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THE COLLECTED MATHEMATICAL PAPERS

OF

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PREFATORY NOTE.

THE object aimed at in this volume has been to present a faithful record of the course of the author's thought, without such additions as recent developments of the subjects treated of might have afforded, and without any alterations other than that considerable number involved in the attempt to make the algebraical symbols read as the writer intended. While, for the reader's convenience, the author's references to his own papers have been accompanied by cross references to the pages of this volume, placed in square brackets.

By far the longest paper in the volume is No. 57, "On the Theory of the Syzygetic Relations of two Rational Integral Functions, comprising an application to the Theory of Sturm's Functions," and to this many of the shorter papers in the volume are contributory.

The volume contains also Sylvester's dialytic method of elimination (No. 9, etc.), his Essay on Canonical Forms (No. 34), and early investigations in the Theory of Invariants (Nos. 42, 43, etc.).

It contains also celebrated theorems as to Determinants (Nos. 37, 39, 48, etc.) and investigations as to the Transformation of Quadratic Forms (the *Law of Inertia*, No. 47, and the recognition of the Invariant factors of a matrix, Nos. 22, 24, 36).

A full table of contents is prefixed.

H. F. BAKER.

ST JOHN'S COLLEGE, CAMBRIDGE.
April, 1904.

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1.

ANALYTICAL DEVELOPMENT OF FRESNEL'S OPTICAL THEORY OF CRYSTALS.

[*Philosophical Magazine*, XI. (1837), pp. 461—469, 537—541; XII. (1838), pp. 73—83, 341—345.]

THE following is, I believe, the first successful attempt to obtain the full development of Fresnel's Theory of Crystals by direct geometrical methods. Hitherto little has been done beyond finding and investigating the properties of the wave surface, a subject certainly curious and interesting, but not of chief importance for ordinary practical purposes. Mr Kelland, in a most valuable contribution to the *Cambridge Philosophical Transactions**, has incidentally obtained the difference of the squares of the velocities of a plane front in terms of the angles made by it with the optic axes. I have obtained each of the velocities *separately*, and in a form precisely the same for biaxal as for uniaxal crystals.

I have also assigned in my last proposition the place of the lines of vibration in terms of the like quantities, and *that* in a shape remarkably convenient for determining the *plane* of polarization when the ray is given. For at first sight there appears to be some ambiguity in selecting *which* of the *two* lines of vibration is to be chosen when the front is known. If p be the perpendicular from the centre of the surface of elasticity let fall upon the front, ι_1, ι_2 the angles made by the front with the optic planes, ϵ_1, ϵ_2 the angles between its *due* line of vibration and the optic axes, I have shown that

$$\cos\epsilon_1 = \sqrt{\left(\frac{b^2-p^2}{a^2-c^2}\cdot\frac{\sin\iota_1}{\sin\iota_2}\right)}, \quad \cos\epsilon_2 = \sqrt{\left(\frac{b^2-p^2}{a^2-c^2}\cdot\frac{\sin\iota_2}{\sin\iota_1}\right)},$$

so that all doubt is completely removed. The equation preparatory to obtaining the wave surface is found in Prop. 6 by common algebra, without any use of the properties of maxima and minima, and various other curious relations are discussed.

Without the most careful attention to preserve pure symmetry, the expressions could never have been reduced to their present simple forms.

* See *Lond. and Edinb. Phil. Mag.* Vol. x. p. 336.

Analytical Reduction of Fresnel's Optical Theory of Crystals.

Index of Contents.

In Proposition 1, a plane front within a crystal being given, the two lines of vibration are investigated.

In Proposition 2 it is shown that the product of the cosines of the inclinations of one of the axes of elasticity to the two lines of vibration, is to the same for either other axis of elasticity in a constant ratio for the same crystal; and the two lines of vibration are proved to be perpendicular to each other.

In Proposition 3, a line of vibration being given, the front to which it belongs is determined; and it is proved that there is only one such, and consequently any line of vibration has but one other line conjugate to it.

In Proposition 4, certain relations are instituted between the positions of, and velocities due to, conjugate lines.

In Proposition 5, the angles made by the front with the planes of elasticity are found in terms of the velocities only.

In Proposition 6, the above is reversed.

In Proposition 7, the position of the planes in which the two velocities are equal (viz. the optic planes) is determined.

In Proposition 8, the position of a front in respect to the optic axes is expressed in terms of the velocities.

In Proposition 9, the problem is reversed, and it is shown that if v_1, v_2 be the two normal velocities with which any front can move perpendicular to itself, and ι_1, ι_2 the angles which it makes with the optic planes, then

$$v_1^2 = a^2\left(\sin\frac{\iota_1+\iota_2}{2}\right)^2 + c^2\left(\cos\frac{\iota_1+\iota_2}{2}\right)^2,$$

$$v_2^2 = a^2\left(\sin\frac{\iota_1-\iota_2}{2}\right)^2 + c^2\left(\cos\frac{\iota_1-\iota_2}{2}\right)^2.$$

In the 10th the angle made by a line of vibration with the axes ot elasticity is expressed in terms of the two velocities of the front to which it belongs.

In the 11th Proposition the velocity due to any line of vibration is expressed in terms of the angles which it makes with the optic axes, viz.

$$v^2 - b^2 = (a^2 - c^2)\cos\epsilon_1\cos\epsilon_2.$$

In the 12th Proposition ϵ_1, ϵ_2 are separately expressed in terms of ι_1, ι_2.

In the Appendix I have given the polar or rather radio-angular equation to the wave surface, from which the celebrated proposition of the ray flows as an immediate consequence.

Proposition 1.

If
$$lx + my + nz = 0 \qquad (a)$$
be the equation to a given front, to determine the lines of vibration therein.

It is clear that if x, y, z be any point in one of these lines, the force acting on a particle placed there when resolved into the plane must tend to the centre. Consequently the line of force at x, y, z must meet the perpendicular drawn upon the front from the origin. Now the equation to this perpendicular is

$$\frac{X}{l} = \frac{Y}{m} = \frac{Z}{n} \qquad (1)$$

and the forces acting at x, y, z are a^2x, b^2y, c^2z parallel to x, y, z, so that the equation to the line of force is

$$\frac{X-x}{a^2x} = \frac{Y-y}{b^2y} = \frac{Z-z}{c^2z}. \qquad (2)$$

From (2) we obtain

$$b^2yX - a^2xY = (b^2 - a^2)\,xy \qquad (3)$$

$$c^2zY - b^2yZ = (c^2 - b^2)\,yz \qquad (4)$$

$$a^2xZ - c^2zX = (a^2 - c^2)\,zx. \qquad (5)$$

Hence

$$(b^2 - a^2)\,xyn + (c^2 - b^2)\,yzl + (a^2 - c^2)\,zxm$$
$$= b^2y\,(nX - lZ) + c^2z\,(lY - mX) + a^2x\,(mZ - nY);$$

but by equations (1)

$$lZ - nX = 0,\quad mX - lY = 0,\quad nY - mZ = 0$$

therefore

$$(b^2 - a^2)\frac{n}{z} + (c^2 - b^2)\frac{l}{x} + (a^2 - c^2)\frac{m}{y} = 0. \qquad (b)$$

Also we have

$$nz + lx + my = 0 \qquad (a)$$

therefore

$$(b^2 - a^2)\,n^2 + (c^2 - b^2)\,l^2 + nl\left((c^2 - b^2)\frac{z}{x} + (b^2 - a^2)\frac{x}{z}\right) = (a^2 - c^2)\,m^2$$

or

$$(c^2 - b^2)\left(\frac{z}{x}\right)^2 + \frac{1}{nl}\{(c^2 - b^2)\,l^2 + (b^2 - a^2)\,n^2 - (a^2 - c^2)\,m^2\}\frac{z}{x} + (b^2 - a^2) = 0.$$

And in like manner interchanging b, y, m with c, z, n

$$(b^2 - c^2)\left(\frac{y}{x}\right)^2 + \frac{1}{ml}\{(b^2 - c^2)\,l^2 + (c^2 - a^2)\,m^2 - (a^2 - b^2)\,n^2\}\frac{y}{x} + (c^2 - a^2) = 0.$$

Hence if $\left(\frac{y_1}{x_1}, \frac{z_1}{x_1}\right)\left(\frac{y_2}{x_2}, \frac{z_2}{x_2}\right)$ be the two systems of values of $\frac{y}{x}, \frac{z}{x}$, then

$$\left(\frac{Y}{X}=\frac{y_1}{x_1},\ \frac{Z}{X}=\frac{z_1}{x_1}\right)\left(\frac{Y}{X}=\frac{y_2}{x_2},\ \frac{Z}{X}=\frac{z_2}{x_2}\right)$$

are the two lines of vibration required.

Proposition 2.

By last proposition it appears that

$$\frac{y_1y_2}{x_1x_2}=\frac{c^2-a^2}{b^2-c^2} \qquad (c)$$

and

$$\frac{z_1z_2}{x_1x_2}=\frac{b^2-a^2}{c^2-b^2} \qquad (d)$$

therefore

$$\frac{y_1y_2+z_1z_2}{x_1x_2}=\frac{c^2-b^2}{b^2-c^2}=-1$$

therefore

$$x_1x_2+y_1y_2+z_1z_2=0.$$

And therefore the two lines of vibration are perpendicular to each other.

N.B. Equations (c) and (d) must not be overlooked.

Proposition 3.

A line of vibration is given $\left(\textit{that is } \frac{y_1}{x_1}, \frac{z_1}{x_1} \textit{ are given}\right)$ *and the position of the front is to be determined.*

Let $lx+my+nz=0$ be the front required, then $lx_1+my_1+nz_1=0$, and

$$(b^2-c^2)\frac{l}{x_1}+(c^2-a^2)\frac{m}{y_1}+(a^2-b^2)\frac{n}{z_1}=0.$$

Eliminating n we get

$$l\left((a^2-b^2)\frac{x_1}{z_1}-(b^2-c^2)\frac{z_1}{x_1}\right)+m\left((a^2-b^2)\frac{y_1}{z_1}-(c^2-a^2)\frac{z_1}{y_1}\right)=0$$

therefore

$$\frac{l}{m}=\frac{x_1}{y_1}\frac{(a^2-b^2)\,y_1^2-(c^2-a^2)\,z_1^2}{(b^2-c^2)\,z_1^2-(a^2-b^2)\,x_1^2}$$

$$=\frac{x_1}{y_1}\frac{a^2(x_1^2+y_1^2+z_1^2)-(a^2x_1^2+b^2y_1^2+c^2z_1^2)}{b^2(x_1^2+y_1^2+z_1^2)-(a^2x_1^2+b^2y_1^2+c^2z_1^2)}.$$

If now we make $x_1^2+y_1^2+z_1^2=1$

$$a^2x_1^2+b^2y_1^2+c^2z_1^2=v_1^2$$

and therefore

$$\frac{l}{m}=\frac{x_1}{y_1}\cdot\frac{a^2-v_1^2}{b^2-v_1^2}$$

and in like manner

$$\frac{l}{n}=\frac{x_1}{z_1}\cdot\frac{a^2-v_1^2}{c^2-v_1^2};$$

therefore

$$(a^2-v_1^2)\,x_1x+(b^2-v_1^2)\,y_1y+(c^2-v_1^2)\,z_1z=0$$

is the equation required.

Proposition 4.

$\frac{l}{m}, \frac{l}{n}$ having each only one value, shows that only one front corresponds to the given line of vibration. Let x_2, y_2, z_2, v_2 correspond to x_1, y_1, z_1, v_1 for the conjugate line of vibration, then the equation to the front may be expressed likewise by

$$(a^2-v_2^2)\,x_2x+(b^2-v_2^2)\,y_2y+(c^2-v_2^2)\,z_2z=0,$$

so that

$$\frac{(a^2-v_1^2)\,x_1}{(a^2-v_2^2)\,x_2}=\frac{(b^2-v_1^2)\,y_1}{(b^2-v_2^2)\,y_2}=\frac{(c^2-v_1^2)\,z_1}{(c^2-v_2^2)\,z_2}.$$

Proposition 5.

To find ω, ϕ, ψ, the angles made by the front with the planes of elasticity in terms of v_1, v_2.

By the last proposition

$$(\cos\omega)^2=\frac{(a^2-v_1^2)^2\,x_1^2}{(a^2-v_1^2)^2\,x_1^2+(b^2-v_1^2)^2\,y_1^2+(c^2-v_1^2)^2\,z_1^2}$$

$$=\frac{(a^2-v_1^2)(a^2-v_2^2)\,x_1x_2}{(a^2-v_1^2)(a^2-v_2^2)\,x_1x_2+(b^2-v_1^2)(b^2-v_2^2)\,y_1y_2+(c^2-v_1^2)(c^2-v_2^2)\,z_1z_2}.$$

Now, by Proposition 2,

$$\frac{x_1x_2}{c^2-b^2}=\frac{y_1y_2}{a^2-c^2}=\frac{z_1z_2}{b^2-a^2}$$

therefore $(\cos\omega)^2$

$$=\frac{(a^2-v_1^2)(a^2-v_2^2)(c^2-b^2)}{(a^2-v_1^2)(a^2-v_2^2)(c^2-b^2)+(b^2-v_1^2)(b^2-v_2^2)(a^2-c^2)+(c^2-v_1^2)(c^2-v_2^2)(b^2-a^2)}$$

$$=\frac{(a^2-v_1^2)(a^2-v_2^2)(c^2-b^2)}{a^4(c^2-b^2)+b^4(a^2-c^2)+c^4(a^2-b^2)}$$

$$=\frac{(a^2-v_1^2)(a^2-v_2^2)}{(a^2-b^2)(a^2-c^2)}.$$

Similarly,

$$(\cos\phi)^2=\frac{(b^2-v_1^2)(b^2-v_2^2)}{(b^2-a^2)(b^2-c^2)},$$

$$(\cos\psi)^2=\frac{(c^2-v_1^2)(c^2-v_2^2)}{(c^2-a^2)(c^2-b^2)}.$$

PROPOSITION 6.

To find v_1, v_2 in terms of ω, ϕ, ψ.

By the last proposition

$$\frac{(\cos\omega)^2}{a^2-v_1^2}=\frac{a^2}{(a^2-b^2)(a^2-c^2)}-v_2^2\cdot\frac{1}{(a^2-b^2)(a^2-c^2)}$$

$$\frac{(\cos\phi)^2}{b^2-v_1^2}=\frac{b^2}{(b^2-a^2)(b^2-c^2)}-v_2^2\cdot\frac{1}{(b^2-a^2)(b^2-c^2)}$$

$$\frac{(\cos\psi)^2}{c^2-v_1^2}=\frac{c^2}{(c^2-b^2)(c^2-a^2)}-v_2^2\cdot\frac{1}{(c^2-a^2)(c^2-b^2)}$$

therefore

$$\frac{(\cos\omega)^2}{a^2-v_1^2}+\frac{(\cos\phi)^2}{b^2-v_1^2}+\frac{(\cos\psi)^2}{c^2-v_1^2}=0.$$

Just in the same way

$$\frac{(\cos\omega)^2}{a^2-v_2^2}+\frac{(\cos\phi)^2}{b^2-v_2^2}+\frac{(\cos\psi)^2}{c^2-v_2^2}=0,$$

so that v_1^2, v_2^2 are the two roots of the equation

$$\frac{(\cos\omega)^2}{a^2-v^2}+\frac{(\cos\phi)^2}{b^2-v^2}+\frac{(\cos\psi)^2}{c^2-v^2}=0.$$

COR. Hence the equation to the wave surface may be obtained by making

$$(\cos\omega)\,x+(\cos\phi)\,y+(\cos\psi)\,z=v,$$

or if we please to apply Prop. 5, we may make

$$\sqrt{\frac{(a^2-v_1^2)(a^2-v_2^2)}{(a^2-b^2)(a^2-c^2)}}\cdot x+\sqrt{\frac{(b^2-v_1^2)(b^2-v_2^2)}{(b^2-a^2)(b^2-c^2)}}\cdot y$$
$$+\sqrt{\frac{(c^2-v_1^2)(c^2-v_2^2)}{(c^2-a^2)(c^2-b^2)}}\cdot z=v_1,$$

or, if we please*,

$$\sqrt{\frac{(a^2u^2-1)(a^2-v^2)}{(a^2-b^2)(a^2-c^2)}}\cdot x+\sqrt{\frac{(b^2u^2-1)(b^2-v^2)}{(b^2-a^2)(b^2-c^2)}}\cdot y$$
$$+\sqrt{\frac{(c^2u^2-1)(c^2-v^2)}{(c^2-a^2)(c^2-b^2)}}\cdot z=1.$$

PROPOSITION 7.

To find when $v_1=v_2$.

By Prop. 4,

$$\frac{x_1(v_1^2-a^2)}{x_2(v_2^2-a^2)}=\frac{y_1(v_1^2-b^2)}{y_2(v_2^2-b^2)}=\frac{z_1(v_1^2-c^2)}{z_2(v_2^2-c^2)}.\qquad(\theta)$$

Hence when $v_1=v_2$ we have, generally speaking,

$$\frac{x_1}{x_2}=\frac{y_1}{y_2}=\frac{z_1}{z_2}.$$

Now $$x_1x_2+y_1y_2+z_1z_2=0;$$

therefore $x_1^2+y_1^2+z_1^2$ would $=0$, which is absurd.

The only case therefore when v_1 can $=v_2$ is when one of those terms of equation (θ) becomes $\frac{0}{0}$: thus suppose $v_1=b$, then we have $\frac{x_1}{x_2}=\frac{z_1}{z_2}=\frac{0}{0}$, and we can no longer infer $\frac{x_1}{x_2}=\frac{y_1}{y_2}$.

Let now $(\omega_1, \phi_1, \psi_1)(\omega_2, \phi_2, \psi_2)$ be the two systems of values which ω, ϕ, ψ assume when $v_1=v_2=b$, then applying the equation of Prop. 5 we have

$$\cos\omega_1=\sqrt{\frac{a^2-b^2}{a^2-c^2}}\qquad\cos\omega_2=\sqrt{\frac{a^2-b^2}{a^2-c^2}}$$

$$\cos\phi_1=0\qquad\cos\phi_2=0$$

$$\cos\psi_1=\sqrt{\frac{b^2-c^2}{a^2-c^2}}\qquad\cos\psi_2=\sqrt{\frac{b^2-c^2}{a^2-c^2}},$$

so that b must correspond to the mean axis.

[* See below, p. 27. ED.]

Proposition 8.

ι_1, ι_2 being the angles made by the front with the optic planes, to find ι_1, ι_2 in terms of v_1, v_2.

By analytical geometry

$$\cos \iota_1 = \cos \omega \,.\, \cos \omega_1 + \cos \phi \,.\, \cos \phi_1 + \cos \psi \,.\, \cos \psi_1$$

$$= \sqrt{\frac{(v_1^2 - a^2)(v_2^2 - a^2)}{(a^2 - b^2)(a^2 - c^2)}} \cdot \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}$$

$$+ \sqrt{\frac{(v_1^2 - c^2)(v_2^2 - c^2)}{(c^2 - a^2)(c^2 - b^2)}} \cdot \sqrt{\frac{c^2 - b^2}{c^2 - a^2}}$$

$$= \frac{\sqrt{\{(v_1^2 - a^2)(v_2^2 - a^2)\}} + \sqrt{\{(v_1^2 - c^2)(v_2^2 - c^2)\}}}{a^2 - c^2},$$

and similarly

$$\cos \iota_2 = \cos \omega \,.\, \cos \omega_2 + \cos \phi \,.\, \cos \phi_2 + \cos \psi \,.\, \cos \psi_2$$

$$= \frac{\sqrt{\{(v_1^2 - a^2)(v_2^2 - a^2)\}} - \sqrt{\{(v_1^2 - c^2)(v_2^2 - c^2)\}}}{a^2 - c^2}.$$

Proposition 9.

To find v_1, v_2 in terms of ι_1, ι_2.

By the last proposition

$$\cos \iota_1 \,.\, \cos \iota_2 = \frac{(v_1^2 - a^2)(v_2^2 - a^2) - (v_1^2 - c^2)(v_2^2 - c^2)}{(a^2 - c^2)^2}$$

$$= \frac{(a^4 - c^4) - (a^2 - c^2)(v_1^2 + v_2^2)}{(a^2 - c^2)^2}$$

$$= \frac{(a^2 + c^2) - (v_1^2 + v_2^2)}{(a^2 - c^2)}$$

therefore

$$v_1^2 + v_2^2 = a^2 + c^2 - (a^2 - c^2) \cos \iota_1 \cos \iota_2.$$

Again, $(\sin \iota_1)^2 \,.\, (\sin \iota_2)^2 = 1 - (\cos \iota_1)^2 - (\cos \iota_2)^2 + (\cos \iota_1)^2 (\cos \iota_2)^2$

$$= 1 - 2 \,.\, \frac{(v_1^2 - a^2)(v_2^2 - a^2) + (v_1^2 - c^2)(v_2^2 - c^2)}{(a^2 - c^2)^2}$$

$$+ \frac{(a^2 + c^2)^2 - 2(a^2 + c^2)(v_1^2 + v_2^2) + (v_1^2 + v_2^2)^2}{(a^2 - c^2)^2}$$

$$= \frac{v_1^4 - 2v_1^2 v_2^2 + v_2^4}{(a^2 - c^2)^2}$$

therefore

$$v_1^2 - v_2^2 = (a^2 - c^2) \sin \iota_1 \,.\, \sin \iota_2$$

but

$$v_1^2 + v_2^2 = (a^2 + c^2) - (a^2 - c^2) \cos \iota_1 \cos \iota_2$$

therefore

$$v_1^2 = \frac{a^2 + c^2}{2} - \frac{a^2 - c^2}{2} \cos (\iota_1 + \iota_2)$$

$$= a^2 \left(\sin \frac{\iota_1 + \iota_2}{2}\right)^2 + c^2 \left(\cos \frac{\iota_1 + \iota_2}{2}\right)^2$$

$$v_2^2 = \frac{a^2 + c^2}{2} - \frac{a^2 - c^2}{2} \cos (\iota_1 - \iota_2)$$

$$= a^2 \left(\sin \frac{\iota_1 - \iota_2}{2}\right)^2 + c^2 \left(\cos \frac{\iota_1 - \iota_2}{2}\right)^2.$$

Thus for uniaxal crystals where $\iota_1 + \iota_2 = 180°$

$$v_1^2 = a^2$$

$$v_2^2 = a^2 (\cos \iota)^2 + c^2 (\sin \iota)^2.$$

Cor. Hence we may reduce the discovery of the two fronts into which a plane front is refracted on entering a crystal to the following trigonometrical problem.

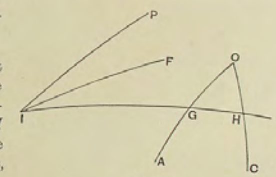


Fig. 1.

Let a sphere be described about any point in the line in which the air front intersects the plane of incidence. Let the great circle PI denote the latter plane, IF the former, OA, OC also great circles, the planes of single velocity. Suppose IGH to be one of the refracted fronts intersecting OA, OC in G and H, then

$$\frac{(a^2 + c^2) - (a^2 - c^2) \cos (G + H)}{2\,(\text{vel. in air})^2} = \frac{(\sin PIF)^2}{(\sin PIGH)^2}.$$

The double sign will give rise to two positions of the refracted front IGH.

The propositions which follow are perhaps more curious than immediately useful.

Proposition 10.

To determine the portion of a line of vibration in terms of the two velocities of its corresponding front.

We have here to determine the quantities $\frac{y_1}{x_1}, \frac{z_1}{x_1}$ (of Prop. 1) in terms of v_1, v_2, or on putting $x_1^2+y_1^2+z_1^2=1$, x_1, y_1, z_1 are to be found in terms of v_1, v_2.

By Prop. 3

$$x_1 : y_1 : z_1 :: \frac{l}{a^2-v_1^2} : \frac{m}{b^2-v_1^2} : \frac{n}{c^2-v_1^2}$$

and by Prop. 5

$$l^2 : m^2 : n^2 :: (b^2-c^2)(a^2-v_1^2)(a^2-v_2^2)$$
$$: (c^2-a^2)(b^2-v_1^2)(b^2-v_2^2)$$
$$: (a^2-b^2)(c^2-v_1^2)(c^2-v_2^2);$$

therefore

$$x_1^2 \quad : \quad y_1^2 \quad : \quad z_1^2$$
$$:: (b^2-c^2)\frac{a^2-v_2^2}{a^2-v_1^2} : (c^2-a^2)\frac{b^2-v_2^2}{b^2-v_1^2} : (a^2-b^2)\frac{c^2-v_2^2}{c^2-v_1^2}.$$

Let α, β, γ be the angles made by the given line of vibration with the elastic axes, then

$$(\cos\alpha)^2 = \frac{x_1^2}{x_1^2+y_1^2+z_1^2}$$
$$= (b^2-c^2)(a^2-v_2^2)(b^2-v_1^2)(c^2-v_1^2)$$

divided by

$$(b^2-c^2)(a^2-v_2^2)(b^2-v_1^2)(c^2-v_1^2) + (c^2-a^2)(b^2-v_2^2)(c^2-v_1^2)(a^2-v_1^2)$$
$$+ (a^2-b^2)(c^2-v_2^2)(a^2-v_1^2)(b^2-v_1^2)$$

and therefore

$$= \frac{(b^2-c^2)(a^2-v_2^2)(b^2-v_1^2)(c^2-v_1^2)}{(v_1^2-v_2^2)(a^2-b^2)(b^2-c^2)(c^2-a^2)}$$

(where it is to be observed that the reduction of the denominator is simply the effect of a vast heap of terms disappearing under the influence of contact with the magic circuit (a^2-b^2), (b^2-c^2), (c^2-a^2), a simpler instance of which was seen in Proposition 5).

In fact the coefficient of $v^4 . v^2$

$$= (b^2-c^2)+(c^2-a^2)+(a^2-b^2)$$
$$= 0$$

that of $v_1^2 . v_2^2$

$$= (c^2+b^2).(c^2-b^2)$$
$$+ (a^2+c^2).(a^2-c^2)$$
$$+ (b^2+a^2).(b^2-a^2)$$
$$= (c^4-b^4)+(a^4-c^4)+(b^4-a^4)$$
$$= 0.$$

The term in which neither v_1 nor v_2 enters

$$= a^2b^2c^2\{(b^2-c^2)+(c^2-a^2)+(a^2-b^2)\}$$
$$= 0.$$

The coefficient of

$$-v_1^2 = a^2.(b^4-c^4)+b^2.(c^4-a^4)+c^2.(a^4-b^4)$$

and that of

$$v_2^2 = b^2c^2.(c^2-b^2)+c^2a^2.(a^2-c^2)+a^2b^2.(b^2-a^2)$$

each of which

$$= (a^2-b^2).(b^2-c^2).(c^2-a^2).$$

Hence

$$(\cos\alpha)^2 = \frac{v_1^2-b^2}{v_1^2-v_2^2}\cdot\frac{(a^2-v_2^2)(c^2-v_1^2)}{(a^2-b^2)(a^2-c^2)},$$

in like manner $(\cos\beta)^2 = \&c.$

and

$$(\cos\gamma)^2 = \frac{v_1^2-b^2}{v_1^2-v_2^2}\cdot\frac{(c^2-v_2^2)(a^2-v_1^2)}{(c^2-b^2)(c^2-a^2)}.$$

Proposition 11.

ϵ_1, ϵ_2 being the angles between any line of vibration and the optic axes, required the velocity due to that line in terms of ϵ_1, ϵ_2.

By analytical geometry,

$$\cos\epsilon_1 = \cos\alpha . \cos\phi_1 + \cos\gamma . \cos\psi_1$$
$$\cos\epsilon_2 = \cos\alpha . \cos\phi_1 - \cos\gamma . \cos\psi_1$$

therefore

$$\cos\epsilon_1 . \cos\epsilon_2 = (\cos\alpha)^2(\cos\phi_1)^2 - (\cos\gamma)^2(\cos\psi_1)^2$$
$$= \frac{v_1^2-b^2}{v_1^2-v_2^2}\cdot\left\{\frac{(a^2-v_2^2).(c^2-v_1^2)-(c^2-v_2^2).(a^2-v_1^2)}{(a^2-c^2)^2}\right\}$$
$$= \frac{v_1^2-b^2}{v_1^2-v_2^2}\cdot\frac{(a^2-c^2)(v_2^2-v_1^2)}{(a^2-c^2)^2}$$
$$= \frac{b^2-v_1^2}{a^2-c^2}.$$

Hence

$$v_1^2 = b^2-(a^2-c^2)\cos\epsilon_1\cos\epsilon_2,$$

and in like manner, for the *conjugate* line of vibration

$$v_2^2 = b^2-(a^2-c^2)\cos\epsilon_1'\cos\epsilon_2'.$$

PROPOSITION 12.

To find ϵ_1, ϵ_2 in terms of ι_1, ι_2.

$$(\cos\epsilon_1)^2 + (\cos\epsilon_2)^2 = 2(\cos\alpha)^2 . (\cos\phi_1)^2 + 2(\cos\gamma)^2 . (\cos\psi_1)^2$$

$$= 2\,\frac{v_1^2 - b^2}{v_1^2 - v_2^2}\left\{\frac{(a^2 - v_2^2).(c^2 - v_1^2) + (c^2 - v_2^2).(a^2 - v_1^2)}{(a^2 - c^2)^2}\right\}$$

but by Prop. 9

$$v_1^2 = a^2\left(\sin\frac{\iota_1 + \iota_2}{2}\right)^2 + c^2\left(\cos\frac{\iota_1 + \iota_2}{2}\right)^2$$

$$v_2^2 = a^2\left(\sin\frac{\iota_1 - \iota_2}{2}\right)^2 + c^2\left(\cos\frac{\iota_1 - \iota_2}{2}\right)^2$$

therefore

$$(\cos\epsilon_1)^2 + (\cos\epsilon_2)^2 = \frac{b^2 - v_1^2}{(a^2 - c^2)\sin\iota_1 . \sin\iota_2}$$

multiplied by

$$\frac{2(a^2 - c^2)^2\left[\left(\cos\frac{\iota_1 + \iota_2}{2}\right)^2\left(\sin\frac{\iota_1 - \iota_2}{2}\right)^2 + \left(\cos\frac{\iota_1 - \iota_2}{2}\right)^2\left(\sin\frac{\iota_1 + \iota_2}{2}\right)^2\right]}{(a^2 - c^2)^2}$$

$$= \frac{b^2 - v_1^2}{(a^2 - c^2)\sin\iota_1 . \sin\iota_2}\{(\sin\iota_1)^2 + (\sin\iota_2)^2\}$$

and we have seen that

$$\cos\epsilon_1 \cos\epsilon_2 = \frac{b^2 - v_1^2}{a^2 - c^2}$$

therefore

$$\cos\epsilon_1 + \cos\epsilon_2 = \sqrt{\left(\frac{b^2 - v_1^2}{a^2 - c^2}\right)} . \frac{\sin\iota_1 + \sin\iota_2}{\sqrt{(\sin\iota_1 . \sin\iota_2)}}$$

$$\cos\epsilon_1 - \cos\epsilon_2 = \sqrt{\left(\frac{b^2 - v_1^2}{a^2 - c^2}\right)} . \frac{\sin\iota_1 - \sin\iota_2}{\sqrt{(\sin\iota_1 . \sin\iota_2)}}$$

therefore

$$\cos\epsilon_1 = \sqrt{\left\{\frac{b^2 - v_1^2}{a^2 - c^2} . \frac{\sin\iota_1}{\sin\iota_2}\right\}}$$

$$\cos\epsilon_2 = \sqrt{\left\{\frac{b^2 - v_1^2}{a^2 - c^2} . \frac{\sin\iota_2}{\sin\iota_1}\right\}}$$

and in like manner

$$\cos\epsilon_1' = \sqrt{\left\{\frac{b^2 - v_2^2}{a^2 - c^2} . \frac{\sin\iota_1}{\sin\iota_2}\right\}}$$

$$\cos\epsilon_2' = \sqrt{\left\{\frac{b^2 - v_2^2}{a^2 - c^2} . \frac{\sin\iota_2}{\sin\iota_1}\right\}}$$

where v_1, v_2 for the sake of neatness are left *unexpressed* in terms of ι_1, ι_2.

This is the simplest form by which the position of the lines of vibration can be denoted.

COR. From the last proposition it appears that

$$\frac{\cos\epsilon_1}{\cos\epsilon_2} = \frac{\sin\iota_1}{\sin\iota_2}.$$

Hence we may construct geometrically for the two planes of polarization.

Let I, K be the projections of the two optic axes on a sphere, E the projection of the normal to the front, P the projection of one line of vibration; then

$$\frac{\cos PK}{\cos PI} = \frac{\sin KE}{\sin IE}.$$

Draw FEG the circle of which P is the pole, meeting PK, PI produced in G and F.

Then $\cos PK = \sin KG$,

and $\cos PI = \sin IF$,

therefore

$$\frac{\sin KG}{\sin FI} = \frac{\sin KE}{\sin IE}$$

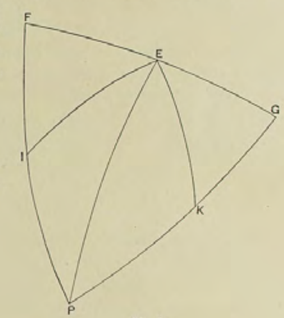


Fig. 2.

therefore

$$\frac{\sin KG}{\sin KE} = \frac{\sin IF}{\sin IE}$$

therefore

$$\sin KEG = \sin IEF$$

therefore $KEG = IEF$ or $180° - IEF$. But $PEF = PEG$, therefore EP bisects either the angle IEK or the supplement to it.

These two positions of EP give the two planes of polarization. The construction is the same as that given in Mr Airy's tracts, and originally proposed, I believe, by Mr MacCullagh.

ADDENDUM.

If in the equation of Prop. 6, viz.

$$\frac{(\cos\omega)^2}{a^2 - v^2} + \frac{(\cos\phi)^2}{b^2 - v^2} + \frac{(\cos\psi)^2}{c^2 - v^2} = 0$$

we change a, b, c, v into $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, $\frac{1}{v}$, and consider v to be the length of a line drawn perpendicular to the plane

$$\cos\omega \,.\, x + \cos\phi \,.\, y + \cos\psi \,.\, z = 0,$$

the equation to the extremity thereof must be

$$\frac{a^2 r^2 (\cos\omega)^2}{a^2 - r^2} + \frac{b^2 r^2 (\cos\phi)^2}{b^2 - r^2} + \frac{c^2 r^2 (\cos\psi)^2}{c^2 - r^2}$$

where ω, ϕ, ψ denote the angles between the radius vector r, and the axes of x, y, z, so that the equation may be written

$$\frac{a^2 x^2}{a^2 - r^2} + \frac{b^2 y^2}{b^2 - r^2} + \frac{c^2 z^2}{c^2 - r^2} = 0,$$

which is that of the wave surface.

But we have seen that

$$v^2 = c^2 \left\{\cos\left(\frac{\iota_1 \pm \iota_2}{2}\right)\right\}^2 + a^2 \left\{\sin\left(\frac{\iota_1 \pm \iota_2}{2}\right)\right\}^2,$$

therefore the equation to the wave surface may be written

$$\frac{1}{r^2} = \frac{\left(\cos\frac{\iota_1 \pm \iota_2}{2}\right)^2}{c^2} + \frac{\left(\sin\frac{\iota_1 \pm \iota_2}{2}\right)^2}{a^2},$$

where ι_1, ι_2 denote the angles between the radius vector v and the two lines which would be the optic axes if a, b, c were changed into $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ so that if e be the inclination of either to the mean axis of elasticity

$$\cos e = \sqrt{\left(\frac{\frac{1}{a^2} - \frac{1}{b^2}}{\frac{1}{a^2} - \frac{1}{c^2}}\right)} = \frac{c}{b}\sqrt{\left(\frac{a^2 - b^2}{a^2 - c^2}\right)}$$

$$\sin e = \sqrt{\left(\frac{\frac{1}{b^2} - \frac{1}{c^2}}{\frac{1}{a^2} - \frac{1}{c^2}}\right)} = \frac{a}{b}\sqrt{\left(\frac{b^2 - c^2}{a^2 - c^2}\right)}.$$

These lines I shall call by way of distinction the prime radii*.

* Upon the authority of Professor Airy I have appropriated the term optic axes to the lines normal to the fronts of single velocity.

Cor. 1. If r_1, r_2 be the two values of r corresponding to the same values of ι_1, ι_2 we have

$$\frac{1}{r_1^2} - \frac{1}{r_2^2} = \frac{1}{c^2}\left\{\left(\cos\frac{\iota_1 - \iota_2}{2}\right)^2 - \left(\cos\frac{\iota_1 + \iota_2}{2}\right)^2\right\}$$
$$+ \frac{1}{a^2}\left\{\left(\sin\frac{\iota_1 - \iota_2}{2}\right)^2 - \left(\sin\frac{\iota_1 + \iota_2}{2}\right)^2\right\}$$
$$= \left(\frac{1}{c^2} - \frac{1}{a^2}\right)\sin\iota_1 \,.\, \sin\iota_2,$$

which proves the celebrated problem of *two rays* having a common direction in a crystal.

Cor. 2. The intersection of any concentric sphere with the wave surface is found by making r constant. Hence $\iota_1 \pm \iota_2$ becomes constant, and therefore $r\iota_1 \pm r\iota_2 = \text{constant}$. Hence the curve of intersection is the locus of points, the sum or difference of whose distances from two poles when measured by the arcs of great circles is constant; the poles being the points in which the prime radii pierce the sphere.

In three cases these spherico-ellipses or spherico-hyperbolas become great circles:

(1) When $\iota_1 \pm \iota_2 =$ the angle between the two poles, in which case the curve of intersection is the great circle which comprises the two poles.

(2) When $\iota_1 - \iota_2 = 0$, when the locus is a great circle perpendicular to the former and bisecting the angle between the optic axes.

(3) When $\iota_1 + \iota_2 = 180°$, when the locus is a great circle perpendicular to the two above, and bisecting the supplemental angle between the two axes.

Various other properties may be with the greatest simplicity deduced from the radio-angular equation. The hurry of the press leaves me time only to subjoin the following

Proposition.

To find the inclination of the radius vector to the tangent plane, in terms of the angles which the radius vector makes with the prime radii.

Let O be the centre of the wave surface, OA, OB the two prime radii, OP any radius vector. Let $OP = v$, $POA = \iota_1$, $POB = \iota_2$, and let the inclination of the planes POA, $POB = \mu$;

then
$$\frac{1}{r^2} = \frac{\left(\sin\frac{\iota_1 + \iota_2}{2}\right)^2}{a^2} + \frac{\left(\cos\frac{\iota_1 + \iota_2}{2}\right)^2}{c^2},$$

(taking only the positive sign for the sake of brevity).

Let OQ, OR be the two adjacent radii vectores, so assumed that

$$QOA = POA, \quad QOB = POB + \delta\iota_2,$$
$$ROB = POB, \quad ROA = POA + \delta\iota_1,$$

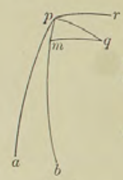


Fig. 3.

and let p, q, r, a, b be the projections of P, Q, R, A, B on a sphere of which O is the centre, then it is clear that

$$qpa = 90^\circ, \quad rpb = 90^\circ,$$

draw qm perpendicular to pb, then $pm = \delta\iota_2$, and therefore

$$pq = \frac{pm}{\sin pqm} = \frac{pm}{\sin apb} = \frac{\delta\iota_2}{\sin\mu}.$$

In like manner

$$pr = \frac{\delta\iota_1}{\sin\mu}.$$

Now the angle QPO

$$= \tan^{-1} . \frac{r . POQ}{OQ - OP} = \tan^{-1} . \frac{r . pq}{\frac{dr}{d\iota_2} . \delta\iota_2};$$

also

$$\frac{d . \frac{1}{r^2}}{d\iota_2} = d\iota_2 \left\{ \left(\frac{1}{c^2} - \frac{1}{a^2}\right) \left(\cos\frac{\iota_1 + \iota_2}{2}\right)^2 \right\}$$
$$= -\left(\frac{1}{c^2} - \frac{1}{a^2}\right) \left(\sin\frac{\iota_1 + \iota_2}{2}\right) \left(\cos\frac{\iota_1 + \iota_2}{2}\right);$$

therefore

$$\frac{dr}{r d\iota_2} = \tfrac{1}{4} r^2 \left(\frac{1}{c^2} - \frac{1}{a^2}\right) \sin(\iota_1 + \iota_2),$$

therefore

$$\cot QPO = \frac{r^2}{4} \left(\frac{1}{c^2} - \frac{1}{a^2}\right) \sin(\iota_1 + \iota_2) \sin\mu.$$

In like manner

$$\cot RPO = \frac{r^2}{4} \left(\frac{1}{c^2} - \frac{1}{a^2}\right) \sin(\iota_1 + \iota_2) \sin\mu,$$

therefore

$$QPO = RPO.$$

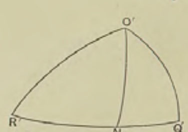


Fig. 4.

Also it is clear that $rpq = apb = \mu$. And to find the inclination of OP to RPQ, we have only to describe a sphere of which P is the centre, and intersecting PQ, PR, PO in Q', R', O'.

Then $R'O'Q' = \mu$, and

$$O'Q' = O'R' = \cot^{-1} \left\{ \frac{r^2}{4} \left(\frac{1}{c^2} - \frac{1}{a^2}\right) \sin(\iota_1 + \iota_2) \sin\mu \right\}.$$

Draw $O'N$ perpendicular to $R'Q'$, then $O'N$ measures the inclination of the radius vector to the tangent plane*.

And

$$Q'O'N = \frac{\mu}{2},$$

therefore

$$\cos\frac{\mu}{2} = \tan O'N . \cot O'Q',$$

therefore

$$\cot O'N = \frac{\cot O'Q'}{\cos\frac{\mu}{2}},$$

and therefore

$$\cot O'N = \tfrac{1}{2} r^2 . \left(\frac{1}{a^2} - \frac{1}{c^2}\right) \sin\frac{\mu}{2} . \sin(\iota_1 + \iota_2).$$

Let AOB the angle between the optic axes $= 2e$, then by mere trigonometry

$$\sin\frac{\mu}{2} = \sqrt{\frac{\sin\left(e + \frac{\iota_1 - \iota_2}{2}\right) \sin\left(e - \frac{\iota_1 - \iota_2}{2}\right)}{\sin\iota_1 . \sin\iota_2}},$$

therefore the tangent of the inclination between the radius vector and the normal

$$= \tfrac{1}{2} r^2 . \left(\frac{1}{a^2} - \frac{1}{c^2}\right) \sin(\iota_1 + \iota_2) . \sqrt{\frac{\sin\left(e + \frac{\iota_1 - \iota_2}{2}\right) \sin\left(e - \frac{\iota_1 - \iota_2}{2}\right)}{\sin\iota_1 . \sin\iota_2}}.$$

In like manner the tangent of the inclination between the same radius vector and the normal at the other point of the wave-surface pierced by it

$$= \tfrac{1}{2} (r_1)^2 \left(\frac{1}{a^2} - \frac{1}{c^2}\right) \sin(\iota_1 - \iota_2) . \sqrt{\frac{\sin\left(e + \frac{\iota_1 + \iota_2}{2}\right) \sin\left(e - \frac{\iota_1 + \iota_2}{2}\right)}{\sin\iota_1 . \sin\iota_2}}.$$

We may, in the same way, find the inclination of the tangent plane to either of the prime radii, and to the plane which contains them both, in terms of ι_1 and ι_2; the former by a remarkably elegant construction; but the final expressions do not present themselves under the same simple aspect.

If we call ϕ the angle between the ray and the front, we may still further reduce by substituting for r^2 its values in terms of ι_1, ι_2 and we shall obtain

$$\cot\phi = \frac{2(c^2 - a^2)}{c^2 \tan\frac{\iota_1 \mp \iota_2}{2} + a^2 \cot\frac{\iota_1 \mp \iota_2}{2}}$$
$$\times \sqrt{\left\{ \sin\left(e + \frac{\iota_1 \pm \iota_2}{2}\right) \sin\left(e - \frac{\iota_1 \pm \iota_2}{2}\right) . \operatorname{cosec}\iota_1 . \operatorname{cosec}\iota_2 \right\}}.$$

* O' is the projection of the ray and $R'O'$ of the tangent plane. Therefore $O'N$ being perpendicular to $R'Q'$ represents their inclination.

