gzx + 2hxy.$$

We have thus reduced the number of constants in the system from eighteen to five; and as it will readily be seen that in any combinant of the system in its reduced form ρ and σ can only enter as factors of the simple quantity, $(\rho\sigma)^i$, for all purposes of comparison of the combinants of the system of like dimensions with one another, ρ and σ might admit of being treated as being each unity, and accordingly, practically speaking, we have only to deal with three in place of eighteen constants, a marvellous simplification, and which makes it obvious, *à priori*, or at least affords a presumption almost amounting to and capable of being reduced to certainty, that the number of fundamental combinants of the system, of which all the rest must be explicit rational functions, will be exactly four in number; which, for the canonical form hereinbefore written, on making ρ and σ each unity, will correspond to

$$1, \quad f^2 + g^2 + h^2, \quad f^2g^2 + g^2h^2 + h^2f^2, \quad fgh,$$

and will be of the 3rd, 6th, 12th, and 9th degrees respectively. The reason why the squares of f, g, h, instead of the simple terms f, g, h, appear in the 2nd and 3rd of these forms is, because, on changing x into $-x$, y into $-y$, or z into $-z$, two of the quantities f, g, h will change their sign, but the forms representing the invariants of even degrees ought to remain absolutely unaltered for such transformations. I shall in the course of the present section set forth the methods for obtaining these four combinants, which, although of the regularly ascending dimensions 3, 6, 9, 12, belong obviously to two different groups, the one of three dimensions forming a class in itself, and the natural order of the three others being that denoted by the sequence 6, 12, and 9, and not that which would be denoted by the sequence 6, 9, 12, the combinant of the ninth degree being properly to be regarded as in some sort an accidentally rational square root of a combinant of 18 dimensions.

Let now
$$\rho(x^2-y^2)=U,$$
$$\sigma(y^2-z^2)=W,$$
$$y^2+2fyz+2gzx+2hxy=V.$$

The resultant will be found by making
$$x=\pm y,$$
$$z=\pm y,$$

when
$$\left.\begin{matrix}x=+y\\ z=+y\end{matrix}\right\},\quad V=(1+2f+2g+2h)\,y^2,$$
$$\left.\begin{matrix}x=+y\\ z=-y\end{matrix}\right\},\quad V=(1-2f-2g+2h)\,y^2,$$
$$\left.\begin{matrix}x=-y\\ z=+y\end{matrix}\right\},\quad V=(1+2f-2g-2h)\,y^2,$$
$$\left.\begin{matrix}x=-y\\ z=-y\end{matrix}\right\},\quad V=(1-2f+2g-2h)\,y^2.$$

Hence the resultant R
$$=\rho^4\sigma^4(1+2f+2g+2h)(1-2f-2g+2h)(1+2f-2g-2h)(1-2f+2g-2h)$$
$$=(\rho\sigma)^4\{(1+2h)^2-4(f+g)^2\}\{(1-2h)^2-4(f-g)^2\}$$
$$=(\rho\sigma)^4\{(1+4h^2-4f^2-4g^2)^2-(4h-8fg)^2\}$$
$$=(\rho\sigma)^4[1-8(f^2+g^2+h^2)+16\{(f^4+g^4+h^4)-2(g^2h^2+h^2f^2+f^2g^2)\}+64fgh].$$

Let now
$$K=\lambda U+\mu V+\nu W,$$
K being what I term a linear conjunctive of U, V, W. The invariant of K, in respect to x, y, z, will be the determinant
$$\begin{vmatrix}\rho\lambda, & h\mu, & g\mu\\ h\mu, & \mu-\rho\lambda+\sigma\nu, & f\mu\\ g\mu, & f\mu, & -\sigma\nu\end{vmatrix},$$