And if π_1, π_2 be the inclinations of the normal to the two prime radii, it may be shown that

$$\cos\pi_1 = \cos\phi\sin\iota_1 \mp \sin\phi\cos\iota_1\sin\frac{\mu}{2},$$

$$\cos\pi_2 = \cos\phi\sin\iota_2 \pm \sin\phi\cos\iota_2\sin\frac{\mu}{2}.$$

Cor. 1. For uniaxal crystals $\frac{\mu}{2} = 90°$ and $\iota_1 + \iota_2 = 180°$, so that the tangent of the inclination of normal to radius vector

$$= \tfrac{1}{2} r^2 . \left(\frac{1}{a^2} - \frac{1}{c^2}\right) \sin 2\iota \text{ for one point,}$$

and $= 0$ for the other.

Cor. 2. For every point in the circular section which passes through the poles $\sin\frac{\mu}{2} = 0$, and for the other two circular sections $\iota_1 \pm \iota_2 = 0$ or $180°$.

Therefore every point in the three circular sections is an apse.

Cor. 3. When a nearly $= c$, $\frac{1}{a^2} - \frac{1}{c^2}$ is very small; and therefore the normal and radius vector very nearly coincide.

Cor. 4. Referring to fig. 4 we see that $O'N$ bisects the angle $R'O'Q'$. Now $R'O$, $Q'O$ are respectively perpendicular to the planes passing through O' and the optic axes; and therefore the meridian plane as we may term it, that is, the plane containing both the ray and the normal, always bisects the angle formed by the two planes drawn through the ray and the two optic axes.

Cor. 5. When

$$\iota_1 \text{ or } \iota_2 = 0,$$

$$\iota_2 \text{ or } \iota_1 = e.$$

And therefore ϕ assumes the form $\frac{0}{0}$, which indicates that the extremities of the four prime radii are singular points.

In concluding for the present it behoves me to state that one step has been omitted in the foregoing paper*, viz. the actual performance of the eliminations which lead to the rectilinear equation to the wave-surface. But Mr Archibald Smith's elegant and brief Memoir in the *Cambridge Philosophical Transactions*† of last year leaves nothing to be desired further on that head.

[* See below, p. 27. Ed.] [† Vol. vi. Also *Phil. Mag.* April, 1838, p. 335. Ed.]

That I have not exhibited it in its proper place (Prop. 6) arises only from my respect to the principle of literary propriety. With this important blank supplied the Analytical Theory may be pronounced to be complete.

For all errors and imperfections in what precedes my excuse must be press of time and a total want of the materials to be derived from consulting works of reference.

Since writing the above I have had an opportunity of reading the paper of our living Laplace inserted as part of the Third Supplement to his System of Rays in the *Transactions* of the Royal Irish Academy, in which the principal foregoing results are obtained by aid of a more refined and transcendental analysis.

The nature of the four singular points is there discussed and the existence of four circles of plane contact demonstrated.

The former may be very easily shown thus: when ι_1 is very small $\iota_2 = 2e - \iota_1 \cos\psi$ very nearly, ψ denoting the inclination of the plane in which e is reckoned to the plane in which ι_1 is reckoned.

Hence

$$\left(\frac{1}{r}\right)^2 = \frac{1}{2}\left(\frac{1}{a^2} + \frac{1}{c^2}\right) - \frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{c^2}\right)\cos\{2e - \iota_1(\cos\psi \pm 1)\}$$

$$= \frac{1}{2}\left(\frac{1}{a^2} + \frac{1}{c^2}\right) - \frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{c^2}\right)\cos 2e - \frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{c^2}\right)\sin 2e(\cos\psi \pm 1)\iota_1$$

$$= \frac{1}{b^2} - \frac{1}{b^2 ac}\sqrt{\{(a^2 - b^2)(b^2 - c^2)\}}(\cos\psi \pm 1)\iota_1,$$

therefore

$$r = b\left\{1 + \frac{1}{2}(\cos\psi \pm 1)\left(1 - \frac{b^2}{a^2}\right)^{\frac{1}{2}}\left(\frac{b^2}{c^2} - 1\right)^{\frac{1}{2}}\iota_1\right\}.$$

Take ψ constant and let the abscissæ and ordinates be reckoned respectively along and perpendicular to the prime ray.

Then $\iota_1 = \frac{y}{x}$ nearly, and $r = \sqrt{(y^2 + x^2)} = x$,

or, if we change the origin to the other extremity of the prime ray,

$$\iota_1 = \frac{y}{b}, \quad r = b - x,$$

so that the equation becomes

$$-\frac{x}{y} = \frac{1}{2}(\cos\psi \pm 1)\sqrt{\left\{\left(1 - \frac{b^2}{a^2}\right)\left(\frac{b^2}{c^2} - 1\right)\right\}}.$$

Hence at each singular point the surface is touched by a cone, the equation to the generating line of which is given by the above, the extreme angle between it and the prime ray being

$$\cot^{-1}\left[\sqrt{\left\{\left(1-\frac{b^2}{a^2}\right)\left(\frac{b^2}{c^2}-1\right)\right\}}\right].$$

When $b=a$, ψ always $=\frac{\pi}{2}$ and the cone returns into a plane.

Again, let us suppose that the position of any perpendicular from the centre is given, and that of the corresponding radius vector required.

Let OA, OB* denote what we have termed the optic axes, but which it will be more agreeable to analogy to term the prime perpendiculars from centre, and let OP be the given normal. Take OQ, OR contiguous perpendiculars from centre in planes POQ, ROP, perpendicular to POA, POB respectively, then the inclination of the two former will be the same as that of the two latter, and may be termed μ.

Let ι_1, ι_2 now denote the angles POA, POB respectively, then

$$QOA=\iota_1, \qquad QOB=\iota_2+\delta\iota_2,$$
$$ROA=\iota_1+\delta\iota_1, \quad ROB=\iota_2.$$

The ray will be found by joining O with the intersection of three planes drawn at P, Q, R, perpendicular to OP, OQ, OR, respectively.

Now from Prop. 9 it appears that

$$OP=\sqrt{\left\{a^2\left(\sin\frac{\iota_1+\iota_2}{2}\right)^2+c^2\left(\cos\frac{\iota_1+\iota_2}{2}\right)^2\right\}},$$

using only one sign for the sake of simplicity, which we may do by throwing the ambiguity upon the *way* in which ι_1 or ι_2 is measured, also

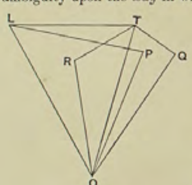


Fig. 5.

$$OQ=OP+\frac{d\,.\,OP}{d\iota_2}\delta\iota_2,$$

$$OR=OP+\frac{d\,.\,OP}{d\iota_1}\delta\iota_1.$$

Let $\delta\iota_1=\delta\iota_2$, then it is clear that $OQ=OR$, and the intersection of the two planes perpendicular to OQ, OR is therefore a line perpendicular to the plane QOR, and to the line which bisects the angle QOR.

In fact if we draw QT, RT perpendicular to OQ, OR respectively in the plane QOR, the intersection in question passes through T and is perpendicular to OT; also

$$OT=OQ\,.\,\sec\left(\tfrac{1}{2}ROQ\right)=OQ$$

to the first order of smallness.

* OA, OB are not expressed in the figure.

Now it is easy to see (just as on p. 16) that

$$ROP=\frac{\delta\iota_1}{\sin\mu},$$

and also

$$QOP=\frac{\delta\iota_2}{\sin\mu},$$

therefore $ROP=QOP$ and therefore POT is perpendicular to QOR.

Hence the problem is reduced to finding L the intersection of two lines TL, PL drawn in the *same* plane POT.

Now because OTL, OPL are each right angles, a circle may be made to pass through L, T, P, O.

Hence the angle

$$PLO=PTO=\tan^{-1}\frac{OP\times POT}{OT-OP}$$

$$=\tan^{-1}\frac{OP\times POR\,.\,\cos\frac{1}{2}\mu}{\frac{d\,.\,OP}{d\iota_2}\delta\iota_2}=\tan^{-1}\frac{OP\times\frac{\delta\iota_2}{\sin\mu}\cos\frac{1}{2}\mu}{\frac{d\,.\,OP}{d\iota_2}\delta\iota_2},$$

and

$$OL=OP\,.\,\sec POL.$$

Also the position of the plane POL is known, and therefore the radius is completely determined in magnitude and position.

It may be worth while also to remark that the above constructions enable us to form a series of equations between the magnitude of the radius and its inclinations to the two prime perpendiculars.

In fact, if we call π_1, π_2 the two inclinations in question

$$\cos\pi_1=\cos POL\cos\iota_1\pm\sin POL\sin\iota_1\,.\,\sin\frac{\mu}{2},$$

$$\cos\pi_2=\cos POL\cos\iota_2\mp\sin POL\sin\iota_2\,.\,\sin\frac{\mu}{2},$$

and of course if we call the angle between the two prime normals $2E$

$$\sin\frac{\mu}{2}=\sqrt{\frac{\sin\left(E+\frac{\iota_1+\iota_2}{2}\right)\sin\left(E-\frac{\iota_1+\iota_2}{2}\right)}{\sin\iota_1\sin\iota_2}}.$$

Cor. 1. When ι_1 or $\iota_2=0$, $\tan POL$ assumes the form $\frac{0}{0}$ which may be interpreted analogously to the method used in the reverse problem, but may be more elegantly illustrated by

Cor. 2. Which is that the meridian plane POT (that is, the plane in which both normal and radius lie) bisects the angle formed by ROP, QOP, and therefore that formed by the planes drawn through the normal and the two prime normals to which these two are perpendicular.

Now we have found (Cor. 4, page 18), that it also bisects the angle formed by the two planes passing through the radius and the two prime radii. Hence when the ray is given, we may find by the easiest geometry the normal and the tangent plane, and *vice versâ*.

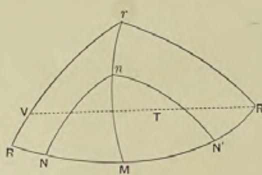


Fig. 6.

Thus suppose (N, N') (R, R') to be the projections of the prime perpendiculars and prime radii on a sphere concentric with the wave surface.

Let n be the projection of any given perpendicular on the same sphere; join nN, nN'; bisect NnN' by nM, which will be the meridian plane.

Draw from R', $R'TV$ perpendicular to nM and make $R'T = TV$. Produce RV to meet Mn in r, then $RrM = R'rM$, and therefore r is the projection of the radius. Just in the same way when r is given we may find n.

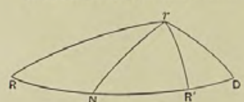


Fig. 7.

Now suppose n to come to N, then the position of the meridian plane nM becomes indeterminate, and r from a point becomes a locus, subject to the condition that $R'rN = RrN$. From r draw rD perpendicular to rN.

Then it is clear that because rN bisects RrR'

$$\frac{\sin RD}{\sin R'D} = \frac{\sin Rr}{\sin R'r} = \frac{\sin RN}{\sin R'N},$$

and therefore D is a fixed point and ND a fixed length, and

$$\cos rND = \tan rN \,.\, \cot ND\,;$$

therefore the projection of the locus of r upon a plane drawn at N perpendicular to the line joining N with the centre O is given by the equation

$$\rho = ON \,.\, \cot ND \,.\, \cos\theta,$$

N being the origin and the projection of ND the prime radius; which is the equation to a circle passing through N, and whose diameter $= ON \cot ND$.

Hence at the extremity of each prime perpendicular the tangent plane meets the surface in a circle passing through that extremity and whose radius $= \frac{1}{2} b \cot a$, a being to be found from the equation

$$\frac{\sin(2E + a)}{\sin a} = \frac{\sin(E + e)}{\sin(E - e)},$$

that is $$\tan(E + a) = (\tan E)^2 \cot e.$$

Just in the same way it may be shown that the trace of the perpendiculars to the tangent planes of the surface at the point where it is pierced by any prime radius upon a plane perpendicular to that radius at its extremity, is also a circle passing through it, and curved in an opposite direction from the circle of plane contact nearest to it.

Hence the enveloping cone at these points may be described as being perpendicular to the circular cone, formed by drawing lines from the centre to the above described circle; that is every generating line of the one will be perpendicular to the generating line which it meets of the other.

More generally it easily appears from fig. 6 that if a series of great circles (representing meridian planes) be taken intersecting the great circle $NRR'N'$ in a fixed point, a plane perpendicular to the radius passing through that point, will intersect the cone of rays as well as the cone of perpendiculars corresponding to those meridian planes, in two *circles*. So that there exist an indefinite number of *circular* cones of rays corresponding to *circular* cones of perpendiculars touching each other in a line lying in the plane containing the extreme axes, and having their circular sections perpendicular to that line.

The cusps are explained by the cone of *rays* degenerating into a right line, and the circles of plane contact by the cone of perpendiculars so degenerating.

Furthermore I observe in conclusion that when a ray is given it follows from the general geometrical construction above that there will be two meridian planes according as we take R with R', or with a point 180° from R', and consequently these two planes will be perpendicular to each other.

And similarly when a *normal* is given there will be two meridian planes perpendicular to each other.

Thus the planes passing through any radius and the two normals at the points where it pierces the wave surface, are *perpendicular* to each other, as are also the two planes passing through any normal and *its* two corresponding radii.

Moreover a glance at fig. 2 will show that the two lines of vibration corresponding to any front lie respectively in the two meridian planes passing through the perpendicular to that front or, in other words, the intersection of a plane drawn through either ray belonging to a front perpendicular thereunto is *always* a line of vibration in that front.

This has been noticed, I think, by Sir William Hamilton for the particular case of the singular points.

As two fronts belong to every ray, so two rays pertain to every front. And from what has been said above it appears that the two lines of vibration in any front are the projections of its two rays upon its own plane.

NOTE 1.

In the paper above, it is shown that the meridian plane, that is, the plane containing the ray and normal, always passes through a line of vibration in the corresponding point. Now the line of force called into action by a displacement in the line of vibration clearly lies in this very plane; for the resolved part of it lies in the line of vibration itself.

Harmony and analogy concur in suggesting that as two of these four lines are perpendicular to each other, so are also the other two, or in other words, that the ray is always perpendicular to the direction of unresolved force.

The following investigation verifies this conjecture.

Let x, y, z be the coordinates of a point taken at distance unity from the origin and in any line of vibration; then the cosines of the angles made by the line of force with the axes are as $a^2x : b^2y : c^2z$ respectively.

Let a be the inclination between the line of vibration and the line of force, then

$$\cos\omega = \frac{a^2x \,.\, x + b^2y \,.\, y + c^2z \,.\, z}{\sqrt{\{(a^4x^2 + b^4y^2 + c^4z^2)(x^2 + y^2 + z^2)\}}} = \frac{a^2x^2 + b^2y^2 + c^2z^2}{\sqrt{(a^4x^2 + b^4y^2 + c^4z^2)}}.$$

Let $\sqrt{(a^4x^2 + b^4y^2 + c^4z^2)} = P,$

then $P^2 = v^4 (\sec\omega)^2.$

Now let α, β, γ be the angles of inclination between the coordinate planes and the front in which the line of vibration lies, and λ some quantity to be determined. I have shown in Prop. 3 that if

$$\lambda \cos\alpha = (a^2 - v^2)\, x,$$

then will $\lambda \cos\beta = (b^2 - v^2)\, y,$

and $\lambda \cos\gamma = (c^2 - v^2)\, z;$

therefore $\lambda^2 = a^4x^2 + b^4y^2 + c^4z^2 - 2v^2 (a^2x^2 + b^2y^2 + c^2z^2) + v^4 = P^2 - v^4.$

Again,

$$\lambda^2 . \left(\frac{(\cos\alpha)^2}{(a^2 - v^2)^2} + \frac{(\cos\beta)^2}{(b^2 - v^2)^2} + \frac{(\cos\gamma)^2}{(c^2 - v^2)^2}\right) = x^2 + y^2 + z^2 = 1;$$

therefore $$\frac{1}{P^2 - v^4} = \frac{(\cos\alpha)^2}{(a^2 - v^2)^2} + \frac{(\cos\beta)^2}{(b^2 - v^2)^2} + \frac{(\cos\gamma)^2}{(c^2 - v^2)^2}.$$

Now $$\frac{1}{P^2 - v^4} = \frac{1}{v^4 (\sec\omega)^2 - v^4} = \frac{1}{v^4} (\cot\omega)^2.$$

And in Mr Smith's investigation of the form of the wave surface (already alluded to*) by great good fortune I find ready to my hand

$$\frac{(\cos\alpha)^2}{(a^2 - v^2)^2} + \frac{(\cos\beta)^2}{(b^2 - v^2)^2} + \frac{(\cos\gamma)^2}{(c^2 - v^2)^2} = \frac{1}{v^2 (r^2 - v^2)},$$

r being the radius vector to the point whose tangent plane is parallel to the point in question.

Hence

$$(\cot\omega)^2 = \frac{v^4}{v^2 (r^2 - v^2)} = \frac{v^2}{r^2 - v^2} = \frac{p^2}{r^2 - p^2},$$

p being the length of the perpendicular from the centre upon the tangent plane, for $p = v$.

Hence $(\cot\omega)^2 =$ the square of the cotangent of the angle between radius vector and normal.

Or, in other words, the line of force is as much inclined to the line of vibration as the ray is to the normal.

Now the normal is perpendicular to the line of vibration, and all four lines lie in one plane.

Therefore the ray is perpendicular to the line of force. Q. E. D.

I may be allowed to conclude this long paper with a summary of some of the most remarkable consequences which I have extricated from Fresnel's hypothesis.

(1) The two meridian planes corresponding to any given radius are perpendicular to each other†.

(2) So are the two corresponding to any given normal.

(3) Every meridian plane bisects the angle formed by two planes drawn through the radius and the two prime radii.

(4) It also bisects the angle formed by two planes drawn through the normal and the two prime normals.

(5) Each meridian plane contains one line of vibration and the corresponding line of force.

(6) The ray is perpendicular to the line of force.

All these conclusions, except the fourth, are, I believe, original.

* See above, p. 18.
† I have defined the meridian plane to be that which contains radius vector and normal belonging to the same point.

The theory of external and internal conical refraction follows immediately as a particular consequence from the third and fourth combined as already shown; the same propositions also enable us to draw a tangent plane to any point of the wave surface by mere Euclidean geometry. May not some of these conclusions serve to suggest to physical inquirers the question, Has the theory been started from the most natural point of view?

NOTE 2. *Investigation* of the Wave Surface.*

Since the appearance of the preceding parts, I have succeeded in completing the self-sufficiency of my method by deducing the equation to the wave surface from the expressions given in Prop. 5 for the angles between a front and the principal planes in terms of its two velocities. If these angles be ω, ϕ, ψ, and the two velocities v_1, v_2 we found

$$\cos\omega = \sqrt{\frac{(a^2-v_1^2)(a^2-v_2^2)}{(a^2-b^2)(a^2-c^2)}},$$

$$\cos\phi = \sqrt{\frac{(b^2-v_1^2)(b^2-v_2^2)}{(b^2-a^2)(b^2-c^2)}},$$

$$\cos\psi = \sqrt{\frac{(c^2-v_1^2)(c^2-v_2^2)}{(c^2-a^2)(c^2-b^2)}}.$$

Let the tangent plane to the wave surface be written

$$\frac{\cos\omega}{v_1}.x + \frac{\cos\phi}{v_1}.y + \frac{\cos\psi}{v_1}.z = 1, \qquad (\alpha)\dagger$$

then

$$\frac{d\frac{\cos\omega}{v_1}\ddagger}{d\left(\frac{1}{v_1}\right)^2}x + \frac{d\frac{\cos\phi}{v_1}}{d\left(\frac{1}{v_1}\right)^2}y + \frac{d\frac{\cos\psi}{v_1}}{d\left(\frac{1}{v_1}\right)^2}z = 0, \qquad (\beta)$$

$$\frac{d\cos\omega}{d(v_2)^2}x + \frac{d\cos\phi}{d(v_2)^2}y + \frac{d\cos\psi}{d(v_2)^2}z = 0. \qquad (\gamma)$$

Let

$$\frac{1}{v_1}\sqrt{\frac{(a^2-v_1^2)}{(a^2-v_2^2)}} = \xi, \qquad \sqrt{\{(a^2-b^2)(a^2-c^2)\}} = \frac{1}{A},$$

$$\frac{1}{v_1}\sqrt{\frac{(b^2-v_1^2)}{(b^2-v_2^2)}} = \eta, \qquad \sqrt{\{(b^2-a^2)(b^2-c^2)\}} = \frac{1}{B},$$

$$\frac{1}{v_1}\sqrt{\frac{(c^2-v_1^2)}{(c^2-v_2^2)}} = \zeta, \qquad \sqrt{\{(c^2-a^2)(c^2-b^2)\}} = \frac{1}{C},$$

* This investigation supplies the step which Mr Tovey was desirous should appear in the *Magazine*. [*Phil. Mag.* March, 1838, p. 261. ED.]

† In lieu of v_1 we might write v_2 in the denominator without affecting the result.

‡ Observe, that $\frac{\cos\omega}{v_1} = \frac{\sqrt{\left\{\left(\frac{a^2}{v_1^2}-1\right)(a^2-v_2^2)\right\}}}{\sqrt{\{(a^2-b^2)(a^2-c^2)\}}}$, and so on for the rest.

then equation (γ) becomes

$$A\xi x + B\eta y + C\zeta z = 0, \qquad (1)$$

and equation (β)

$$\frac{Aa^2}{\xi}x + \frac{Bb^2}{\eta}y + \frac{Cc^2}{\zeta}z = 0, \qquad (2)$$

and equation (α) may be written under two forms, viz.

$$(a^2-v_2^2)A\xi x + (b^2-v_2^2)B\eta y + (c^2-v_2^2)C\zeta z = 1, \qquad (3)$$

or

$$\left(\frac{a^2}{v_1^2}-1\right)\frac{A}{\xi}x + \left(\frac{b^2}{v_1^2}-1\right)\frac{B}{\eta}y + \left(\frac{c^2}{v_1^2}-1\right)\frac{C}{\zeta}z = 1. \qquad (4)$$

From (1)

$$A\xi x + B\eta y = -C\zeta z. \qquad (5)$$

From (2)

$$\frac{Aa^2}{\xi}x + \frac{Bb^2}{\eta}y = -\frac{Cc^2}{\zeta}z. \qquad (6)$$

From (3) and (1)

$$A(a^2-c^2)\xi x + B(b^2-c^2)\eta y = 1. \qquad (7)$$

From (2) and (4)

$$A(a^2-c^2)\frac{x}{\xi} + B(b^2-c^2)\frac{y}{\eta} = c^2. \qquad (8)$$

From (5) and (6)

$$C^2c^2z^2 - B^2b^2y^2 - A^2a^2x^2 = ABxy\left(a^2\frac{\eta}{\xi} + b^2\frac{\xi}{\eta}\right). \qquad (9)$$

From (7) and (8)

$$c^2 - B^2(b^2-c^2)^2y^2 - A^2(a^2-c^2)^2x^2 = ABxy\left(\frac{\eta}{\xi}+\frac{\xi}{\eta}\right)\times(a^2-c^2)(b^2-c^2). \qquad (10)$$

From (9) and (10)

$$AB(a^2-b^2)(a^2-c^2)(b^2-c^2)xy\frac{\xi}{\eta} = a^2c^2 - (a^2-c^2)(b^2-c^2)C^2c^2z^2$$
$$-\{a^2(b^2-c^2)^2 - b^2(a^2-c^2)(b^2-c^2)\}B^2y^2$$
$$-\{a^2(a^2-c^2)^2 - a^2(a^2-c^2)(b^2-c^2)\}A^2x^2 = a^2c^2 - c^2z^2 - c^2y^2 - a^2x^2. \qquad (11)$$

From (11), interchanging (a, x, ξ) with (b, y, η) we have

$$AB(b^2-a^2)(b^2-c^2)(a^2-c^2)xy\frac{\eta}{\xi} = b^2c^2 - c^2z^2 - c^2x^2 - b^2y^2. \qquad (12)$$

Finally, from (11) and (12) we have

$$\{a^2c^2 - (a^2-c^2)x^2 - c^2(x^2+y^2+z^2)\}\{b^2c^2 - (b^2-c^2)y^2 - c^2(x^2+y^2+z^2)\}$$
$$= (a^2-c^2)(b^2-c^2)x^2y^2,$$

that is

$$(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2) - a^2(b^2+c^2)x^2$$
$$-b^2(a^2+c^2)y^2 - c^2(b^2+a^2)z^2 + a^2b^2c^2 = 0$$

the equation required.

2.

ON THE MOTION AND REST OF FLUIDS.

[*Philosophical Magazine*, XIII. (1838), pp. 449—453.]

M. OSTROGRADSKY'S memoir on this subject inserted in the *Scientific Memoirs* seems to have excited much attention, and has been made the occasion of some annotations* by a distinguished writer in the *Philosophical Magazine*. Mr Ivory's recent papers in the same periodical must still more tend to invest with a new interest all such speculations. It seems to me desirable therefore to present the theory of fluids in all the simplicity of which it is susceptible.

I consider a fluid as a collection of particles subject to some law of relative position other than that of rigidity. These particles by their mutual actions maintain the connections of the system. As to the law of force between them we know nothing; but I assume it is a general principle of nature, that for each instant of time the sum of the internal actions (reckoned by the product of each particle into the square of the space due to the internal force acting on it) is a minimum. This in fact is Gauss's principle of least restraint. We may if we please split this principle into two parts; that is to say, assume that the internal system of forces is always such as if acting alone would keep the fluid at rest; and then again assume that any equilibrating system of forces must be subject to the law of virtual velocities. I say *assume*, because it is impossible *à priori* to prove this.

Lagrange's so-called demonstration is unworthy of his name, and (albeit sanctioned by the powerful oral authority of an ex-Cambridge Professor) contrary alike to sense and honesty. It is better therefore at once to proceed upon Gauss's principle. It might easily be shown that this is in effect tantamount in all cases to D'Alembert's and Lagrange's principles combined.

Before entering upon the investigation I may call attention to one point of great analytical interest, and relating to the difficult subject of the algebraical sign, viz. that if the density of a point (x, y) in any circumscribed space be expressed by the quantity $\frac{du}{dx} + \frac{dv}{dy}$ so that the mass is

$$\iint dx\,dy\left(\frac{du}{dx}\right) + \iint dx\,dy\left(\frac{dv}{dy}\right),$$

[* *Phil. Mag.* May, 1838, p. 385. ED.]

that is not equivalent to

$$\int (u\,dy + v\,dx),$$

that is if we please

$$\int\left(u\frac{dy}{ds} + v\frac{dx}{ds}\right) ds,$$

(where s is for clearness' sake and to avoid double limits taken an element of the bounding curve) as at first sight it might appear to be, but is in fact equal to

$$\int\left(u\frac{dy}{ds} - v\frac{dx}{ds}\right) ds.$$

I shall demonstrate this point in the next number* of the *Magazine*. It at first caused me some trouble in conducting the annexed inquiry. I shall also take occasion at some other time to revert to a new species (as I believe) of partial differential equations; that is to say, where there are fewer of them than of the principal variables, which may be called therefore Indeterminate Partial Differential Equations. A complete solution of one of these appears in the subjoined

Investigation.

For the sake of simplicity I take an incompressible fluid. The method is nowise different for a fluid of varying density.

Let Δx, Δy, Δz be any displacement undergone by a particle at the point x, y, z parallel to the axes x, y, z respectively; it is easily shown that to satisfy the condition of invariability of mass we must have

$$\frac{d\Delta x}{dx} + \frac{d\Delta y}{dy} + \frac{d\Delta z}{dz} = 0. \qquad (\alpha)$$

One relation between u, v, w the velocities parallel to x, y, z is obtained immediately by putting $u\delta t$, $v\delta t$, $w\delta t$, for Δx, Δy, Δz, which gives

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0, \qquad (1)$$

as usual.

Again, if X, Y, Z be the impressed forces, and X_1, Y_1, Z_1 the internal forces acting on any particle parallel to the axes, we have

$$X_1 + X = \frac{du}{dt} + \frac{du}{dx}u + \frac{du}{dy}v + \frac{du}{dz}w, \qquad (2)$$

$$Y_1 + Y = \frac{dv}{dt} + \frac{dv}{dx}u + \frac{dv}{dy}v + \frac{dv}{dz}w, \qquad (3)$$

$$Z_1 + Z = \frac{dw}{dt} + \frac{dw}{dx}u + \frac{dw}{dy}v + \frac{dw}{dz}w, \qquad (4)$$

from the mere geometry of the question.

[* p. 36, below. ED.]

Finally, Gauss's principle teaches us that

$$\iiint dxdydz\,\{X_1\Delta X_1 + Y_1\Delta Y_1 + Z_1\Delta Z_1\} = 0. \qquad (\beta)$$

Now
$$\frac{d(X+X_1)}{dx} + \frac{d(Y+Y_1)}{dy} + \frac{d(Z+Z_1)}{dz}$$
$$= \left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dy}\right)^2 + \left(\frac{dw}{dz}\right)^2 + 2\left\{\frac{dv}{dz}\frac{dw}{dy} + \frac{dw}{dx}\frac{du}{dz} + \frac{du}{dy}\frac{dv}{dx}\right\},$$

as appears from the equations (1), (2), (3), (4); and hence

$$\frac{d\Delta X_1}{dx} + \frac{d\Delta Y_1}{dy} + \frac{d\Delta Z_1}{dz} = 0,$$

the complete solution of which, free from the sign of integration, is

$$\Delta X_1 = \frac{d\psi}{dy} - \frac{d\phi}{dz},$$

$$\Delta Y_1 = \frac{d\omega}{dz} - \frac{d\psi}{dx},$$

$$\Delta Z_1 = \frac{d\phi}{dx} - \frac{d\omega}{dy},$$

ω, ϕ, ψ being any three independent functions of x, y, z.

On substituting these values in equation (β) we obtain

$$\iiint dxdydz\left\{X_1\frac{d\psi}{dy} - Y_1\frac{d\psi}{dx}\right\} + \iiint dxdydz\left\{Y_1\frac{d\omega}{dz} - Z_1\frac{d\omega}{dy}\right\}$$
$$+ \iiint dxdydz\left\{Z_1\frac{d\phi}{dx} - X_1\frac{d\phi}{dz}\right\} = 0.$$

This may be put under the form

$$\int dz \iint dxdy\left\{\frac{d}{dy}(\psi X_1) - \frac{d}{dx}(\psi Y_1)\right\}$$
$$+ \int dx \iint dydz\left\{\frac{d}{dz}(\omega Y_1) - \frac{d}{dy}(\omega Z_1)\right\}$$
$$+ \int dy \iint dzdx\left\{\frac{d}{dx}(\phi Z_1) - \frac{d}{dz}(\phi X_1)\right\}$$
$$- \iiint dxdydz\,.\,\psi\left(\frac{dX_1}{dy} - \frac{dY_1}{dx}\right)$$
$$- \iiint dxdydz\,.\,\omega\left(\frac{dY_1}{dz} - \frac{dZ_1}{dy}\right)$$
$$- \iiint dxdydz\,.\,\phi\left(\frac{dZ_1}{dx} - \frac{dX_1}{dz}\right) = 0.$$

Here it must be remembered that ω, ϕ, ψ are perfectly independent of each other. Also the values of the three first written quantities depend upon the values of X_1, Y_1, Z_1 at the bounding surface; the values of the three last-written depend upon the general values of X_1, Y_1, Z_1. It is clear therefore that each system of three equations and each member of each system must be separately zero.

The three latter equations give

$$\left.\begin{aligned}\frac{dX_1}{dy} - \frac{dY_1}{dx} &= 0\\ \frac{dY_1}{dz} - \frac{dZ_1}{dy} &= 0\\ \frac{dZ_1}{dx} - \frac{dX_1}{dz} &= 0\end{aligned}\right\}. \qquad (\gamma)$$

The three former require that for each section of the surface parallel to the plane xy

$$\int \psi(X_1dx + Y_1dy) = 0,$$

for each section parallel to yz

$$\int \omega(Y_1dy + Z_1dz) = 0,$$

for each section parallel to zx

$$\int \phi(Z_1dz + X_1dx) = 0 \qquad (\delta)^*$$

and these equations are to hold good whatever ψ, ϕ, ω may be. From the equations (γ) we derive

$$X_1dx + Y_1dy + Z_1dz = df, \qquad (5)$$

from equations (δ) we obtain

f = constant for all points in any section of the bounding surface parallel to the plane of xy,

f = constant for all points in any section of the bounding surface parallel to the plane of yz,

f = constant for all points in any section of the bounding surface parallel to the plane of zx.

Now by drawing through all the points in a plane parallel to xy, planes parallel to yz, we may cover the whole surface; hence f is constant all over the surface bounding the fluid.

* See remark at introduction.

Therefore $$X_1 dx + Y_1 dy + Z_1 dz = 0, \qquad (6)$$

for all variations of dx, dy, dz taken upon the surface.

The equations (1, 2, 3, 4, 5, 6) are coincident with those obtained by the usual method; with this difference, that X_1, Y_1, Z_1 here take the place of

$$-\frac{dp}{dx},\ -\frac{dp}{dy},\ -\frac{dp}{dz}.$$

Thus then we have obtained all the conditions requisite for determining the motion of fluids from the universal principle of least constraint conjoined with the specific character of the system in question.

General Remarks.

In the case of equilibrium, that is in the case where no particle moves, we have $X_1 + X = 0$, $Y_1 + Y = 0$, $Z_1 + Z = 0$. Hence $Xdx + Ydy + Zdz$ is a complete differential always and zero for the surface.

The above results have been obtained upon the principles of the differential calculus, and the continuity of the forces has been tacitly assumed. If now we were to suppose forces of finite magnitude (as compared with the *whole sum* acting upon the entire system) to be applied to a layer of single particles or to a layer of a thickness of the same order of magnitude as the distances between the particles themselves, (which has been treated as an infinitesimal) it would appear that our results would be no longer applicable, just in the same manner as it would be erroneous to apply the principle of *vis-viva* (for example) without modification, to the case of impulsive forces, because we had deduced it by the calculus in the case of the motion being continuous. Hence the above equations ought not strictly to apply to the motion or rest of a fluid *contained between physical surfaces*; for the pressure afforded by these surfaces, whatever its actual value may be, we know *à priori* is commensurable with the whole amount of force acting on the fluid; but the immediate application of this pressure (*alias* repulsive force) is confined to the bounding layer of fluid particles, or at most extends to a distance bearing a low ratio to the distances between the particles themselves.

Accordingly, to the non-applicability of the equations for free fluids to the case of fluids confined at the boundaries, and to an independent investigation upon the minimum principle for this class of problems, it is, that I look for the true explanation of the phenomena of capillary attraction (vulgarly so called).

3.

ON THE MOTION AND REST OF RIGID BODIES.

[*Philosophical Magazine*, XIV. (1839), pp. 188—190.]

In the subjoined investigation, which, as far as I know, is my own, I apply the same method to rigid as in the preceding paper I applied to fluid systems.

Let x, y, z be the coordinates of any particle in a rigid body; x', y', z' the coordinates of some other particle, and let

$$x' = x + h,\quad y' = y + k,\quad z' = z + l.$$

Call Δx, Δy, Δz the increments which x, y, z receive after the lapse of a small interval of time; so that terms in which they enter in two or more dimensions may be neglected.

Then
$$\Delta(x') = \Delta x + \frac{d\Delta x}{dx}h + \frac{d\Delta x}{dy}k + \frac{d\Delta x}{dz}l + P,$$
$$\Delta(y') = \Delta y + \frac{d\Delta y}{dx}h + \frac{d\Delta y}{dy}k + \frac{d\Delta y}{dz}l + Q,$$
$$\Delta(z') = \Delta z + \frac{d\Delta z}{dx}h + \frac{d\Delta z}{dy}k + \frac{d\Delta z}{dz}l + R,$$

P, Q, R containing binary and higher combinations of h, k, l, which we shall have no occasion to express.

At the commencement of the interval the squared distance of the two particles was $(x'-x)^2 + (y'-y)^2 + (z'-z)^2$; at the end of the interval the distance squared is

$$(x' - x + \Delta(x') - \Delta x)^2 + (y' - y + \Delta(y') - \Delta y)^2 + (z' - z + \Delta(z') - \Delta z)^2,$$

and these two expressions must be the same by the conditions of rigidity whatever h, k, and l may be; that is

$$h^2 + k^2 + l^2 = \left(h + \frac{d\Delta x}{dx}h + \frac{d\Delta x}{dy}k + \frac{d\Delta x}{dz}l + P\right)^2$$
$$+ \left(k + \frac{d\Delta y}{dx}h + \frac{d\Delta y}{dy}k + \frac{d\Delta y}{dz}l + Q\right)^2$$
$$+ \left(l + \frac{d\Delta z}{dx}h + \frac{d\Delta z}{dy}k + \frac{d\Delta z}{dz}l + R\right)^2,$$

for all values of h, k, and l.

Hence rejecting infinitesimals of the second order and equating to zero separately the coefficients of h^2, k^2, l^2, and of kl, lh, hk, we have

$$\frac{d\Delta x}{dx} = 0. \quad (a) \qquad \frac{d\Delta y}{dz} + \frac{d\Delta z}{dy} = 0. \quad (d)$$

$$\frac{d\Delta y}{dy} = 0. \quad (b) \qquad \frac{d\Delta z}{dx} + \frac{d\Delta x}{dz} = 0. \quad (e)$$

$$\frac{d\Delta z}{dz} = 0. \quad (c) \qquad \frac{d\Delta x}{dy} + \frac{d\Delta y}{dx} = 0. \quad (f)$$

By differentiating (d), (e), (f) with respect to z, x, y respectively, and substituting from (a), (b), (c), we obtain

$$\frac{d^2\Delta y}{dz^2} = 0, \quad \frac{d^2\Delta z}{dx^2} = 0, \quad \frac{d^2\Delta x}{dy^2} = 0.$$

By differentiating the same with respect to y, z, x respectively, and proceeding as before, we have

$$\frac{d^2\Delta z}{dy^2} = 0, \quad \frac{d^2\Delta x}{dz^2} = 0, \quad \frac{d^2\Delta y}{dx^2} = 0.$$

Thus, then, we have

$$\frac{d\Delta x}{dx} = 0, \quad \frac{d^2\Delta x}{dy^2} = 0, \quad \frac{d^2\Delta x}{dz^2} = 0,$$

$$\frac{d\Delta y}{dy} = 0, \quad \frac{d^2\Delta y}{dz^2} = 0, \quad \frac{d^2\Delta y}{dx^2} = 0,$$

$$\frac{d\Delta z}{dz} = 0, \quad \frac{d^2\Delta z}{dx^2} = 0, \quad \frac{d^2\Delta z}{dy^2} = 0,$$

therefore

$$\Delta x = A + By + Cz, \quad (o)$$

$$\Delta y = D + Ez + Fx, \quad (p)$$

$$\Delta z = G + Hx + Ky, \quad (q)$$

A, B, C, D, E, F, being constant for a *given instant* of time; between which by virtue of the equations (d), (e), (f), we have the relations

$$E + K = 0, \qquad H + C = 0, \qquad B + F = 0.$$

If we call u, v, w the three component velocities of the particles at x, y, z parallel to the three axes, and X_1, Y_1, Z_1, the three internal forces, it is at once seen that u, v, w, as also ΔX_1, ΔY_1, ΔZ_1 must be subject to the same equations as limit Δx, Δy, Δz; so that

$$u = a + \gamma y - \beta z, \quad (1)$$

$$v = b + \alpha z - \gamma x, \quad (2)$$

$$w = c + \beta x - \alpha y, \quad (3)$$

$$\Delta X_1 = a_1 + \gamma_1 y - \beta_1 z, \quad (h)$$

$$\Delta Y_1 = b_1 + \alpha_1 z - \gamma_1 x, \quad (j)$$

$$\Delta Z_1 = c_1 + \beta_1 x - \alpha_1 y. \quad (k)$$

Also if X, Y, Z be the impressed forces, we have

$$X_1 + X = \frac{du}{dt}, \quad (4)$$

$$Y_1 + Y = \frac{dv}{dt}, \quad (5)$$

$$Z_1 + Z = \frac{dw}{dt}. \quad (6)$$

And by Gauss's principle, calling m the mass of the particle at x, y, z,

$$\Delta\Sigma m\,(X_1^2 + Y_1^2 + Z_1^2) = 0.$$

Hence equating separately to zero the coefficients of a_1, b_1, c_1 and of α_1, β_1, γ_1 in the quantity $\Sigma m\,(X_1\Delta X_1 + Y_1\Delta Y_1 + Z_1\Delta Z_1)$ we have

$$\left.\begin{aligned} \Sigma m\, X_1 &= 0 \\ \Sigma m\, Y_1 &= 0 \\ \Sigma m\, Z_1 &= 0 \\ \Sigma m\,(Z_1 y - Y_1 z) &= 0 \\ \Sigma m\,(X_1 z - Z_1 x) &= 0 \\ \Sigma m\,(Y_1 x - X_1 y) &= 0 \end{aligned}\right\}. \quad (7\text{—}12)$$

Lastly, we have the equations

$$u = \frac{dx}{dt}, \quad (13)$$

$$v = \frac{dy}{dt}, \quad (14)$$

$$w = \frac{dz}{dt}. \quad (15)$$

From the fifteen equations marked (1) to (15), the motion may be determined by assigning the position of each particle at the end of the time t in terms of its three initial coordinates, its three initial velocities, and the initial values of the nine quantities

$$\Sigma mx, \quad \Sigma myz, \quad \Sigma mx^2,$$

$$\Sigma my, \quad \Sigma mzx, \quad \Sigma my^2,$$

$$\Sigma mz, \quad \Sigma mxy, \quad \Sigma mz^2.$$

In the case of rest $X_1 = -X$, $Y_1 = -Y$, $Z_1 = -Z$, and the equations (7) to (12) inclusively taken, express the conditions of equilibrium.

The equations (o), (p), (q), which have been obtained from conditions *purely geometrical*, establish the well-known but interesting and *not obvious* fact, that any *small* motion of a rigid body may be conceived as made up of a motion of translation and a motion about *one* axis.

4.

ON DEFINITE DOUBLE INTEGRATION, SUPPLEMENTARY TO A FORMER PAPER ON THE MOTION AND REST OF FLUIDS.

[*Philosophical Magazine*, XIV. (1839), pp. 298—300.]

IN a paper on Fluids which appeared in the December Number of this *Magazine*, I had occasion to remark, that the mass of an area having at the point (x, y) a density $\frac{du}{dx} + \frac{dv}{dy}$ could be expressed by the simple formula

$$\int_l^0 \left\{u\frac{dy}{ds} - v\frac{dx}{ds}\right\} ds;$$

l being the length, and ds an element of the bounding curve: this may be thought to require some explanation.

(1) Let $APBq$ represent any oval; PpL, QqM any two contiguous ordinates cutting the curve in Pp, Qq respectively, AC, BD the two extreme tangents parallel to Oy, and ρ the density at any point (x, y). The expression $\iint\rho dx dy$ will serve to denote the mass of the oval area $APBq$, and the limits may be twice taken, that is, (i) the two values of y corresponding to any one of x; and (ii) the two values of x corresponding to C and D. This method is in fact tantamount to taking the sum of the columns Pp qQ; but this is not necessary, for $APBq$ may be considered as the *algebraical* sum of the mixtilinear area $APQBDC$, and the mixtilinear area $BDCApq$, or (if any line $O'C'D'$ be drawn parallel to $OCLMD$) of $APQBD'C'$ and $BD'C'Apq$.

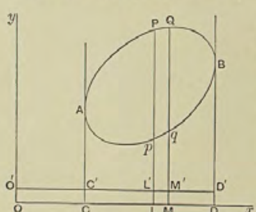


Fig. 1.

Thus then the mass $= \int dx\,(\int\rho dy)$, $\int\rho dy$ being left indeterminate, and the extremity of x travelled round from C to D, and back again from D to C.

This will be better expressed by transforming the variable, and summing with respect to some quantity, such as the arc of the curve, which continuously increases, or if we please, with respect to θ, the angle subtending any point taken within the curve.

The mass is then

$$= \pm\int_{2\pi}^0 d\theta \left\{(\int\rho dy)\frac{dx}{d\theta}\right\};$$

always remembering that *no* constant need be added to $\int\rho dy$, and that the doubtful sign arises from the choice of ways in which θ may be measured round. If the area be not included by one line; but by several, as for example, by a curve and a right line, the above integral, if broken up into as many parts as there are breaches of continuity, will still apply.

(2) Let us suppose that we have two areas exactly coinciding with, and overlapping one another; but the density of the one at (x, y) to be ρ, and of the other ρ'.

Let the mass of the first be treated as the sum of columns parallel to Oy, and that of the second as the sum of columns parallel to Ox.

The one will be represented by

$$\pm\int_{2\pi}^0 d\theta\,(\int\rho dy)\frac{dx}{d\theta},$$

the other will be represented by

$$\pm\int_{2\pi}^0 d\theta\,(\int\rho' dx)\frac{dy}{d\theta},$$

and the sum of the two, or the joint mass, by

$$\pm\int_{2\pi}^0 d\theta\,\{\int\rho dy\}\frac{dx}{d\theta} \pm \int_{2\pi}^0 d\theta\,\{\int\rho' dx\}\frac{dy}{d\theta}.$$

So long as these two operations are performed separately, the doubtful signs may be preserved in each term, because s need not be travelled round in the same direction for the two summations; but if we perform the second integration conjointly for the two masses, their sum

$$= \pm\int_{2\pi}^0 d\theta \left\{(\int\rho dy)\frac{dx}{d\theta} \pm ?\,(\int\rho' dx)\frac{dy}{d\theta}\right\},$$

the mark of interrogation denoting that *one or the other*, but not *either* of the signs $\pm$ must be used, and the question is, which?

This will be answered by taking different points in the bounding line which may be continuous or not. Now every line returning into itself, whether continuous or not, will naturally divide with respect of any given system of axes, into at most four parts, or sets of parts; two in which dx and dy both increase or both decrease, and two in which one increases and the other decreases.

Take P_1, P_2, P_3, P_4, any points in the four quadrants respectively, it will be observed that,

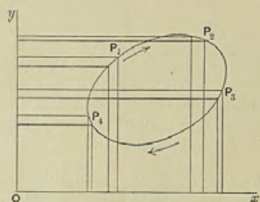


Fig. 2.

At P_1 the ρ column enters additively, and the ρ' column subtractively.

At P_2 both columns are additive.

At P_3 the ρ' column is additive and the ρ column subtractive.

At P_4 both columns enter subtractively.

Again, reckoning round in the direction of the arrows,

At P_1, x and y are both increasing.

At P_2, x is increasing and y decreasing.

At P_3, x and y both decrease.

At P_4, x is decreasing and y increasing.

Thus when $\int\rho dy$ and $\int\rho' dx$ are affected with the same signs, dx and dy are of opposite signs; and when $\int\rho dy$, $\int\rho' dx$ are of opposite signs, dx and dy are of the same sign.

Hence it appears that the mass of the area, whose density at (x, y) is $\rho + \rho'$, is capable of being represented by

$$\pm \int_{2\pi}^{0} d\theta \left\{ (\int \rho dy) \frac{dx}{d\theta} - (\int \rho' dx) \frac{dy}{d\theta} \right\}.$$

5.

ON AN EXTENSION OF SIR JOHN WILSON'S THEOREM TO ALL NUMBERS WHATEVER.

[*Philosophical Magazine*, XIII. (1838), p. 454.]

THE annexed original theorem in numbers will serve as a pendant to the elegant discovery announced by the ever-to-be-lamented and commemorated Horner*, with his dying voice, in your valued pages†.

THEOREM.

If N be any number whatever and

$$p_1, p_2, p_3 \ldots\ldots p_c$$

be all the numbers less than N and prime to it, then either

$$p_1 \cdot p_2 \cdot p_3 \ldots\ldots p_c + 1,$$

or else

$$p_1 \cdot p_2 \cdot p_3 \ldots\ldots p_c - 1,$$

is a multiple of N.

6.

NOTE TO THE FOREGOING.

[*Philosophical Magazine*, XIV. (1839), pp. 47, 48.]

I HAVE to apologize for calling "original" (in the last Number of the *Magazine*) the theorem of numbers which I termed "a pendant to Horner's theorem." This Mr Ivory has done me the honour to inform me may be found in Gauss's *Disquisitiones Arithmeticæ*, p. 76. As Horner's extension of Fermat's theorem suggested this extension of Sir John Wilson's to me, so I concluded that had this extension of Wilson's been known to the world it would naturally have suggested his to Horner. No acknowledgment of this kind having been made, I took it for granted that the theorem I gave was new. Undoubtedly had Mr Horner been aware of Gauss's theorem he would have made mention of it.

I take this opportunity of adding that my acquaintance with Gauss's principle‡ has not been derived from the study of his works, but from a casual statement of it in an English work, *Dynamics*, by Mr Earnshaw, of St John's College, Cambridge.

* Horner's proof is highly valuable as a novel and highly ingenious form of reasoning, but his theorem may be deduced with infinitely more ease and brevity from Fermat's than he seems to have been aware of.

[† *Phil. Mag.* Vol. XI. p. 456. ED.]

[‡ See p. 28 above. ED.]

7.

ON RATIONAL DERIVATION FROM EQUATIONS OF COEXISTENCE, THAT IS TO SAY, A NEW AND EXTENDED THEORY OF ELIMINATION*. PART I.

[*Philosophical Magazine*, XV. (1839), pp. 428—435.]

ANY number of equations existing at the same time and having the same quantities repeated, may be termed equations of coexistence: in the *present* paper we consider only the case of *two* algebraical equations:

$$x^m + a_1x^{m-1} + a_2x^{m-2} + \ldots\ldots + a_m = 0,$$

$$x^n + b_1x^{n-1} + b_2x^{n-2} + \ldots\ldots + b_n = 0.$$

The above being "equations of coexistence," x is called "the repeating term."

If we suppose the equation

$$c_0x^r + c_1x^{r-1} + c_2x^{r-2} + \ldots\ldots + c_r = 0$$

to be capable of being deduced from the two above, and, therefore, necessarily implied by them, this will be called "a Particular Derivative" from the equations of coexistence, of the rth degree, (r being supposed less than m and n†, and the coefficients being *rational* functions of the coefficients of the equations of coexistence).

There will be an *indefinite* number in general of such derivatives, and the form involving arbitrary quantities which includes them *all* is called "the general derivative of the rth degree."

Any "Particular Derivative," in which the terms are all integral, numerically as well as literally speaking, is called an "Integral Derivative."

That "Integral Derivative" of any given degree in which the literal parts of the coefficients are of the *lowest possible dimensions*‡, and the numerical parts as low as they can be made, is called the "Prime Derivative"

[* The results of this and some following papers were repeated, with demonstrations, in the paper "On a Theory of the Syzygetic Relations of two rational integral functions comprising an application to the Theory of Sturm's Functions, and that of the greatest Algebraical Common Measure," *Phil. Trans. Royal Soc.* Vol. CXLIII., Part I. pp. 407—548, 1853. See below Section II. Art. (16) of that paper. ED.]

† This restriction upon the value of r is not essentially requisite, and is only introduced to keep the attention fixed upon the particular objects of this first Part.

‡ Of course the dimensions of the coefficients in the equations of coexistence are to be understood as denoted by the indices subscribed.

of that degree. So that there is nothing left ambiguous in the prime derivative save the sign.

The "Derivative by succession" is that particular derivative which is obtained by performing upon the equations of coexistence the process commonly employed for the discovery of the greatest common measure, and equating the *successive remainders* to zero.

To express the product of the sums formed by adding each of one row of quantities to each of another row, we simply write the one row above the other; a notation clearly capable of extension to any number of rows, which would not be the case if we spoke of *differences* instead of *sums**.

THEOREM 1.

Let $h_1, h_2, \ldots h_m$, be the roots of one equation of coexistence, $k_1, k_2, \ldots k_n$, the roots of the other. The general derivative of the rth degree is represented by

$$\Sigma\left(SR(h_1, h_2, h_3 \ldots h_r)\{(x-h_1)(x-h_2)\ldots(x-h_r)\} \times \begin{Bmatrix} h_{r+1}, h_{r+2} \ldots h_m \\ -k_1, -k_2 \ldots -k_n \end{Bmatrix}\right) = 0.$$

$SR(h_1, h_2, h_3 \ldots h_r)$ denoting any *symmetrical rational* (integral or fractional) function of $h_1, h_2 \ldots h_r$;

$$\begin{Bmatrix} h_{r+1}, h_{r+2} \ldots h_m \\ -k_1, -k_2 \ldots -k_n \end{Bmatrix}$$

being to be interpreted as above explained, and Σ of course including as many terms as there are ways of putting n things r and r together†.

A form tantamount to the above, and which may be substituted for it, is its analogue,

$$\Sigma\left(SR(k_1, k_2 \ldots k_r)\{(x-k_1)(x-k_2)\ldots(x-k_r)\} \times \begin{Bmatrix} k_{r+1}, k_{r+2} \ldots k_n \\ -h_1, -h_2 \ldots -h_m \end{Bmatrix}\right) = 0.$$

When $r = 0$ the theorem gives simply

$$\begin{Bmatrix} h_1, h_2 \ldots h_m \\ -k_1, -k_2 \ldots -k_n \end{Bmatrix} = 0,$$

and is coincident with that given by Bezout in his Theory of *Elimination*.

* The wider views which I have attained since writing the above, and which will be developed in a future paper, lead me to request that this notation may be considered only as temporary. It would have been more in accordance with these views to have used the two rows to denote products of differences than of sums. But a change now in the text would be very apt to cause errors in printing.

† The general derivative may clearly be expressed also by the sum of any two particular derivatives affected respectively with arbitrary rational coefficients. The equivalency of an arbitrary *function* to two arbitrary *multipliers* is very remarkable, and analogous to what occurs in the solution of certain differential equations.

Subsidiary Theorem (A).

If $h_1, h_2 \ldots h_m$ be the roots of the equation

$$x^m + a_1 x^{m-1} + a_2 x^{m-2} + \ldots\ldots + a_m = 0,$$

and if

$$e^m + a_1 e^{m-1} + a_2 e^{m-2} + \ldots\ldots + a_m - u = 0,$$

then

$$\Sigma \frac{h_1^r}{(h_1 - h_2)(h_1 - h_3) \ldots (h_1 - h_m)} = \frac{1}{r+1} \frac{d}{du} \Sigma (e^{r+1}),$$

u being made zero after differentiation.

Cor. If $R(h_1)$ denote any integral rational function of h_1, then

$$\Sigma \frac{R(h_1)}{(h_1 - h_2)(h_1 - h_3) \ldots (h_1 - h_m)}$$

is *always* integral and is *zero* when the dimensions of $R(h_1)$ fall short of $(m-1)$.

Subsidiary Theorem (B).

$$\Sigma \frac{SR(h_1, h_2 \ldots h_r)}{\left\{\begin{matrix} h_1, h_2 \ldots h_r \\ -h_{r+1}, -h_{r+2} \ldots -h_m \end{matrix}\right\}}$$

can be expressed by the sum of terms, each of which is the product of series of the form

$$\Sigma \frac{R(h_1)}{(h_1 - h_2)(h_1 - h_3) \ldots (h_1 - h_m)},$$

it is *always* integral, and when the dimensions of the numerator fall short of $(m-r)r$ it vanishes*.

Subsidiary Theorem (C).

The *only* modes of satisfying the equation

$$\Sigma \{f(h_1, h_2 \ldots h_r) \times SR(h_1, h_2 \ldots h_r)\} = 0,$$

for all forms of the latter factors short of $(m-r)(n-r)$ dimensions, are to put $f(h_1, h_2 \ldots h_r) = 0$, or else

$$f(h_1, h_2 \ldots h_r) = \frac{\text{constant}}{\left(\begin{matrix} h_1, h_2 \ldots h_r \\ -h_{r+1}, -h_{r+2} \ldots -h_m \end{matrix}\right)}.$$

* It may be remarked also in passing, that any term in the numerator which contains any one power not greater than $m - 2r$ may be neglected and thrown out of calculation. Moreover, an analogous proposition may be stated of fractions in the denominators of which *any number* of rows are written one under the other; see the first note, page 41.

Theorem 2.

By virtue of the subsidiary theorem (B), the two equations

$$\pm \Sigma \left((x - h_1)(x - h_2) \ldots (x - h_r) \times \frac{\left\{\begin{matrix} h_{r+1}, h_{r+2} \ldots h_m \\ -k_1, -k_2 \ldots -k_n \end{matrix}\right\}}{\left\{\begin{matrix} h_{r+1}, h_{r+2} \ldots h_m \\ -h_1, -h_2 \ldots -h_r \end{matrix}\right\}} \right) = 0,$$

$$\pm \Sigma \left((x - k_1)(x - k_2) \ldots (x - k_r) \times \frac{\left\{\begin{matrix} k_{r+1}, k_{r+2} \ldots k_n \\ -h_1, -h_2 \ldots -h_m \end{matrix}\right\}}{\left\{\begin{matrix} k_{r+1}, k_{r+2} \ldots k_n \\ -k_1, -k_2 \ldots -k_r \end{matrix}\right\}} \right) = 0,$$

are each integer derivatives of the rth degree.

Theorem 3.

And by virtue of the subsidiary theorem (C), the two above equations are the "Prime Integer Derivatives," and are exactly identical with each other.

Cor. 1. The leading coefficient of the "prime derivative" of the rth degree is always of $(m-r)(n-r)$ dimensions.

Cor. 2. If P_r be the prime derivative of the rth degree and if $(X=0, Y=0)$ be the two equations of coexistence, and λ_r, μ_r the two "prime constituents of multiplication" to the said derivative, that is if λ_r and μ_r satisfy the equation $\lambda_r X + \mu_r Y = P_r$, then the coefficient of the leading terms in λ_r and in μ_r is of $(m-r-1)(n-r-1)$ dimensions.

Theorem 4.

The "Prime Derivative" of any given degree is an exact factor of the "derivative by succession," of the same degree. The quotient resulting from striking out this factor is called "the quotient of succession."

Theorem 5.

If L_1, L_2, L_3, &c., be the leading coefficients of the derivatives occurring first, second, third, &c., in order after the equations of coexistence, and if Q_1, Q_2, Q_3, &c., represent the first, second, third, "quotients of succession" reckoned in the same order, then

$$Q_1 = 1,$$

$$Q_2 = \frac{1}{L_1^2},$$

$$Q_3 = \frac{L_1^4}{L_2^2},$$

$$Q_4 = \frac{L_2^4}{L_1^4 L_3^2},$$

and in general

$$Q_{2n} = \frac{L_2^4 L_4^4 \dots L_{2n-4}^4 L_{2n-2}^4}{L_1^4 L_3^4 \dots L_{2n-3}^4 L_{2n-1}^2} *,$$

$$Q_{2n+1} = \frac{L_1^4 L_3^4 \dots L_{2n-3}^4 L_{2n-1}^4}{L_2^4 L_4^4 \dots L_{2n-2}^4 L_{2n}^2} \dagger.$$

COR. Hence, in place of Sturm's auxiliary functions, we may substitute the functions derived from the equations of coexistence $\left(fx = 0, \frac{dfx}{dx} = 0\right)$ according to Theorem 2, due regard being had to the sign.

Scholium. Hitherto it has been supposed that the values of the coefficients in the equations of coexistence are independent of one another, but particular relations may be supposed to exist which shall cause the leading terms given by Theorem 2 to vanish, giving rise to anormal or singular primes, as they may be called, of the degree r of fewer than $(m-r)(n-r)$ dimensions. The theory of this, the failing case (so to say), is highly interesting, and I have already discovered the law of formation for the quotients of succession on the supposition of *any number* of primes vanishing consecutively; but I forbear to vex the patience of my reader further, the more so, as I hope soon to be able to present a complete memoir, with all the steps here indicated filled up, and numerous important additions, (the perfect image of which this is but a rough mould), as homage to the learned and illustrious society which has lately done me the honour of admitting me into its ranks.

Why this has not already been done must be excused, by the fact of the theory having suggested itself abroad in the intervals of sickness‡. Yet thus much will I add in general terms, namely, that as many primes as vanish consecutively, so many units must be added to the index 2 of the accessions

* That the appearance of the index 4 may not startle, let my reader bear in mind that there are what may be termed secondary derivatives of succession for every degree appearing in the process of successive division.

† The prime derivatives must be capable of yielding an *internal* evidence of the truth of Sturm's theorem. In fact, for the case of all the roots being possible, a little consideration will serve to show that the leading term of each prime derivative of the equation $\left\{fx \frac{dfx}{dx}\right\} = 0$ will consist of a series of fractions, each of which fractions is, *numerically* speaking, of the *same sign*.

‡ The reflections which Sturm's memorable theorem had originally excited, were revived by happening to be present at a sitting of the French Institute, where a letter was read from the Minister of Public Instruction, requesting an opinion upon the expediency of forming tables of elimination between two equations as high as the 5th or 6th degree containing one repeating term. The offer was rejected, on the ground of the excessive labour that would be required. I think that this has been very much overrated; and probably many will be of the same opinion who have dwelt upon the fact that no numerical quantity will occur in the result higher than the highest index of the repeating term. Would it not redound to the honour of British science that some painstaking ingenious person should gird himself to the task? and would not this be a proper object to meet with encouragement from the Scientific Association of Great Britain?

received in the numerator and denominator of the subsequent quotient; and in the quotient after that, it is not the square of the leading term of the penultimate prime,—but the product of this term by the leading term of that anormal prime of the same degree which has the lowest dimensions,—that finds its way into the numerator. The rest of the formation remaining undisturbed, *unless* and *until* a new failure have taken place.

NOTE ON STURM'S THEOREM.

When one of the equations of coexistence is the differential coefficient with respect to the repeated term of the other, the prime derivatives given in Theorem 2 which coincide in this case with Sturm's auxiliary functions *reduced* to their lowest terms, may be exhibited under an integral *aspect*.

Let SPD intimate that the squared product of the differences is to be taken of the quantities which follow it.

Let S_1 indicate the sum of the quantities to which it is prefixed.

S_2 the sum of the binary products.

S_3 the sum of the ternary products, and so on

Let $h_1, h_2 \dots h_n$ be the roots of any equation.

Then Sturm's last auxiliary function may be *replaced* by

$$SPD(h_1, h_2 \dots h_n).$$

The last but one may be replaced by

$$\Sigma SPD(h_1, h_2 \dots h_{n-1})\, x + \Sigma S_1(h_2, h_3 \dots h_{n-1})\, SPD(h_1, h_2 \dots h_{n-1}).$$

The one preceding by

$$\Sigma SPD(h_1, h_2 \dots h_{n-2})\, x^2 + \Sigma S_1(h_1, h_2 \dots h_{n-2})\, SPD(h_1, h_2 \dots h_{n-2})\, x + \Sigma S_2(h_1, h_2 \dots h_{n-2})\, SPD(h_1, h_2 \dots h_{n-2}),$$

and so on.

Thus then Sturm's rule for determining the absolute number of real roots in an equation is based wholly and solely upon the following

ALGEBRAICAL PROPOSITION.

If there be n quantities, real and imaginary, the imaginary ones entering in pairs, as many changes of sign as there are in the terms

$$\Sigma SPD(h_1, h_2),$$

$$\Sigma SPD(h_1, h_2, h_3),$$

$$\dots\dots\dots\dots$$

$$\Sigma SPD(h_1, h_2 \dots h_{n-1}),$$

$$\Sigma SPD(h_1, h_2 \dots h_n),$$

so many in number are these pairs.

Query (1). Is there no proposition applicable to any n quantities *whatever*?

Query (2). Is there no faintly analogous proposition applicable to higher powers than the squares?

Query (3). Seeing that in forming the coefficients in the equation of the squares of the differences, we pass from n functions of the roots to $n\frac{n-1}{2}$ and not n functions, of their squared differences, does not a natural passage to the former lie through n functions of the squared differences?

In other words, may not the quantities $\Sigma SPD(h_1, h_2 \dots h_n)$, &c., serve as natural and valuable intermediaries between the coefficients of an equation involving simple quantities and the coefficients of the equation involving the squares of their differences?

P.S. In the next part I trust to be able to present the readers of this *Magazine* with a *direct* and *symmetrical* method of eliminating any number of unknown quantities between any number of equations of any degree, by a newly invented process of symbolical multiplication, and the use of *compound* symbols of notation.

I must not omit to state that the constituents of multiplication λ_r and μ_r explained in Cor. 2 to Theorem 3 are equal to the expression

$$\Sigma(x-k_1)(x-k_2)\dots(x-k_{n-r-1})\frac{\begin{pmatrix}k_1, k_2 \dots k_{n-r-1} \\ -h_1, -h_2 \dots -h_m\end{pmatrix}}{\begin{pmatrix}k_1, k_2 \dots k_{n-r-1} \\ -k_{n-r} \dots -k_n\end{pmatrix}},$$

and its analogue respectively.

8.

ON DERIVATION OF COEXISTENCE. PART II. BEING THE THEORY OF SIMULTANEOUS SIMPLE HOMOGENEOUS EQUATIONS.

[*Philosophical Magazine*, XVI. (1840), pp. 37—43.]

Art. (1). We shall have constant occasion in this paper to denote different quantities by the same letter affected with different subscribed numerical indices.

Such a letter is to be termed a "Base."

Every character consisting of a base and an inferior index, this index is called an argument of the base, namely, the first, second, or nth argument, according as 1, 2, or in general n, be the number subscribed.

Art. (2). I use the symbol PD to denote the product of the differences of the quantities to which it is prefixed (each being to be subtracted *from* each that follows); thus

$PD(a, b, c)$ indicates $(b-a)(c-a)(c-b)$.

$PD(0, a, b, c)$ indicates $abc(b-a)(c-a)(c-b)$.

$PD(0, a, b, c \dots l)$ indicates $abc \dots l \times PD(a, b, c \dots l)$.

Art. (3). For want of a better symbol I use the Greek letter ζ to denote that the product of factors to which it is prefixed is to be effected after a certain symbolical manner. This I shall distinguish as the zeta-ic product.

The symbol ζ will never be prefixed except to factors, each of which is made up of one or more terms, consisting solely of linear arguments of different bases, that is, characters bearing indices below but none above.

I am thereby enabled to give this short rule for zeta-ic multiplication: "Imagine all the inferior indices to become superior, so that each argument is transformed into a *power* of its base; multiply according to the rules of ordinary algebra; after the multiplication has been *done fully out* depress all the indices into their original position; the result is the zeta-ic product*."

* It is scarcely necessary to add that an analogous interpretation may be extended to any zeta-ic function whatever. Thus

$$\zeta(a_1+b_1)^2 = a_2 + 2a_1b_1 + b_2,$$

$$\zeta\cos(a_1) = 1 - \frac{a_2}{1.2} + \frac{a_4}{1.2.3.4}, \text{ \&c.}$$

Thus for example $\zeta(a_r, b_s)$ is the same as simply $a_r b_s$, but $\zeta(a_r, a_s)$ represents not $a_r a_s$ but a_{r+s}.

So in like manner

$$\zeta\{(a_h - b_k)(a_l - b_m)\}$$
$$= a_{h+l} - a_h b_m - b_k a_l + b_{m+k},$$
$$\zeta\{(a_1 - b_1)(a_1 - c_1)(b_1 - c_1)\}$$
$$= \text{the } \textit{depressed product} \text{ of } (a-b)(a-c)(b-c)$$
$$= \text{the } \textit{depressed} \text{ value of } a^2(b-c) + b^2(c-a) + c^2(a-b),$$

that is, $= a_2 b_1 - a_2 c_1 + b_2 c_1 - b_2 a_1 + c_2 a_1 - c_2 b_1$.

Art. (4). We shall have occasion in this part to combine the two symbols ζ, PD: thus we shall use

$$\zeta PD(a_1 b_1) \text{ to denote } \zeta(b_1 - a_1),$$
$$\zeta PD(a_1 b_1 c_1) \text{ to denote } \zeta\{(b_1 - a_1)(c_1 - a_1)(c_1 - b_1)\}.$$

Art. (5). For the sake of elegance of diction I shall in future sometimes omit to insert the inferior index when it is unity; but the reader must always bear in mind that it is to be *understood* though not expressed.

I shall thus be able to speak of the zeta-ic product of such and such bases mentioned by name.

Art. (6). We are not yet come to the limit of the powers of our notation. The zeta-ic product of the sum of arguments will consist of the sum of products of arguments, each argument being (as I have defined) made up of a base and an inferior index. Now we may imagine each index of every term of the zeta-ic product *after it is fully expanded* to be increased or diminished by unity, or each at the same time to be increased or diminished by 2, or each in general to be increased or diminished by r. I shall denote this alteration by affixing an r with the positive or negative sign to the ζ. Thus

$$\zeta(a_1 - b_1)(a_1 - c_1) \text{ being equal to } a_2 - a_1 c_1 + b_1 c_1 - b_1 a_1,$$
$$\zeta_{+1}(a_1 - b_1)(a_1 - c_1) \text{ is equal to } a_3 - a_2 c_2 + b_2 c_2 - b_2 a_2,$$
$$\zeta_{-1}(a_1 - b_1)(a_1 - c_1) \text{ is equal to } a_1 - a_0 c_0 + b_0 c_0 - b_0 a_0.$$

In like manner $\zeta PD(a, b, c)$ indicating

$$b_2 a_1 - b_2 c_1 + c_2 b_1 - c_2 a_1 + a_2 c_1 - a_2 b_1,$$

$\zeta_{\pm r} PD(a, b, c)$ indicates

$$b_{2\pm r} a_{1\pm r} - b_{2\pm r} c_{1\pm r} + c_{2\pm r} b_{1\pm r} - c_{2\pm r} a_{1\pm r} + a_{2\pm r} c_{1\pm r} - a_{2\pm r} b_{1\pm r}.$$

I shall in general denote $\zeta_{+r} PD(a, b, c \ldots l)$ *actually expanded* as the zeta-ic product of $a, b, c, \ldots l$ in its rth *phase*.

Art. (7). *General Properties of Zeta-ic Products of Differences.*

If there be made one interchange in the order of the bases to which ζ is prefixed, the zeta-ic product, in whatever phase it be taken, remains unaltered in magnitude, but changes its sign.

If in any *phase* of a zeta-ic product two of the bases be made to coincide, the expansion vanishes.

Let $\int_1$ be used, agreeably to the ordinary notation, to denote the sum of the quantities to which it is prefixed, $\int_2$ to denote the sum of the binary products, $\int_3$ of the ternary ones, and so on.

Thus let $\int_1(a_1 b_1 c_1)$ or $\int_1(a, b, c)$ indicate $a_1 + b_1 + c_1$,

and $\int_2(a_1 b_1 c_1)$ or $\int_2(a, b, c)$ indicate $a_1 b_1 + a_1 c_1 + b_1 c_1$,

and $\int_3(a_1 b_1 c_1)$ or $\int_3(a, b, c)$ indicate $a_1 b_1 c_1$,

we shall be able now to state the following remarkable proposition connecting the several phases of certain the same zeta-ic products.

Art. (8). Let $a, b, c, \ldots l$, denote any number of independent bases, say $(n-1)$; but let the arguments of each base be periodic, and the number of terms in each period the same for every base, namely n, so that

$$a_r = a_{r+n} = a_{r-n}, \quad a_n = a_0 = a_{-n},$$
$$b_r = b_{r+n} = b_{r-n}, \quad b_n = b_0 = b_{-n},$$
$$c_r = c_{r+n} = c_{r-n}, \quad c_n = c_0 = c_{-n},$$
$$\cdots\cdots\cdots \quad \cdots\cdots\cdots$$
$$l_r = l_{r+n} = l_{r-n}, \quad l_n = l_0 = l_{-n},$$

r being any number whatever. Then

$$\zeta_{-1} PD(0, a, b, c \ldots l) = \zeta\{\textstyle\int_1(a, b, c \ldots l)\ \zeta PD(0, a, b, c \ldots l)\},$$
$$\zeta_{-2} PD(0, a, b, c \ldots l) = \zeta\{\textstyle\int_2(a, b, c \ldots l)\ \zeta PD(0, a, b, c \ldots l)\},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\zeta_{-r} PD(0, a, b, c \ldots l) = \zeta\{\textstyle\int_r(a, b, c \ldots l)\ \zeta PD(0, a, b, c \ldots l)\}.$$

This proposition admits of a great generalization*, but we have now all that is requisite for enabling us to arrive at a proposition exhibiting under one *coup d'œil* every combination and every effect of every combination that can possibly be made with any number of coexisting equations of the first degree, containing any number of *repeated*, or to use the ordinary language of analysts, (variable or) unknown quantities.

* See the Postscript to this paper for *one* specimen.

For the sake of symmetry I make every equation homogeneous; so that to eliminate n repeated terms, no more than n equations will be required.

In like manner the problem of determining n quantities from n equations will be here represented by the case in which we have to determine the *ratios* of $(n+1)$ quantities from n equations.

Art. (9). *Statement of the Equations of Coexistence.*

Let there be any number of bases $(a, b, c \dots l)$, and as many repeated terms $(x, y, z \dots t)$, and let the number of equations be any whatever, say n. The system may be represented by the *type* equation

$$a_r x + b_r y + c_r z + \dots + l_r t = 0,$$

in which r can take up all integer values from $-\infty$ to $+\infty$. The specific number of equations given will be represented by making the arguments of each base *periodic*, so that

$$a_r = a_{\mu n + r},\ b_r = b_{\mu n + r},\ c_r = c_{\mu n + r},\ \dots\ l_r = l_{\mu n + r},$$

μ being any integer whatever.

Art. (10). *Combination of the given Equations.—Leading Theorem.*

Take $f, g, \dots k$ as the *arbitrary* bases of new and absolutely independent but periodic arguments, having the same index of periodicity (n) as $a, b, c \dots l$, and being in number $(n-1)$, that is, one fewer than there are units in that index.

The number of *differing* arbitrary constants thus *manufactured* is $n(n-1)$.

Let $Ax + By + Cz + \dots + Lt = 0$ be the general *prime* derivative from the given equations, then we may make

$$A = \zeta PD(0, a, f, g \dots k),$$
$$B = \zeta PD(0, b, f, g \dots k),$$
$$C = \zeta PD(0, c, f, g \dots k),$$
$$\dots\dots\dots\dots\dots\dots$$
$$L = \zeta PD(0, l, f, g \dots k).$$

Art. (11). Cor. 1. *Inferences from the Leading Theorem.*

Let the number of equations, or, which is the same thing, the index of periodicity (n), be the same as the number of repeated terms $(x, y, z \dots t)$, then one relation exists between the coefficients: this is found by making the $(n-1)$ new bases coincide with $(n-1)$ out of the old bases. We get accordingly, as the result of elimination,

$$\zeta PD(0, a, b, c \dots l) = 0.$$

Art. (11). Cor. 2. Let the number of equations be one more than that of the given bases, there will then be two equations of condition. These are represented by preserving one new arbitrary base, as λ. The result of elimination being in this case

$$\zeta PD(0, a, b, c \dots l, \lambda) = 0.$$

Example. The result of eliminating between

$$a_1 x + b_1 y = 0,$$
$$a_2 x + b_2 y = 0,$$
$$a_3 x + b_3 y = 0,$$

is $\zeta PD(0, a, b, \lambda) = 0$, that is

$$\lambda_3 b_2 a_1 - \lambda_3 b_1 a_2 + \lambda_1 b_3 a_2 - \lambda_1 b_2 a_3 + \lambda_2 b_1 a_3 - \lambda_2 b_3 a_1 = 0,$$

from which we infer, seeing that $\lambda_3, \lambda_2, \lambda_1$ are independent,

$$b_2 a_1 - b_1 a_2 = 0,$$
$$b_3 a_2 - b_2 a_3 = 0,$$
$$b_1 a_3 - b_3 a_1 = 0,$$

any *two* of which imply the third.

In like manner, in general, if the number of equations exceed in any manner the number of bases or repeated terms, the rule is to introduce so many *new* and *arbitrary* bases as together with the old bases shall make up the number of equations, and then equate the zeta-ic product of the differences of zero, the old bases and the new bases, to nothing.

Art. (12). Cor. 3. Let the number of equations be *one* fewer than the number (n) of bases or repeated terms; the number of introduced bases in the general theorem is here $(n-2)$. Make these $(n-2)$ bases equal severally to the bases which in the type equation are affixed to $z, u \dots t$, then

$$C = 0,$$
$$D = 0,$$
$$\dots\dots$$
$$L = 0,$$

and we have left simply

$$\zeta PD(0, a, c, d \dots kl)\, x + \zeta PD(0, b, c, d \dots kl)\, y = 0.$$

In like manner we may make to vanish all but A and C, and thus get

$$\zeta PD(0, a, b, d \dots kl)\, x + \zeta PD(0, c, b, d \dots kl)\, z = 0,$$

and similarly

$$\zeta PD(0, a, b \ldots k)x + \zeta PD(0, b, c \ldots l)t = 0.$$

Hence $\left.\begin{matrix} x \\ y \\ z \\ \cdot \\ \cdot \\ \cdot \\ t \end{matrix}\right\}$ are severally as $\left\{\begin{matrix} \zeta PD(0, b, c \ldots l) \\ \zeta PD(a, 0, c \ldots l) \\ \zeta PD(a, b, 0 \ldots l) \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \zeta PD(a, b, c \ldots 0). \end{matrix}\right.$

This is the symbolical representation as a *formula* of the remarkable *method* discovered by Cramer, perfected by Bezout and demonstrated by Laplace for the solution of simultaneous simple equations.

Art. (13). COR. 4. In like manner if the number of repeated terms be two greater than the number of equations, we have for the relation between any *three* of them, taken at pleasure, for instance, x, y, z,

$$\zeta PD(0, a, d \ldots l)x + \zeta PD(0, b, d \ldots l)y + \zeta PD(0, c, d \ldots l)z = 0.$$

And in like manner we may proceed, however much in excess the number of repeated terms (unknown quantities) is over the number of equations.

Art. (14). *Subcorollary to Corollary* 3.

If there be any number of bases $(a, b, c \ldots l)$, and any other two fewer in number $(f, g \ldots k)$

$$\left.\begin{aligned} &\zeta PD(a, f, g \ldots k) \times \zeta PD(b, c \ldots l) \\ +&\zeta PD(b, f, g \ldots k) \times \zeta PD(a, c \ldots l) \\ +&\zeta PD(c, f, g \ldots k) \times \zeta PD(b, c \ldots l) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ +&\zeta PD(l, f, g \ldots k) \times \zeta PD(a, b, c \ldots \quad) = 0 \end{aligned}\right\}*,$$

a formula that from its very nature suggests and *proves* a wide extension of itself.

In conclusion I feel myself bound to state that the principal substance of *Corollaries* (1), (2) and (3) may be found in Garnier's *Analyse Algébrique*, in the chapter headed "Dévéloppement de la Théorie donnée par M. Laplace, &c." But I am not aware of having been anticipated either in the fertile notation which serves to express them nor in the general theorems to which it has given birth.

P.S. I shall content myself for the present with barely enunciating a theorem, one of a class destined it seems to the author to play no secondary part in the development of some of the most curious and interesting points of analysis.

* The cross is used to denote *ordinary* algebraical multiplication.

Let there be $(n-1)$ bases $a, b, c \ldots l$, and let the arguments of each be "recurrents of the nth order*," that is to say let

$$a_\iota = \phi\left(\cos\frac{2\pi\iota}{n}\right),\ b_\iota = \psi\left(\cos\frac{2\pi\iota}{n}\right),\ c_\iota = \chi\left(\cos\frac{2\pi\iota}{n}\right),\ \ldots\ldots l_\iota = \omega\left(\cos\frac{2\pi\iota}{n}\right).$$

Let R_r denote that any symmetrical function of the rth degree is to be taken of the quantities in a parenthesis which come after it, and let ϑ indicate any function whatever. Then the zeta-ic product

$$\zeta\{\zeta R_r(a, b, c \ldots l) \times \zeta_\rho \vartheta PD(0, a, b, c \ldots l)\}$$

is equal to the product of the *number*

$$R_r\left\{\left(\cos\frac{2\pi}{n} + \sqrt{(-1)}\sin\frac{2\pi}{n}\right), \left(\cos\frac{4\pi}{n} + \sqrt{(-1)}\sin\frac{4\pi}{n}\right), \left(\cos\frac{6\pi}{n} + \sqrt{(-1)}\sin\frac{6\pi}{n}\right),\right.$$

$$\left.\ldots\ldots, \left(\cos\frac{2(n-1)\pi}{n} + \sqrt{(-1)}\sin\frac{2(n-1)\pi}{n}\right)\right\},$$

multiplied by the zeta-ic phase

$$\zeta_{\rho-r}\vartheta PD(0, a, b, c \ldots l)!!$$

* I am indebted for this term to Professor De Morgan, whose pupil I may boast to have been. I have the sanction also of his authority, and that of another profound analyst, my colleague Mr Graves, for the use of the arbitrary terms zeta-ic, zeta-ically. I take this opportunity of retracting the symbol *SPD* used in my last paper, the letter *S* having no meaning except for English readers. I substitute for it *QDP*, where *Q* represents the Latin word Quadratus. On some future occasion I shall enlarge upon a new method of notation, whereby the language of analysis may be rendered much more expressive, depending essentially upon the use of similar figures inserted within one another, and containing numbers or letters, according as quantities or operations are to be denoted. This system to be carried out would require special but very simple printing types to be founded for the purpose.

In the next part of this paper an easy and *symmetrical* mode will be given of representing any polynomial either in its developable or expanded form.

9.

A METHOD OF DETERMINING BY MERE INSPECTION THE DERIVATIVES FROM TWO EQUATIONS OF ANY DEGREE.

[*Philosophical Magazine*, XVI. (1840), pp. 132—135.]

LET there be two equations, one of the nth, the other of the mth degree in x; let the coefficients of the first equation be $a_n, a_{n-1}, a_{n-2} \ldots a_0$, each power of x having a coefficient attached to it, a_n belonging to x^n and a_0 to the constant term.

In like manner let $b_m, b_{m-1} \ldots b_0$ be the coefficients of the second equation.

I begin with

A Rule for absolutely eliminating x.

Form out of the (a) progression of coefficients m lines, and in like manner out of the (b) progression of coefficients form n lines in the following manner:

1. (a) Attach $(m-1)$ zeros all to the *right* of the terms in the (a) progression; next attach $(m-2)$ zeros to the right and carry over to the left; next attach $(m-3)$ zeros to the right and carry over 2 to the left. Proceed in like manner until all the $(m-1)$ zeros are carried over to the left and none remain on the right.

The m lines thus formed are to be written under one another.

1. (b) Proceed in like manner to form n lines out of the (b) progression by scattering $(n-1)$ zeros between the right and left.

2. If we write these n lines under the m lines last obtained, we shall have a solid square $(m+n)$ terms *deep* and $(m+n)$ terms *broad*.

3. Denote the lines of this square by arbitrary characters, which write down in vertical order and permute in every possible way, but separate the permutations that can be derived from one another by an even number of interchanges (effected between *contiguous* terms) from the rest; there will thus be half of one kind and half of another.

4. Now arrange the $(m+n)$ lines accordingly, so as to obtain

$$\tfrac{1}{2}\{(m+n)(m+n-1)\ldots 2 \,.\, 1\}$$

squares of one kind which shall be called positive squares, and an equal number of the opposite kind which shall be called negative.

Draw diagonals in the same direction in all the squares; multiply the coefficients that stand in any diagonal line together: take the sum of the diagonal products of the *positive* squares, and the sum of the diagonal products of the *negative* squares; the difference between these two sums is the prime derivative of the zero degree, that is, is the result of elimination between the two given equations reduced to its ultimate state of simplicity, there will be no irrelevant factors to reject, and no terms which mutually destroy.

Example. To *eliminate* between

$$ax^2 + bx + c = 0,$$

$$lx^2 + mx + n = 0,$$

I write down

$$a,\ b,\ c,\ 0, \qquad (1)$$

$$0,\ a,\ b,\ c, \qquad (2)$$

$$l,\ m,\ n,\ 0, \qquad (3)$$

$$0,\ l,\ m,\ n. \qquad (4)$$

I permute the four characters (1), (2), (3), (4), distinguishing them into positive and negative; thus I write together

Positive Permutations.

1	2	3	1	2	3	2	1	3	4	4	4
2	3	1	4	4	4	1	3	2	2	1	3
3	1	2	2	3	1	4	4	4	1	3	2
4	4	4	3	1	2	3	2	1	3	2	1

and again

Negative Permutations.

1	2	3	4	4	4	2	1	3	2	1	3
2	3	1	1	2	3	4	4	4	1	3	2
4	4	4	2	3	1	1	3	2	3	2	1
3	1	2	3	1	2	3	2	1	4	4	4

I reject from the permutations of each species all those where 1 or 3 appear in the fourth place, and also those where 2 or 4 appear in the first place, for these will be presently seen to give rise to diagonal products which are zero.

The permutations remaining are

Positive effectual permutations.

1	3	3	1
2	1	4	3
3	2	1	4
4	4	2	2

Negative effectual permutations.

3	1	1	3
1	4	3	2
4	3	2	1
2	2	4	4

I now accordingly form four positive squares, which are

$a, b, c, 0,$	$l, m, n, 0,$	$l, m, n, 0,$	$a, b, c, 0,$
$0, a, b, c,$	$a, b, c, 0,$	$0, l, m, n,$	$l, m, n, 0,$
$l, m, n, 0,$	$0, a, b, c,$	$a, b, c, 0,$	$0, l, m, n,$
$0, l, m, n,$	$0, l, m, n,$	$0, a, b, c,$	$0, a, b, c.$

Drawing diagonal lines from left to right, and taking the sum of the diagonal products, I obtain $a^2n^2 + lb^2n + l^2c^2 + am^2c$. Again, the four negative squares

$l, m, n, 0,$	$a, b, c, 0,$	$a, b, c, 0,$	$l, m, n, 0,$
$a, b, c, 0,$	$0, l, m, n,$	$l, m, n, 0,$	$0, a, b, c,$
$0, l, m, n,$	$l, m, n, 0,$	$0, a, b, c,$	$a, b, c, 0,$
$0, a, b, c,$	$0, a, b, c,$	$0, l, m, n,$	$0, l, m, n,$

give as the sum of the diagonal products

$$lbmc + alnc + ambn + lacn,$$

that is,

$$lbmc + ambn + 2acln.$$

Thus the result of eliminating between

$$ax^2 + bx + c = 0,$$
$$lx^2 + mx + n = 0,$$

ought to be, and is

$$a^2n^2 + l^2c^2 - 2acln + lb^2n + am^2c - lbmc - ambn = 0.$$

Rule for finding the prime derivative of the first degree, which is of the form $Ax - B$.

Begin as before, only attach one zero less to each progression; we shall thus obtain *not* a square, but an oblong broader than it is deep, containing $(m + n - 2)$ rows, and $(m + n - 1)$ terms in each row: in a word, $(m + n - 2)$ rows, and $(m + n - 1)$ *columns.*

To find A reject the column at the extreme right, we thus recover a square arrangement $(m + n - 2)$ terms broad and deep.

Proceed with this new square as with the former one; the difference between the sums of the positive and negative diagonal products will give A.

To find B, do just the same thing, with the exception of striking off not the last column, but the last but *one.*

Rule for finding the prime derivative of any degree, say the rth, namely,
$$A_r x^r - A_{r-1} x^{r-1} + \ldots\ldots \pm A_0.$$

Begin with adding zeros as before, but the number to be added to the (a) progression is $(m - r)$ and to the (b) progression $(n - r)$.

There will thus be formed an oblong containing $(m + n - 2r)$ rows, and $(m + n - r)$ terms in each row, and therefore the same number of *columns.*

To find any coefficient as A_s, strike off all the last $(r + 1)$ columns except that which is (s) places distant from the extreme right, and proceed with the resulting squares as before.

Through the well-known ingenuity and kindly proferred help of a distinguished friend, I trust to be able to get a machine made for working Sturm's theorem, and indeed all problems of derivation, after the method here expounded; on which subject I have a great deal more yet to say, than can be inferred from this or my preceding papers.

10.

NOTE ON ELIMINATION.

[*Philosophical Magazine*, XVII. (1840), pp. 379, 380.]

THE object of this brief note is to generalise Theorem 2 in my paper on Elimination* which appeared in the last December number of this *Magazine*. The theorem so generalised presents a symmetry which before was wanting. Here, as in so many other instances, the whole occupies in the memory a *less* space than the part.