that is
$$=(2fgh-g^2)\,\mu^3+\sigma(h^2-g^2)\,\mu^2\nu-\rho(f^2-g^2)\,\mu^2\lambda-\rho\sigma\mu\lambda\nu+\rho^2\sigma\lambda^2\nu-\rho\sigma^2\lambda\nu^2;$$
or, multiplying by 6, we may write
$$I_{x,y,z}K=6d\lambda\mu\nu+3b_3\mu^2\nu+3b_1\mu^2\lambda+3a_3\lambda^2\nu+3c_1\lambda\nu^2+b_2\mu^3,$$
where
$$d=-\rho\sigma,\qquad b_2=12fgh-6g^2,$$
$$b_1=-2\rho(f^2-g^2),\qquad b_3=2\sigma(h^2-g^2),$$
$$a_3=\rho^2\sigma,\qquad c_1=-2\rho\sigma^2,$$
the notation being accommodated to that employed by Mr Salmon in *The Higher Plane Curves*, λ, μ, ν in IK being correspondent to x, y, z in Mr Salmon's form. If now we employ Mr Salmon's expression for the S (the biquadratic Aronhòldian of IK), observing that
$$a_2=0,\quad c_2=0,\quad a_1=0,\quad c_3=0,$$
we have the complex combinant
$$S_{\lambda,\mu,\nu}I_{x,y,z}K=d^4-2d^2(b_1c_1+a_3b_3)+da_3b_2c_1-a_3c_1b_1b_3+b_1^2c_1^2+a_3^2b_3^2$$
$$=\rho^4\sigma^4\begin{pmatrix}1-8(f^2+h^2-2g^2)+4(12fgh-6g^2)\\ -16(f^2-g^2)(h^2-g^2)+16(f^2-g^2)^2+(h^2-g^2)^2\end{pmatrix}$$
$$=\rho^4\sigma^4\{1-8(f^2+g^2+h^2)+16(f^4+g^4+h^4-h^2g^2-g^2f^2-f^2h^2)+48fgh\}.$$

Hence, calling the resultant R, we have
$$-3R+4S_{\lambda,\mu,\nu}I_{x,y,z}K=1-8(f^2+g^2+h^2)+16(f^4+g^4+h^4)$$
$$+32(f^2g^2+g^2h^2+h^2f^2)=\{1-4(f^2+g^2+h^2)\}^2=P^2.$$

Let Ω be taken the polar reciprocal to the conjunctive
$$-\lambda U+\mu V+\nu W;$$
and for greater simplicity, as we know, *à priori*, from the fundamental definition of a combinant, which (save as to a factor) must remain unaltered by any linear modification impressed upon the functions to which it appertains, that ρ and σ can enter factorially only in any combinant, let ρ and σ be each taken equal to unity in performing the intermediary operations.

Then
$$\Omega=\begin{vmatrix}-\lambda, & h\mu, & g\mu, & \xi\\ h\mu, & \lambda+\mu+\nu, & f\mu, & \eta\\ g\mu, & f\mu, & -\nu, & \zeta\\ \xi, & \eta, & \zeta, & 0\end{vmatrix}$$
$$=\left\{\begin{matrix}\xi^2(\nu^2+\nu\mu+\nu\lambda+f^2\mu^2)\\ +\eta^2(-\lambda\nu+g^2\mu^2)\\ +\zeta^2(\lambda^2+\lambda\mu+\lambda\nu+h^2\mu^2)\\ -2\eta\zeta(f\lambda\mu+hg\mu^2)\\ +2\zeta\xi\{g(\mu\lambda+\mu\nu)+(g-fh)\mu^2\}\\ -2\xi\eta(h\mu\nu+fg\mu^2)\end{matrix}\right\}.$$

Upon Ω, which is a quadratic function in respect of each of the two unrelated systems $\xi, \eta, \zeta; \lambda, \mu, \nu$, and also in respect of the coefficients in (U, V, W), we may operate with the commutantive symbol

$$\left.\begin{array}{ccc} \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\lambda}, & \frac{d}{d\mu}, & \frac{d}{d\nu} \\ \frac{d}{d\lambda}, & \frac{d}{d\mu}, & \frac{d}{d\nu} \end{array}\right\},$$

which, for facility of reference, I shall term $8E$.