To avoid the ill-looking and slippery negative symbols, I warn my reader that I now use two rows of quantities written one over the other, to denote the product of the terms resulting from *taking away* each quantity in the under from each in the upper row.

Let $h_1, h_2 \ldots h_m$ be the roots of one equation of coexistence,

$k_1, k_2 \ldots k_n$ of the other,

and let the prime derivative of the degree r be required. Take *any* two integers p and q, such that $p+q=r$. The derivative in question may be written

$$\Sigma\left\{(x-h_1)\ldots(x-h_p)(x-k_1)\ldots(x-k_q)\frac{\begin{pmatrix}h_1h_2\ldots h_p\\ k_1k_2\ldots k_q\end{pmatrix}\cdot\begin{pmatrix}h_{p+1}h_{p+2}\ldots h_m\\ k_{q+1}k_{q+2}\ldots k_n\end{pmatrix}}{\begin{pmatrix}h_1\;\; h_2\;\;\ldots h_p\\ h_{p+1}h_{p+2}\ldots h_m\end{pmatrix}\cdot\begin{pmatrix}k_1\;\; k_2\;\;\ldots k_q\\ k_{q+1}k_{q+2}\ldots k_n\end{pmatrix}}\right\}.$$

N.B. Whatever p and q be taken, so long only as $p+q=r$, the above expression changes nothing but its sign; which, therefore, upon transcendental grounds, it is easy to see is of one name or another, according as p is odd or even.

In the original paper, I asserted this theorem only for the case of $p=0$, or $q=0$.

[* p. 43 above. ED.]

11.

ON THE RELATION OF STURM'S AUXILIARY FUNCTIONS TO THE ROOTS OF AN ALGEBRAIC EQUATION.

[*Plymouth British Association Report* 1841, (Pt II.), pp. 23, 24.]

THE author availed himself of the present meeting of the British Association to bring under the more general notice of mathematicians his discovery, made in the year 1839, of the real nature and constitution of the auxiliary functions (so-called) which Sturm makes use of in *locating* the roots of an equation: these are obtained by proceeding with the left-hand side of the equation and its first differential coefficient as if it were our object to obtain their greatest common factor; the successive remainders, with their signs *alternately* changed and preserved, constitute the functions in question. Each of these may be put under the form of a fraction, the denominator of which is a perfect square, or in fact the product of *many*: likewise the numerator contains a huge heap of factors of a similar form.

These therefore, as well as the denominator, since they cannot influence the series of *signs*, may be rejected; and furthermore we may, if we please, again make every other function, beginning from the last but one, change its sign, if we consent to use changes wherever Sturm speaks of continuations of sign, and *vice versâ*.

The functions of Sturm, thus modified and purged of irrelevancy, the author, by way of distinction, and still to attribute honour where it is really most due, proposes to call "Sturm's Determinators"; and he proceeds to lay bare the internal anatomy of these remarkable forms.

He uses the Greek letter "ζ" to indicate that the squared product of the differences of the letters before which it is prefixed is to be taken.

Let the roots of the equation be called respectively $a, b, c, e \ldots l$, the determinators taken in the inverse order are as follows:—

$$\zeta(a, b, c, e \ldots l).$$

$$\Sigma\zeta(b, c, e \ldots l)\,x - \Sigma a\zeta(b, c, e \ldots l).$$

$$\Sigma\zeta(c, e \ldots l)\,x^2 - \Sigma(a+b)\,.\,\zeta(c, e \ldots l)\,x + \Sigma ab\,.\,\zeta(c, e \ldots l).$$

* * * * * * *

$$\Sigma\{\zeta(k, l)(x-a)(x-b)(x-c)(x-e)\ldots(x-h)\}.$$

It may be here remarked, that the work of assigning the total number of real and of imaginary roots falls exclusively upon the coefficients of the leading terms, which the author proposes to call "Sturm's Superiors": these superiors are only *partial* symmetric functions of the *squared differences*, but *complete* symmetric functions of the *roots themselves*, differing in the former respect from those other (at first sight similar-looking) functions of the squared differences of the roots, in which, from the time of Waring downwards, the conditions of reality have been sought for. It seems to have escaped observation, that the series of terms constituting any one of the coefficients in the equation of the squares of the differences (with the exception of the first and last) each admit of being separated and classified into various subordinate groups in such a way, that instead of being treated as a single symmetric function of the *roots*, they ought to be viewed as aggregates of many. In fact, Sturm's superior No. 1 is identical with Waring's coefficient No. 1; Sturm's superior No. 2 is a *part* of Waring's coefficient No. 3; Sturm's superior No. 3 is a *part* of Waring's coefficient No. 6; and so forth till we come to Sturm's final superior, which is again coextensive and identical with the last coefficient in the equation of the squares of the differences. The theory of symmetric functions of forms which are themselves symmetric functions of simple letters, or even of other forms, the author states his belief is here for the first time shadowed forth, but would be beside his present object to enter further into. He would conclude by calling attention to the importance to the general interests of algebraical and arithmetical science that a searching investigation should be instituted for showing, *à priori*, how, when a set of quantities is known to be made up partly of possible and partly of *pairs* of impossible values, symmetrical functions of these, one less in number than the quantities themselves, may be formed, from the signs of the ratios of which to unity and to one another the respective amounts of possible and impossible quantities may at once be inferred: in short, we ought not to rest satisfied, until, from the very *form* of Sturm's Determinators, without caring to know how they have been obtained, we are able to pronounce upon the uses to which they may be applied.

12.

EXAMPLES OF THE DIALYTIC METHOD OF ELIMINATION AS APPLIED TO TERNARY SYSTEMS OF EQUATIONS.

[*Cambridge Mathematical Journal*, II. (1841), pp. 232—236.]

THIS method is of universal application, and at once enables us to reduce any case of elimination to the form of a problem, where that operation is to be effected between quantities linearly involved in the equations which contain them.

As applied to a binary system, $fx = 0$, $\phi x = 0$, the method furnishes a rule by which we may unfailingly arrive at *the determinant*, free from every species of irrelevancy, whether of a linear, factorial, or numerical kind.

The rule itself is given in the *Philosophical Magazine* (London and Edinburgh, Dec. 1840). The principle of the rule will be found correctly stated by Professor Richelot, of Königsberg, in a late number of *Crelle's Journal*, at the commencement of a memoir in Latin bordering on the same subject ("Nota ad Eliminationem pertinens").

My object at present is to supply a few instances of its application to ternary systems of equations.

Ex. 1. To eliminate x, y, z, between the three homogeneous equations

$$Ay^2 - 2C'xy + Bx^2 = 0, \qquad (1)$$

$$Bz^2 - 2A'yz + Cy^2 = 0, \qquad (2)$$

$$Cx^2 - 2B'zx + Az^2 = 0. \qquad (3)$$

Multiply the equations in order by $-z^2$, x^2, y^2, add together, and divide out by $2xy$; we obtain

$$C'z^2 + Cxy - A'xz - B'yz = 0. \qquad (4)$$

By similar processes we obtain

$$A'x^2 + Ayz - B'yx - C'zx = 0, \qquad (5)$$

$$B'y^2 + Bzx - C'zy - A'xy = 0. \qquad (6)$$

Between these six, treated as simple equations, the six functions of x, y, z, namely, $x^2, y^2, z^2, xy, xz, yz$, treated as *independent* of each other, may be eliminated; the results may be seen, by mere inspection, to come out

$$ABC(ABC - AB'^2 - BC'^2 - CA'^2 + 2A'B'C') = 0,$$

or rejecting the special (N.B. not *irrelevant*) factor ABC, we obtain

$$ABC - AB'^2 - BC'^2 - CA'^2 + 2A'B'C' = 0.$$

I may remark, that the equations (1), (2), (3), or (4), (5), (6), express the condition of

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy,$$

having a factor $\lambda x + \mu y + \nu z$; a general symbolical formula of which I am in possession for determining in general the condition of any polynomial of any degree having a factor, furnishes me at once with either of the two systems indifferently. The aversion I felt to reject *either*, led me to employ both, and thus was the occasion of the Dialytic Principle of Solution manifesting itself.

Ex. 2.

$$Ax^2 + ayz + bzx + cxy = 0, \qquad (1)$$

$$My^2 + lyz + mzx + nxy = 0, \qquad (2)$$

$$Rz^2 + pyz + qzx + rxy = 0. \qquad (3)$$

Multiply equation (1) by $\beta y + \gamma z$, equations (2) and (3) by νz and κy respectively, and add the products together, we obtain terms of which y^2z and yz^2 are the only two into which x does not enter.

Make now the coefficients of each of these zero, and we have

$$a\gamma + l\nu + R\kappa = 0,$$

$$a\beta + M\nu + p\kappa = 0.$$

Let $\nu = a$, $\kappa = a$, then $\gamma = -(l + R)$, $\beta = -(M + p)$.

Hence, multiplying as directed, and then dividing out by x, we obtain

$$(m\nu + b\gamma) z^2 + (r\kappa + c\beta) y^2 + (b\beta + c\gamma + n\nu + q\kappa) yz + A\beta xy + A\gamma xz = 0,$$

or by substitution,

$$\{ra - c(M + p)\} y^2 + \{ma - b(l + R)\} z^2 + \{an + aq - b(M + p) - c(l + R)\} yz$$
$$- A(M + p) xy - A(M + p) xz = 0. \qquad (4)$$

Similarly, by preparing the equations so as to admit in turn of y and z as a divisor, we obtain

$$\{ma - l(R + b)\} z^2 + \{mr - n(A + q)\} x^2 + \{mc + mp - n(R + b) - l(A + y)\} xz$$
$$- M(R + b) yz - A(A + q) xy = 0. \qquad (5)$$

$$\{rm - q(A + n)\} x^2 + \{ra - p(M + c)\} y^2 + \{rl + rb - p(A + n) - q(M + c)\} xy$$
$$- R(A + n) xz - R(M + c) yz = 0. \qquad (6)$$

Between the six equations (1), (2), (3), (4), (5), (6), $x^2, y^2, z^2, xy, xz, yz$, may be eliminated; the result will be a function of nine letters {three out of each equation (1), (2), (3)} equated to zero. *Perhaps* the determinant may be found to contain a special factor of three letters; and if so, may be replaced by a simpler function of six letters only.

Ex. 3. To eliminate between the three general equations

$$Ax^2 + By^2 + Cz^2 + 2Dyz + 2Ezx + 2Fxy = 0,$$

$$Lx^2 + My^2 + Nz^2 + 2Pyz + 2Qzx + 2Rxy = 0,$$

$$fx + gy + hz = 0.$$

By virtue of *one* of the two canons which limit the forms in which the letters can appear combined in the determinant of a general system of equations, we know that the determinant in this case (freed of irrelevant factors) ought to be made up in every term of eight letters (powers being counted as repetitions), namely, (A, B, C, D, E, F) must enter in binary combinations, (L, M, N, P, Q, R) the same, whereas f, g, h must enter in *quaternary* combinations.

To obtain the determinant, write

$$Ax^2 + By^2 + Cz^2 + Dyz + Ezx + Fxy = 0, \qquad (1)$$

$$Lx^2 + My^2 + Nz^2 + Pyz + Qzx + Rxy = 0, \qquad (2)$$

$$fx^2 + gyx + hzx = 0, \qquad (3)$$

$$fxy + gy^2 + hzy = 0, \qquad (4)$$

$$fxz + gyz + hz^2 = 0. \qquad (5)$$

We want one equation more of *three* letters between $x^2, y^2, z^2, xy, xz, yz$. To obtain this, write

$$(Ax + Ez + Fy) x_1 + (By + Fx + Dz) y_1 + (Cz + Dy + Ex) z_1 = 0,$$

$$(Lx + Qz + Ry) x_1 + (My + Rx + Pz) y_1 + (Nz + Py + Qx) z_1 = 0,$$

$$fx_1 + gy_1 + hz_1 = 0.$$

Forget that $x_1 = x$, $y_1 = y$, $z_1 = z$, and eliminate x_1, y_1, z_1, we obtain

$$h \begin{Bmatrix} (Ax + Ez + Fy)(My + Rx + Pz) \\ -(By + Fx + Dz)(Lx + Qz + Ry) \end{Bmatrix}$$
$$+ g \begin{Bmatrix} (Cz + Dy + Ex)(Lx + Qz + Ry) \\ -(Nz + Py + Qx)(Ax + Ez + Fy) \end{Bmatrix}$$
$$+ f \begin{Bmatrix} (Nz + Py + Qx)(By + Fx + Dz) \\ -(Cz + Dy + Ex)(My + Rx + Pz) \end{Bmatrix} = 0.$$

This may be put under the form

$$\alpha x^2+\beta y^2+\gamma z^2+\alpha' yz+\beta' zx+\gamma' xy=0, \qquad (6)$$

where the coefficients are of the first order in respect to $f, g, h, L, M, N, P, Q, R, A, B, C, D, E, F$; in all of the third order.

Between the equations marked from (1) to (6), the process of linear elimination being gone through, we obtain as equated to zero a function of $5+3$, or of eight letters, two belonging to the first equation, two to the second, and four to the third; so that the determinant is clear of all factorial irrelevancy.

Ex. 4. To eliminate x, y, z between the three equations

$$Ax^2+By^2+Cz^2+2A'yz+2B'zx+2C'xy=0,$$
$$Lx^2+My^2+Nz^2+2L'yz+2M'zx+2N'xy=0,$$
$$Px^2+Qy^2+Rz^2+2P'yz+2Q'zx+2R'xy=0.$$

Call these three equations $U=0$, $V=0$, $W=0$, respectively. Write

$xU=0$,	(1)	$yU=0$,	(2)	$zU=0$,	(3)
$xV=0$,	(4)	$yV=0$,	(5)	$zV=0$,	(6)
$xW=0$,	(7)	$yW=0$,	(8)	$zW=0$.	(9)

We have here nine unilateral equations: one more is wanted to enable us to eliminate *linearly* the ten quantities

$$x^3,\quad y^3,\quad z^3,\quad x^2y,\quad x^2z,\quad xy^2,\quad xz^2,\quad xyz,\quad y^2z,\quad yz^2.$$

This tenth may be found by eliminating x, y, z between the three equations

$$x(Ax+B'z+C'y)+y(By+C'x+A'z)+z(Cz+A'y+B'x)=0,$$
$$x(Lx+M'z+N'y)+y(My+N'x+L'z)+z(Nz+L'y+M'x)=0,$$
$$x(Px+Q'z+R'y)+y(Qy+R'x+P'z)+z(Rz+P'y+Q'x)=0;$$

for, by forgetting the relations between the bracketed and unbracketed letters, we obtain

$$(Ax+B'z+C'y)\left\{\begin{matrix}(My+N'x+L'z)(Rz+P'y+Q'x)\\-(Qy+R'x+P'z)(Nz+L'y+M'x)\end{matrix}\right\}$$
$$+\ \&c. + \&c. = 0,$$

which may be put under the form

$$\alpha x^3+\beta y^3+\gamma z^3+\delta x^2y+\ldots\ldots=0^*. \qquad (10)$$

* We might dispense with a 10th equation, using the nine above given, to determine the ratios of the ten quantities involved to one another; and then by means of any such relations as

$$x^3y\times xy^3=x^2y^2\times x^2y^2, \text{ or } x^3\times y^3=x^2y\times xy^2, \&c.$$

obtain a determinant. But it is easy to see that this would be made up of terms, each containing literal combinations of the 18th order.

Again, we might use five out of the nine equations to obtain a new equation free from y^3, y^2z, yz^2, z^3; that is, containing x in every term: which being divided by x, and multiplied by y, or by z, would furnish a 10th equation no longer linearly involved in the 9 already found. The determinant, however, found in this way, would consist of 14-ary combinations of letters.

Finally, we might, instead of a system of ten equations, employ a system of 15, obtained by multiplying each of the given three by any 5 out of the 6 quantities $x^2, y^2, z^2, xy, xz, yz$; but the determinant, besides being not *totally* symmetrical, would contain combinations of the 15th order.

I may take this opportunity of just adverting to the fact, that the method in the text does in fact contain a solution of the equation

$$\lambda U+\mu V+\nu W=x^r y^s z^t,$$

where $r+s+t=4$, and λ, μ, ν are functions of the second degree in regard to x, y, z to be determined.

By eliminating linearly between the equations marked from (1) to (10), we obtain as zero a quantity of the twelfth order in all, being of the fourth order in respect to the coefficients of each of the three equations, which is therefore the determinant in its simplest form.

I have purposely, in this brief paper, avoided discussing any theoretical question. I may take some other opportunity of enlarging upon several points which have hitherto been little considered in the theory of elimination, such as the Canons of Form,—the Doctrine of Special Factors,—the Method of Multipliers as extended to a system of any order,—the Connexion between the method of Multipliers and the Dialytic Process,—the Idea of Derivations and of Prime Derivatives extended to ultra-binary Systems. For the present I conclude with the expression of my best wishes for the continued success of this valuable Journal.

13.

INTRODUCTION TO AN ESSAY ON THE AMOUNT AND DISTRIBUTION OF THE MULTIPLICITY OF THE ROOTS OF AN ALGEBRAIC EQUATION.

[*Philosophical Magazine*, XVIII. (1841), pp. 136—139.]

I USE the word *multiplicity* to denote a number, and distinguish between the total and partial multiplicities of the roots of an algebraic equation.

There may be r different roots repeated respectively $h_1, h_2 \ldots h_r$ times.

r is the index of distribution.

$h_1, h_2 \ldots h_r$ are the partial multiplicities, and if $h = h_1 + h_2 + \ldots + h_r$

h is the *total* multiplicity.

The total multiplicity it is clear may be defined as the difference between the index of the equation and the number of its roots distinguishable from one another.

In this Introduction, I propose merely to consider how existing methods may be applied to determine the amount and distribution of multiplicity in a given equation, and conversely, how equations of condition can be formed which shall imply a *given* distribution and amount.

Let the greatest common factor between fx (the argument of the proposed equation) and $\frac{dfx}{dx}$ be called f_1x.

And in like manner, let the greatest common factor of f_1x and $\frac{df_1x}{dx}$ be called f_2x and so on, till in the end we come to f_rx, which has no common factor with $\frac{df_rx}{dx}$.

Let $k_1, k_2 \ldots k_r$ denote the degrees in x of $fx, f_1x \ldots f_rx$ respectively.

It is easy to see that

$k_1 - k_2$, partial multiplicities, are less than 2, that is, are each units.

$k_2 - k_3$, partial multiplicities, will be less than 3, and therefore either 1 or 2 in value respectively, and so on till we come to

$k_{r-1} - k_r$ which will severally be between zero and $r - 1$, and

$k_r - 0$ of values intermediate between zero and r.

Hence there will be

$k_1 - 2k_2 + k_3$ multiplicities each of the value 1,

$k_2 - 2k_3 + k_4$ " " " 2,

....

$k_{r-1} - 2k_r \ldots$ of the value $r - 1$,

and $k_r \ldots\ldots\ldots\ldots$ of the value r.

In place of fx with $\frac{dfx}{dx}$ we might employ $\frac{dfx}{dx}$ with $\frac{d^2fx}{dx^2}$ and so on for the rest; the values of $k_2, k_3 \ldots k_r$ will remain unaffected by this change; but the former method would be more expeditious in practice.

The total multiplicity is, of course, $= k_1$.

Suppose now that we propose to ourselves the converse problem to determine the conditions that an algebraic equation may have a given amount of multiplicity distributed in a given manner.

If $h_1, h_2, h_3 \ldots h_r$ be used to denote the given number of partial multiplicities which are respectively of the values $1, 2, 3 \ldots r$, it is easy to see that the quantities derived above by $k_1, k_2 \ldots k_r$ are respectively equal to

$$h_1 + 2h_2 + \ldots\ldots + rh_r,$$
$$h_2 + 2h_3 + \ldots\ldots + rh_{r-1},$$
$$h_3 + 2h_4 + \ldots\ldots + rh_{r-2},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$h_r.$$

Now from $\frac{dfx}{dx}$ having a factor of the degree k_1 common with fx we obtain k_1 conditions, from $\frac{df_1x}{dx}$ having a factor of the degree k_2 common with f_1x we obtain k_2 more, and so on. So that altogether we obtain in this way

$$k_1 + k_2 + \ldots\ldots + k_r \text{ conditions.}$$

But it may easily be seen that the total multiplicity being k_1, the number of conditions *need* never to exceed k_1 in number, no matter what its distribution may be. Hence, besides the enormous labour of the process, and the extreme complexity of the results, we obtain by this method more equations by far than are necessary, and it requires some caution to know which to reject.

In my forthcoming paper (to appear in *Philosophical Magazine* of next month) I shall show, by a most simple means, how without the use of derived or other subsidiary functions, to obtain the simplest equations of condition which correspond to a given distribution of a given amount of multiplicity.

The total multiplicity, say m, being given in as many ways as that number can be broken into parts, so many different systems of m equations can be formed differing each from the other in the dimensions of the terms.

These systems may be arranged in order so that each in the series shall imply all those that follow it, and be implied in all those that go before, without the converse being satisfied.

The subject of the unreciprocal implication of systems of equations is a very curious one, upon which the limits assigned to me prevent me from enlarging at present. It is closely connected with a part of the theory of elimination, which, as far as I am aware, has either been overlooked, or has not met with the attention which it deserves; I mean the theory of *Special Factors.*

An *example* may make what I mean by these clear.

Let C be a function (if my reader please) void of x, which equivalent to zero implies two given equations in x having a common root.

Let C be rid of all irrelevant factors, that is, let C be the simplest form of the determinant, when the coefficients of the two equations are perfectly independent qualities. Now suppose, as is *quite possible in a variety of ways,* that such relations are instituted between the coefficients alluded to as make C split up into factors, so that $C = L \times M \times N = 0$.

Only one of the factors L, M, N will satisfy the condition of the co-existence of the two given equations: the others are clearly, however, not to be confounded with factors of solution, or irrelevant factors, as they are termed, but are of quite a different nature, and enjoy remarkable properties, which point to an enlarged theory of elimination, and constitute what I call special or singular factors.

I shall feel much obliged to any of the readers of your widely circulated Journal, interested in the subject of this paper, who would do me the honour of communicating with me upon it, and especially if they would (between now and the next coming out of the *Magazine*) inform me whether any-thing, and if so how much, different from what is here stated has been done in the matter of determining the relations between the coefficients of an equation corresponding to a given amount and distribution of multiplicity in its roots.

I ought to add, that my method enables me not merely to determine the conditions of multiplicity, but also to decompose the equations containing multiple roots into others free of multiplicity, that is, to find, *à priori*, the values of the several quantities

$$\frac{fx f_2x}{(f_1x)^2},\ \frac{f_1x f_3x}{(f_2x)^2},\ \ldots\ldots,\ \frac{f_{r-1}x}{(f_rx)^2},\ f_rx.$$

Moreover, other decompositions, not necessary to be enlarged upon in this place, may be obtained with equal facility.

14.

A NEW AND MORE GENERAL THEORY OF MULTIPLE ROOTS.

[*Philosophical Magazine*, XVIII. (1841), pp. 249—254.]

I SHALL begin with developing the theory of polynomials containing perfect *square factors*, one or more.

First, let us proceed to determine the relations which must exist between the coefficients of such polynomials, and afterwards show how they may be broken up into others of an inferior degree.

A parallelogram filled with letters standing in *one* row is intended to express the product of the squared difference of the quantities contained. Thus $(\overline{ab})$ indicates $(a-b)^2$, $(\overline{abc})$ is used to indicate $(a-b)^2(a-c)^2(b-c)^2$, and so forth.

Suppose now that two of the roots $e_1, e_2 \ldots e_n$ belonging to the equation $fx = 0$ are equal to one another, it is clear that $(\overline{e_1, e_2 \ldots e_n}) = 0$; and moreover is a symmetric function, and can be calculated in terms of the coefficients of fx.

Next let us suppose that we have two couples of equals (as for instance a and b, two of the roots equal, as also c and d two others), it is clear, that on leaving any one of the roots out, the $(n-1)$ that are left will still contain one equality, and therefore we have

$$(\overline{e_2, e_3 \ldots e_n}) = 0,\quad (\overline{e_1, e_3 \ldots e_n}) = 0 \ldots (\overline{e_1, e_2 \ldots e_{n-1}}) = 0.$$

None of the parallelogrammatic functions above taken *singly*, are symmetric functions of the coefficients, but their sum is; so also is the sum of the product of each into the quantity left out.

Now in general, suppose that the polynomial fx contains r perfect square factors, so that we have r couples of equal roots belonging to the equation $fx = 0$, it is clear that $(\overline{e_r, e_{r+1} \ldots e_n})$ and all the other $\dfrac{n(n-1)\ldots(n-r+2)}{1.2\ldots(r-1)}$ functions of which it is the type are severally zero. Moreover, the sum of these or the sum of the products of each by *any* symmetrical function of the $(r-1)$ letters left out will be a symmetrical function of the coefficients of the powers of x in fx. To express now the *affirmative** conditions corresponding to the case of there being r pairs of equal roots, we *might* employ the r equations,

$$\left(\overline{\underline{e_1, e_2 \dots e_n}}\right) = 0,$$
$$\Sigma \left(\overline{\underline{e_2, e_1 \dots e_n}}\right) = 0,$$
$$\Sigma \left(\overline{\underline{e_3 \dots e_n}}\right) = 0,$$
$$\dots\dots\dots\dots\dots\dots$$
$$\Sigma \left(\overline{\underline{e_r, e_{r+1} \dots e_n}}\right) = 0.$$

But these, except the last, are not the *simplest* that can be employed; that is to say, we can write down r others, the terms of which shall be of lower dimensions in respect to the roots.

Let $\int_\mu$ denote that any rational symmetrical function of the μth degree is to be taken of the quantities which it precedes.

Then the r equations in question are all contained in the general equation

$$\Sigma \left\{ \int_\mu (e_1, e_2 \dots e_{r-1}) \times \left(\overline{\underline{e_r, e_{r+1} \dots e_n}}\right) \right\} = 0;$$

μ being taken from 0 up to $(r-1)$ we obtain r equations, which in respect to the roots are respectively of all degrees between

$$\frac{n(n-1)\dots(n-r+2)}{1.2\dots(r-1)} \text{ and } \frac{n(n-1)\dots(n-r+2)}{1.2\dots(r-1)} + (r-1)$$

reckoned inclusively.

Now at this stage it is important to remark that the above r equations, although *necessary*, are not *sufficient*; and indeed, no mere affirmations of equality can be sufficient to ensure there being r pairs of equal roots.

To make this manifest, suppose $r=2$. Then in order that an equation *may* have two pairs of equal roots, we must have by the above formula

$$\Sigma \left(\overline{\underline{e_2, e_3 \dots e_n}}\right) = 0, \quad \Sigma \left\{ e_1 \left(\overline{\underline{e_2, e_3 \dots e_n}}\right) \right\} = 0.$$

But if instead of there being two perfect square factors there be one perfect *cube* factor in fx, it may be shown by the same reasoning as above, that the very same two equations apply. In fact, it may be shown in general that no such equations as those given above can be *affirmed* in consequence of there being an amount r of multiplicity consisting of unit parts which may not be affirmed with equal truth as necessary consequences of the same

* The importance of the restriction hinted at by the use of the word affirmative will appear hereafter.

amount distributed *in any other manner* whatever. How to obtain affirmative equations sufficient as well as necessary (under certain limitations) will appear at the close of this present paper.

It is worthy of being remarked, that if we make $\int_\mu$ denote the sum of the products of the quantities to which it is prefixed, taken μ and μ together, the equations of affirmation become identical with those obtained by eliminating between fx and $\frac{dfx}{dx}$*.

It can scarcely be doubted that the illustrious Lagrange, had he chosen to perfect the incomplete theory of equal roots given in the *Résolution Numérique*, by applying to it his own favourite engine of symmetric functions, could scarcely have failed of stumbling by a back passage upon Sturm's memorable theorem.

Let us now proceed to show how a polynomial known to contain one or more perfect square factors may be decomposed.

Let us begin with supposing that it contains but one such factor; so that $fx = \phi x (x-a)^2$.

I shall show how to obtain the equations

$$C(x-a) = 0, \quad D\phi x (x-a) = 0, \quad E(x-a)^2 = 0, \quad F(\phi x) = 0,$$

each in its lowest terms.

1. To form the equation $Lx + M = 0$, where $x = a$, it is easy to see that if we write down in general the expression $(x-e_1)\left(\overline{\underline{e_2, e_3 \dots e_n}}\right)$ this will become zero whenever the root e_1 left out is not one of the equal roots (a): so that in fact (calling the two equal roots e_1, e_2 respectively)

$$\Sigma \left\{ (x-e_1) \times \left(\overline{\underline{e_2, e_3 \dots e_n}}\right) \right\} = (x-e_1) \times \left(\overline{\underline{e_2, e_3 \dots e_n}}\right) + (x-e_2) \times \left(\overline{\underline{e_1, e_3 \dots e_n}}\right),$$

or simply

$$= 2(x-a)\left(\overline{\underline{e_2, e_3 \dots e_n}}\right).$$

Hence by making

$$x\Sigma \left(\overline{\underline{e_2, e_3 \dots e_n}}\right) - \Sigma \left\{ e_1 \times \left(\overline{\underline{e_2, e_3 \dots e_n}}\right) \right\} = 0,$$

we have an equation for finding the equal roots e_1, e_2.

Again, it is easily seen upon the same hypothesis, that

$$\Sigma \left\{ (x-e_2)(x-e_3)(x-e_4) \dots (x-e_n) \times \left(\overline{\underline{e_2, e_3 \dots e_n}}\right) \right\}$$
$$= 2(x-e_2)(x-e_3)\dots(x-e_n) \times \left(\overline{\underline{e_2, e_3 \dots e_n}}\right).$$

* See my note on Sturm's Theorem, *Phil. Mag.*, December, 1839 [p. 45 above. Ed.].

Hence, to form the equation having the same roots as $(x-a)\,\phi x$, we have only to make

$$x^{n-1}\Sigma\left(\overline{e_2, e_3 \ldots e_n}\right) - x^{n-2}\Sigma\left\{(e_2+e_3+\ldots e_n)\times\left(\overline{e_2, e_3 \ldots e_n}\right)\right\}\ldots\ldots$$
$$\pm\,\Sigma\left\{(e_2 e_3 \ldots e_n)\times\left(\overline{e_2, e_3 \ldots e_n}\right)\right\} = 0.$$

Suppose now in general that we have r perfect square factors, so that

$$fx = \phi x\,(x-a_1)^2\,(x-a_2)^2 \ldots (x-a_r)^2.$$

To form the equation $C(x-a_1)(x-a_2)\ldots(x-a_r) = 0$, we have only to make

$$\Sigma\left\{(x-e_1)(x-e_2)\ldots(x-e_r)\times\left(\overline{e_{r+1}, e_{r+2} \ldots e_n}\right)\right\} = 0.$$

And to obtain

$$D\phi x \times (x-a_1)(x-a_2)\ldots(x-a_r) = 0,$$

we must make

$$\Sigma\left\{(x-e_{r+1})(x-e_{r+2})\ldots(x-e_n)\times\left(\overline{e_{r+1}, e_{r+2} \ldots e_n}\right)\right\} = 0.$$

The theory of perfect square factors is not yet complete until it has been shown how to obtain constructively ϕx, and, as analogy suggests, the complementary part $D'(x-a_1)^2(x-a_2)^2\ldots(x-a_r)^2$, each in its lowest terms. To effect the latter it might be said that it is only necessary to take the square of $C(x-a_1)(x-a_2)\ldots(x-a_r)$. It is true the polynomial so formed would contain every pair of equal factors, but not in the lowest terms as regards the coefficients (as we shall presently show).

To solve this last part of the problem, let it be agreed that two rows of letters inclosed in a parenthesis shall indicate the product of the squares of the differences got by subtracting each in the row from each in the other, so that

$$\begin{pmatrix} a \\ b \end{pmatrix} = (a-b)^2, \quad \begin{pmatrix} a & \\ b & c \end{pmatrix} = (a-b)^2(a-c)^2, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a-c)^2(a-d)^2(b-c)^2(b-d)^2.$$

Let us begin with supposing that fx has one pair only of equal roots; to form the simplest quadratic equation containing this pair, write down

$$(x-e_1)(x-e_2)\times\left(\overline{e_3, e_4 \ldots e_n}\right)\times\begin{pmatrix} e_1, e_2 & \\ e_3, e_4 \ldots e_n \end{pmatrix}.$$

Now if e_1 and e_2 are the two equal roots in question neither of the multipliers of $(x-e_1)(x-e_2)$ vanishes.

If e_1 and e_2 are neither of them equal roots $\left(\overline{e_3, e_4 \ldots e_n}\right) = 0$.

If one of the two only belong to the pair of equal roots

$$\begin{pmatrix} e_1, e_2 & \\ e_3, e_4 \ldots e_n \end{pmatrix} = 0.$$

Hence it is clear that

$$\Sigma\left\{(x-e_1)(x-e_2)\times\left(\overline{e_3, e_4 \ldots e_n}\right)\times\begin{pmatrix} e_1, e_2 & \\ e_3, e_4 \ldots e_n \end{pmatrix}\right\} = 0$$

is the equation desired.

In like manner if there be r pairs of equal roots the equation of the $(2r)$th degree which contains them all may be written

$$\Sigma\left\{(x-e_1)(x-e_2)\ldots(x-e_{2r})\times\left(\overline{e_{2r+1} \ldots e_n}\right)\times\begin{pmatrix} e_1, e_2 \ldots e_{2r} \\ e_{2r+1} \ldots e_n \end{pmatrix}\right\} = 0.$$

The coefficient of x^{2r} in this equation is clearly of

$$(n-2r)(n-2r-1) + 4r(n-2r),$$

that is, of $(n+2r-1)(n-2r)$ dimensions. The coefficient of x^r in the equation which contains the r equal roots unyoked together is of $(n-r)(n-r-1)$ dimensions, and consequently the coefficient of x^{2r} in the square of this equation would be of $2(n-r)(n-r-1)$ dimensions, that is, would be $n^2+6r^2-(4r+1)\,n$ dimensions higher than needful.

Finally, to obtain an equation clear of *simple* as well as double appearances of the equal roots, we have only to write the complementary form

$$\Sigma\left\{(x-e_{2r+1})(x-e_{2r+2})\ldots(x-e_n)\times\left(\overline{e_{2r+1}+e_n}\right)\times\begin{pmatrix} e_1, e_2 \ldots e_{2r} \\ e_{2r+1} \ldots e_n \end{pmatrix}\right\} = 0.$$

Let us, now that we are more familiarized with the notation essential to this method, revert to the question with which we set out, and endeavour to obtain r such equations as shall imply *unambiguously* the existence of r pairs of equal roots.

The existence of r such pairs enables us to assert the following disjunctive proposition, which cannot be asserted when the *same amount* of multiplicity is distributed in any other way.

To wit, on selecting any r roots out of the entire number, either these r will all be found again in those that are left, *or* those that are left will contain *inter se*, one repetition at least; so that except on the latter supposition any $(r-1)$ may be absolutely sunk out of those that are left, and there will still be *one* root common to the $(n-2r+1)$ remaining, and to the r originally selected to be left out.

Wherefore calling the roots $e_1, e_2 \ldots e_n$, and giving μ any value whatever, we have

$$\Sigma\left\{\int_\mu (e_1, e_2 \ldots e_r)\times\left(\overline{e_{r+1}, e_{r+2} \ldots e_n}\right)\times\Sigma\begin{pmatrix} e_1, e_2 \ldots e_r & \\ e_{2r}, e_{2r+1} \ldots e_n \end{pmatrix}\right\} = 0.$$

Hence the simplest distinctive equations indicative of the existence of r pairs of equal roots are to be found by putting μ equal in succession to all values from 0 up to $(r-1)$.

For instance, if we require that an equation of the seventh degree shall have three pairs of equal roots, we need only to call the seven roots respectively a, b, c, d, e, f, g, and then our type equation becomes

$$\Sigma\left\{\int_{\mu}(a\,b\,c)\times(\overline{d\,e\,f\,g})\times\left\{\begin{matrix}\binom{d\,e}{a\,b\,c}+\binom{d\,f}{a\,b\,c}+\binom{d\,g}{a\,b\,c}\\+\binom{e\,f}{a\,b\,c}+\binom{e\,g}{a\,b\,c}+\binom{f\,g}{a\,b\,c}\end{matrix}\right\}\right\}=0.$$

From this it appears that the r distinctive equations for r pairs of equal roots are of different dimensions from the r general or overlying ones corresponding to the multiples r, anyhow distributed; the lowest of the latter being of $(n-r+1)(n-r)$, the lowest of the former of

$$(n-r)(n-r-1)+2r(n-2r+1),$$

that is, of $n(n-1)-3r(n-1)$ dimensions. In general we shall find that the more unequally distributed the multiplicity may be the lower are the dimensions of the distinctive equations, and are accordingly lowest when the multiplicity is absolutely undistributed*.

* It must not, however, be overlooked, that the equations above given, although decisive as to the existence of r pairs of equal roots *when* the multiplicity is known to be not greater than r, do not enable us to affirm with certainty their existence when this limitation is absent: for should the multiplicity exceed r, then inevitably (no matter how it may be distributed) $(\overline{e_{r+1}, e_{r+2}\dots e_n})$ is always zero, and consequently nullifies each term of every one of the equations in question. In fact (repugnant as it may appear to be to the ordinary assumptions of analytical reasoning), it is not possible to express with *absolute* unambiguity the conditions of there being a multiplicity (r) distributed in any assigned manner by means of r *affirmative equations alone*.

15.

ON A LINEAR METHOD OF ELIMINATING BETWEEN DOUBLE, TREBLE, AND OTHER SYSTEMS OF ALGEBRAIC EQUATIONS.

[*Philosophical Magazine*, XVIII. (1841), pp. 425—435.]

PART I. BINARY SYSTEMS.

LET U and V be two integer complete homogeneous functions of x and y, one of the mth, the other of the nth degree; and let it be required to express the condition of the coexistence of the two equations $U=0$, $V=0$ by means of the equation $C=0$, where C is free from all appearances of x or y.

This equation, according to the system of notation developed in a preceding paper, and which has been since adopted and sanctioned by the high authority of M. Cauchy, I call the final derivative: the quantity C is designated the final derivee: and it is our present object to show how this may be obtained in a *prime* form, that is to say, divested of irrelevant factors: in this state it must consist of terms, each containing $m+n$ letters, of which n belong to the coefficients of U, and m to those of V.

Of course in applying this rule it is to be understood that every combination of powers in U or V has a single letter prefixed for its coefficient, and that in the final derivee powers are represented by repetitions of the same character.

Every term in U or V being of the form Cx^py^q, x^py^q is called an argument, C its prefix.

Assume two integer positive numbers r and r', and also two others s and s', such that $r+r'=n-1$, $s+s'=m-1$, and form from $U=0$, $V=0$ two new equations,

$$x^ry^{r'}U=0,\quad x^sy^{s'}V=0.$$

Such equations are termed the augmentatives of the two given ones respectively; also $x^ry^{r'}U$ and its fellow are termed the augmentees of U and V.

r and r' are termed the indices of augmentation belonging to U, s and s' the same belonging to V.

Finally, it will be useful hereafter to call the given polynomials U and V themselves the proposees, and the given equations which assert their nullity, the propositive equations, or, briefly, the propositives.

Now as many augmentees of either proposee can be formed as there are ways of stowing away between two lockers (vacancies admissible) a number of things equal to the index of the other*; hence we shall have n augmentees of U, and m of V: thus there will be $m+n$ augmentatives each of the degree $m+n-1$, and the number of arguments is clearly $m+n$ also, so that they can be eliminated linearly, and the final derivee thus found, containing $m+n$ letters (properly aggregated) in each term, will be in its prime form, that is, incapable of further reduction, and void of irrelevant factors.

It is worthy of remark, that the final derivee obtained by arranging in square battalion the prefixes of the augmentees, permuting the rows or columns, and reading off diagonal products, affected each with the proper sign (according to the well known rule of Duality), will not only be free from factorial irrelevancy, but also of linear redundancy, which latter term I use to signify the reappearance of the same combination of prefixes, sometimes with positive and sometimes with negative signs: furthermore, it follows obviously from the nature of the process that no numerical quantity in the final derivee will be greater than the higher of the indices of the two given polynomials.

PART II. TERNARY SYSTEMS.

CASE A. *Indices all equal.*

Method 1.

Let there be now three proposees, U, V, W, integer complete homogeneous functions of x, y, z, each of the degree n: let

$$r+r'+r''=n-1, \quad s+s'+s''=n-1, \quad t+t'+t''=n-1,$$

$$x^r y^{r'} z^{r''} U, \; x^s y^{s'} z^{s''} V, \; x^t y^{t'} z^{t''} W,$$

will, as above, be called the augmentees of U, V, W, and every other part of the notation previously described is to be preserved.

* "Tot Augmenta utriusvis ex æquationibus propositis formari possunt quot modi sint inter duo receptacula (utrivis vel ambobus omnino vacare licet) rerum, quarum numerus indicem alterius æquat, distributionem faciendi."

Suppose now

$$U=0, \quad V=0, \quad W=0,$$

we shall have as many augmentative equations formed from each proposee as there are ways of stowing away n things between *three* lockers (vacancies admissible)*, that is, $n\dfrac{n+1}{2}$ of each kind; in all, therefore, $3\dfrac{n(n+1)}{2}$, and every one of these will be of the degree $2n-1$, so that the number of arguments to be eliminated is equal to the number of ways of stowing away $2n-1$ things between three lockers (empty ones counting), that is

$$\frac{2n(2n+1)}{2}.$$

As yet, then, we have not *enough* equations for eliminating these linearly.

Make, however,

$$\alpha+\beta+\gamma=n+1,$$

and write

$$U=x^\alpha F+y^\beta F'+z^\gamma F''=0,$$
$$V=x^\alpha G+y^\beta G'+z^\gamma G''=0,$$
$$W=x^\alpha H+y^\beta H'+z^\gamma H''=0,$$

it will always be possible to make the multipliers of x^α, y^β, z^γ integer functions: for if we look to any argument in U, V, or W, it is of the form $x^a y^b z^c$, and one of the letters a, b, c must be *not less* than its correspondent α, β, γ, for otherwise $a+b+c$ would be not greater than $\alpha+\beta+\gamma-3$, that is, n would be not greater than $(n+1)-3$, or $n-2$, which is absurd: if now any one, as a, be equal to or greater than α, it may be made to supply an integer part to the multiplier of x^α.

Here it may be asked what is to be done with such terms as $Kx^a y^b z^c$, when two letters a, b are each not less than their correspondents α, β: the answer is, such terms may be made to enter under the multiplier of x^α, or of x^β, or to supply a part to both in any proportion at pleasure†.

From the equations above we get, by linear elimination,

$$FG'H''+GH'F''+HF'G''-GF'H''-HG'F''-FH'G''=0.$$

This may be denoted thus: $\Pi(\alpha, \beta, \gamma)=0$, which equation I call a *secondary* derivative, and the left side of it a *secondary* derivee; α, β, γ may likewise be termed the indices of derivation (as r, s, t, &c. are of augmentation).

Now since $\alpha+\beta+\gamma=n+1$, it is clear that the index of $\Pi(\alpha, \beta, \gamma)$ is always $n+n+n-(n+1)$; that is, $2n-1$.

* See for Latin translation the preceding note.

† The prefixes of any such terms (say K) may be conceived as made up of two parts, an arbitrary constant, as e and $(K-e)$; e will disappear spontaneously from the final derivee.

1st. Let any two of the indices of derivation be taken zero, then it is easily seen that all the terms in $\Pi(\alpha, \beta, \gamma)$ vanish, and consequently the secondary derivative equations obtained upon this hypothesis become mere identities, and are of no use.

2nd. Let any *one* of them become zero.