Considering the first line as stationary, we shall obtain, for the value of $8E(\Omega)$, 216 commutantives, which may be expressed under the following forms:

$$\left|\begin{array}{ccc} \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \left[\frac{d^2}{d\lambda^2},\right. & \frac{d^2}{d\mu^2}, & \left.\frac{d^2}{d\nu^2}\right] \end{array}\right|,$$

$$-\left|\begin{array}{ccc} \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \left[\frac{d^2}{d\lambda^2},\right. & \frac{d}{d\mu}\frac{d}{d\nu}, & \left.\frac{d}{d\mu}\frac{d}{d\nu}\right] \end{array}\right|,$$

$$-\left|\begin{array}{ccc} \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \left[\frac{d}{d\lambda}\frac{d}{d\nu},\right. & \frac{d^2}{d\mu^2}, & \left.\frac{d}{d\lambda}\frac{d}{d\nu}\right] \end{array}\right|,$$

$$-\left|\begin{array}{ccc} \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \left[\frac{d}{d\lambda}\frac{d}{d\mu},\right. & \frac{d}{d\lambda}\frac{d}{d\mu}, & \left.\frac{d^2}{d\nu^2}\right] \end{array}\right|,$$

$$2\left|\begin{array}{ccc} \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta}, & \frac{d}{d\zeta} \\ \left[\frac{d}{d\lambda}\frac{d}{d\mu},\right. & \frac{d}{d\mu}\frac{d}{d\nu}, & \left.\frac{d}{d\nu}\frac{d}{d\lambda}\right] \end{array}\right|.$$

In this expression the first lines may be considered stationary, the second lines are subject to the usual process of commutation, which makes three of the six permutations positive and three negative; and the third or bracketed lines are subject to the simple process which makes all the permutations of the same sign. In the three middle groups two of the terms in the final line are always identical; it will therefore be more convenient to introduce the multiplier 2, and then to consider each such line to represent the three distinct permutations, taken singly.

Let now

$$\frac{1}{8}\left\{\frac{d^2}{d\xi^2}, \frac{d^2}{d\eta^2}, \frac{d^2}{d\zeta^2}\right\}\Omega = (\Omega),$$

$$\frac{1}{8}\left\{\frac{d^2}{d\xi^2}, \frac{d}{d\eta}\frac{d}{d\zeta}, \frac{d}{d\eta}\frac{d}{d\zeta}\right\}\Omega = (\Omega)',$$

$$\frac{1}{8}\left\{\frac{d}{d\xi}\frac{d}{d\zeta}, \frac{d^2}{d\eta^2}, \frac{d}{d\xi}\frac{d}{d\zeta}\right\}\Omega = (\Omega)'',$$

$$\frac{1}{8}\left\{\frac{d}{d\eta}\frac{d}{d\xi}, \frac{d}{d\eta}\frac{d}{d\xi}, \frac{d^2}{d\zeta^2}\right\}\Omega = (\Omega)''',$$

$$\left\{\frac{d}{d\xi}\frac{d}{d\eta}, \frac{d}{d\eta}\frac{d}{d\zeta}, \frac{d}{d\zeta}\frac{d}{d\xi}\right\}\Omega = (\Omega)_1.$$

And let

$$\left[\frac{d^2}{d\lambda^2}, \frac{d^2}{d\mu^2}, \frac{d^2}{d\nu^2}\right] = L,$$

$$\left[\frac{d^2}{d\lambda^2}, \frac{d}{d\mu}\frac{d}{d\nu}, \frac{d}{d\mu}\frac{d}{d\nu}\right] = L',$$

$$\left[\frac{d}{d\lambda}\frac{d}{d\nu}, \frac{d^2}{d\mu^2}, \frac{d}{d\lambda}\frac{d}{d\nu}\right] = L'',$$

$$\left[\frac{d}{d\lambda}\frac{d}{d\mu}, \frac{d}{d\lambda}\frac{d}{d\mu}, \frac{d^2}{d\nu^2}\right] = L''',$$

$$\left[\frac{d}{d\lambda}\frac{d}{d\mu}, \frac{d}{d\mu}\frac{d}{d\nu}, \frac{d}{d\nu}\frac{d}{d\lambda}\right] = L_1.$$