It is manifest, from the doctrine of simple equations, that $\Pi(\alpha, \beta, \gamma)$ may be made equal to

$$\left\{\lambda U + \mu V + \nu W\right\} \frac{1}{x^{\alpha}},$$

or

$$\left\{\lambda' U + \mu' V + \nu' W\right\} \frac{1}{x^{\beta}},$$

or

$$\left\{\lambda'' U + \mu'' V + \nu'' W\right\} \frac{1}{x^{\gamma}},$$

upon the understanding that

$$\lambda = G'H'' - G''H', \quad \mu = H'F'' - H''F', \quad \nu = F'G'' - F''G',$$
$$\lambda' = G''H - GH'', \quad \mu' = H''F - HF'', \quad \nu' = F''G - FG'',$$
$$\lambda'' = GH' - G'H, \quad \mu'' = HF' - H'F, \quad \nu'' = FG' - F'G.$$

The three rows of coefficients will be respectively of the degrees

$$(n-\beta)+(n-\gamma), \quad (n-\gamma)+(n-\alpha), \quad (n-\alpha)+(n-\beta).$$

Thus if any one of the indices α, β, γ be zero, $\Pi(\alpha, \beta, \gamma)$ becomes identical with $\lambda' U + \mu' V + \nu' W$, where the multipliers of U, V, W are of $2n-(\alpha+\beta+\gamma)$ dimensions, that is of $(n-1)$ dimensions, and may accordingly be put under the form

$$\Sigma A x^r y^{r'} z^{r''} U + \Sigma B x^s y^{s'} z^{s''} V + \Sigma C x^t y^{t'} z^{t''} W,$$

that is to say, becomes a *linear function* of the augmentatives, and therefore if combined with them in the process of linear elimination would *give rise* to the identity $0 = 0$.

Hence we must reject all such secondary derivatives as have zero for one of the indices of derivation. But all others, it may be shown, will be *linearly* independent of one another, and of the augmentees previously found. Hence, besides $3\frac{n(n+1)}{2}$ equations of augment of the degree $2n-1$, we shall have of the same degree so many equations of derivation as there are ways of stowing away between three lockers $(n+1)$ things, under the condition that no locker shall ever be left *empty*, that is $\frac{n(n-1)}{2}$*.

Thus, then, in all we have $n\frac{n-1}{2} + 3\frac{n(n+1)}{2} = \frac{2n(2n+1)}{2}$ equations, which is exactly equal to the number of arguments to be eliminated. Hence

* Vide page 76 for the Latin version.

the final derivee can be obtained by the usual explicit rule of permutation, and moreover will be *its lowest form*, for it will contain in each term $\frac{n(n+1)}{2}$ prefixes belonging to the augmentatives of U, and a like number pertaining to those of V and of W, as well as $n\frac{n-1}{2}$ belonging to the secondary derivatives, each prefix in any one of which is triliteral, containing a prefix drawn out of those belonging to each of the proposees.

Thus every member containing $n\frac{n+1}{2} + n\frac{n-1}{2}$, that is n^2 of the original prefixes belonging to U, V, W, singly and respectively, the final derivee evolved by this process will be in its lowest terms; as was to be proved.

Case A. *Indices all equal.*

Method 2.

It is remarkable that we may vary the method just given by making

$$r + r' + r'' = n-2, \quad s + s' + s'' = n-2, \quad t + t' + t'' = n-2.$$

The augmentatives will thus be of the degree $2n-2$.

Furthermore, we must make $\alpha + \beta + \gamma = n+2$. It will still be possible to satisfy by integer multipliers the equations

$$U = x^{\alpha}F + y^{\beta}F' + z^{\gamma}F'',$$
$$V = x^{\alpha}G + y^{\beta}G' + z^{\gamma}G'',$$
$$W = x^{\alpha}H + y^{\beta}H' + z^{\gamma}H'',$$

[these it will be useful in future to term the *equations*, $x^{\alpha}, y^{\beta}, z^{\gamma}$ being the *arguments*, and F, G, H, &c. the *factors* of decomposition] for otherwise calling the indices of x, y, z in any original argument a, b, c, their sum or n would be not greater than $(n+2)-3$, that is $(n-1)$, which is absurd.

For the same reasons as in the last case no index of augmentation must be made zero: the degree of each will be $(n-\alpha)+(n-\beta)+(n-\gamma)$, that is $(2n-2)$, and their number $\frac{(n+1)n}{2}$; the number of augmentatives will be $\frac{3(n-1)n}{2}$ linearly uninvolved, each of the degree $2n-2$, and therefore containing $\frac{(2n-1)2n}{2}$ arguments.

Now $$\frac{(n+1)n}{2} + \frac{3(n-1)n}{2} = \frac{(2n-1)2n}{2}.$$

Hence the final derivee may be found, and it will be in its *lowest terms*, for every member will contain $\frac{3(n-1)n}{2}$ letters due to the augmentative, and $\frac{3(n+1)n}{2}$ due to the partial derivative equations; in all then there will be $3n^2$ letters in each term.

This second method being applied to three quadratic equations of the most general form, leads to the problem of eliminating between six simple equations which lies within the limits of practical feasibility, and it is my intention to register the final derivee upon the pages of some one of our scientific Transactions as a standing monument for the guidance of hereafter coming explorers*.

Scholium to Case A.

If we attempt to carry forward these processes to quaternary systems, it becomes necessary to make

$$\alpha+\beta+\gamma+\delta=(r-2)n+1$$

or else

$$\alpha+\beta+\gamma+\delta=(r-2)n+2,$$

where r is the number of proposees.

Now if the factors in the equations of decomposition are all integer, one of the indices of derivation must be not greater than the corresponding index in any of the original arguments, which may easily be shown to be always impossible for a system of equations, *complete* in *all* their terms, whenever their number r is greater than three, if $\alpha+\beta+\gamma+\delta=(r-2)n+2$; but if $\alpha+\beta+\gamma+\delta=(r-2)n+1$ only possible for the case of $n=2$.

Particular method applicable to *four Quadratics*.

Let $U=0$, $V=0$, $W=0$, $Z=0$, be four quadratic equations existing between x, y, z, t.

Make

$$\begin{array}{llll} xU=0, & xV=0, & xW=0, & xZ=0, \\ yU=0, & yV=0, & yW=0, & yZ=0, \\ zU=0, & zV=0, & zW=0, & zZ=0, \\ tU=0, & tV=0, & tW=0, & tZ=0. \end{array}$$

* Elimination between *two* quadratics leads to a final derivee made up of *seven* terms only; the final derivee of *three* quadratics is made up of at least several *thousand*; nay, I believe I may safely say, several *myriads* of terms!

Also write

$$U=x^2F+yF'+zF''+tF'''=0,$$
$$V=x^2G+yG'+zG''+tG'''=0,$$
$$W=x^2H+yH'+zH''+tH'''=0,$$
$$Z=x^2K+yK'+zK''+tK'''=0.$$

By eliminating linearly we get

$$\Sigma\{F\Sigma G'(H''K'''-H'''K'')\}=0,$$

which will be of the third degree, since the factors represented by the *unmarked* letters F, G, H, K are of zero, and all the rest of *unit* dimensions.

Similarly we may obtain other equations, so that besides the *sixteen* augmentatives already written down, we have four secondary derivatives, namely,

$$\Pi(2\ 111)=0, \quad \Pi(12\ 11)=0, \quad \Pi(11\ 21)=0, \quad \Pi(111\ 2)=0.$$

Thus we have *twenty* equations and as many arguments to eliminate, since a perfect cubic function of four letters contains twenty terms.

The final derivee will contain $16+4.4$ letters, that is 32, 8 or 2^3 belonging to each system of original prefixes in each member, and will therefore be in its lowest terms: for one of the canons of form teaches us, *à priori*, that every member of the derivee deduced from any number of assumed equations must contain in each member as many prefixes belonging to one equation of the system as there are units in the product of the indices of all the rest taken together.

Corollary to Case A.

Either of the two methods given as applicable to this case enables us to determine integer values of X, Y, Z, which shall satisfy the equation

$$XU+YV+ZW=Fx^py^qz^r,$$

where F is the final derivee and $p+q+r=3n-2$. For by the doctrine of simple equations we know how to express F in terms of the linear functions, out of which it is obtained by permutation, that is we are able to assign values of A, B, C, and their antitypes, as also of L and its antitype, which shall satisfy the equation

$$\Sigma(Ax^ry^{r'}z^{r''}U)+\Sigma(Bx^sy^{s'}z^{s''}V)+\Sigma(Cx^ty^{t'}z^{t''}W)$$
$$+\Sigma\{L\Pi(\alpha,\beta,\gamma)\}=Fx^fy^gz^h, \qquad (1)$$

where A, B, C, as well as L and all the quantities formed after them, are made up of integer combinations of the original prefixes.

Now the functions $\Pi(\alpha,\beta,\gamma)$ may be expressed in three ways in terms of U, V, W, as has been already shown.

We may therefore suppose these functions to be divided into three groups, and make

$$\Sigma L \Pi (\alpha\beta\gamma) = \Sigma \frac{QU + Q'V + Q''W}{x^{\alpha}} + \Sigma \frac{RU + R'V + R''W}{x^{\beta}} + \Sigma \frac{SU + S'V + S''W}{x^{\gamma}}. \quad (2)$$

And it is evident that the equations (1) and (2) lead immediately to the equation

$$XU + YV + ZW = Fx^{a+f}y^{b+g}z^{c+h},$$

if we call a, b, c the *greatest values* attributed respectively to α, β, γ.

Now if we suppose the first method to be followed,

$$f + g + h = 2n - 1.$$

And it will always be possible to make a, b, c of what values we please *subject* to the condition of $a + b + c = n - 1$; for *one* at least of the indices of derivation in $\Pi(\alpha, \beta, \gamma)$ must be not greater than its correspondent among a, b, c; otherwise $\alpha + \beta + \gamma$ would be not less than $(a + b + c) + 3$; but

$$\alpha + \beta + \gamma = n + 1$$

$$a + b + c = n - 1,$$

which is absurd.

Hence we can satisfy $XU + YV + ZW = Fx^p y^q z^r$, p, q, r being subject to the condition of $p + q + r = 3n - 2$, but otherwise arbitrary.

Moreover, we can *not* do so if $p + q + r$ be *less* than $3n - 2$, for that would require $a + b + c$ to be less than $n - 1$. Now if two of the indices of derivation, as α and β, be made equal to $a + 1$, $b + 1$ respectively, the third $\gamma = (n + 1) - (a + b + 2) = (n - 1) - (a + b)$, and is therefore greater than c: so that $\alpha + \beta + \gamma$ for this case becomes greater than $a + b + c$, and the method falls to the ground.

In fact, I have discovered a theorem which lets me know this, *à priori*, a law which serves as a staff to guide my feet from falling into error in devising linear methods of solution, and the importance of which all candid judges who have studied the general theory of elimination cannot fail to recognize. To wit, if $X_1, X_2, X_3 \ldots X_n$ be n integer complete polynomial functions of n letters $x_1, x_2 \ldots x_n$, and severally of the degree $b_1, b_2, b_3 \ldots b_n$; then it is always possible to satisfy the identity

$$P_1 X_1 + P_2 X_2 + P_3 X_3 + \ldots + P_n X_n = F x_1^{a_1} x_2^{a_2} x_3^{a_3} \ldots x_n^{a_n},$$

if $a_1 + a_2 + a_3 + \ldots + a_n$ be equal to or greater than $b_1 + b_2 + b_3 + \ldots + b_n - n + 1$, but otherwise *not**.

This again is founded immediately upon a simple proposition, of which I have obtained a very interesting and instructive demonstration, shortly to appear, and which may be enumerated thus: "*The number of augmentees of the same degree that can be formed, linearly independent of one another, out of any number of polynomial functions of as many variables, may be either equal to or less than the number of distinct arguments contained in such augmentees, but* never *greater. The latter will be the case when the index of the augmentees diminished by unity is less than the sum of the indices of the original unaugmented polynomials each so diminished; the former, when the aforesaid index is equal to or greater than the aforesaid sum.*"

To return to the particular case of finding X, Y, Z to satisfy

$$XU + YV + ZW = Fx^p y^q z^r.$$

This has been already done according to the first method; if we employ the *second* method of elimination we shall have

$$f + g + h = 2n - 2.$$

But, now since $\alpha + \beta + \gamma = n + 2$, we shall easily see by the same method as above, that the least value of $a + b + c$ {where a, b, c denote respectively the greatest values of α, β, γ, appearing in the denominator of the fractional *forms* used to express $\Pi(\alpha, \beta, \gamma)$}, will be one greater than before, or n; so that $f + g + h + a + b + c$ will still be equal to $3n - 2$, as we might, *à priori*, by virtue of our rule, have been assured.

Ternary Systems.

Case B. *Two of the indices equal; the third* less *by a unit.*

Let $U = 0$, $V = 0$, $W = 0$, be the three given equations severally of the degree $n, n, (n - 1)$.

* Hence it is apparent, that in applying the method of multipliers, a curious and important distinction exists between the cases of there being two equations, and there being a greater number to eliminate from: for in the first case the element of arbitrariness needs never to appear; in the *latter* it cannot possibly be excluded from appearing in the multipliers.

This will explain how it comes to pass that the method of the text may be employed to give *various* solutions of the $XU + YV + ZW = Fx^p y^q z^r$; thus not only can p, q and r be variously made up of $(f + a), (g + b), (h + c)$, but also $\Pi(\alpha, \beta, \gamma)$ when two of the indices (α, β suppose) are each not greater than the assigned greatest values a, b may be made to figure indifferently either under the form

$$\frac{\lambda U + \mu V + \nu W}{x^{\alpha}} \text{ or that of } \frac{\lambda' U + \mu' V + \nu' W}{x^{\beta}}.$$

Make $r + r' + r'' = n - 2,\quad s + s' + s'' = n - 2,\quad t + t' + t'' = n - 1,$
by multiplying U into $x^r y^{r'} z^{r''}$, V into $x^s y^{s'} z^{s''}$, W into $x^t y^{t'} z^{t''}$, we obtain augmentees each of the same, namely, the $(2n-2)$th degree.

The number of these is

$$\frac{(n-1)\,n}{2} + \frac{(n-1)\,n}{2} + \frac{n\,(n+1)}{2}.$$

Again, make $$\alpha + \beta + \gamma = n + 1.$$

It will still be possible, as before, to form equations of decomposition in which x^α, y^β, z^γ are the arguments, and affected with *integer* factors. For if we look to W *even*, all its arguments are of the form $x^a y^b z^c$, where $a + b + c = (n-1)$, and each of these cannot be less than its correspondent, for that would be to say that $(n-1)$ is not greater $(n+1) - 3$, *à fortiori*, U and V can be decomposed in the manner described. Thus, then, we shall obtain as many secondary derivees as in the last case (Method 1), that is, $\frac{n\,(n-1)}{2}$ {since $\alpha + \beta + \gamma$ is still equal to $(n+1)$}, as before. Moreover, each of these will be of $(n-\alpha) + (n-\beta) + (n-1-\gamma)$, that is of $2n-2$ dimensions.

Altogether, therefore, we have

$$\left\{\frac{(n-1)\,n}{2} + \frac{(n-1)\,n}{2} + \frac{n\,(n+1)}{2}\right\} + \frac{(n-1)\,n}{2}$$

linear independent equations of the degree $2n-2$, and the number of arguments to eliminate is $\frac{(2n-1)\,2n}{2}$. Now these two numbers are equal. Thus we obtain a final derivee containing of U's coefficients $\frac{(n-1)\,n}{2} + \frac{(n-1)\,n}{2}$, an equal number of V's, but of W's $\frac{n\,(n+1)}{2} + \frac{(n-1)\,n}{2}$; now $n\,(n-1)$, $n\,(n-1)$ and n^2 exactly express the number that ought to appear of each of these respectively: hence the final derivee is clear *of irrelevant factors*.

Ternary Systems.

Case C. *Two of the indices equal; the third one* greater *by a unit.*

Here, calling n the highest index, the augmentees must each be made of the degree $(2n-3)$, their number will evidently be

$$\frac{(n-2)\,(n-1)}{2} + \frac{(n-1)\,n}{2} + \frac{(n-1)\,n}{2},$$

making the sum of the indices of derivation now, as before, equal to $(n+1)$; it will be still possible to form integer equations of decomposition, which will give rise to augmentatives of the degree $(n-\alpha) + (n-1) - \beta + (n-1) - \gamma$, that is, of $(2n-3)$ dimensions. The total number of equations, what with augmentatives and secondary derivatives, will be

$$\left\{\frac{(n-2)(n-1)}{2} + \frac{(n-1)n}{2} + \frac{(n-1)n}{2}\right\} + \frac{n(n-1)}{2} = \frac{4n^2-4n+2}{2} = \frac{(2n-2)\,(2n-1)}{2},$$

that is, is equal to the exact number of distinct arguments contained between them.

Also the final derivative will contain in each member

$$\frac{(n-2)\,(n-1)}{2} + \frac{n\,(n-1)}{2},$$

that is, $(n-1)(n-1)$, letters belonging to the first equation, and

$$\frac{(n-1)\,n}{2} + \frac{n\,(n-1)}{2},$$

that is, $n\,(n-1)$ belonging to those of the second and of the third, and will therefore be in its *lowest terms.*

Corollary to Cases B and C.

It is not necessary, after all that has been already said, to do more than just point out that the processes applicable to these cases enable us to determine X, Y, Z, which satisfy the equation

$$XU + YV + ZW = F x^f y^g z^h,$$

where $$f + g + h = 3n - 3 \text{ for Case B},$$

and $$f + g + h = 3n - 4 \text{ for Case C}.$$

16.

MEMOIR ON THE DIALYTIC METHOD OF ELIMINATION. PART I.

[*Philosophical Magazine*, XXI. (1842), pp. 534—539*.]

THE author confines himself in this part to the treatment of two equations, the final and other derivees of which form the subject of investigation.

The author was led to reconsider his former labours in this department of the general theory by finding certain results announced by M. Cauchy in *L'Institut*, March Number of the present year, which flow as obvious and immediate consequences from Mr Sylvester's own previously published principles and method.

Let there be two equations in x,

$$U = ax^n + bx^{n-1} + cx^{n-2} + ex^{n-3} + \&c. = 0,$$

$$V = \alpha x^m + \beta x^{m-1} + \gamma x^{m-2} + \&c. \qquad = 0,$$

and let $n = m + \iota$, where ι is zero or any positive value (as may be).

Let any such quantities as $x^r U$, $x^s V$, be termed augmentatives of U or V.

To obtain the derivee of a degree s units lower than V, we must join s augmentatives of U with $s + \iota$ of V. Then out of $2s + \iota$ equations

$$x^0 U = 0, \quad x^1 U = 0, \quad x^2 U = 0, \ldots\ldots x^{s-1} U = 0,$$

$$x^0 V = 0, \quad x^1 V = 0, \quad x^2 V = 0, \ldots\ldots x^{\iota+s-1} V = 0,$$

we may eliminate linearly $2s + \iota - 1$ quantities.

Now these equations contain no power of x higher than $m + \iota + s - 1$; accordingly, all powers of x, superior to $m - s$, may be eliminated, and the derivee of the degree $(m - s)$ obtained in its prime form.

Thus to obtain the final derivee (which is the derivee of the degree zero), we take m augmentatives of U with n of V, and eliminate $(m + n - 1)$ quantities, namely,

$$x, \; x^2, \; x^3, \ldots\ldots \text{ up to } x^{m+n-1}.$$

* Reprinted from *Proc. Roy. Irish Acad.*, Vol. II. (1840—1844), p. 130.

This process, founded upon the dialytic principle, admits of a very simple modification. Let us begin with the case where $\iota = 0$, or $m = n$. Let the augmentatives of U be termed $U_0, U_1, U_2, U_3, \ldots$ and of V, $V_0, V_1, V_2, V_3, \ldots$, the equations themselves being written

$$U = ax^n + bx^{n-1} + cx^{n-2} + \&c.$$

$$V = a'x^n + b'x^{n-1} + c'x^{n-2} + \&c.$$

It will readily be seen that

$$a'U_0 - aV_0,$$

$$(b'U_0 - bV_0) + (a'U_1 - aV_1),$$

$$(c'U_0 - cV_0) + (b'U_1 - bV_1) + (a'U_2 - aV_2), \; \&c.$$

will be each linearly independent functions of $x, x^2, \ldots x^{m-1}$, no *higher* power of x remaining. Whence it follows, that to obtain a derivee of the degree $(m - s)$ in its prime form, we have only to employ the s of those which occur first in order, and amongst them eliminate $x^{m-1}, x^{m-2}, \ldots x^{m-s+1}$. Thus, to obtain the *final* derivee, we must make use of n, that is, the entire number of them.

Now, let us suppose that ι is not zero, but $m = n - \iota$. The equation V may be conceived to be of n instead of m dimensions, if we write it under the form

$$0x^n + 0x^{n-1} + 0x^{n-2} + \ldots + 0x^{m+1} + \alpha x^m + \beta x^{m-1} + \&c. = 0,$$

and we are able to apply the same method as above; but as the first ι of the coefficients in the equation above written are zero, the first ι of the quantities

$$(a'U_0 - aV_0), \quad (b'U_0 - bV_0) + (a'U_1 - aV_1), \; \&c.$$

may be read simply

$$-aV_0, \quad -bV_0 - aV_1, \quad -cV_0 - bV_1 - aV_2, \; \&c.$$

and evidently their office can be supplied by the simple augmentatives themselves,

$$V_0 = 0, \quad V_1 = 0, \quad V_2 = 0 \ldots V_{\iota-1} = 0;$$

and thus ι letters, which otherwise would be *irrelevant*, fall out of the several derivees.

The author then proceeds with remarks upon the general theory of simple equations, and shows how by virtue of that theory his method contains a solution of the identity

$$X_r U + Y_r V = D_r$$

where D_r is a derivee of the rth degree of U and V, and accordingly, X_r of the form

$$\lambda + \mu x + \nu x^2 + \ldots + \theta x^{m-r-1},$$

and Y_r of the form

$$l + mx + \ldots + tx^{n-r-1},$$

and accounts *à priori* for the fact of not more than $(n-r)$ simple equations being required for the determination of the $(m+n-2r)$ quantities λ, μ, ν, &c. l, m, n, &c., by exhibiting these latter as *known* linear functions of no more than $(n-r)$ unknown quantities left to be determined.

Upon this remarkable relation may be constructed a method well adapted for the expeditious computation of numerical values of the different derivees.

He next, as a point of curiosity, exhibits the values of the secondary functions,

$$a'U_0 - aV_0,$$
$$b'U_0 - bV_0 + a'U_1 - aV_1,$$
$$c'U_0 - cV_0 + b'U_1 - bV_1 + a'U_2 - aV_2, \text{ \&c.}$$

under the form of symmetric functions of the roots of the equations $U=0$, $V=0$, by aid of the theorems developed in the London and Edinburgh *Philosophical Magazine*, December 1839*, and afterwards proceeds to a more close examination of the final derivee resulting from two equations each of the same (any given) degree.

He conceives a number of cubic blocks each of which has two numbers, termed its *characteristics*, inscribed upon one of its faces, upon which the value of such a block (itself called an *element*) depends.

For instance, the value of the *element*, whose *characteristics* are r, s, is the difference between two products: the one of the coefficient rth in order occurring in the polynomial U, by that which comes sth in order in V; the other product is that of the coefficient sth in order of the polynomial U, by that rth in order of V; so that if the degree of each equation be n, there will be altogether $\frac{1}{2}n(n+1)$ such elements.

The blocks are formed into squares or flats (*plafonds*) of which the number is $\frac{n}{2}$ or $\frac{n+1}{2}$, according as n is even or odd. The first of these contains n blanks in a side, the next $(n-2)$, the next $(n-4)$, till finally we reach a square of four blocks or of one, according as n is even or odd. These flats are laid upon one another so as to form a regularly ascending pyramid, of which the two diagonal planes are termed the planes of separation and symmetry respectively. The former divides the pyramid into two halves, such that no element on the one side of it is the same as that of any block in the other. The plane of symmetry, as the name denotes, divides the pyramid into two exactly *similar* parts; it being a rule, that *all elements lying in any given line of a square (plafond) parallel to the plane of separation are identical*; moreover, the sum of the characteristics is the same, for *all* elements lying *anywhere* in a *plane* parallel to that of separation.

[* p. 40 above. Ed.]

All the terms in the final derivee are made up by multiplying n elements of the pile together, under the sole restriction, that no two or more terms of the said product shall lie in any one plane out of the two *sets* of planes perpendicular to the sides of the squares. The *sign* of any such product is determined by the places of either set of planes parallel to a side of the squares and to one another, in which the elements composing it may be conceived to lie.

The author then enters into a disquisition relating to the *number* of terms which will appear in the final derivee, and concludes this first part with the statement of two general canons, each of which affords as many tests for determining whether a prepared combination of coefficients can enter into the final derivee of any number of equations as there are units in that number, but so connected as together only to afford double that number, *less* one, of independent conditions.

The first of these canons refers simply to the number of letters drawn out of *each of the given equations* (supposed homogeneous); the second to what he proposes to call the *weight* of every term in the derivee in respect to *each of the variables* which are to be eliminated.

The author subjoins, for the purpose of conveying a more accurate conception of his Pyramid of derivation, examples of the mode in which it is constructed.

When $n=1$ there is one flat, viz.

1, 2

When $n=2$ there is one flat, viz.

2, 3	2, 4
2, 4	3, 4

Let $n=3$, there will be two flats:

2, 3

1, 2	1, 3	1, 4
1, 3	1, 4	2, 4
1, 4	2, 4	3, 4

Let $n=4$, there will still be two flats only:

2, 3	2, 4
2, 4	3, 4

1, 2	1, 3	1, 4	1, 5
1, 3	1, 4	1, 5	2, 5
1, 4	1, 5	2, 5	3, 5
1, 5	2, 5	3, 5	4, 5

Let $n = 5$, there will be three flats:

3, 4	

2, 3	2, 4	2, 5
2, 4	2, 5	3, 5
2, 5	3, 5	4, 5

1, 2	1, 3	1, 4	1, 5	1, 6
1, 3	1, 4	1, 5	1, 6	2, 6
1, 4	1, 5	1, 6	2, 6	3, 6
1, 5	1, 6	2, 6	3, 6	4, 6
1, 6	2, 6	3, 6	4, 6	5, 6

Let $n = 6$, there will be three flats:

3, 4	3, 5
3, 5	4, 5

2, 3	2, 4	2, 5	2, 6
2, 4	2, 5	2, 6	3, 6
2, 5	2, 6	3, 6	4, 6
2, 6	3, 6	4, 6	5, 6

1, 2	1, 3	1, 4	1, 5	1, 6	1, 7
1, 3	1, 4	1, 5	1, 6	1, 7	2, 7
1, 4	1, 5	1, 6	1, 7	2, 7	3, 7
1, 5	1, 6	1, 7	2, 7	3, 7	4, 7
1, 6	1, 7	2, 7	3, 7	4, 7	5, 7
1, 7	2, 7	3, 7	4, 7	5, 7	6, 7

Thus the work of computation reduces itself merely to calculating $n\frac{n+1}{2}$ elements, or the $n(n+1)$ cross-products out of which they are constituted, and combining them factorially after that law of the pyramid, to which allusion has been already made.

17.

ELEMENTARY RESEARCHES IN THE ANALYSIS OF COMBINATORIAL AGGREGATION.

[*Philosophical Magazine*, XXIV. (1844), pp. 285—296.]

THE ensuing inquiries will be found to relate to combination-systems, that is, to combinations viewed in an aggregative capacity, whose species being given, we shall have to discover rules for ranging or evolving them in classes amenable to certain prescribed conditions. The question of numerical amount will only appear incidentally, and never be made the primary object of investigation*.

The number of things combined will be termed the modulus of the system to which they belong. The elements taken singly, or combined in twos, threes, &c., will be denominated accordingly the monadic, duadic, triadic elements, or simply the monads, duads, or triads of the system.

Let us agree to denote by the word syntheme† any aggregate of combinations in which all the monads of a given system appear once, and once only.

It is manifest that many such synthemes totally diverse in every term may be obtained for a given system to any modulus, and for any order of combination.

Let us begin with considering the case of duad synthemes. Take the modulus 4 and call the elements a, b, c, d.

(ab, cd), (ac, bd), (ad, cb) constitute three perfectly independent synthemes, and these three synthemes include between them all the duad elements, so that no more independent synthemes can be obtained from them.

* The present theory may be considered as belonging to a part of mathematics which bears to the combinatorial analysis much the same relation as the geometry of position to that of measure, or the theory of numbers to computative arithmetic; number, place, and combination (as it seems to the author of this paper) being the three intersecting but distinct spheres of thought to which all mathematical ideas admit of being referred.

† From σύν and τίθημι.

Again, let a, b, c, d, e, f be the monads; we can write down five independent synthemes, to wit,

$$\left.\begin{matrix} ab,\ cd,\ ef \\ ad,\ cf,\ eb \\ ac,\ de,\ fb \\ af,\ bd,\ ce \\ ae,\ df,\ bc \end{matrix}\right\}.$$

We can write no more than these without repeating duads which have already appeared*.

We propose to ourselves this problem:—*A system to any even*† *modulus being given, to arrange the whole of its duads*‡ *in the form of synthemes*; or in other words, *to evolve a Total of duad synthemes to any given even modulus*§.

When the modulus is odd, as before remarked, the formation of a duad syntheme is of course impossible, for any number of duads must necessarily contain an even number of monadic elements; but there is nothing to prevent us from forming in *all* cases what may be termed a bisyntheme or diplotheme, that is, an aggregate of combinations, where each element occurs twice and no more.

For instance, if the elements be called after the letters of the alphabet, we have $\begin{pmatrix} ab, & bc, & cd, & de, & ea \\ ac, & ce, & eb, & bd, & da \end{pmatrix}$, the bisynthematic total to modulus 5; and in

* Such an *aggregate* of synthemes may be therefore termed a Total.

† The modulus must be even, as otherwise it is manifest no single syntheme can be formed. We shall before long extend the scope of our inquiry so as to take in the case of odd moduli.

‡ Triadic systems will be treated of hereafter.

§ It is scarcely necessary to advert here to the fact of the problem being in general indeterminate and admitting of a great variety of solutions.

When the modulus is four there is only one synthematic arrangement possible, and there is no indeterminateness of any kind; from this we can infer, *à priori*, the reducibility of a biquadratic equation; for using ϕ, f, F to denote rational symmetrical forms of function, it follows that

$$F\left\{\begin{matrix} f\{\phi(a, b),\ \phi(c, d)\} \\ f\{\phi(a, c),\ \phi(b, d)\} \\ f\{\phi(a, d),\ \phi(b, c)\} \end{matrix}\right\} \text{ is itself a rational symmetric function of } a, b, c, d.$$

Whence it follows that if a, b, c, d be the roots of a biquadratic equation, $f\{\phi(a, b), \phi(c, d)\}$ can be found by the solution of a cubic: for instance, $(a+b)\times(c+d)$ can be thus determined, whence immediately the sum of any two of the roots comes out from a quadratic equation.

To the modulus 6 there are fifteen different synthemes capable of being constructed; at first sight it might be supposed that these could be classed in natural families of three or of five each, on which supposition the equation of the sixth degree could be depressed; but on inquiry this hope will prove to be futile, not but what natural affinities do exist between the totals; but in order to separate them into families each will have to be taken twice over, or in other words, the fifteen synthemes to modulus 6 being reduplicated subdivide into six natural families of five each. Again, it is true that the triads to modulus 6 (just like the duads to modulus 4) admit of being thrown into but one synthematic total, but then this will contain ten synthemes, a number greater than the modulus itself.

like manner

$$\left.\begin{matrix} ab,\ bc,\ cd,\ de,\ ef,\ fg,\ ga \\ ac,\ ce,\ eg,\ gb,\ bd,\ df,\ fa \\ ad,\ dg,\ gc,\ cf,\ fb,\ be,\ ea \end{matrix}\right\} \text{ the total to modulus 7.}$$

In general, if n be the modulus, the number of duads is $n\frac{n-1}{2}$; n being even, $\frac{n}{2}$ duads go to each syntheme, and therefore the total contains $(n-1)$ of these. If n be odd, then, since always n duads go to a bisyntheme, the number of such in the total is $\frac{n-1}{2}$.

Before proceeding to the solution of the problem first proposed, let us investigate the theory of diplothematic arrangement. Here we shall find another term convenient to employ. By a cyclotheme, I designate a fixed arrangement of the elements in one or more circles, in which, although for typographical purposes they are written out in a straight line, the last term is to be viewed as contiguous and antecedent to the first; the recurrence may be denoted by laying a dot upon the two opened ends of the circle; $\dot{a}.b.c.d.\dot{e}$ will thus denote a cyclotheme to modulus 5; $\dot{a}.b.c.d.e.f.g.h.\dot{k}$ the same to modulus 9; so also is $\dot{a}.b.\dot{c}$, $\dot{d}.e.\dot{f}$, $\dot{g}.h.\dot{k}$ a cyclotheme of another species to the same modulus. In general the number of terms will be alike in each division of a cyclotheme.

Now it is evident that every cyclotheme, on taking together the elements that lie in conjunction, may be developed into a diplotheme. Thus

$$\dot{1}.2.\dot{3} = 12,\ 23,\ 31,$$

$$\dot{1}.2.3.\dot{4} = 12,\ 23,\ 34,\ 41,$$

$$(\dot{1}.2.\dot{3};\ \dot{4}.5.\dot{6};\ \dot{7}.8.\dot{9}) = \begin{pmatrix} 12, & 23, & 31 \\ 45, & 56, & 64 \\ 78, & 89, & 97 \end{pmatrix}.$$

Hence we shall derive a rule for throwing the duads of any system into bisynthemes.

Let $m=3$, we have simply $\dot{a}b\dot{c}$,

$m=5$, we write $\qquad \dot{a}.b.c.d.\dot{e}$,

$\qquad\qquad\qquad\qquad \dot{a}.c.e.b.\dot{d}$,

the second being derived from the first by omitting every alternate term; similarly below, the lines are derived each from its antecedent.

$m=7$, we have $\qquad \dot{a}.b.c.d.e.f.\dot{g}$,

$\qquad\qquad\qquad\qquad \dot{a}.c.e.g.b.d.\dot{f}$,

$\qquad\qquad\qquad\qquad \dot{a}.e.b.f.c.g.\dot{d}$.

A very little consideration will serve to prove that in this way, m being a *prime number*, $\frac{m-1}{2}$ cyclothemes may be formed, such that no element will ever be found more than once in contact on either side with any other; whence the rule for obtaining the diplothematic total to any prime-number modulus is apparent.

For example, to modulus 7 the total reads thus:—

$$\left.\begin{array}{ll}\text{1st.} & ab,\ bc,\ cd,\ de,\ ef,\ fg,\ ga \\ \text{2nd.} & ac,\ ce,\ eg,\ gb,\ bd,\ df,\ fa \\ \text{3rd.} & ae,\ eb,\ bf,\ fc,\ cg,\ gd,\ da\end{array}\right\},$$

and no more remains to be said on this special case.

Let us now return to the theory of even moduli, and show how to apply what has been just done to constructing a synthematic total to a modulus which is the double of a prime number.

Suppose the modulus to be six, the number of synthemes is five. Let the six elements, a, b, c, d, e, f, be taken in three parts, so that each part contains two of them; let these parts be called A, B, C, where A denotes ab, B, cd, and C, ef.

Now the duads will evidently admit of a distinction into two classes, those that lie in one part, and those that lie between two; thus ab, cd, ef will be each *unipartite* duads, the rest will be *bipartite*.

The unipartite duads may be conveniently formed into a syntheme by themselves; it only remains to form the four remaining bipartite duad synthemes.

Write the parts in cyclothematic order, as below:

$$\dot{A}B\dot{C}.$$

It will be observed that each part may be written in two positions; thus

$$\begin{array}{llllll} A & \text{may be expressed by} & \begin{smallmatrix}a\\b\end{smallmatrix} & \text{or by} & \begin{smallmatrix}b\\a\end{smallmatrix}, \\ B & \text{"} \quad \text{"} & \begin{smallmatrix}c\\d\end{smallmatrix} & \text{"} & \begin{smallmatrix}d\\c\end{smallmatrix}, \\ C & \text{"} \quad \text{"} & \begin{smallmatrix}e\\f\end{smallmatrix} & \text{"} & \begin{smallmatrix}f\\e\end{smallmatrix}.\end{array}$$

Now we may form a cyclic table of positions as below:

$\dot{A}$	B	$\dot{C}$
1	1	1
1	2	2
2	1	2
2	2	1

Here the numbers in each horizontal line denote the synchronic positions of the parts.

On inspection it will be discovered that A will be found in each of its two positions, with B in each of its two; similarly B with C, and C with A. In fact the four permutations, 11, 12, 21, 22, occur, though in different orders, in any two assigned vertical columns.

Now develope the preceding table, and we have

$$\dot{a}c\dot{e} \quad \dot{a}d\dot{f} \quad \dot{b}c\dot{f} \quad \dot{b}d\dot{e},$$
$$bdf \quad bce \quad ade \quad acf;$$

and these being read off (the *superior* of each antecedent with the *inferior* of each consequent*) must manifestly give the four independent bipartite synthemes which we were in quest of, *videlicet*

$$(ad,\ cf,\ eb),\quad (ac,\ de,\ fb),\quad (bd,\ ce,\ fa),\quad (bc,\ df,\ ea);$$

these four, together with the syntheme first described (ab, cd, ef), constitute a duad synthematic total to modulus 6.

Before proceeding further let us take occasion to remark that the foregoing table of positions may evidently be extended to any odd number of terms by repetition of the second and third places, as seen in the annexed tables of position.

$$\begin{array}{ll} \dot{1}.1.1.1.\dot{1} & \dot{1}.1.1.1.1.1.\dot{1}, \\ \dot{1}.2.2.2.\dot{2} & \dot{1}.2.2.2.2.2.\dot{2}\dagger, \\ \dot{2}.1.2.1.\dot{2} & \dot{2}.1.2.1.2.1.\dot{2}, \\ \dot{2}.2.1.2.\dot{1} & \dot{2}.2.1.2.1.2.\dot{1}. \end{array}$$

Now let 10 be the modulus.

As before divide the elements into five parts, which call A, B, C, D, E.

The unipartite duads fall into a single syntheme; the eight remaining bipartite synthemes may be found as follows:—

Arrange in cyclothemes $\left(\frac{n-1}{2}\text{ in number}\right)$ the odd modulus system A, B, C, D, E. We have thus

$$\dot{A}BCD\dot{E},$$
$$\dot{A}CEB\dot{D}.$$

* Any other *fixed* order of successive conjunction would answer equally well.

† It will not fail to be borne in mind that in operating with these tables only *contiguous* elements are taken in conjunction: the first with the second, the second with the third, the third with the fourth, &c., and the last with the first; no two terms but such as lie together are in any manner conjugated with one another.

Let each cyclotheme be taken in the four positions given in the table above, we have thus 2×4, that is, eight arguments.

$$\dot{a}\,b\,c\,d\,\dot{e}\,.\,\dot{a}\,\beta\,\gamma\,\delta\,\dot{\epsilon}\,.\,\dot{a}\,b\,\gamma\,d\,\dot{\epsilon}\,.\,\dot{\alpha}\,\beta\,c\,\delta\,\dot{e},$$
$$\alpha\,\beta\,\gamma\,\delta\,\epsilon,\ \alpha\,b\,c\,d\,e,\ a\,\beta\,c\,\delta\,e,\ a\,b\,\gamma\,d\,\epsilon,$$
$$\dot{a}\,c\,e\,b\,\dot{d}\,.\,\dot{a}\,\gamma\,\epsilon\,\beta\,\dot{\delta}\,.\,\dot{\alpha}\,c\,\epsilon\,b\,\dot{\delta}\,.\,\dot{a}\,\gamma\,e\,\beta\,\dot{d},$$
$$\alpha\,\gamma\,\epsilon\,\beta\,\delta,\ \alpha\,c\,e\,b\,d,\ a\,\gamma\,e\,\beta\,d,\ a\,c\,\epsilon\,b\,\delta.$$

And each of these arguments will furnish one bipartite syntheme, by reading off, as before, the *superior* of each antecedent with the *inferior* of each consequent; and the least reflection will serve to show that the same duad can never appear in two distinct arguments.

In like manner, if the modulus be 14 and seven parts be taken, the bipartite synthemes, twelve in number, may be expressed symbolically thus:

$$\left\{\begin{array}{l} \quad \dot{1}.1.1.1.1.1.\dot{1} \\ +\,\dot{1}.2.2.2.2.2.\dot{2} \\ +\,\dot{2}.1.2.1.2.1.\dot{2} \\ +\,\dot{2}.2.1.2.1.2.\dot{1} \end{array}\right\} \times \left\{\begin{array}{l} \quad \dot{A}.B.C.D.E.F.\dot{G} \\ +\,\dot{A}.C.E.G.B.D.\dot{F} \\ +\,\dot{A}.E.B.F.C.G.\dot{D} \end{array}\right\}.$$

Nay more, from the above table, if we agree to name the elements $\frac{A_1\, B_1}{A_2\, B_2}$, &c., we can at once proceed to calculate each of the twelve synthemes in question by an easy algorithm. For instance,

$$(\dot{1}.2.2.2.2.2.\dot{2}) \times (\dot{A}.C.E.G.B.D.\dot{F})$$
$$= (A_1C_1,\ C_2E_1,\ E_2G_1,\ G_2B_1,\ B_2D_1,\ D_2F_1,\ F_2A_2).$$

And again

$$(\dot{2}.1.2.1.2.1.\dot{2}) \times (\dot{A}.E.B.F.C.G.\dot{D})$$
$$= A_2E_2,\ E_1B_1,\ B_2F_2,\ F_1C_1,\ C_2G_2,\ G_1D_1,\ D_2A_1;$$

each figure occurring once unchanged as an antecedent and once changed as a consequent.

If it were thought worth while it would not be difficult, by using numbers instead of letters, to obtain a general analytical formula, from which all similarly constituted synthemes to any modulus might be evolved.

But the rule of proceeding must be now sufficiently obvious; the modulus being $2p$, we divide the elements into p classes; these may be arranged into $\frac{p-1}{2}$ distinct forms of cyclothematic arrangement, and each of the cyclothemes taken in four positions, thus giving $4 \times \frac{p-1}{2}$, that is, $2p-2$ bipartite synthemes, the whole number that can be formed to the given modulus $2p$.

I shall now proceed to the theory of bipartite synthemes to the modulus $2m \times p$, by which it is to be understood that we have p parts each containing $2m$ terms, and p is at present supposed to be a prime number; the total number of synthemes to the modulus $2mp$ being $2mp-1$, and $2m-1$ of these evidently being capable of being made unipartite; the remainder, $2mp-2m$, that is, $(p-1)\,2m$, will be the number of bipartites to be obtained*:

$$2m\,(p-1) = \frac{p-1}{2} \times 4m\,;$$

$\frac{p-1}{2}$ denotes the total number of cyclothemes to modulus p; $4m$, as will be presently shown, the number of lines or syzygies in the *Table of position*.

To fix our ideas let the modulus be 4×3, and let A, B, C be three parts:

$$\left.\begin{array}{l} a_1\, a_2\, a_3\, a_4 \\ b_1\, b_2\, b_3\, b_4 \\ c_1\, c_2\, c_3\, c_4 \end{array}\right\} \text{ their constituents respectively.}$$

Give a *fixed* order to the constituents of each part, then each of them may be taken in four positions; thus A may be written

$$a_1\, a_2\, a_3\, a_4,$$
$$a_2\, a_3\, a_4\, a_1,$$
$$a_3\, a_4\, a_1\, a_2,$$
$$a_4\, a_1\, a_2\, a_3.$$

Assume some particular position for each, as, for instance,

$$a_1\, b_1\, c_1,$$
$$a_2\, b_2\, c_2,$$
$$a_3\, b_3\, c_3,$$
$$a_4\, b_4\, c_4,$$

and read off by coupling the first and third vertical places of each antecedent with the second and fourth respectively of each consequent; we have accordingly,

$$a_1b_2,\ b_1c_2,\ c_1a_2,$$
$$a_3b_4,\ b_3c_4,\ c_3a_4.$$

It is apparent that the same combinations will recur if any two contiguous parts revolve simultaneously through two steps; or in other words, that $A_rB_s = A_{r+2}B_{s+2}$, where μ is any number, odd or even.

* In general, if there be π parts of μ terms each, and $\mu\pi$ be even, the number of bipartite synthemes is $(\pi-1)\,\mu$, as is easily shown from dividing the whole number of bipartite duads by the semi-modulus.