Then, attending to the convention just previously explained, we shall have

$$E(\Omega) = (L - 2L' - 2L'' - 2L''' + 2L_1) \\ \times \{(\Omega) - 2(\Omega)' - 2(\Omega)'' - 2(\Omega)''' + 2(\Omega)_1\},$$

a symbolical product, any term in which such as $L'\Omega''$ will mean

$$\left\{\begin{matrix}\left[\dfrac{d^2}{d\lambda^2}, & \dfrac{d}{d\mu}\dfrac{d}{d\nu}, & \dfrac{d}{d\mu}\dfrac{d}{d\nu}\right] \\ \dfrac{d}{d\xi}\dfrac{d}{d\zeta}, & \dfrac{d^2}{d\eta^2}, & \dfrac{d}{d\xi}\dfrac{d}{d\zeta}\end{matrix}\right\}\tfrac{1}{8}\Omega,$$

and a similar interpretation must be extended to each of the 25 partial products; we have then

$$\begin{gathered}
L(\Omega)=8g^2,\quad -2L'(\Omega)=0,\quad -2L'''(\Omega)=0,\\
-2L''(\Omega)=-4g^2,\quad 2L_1(\Omega)=-2,\\
-2L(\Omega)'=0,\quad -2L(\Omega)'''=0,\\
4L'(\Omega)'=0,\quad 4L''(\Omega)'''=0,\\
4L''(\Omega)'=0,\quad 4L'''(\Omega)'''=0,\\
4L'''(\Omega)'=8f^2,\quad 4L'(\Omega)'''=8h^2,\\
-2L(\Omega)''=0,\quad 4L'(\Omega)''=0,\quad 4L''(\Omega)''=0,\quad 4L'''(\Omega)''=0,\\
-4L_1(\Omega)'=0,\quad -4L_1(\Omega)'''=0,\\
-4L_1(\Omega)''=4g^2;
\end{gathered}$$

and, finally, the five terms comprised in

$$2L(\Omega)_1,\ \ldots,\ 4L_1(\Omega)_1,$$

each $=0$. All the above equations can be easily verified by direct inspection, it being observed that $8(\Omega)$ represents

$$\nu^2+\lambda\nu+\mu\nu+f^2\mu^2,\quad -\lambda\nu+g^2\mu^2,\quad \lambda^2+\lambda\mu+\lambda\nu+h^2\mu^2,$$

that $8(\Omega)'$ represents

$$\nu^2+\mu\nu+\lambda\nu+f^2\mu^2,\quad -f\lambda\mu-hg\mu^2,\quad -f\lambda\mu-hg\mu^2,$$

that $8(\Omega)''$ represents

$$-\lambda\nu+g^2\mu^2,\quad g(\mu\lambda+\mu\nu)+(g-fh)\mu^2,\quad g(\mu\lambda+\mu\nu)+(g-fh)\mu^2,$$

that $8(\Omega)'''$ represents

$$\lambda^2+\mu\lambda+\nu\lambda+h^2\mu^2,\quad -h\mu\nu-fg\mu^2,\quad -h\mu\nu-fg\mu^2,$$

and that $(\Omega)_1$ represents

$$-f\lambda\mu-hg\mu^2,\quad g(\mu\lambda+\mu\nu)+(g-fh)\mu^2,\quad -h\mu\nu-fg\mu^2.$$

We have thus

$$\begin{aligned}E(\Omega)&=8g^2-4g^2-2+8f^2+8h^2+4g^2\\&=2\{4f^2+4g^2+4h^2-1\}.\end{aligned}$$

Hence

$$3R=4S_{\lambda,\mu,\nu}I_{x,y,z}K-\tfrac{1}{4}\{E\Omega\}^2. \qquad \text{(A)}$$

If we restore to U, V, W their general values, and make

$$\begin{aligned}
U&=ax^2+by^2+cz^2+2fyz+2gzx+2hxy,\\
V&=a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy,\\
W&=a''x^2+b''y^2+c''z^2+2f''yz+2g''zx+2h''xy,
\end{aligned}$$

and construct the cubic function

$$\begin{aligned}
\vartheta=&(ax+a'y+a''z)(bx+b'y+b''z)(cx+c'y+c''z)\\
&-(ax+a'y+a''z)(fx+f'y+f''z)^2-(bx+b'y+b''z)(gx+g'y+g''z)^2\\
&-(cx+c'y+c''z)(hx+h'y+h''z)^2\\
&+2(fx+f'y+f''z)(gx+g'y+g''z)(hx+h'y+h''z),
\end{aligned}$$