Symbolically speaking, therefore, as regards our table of position,

$$r:s = r+2:s+2,$$

or more generally,

$$= r+2\pm 4i : s+2\pm 4i.$$

So that

$$\begin{array}{ll} 1:1=3:3, & 2:1=4:3, \\ 1:2=3:4, & 2.2=4:4, \\ 1:3=3.1, & 2.3=4:1, \\ 1:4=3.2, & 2.4=4:2. \end{array}$$

There are therefore no more than eight independent unequivalent permutations to every pair of parts. Now inspect the following table of position:—

$$\begin{array}{ll} \dot{1}.1.\dot{1}, & \dot{2}.1.\dot{2}, \\ 1.2.3, & 2.2.4, \\ 1.3.2, & 2.3.1, \\ 1.4.4, & 2.4.3. \end{array}$$

It will be seen that in the first and second, second and third, third and first places, all the eight independent permutations occur under different *names*; the law of formation of such and similar tables will be explained in due time; enough for our present object to see how, by means of this table, we are able to obtain the bipartite synthemes to the given modulus 4×3; the number according to our formula is $2\times 4\times \frac{3-1}{2}=8$, and they may be denoted symbolically as follows:—

$$(\dot{A}.B.\dot{C})\begin{pmatrix}1.1.1+1.2.3+1.3.2+1.4.4\\+2.1.2+2.2.4+2.3.1+2.4.3\end{pmatrix}.$$

Each of the eight terms connected by the sign of + gives a distinct syntheme; for example, let us operate on

$$\dot{A}.B.\dot{C}\times(2.3.1).$$

2.3.1 denotes 2.3, 3.1, 1.2.

2.3 gives rise to $2(3+1)+(2+2).(3+3)=2.4+4.2.$

3.1 gives rise to $3(1+1)+(3+2).(1+3)=3.2+1.4.$

1.2 gives rise to $1(2+1)+(1+2).(2+3)=1.3+3.1.$

The syntheme in question is therefore

$$A_2B_4,\ A_4B_2,\ B_3C_2,\ B_1C_4,\ C_1A_3,\ C_3A_1,$$

and so on for all the rest, the rule being that

$$r:s=r(s+1)+(r+2)(s+3).$$

Now, as before, it is evident that if we look only to contiguous terms, the above table of position may be extended to any number of odd terms, simply by repetition of the second and third figures in each syzygy; and hence the rule for obtaining the bipartite synthemes to the modulus $4\times p$ is apparent. For instance, let $p=7$, there will be $8\times\frac{7-1}{2}$, that is, 8×3 of them denoted as follows:—

$$\begin{Bmatrix}\dot{A}.B.C.D.E.F.\dot{G}\\+\dot{A}.C.E.G.B.D.\dot{F}\\+\dot{A}.E.B.F.C.G.\dot{D}\end{Bmatrix}\times\begin{Bmatrix}1.1.1.1.1.1.1+2.1.2.1.2.1.2\\+1.2.3.2.3.2.3+2.2.4.2.4.2.4\\+1.3.2.3.2.3.2+2.3.1.3.1.3.1\\+1.4.4.4.4.4.4+2.4.3.4.3.4.3\end{Bmatrix}.$$

As an example of the mode of development, let us take the term

$$\dot{A}.E.B.F.C.G.\dot{D}\times\dot{2}.4.3.4.3.4.\dot{3},$$

$$\dot{2}.4.3.4.3.4.\dot{3}=(2:4,\ 4:3,\ 3:4,\ 4:3,\ 3:4,\ 4:3,\ 3:2)$$

$$=\left(\begin{matrix}2.1\\+4.3\end{matrix}\right\}\begin{matrix}4.4\\+2.2\end{matrix}\Big\}\begin{matrix}3.1\\+1.3\end{matrix}\Big\}\begin{matrix}4.4\\+2.2\end{matrix}\Big\}\begin{matrix}3.1\\+1.3\end{matrix}\Big\}\begin{matrix}4.4\\+2.2\end{matrix}\Big\}\left.\begin{matrix}3.3\\+1.1\end{matrix}\right\}\Big),$$

$$\dot{A}.E.B.F.C.G.\dot{D}=A.E,\ E.B,\ B.F,\ F.C,\ C.G,\ G.D,\ D.A,$$

and the product

$$=\begin{pmatrix}A_2E_1,\ E_4B_4,\ B_3F_1,\ F_4C_4,\ C_3G_1,\ G_4D_4,\ D_3A_3\\A_4E_3,\ E_2B_2,\ B_1F_3,\ F_2C_2,\ C_1G_3,\ G_2D_2,\ D_1A_1\end{pmatrix}.$$

Let the modulus be 6×3; as before, give a *fixed* cyclic order to the constituents of each part, and each will admit of being exhibited in six positions.

Write similarly as before,

$$\begin{array}{c}\dot{a}_1\dot{b}_1\dot{c}_1,\\a_2b_2c_2,\\a_3b_3c_3,\\a_4b_4c_4,\\a_5b_5c_5,\\a_6b_6c_6,\end{array}$$

and take the odd places of each antecedent with the even places of each consequent; it will now be seen that

$$r:s=r+2:s+2=r+4:s+4,$$

and the number of independent permutations is $\frac{6.6}{3}=2.6$; and so in general, if there be $2m$ constituents in a part, the number of independent permutations is $\frac{2m.2m}{\frac{m}{2}}=4m.$

The rule for the formation of the table will be apparent on inspection. I suppose only three parts, as the rule may always be extended to any number by reiteration of the second and third terms. The table will be found to resolve itself naturally into four parts, each containing m lines.

Let $m=1$, we have

$$\begin{array}{ll} 1.1.1 & 2.1.2 \\ 1.2.2 & 2.2.1 \end{array}$$

$m=2$, we have

$$\begin{array}{ll} 1.1.1 & 2.1.2 \\ 1.2.3 & 2.2.4 \\ 1.3.2 & 2.3.1 \\ 1.4.4 & 2.4.3 \end{array}$$

$m=3$, we have

$$\begin{array}{ll} 1.1.1 & 2.1.2 \\ 1.2.3 & 2.2.4 \\ 1.3.5 & 2.3.6 \\ 1.4.2 & 2.4.1 \\ 1.5.4 & 2.5.3 \\ 1.6.6 & 2.6.5 \end{array}$$

$m=4$, we have

$$\begin{array}{ll} 1.1.1 & 2.1.2 \\ 1.2.3 & 2.2.4 \\ 1.3.5 & 2.3.6 \\ 1.4.7 & 2.4.8 \\ 1.5.2 & 2.5.1 \\ 1.6.4 & 2.6.3 \\ 1.7.6 & 2.7.5 \\ 1.8.8 & 2.8.7 \end{array}$$

So that x, going through all its values from 1 to m, the general expression for the four parts is

$$\Sigma \begin{Bmatrix} 1\,.\,x\,(2x-1) + 1\,(m+x)\,2x \\ +2\,.\,x\,.\,2x \quad + 2\,(m+x)\,(2x-1) \end{Bmatrix}.$$

To show the use of this formula, let us suppose that we have seven parts, each containing ten terms, the general expression for the bipartite duad synthemes is

$$\begin{Bmatrix} \dot{A}.B.C.D.E.F.\dot{G} \\ + A.C.E.G.B.D.F \\ + A.E.B.F.C.G.D \end{Bmatrix} \times \Sigma \begin{Bmatrix} 1\,.\,x\,(2x-1)\,x\,(2x-1)\,x\,(2x-1) \\ +2\,.\,x\,.\,2x\,.\,x\,.\,2x\,.\,x\,.\,2x \\ +1\,(5+x)\,2x\,(5+x)\,2x\,(5+x)\,2x \\ +2\,(5+x)(2x-1)(5+x)(2x-1)(5+x)(2x-1) \end{Bmatrix}.$$

Make, for example, $x=3$, one of the synthemes in question out of the twelve corresponding to this value will be

$$A.C.E.G.B.D.F \times 2.3.6.3.6.3.6.$$

Here

$$A.C.E.G.B.D.F = AC,\ CE,\ EG,\ GB,\ BD,\ DF,\ FA,$$

$$\dot{2}.3.6.3.6.3.\dot{6} =$$

$$= \left.\begin{matrix} 2.4 \\ +4.6 \\ +6.8 \\ +8.10 \\ +10.2 \end{matrix}\right\} \left.\begin{matrix} 3.7 \\ +5.9 \\ +7.1 \\ +9.3 \\ +1.5 \end{matrix}\right\} \left.\begin{matrix} 6.4 \\ +8.6 \\ +10.8 \\ +2.10 \\ +4.2 \end{matrix}\right\} \left.\begin{matrix} 3.7 \\ +5.9 \\ +7.1 \\ +9.3 \\ +1.5 \end{matrix}\right\} \left.\begin{matrix} 6.4 \\ +8.6 \\ +10.8 \\ +2.10 \\ +4.2 \end{matrix}\right\} \left.\begin{matrix} 3.7 \\ +5.9 \\ +7.1 \\ +9.3 \\ +1.5 \end{matrix}\right\} \left.\begin{matrix} 6.3 \\ +8.5 \\ +10.7 \\ +2.9 \\ +4.1 \end{matrix}\right\}$$

and the product

$$\begin{array}{l} = A_2C_4,\ C_3E_7,\ E_6G_4,\ G_3B_7,\ B_6D_4,\ D_3F_7,\ F_6A_3 \\ \quad A_4C_6,\ C_5E_9,\ E_8G_6,\ G_5B_9,\ B_8D_6,\ D_5F_9,\ F_8A_5, \\ \qquad \&c. \qquad \&c. \qquad \&c. \end{array}$$

To *prove* the rule for the table of formation, it will be sufficient to show that no two contiguous duads ever contain the same or *equivalent* permutations; the equation of equivalence it will be remembered is

$$r:s = r+2i \pm 2m : s+2i \pm 2m.$$

Now, as regards the first and second terms, it is manifest that $1:x$ cannot be equivalent, either to $1:x'$ nor to $2:x$, nor to $2:x'$, where x' is any number differing from x.

Similarly, as regards the last and first terms, $x:1$ cannot be equivalent to $x':1$, nor to $x:2$, nor to $x':2$; therefore there is no danger as far as the first term is concerned, either as antecedent or consequent.

Again, it is clear that $x:(2x-1)$ cannot interfere with $x':2x'$, nor $(m+x):2x$ with $(m+x'):(2x'-1)$; neither can $(2x-1):x$ with $2x':x'$, nor $2x:(m+x)$ with $(2x'-1):(m+x')$.

Again, if possible, let

$$x:(2x-1) = (m+x'):(2x'-1);$$

then

$$m+x'-x = 2i,$$

and

$$2x'-2x = 2i,$$

therefore

$$2m = 2i,$$

or

$$m = i,$$

which is impossible, since $+i$ is the difference between two indices, each less than m.

Similarly,

$$m + x : 2x \text{ cannot} = x' : 2x',$$

and *vice versâ* with the terms changed

$$2x : (m + x) \text{ cannot} = 2x' : x',$$

and

$$(2x - 1) : x \text{ cannot} = (2x' - 1) : (m + x'),$$

which proves the rule for the table of formation.

So much for the bipartite duad synthemes. As regards the unipartite synthemes little need be said, for every part may be treated as a separate system, and as each will produce an equal number of synthemes, these being taken one with another, will furnish just as many unipartite synthemes of the whole system as there are synthemes due to each part. Thus then the synthematic resolution of the modulus $2m \times p$ may be made to depend on the synthematization of $2m$ and the cyclothematization of p. This has been already shown (whatever m may be) for the case of p being a prime number; but I proceed now to extend the rule to the more general case of p being any number whatever.

18.

ON THE EXISTENCE OF ABSOLUTE CRITERIA FOR DETERMINING THE ROOTS OF NUMERICAL EQUATIONS.

[*Philosophical Magazine*, XXV. (1844), pp. 442—445.]

I WISH to indicate in this brief notice a fact which I believe has escaped observation hitherto, that there exist, certainly in some cases, and probably in all, infallible criteria for determining whether a given equation has all its roots rational or not.

In the equation of the second degree it is enough, in order that this may be the case, that the expression for the square of the difference of the roots shall be a perfect square; in other words, if $x^2 - px + q = 0$ have its roots *rational*, $p^2 - 4q$ must be not only a *positive* number (the condition of the roots being real), but that number must also be a complete square. In this case it is further evident that p must be either prime to q, or if not, the greatest common measure of p^2 and q must be a perfect square; but this condition is contained in the former, which is a sufficient criterion in itself.

If we now consider the equation of the third degree,

$$x^3 - px^2 + qx - r = 0,$$

one condition is, that the product of the squared differences shall be a perfect square; in other words, the equation cannot have all its roots rational unless

$$p^2q^2 - 4q^3 - 18pqr - 4p^3r - 27r^2$$

be a positive square number.

This remark is made at the end of the second supplement of Legendre's *Theory of Numbers*, and is indeed self-evident; and in like manner one condition may be obtained for an equation of any degree which is to have all its roots rational; but this is far from being the sole condition required.

In the equation of the third degree, however, one other condition, conjoined with that above expressed, will serve to determine infallibly whether all the roots are rational or not.

To obtain this condition, let us suppose that by making $3x = y + p$ we obtain the equation

$$y^3 - Qx - R = 0.$$

Calling the three roots of this new equation α, β, γ (all of which it is evident must be rational if those of the first equation are so), we have

$$\alpha + \beta + \gamma = 0,$$

$$Q = -(\alpha\beta + \alpha\gamma + \beta\gamma) = \alpha^2 + \alpha\beta + \beta^2,$$

$$R = \alpha\beta\gamma.$$

From the last two equations it is easily seen that if k be any prime factor common to Q and R, k^2 will be contained in Q, and k^3 in R; or, in other words, k will be a common measure of α, β, γ.

We have therefore a *second condition*, that $9q - 3p^2$ shall be a negative quantity, which is either prime to $2p^3 - 9qp + 27r$, or else so related to it, that the greatest common measure of the cube of the first and the square of the second is a perfect sixth power.

I now proceed to show the converse, that if these two conditions be both satisfied (and it will appear in the course of the inquiry that the first does *not* involve the second), the roots cannot help being all rational.

It is evident that the two conditions in question are tantamount to supposing that the roots of the proposed equation are linearly connected with those of another $z^3 - Qz - R = 0$ (by virtue of the assumption $3x = kz + p$), where Q may be considered as *prime* to R; and where $4Q^3 - 27R^2$ is a perfect square.

Let now $4Q^3 - 27R^2 = D^2$, then $D^2 + 27R^2 = 4Q^3$, or $D^2 + 3(3R)^2 = 4Q^3$.

Here, as Q is prime to R, D can have no common measure but 3, with $3R$.

Firstly, let Q be prime to $3R$.

Then putting $f^2 + 3g^2 = Q^3$, the complete solution of the equation immediately preceding is contained in the two systems:

1st. $D = 2f$, $3R = 2g$.

2nd. $D = (f \pm 3g)$, $3R = f \mp g$,

and for both systems,

$$f \pm g\sqrt{(-3)} = \{h \pm 3k\sqrt{(-3)}\}^3.$$

The second system must therefore be rejected, for g evidently contains 3, and therefore $f = 3R \pm g$ will contain 3, and therefore D and therefore Q will do the same, contrary to supposition.

Hence

$$\sqrt[3]{\left[\frac{R}{2} \pm \sqrt{\left\{-\left(\frac{Q^3}{27} - \frac{R^2}{4}\right)\right\}}\right]}$$

$$= \sqrt[3]{\left\{\frac{R}{2} \pm \frac{D}{2}\sqrt{\left(-\frac{1}{27}\right)}\right\}}$$

$$= \sqrt[3]{\left\{\frac{g}{3} + f\sqrt{\left(-\frac{1}{27}\right)}\right\}}$$

$$= \mp \frac{1}{3\sqrt{(-3)}} \sqrt[3]{\{f \pm g\sqrt{(-3)}\}}$$

$$= -K \pm \frac{h}{3\sqrt{(-3)}} = \lambda \pm \mu\sqrt{(-3)};$$

and the three roots of the equation being

$$\begin{cases} \{\lambda + \mu\sqrt{(-3)}\} + \{\lambda - \mu\sqrt{(-3)}\}, \\ \dfrac{1 \pm \sqrt{(-3)}}{2}\{\lambda + \mu\sqrt{(-3)}\} + \dfrac{1 \mp \sqrt{(-3)}}{2}\{\lambda - \mu\sqrt{(-3)}\}, \end{cases}$$

will evidently be all rational, which of course includes the necessity of their being also integer.

Again, secondly, if we suppose that Q does contain 3, D^2 will contain 27, and consequently D will contain 9; and we shall have

$$R^2 + 3\left(\frac{D}{9}\right)^2 = 4\left(\frac{Q}{3}\right)^3.$$

Here R being prime to $\frac{D}{9}$, it may be shown, as in the last case, that the complete solution is

$$\frac{R}{2} \pm \frac{D}{18}\sqrt{(-3)} = \{h \pm k\sqrt{(-3)}\}^3,$$

consequently

$$\sqrt[3]{\left\{\frac{R}{2} \pm \sqrt{\left(\frac{R^2}{4} - \frac{Q^3}{27}\right)}\right\}} = h \pm k\sqrt{(-3)};$$

and the three roots of the equation are

$$2h, \quad h - 3k, \quad h + 3k$$

respectively, and are therefore all rational.

Here it may be observed that the condition of R being an even number, which we know, *à priori*, is the case when all the roots are rational, is involved in the two more general conditions already expressed. It will now be evident that the first condition by no means involves the second, as it is perfectly easy to satisfy the equation $f^2 + 3g^2 = Q^3$ without supposing anything relative to k, the common measure of f, g, Q, except that it be itself of the form $\lambda^2 + 3\mu^2$, which will give

$$\left(\frac{f}{k}\right)^2 + 3\left(\frac{g}{k}\right)^2 = (\lambda^2 + 3\mu^2)(r^2 + 3s^2)^3,$$

an equation which can be solved in rational terms for all values of λ, μ, r, s; and consequently the product of the squares of the differences of the roots may be a square, and at the same time the roots themselves may be irrational*.

I believe it will be found on inquiry that the equation $x^n - qx + r = 0$ will always have two *rational* roots if

$$(n-1)^{n-1} \cdot q^n - n^n \cdot r^{n-1}$$

be a complete square, provided that q be prime to r.

Furthermore, viewing the striking analogy of the general nature of the conditions of rationality already obtained, to those which serve to determine the reality of the roots of equations, I am strongly of opinion that a theorem remains to be discovered, which will enable us to pronounce on the existence of integer, as Sturm's theorem on that of possible roots of a complete equation of any degree: the analogy of the two cases fails however in this respect, that while imaginary roots enter an equation in pairs, irrational roots are limited to entering in groups, each containing *two* or MORE.

* Thus then it appears that the *total* rationality of the roots of the equation $x^3 - qx - r = 0$ may be determined by a direct method without having recourse to the method of divisors to determine the roots themselves; the two conditions being that $4q^3 - 27r^2$ shall be a perfect *square*, and the greatest common measure of q^3 and r^2 a perfect *sixth* power.

19.

AN ACCOUNT OF A DISCOVERY IN THE THEORY OF NUMBERS RELATIVE TO THE EQUATION $Ax^3 + By^3 + Cz^3 = Dxyz$.

[*Philosophical Magazine*, XXXI. (1847), pp. 189—191.]

FIRST GENERAL THEOREM OF TRANSFORMATION.

IF in the equation

$$Ax^3 + By^3 + Cz^3 = Dxyz, \qquad (1)$$

A and B are equal, or in the ratio of two cube numbers to one another, and if $27ABC - D^3$ (which I shall call the Determinant) is free from all single or square prime positive factors of the form $6n+1$, but without exclusion of *cubic* factors of such form, and if A and B are each odd, and C the double or quadruple of an odd number, or if A and B are each even and C odd, then, I say, the given equation may be made to depend upon another of the form

$$A'u^3 + B'v^3 + C'w^3 = D'uvw;$$

where

$$A'B'C' = ABC,$$
$$D' = D,$$
$$uvw = \text{some factor of } z.$$

The following are some of the consequences which I deduce from the above theorem. In stating them it will be convenient to use the term Pure Factorial to designate any number into the composition of which no single or square prime positive factor of the form $6n+1$ enters.

The equations

$$x^3 + y^3 + 2z^3 = Dxyz,$$
$$x^3 + y^3 + 4z^3 = Dxyz,$$
$$2x^3 + 2y^3 + z^3 = Dxyz,$$

are insoluble in integer numbers, provided that the Determinant in each case is a Pure Factorial.

The equation

$$x^3 + y^3 + Az^3 = 9Bxyz$$

is insoluble in integer numbers, provided that the Determinant, for which in this case we may substitute $A - 27B^3$, is a pure factorial whenever A is of the form $9n \pm 1$, and equal to $2p^{3i\pm1}$ or $4p^{3i\pm1}$, p being any prime number whatever.

I wish however to limit my assertion as to the insolubility of the equations above given. The theorem from which this conclusion is deduced does not preclude the possibility of two of the three quantities x, y, z being taken positive or negative *units*, either in the given equation itself or in one or the other of those into which it may admit of being transformed. Should such values of two of the variables afford a particular solution, then instead of affirming that the equations are insoluble, I should affirm that the *general solution* can be obtained by equations in finite differences*.

Second General Theorem of Transformation.

The equation

$$f^3x^3 + g^3y^3 + h^3z^3 = Kxyz \qquad (2)$$

may always be made to depend upon an equation of the form

$$Au^3 + Bv^3 + Cw^3 = Duvw,$$

where

$$ABC = R^3 - S^3,$$

$$D = 3R;$$

and uvw = some factor of $fx + gy + hz$.

R representing $K + 6fgh$,

S " $K - 3fgh$.

* Take for instance the equation $x^3+y^3+2z^3=9xyz$. The Determinant 27.25 is a Pure Factorial: consequently if the solution be possible, since in this case the transformed must be identical with the given equation, this latter must be capable of being satisfied by making x and y positive or negative units. Upon trial we find that $x=1$, $y=1$, $z=2$ will satisfy the equation. I believe, but have not fully gone through the work of verification, that these are the only possible values (prime to one another) which will satisfy the equation. Should they not be so, my method will infallibly enable me to discover and to give the law for the formation of all the others.

Here, then, under any circumstances, is an example, the first on record, of the complete resolution of a numerical equation of the third degree between three variables.

I have not leisure to show the consequences of this theorem of transformation in connexion with the one first given, but shall content myself with a single numerical example of its applications:

$$x^3 + y^3 + z^3 = -6xyz$$

may be made to depend on the equation

$$u^3 + v^3 + w^3 = 0,$$

and is therefore insoluble.

It is moreover apparent that the Determinant of equation (2) transformed is in general $-27R^3$, and is therefore always a Pure Factorial, and consequently the equation

$$f^3x^3 + g^3y^3 + h^3z^3 = Kxyz$$

will be itself insoluble, being convertible into an insoluble form, provided that $K+6fgh$ is divisible by 9, and provided further that $(K+6fgh)^3 - (K-3fgh)^3$ belongs to the form m^3Q, where Q is of the form $9n \pm 1$, and also of one or the other of the two forms $2p^{3i\pm1}$, $4p^{3i\pm1}$, p being any prime number whatever.

Pressing avocations prevent me from entering into further developments or simplifications at this present time.

It remains for me to state my reasons for putting forward these discoveries in so imperfect a shape. They occurred to me in the course of a rapid tour on the continent, and the results were communicated by me to my illustrious friend M. Sturm in Paris, who kindly undertook to make them known on my part to the Institute.

Unfortunately, in the heat of invention I got confused about the law of oddness and evenness, to which the coefficients of the given equation are in the first theorem *generally* (in order for the successful application of my method as far as it is yet developed) required to be subject. I stated this law erroneously, and consequently drew erroneous conclusions from my Theorems of Transformation, which I am very anxious to seize the earliest opportunity of correcting. I venture to flatter myself that as opening out a new field in connexion with Fermat's renowned Last Theorem, and as breaking ground in the solution of equations of the third degree, these results will be generally allowed to constitute an important and substantial accession to our knowledge of the Theory of Numbers.

20.

ON THE EQUATION IN NUMBERS $Ax^3 + By^3 + Cz^3 = Dxyz$, AND ITS ASSOCIATE SYSTEM OF EQUATIONS.

[*Philosophical Magazine*, XXXI. (1847), pp. 293—296.]

IN the last Number of this *Magazine* I gave an account of a remarkable transformation to which the equation

$$Ax^3 + By^3 + Cz^3 = Dxyz$$

is subject when certain conditions between the coefficients A, B, C, D are satisfied; which conditions I shall begin by expressing with more generality and precision than I was enabled to do in my former communication.

1. Two of the quantities A, B, C are to be to one another in the ratio of two cubes.

2. $27ABC - D^3$ must contain no positive prime factor whatever of the form $6n+1$. I erred in my former communication in not excluding cubic factors of this form.

3. If 2^m is the highest power of 2 which enters into ABC, and 2^n the highest power of 2 which enters into D, then either m must be of the form $3n \pm 1$, or if not, then m must be greater than $3n$.

These three conditions being satisfied, the given equation can always be transformed into another,

$$A'u^3 + B'v^3 + C'w^3 = D'uvw,$$

where

$$A'B'C' = ABC, \quad D' = D, \quad uvw = \text{a factor of } z.$$

The consequence of this is, as stated in my former paper, that wherever A, B, C, D, besides satisfying the conditions above stated, are taken so as likewise to satisfy the condition,—firstly, of ABC being equal to $2^{3m\pm1}$, or secondly, of ABC being equal to $2^{3m\pm1} \cdot p^{3n\pm1}$, provided in the second case that ABC is of the form $9m \pm 1$, and that D is divisible by 9, p being in

both cases a prime, then the given equation will be *generally* insoluble. And I am now enabled to add that the *only* solution of which it will in any case admit, is the solitary one found by making two of the terms Ax^3, By^3, Cz^3 equal to one another; so that, for instance, if the given equation should be of the form

$$x^3 + y^3 + ABCz^3 = Dxyz,$$

then the above conditions being satisfied, the one solitary solution of which the equation can possibly admit, is $x = 1$, $y = 1$,

$$Az^3 - Dz + 2 = 0,$$

which may or may not have possible roots. I call this a *solitary* or singular solution, because it exists alone and no other solution can be deduced from it; whereas in general I shall show that any one solution of the equation

$$Ax^3 + By^3 + Cz^3 = Dxyz$$

can be made to furnish an infinity of other solutions independent of the one supposed given, that is, not reducible thereto by expelling a common factor from the new system of values of x, y, z deduced from the given system.

The following is the Theorem of Derivation in question:

Let

$$A\alpha^3 + B\beta^3 + C\gamma^3 = D\alpha\beta\gamma.$$

Then if we write

$$F = A\alpha^3, \quad G = B\beta^3, \quad H = C\gamma^3,$$

and make

$$x = F^2G + G^2H + H^2F - 3FGH,$$

$$y = FG^2 + GH^2 + HF^2 - 3FGH,$$

$$z = \frac{1}{D}\{F^3 + G^3 + H^3 - 3FGH\},$$

or

$$= \alpha\beta\gamma\{F^2 + G^2 + H^2 - FG - FH - GH\},$$

we shall have

$$x^3 + y^3 + ABCz^3 = Dxyz.$$

I am hence enabled to show that whenever $x^3 + y^3 + Az^3 = Dxyz$ is insoluble, there will be a whole family of allied equations equally insoluble. For instance, because $x^3 + y^3 + z^3 = 0$ is insoluble in integer numbers, I know likewise that

$$x^6 + y^6 + z^6 = x^3y^3 + x^3z^3 + y^3z^3$$

$$x^6 + y^6 + z^6 = x^3y^3 + x^3z^3 - 2y^3z^3$$

are each equally insoluble.

In fact

$$\begin{aligned}(x^3+y^3+z^3) &\times (x^6+y^6+z^6-x^3y^3-x^3z^3-y^3z^3)\\ &\times (x^6+y^6+z^6-x^3y^3-x^3z^3+2y^3z^3)\\ &\times (x^6+y^6+z^6-y^3z^3-y^3x^3+2x^3z^3)\\ &\times (x^6+y^6+z^6-x^3z^3-z^3y^3+2y^3x^3)\\ &= u^3+v^3+w^3,\end{aligned}$$

where u, v, w are rational integral functions of x, y, z.

Hence each of the factors must be incapable of becoming zero*.

As a particular instance of my general theory of transformation and elevation, take the equation

$$x^3+y^3+2z^3 = Mxyz.$$

Then, with the exception of the singular or solitary solution $x=1$, $y=1$, of which I take no account, I am able to affirm that for all values of M between 7 and -6, both inclusive, with the exception of $M=-2$, the equation is insoluble in integer numbers.

Take now the equation where $M=-2$, namely

$$x^3+y^3+2z^3+2xyz=0.$$

One particular solution of this is

$$x=1, \quad y=-1, \quad z=1.$$

Another, which I shall call the second†, is

$$x=1, \quad y=3, \quad z=-2.$$

From the first solution I can deduce in succession the following:

$$\begin{aligned}&x=11, && y=5, && z=-7,\\ &x=-793269121, && y=1179490001, && z=-1189735855,\\ &\&c. && \&c. && \&c.\end{aligned}$$

From the second,

$$\begin{aligned}&x=-10085, && y=8921, && z=-8442,\\ &x=\&c. && y=\&c. && z=\&c.\end{aligned}$$

As another example, take the equation

$$x^3+y^3+6z^3=6xyz.$$

* It is however sufficiently evident from their intrinsic form, which may be reduced to $\frac{1}{4}(M^2+3N^2)$, that this impossibility exists for all the factors except the first.

† See Postscript.

One solution of the transformed equation

$$u^3+2v^3+3w^3=6uvw$$

is evidently

$$u=1, \quad v=1, \quad w=1.$$

Hence I can deduce an infinite series of solutions of the given equation, of which the first in order of ascent will be

$$x=5, \quad y=7, \quad z=3.$$

Again, the lowest possible solution in integers of the equation

$$x^3+y^3+6z^3=0$$

will be

$$x=17, \quad y=37, \quad z=-21.$$

The equation

$$x^3+y^3+9z^3=0$$

admits of the solutions

$$\begin{aligned}&x=1, && y=2, && z=-1,\\ &x=-271, && y=919, && z=-438.\end{aligned}$$

I trust that my readers will do me the justice to believe that I am in possession of a strict demonstration of all that has been here advanced without proof. Certain of the writer's friends on the continent have, in their comments upon one of his former papers which appeared in this *Magazine*, complimented his powers of divination at the expense of his judgment, in rather gratuitously assuming that the author of the Theory of Elimination was unprovided with the demonstrations, which he was too inert or too beset with worldly cares and distractions to present to the public in a sufficiently digested form. The proof of whatever has been here advanced exists not merely as a conception of the author's mind, but fairly drawn out in writing, and in a form fit for publication.

P.S. It must not be supposed that the two primary or basic solutions above given of the equation

$$x^3+y^3+2z^3+2xyz=0,$$

namely,

$$\begin{aligned}&x=1, && y=-1, && z=1,\\ &x=1, && y=3, && z=-2,\end{aligned}$$

are independent of one another. The second may be derived from the first, as I shall show in a future communication. In fact there exist *three* independent processes, by combining which together, one particular solution may be made to give rise to an infinite series of infinite series of infinite series of correlated solutions, which it may possibly be discovered contain between them the *general* complete solution of the equation

$$x^3+y^3+Az^3=Dxyz.$$

21.

ON THE *GENERAL* SOLUTION (IN CERTAIN CASES) OF THE EQUATION $x^3 + y^3 + Az^3 = Mxyz$, &c.

[*Philosophical Magazine*, XXXI. (1847), pp. 467—471.]

I SHALL restrict the enunciation of the proposition I am about to advance to much narrower limits than I believe are necessary to the truth, with a view to avoid making any statement which I may hereafter have occasion to modify. Let us then suppose in the equation

$$x^3 + y^3 + Az^3 = Mxyz$$

that A is a *prime* number, and that $27A - M^3$ is *positive*, but exempt from positive prime factors of the form $6i + 1$. Then I say, and have succeeded in demonstrating, that all the possible solutions in integer numbers of the given equation may be obtained by explicit processes from one particular solution or system of values of x, y, z, which may be called the Primitive system.

This system of roots or of values of x, y, z is that system in which the value of the greatest of the three terms x, y, $A^{\frac{1}{3}}.z$ (which may be called the Dominant) is the least possible of all such dominants. I believe that in general the system of the least *Dominant* is identical with the system of the least *Content*, meaning by the latter term the product of the three terms out of which the *Dominant* is elected. I proceed to show the law of derivation.

To express this simply, I must premise that I shall have to employ such an expression as $S' = \phi(S)$ to indicate, not that a certain quantity, S', is a function of S, but that a certain system of quantities disconnected from one another, denoted by S', are severally functions of a certain other system of quantities denoted by S; and, as usual, I shall denote $\phi\phi S$ by $\phi^2 S$, $\phi\phi^2 S$ by $\phi^3 S$, and so forth.

Let now P be the Primitive system of solution of the equation

$$x^3 + y^3 + Az^3 = Mxyz,$$

P denoting a certain system of values of and written in the order of the

letters x, y, z, which may always be found by a limited number of trials (provided that the equation admits of any solution). That this is the case is obvious, since we have only to give the Dominant every possible value from the integer next greatest to $A^{\frac{1}{3}}$ upwards, and combine the values of x^3, y^3, Az^3 so that none shall ever exceed at each step the cube of such dominant, and we must at last, if there *exist any solution*, arrive at the System of the Least Dominant.

Now, every system of solution is of one or the other of two characters. Either x and y must be odd and z even, or x and y must be one odd and the other even and z odd. That all three should be odd is inconsistent with the given conditions as to A being odd and M even; and if all three were even, by driving out the common factor we should revert to one or the other of the foregoing cases.

The systems of solution where z is even may be termed Reducible, those where z is odd Irreducible. Let ϕ denote a certain symbol of transformation hereafter to be explained.

Then the Reducible systems of the first order may be expressed by

$$\phi P,\ \phi^2 P,\ \phi^3 P,\ \textit{ad infinitum};$$

or in general by $\phi^{n_1} P$, n_1 being absolutely arbitrary. I will anticipate by stating that the function ϕ involves no *variable* constants; that is to say, $\phi(S)$ may be found explicitly from S without any reference to the particular equation to which S belongs. Let now ψ denote another symbol of transformation, also hereafter to be defined, and differing from ϕ insofar as it does involve as *constants* the three values of x, y, z contained in P: then the general representations of Irreducible systems of the first order will be denoted by $\psi\phi^{n_1} P$.

It is proper to state here that the symbol ψ is ambiguous; and $\psi\phi^{n_1} P$, when P and n_1 are given, will have two values, according to the way in which the terms represented by P are compared with x, y, z in the given equation

$$x^3 + y^3 + Az^3 = Mxyz;$$

for it is obvious that if $x = a$, $y = b$, $z = c$ satisfies the equation, so likewise will

$$x = b, \quad y = a, \quad z = c.$$

Each however of these values of $\psi\phi^{n_1} P$ gives a solution of the kind above designated.

Proceeding in like manner as before, the Reducible system of the second order may be designated by $\phi^{n_2}.\psi\phi^{n_1}.P$, the Irreducible by $\psi\phi^{n_2}.\psi\phi^{n_1}.P$; and in general *every possible system* of values of x, y, z satisfying the proposed equation, in which z is even, is comprised under the form

$$\phi^{n_r}.\psi\phi^{n_{r-1}}.\psi\ldots\phi^{n_2}.\psi\phi^{n_1}.P;$$

and every possible system of such values, in which z is odd, is comprised under the form

$$\psi\phi^{n_r}.\psi\phi^{n_{r-1}}.\psi\ldots\psi\phi^{n_1}.P:$$

the quantities $n_1, n_2 \ldots n_r$ being of course all independent of one another, and unlimited in number and value.

Thus then we may be said to have the general solution of the given equation in the same sense as an arbitrary sum of terms, each of a certain form, is in certain cases accepted as the complete solution of a partial differential equation.

As regards the value of the symbols ψ and ϕ, ϕ indicates the process by which a, b, c becomes transformed into α, β, γ, the relations between the two sets of elements being contained in the following equations:

$$a' = a^3, \quad b' = b^3, \quad c' = Ac^3,$$

$$\alpha = a'^2b' + b'^2c' + c'^2a' - 3a'b'c',$$

$$\beta = a'b'^2 + b'c'^2 + c'a'^2 - 3a'b'c',$$

$$\gamma = abc\,\{a'^2 + b'^2 + c'^2 - a'b' - a'c' - b'c'\}.$$

Next, as to the effect of the Duplex symbol ψ. Let e, g, ι be the elements of the Primitive system P: ι being the value of z and e, g of x and y taken in either mode of combination, each with each, which satisfy the proposed equation

$$x^3 + y^3 + Az^3 = Mxyz.$$

Let l, m, n represent any system S,

λ, μ, ν represent any system $\psi(S)$,

ψS has two values, which we may denote by $\psi' S$, $'\psi S$ respectively, and accentuating the elements λ, μ, ν accordingly to correspond, we shall have

$$\lambda' = 3gm(gl - em) + 3A\iota n(\iota l - en) - M(g\iota l^2 - e^2lm),$$

$$\mu' = 3A\iota n(\iota m - gl) + 3el(em - gl) - M(e\iota m^2 - g^2lm),$$

$$\nu' = 3el(en - \iota l) + 3gm(gn - \iota m) - M(egn^2 - \iota^2lm):$$

we have then

$$\psi' S \equiv \lambda', \mu', \nu',$$

and in like manner

$$'\psi S \equiv {}'\lambda, {}'\mu, {}'\nu,$$

$'\psi S$ being derived from $\psi' S$ by the mere interchange of e and g one with the other.

I have stated that every possible solution of the proposed equation comes under one or the other of the orders, infinite in number and infinite to the power of infinity in variety of degree, above given: this is not strictly true, unless we understand that all systems of solution are considered to be equivalent which differ only in a multiplier common to all three terms of each; that is to say, which may be rendered identical by the expulsion of a common factor. So that $m\alpha, m\beta, m\gamma$ as a system is treated as identical with α, β, γ, which of course substantially it is; and it should be remarked that there is nothing to prevent the operations denoted by ϕ and ψ introducing a common factor into the systems which they serve to generate, and the latter in particular will have a strong tendency so to do.

I believe that this theorem may be extended with scarcely any modification to the case where A, instead of being a prime, is any power of the same, and to suppositions still more general. I believe also that, subject to certain very limited restrictions, the theorem *may* prove to apply to the case where the determinant $27A - M^3$ becomes negative.

The peculiarity of this case which distinguishes it from the former, is that it admits of all the three variables x, y, z in the equation

$$x^3 + y^3 + Az^3 = Mxyz$$

having the same sign, which is impossible when the determinant is positive; or in other words, the curve of the third degree represented by the equation $Y^3 + X^3 + 1 = \frac{M}{A^{\frac{1}{3}}}XY$ (in which I call the coefficient of XY the characteristic), which, as long as the quantity last named is less than 3, is a single continuous curve extending on both sides to infinity, as soon as the characteristic becomes equal to 3 assumes to itself an isolated point, the germ of an oval or closed branch, which continues to swell out (always lying apart from the infinite branch) as the characteristic continues indefinitely to increase.

I ought not to omit to call attention to the fact that the theorem above detailed is always applicable to the case of the equation

$$x^3 + y^3 + Az^3 = 0,$$

when A is *any* power of a prime number *not* of the form $6i + 1$; in other words, the above always belongs to the class of equations having Monogenous solutions, which for the sake of brevity may be termed themselves Monogenous Equations*.

* Thus the equation $x^3 + y^3 + 9z^3 = 0$ alluded to by Legendre is Monogenous, and the Primitive system of solution is $x = 1$, $y = 2$, $z = -1$, from which every other possible solution in Integers may be deduced.

On the probable existence of such a class of equations I hazarded a conjecture at the conclusion of my last communication to this *Magazine*. As I hope shortly to bring out a paper on this subject in a more complete form, I shall content myself at this time with merely stating a theorem of much importance to the completion of the theory of insoluble and of Monogenous equations of the third degree; to wit, that the equation in integers

$$a(x^3+y^3+z^3)+c(x^2y+y^2z+z^2x+xy^2+yz^2+zx^2)+exyz=0$$

may always be transformed so as to depend upon the equation

$$fu^3+gv^3+hw^3=(6a-e)\,uvw,$$

wherein $$fgh=ae^2-(c^2+3a^2)\,e+9a^3-3ac^2-2c^3.$$

By means of the above theorem, among other and more remarkable consequences, we are enabled to give a theory of the irresoluble and monogenous cases of the equation

$$x^3+y^3+m^3z^3=Mxyz,$$

when m is some power of 2, or of certain other numbers.

22.

ON THE INTERSECTIONS, CONTACTS, AND OTHER CORRELATIONS OF TWO CONICS EXPRESSED BY INDETERMINATE COORDINATES.

[*Cambridge and Dublin Mathematical Journal*, v. (1850), pp. 262—282.]

Let $U=0$, $V=0$ be two homogeneous equations of the second degree with real coefficients, between the same three variables ξ, η, ζ.

The direct and most general mode of determining the intersections of the conics expressed by these equations would be to make

$$a\xi+b\eta+c\zeta=t,$$

$$a'\xi+b'\eta+c'\zeta=u:$$

eliminating ξ, η, ζ between the four equations in which they appear, there results a biquadratic equation between t and u. The nature of the intersections will depend upon the nature of the roots of this biquadratic; and thus the conditions may be expressed analytically, which will represent the several cases of all the intersections being real or all imaginary, or one pair real and the other imaginary. These analytical conditions will depend upon the signs of certain functions of the coefficients of the given and the *assumed* equations being of an assigned character; my endeavour has been to obtain conditions of a character perfectly symmetrical and free from the coefficients arbitrarily introduced.

In this research I have only partially succeeded, but the method employed, and some of the collateral results, will, I think, be found of sufficient interest to justify their appearance in the pages of this *Journal*.

Adopting Mr Cayley's excellent designation, let the four points of intersection of the two conics be called a quadrangle. This quadrangle will have three pairs of sides; the intersections of each pair, from principles of analogy, I call the vertices of the quadrangle. Then, inasmuch as the four sets of ratios $\xi : \eta : \zeta$, corresponding with the four sets of the ratio $t : u$, must be so related that we may always make

$$\frac{\xi_1}{\zeta_1} = a + b\sqrt{(-1)}, \quad \frac{\eta_1}{\zeta_1} = c + d\sqrt{(-1)},$$

$$\frac{\xi_2}{\zeta_2} = a - b\sqrt{(-1)}, \quad \frac{\eta_2}{\zeta_2} = c - d\sqrt{(-1)},$$

$$\frac{\xi_3}{\zeta_3} = \alpha + \beta\sqrt{(-1)}, \quad \frac{\eta_3}{\zeta_3} = \gamma + \delta\sqrt{(-1)},$$

$$\frac{\xi_4}{\zeta_4} = \alpha - \beta\sqrt{(-1)}, \quad \frac{\eta_4}{\zeta_4} = \gamma - \delta\sqrt{(-1)},$$

we may easily draw the following conclusions.

If all the four points of the quadrangle of intersection are real, the three vertices and the three pairs of sides are all real. If only two points of the quadrangle are real, one vertex and one of the three pairs of sides will be real; the other two vertices and two pairs of sides being imaginary. If all four points of the quadrangle are unreal, one pair of sides will be real and the other two pairs imaginary, as in the last case; but all the three vertices will remain real, as in the first case. Hence we have a direct and simple criterion for distinguishing the case of *mixed* intersection from intersection wholly real or wholly imaginary; namely, that the cubic equation of the roots of which the coordinates of the vertices are real linear functions shall have *a pair of imaginary roots.* This is the sole and unequivocal condition required.

The equation in question is, or ought to be, well known to be the determinant in respect to ξ, η, ζ of $\lambda U + \mu V$. In fact, if we write

$$U = a\xi^2 + b\eta^2 + c\zeta^2 + 2a'\eta\zeta + 2b'\zeta\xi + 2c'\xi\eta,$$

$$V = \alpha\xi^2 + \beta\eta^2 + \gamma\zeta^2 + 2\alpha'\eta\zeta + 2\beta'\zeta\xi + 2\gamma'\xi\eta,$$

$$\lambda U + \mu V = (a\lambda + \alpha\mu)\,\xi^2 + \&c. = A\xi^2 + B\eta^2 + C\zeta^2 + 2A'\eta\zeta + 2B'\zeta\xi + 2C'\xi\eta,$$

the ratios of the coordinates ξ, η, ζ of the vertex of $\lambda U + \mu V$ may easily be shown to be identical with

$$AB - C^2 : C'A' - B'B : B'C' - A'A,$$

and will be real or imaginary as $\lambda : \mu$ is one or the other.

If then the cubic equation in $\lambda : \mu$, namely, $\underset{\xi\eta\zeta}{\square}(\lambda U + \mu V) = 0$, has a pair of imaginary roots, that is, if $\underset{\lambda\mu}{\square}\underset{\xi\eta\zeta}{\square}(\lambda U + \mu V)$ is a positive quantity, the intersections of U and V are of a mixed kind, that is, the two conics have two real points in common.

I may remark here, *en passant*, that if we form the biquadratic equation in t and u, $\phi(t, u) = 0$ from the equations

$$U = 0,$$

$$V = 0,$$

$$a\xi + b\eta + c\zeta = t,$$

$$a'\xi + b'\eta + c'\zeta = u,$$

and if any reducing cubic of this equation be $P(\theta, \omega) = 0$, the determinant of $P(\theta, \omega)$ must, from what has been shown above, be identical with $\underset{\lambda\mu}{\square}\underset{\xi\eta\zeta}{\square}(\lambda U + \mu V)$ multiplied by some *squared function* of the extraneous coefficients

$$a, b, c;\ a', b', c'.$$

If $\square\square(\lambda U + \mu V)$ is a negative quantity, it remains to distinguish between the cases of the conics intersecting really in four points or not at all.

The most obvious mode of proceeding to distinguish between purely real and purely imaginary intersections would be as follows. Let $\lambda_1, \mu_1;\ \lambda_2, \mu_2;\ \lambda_3, \mu_3$, be the three sets of values of λ, μ which satisfy the equation

$$\square(\lambda U + \mu V) = 0$$

and make

$$\begin{array}{lll} A_1 = a\lambda_1 + \alpha\mu_1, & A_2 = a\lambda_2 + \alpha\mu_2, & A_3 = a\lambda_3 + \alpha\mu_3, \\ C_1 = c\lambda_1 + \gamma\mu_1, & C_2 = c\lambda_2 + \gamma\mu_2, & C_3 = c\lambda_3 + \gamma\mu_3, \\ B_1' = b'\lambda_1 + \beta'\mu_1, & B_2' = b'\lambda_2 + \beta'\mu_2, & B_3' = b'\lambda_3 + \beta'\mu_3, \\ A_1C_1 - B_1'^2 = e_1, & A_2C_2 - B_2'^2 = e_2, & A_3C_3 - B_3'^2 = e_3. \end{array}$$

Now if the equation

$$A\xi^2 + B\eta^2 + C\zeta^2 + 2A'\eta\zeta + 2B'\zeta\xi + 2C'\xi\eta = 0$$

represent a pair of straight lines, it may be thrown into the form

$$Au^2 + \frac{AC - B'^2}{A}v^2 = 0,$$

where u and v are linear functions of ξ, η, ζ, and the straight lines will be real or imaginary, according as $B'^2 - AC$ is positive or negative; hence one or else all of the quantities e_1, e_2, e_3, will be necessarily negative, and the intersections will be all real or all imaginary, according as all three are negative or only one is so. A cubic equation in e may be formed containing e_1, e_2, e_3 as its roots by eliminating between the equations

$$e = AC - B'^2;\quad \square(\lambda U + \mu V) = 0,$$

and the conditions for the reality of the intersections will be that all four coefficients of this cubic shall be of the same sign, which in reality amount only to two, since the first and last must in all cases have the same sign.

The same objection however of want of symmetry and consequent irrelevancy and complexity attaches to this as much as to the method originally proposed. The following treatment of the question relieves the objection of want of symmetry as far as the coefficients of the same equation are concerned, but in its practical application necessitates an arbitrary and therefore unsymmetrical election to be made between the two sets of coefficients appertaining to the two equations. It is however, I think, too curious and suggestive to be suppressed.

I observe that if the four intersections are all real, an imaginary conic cannot be drawn through them; for the equation to an imaginary conic may always be reduced to the form $Ax^2+By^2+Cz^2=0$, where A, B, C are all positive and can therefore have at utmost one real point. Consequently the case of total non-intersection is distinguishable from that of complete intersection by the peculiarity that in the one case μ may be so taken that $U+\mu V=0$ shall represent an imaginary conic, that is, $U+\mu V$ will be a function whose sign never changes for real values of ξ, η, ζ, whereas in the latter case no value of μ will make $U+\mu V=0$ the equation to an imaginary conic, and therefore $U+\mu V$ will have values on both sides of zero. On the other hand, it is obvious that an infinite number of real as well as unreal conics may be drawn through four imaginary points of intersection. Consequently if we make $U+\mu V=0$ (supposing the intersections of U and V to be imaginary), there will be a range or ranges of values of μ consistent, and another range or ranges of values of μ inconsistent with real values of ξ, η, ζ; in other words, $U\pm\mu V=0$ treated as an equation between the four variables ξ, η, ζ, μ, will give one or more maxima or minima values of μ in the case supposed, but no such values when the intersections are two or all of them real.

To determine these values of μ, let $d\mu=0$; then we have

$$\frac{d}{d\xi}(U-\mu V)=0,$$

$$\frac{d}{d\eta}(U-\mu V)=0,$$

$$\frac{d}{d\zeta}(U-\mu V)=0,$$

that is

$$\underset{\xi\eta\zeta}{\square}(U-\mu V)=0.$$

In order that any value of μ found from this equation may be a maximum or minimum, Lagrange's condition requires that

$$\left(h\frac{d}{d\xi}+k\frac{d}{d\eta}+l\frac{d}{d\zeta}\right)^2\mu$$

may be a function of unchangeable sign.

Now

$$\frac{dU}{d\xi}=\mu\frac{dV}{d\xi}+V\frac{d\mu}{d\xi},$$

therefore since $d\mu=0$,

$$\frac{d^2U}{d\xi^2}=\mu\frac{d^2V}{d\xi^2}+V\frac{d^2\mu}{d\xi^2}.$$

Hence

$$\frac{d^2\mu}{d\xi^2}=\frac{1}{V}\left(\frac{d}{d\xi}\right)^2\{U-\mu V\};$$

similarly

$$\frac{d}{d\xi}\cdot\frac{d}{d\eta}=\frac{1}{V}\frac{d}{d\xi}\cdot\frac{d}{d\eta}\{U-\mu V\},$$

&c. &c. &c.

Making now as before

$$U=a\xi^2+b\eta^2+\&c.,$$

$$V=\alpha\xi^2+\beta\eta^2+\&c.,$$

$$a-\mu\alpha=A,\quad b-\mu\beta=B,\ \&c.,$$

the condition for μ, a root of $\square\{U-\mu V\}=0$, giving μ a maximum or minimum, may be expressed by saying that

$$Ah^2+Bk^2+Cl^2+2A'kl+2B'hl+2C'hk$$

shall be unchangeable in sign for all real values of h, k, l.

The above quantity, by virtue of the equation $\square=0$, is always the product of two linear functions. Hence we see, as above indicated, that if all these pairs are real, that is, if all the points of intersection of U and V are real, there is no maximum or minimum value of μ; but if only one pair be real and the other two pairs be imaginary, that is, if all the four intersections are imaginary, then two of the values of μ, namely those corresponding to the imaginary pairs, are real maxima or minima values of μ, but the third is illusory.

Now I shall show that if $V=0$ is a *real* conic, but the intersections of U and V are all unreal, the value of μ which makes $U+\mu V$ the product of real linear functions of ξ, η, ζ, is always one or the other *extreme* of the three values of μ which satisfy the equation

$$\square(U-\mu V)=0.$$

Assume as the three axes of coordinates the three lines joining the vertices of the quadrangle each with each, the two non-intersecting conics may evidently be written under the form

$$U=c(x^2+y^2)-e(y^2+z^2)=0,$$

$$V=-\gamma(x^2+y^2)+\epsilon(y^2+z^2)=0;$$

these equations being only other modes of writing

$$U = Ax^2 + By^2 + Cz^2,$$
$$V = A'x^2 + B'y^2 + C'z^2,$$

in which $A, B, C; A', B', C'$ will be real, because by hypothesis $\square(U+\mu V)=0$ has all its roots real.

Hence x, y, z are linear functions of ξ, η, ζ, and consequently, by a simple inference from a theorem of Prof. Boole*, the roots of $\underset{\xi\eta\zeta}{\square}\{U+\mu V\}$ are identical with those of

$$\underset{xyz}{\square}\{U+\mu V\}=0.$$

These latter are evidently $\frac{c}{\gamma}, \frac{e}{\epsilon}, \frac{c-e}{\gamma-\epsilon}$; the third of which is the one which makes $U + \mu V$ the product of two *real* linears, for we have

$$\gamma U + cV = (c\epsilon - \gamma e)(y^2 + z^2),$$
$$\epsilon U + eV = (\epsilon c - e\gamma)(x^2 + y^2),$$
$$(\gamma - \epsilon)U + (c - e)V = (c\epsilon - e\gamma)(z^2 - x^2)\dagger.$$

Now
$$\frac{c}{\gamma} - \frac{c-e}{\gamma-\epsilon} = \frac{e\gamma - c\epsilon}{\gamma(\gamma-\epsilon)},$$
$$\frac{e}{\epsilon} - \frac{c-e}{\gamma-\epsilon} = \frac{e\gamma - c\epsilon}{\epsilon(\gamma-\epsilon)};$$

and ϵ, γ are supposed to have the same sign, as otherwise V would be an unreal conic; hence the ascending or descending order of magnitudes of the three values of λ follows the scale $\frac{c}{\gamma}, \frac{e}{\epsilon}, \frac{c-e}{\gamma-\epsilon}$, as was to be shown.

Imagine now lengths reckoned on a line corresponding to all values of μ from $-\infty$ to $+\infty$, and mark off upon this line by the letters A, B, C, the lengths corresponding with the three roots of $\square(U+\mu V)=0$. Then observing that when $\mu = \pm\infty$, $U+\mu V$ is of the same nature as V, and is therefore a possible conic by hypothesis, and agreeing to understand by a possible and impossible region of μ, a range of values for which $U+\mu V$ corresponds to a possible and impossible conic respectively, one or the other of the annexed schemes will represent the circumstances of the case supposed:

$-\infty$ ☜ Poss. Reg. A Imposs. Reg. B Poss. Reg. C Poss. Reg. ☞ $+\infty$

$-\infty$ ☜ Poss. Reg. A Poss. Reg. B Imposs. Reg. C Poss. Reg. ☞ $+\infty$

But in either scheme it is essential to observe that the *middle* root of $\square(U+\mu V)=0$ divides a possible from an impossible region; and therefore

* See Postscript.

† $z^2 - x^2 = 0$ of course represents a *real* pair of lines.

if we can find n, ν, any two values lying between the first and second and second and third roots of the above equation arranged in order of their magnitude, one of the two equations $U + \nu V = 0$, $U + nV = 0$, will represent a possible and the other an impossible conic: one such couple of values may always be found by taking the roots of the quadratic equation

$$\frac{d}{d\mu}\square\{U+\mu V\}=0.$$

Hence calling the two roots thereof m and M, we see (which is in itself a theorem) that one at least of the conics $U + mV = 0$, $U + MV = 0$, must be a possible conic, provided only that $V = 0$ be a possible conic: if both $U + mV$ and $U + MV$ are possible conics, the intersections of U and V are all real, and if not, not*. The criteria for distinguishing possible from impossible conics being well known need not be stated in this place.

We may of course proceed analogously by forming the two conics $lU + V$, $LU + V$, where l and L are roots of $\frac{d}{d\lambda}\square\{\lambda U + V\} = 0$ upon the supposition of $U = 0$ being a possible conic.

If either of the two U and V be not possible, their intersections are of course impossible, and the question is already decided.

It will be seen as pre-indicated that this method only fails in symmetry because of the choice between the couples m, M, and l, L. But moreover a perfect method for the discrimination of the two cases of *unmixed* intersection one from the other should (perhaps?) require the application of only a single test (in lieu of the two conditions which the above method supposes), over and above the condition which expresses the fact of the intersections being so unmixed. Such more perfect method I have not yet been able to achieve.

Another interesting question of intersections remains to be discussed, namely, supposing the two conics are known to be non-intersecting, how are we to ascertain if they are external to one another, or if one contains the other? In order to settle this point we must first establish a criterion for determining whether a given *point* is internal or external to a given conic; the point being in general said to be external when two real tangents can be drawn from it to the curve, and internal when this cannot be done.

* It must be well observed however that the possibility of the conics $U+mV$ and of $U+MV$ does not imply the reality of the intersections unless the conic V is known to be possible.

For if V be impossible ϵ and γ have opposite signs, and therefore $\frac{c-e}{\gamma-\epsilon}$ is intermediate between $\frac{e}{\epsilon}$ and $\frac{\gamma}{\epsilon}$, and the scheme for μ will be as here annexed:

$-\infty$ ☜ Impossible. A Possible. B Possible. C Impossible. ☞ $+\infty$

so that $U+mV$ and $U+MV$ will both represent possible conics.

Let now

$$\phi(x, y, z) = ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy = 0,$$

be the equation to any conic: l, m, n the coordinates of any point. Let

$$A = bc - a'^2, \quad B = ca - b'^2, \quad C = ab - c'^2,$$
$$A' = aa' - b'c', \quad B' = bb' - c'a', \quad C' = cc' - a'b'.$$

Then the reciprocal equation to the conic is

$$A\xi^2 + B\eta^2 + C\zeta^2 + 2A'\eta\zeta + 2B'\zeta\xi + 2C'\xi\eta = 0,$$

and in making $l\xi + m\eta + n\zeta = 0$, the ratios of ξ, η, ζ must be real if the tangents drawn from l, m, n are real: this will be found to imply that the determinant

$$\begin{vmatrix} A, & C', & B', & l \\ C', & B, & A', & m \\ B', & A', & C, & n \\ l, & m, & n, & 0 \end{vmatrix}$$

shall be negative*. This determinant may be shown† to be equal to the product of the determinant

$$\begin{vmatrix} a, & c', & b' \\ c', & b, & a' \\ b', & a', & c \end{vmatrix}$$

by the quantity

$$al^2 + bm^2 + cn^2 - 2a'mn - 2b'ln - 2c'lm,$$

that is, equal to $\phi(l, m, n) \times \square$.

Hence l, m, n is internal or external to $\phi(x, y, z)$ according as $\phi(l, m, n)$ and $\square\phi$ have the same or contrary sign.

If $\phi(l, m, n) = 0$, the point lies on the conic, and the point is neither internal nor external; if $\square\phi = 0$, the conic becomes a pair of straight lines, and no point can be said either to be within or without such a system. Hence our criterion fails, as it *ought to do*, just in the very two cases where the distinction vanishes. I believe that this criterion is here given for the first time.

* See theorem of the "Diminished Determinant" in Postscript to this paper.

† As we know *à priori* by virtue of a theorem given by M. Cauchy, and which is included as a particular case in a theorem of my own, relating to Compound Determinants, that is, Determinants of Determinants, which will take its place as an immediate consequence of my fundamental Theorem given in a Memoir about to appear. The well-known rule for the multiplication of Determinants is also a direct and simple consequence from my theorem on Compound Determinants, which indeed comprises, I believe, in one glance, all the heretofore existing Doctrine of Determinants.

To return to the two non-intersecting conics. Let us again throw them under the form

$$U = (x^2 + y^2) - e^2(z^2 + y^2),$$
$$V = k(x^2 + y^2) - k\epsilon^2(z^2 + y^2),$$

e and ϵ being real, that is, U and V being both functions corresponding to possible conics. Suppose U external to V; then *any point* in U is an external point to V.

Take in U either of the two points represented by the equations $y = 0$, $x^2 = e^2z^2$; substituting these values of y and x, V becomes $k(e^2 - \epsilon^2)z^2$, and $\square V$ becomes $-k^3\epsilon^2(1 - \epsilon^2)$; therefore $(1 - \epsilon^2)(e^2 - \epsilon^2)$ must be positive, that is, ϵ^2 must be one of the extremes of the three values $1, e^2, \epsilon^2$. In like manner, if V is external to U, e will be also one of the extremes of the same three quantities; and hence, if the two conics are mutually external, unity will be the middle magnitude of the group $e^2, 1, \epsilon^2$.

Now the three roots of $\square(V + \lambda U) = 0$, are

$$\lambda = -k, \quad \lambda = -k\frac{\epsilon^2}{e^2}, \quad \lambda = -k\frac{1 - \epsilon^2}{1 - e^2}.$$

Hence if U and V be without one another, or, as it may be termed, are extra-spatial, the third value of λ will be of a different sign from the first two; but if the two conics be co-spatial, that is, if one includes the other, all the three values of λ will have the same sign. Hence we have the following elegant criterion of co-spatiality of two possible conics expressed by the equations $U = 0$, $V = 0$, between indeterminate coordinates ξ, η, ζ; the coefficients of the cubic function $\underset{\xi\eta\zeta}{\square}(\lambda U + \mu V)$ must give only changes or only continuations of sign.

If this test be not satisfied, it will remain to determine which of the two conics contains, and which is contained by the other. Let U contain V, then the order of magnitudes will be $1, e^2, \epsilon^2$; therefore $k\frac{1 - \epsilon^2}{1 - e^2}$ is greater than k, and therefore $k\frac{1 - \epsilon^2}{1 - e^2}$, which is that root of the equation $\square(V + \lambda U) = 0$ which is always one or the other of the extremes, is the *greatest* of the three. Hence the scheme for the impossible and possible regions of λ will be as below:

$-\infty$ ☜ Poss. *A* Imposs. *B* Poss. *C* Poss. ☞ $+\infty$

Hence if the two roots of $\frac{d}{d\lambda}\{V + \lambda U\} = 0$ be l and L, and of the two conics $V + lU = 0$, $V + LU = 0$, the former be the possible, and the latter the impossible one, U contains V or is contained in it according as l is greater or less than L.

Observe that if U and V be non-cospatial, so that the three values of μ in $\Box(U+\mu V)=0$ have not all the same sign and consequently zero lies between the greatest and least of them, it will not be necessary to make trial of the characters of the two curves $U+mV=0$, and $U+MV=0$, in order to ascertain whether U and V intersect or not; for it will be sufficient to find which of the two quantities m and M substituted for μ in $\Box(U+\mu V)$ causes it to have the opposite sign to $\Box(U+0V)$, that is, $\Box U$, and this one of the two it is, if either, which will make $U+\mu V$ an impossible conic, and will thus alone serve to determine whether the intersections of U and V are unreal, or the contrary.

It might be a curious question to consider whether, in a certain sense, conics not both possible may not be said to lie one within or without the other. Upon general logical grounds, I think it not improbable that two impossible conics might be discovered *each to contain the other;* but this is an inquiry which I have not had leisure to enter upon.

I have thus far supposed the roots of $\Box(\lambda U+V)=0$ to be all distinct from one another. I now approach the discussion of the contact of two conics, in which event two or more of the roots will be equal. The condition for simple contact is evidently $\underset{\lambda\mu}{\Box}\ \underset{\xi\eta\zeta}{\Box}(\lambda U+\mu V)=0$.

The unpaired value of λ in $\Box(\lambda U+V)$ makes $\lambda U+V$ an impossible pair of lines, and therefore, in the scheme for λ drawn as above, will separate the possible from the impossible region.

Whether the conics intersect in two real or two unreal points, besides the point of contact, will be known at once by ascertaining whether $U+\mu V=0$ represents two real or two imaginary lines. If the latter, the two curves lie *dos-à-dos* or one within the other, according as the successions of sign in $\Box(\lambda U+V)$ are all of the same kind or not; if they be all of the same kind, one will include the other, namely, U will include V if the equal roots are greater, and be included in it if they be less than the unequal one. This last conclusion however, it should be observed, is inferred upon the principle of continuity, by making two values of λ approach indefinitely near to one another, but cannot be strictly deduced from the equations given for U and V applicable to the general case, in which the axes of coordinates are the three axes joining the vertices; since these latter, in the case supposed, reduce to two only, and consequently such representation of U and V becomes illusory.

If all three values of λ are equal, the three vertices come together, and hence the two conics will have three consecutive points in common, that is, will have the same circle of curvature. On this supposition the two curves cut at the point of contact, and all four points of intersection are of course real.

The classification of contacts between two conics may be stated as follows:

Simple contact = one case.

Second degree contact = two cases, namely, common curvature or double contact.

Third degree contact = one case, namely, contact in four consecutive points.

These four cases of course correspond to the several suppositions of there being two equal roots, three equal roots, two pairs of equal roots, or four equal roots in the biquadratic equation obtained between two variables by elimination performed in any manner between the given equations in the two conics.

The first species and the first case of the second species have been already disposed of. I proceed to assign the conditions appertaining to the second case of the second species, when U and V have a double contact.

Let A, A', B, B' be the two pairs of coincident points in which the conics are supposed to meet; either pair of lines AB, $A'B'$, and AB', $A'B$, becomes a coincident pair. Hence such a value of μ can be found as will make $U+\mu V$ the *square of a linear function of* ξ, η, ζ. If therefore we make $U+\mu V=W$, and form the determinant

$$\begin{vmatrix} \frac{d^2W}{d\xi^2}, & \frac{d^2W}{d\xi d\eta}, & \frac{d^2W}{d\xi d\zeta}, & p \\ \frac{d^2W}{d\eta d\xi}, & \frac{d^2W}{d\eta^2}, & \frac{d^2W}{d\eta d\zeta}, & q \\ \frac{d^2W}{d\zeta d\xi}, & \frac{d^2W}{d\zeta d\eta}, & \frac{d^2W}{d\zeta^2}, & r \\ p, & q, & r, & 0 \end{vmatrix}$$

$$=Ap^2+Bq^2+Cr^2+2Fqr+2Grp+2Hpq,$$

where all the coefficients are quadratic functions of μ, and make

$$A=0,\ B=0,\ C=0,\ F=0,\ G=0,\ H=0,$$

each of these six equations in μ will have one and the same root in common.

It is, however, enough to select any three; if these vanish together for any value of μ, the remaining three must also vanish. This is a simple application of a general law* which will appear in a forthcoming memoir on "Determinants and Quadratic Forms," of which this paper is to be considered as an accidental episode.

* For statement of this law called the Homaloidal Law, see *Philosophical Magazine* of this month "On Certain Additions, &c." [p. 150 below. Ed.]

Take now any three of the six equations which for the sake of generality call $P=0$, $Q=0$, $R=0$. The hypothesis of double contact requires that P and Q, Q and R, R and P shall have a factor in common; but these conditions are not sufficiently explicit for our present object, since P, Q, R might be of the form

$$\kappa(\lambda-a)(\lambda-b),\quad \kappa'(\lambda-b)(\lambda-c),\quad \kappa''(\lambda-c)(\lambda-a),$$

and would thus satisfy the conditions above stated, without P, Q, R having a common factor. A sufficient criterion is that $fQ+gR$ and P shall have a common factor for all values of f and g.

Let then the resultant of $fQ+gR$ and P be

$$Lf^2+Mfg+Ng^2,$$

we must have

$$L=0,\quad M=0,\quad N=0,$$

where

L is the resultant of P and Q,

N " " " R and Q;

and M is a new function, which if we call $Q=\phi(\lambda)$, $R=\psi(\lambda)$, and suppose a and b to be the two roots of $P=0$, is easily seen to be equal to

$$\phi a\,.\,\psi b+\phi b\,.\,\psi a.$$

This I call the connective of $P\,.\,Q$ and $P\,.\,R$.

L, M, N may conveniently be denoted by the forms

$$P\,.\,Q,\quad P\,.\,R,\quad Q\,.\,P\,.\,R.$$

We may now take more generally

$$aP+bQ+cR,$$
$$\alpha P+\beta Q+\gamma R,$$

which will have a factor in common for all values of a, b, c, α, β, γ.

I am indebted to Mr Cayley for the remark that the resultant of these two functions is a new quadratic function, which, according to my notation just given, may be put under the form

$$PQ(a\beta-b\alpha)^2+QR(b\gamma-c\beta)^2+RP(c\alpha-a\gamma)^2$$
$$+PRQ(b\gamma-c\beta)(c\alpha-a\gamma)+QPR(c\alpha-a\gamma)(a\beta-b\alpha)+RQP(a\beta-b\alpha)(b\gamma-c\beta).$$

Ternary systems of the six coefficients formed upon the type of (PQ, PQR, QR), I call *complete* systems, because the three functions included in such a system equated severally to zero, imply that the remaining three coefficients are all zero. Such a system as (PQ, QR, RP) I term an *incomplete* ternary system as not drawing with it the like implication. Probably we should find on investigation that PRQ, QPR, RQP, would also be an

incomplete system, but that systems formed after the type of PRQ, RQ, RQP are complete. This however is only matter of conjecture, as I have been too much occupied with other things to enter upon the inquiry. The distinct types of ternary systems are altogether six in number, namely, four of a symmetrical species,

$$PQ,\quad QR,\quad RP,$$
$$PRQ,\quad QPR,\quad RQP,$$
$$PQ,\quad PQR,\quad QR,$$
$$PRQ,\quad RQ,\quad RQP;$$

and two of an unsymmetrical species, namely,

$$PQ,\quad PQR,\quad PR,$$
$$PRQ,\quad RQ,\quad QPR.^*$$

If instead of confining ourselves to three out of the six original quantities, A, B, C; F, G, H, we take them all into account, and write down the resultant of

$$aA+bB+cC+fF+gG+hH,$$
$$\alpha A+\beta B+\gamma C+\phi F+\chi G+\eta H;$$

we shall obtain a quadratic function of 15 variables (not however all independent) having 120 coefficients, all of which must be zero. It would be extremely interesting to determine how many *complete* ternary groups can be formed out of these 120 terms.

It will be recollected that we have assigned as the condition of contact in three consecutive points, that a certain cubic equation shall have all its roots real. Now, as well remarked by Mr Cayley, we cannot express this fact by less than three equations in integral terms of the coefficients. Thus if the cubic be written

$$a\lambda^3+3b\lambda^2+3c\lambda+d=0,$$

we have as one of such ternary systems,

$$U=ac-b^2=0,\quad V=bd-c^2=0,\quad W=bc-ad=0.$$

The significant parts of these equations are of course, however, capable of being connected by integral multipliers U', V', W', such that

$$U'U+V'V+W'W=0.$$

* PQ, QR, RP, may be compared in a general way with the angles, and PRQ, QPR, RQP, with the sides of a triangle.

Any number of functions U, V, W so related, I call *syzygetic* functions, and U', V', W' I term the *syzygetic multipliers**. These in the case supposed are c, a, b, respectively.

In like manner it is evident that the members of any group of functions, more than two in number, whose nullity is implied in the relation of double contact, whether such group form a complete system or not, must be in syzygy.

Thus PQ, PQR, QR, must form a syzygy; nor is there any difficulty in assigning a system of multipliers to exhibit such syzygy. Calling $P=\phi(\lambda)$, $R=\psi(\lambda)$, a and b the two roots of $Q=0$, I have found that

$$\{(\psi a)^2+(\psi b)^2\}\, PQ-(\phi a\,.\,\psi a+\phi b\,.\,\psi b)\, PQR+\{(\phi a)^2+(\phi b)^2\}\, QR=0.$$

Again, if we take the *incomplete* system

$$(PQ),\quad (QR),\quad (RP),$$

it will be found that

$$L\,(QR)+M\,(RP)+N\,(PQ)=0,$$

provided that, calling a, b; c, d; e, f, the roots of $P=0$, $Q=0$, $R=0$, respectively, we make

$$L=(k_0+k_1a+k_2a^2+k_3a^3+k_4a^4)\frac{(a-c)(a-d)(a-e)(a-f)}{a-b}$$
$$+(k_0+k_1b+k_2b^2+k_3b^3+k_4b^4)\frac{(b-c)(b-d)(b-e)(b-f)}{b-a},$$

$$M=(k_0+k_1c+k_2c^2+k_3c^3+k_4c^4)\frac{(c-a)(c-b)(c-d)(c-e)(c-f)}{c-d}$$
$$+(k_0+k_1d+k_2d^2+k_3d^3+k_4d^4)\frac{(d-a)(d-b)(d-c)(d-e)(d-f)}{d-c},$$

$$N=(k_0+k_1e+k_2e^2+k_3e^3+k_4e^4)\frac{(e-a)(e-b)(e-c)(e-d)}{e-f}$$
$$+(k_0+k_1f+k_2f^2+k_3f^3+k_4f^4)\frac{(f-a)(f-b)(f-c)(f-d)}{f-e};$$

k_0, k_1, k_2, k_3, k_4 being quite arbitrary, and L, M, N, although presented in a fractional form, being essentially integral.

This fact of L, M, N constituting a system of multipliers to the syzygy QR, RP, PQ, is easily demonstrated; for

$$QR=(c-e)(c-f)(d-e)(d-f),$$
$$RP=(e-a)(e-b)(f-a)(f-b),$$
$$PQ=(a-c)(a-d)(b-c)(b-d).$$

* There will be in general various such systems of multipliers.

Hence $$L\,(QR)+M\,(RP)+N\,(PQ)$$
$$=(a-c)(a-d)(a-e)(a-f)(b-c)(b-d)(b-e)(b-f)(c-e)(c-f)(d-e)(d-f)$$
$$\times\Sigma\frac{k_0+k_1a+k_2a^2+k_3a^3+k_4a^4}{(a-b)(a-c)(a-d)(a-e)(a-f)}=0.$$

My theory of elimination enables me to explain exactly the nature of L, M, N, and the *reason* of their appearance as syzygetic factors.

Let L_r, M_r, N_r signify what L, M, N become, when all the k's except k_r are taken zero. Then the theory given by me in the *Philosophical Magazine* for the year 1838, or thereabouts†, shows that $L_0\lambda+L_1$ is the *prime derivee* of the first degree between the two equations P and $Q\times R$, or, in other words, will be the remainder integralized of $\frac{QR}{P}$.

In like manner $M_0\lambda+M_1$, $N_0\lambda+N_1$ are the integralized remainders of $\frac{RP}{Q}$ and of $\frac{PQ}{R}$ respectively.

If now the resultant of P, Q and of Q, R are each zero, but the resultant of P and R is not zero, it will be evident that P, Q, R must be of the form

$$f(\lambda+a)(\lambda+c),\quad g(\lambda+c)(\lambda+d),\quad h(\lambda+d)(\lambda+b);$$

and therefore $P\times R$ will contain Q, and consequently we must have

$$M_0=0,\quad M_1=0.$$

More generally, if we write

$$Q=0,$$
$$\lambda Q=0,$$
$$\lambda^2 Q=0,$$
$$P\times R=0,$$

and eliminate dialytically, that is, treating λ^4, λ^3, λ^2, λ as distinct quantities, we shall obtain*

$$\lambda^4:\lambda^3:\lambda^2:\lambda:1::M_4:M_3:M_2:M_1:M_0;$$

and therefore when $P\times R$ contains Q,

$$M_0=0,\quad M_1=0,\quad M_2=0,\quad M_3=0,\quad M_4=0.$$

* This cannot be obtained directly from what is stated in the paper referred to, although contained in the general theory of derivation there given. The arbitrary functions which enter into the expression for the general derivees have been in that paper evaluated only for the prime derivees, which however are only particular phenomena, with reference to the general results of Dialytic Elimination. Hereafter I may give a more general exposition of this remarkable, although ignored or neglected theory. The prime derivees of fx and $f'x$ are Sturm's Functions, cleared of quadratic factors, and are expressed by virtue of the general theorems there laid down as functions of x and of symmetrical functions of the roots of fx. [† p. 40 above. Ed.]

In like manner, when $Q \times P$ contains R,

$$N_0=0, \quad N_1=0, \quad N_2=0, \quad N_3=0, \quad N_4=0;$$

and when $R \times Q$ contains P,

$$L_0=0, \quad L_1=0, \quad L_2=0, \quad L_3=0, \quad L_4=0.$$

Accordingly, we see from the equation

$$L(QR)+M(RP)+N(PQ)=0,$$

that if $QR=0$, $RP=0$; but PQ *not* $=0$, then $N=0$; and therefore

$$N_0=0, \quad N_1=0, \quad N_2=0, \quad N_3=0, \quad N_4=0,$$

and so in like manner for the remaining corresponding two suppositions*.

Before proceeding to consider the remaining case of the highest species of contact, I must observe that besides the equations involved in the condition that $A, B, C; F, G, H$, or, which is the same thing, that any three of them shall all have a factor in common, we must have $\square(U+\lambda V)$ containing the square of such common factor. In the memoir before adverted to a general theorem will be given and proved, which shows that this latter condition is involved in the former one; in fact, more generally (but still only as a particular case) that when U and V are quadratic functions of n letters, but $U+\epsilon V$ admits of being represented as a complete function of $(n-2)$ quantities only, which are themselves linear functions of the n letters, then $\square(U+\lambda V)$, which is of course a function of λ of the nth degree, will contain the factor $(\lambda-\epsilon)^2$.

When the two conics have four consecutive points in common, the characters of double-point contact and of contact in three consecutive points must exist simultaneously; and consequently the factor common to $A, B, C; F, G, H$, will enter not as a binary but as a ternary factor into $\square(U+\lambda V)$. This gives the extra condition required. As an example take the two conics,

$$U=\frac{y^2}{1-k}+x^2-z^2=0,$$

$$V=y^2+x^2-2kxz+(2k-1)z^2=0,$$

$$U+\lambda V=\left(\frac{1}{1-k}+\lambda\right)y^2+(1+\lambda)x^2-\{1+\lambda(1-2k)\}z^2-2k\lambda xz.$$

* Since we are able to assign the values of the syzygetic multipliers in the equations

$$L\ (PQ)+M\ (QR)+N\ (RP)=0,$$
$$L'\ (PQ)+M'\ (PQR)+N'\ (QR)=0,$$
$$L''\ (QR)+M''\ (QRP)+N''\ (RP)=0,$$
$$L'''\ (RP)+M'''\ (RPQ)+N'''\ (PQ)=0,$$

it follows that we may eliminate between these four equations any three of the six quantities (PQ), (PRQ), &c., and thus express any one of them in terms of any two others: this method, however, is not practically convenient. I may probably hereafter return to this subject.

The complete determinant of $U+\lambda V$ is then

$$\frac{-1}{1-k}\{1+(1-k)\lambda\}\{(1+\lambda)^2-2k\lambda(1+\lambda)+k^2\lambda^2\}=-\frac{1}{1-k}\{1+(1-k)\lambda\}^3.$$

A, B, C are the determinants of $U+\lambda V$, when $x=0$, $y=0$, $z=0$, respectively. Thus

$$A=\left(\frac{1}{1-k}+\lambda\right)(1+\lambda),$$

$$B=\left(\frac{1}{1-k}+\lambda\right)\{1+\lambda(1-2k)\},$$

$$C=k^2\lambda^2-(1+\lambda)\{1+\lambda(1-2k)\}=\lambda^2(1-k)^2-2\lambda(1-k)-1;$$

$\lambda=-\dfrac{1}{1-k}$ makes $A=0$, $B=0$, $C=0$, and the factor $\lambda+\dfrac{1}{1-k}$ enters cubed into $\square(U+\lambda V)$.

Hence the two conics have a contact of the third order.

This is easily verified; for if we pass from general to Cartesian and rectangular coordinates, and make z unity; $U=0$ will represent an ellipse with centre at the origin, eccentricity $\sqrt{k}$, and mean focal distance 1, and $V=0$ the circle of curvature at the extremity of the axis major*.

I had intended to have added some other remarks connected with the present discussion, and also to have appended an *à posteriori* proof of the propositions relative to the reality and otherwise of the vertices and chordal pairs of intersection which I have, at the commencement of this paper, deduced quite legitimately, but in a manner not at first sight perhaps easily intelligible, from the general principles of conjugate forms; but this discussion has run on already to a length so much greater than I had anticipated and than the importance of the inquiry may seem to justify, that I must reserve for a future number of the *Journal* what further matter I may have to communicate concerning it.

POSTSCRIPT.—As I have alluded to Professor Boole's theorem relative to Linear Transformations, it may be proper to mention my theorem on the subject, which is of a much more general character, and includes Mr Boole's (so far as it refers to *Quadratic* Functions) as a corollary to a particular case. The demonstration will be given in the forthcoming memoir above alluded to.

Let U be a quadratic function of any number of letters $x_1, x_2 \ldots x_n$, and let any number r of linear equations of the general form

$${}_1a_r x_1 + {}_2a_r x_2 + \ldots\ldots + {}_na_r x_n = 0,$$

* We have thus discussed all the four cases of biconical contact: for an exactly parallel discussion of the theory of contact of a plane with the curve of double curvature in which two surfaces of the second order intersect, see the paper in the *Philosophical Magazine* for this month, before referred to. [p. 148 below. ED.]

be instituted between them: and by means of these equations let U be expressed as a function of any $(n-r)$ of the given letters, say of $x_{r+1}, x_{r+2} \ldots\ldots x_n$, and let U, so expressed, be called M. Let

$$ {}_1a_r x_1 + {}_2a_r x_2 + \ldots\ldots + {}_na_r x_n $$

be called L_r. Then the determinant of M in respect to the $(n-r)$ letters above given is equal to the determinant of

$$ U + L_1 x_{n+1} + L_2 x_{n+2} + \ldots\ldots + L_r x_{n+r}, $$

considered as a function of the $(n+r)$ letters

$$ x_1 x_2 \ldots\ldots x_{n+r}, $$

divided by the square of the determinant

$$ \begin{vmatrix} {}_1a_1, & {}_2a_1 \ldots\ldots {}_ra_1 \\ {}_1a_2, & {}_2a_2 \ldots\ldots {}_ra_2 \\ \ldots & \ldots\ldots\ldots \\ {}_1a_r, & {}_2a_r \ldots\ldots {}_ra_r \end{vmatrix}. $$

This I call the theorem of Diminished Determinants.

If now we have U a function of r letters, and V of r other letters, and V is derived from U by linear transformations, that is, by r equations connecting the $2r$ letters; then, since U may be considered as a function of *all the* $2r$ letters with abortive coefficients for all the terms where any of the second set of r letters enter, we may apply our theorem of diminished determinants to the question so considered, and the result may be found to represent Mr Boole's theorem in a form rather more general and symmetrical, but substantially identical with that given by Mr Boole.

Thus suppose $\frac{1}{2}ax^2 + bxy + \frac{1}{2}cy^2$ say P, and $\frac{1}{2}\alpha u^2 + \beta uv + \frac{1}{2}\gamma v^2$ say Q, are mutually transformable by virtue of the linear equations

$$ lx + my = \lambda u + \mu v, $$
$$ l'x + m'y = \lambda' u + \mu' v, $$

P may be considered as a function of x, y, u, v, and Q as the value of P, when we eliminate x and y by virtue of the two linear equations

$$ L_1 = lx + my - \lambda u - \mu v = 0, $$
$$ L_2 = l'x + m'y - \lambda' u - \mu' v = 0; $$

we have therefore by our theorem the determinant of Q equal to the squared reciprocal of the determinant $\begin{vmatrix} l, & m \\ l', & m' \end{vmatrix}$ multiplied by the determinant

$$ \begin{vmatrix} a, & b, & 0, & 0, & l, & l' \\ b, & c, & 0, & 0, & m, & m' \\ 0, & 0, & 0, & 0, & -\lambda, & -\lambda' \\ 0, & 0, & 0, & 0, & -\mu, & -\mu' \\ l, & m, & -\lambda, & -\mu, & 0, & 0 \\ l', & m', & -\lambda', & -\mu', & 0, & 0 \end{vmatrix}, $$

which last determinant is evidently equal to the determinant of P multiplied by the square of the determinant $\begin{vmatrix} \lambda, & \mu \\ \lambda', & \mu' \end{vmatrix}$. Whence we see that the determinant of Q divided by the square of $\begin{vmatrix} \lambda, & \mu \\ \lambda', & \mu' \end{vmatrix}$, is equal to the determinant of P divided by the square of $\begin{vmatrix} l, & m \\ l', & m' \end{vmatrix}$. There is also another way more simple, but less direct, by means of which the theorem of diminished determinants may be made to yield Mr Boole's theorem of transformation*. Some unavowed use has been made in the foregoing pages of this former theorem, one of the highest importance in the analytical and geometrical theory of quadratic functions. It has been nearly a year in my possession, and I trust and believe that I am committing no act of involuntary misappropriation in announcing it as a result of my own researches.

* Namely, by considering P and Q as *each* derived from some common function of x, y, u, v, w, by means of the equations $L_1 = 0$, $L_2 = 0$; the law of Diminished Determinants will then indicate the determinants of P and Q, each under the form of fractions having the *same numerator*, but whose denominators will be $\begin{vmatrix} \lambda, & \mu \\ \lambda', & \mu' \end{vmatrix}^2$ and $\begin{vmatrix} l, & m \\ l', & m' \end{vmatrix}$ respectively.

23.

AN INSTANTANEOUS DEMONSTRATION OF PASCAL'S THEOREM BY THE METHOD OF INDETERMINATE COORDINATES.

[*Philosophical Magazine*, XXXVII. (1850), p. 212.]

THE new analytical geometry consists essentially of two parts—the one determinate, the other indeterminate.

The determinate analysis comprehends that class of questions in which it is necessary to assume *independent* linear coordinates, or else to take cognizance of the equations by which they are connected if they are not independent. The indeterminate analysis assumes at will any number of coordinates, and leaves the relations which connect them more or less indefinite, and reasons chiefly through the medium of the general properties of algebraic forms, and their correspondencies with the objects of geometrical speculation. Pascal's theorem of the mystic hexagon, and the annexed demonstration of its fundamental property, belong to this branch of the subject, and afford an instructive and striking example of the application of the pure method of indeterminate coordinates.

Let x, y, z, t, u, v be the sides of a hexagon inscribed in the conic U. Let the hexagon be divided by a new line ϕ in any manner into two quadrilaterals, say $xyz\phi$, $tuv\phi$.

Then $$ay\phi + bxz = U = \alpha u\phi + \beta tv;$$

therefore $$(ay - \alpha u)\phi = \beta tv - bxz;$$

therefore $ay - \alpha u$ and ϕ are the diagonals of the quadrilateral $txvz$.

By construction, ϕ is the diagonal joining x, v (that is, the intersection of x and v) with z, t; and thus we see that $ay - \alpha u$ is the line joining t, x with v, z; but this line passes through y, u. Therefore x, t; y, u; z, v lie in one and the same right line. Q.E.D.

24.

ON A NEW CLASS OF THEOREMS IN ELIMINATION BETWEEN QUADRATIC FUNCTIONS.

[*Philosophical Magazine*, XXXVII. (1850), pp. 213—218.]

IN a forthcoming memoir on determinants and quadratic functions, I have demonstrated the following remarkable theorem as a particular case of one much more general, also there given and demonstrated.

Let U and V be respectively quadratic functions of the same $2n$ letters, and let it be supposed possible to institute n such linear equations between these letters as shall make U and V both simultaneously become identically zero*. Then the determinant of $\lambda U + \mu V$, which is of course a function of λ and μ of the $2n$th degree, will become the *square* of a function of λ and μ of the nth degree; and conversely, if this determinant be a perfect square, U and V may be made to vanish simultaneously by the institution of n linear equations between the $2n$ letters†.

Let now P and Q be respectively quadratic functions of three letters only, say x, y, z; and let

$$U = P + (lx + my + nz)t,$$

$$V = Q + k(lx + my + nz)t.$$

The determinant of $\lambda U + \mu V$ in respect to x, y, z, t is easily seen to be $(\lambda + k\mu)^2 \times$ the determinant of

$$\lambda P + \mu Q + (lx + my + nz)t$$

in respect to x, y, z, t. Hence if we call

$$\lambda P + \mu Q + (lx + my + nz)t = W,$$

and make $\underset{xyzt}{\square} W$ a squared function of λ, μ or which is the same thing, if

$$\underset{\lambda\mu}{\square}\ \underset{xyzt}{\square} \{W\} = 0,$$

* In the more general theorem above alluded to, the number of letters is any number m, the number of linear equations being any number not exceeding $\frac{m}{2}$.

† When $n = 1$, we obtain a theorem of elimination between two quadratics, which has been already given by Professor Boole.