that is

$$\begin{aligned}
&\Sigma(abc-af^2-bg^2-ch^2+2fgh)x^3\\
&+\Sigma\{a'bc+ab'c+abc'-(a'f^2+2aff')-(b'g^2+2bgg')-(c'h^2+2chh')\\
&\qquad+2f'gh+2fg'h+2fgh'\}x^2y\\
&+\{a'b''c+a'bc''+a''b'c+a''bc'+ab'c''+ab''c'-2a'ff''-2af'f''-2a''ff'\\
&\qquad-2b'gg''-2bg'g''-2b''gg'-2c'hh''-2ch'h''-2c''hh'\\
&\qquad+2f''g'h+2f'g''h+2fg'h''+2f''gh'+2f''gh'+2fg''h'\}xyz,
\end{aligned}$$

$S_{\lambda,\mu,\nu}I_{x,y,z}K$ in the preceding equation becomes simply the Aronholdian S to ϑ, which may be calculated by Mr Salmon's formula previously quoted.

Ω may be taken equal to the determinant

$$\begin{vmatrix}
ax+a'y+a''z, & hx+h'y+h''z, & gx+g'y+g''z, & \xi\\
hx+h'y+h''z, & bx+b'y+b''z, & fx+f'y+f''z, & \eta\\
gx+g'y+g''z, & fx+f'y+f''z, & cx+c'y+c''z, & \zeta\\
\xi, & \eta, & \zeta, & 0
\end{vmatrix}.$$

And the cubic commutant of this, obtained by affecting it with the commutantive operator,

$$\left.\begin{matrix}
\dfrac{d}{dx}, & \dfrac{d}{dy}, & \dfrac{d}{dz}\\
\dfrac{d}{dx}, & \dfrac{d}{dy}, & \dfrac{d}{dz}\\
\dfrac{d}{d\xi}, & \dfrac{d}{d\eta}, & \dfrac{d}{d\zeta}\\
\dfrac{d}{d\xi}, & \dfrac{d}{d\eta}, & \dfrac{d}{d\zeta}
\end{matrix}\right\}$$

will give $48E(\Omega)$ if each of the four lines of the operator undergoes permutation, or $8E(\Omega)$, if one of the four lines is kept stationary. Thus it falls within the limits of practical possibility to calculate explicity, by the formula (A), the value of the resultant. I give to the S of ϑ the appellation of the Hebrew letter ש (*shin*), and to the commutant of Ω the appellation of the Hebrew letter ט (*teth*). These letters are chosen with design; for I shall presently show that when the three given quadratic functions are the differential derivatives of the same cubic function ψ, the ט becomes the Aronholdian T to the cubic function, or, as we may write it, $T\psi$, and the ש becomes the Aronholdian S of the Hessian thereto, that is $SH\psi$.

Thus for the first time the true inward constitution of the resultant of three quadratics is brought to light. The methods anteriorly given by me, and the one subsequently added by M. Hesse for finding this resultant, adverted to in Section II., lead, it is true, to the construction of the form, but throw no light upon the essential mode of its composition.

55.

THÉORÈME SUR LES LIMITES DES RACINES REELLES DES ÉQUATIONS ALGÉBRIQUES.

[*Nouvelles Annales de Mathématiques*, XII. (1853), pp. 286—287.]

Soit
$$f(x)=0$$
une équation algébrique de degré n, et supposons qu'en opérant sur $f(x)$ et $f'(x)$ comme dans le théorème de M. Sturm, on obtienne les n quotients

$$a_1x+b_1,\quad a_2x+b_2,\quad a_3x+b_3\ldots a_nx+b_n\,;$$

il faut remarquer seulement qu'on obtient le $n^{\text{ième}}$ quotient, a_nx+b_n, en divisant l'avant-dernier résidu par le dernier résidu.

Formons la série de $2n$ quantités

$$\frac{\pm 2-b_1}{a_1},\ \frac{\pm 2-b_2}{a_2},\ \frac{\pm 2-b_3}{a_3}\ldots\frac{\pm 2-b_n}{a_n};$$

il n'y a aucune racine de l'équation

$$f(x)=0$$

entre la plus grande de ces quantités et $+\infty$, ni entre la plus petite de ces quantités et $-\infty$ *.

* Prochainement, une démonstration de ce théorème *généralisé*. [p. 424 below.]