U and V will vanish simultaneously when two linear relations are instituted between the quantities (all or some of them) x, y, z, t.

In order that this may be the case, it will be seen to be sufficient that

$$P = 0, \quad Q = 0, \quad (lx + my + nz) = 0,$$

shall coexist; for then two equations between x, y, z of which $lx+my+nz=0$ will be one, will suffice to make U and V each identically zero. Hence we have the following theorem:

$$\underset{\lambda\mu}{\square}\ \underset{xyzt}{\square}\{\lambda U + \mu V + (lx + my + nz)\, t\}$$

is a factor of the resultant of

$$P = 0, \quad Q = 0, \quad lx + my + nz = 0.$$

A comparison of the orders of the resultant and the determinant shows that they must be identical, *à-ci-près*, of a numerical factor, which, if the resultant be taken in its *general* lowest terms, may no doubt be easily shown to be unity.

As an illustration of our theorem, let

$$P = xy + yz + zx,$$
$$Q = cxy + ayz + bzx.$$

Then

$$\underset{xyzt}{\square}\{\lambda P + \mu Q + (lx + my + nz)\, t\} = \begin{vmatrix} 0, & \lambda + c\mu, & \lambda + b\mu, & l \\ \lambda + c\mu, & 0, & \lambda + a\mu, & m \\ \lambda + b\mu, & \lambda + a\mu, & 0, & n \\ l, & m, & n, & 0 \end{vmatrix}$$

$$= n^2(\lambda + c\mu)^2 + m^2(\lambda + b\mu)^2 + l^2(\lambda + a\mu)^2$$
$$- 2lm(\lambda + b\mu)(\lambda + a\mu) - 2mn(\lambda + c\mu)(\lambda + b\mu) - 2nl(\lambda + a\mu)(\lambda + c\mu)$$
$$= \lambda^2\{n^2 + m^2 + l^2 - 2lm - 2mn - 2nl\}$$
$$+ 2\lambda\mu\{cn^2 + bm^2 + al^2 - lm(a + b) - mn(b + c) - nl(c + a)\}$$
$$+ \mu^2\{c^2n^2 + b^2m^2 + a^2l^2 - 2ablm - 2bcmn - 2canl\}.$$

And we thus obtain, finally,

$$\underset{\lambda\mu}{\square}\ \underset{xyzt}{\square}\{\lambda P + \mu Q + (lx + my + nz)\, t\}$$
$$= (n^2 + m^2 + l^2 - 2lm - 2mn - 2nl)$$
$$\times (c^2n^2 + b^2m^2 + a^2l^2 - 2ablm - 2bcmn - 2canl)$$
$$- \{(cn^2 + bm^2 + al^2 - lm(a + b) - mn(b + c) - nl(c + a)\}^2$$
$$= -4lmn\{(a - b)(a - c)\, l + (b - a)(b - c)\, m + (c - a)(c - b)\, n\}.$$

Now to obtain the resultant of

$$xy + yz + zx = 0,$$
$$cxy + azy + bxz = 0,$$
$$lx + my + nz = 0,$$

we need only find the four systems *in their lowest terms* of $x : y : z$, which satisfy the first two equations, and multiply the four linear functions obtained by substituting these values of x, y, z in the fourth: the product will contain the resultant of the system affected with some numerical factor. In the present case, the four systems of x, y, z are

$$x = 0, \quad y = 0, \quad z = 1,$$
$$y = 0, \quad z = 0, \quad x = 1,$$
$$z = 0, \quad x = 0, \quad y = 1,$$
$$x = (a - b)(a - c), \quad y = (b - a)(b - c), \quad z = (c - a)(c - b).$$

and accordingly the product of

$$lx_1 + my_1 + nz_1,$$
$$lx_2 + my_2 + nz_2,$$
$$lx_3 + my_3 + nz_3,$$
$$lx_4 + my_4 + nz_4,$$

becomes

$$lmn\{(a - b)(a - c)\, l + (b - a)(b - c)\, m + (c - a)(c - b)\, n\},$$

agreeing with the result obtained by my theorem,—a *special* numerical factor 4, arising from the peculiar form of the equations, having disappeared from the resultant.

A geometrical demonstration may be given of the theorem which is instructive in itself, and will suggest a remarkable extension of it to functions containing more than three letters; the equation

$$\underset{xyzt}{\square}\{\lambda U + \mu V + (lx + my + nz)\, t\} = 0,$$

which is a quadratic equation in $\lambda : \mu$, may easily be shown to imply that the conic $\lambda U + \mu V$ is touched by the straight line

$$lx + my + nz = 0.$$

And we thus see that in general two conics,

$$\lambda U + \mu V = 0,$$

passing through the intersections of two given conics,

$$U = 0, \quad V = 0,$$

may be drawn to touch a given line. If, however, the given line passes through any of the four points of intersection, in such case only one conic can be drawn to touch it; accordingly

$$\square\,\square\,\{\lambda U+\mu V+(lx+my+nz)\,t\}$$

must be zero when l, m, n are so taken as to satisfy this condition, that is, if

$$lx_1+my_1+nz_1=0,$$

or
$$lx_2+my_2+nz_2=0,$$

or
$$lx_3+my_3+nz_3=0,$$

or
$$lx_4+my_4+nz_4=0,$$

whence the theorem.

Now suppose U and V to be each functions of four letters, x, y, z, t; when

$$\underset{xyztu}{\square}\,\{\lambda U+\mu V+(lx+my+nz+pt)\,u\}=0,$$

the conoid $\lambda U+\mu V$ touches the plane

$$lx+my+nz+pt=0;$$

and $\square=0$ being a cubic equation, in general three such conoids can be drawn.

Considerations of analogy make it obvious to the intuition, that in the particular case of two of these becoming coincident, the given plane

$$lx+my+nz+pt$$

must be a tangent plane to those two coincident conoids at one of the points where it meets the intersections of $U=0$, $V=0$; that is

$$lx+my+nz+pt=0$$

will pass through a tangent line to, or in other words, may be termed a tangent plane to the intersections. Hence the following analytical theorem, derived from supposing q, r, s, t to be proportional to the areas of the triangular faces of the pyramid cut out of space by the four coordinate planes to which x, y, z, t refer. As these planes are left indefinite, q, r, s, t are perfectly arbitrary.

Theorem. The resultant of

1. $\left.\begin{matrix}U=0\\ V=0\end{matrix}\right\}$, where U and V are functions of x, y, z, t;
2.
3. $lx+my+nz+pt=0$;
4. $$\begin{vmatrix} \frac{dU}{dx}, & \frac{dU}{dy}, & \frac{dU}{dz}, & \frac{dU}{dt} \\ \frac{dV}{dx}, & \frac{dV}{dy}, & \frac{dV}{dz}, & \frac{dV}{dt} \\ l, & m, & n, & p \\ q, & r, & s, & t \end{vmatrix}=0;$$

which system, it will be observed, consists of three quadratic functions, and one linear function of x, y, z, t, *contains* the factor

$$\underset{\lambda\mu}{\square}\,\underset{xyzt}{\square}\,\{\lambda U+\mu V+(lx+my+nz+pt)\,u\}.$$

This last quantity is of the 4×3th, that is, the 12th order in respect of the coefficients in U and V combined; of the 4×2th, that is, the 8th order in respect of l, m, n, p; and of the zero order in respect of q, r, s, t.

The resultant which contains it is of the $(4+4+2.4)$th, that is, 16th order in respect to the coefficients in U and V; of the $(4+8)$th, that is, the 12th, in respect of l, m, n, p; and of the 4th in respect of q, r, s, t. Hence the special (and, as far as the geometry of the question is concerned, the unnecessary, I may not say extraneous or irrelevant) factor which enters into the resultant is of the 4th order in respect to the combined coefficients of U and V*; and of the same order in respect to l, m, n, p, and in respect to q, r, s, t.

I have not yet succeeded in divining its general value.

In the very particular example, of the system,

$$ax^2+\beta y^2=0,$$
$$cz^2+dt^2=0,$$
$$lx+my+nz+pt=0,$$
$$\begin{vmatrix} ax, & \beta y, & 0, & 0 \\ 0, & 0, & cz, & dt \\ l, & m, & n, & p \\ q, & 0, & 0, & 0 \end{vmatrix}=0,$$

I find that the double determinant is

$$c^2d^2a^2\beta^2\,(cp^2+dn^2)^2\,(m^2a+l^2\beta)^2,$$

and the resultant is

$$q^4c^2d^2a^2\beta^4\,(cp^2+dn^2)^4\,(m^2a+l^2\beta)^2,$$

giving as the special factor

$$q^4\beta^2\,(cp^2+dn^2)^2.$$

I believe that the theorem which I have here given for determining the condition that $lx+my+nz+pt$ shall be a tangent plane to the intersection of two conoids U and V, namely, that the determinant of

$$\lambda U+\mu V+(lx+my+nz+pt)\,u$$

shall have two equal roots, is altogether novel.

* And consequently of the second in respect to the separate coefficients of each.

What is the meaning of all three roots of this determinant becoming equal, that is, of only one conoid being capable of being drawn through the intersection of U and V to touch the plane

$$lx + my + nz + pt\ ?$$

Evidently (*ex vi analogiæ*) that this plane shall pass through three consecutive points of the curve of intersection, that is, that it shall be the osculating plane to the curve.

If we return to the intersection of two co-planar conics, and if we suppose a line to be drawn through two of the points of intersection, the conics capable of being drawn through the four points of intersection to touch the line, besides becoming coincident, evidently degenerate each into a pair of right lines. It would seem, therefore, by analogy, that if a plane be drawn including any two tangent lines to the curve of intersection of two surfaces of the second degree, this should be touched by two coincident cones drawn through the curve of intersection, and consequently every such double tangent plane to the intersection of two conoids (and it is evident that one or more of these can be taken at every point of the curve) must pass through one of the vertices of the four cones in which the intersection may also be considered to lie; and it would appear from this, that in general four double tangent planes admit of being drawn to the curve, which is the intersection of two conoids, at each point thereof. At particular points a tangent plane may be drawn passing through more than one of the vertices, and then of course the number of double tangent planes that can be drawn will be lessened. These results, indicated by analogy, become immediately apparent on considering the curve in question as traced upon any one of the four containing cones. For the plane drawn through a tangent at any point, and the vertex of the cone being a tangent plane to the cone, must evidently touch the curve again where it meets it. We thus have an additional confirmation of the analogy between a point of intersection of two curves and the tangent at any point of the intersection of two surfaces.

I might extend the analytical theorems which have been established for functions of three and four to functions of a greater number of variables; but enough has been done to point out the path to a new and interesting class of theorems at once in elimination and in geometry, which is all that I have at present leisure or the disposition to undertake.

25.

ADDITIONS TO THE ARTICLES*, "ON A NEW CLASS OF THEOREMS," AND "ON PASCAL'S THEOREM."

[*Philosophical Magazine*, XXXVII. (1850), pp. 363—370.]

First addition.—I have alluded in the second of the above articles to a more general theorem, comprising, as a particular case, the theorem there given for the simultaneous evanescence of two quadratic functions of $2n$ letters, on n linear equations becoming instituted between the letters.

In order to make this generalization intelligible, I must premise a few words on the Theory of Orders, a term which I have invented with particular reference to quadratic functions, although obviously admitting of a more extended application. A linear function of all the letters entering into a function or system of functions under consideration I call an order of the letters, or simply an order. Now it is clear that we may always consider a function of any number of letters as a function of as many orders as there are letters; but in certain cases a function may be expressed in terms of a fewer number of orders than it has letters, as when the general characteristic function of a conic becomes that of a pair of crossing lines or a pair of coincident lines, in which event it loses respectively one and two orders, and so for the characteristic of a conoid becoming that of a cone, a pair of planes or two coincident planes, in which several events, a function of four letters becomes that of only three orders, or two orders, or one order, respectively. When a function may be expressed by means of r orders less than it contains letters, I call it a function minus r orders. I now proceed to state my theorem.

Let U and V be functions each of the same m letters, and suppose that the determinant in respect of those letters of $U + \mu V$ contains i pairs of

[* pp. 138, 139 above. Ed.]

equal linear factors of μ; then it is possible, by means of i linear equations instituted between the letters, to make U and V each become functions of *the same* $m-2i$ orders; and conversely, if by i equations between the letters U and V may be made functions of *the same* $m-2i$ orders, the determinant of $U+\mu V$ considered as a function of μ will contain i square factors.

Thus when $m=2n$ and $i=n$, U and V will each become functions of zero orders, that is, will both disappear, provided that on the institution of a certain system of n linear equations, among the letters of which U and V are functions, the determinant of $(U+\mu V)$ is a perfect square,—which is the theorem given in the article referred to.

So for example if U and V be quadratic functions of four letters, and therefore the characteristics of two conoids, $\square(U+\mu V)$ being a perfect square, expresses that these conoids have a straight line in common lying upon each of their surfaces.

If U and V be quadratic functions of three letters only, and admit therefore of being considered as the characteristics of two conics, $\square(U+\mu V)$ containing a square factor, is indicative of these conics having a common tangent at a common point, that is, of their touching each other at some point; for it is easily shown that the disappearance of two orders from any quadratic function by virtue of one linear function of its letters being zero, indicates that the line, plane, &c. of which the linear function is the characteristic is a tangent to the curve, surface, &c. of which the quadratic function is the characteristic.

I pass now to a generalization of the theorem which shows how to express, under the form of a double determinant, the resultant of one linear and two quadratic homogeneous functions of three letters (which I should have given in the original paper, had I not there been more intent upon developing an ascending scale than of expatiating upon a superficial ramification of analogies), and which constitutes my *Second addition* to that paper, to wit—

If U and V be homogeneous quadratic, and $L_1, L_2 \ldots L_n$ homogeneous linear functions of $(n+2)$ letters $x_1, x_2 \ldots x_{n+2}$, the determinant of the entire system of $n+2$ functions is equal to

$$\underset{\lambda,\,\mu}{\square}\ \underset{x_1,\,x_2\ldots x_{n+2}\,t_1,\,t_2\ldots t_n}{\boxed{}}\{\lambda U+\mu V+L_1t_1+L_2t_2+\ldots+L_nt_n\};$$

the demonstration is precisely similar to the analytical one given in the September Number* for the particular case of $n=1$.

When $n=0$, we revert to Mr Boole's theorem of elimination between U and V already adverted to. The proof, it will be easily recognized, does not require the application of the more general theorem relative to the simul-

[* p. 140 above.]

taneous depression of orders of two quadratic functions, but only the limited one before given, which supplies the conditions of their simultaneous disparition. I now proceed to develope more particularly certain analogies between the theory of the mutual contacts of two conics, and that of the tangencies to the intersection of two conoids.

But here again I must anticipate some of the results which will be given in my forthcoming memoir on Determinants and Quadratic Functions, by explaining what is to be understood by minor determinants, and the relation in which they stand to the complete determinant in which they are included. This preliminary explanation, and the statement of the analogies above alluded to, will constitute my *Third* and last *addition.*

Imagine any determinant set out under the form of a square array of terms. This square may be considered as divisible into lines and columns. Now conceive any one line and any one column to be struck out, we get in this way a square, one term less in breadth and depth than the original square; and by varying in every possible manner the selection of the line and column excluded, we obtain, supposing the original square to consist of n lines and n columns, n^2 such minor squares, each of which will represent what I term a First Minor Determinant relative to the principal or complete determinant. Now suppose two lines and two columns struck out from the original square, we shall obtain a system of $\left\{\frac{n(n-1)}{2}\right\}^2$ squares, each two terms lower than the principal square, and representing a determinant of one lower order than those above referred to. These constitute what I term a system of Second Minor Determinants; and so in general we can form a system of rth minor determinants by the exclusion of r lines and r columns, and such system *in general* will contain

$$\left\{\frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}\right\}^2$$

distinct determinants.

I say "*in general*"; because if the principal determinant be totally or partially symmetrical in respect to either or each of its diagonals, the number of distinct determinants appertaining to each system of minors will undergo a material diminution, which is easily calculable.

Now I have established the following law:—

The whole of a system of rth minors being zero, implies only $(r+1)^2$ equations, that is, by making $(r+1)^2$ of these minors zero, all will become zero; and this is true, no matter what may be the dimensions or form of the complete determinant. But furthermore, if the complete determinant be formed from a quadratic function, so as to be symmetrical about one of its diagonals, then $\frac{1}{2}(r+1)(r+2)$ only of the rth minors being zero, will serve to imply that all these minors are zero. Of course, in applying these theorems, care must be taken that the $(r+1)^2$ or $\frac{1}{2}(r+1)(r+2)$ selected equations must be mutually non-implicative, and shall constitute independent conditions.

In the application I am about to make of these principles, we shall have only to deal with a system of *first* minors and of a *symmetrical* determinant. If three of these properly selected be zero, from the foregoing it appears that all must be zero.

Now let U and V be characteristics of two conics, that is, let each be a function of only three letters, it may be shown (see my paper* in the *Cambridge and Dublin Mathematical Journal* for November, 1850) that the different species of contacts between these two conics will correspond to peculiar properties of the compound characteristic $U+\mu V$.

If the determinant of this function have two equal roots, the conics simply touch; if it have three equal roots, the conics have a single contact of a higher order, that is, the same curvature; if its six first minors become zero simultaneously for the same value of μ, the conics have a double contact. If the same value of μ, which makes all these first minors zero, be at the same time not merely a double root (as of analytical necessity it always must be) but a treble root of

$$\square(U+\mu V)=0,$$

then the conics have a single contact of the highest possible order short of absolute coincidence, that is, they meet in four consecutive points.

The parallelism between this theory and that of two quadratic functions P, Q, and one linear function L† of four letters, say x, y, z, t, is exact‡. For let $P+Lu+\mu Q$ be now taken as our compound characteristic (a function, it will be observed, of five letters, x, y, z, t, u); if its determinant have two equal roots, L has two consecutive points in common with the intersection of P and Q, that is, passes through a tangent to that intersection; if it have three equal roots, L has three consecutive points in common with the said intersection, that is, is an osculating plane thereto; if its fifteen first minors admit of all being made simultaneously zero, L has a double contact with the intersection of P and Q, that is, it is a tangent plane to some one of the four cones of the second order containing this intersection;

[* p. 119 above.]

† Observe that $P=0$, $Q=0$, $L=0$ now express the equations to two conoids and a plane respectively.

‡ This parallelism may be easily shown analytically to imply, and be implied, in the geometrical fact, that the contact of the plane L with the intersection of the two surfaces P and Q, is of exactly the same kind as the contact (which must exist) between the two conics which are the intersections of P and Q respectively with the plane L.

if the same linear function of μ which enters into all these first minors be contained cubically in the complete determinant, then the plane L passes through four consecutive points of the intersection of P and Q, and the points where it meets the curve will be points of contrary plane flexure; and, as it seems to me, at such points the tangential direction of the curve must point to the summit of one or other of the four cones above alluded to*. In assigning the conditions for L being a double tangent plane to the intersection of P and Q, we may take any three independent minors at pleasure equal to zero. One of these may be selected so as to be clear of the coefficients of L; in fact, the determinant of $P+\mu Q$ will be a first minor of $P+\mu Q+Lu$; μ may thus be determined by a biquadratic equation; and then, by properly selecting the two other minors, we may obtain two equations in which only the first powers of the coefficients of x, y, z, t in L appear, and may consequently obtain L under the form of

$$(ae+\alpha)x+(be+\beta)y+(ce+\gamma)z+(de+\delta)t,$$

where a, α; b, β; c, γ; d, δ will be known functions of any one of the four values of μ. The point of contact being given will then serve to determine e, and we shall thus have the equation to each of the four double tangent planes at any given point fully determined.

In the foregoing discussions I have freely employed the word *characteristic* without previously defining its meaning, trusting to that being apparent from the mode of its use. It is a term of exceeding value for its significance and brevity. The characteristic of a geometrical figure† is the function which, equated to zero, constitutes the equation to such figure. Plücker, I think, somewhere calls it the line or surface function, as the case may be. Geometry, analytically considered, resolves itself into a system of rules for the construction and interpretation of characteristics. One more remark, and I have done. A very comprehensive theorem has been given at the commencement of this commentary, for interpreting the effect of a complete determinant of a linear function of two quadratic functions $(U+\mu V)$, having

* If this be so, then we have the following geometrical theorem:—"*The summit of one of the four cones of the second degree which contain the intersections of two surfaces of the second order drawn in any manner respectively through two given conics lying in the same plane, and having with one another a contact of the third degree, will always be found in the same right line, namely in the tangent line to the two given conics at the point of contact.*"

† More generally, the *characteristic* of any fact or existence is the function which, equated to zero, expresses the condition of the actuality of such fact or existence.

Perhaps the most important pervading principle of modern analysis, but which has never hitherto been articulately expressed, is *that*, according to which we infer, that when one fact of whatever kind is implied in another, the characteristic of the first must contain as a factor the characteristic of the second; and that when two facts are mutually involved, their characteristics will be powers of the same integral function.

The doctrine of characteristics, applied to dependent *systems* of facts, admits of a wide development, logical and analytical.

one or more pairs of equal factors $(e+\epsilon\mu)$. But here a far wider theory presents itself, of which the aim should be to determine the effect and meaning of this determinant, having any amount and distribution of multiplicity whatsoever among its roots. Nor must our investigations end at that point; but we must be able to determine the meaning and effect of common factors, one or more entering into the successive systems of *minor* determinants derived from the complete determinant of $U+\mu V$.

Nor are we necessarily confined to two, but may take several quadratic functions simultaneously into account.

Aspiring to these wide generalizations, the analysis of quadratic functions soars to a pitch from whence it may look proudly down on the feeble and vain attempts of geometry proper to rise to its level or to emulate it in its flights.

The law which I have stated for assigning the number of independent, or to speak more accurately, non-coevanescent determinants belonging to a given system of minors, I call the Homaloidal law, because it is a corollary to a proposition which represents analytically the indefinite extension of a property common to lines and surfaces to all loci (whether in ordinary or transcendental space) of the first order, all of which loci may, by an abstraction derived from the idea of levelness common to straight lines and planes, be called Homaloids. The property in question is, that neither two straight lines nor two planes can have a common segment; in other words, if n independent relations of rectilinearity or of coplanarity, as the case may be, exist between triadic groups of a series of $n+2$, or between tetradic groups of a series of $n+3$ points respectively, then every triad or tetrad of the series, according to the respective suppositions made, will be in rectilinear or in plane order. So, too, if n independent relations of *coincidence* exist between the duads formed out of $n+1$ points, every duad will constitute a coincidence.

This homaloidal law has not been stated in the above commentary in its form of greatest generality. For this purpose we must commence, not with a square, but with an oblong arrangement of terms consisting, suppose, of m lines and n columns. This will not in itself represent a determinant, but is, as it were, a Matrix out of which we may form various systems of determinants by fixing upon a number p, and selecting at will p lines and p columns, the squares corresponding to which may be termed determinants of the pth order. We have, then, the following proposition. The number of uncoevanescent determinants constituting a system of the pth order derived from a given matrix, n terms broad and m terms deep, may equal, but can never exceed the number

$$(n-p+1)(m-p+1).$$

Remark on PASCAL'S *and* BRIANCHON'S *Theorems.*

I omitted to state, in the September Number of the *Journal* *, that the demonstration there given by me for Pascal's, applied equally to Brianchon's theorem. This remark is of the more importance, because the fault of the analytical demonstrations hitherto given of these theorems has been, that they make Brianchon's consequence of Pascal's, instead of causing the two to flow simultaneously from the application of the same principles. No demonstration can be held valid in *method*, or as touching the essence of the subject-matter, in which the indifference of the duadic law is departed from. Until these recent times, the analytic method of geometry, as given by Descartes, had been suffered to go on halting as it were on one foot. To Plücker was reserved the honour of setting it firmly on its two equal supports by supplying the complementary system of coordinates. This invention, however, had become inevitable, after the profound views promulgated by Steiner, in the introduction to his Geometry, had once taken hold of the minds of mathematicians. To make the demonstration in the article referred to apply, *totidem literis*, to Brianchon's theorem (recourse being had to the correlative system of coordinates), it is only needful to consider U as the characteristic of the tangential envelope of the conic, x, y, z, t, u, v as the characteristics of the six points of the *circumscribed* hexagon, ϕ the characteristic of the point in which the line x, v meets the line z, t; $ay-\alpha u$ will then be shown to characterize the point in which t, x meets v, z; and thus we see that y, u; t, x; v, z, the three pairs of opposite sides of the hexagon, will meet in one and the same point, which is Brianchon's theorem.

[* p. 138 above.]

26.

ON THE SOLUTION OF A SYSTEM OF EQUATIONS IN WHICH THREE HOMOGENEOUS QUADRATIC FUNCTIONS OF THREE UNKNOWN QUANTITIES ARE RESPECTIVELY EQUATED TO NUMERICAL MULTIPLES OF A FOURTH NON-HOMOGENEOUS FUNCTION OF THE SAME.

[*Philosophical Magazine*, XXXVII. (1850), pp. 370—373.]

LET U, V, W be three homogeneous quadratic functions of x, y, z, and let ω be any function of x, y, z of the nth degree, and suppose that there is given for solution the system of equations

$$U = A\omega,$$
$$V = B\omega,$$
$$W = C\omega.$$

Theorem. The above system can be solved by the solution of a cubic equation, and an equation of the nth degree.

For let D be the determinant in respect to x, y, z of

$$fU + gV + hW,$$

then D is a cubic function of f, g, h. Now make

$$D = 0, \quad Af + Bg + Ch = 0;$$

the ratios of $f:g:h$ which satisfy the last two equations can be determined by the solution of a cubic equation, and there will accordingly be three systems of f, g, h which satisfy the same, as

$$f_1,\ g_1,\ h_1,$$
$$f_2,\ g_2,\ h_2,$$
$$f_3,\ g_3,\ h_3.$$

Now $D = 0$ implies that $fU + gV + hW$ breaks up into two linear factors; accordingly we shall find

$$(l_1x + m_1y + n_1z)(\lambda_1x + \mu_1y + \nu_1z) = 0,$$
$$(l_2x + m_2y + n_2z)(\lambda_2x + \mu_2y + \nu_2z) = 0,$$
$$(l_3x + m_3y + n_3z)(\lambda_3x + \mu_3y + \nu_3z) = 0,$$

in which the several sets of l, m, n; λ, μ, ν can be expressed without difficulty in terms of the several values of $\sqrt{f}$, $\sqrt{g}$, $\sqrt{h}$.

Let the above equations be written under the form

$$PP' = 0,$$
$$QQ' = 0,$$
$$RR' = 0.$$

Since the given equations are perfectly general, it is readily seen that the equations

$$(P = 0,\ P' = 0), \quad (Q = 0,\ Q' = 0), \quad (R = 0,\ R' = 0),$$

will severally represent pairs of opposite sides of a quadrangle expressed by general coordinates x, y, z; so that one of the two functions R, R' will be a linear function of P and Q and also of P' and Q', and the other will be a linear function of P and Q' and also of P' and Q*.

In order to solve the equations, we need only consider two such pairs as $PP' = 0$, $QQ' = 0$; we then make

$$P = 0, \quad Q = 0,$$
or
$$P = 0, \quad Q' = 0,$$
or
$$P' = 0, \quad Q = 0,$$
or
$$P' = 0, \quad Q' = 0.$$

Any one of these four systems will give the ratios of $x:y:z$; and then, by substitution in any one of the given equations, we obtain the values of x, y, z by the solution of an ordinary equation of the nth degree. The number of systems x, y, z is therefore always $4n$.

The equations connected with the solution of Malfatti's celebrated problem, "In a given triangle to inscribe three circles such that each circle touches the remaining two circles and also two sides of the triangle," given by Mr Cayley in the November Number for 1849 of the *Cambridge and Dublin Mathematical Journal*, to wit,

$$by^2 + cz^2 + 2fyz = \theta^2 a\,(bc - f^2) = A,$$
$$cz^2 + ax^2 + 2gzx = \theta^2 b\,(ca - g^2) = B,$$
$$ax^2 + by^2 + 2hxy = \theta^2 c\,(ab - h^2) = C,$$

come under the general form which has just been solved. It so *happens*, however, that in this particular case

$$\left.\begin{matrix} f_1, & g_1, & h_1 \\ f_2, & g_2, & h_2 \\ f_3, & g_3, & h_3 \end{matrix}\right\},$$

* Were it not for this being the case, the number of solutions would be n times the number of ways of obtaining duads out of three sets of two things, excluding the duads forming the sets, that is, the number of solutions would be $12n$ in place of $4n$, the true number.

become respectively

$$\left.\begin{matrix} 0, & \frac{1}{B}, & -\frac{1}{C} \\ -\frac{1}{B}, & 0, & \frac{1}{C} \\ -\frac{1}{C}, & \frac{1}{B}, & 0 \end{matrix}\right\},$$

and the cubic equation is resolved without extraction of roots.

It follows from my theorem that the eight intersections of three concentric surfaces of the second order can be found by the solution of one cubic and one quadratic equation; and in general, if we have ϕ, ψ, θ any three quadratic functions of x, y, z, and $\phi = 0$, $\psi = 0$, $\theta = 0$ be the system of equations to be solved, provided that we can by linear transformations express ϕ, ψ, θ under the form of

$$U - aw,$$
$$V - bw,$$
$$W - cw,$$

U, V, W being homogeneous functions, and w a non-homogeneous function of three new variables, x', y', z', we can find the eight points of intersection of the three surfaces, of which U, V, W are the characteristics, by the solution of one cubic and one quadratic. But (as I am indebted to Mr Cayley for remarking to me) that this may be possible, implies the coincidence of the vertices of one cone of each of the systems of four cones in which the intersections of the three surfaces taken two and two are contained.

I may perhaps enter further hereafter into the discussion of this elegant little theory. At present I shall only remark, that a somewhat analogous mode of solution is applicable to two equations,

$$U = aP^2,$$
$$V = bP^2,$$

in which U, V are homogeneous quadratic functions, and P some non-homogeneous function of x, y.

We have only to make the determinant of $fU + gV$ equal to zero, and we shall obtain two systems of values of f, g, wherefrom we derive

$$l_1 x + m_1 y = \pm \sqrt{(af_1 + bg_1)}\, P,$$
$$l_2 x + m_2 y = \pm \sqrt{(af_2 + bg_2)}\, P,$$

from which x and y may be determined.

27.

ON A PORISMATIC PROPERTY OF TWO CONICS HAVING WITH ONE ANOTHER A CONTACT OF THE THIRD ORDER.

[*Philosophical Magazine*, XXXVII. (1850), pp. 438, 439.]

If two conics have with one another a contact of the third order, that is, if they intersect in four consecutive points, it will easily be seen that their *characteristics* referred to coordinate axes in the plane containing them must be of the relative forms $x^2 + yz$, $k(y^2 + x^2 + yz)$ respectively, y characterizing their common tangent at the point of contact*.

Hence if we take planes of reference in space, and call t the characteristic of the plane of the conics, the equations to any two conoids drawn through them respectively will be of the relative forms

$$U = x^2 + yz + tu = 0,$$
$$V = y^2 + x^2 + yz + tv = 0.$$

Using W to denote $V - U$, and (W) to denote what W becomes when ey is substituted for t, we see that W and (W) are of the respective forms $y^2 + tw$ and $y\theta$; showing that the former is the characteristic of a cone which will be cut by any plane $t - ey$ drawn through the line (t, y) in a pair of right lines; or, in other words, that one of the cones containing the intersection of the two variable conoids (V and U) will have its vertex in the *invariable line* which is the common tangent to the two fixed conics: this proves the theorem stated by me hypothetically in a foot-note in one of my papers in the last number of the *Magazine*†. The steps of the geometrical proof there hinted at are as follows.

* These relative or conjugate forms are taken from a table which I shall publish in a future number of this *Magazine*, exhibiting the conjugate characteristics in their simplest forms, correspondent to all the various species of contacts possible between lines and surfaces of the second degree. This table is as important to the geometer as the fundamental trigonometrical formulæ to the analyst, or the multiplication table to the arithmetician; and it is surprising that no one has hitherto thought of constructing such.

[† p. 149 above.]

The four consecutive points in which the two conics intersect will be consecutive points in the curve of intersection of the two variable conoids. This curve lies in each of four cones of the second degree. Every double tangent plane to it passes through the vertex of one amongst these. The plane containing four, that is, two (consecutive) pairs of consecutive points, is a double tangent plane, and will therefore pass through a vertex; but four consecutive points of a curve of the fourth order described upon a cone, and lying in one tangent plane thereto, can only be *conceived* generally as disposed in the form of an *ʃ*, of which the belly part will point to the vertex; or, in other words, at any point where two consecutive osculating planes coincide so that the *spherical* curvature vanishes, the linear curvature will also vanish, that is, there will be a point of inflexion at which, of course, the tangent line must pass through the vertex of the cone. This is the assumption felt to be true, but stated by me hypothetically in the paper referred to, because a ready demonstration did not at the moment occur to me. The legitimacy of this inference is now vindicated by the above analytical demonstration.

The methods of general and correlative coordinates and of determinants combined possess a perfectly irresistible force (to which I can only compare that of the steam-hammer in the physical world) for bringing under the grasp of intuitive perception the most complicated and refractory forms of geometrical truth.

28.

ON THE ROTATION OF A RIGID BODY ABOUT A FIXED POINT.

[*Philosophical Magazine*, XXXVII. (1850), pp. 440—444.]

In the *Cambridge and Dublin Mathematical Journal* for March 1848, an article by Professor Stokes, of the University of Cambridge, is ushered in with the words following:—

"The most general *instantaneous** motion of a rigid body moveable in all directions about a fixed point consists in a motion of rotation about an axis passing through that point. This elementary proposition is sometimes assumed as self-evident, and sometimes deduced as the result of an analytical process. It ought hardly *perhaps* to be assumed, but it does not seem desirable to refer to a long algebraical process for the demonstration of a theorem so simple. Yet I am not aware of a geometrical proof anywhere published which might be referred to."

The learned and ingenious professor is indubitably right, and might have trusted himself to assert less hesitatingly the necessity of demonstrating this proposition, which possesses none of the characters of a self-evident truth; but it is to be regretted that he should have stated it in such a form as naturally to lead the incautious reader to mistake the nature and grounds of its existence, which consist in this fact—that any kind of displacement of a body moveable about a fixed axis, whether instantaneous and infinitesimal, or secular and finite, is capable of being effected by a single rotation about a single axis.

The annexed simple proof of this capital law has the advantage of affording a rule for compounding into one any two (and therefore any number of) rotations given in direction, magnitude and *order of succession*.

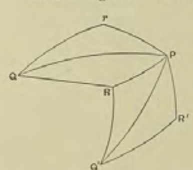
* The italics do not exist in the original.

It will somewhat conduce to simplicity if we fix our attention upon a spherical surface rigidly connected with the rotating body, and having its centre at the fixed point thereof. When the positions of two points in this are given, the position of the body is completely determined.

Now evidently two points A, B may be brought respectively to $A'B'$ (if $AB = A'B'$) by two rotations; the first taking place about a pole situated anywhere in the great circle bisecting AA' at right angles, the second about A', the position into which it is brought by the first rotation. This view leads us to consider the effect of two rotations taking place successively about two axes fixed in *the rotating body*. Or again, we may make the plane $A'B'$ revolve into the position AB round a pole taken at the node in which the two planes intersect, and then the points A, B swing into their new positions A', B' by means of a rotation about the pole of the great circle, of which $A'B'$ forms a part. This mode of effecting the displacement naturally suggests the consideration of the effect of rotations taking place successively about two axes fixed in *space*.

First, then, let us study the effect of the combination of a rotation (α) having P for its pole, followed by another (β), of which Q is the pole, P and Q being points in the surface of the revolving sphere.

In drawing the annexed figure, I have supposed that the two rotations are of the same kind, each tending, when a spectator is standing with his head to the respective poles and his feet to the centre, to make a point at his right-hand pass *in front of his face* towards his left-hand. Let now PQ revolve through $\frac{\alpha}{2}$ positively into the position of PR, and through $\frac{\beta}{2}$ negatively into that of QR.



Then I say that the two impressed rotations α and β about P and Q will be equivalent to a single rotation about R, equal to twice the acute angle between QR, RP.

Let the first rotation about P bring Q to Q' and R to R'; it is clear that QPR, $Q'PR$, $Q'PR'$ are all equal triangles. Therefore $R'Q'R = 2PQR = \beta$. Consequently the positive rotation β about Q' (the new position of Q) will carry R' back again to R, its original position. Hence the actual motion which results from the successive rotations combined being consistent with R remaining at rest, must be equivalent to a single rotation about R.

To find its magnitude, let the second rotation carry P to P'*; then the angular displacement PRP' (which is the required rotation of the whole

* The reader is requested to fill in the point P' and join $P'R$.

body) is equal to twice the acute angle between $Q'R$, RP, which is the same as that between QR, RP, as was to be shown. Thus we see that the semi-rotations about three poles (considered as the angular points of a spherical triangle), which, taken in order, would bring the sphere back to its first undisturbed position, are equal to the included angles at such poles respectively.

If in our figure the order of the rotations had been reversed, PQr, QPr would have been taken respectively equal to PQR, QPR, but on the opposite side of PQ, and r would have been the resultant pole, the resultant rotation remaining in amount the same as before.

If either of the rotations had been negative, the resultant pole would be found in QR produced, namely, at the intersection of rQ or rP with PQ.

Calling the resultant rotation γ, we have always

$$\sin\frac{\alpha}{2} : \sin\frac{\beta}{2} : \sin\frac{\gamma}{2} :: \sin QR : \sin RP : \sin PQ.$$

When the component rotations are infinitesimal in amount, R and r will come together in QP; the order of succession of the rotations will be indifferent, and we shall have

$$\alpha : \beta : \gamma :: \sin\frac{\alpha}{2} : \sin\frac{\beta}{2} : \sin\frac{\gamma}{2} :: \sin QR : \sin RP : \sin PQ,$$

which gives the rule for the parallelogrammatic composition of two simultaneously impressed rotations*.

If, next, we consider the effect of rotations about two poles, P and Q, fixed in space (supposing, as above, that they take place first about P and then about Q), we must take QPr equal to half the *contrary* of the rotation about P, and PQr to half the *direct* rotation about Q (the angle being now taken positive which was on the first supposition negative, and *vice versâ*); so that, retaining the original figure, the first rotation will bring r to R, and the second carry R back to r; showing that r is the resultant pole, and that† $P'rP$, the resultant rotation, will be double the acute angle between Qr, rP, as in the former case.

To popular apprehension the important doctrine of uniaxial rotation may be made intelligible by the following mode of statement. Take a pocket-globe, open the case and roll about the sphere within it in any manner whatever; then closing the case, there will unavoidably remain two points on the terrestrial surface touching the same two points on the celestial surface as they were in apposition with before the sphere was so turned about in its case.

* Compare Mr Airy's *Tracts*, Art. "On Precession and Nutation."

† P' is not expressed in the figure given.

It is right to bear in mind that the whole of this doctrine is comprised in, and convertible with, the following easy geometrical proposition relative to arcs of great circles on any spherical surface, including the plane as an extreme case.

"The arcs joining the extremities (each with each in *either* order) of two other equal arcs, subtend equal angles at either of the points of intersection of two great circles bisecting at right angles the first-named connecting arcs*."

The spherico-triangular mode of compounding rotations given in the above simple disquisition may easily be made the parent of a whole brood of geometrical consequences, which, however, I must leave to the ingenuity and care of those who have a turn for this kind of invention.

But I ought not to omit to invite attention to a remarkable form, which may be imparted to the theorems above stated for the composition of finite rotations, or rather to a theorem which may be derived from them by an obvious process of inference.

Let $P, Q, R \ldots X, Z$ be any number of points on a sphere capable of moving about its centre, joined together by arcs of great circles so as to form a spherical polygon. Imagine any number of rotations to take place about these points in succession as poles. It matters not which is considered the first pole of rotation, but the *order* of the circulation must be supposed given, as, for instance, $PQR \ldots XZ$, or $QR \ldots XZP$, or $R \ldots XZPQ$, &c. This will be one order; the reverse order would be $PZX \ldots RQ$, or $QPZX \ldots R$, &c.

I shall suppose the circulation to be of the kind first above written. Now we may make two hypotheses:—

1. That the poles are fixed in space.
2. That they are fixed in the rotating body.

In the first case, let the rotations about the given poles $P, Q, R, S \ldots X, Z$ be double the amounts which would serve to transport PQ to QR, QR to $RS \ldots XZ$ to ZP respectively.

In the second case, let the rotations be double the amounts which would carry PZ to $ZX \ldots SR$ to RQ, RQ to QP respectively. Then, on either supposition, the sum of the combined rotations is zero; or, to use a more convenient and suggestive form of expression, if the poles of rotation form a closed spherical polygon whose angles are respectively equal to the semi-rotations about the poles, the resultant rotation is zero.

* This proposition will be seen to be immediately demonstrable, by the comparison of equal triangles, when viewed as the converse of this other. "The arcs (or right lines) joining the correspondent extremities of the bases of two similar isosceles spherical (or plane) triangles having a common vertex, are equal to each other."

This proposition is immediately derivable from the fundamental one relative to three poles, given above, by dividing the polygon into triangles by arcs, joining any one of the poles with all the rest, or (as pointed out to me by my eminent friend Prof. W. Thomson) it becomes apparent as a particular case of a more general proposition, on representing the motion about the successive axes as effected by two equal pyramids having a common vertex at the centre of motion, of which the one is fixed in space, and the other is fixed in the revolving body and rolls over the first, so that the corresponding equal faces are successively brought into coincident apposition.

P.S. To find the pole of rotation whereby PQ may be brought into the position $P'Q'$, we may use the following simple construction.

Measure off from O the node of the great circles (or right lines) containing PQ and $P'Q'$, two distances in the proper direction upon each (four distinct assumptions may be made), say OR and OS equal to one another and to the difference between OP and OP', then the pole of rotation required, say E, is the centre of the circle described about ROS, and the amount of rotation is the angle subtended by OR or OS at E. The writer of this paper suggests that *axis of displacement* would be a convenient term for designating the line whereby any finite change in the position of a body moveable about a fixed centre may be brought about; a geometrical theory of rotation leading to the investigation of a very curious species of correlation, now opens upon the view, the general object of which may be stated as follows:

"Given upon a sphere or plane any curve considered as the locus of successive poles of instantaneous rotation, and the ratio of the rotation about each pole to its distance from the one that follows*, to construct the curve of the poles of displacement, and to determine the amount of rotation corresponding to each such pole."

The discussion of this question offers a fine field for the exercise of geometrical taste and skill.

* Which by analogy may be termed the "density of rotation."

29.

ON THE INTERSECTIONS OF TWO CONICS.

[*Cambridge and Dublin Mathematical Journal*, VI. (1851), pp. 18—20.]

LET the two conics be written

$$U = ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy = 0,$$
$$V = \alpha x^2 + \beta y^2 + \gamma z^2 + 2\alpha' yz + 2\beta' zx + 2\gamma' xy = 0,$$

and make

$$U + \lambda V = Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy.$$

In my paper in the last number of the *Journal**, I showed that the case of intersection of the two conics in two points was distinguishable from all other cases by the equation $\square(U + \lambda V) = 0$ having two imaginary roots. When all the roots are real, the curves either intersect in four points or not at all.

On the former supposition,

$$-C'^2 + AB, \quad -A'^2 + BC, \quad -B'^2 + CA,$$

which are quadratic functions of λ, will be negative for all three values of λ. On the contrary supposition, one value of λ will make all these three quantities negative, but the other two values with each make them all three positive.

Hence we obtain a symmetrical criterion (which I strangely omitted to state in my former paper) by forming the quantity

$$A'^2 + B'^2 + C'^2 - AB - AC - BC.$$

A cubic equation

$$Ly^3 + My^2 + Ny + P = 0$$

may be then constructed, of which the three values of the above function corresponding to three values of λ will be the roots.

The condition for *real* intersection is that L, M, N, P should be all of the same sign. The conics being supposed real, L and P are necessarily in both cases of the same sign. The condition is therefore satisfied if either $L, M,$

[* p. 119 above.]

N, or M, N, P be of the same sign, and is consequently equivalent to the condition that $\frac{M}{L}$ and $\frac{N}{L}$ shall be both positive, or $\frac{N}{P}$ and $\frac{M}{P}$ both positive. It does not appear to be possible in the nature of the question to find a criterion for distinguishing between the two cases, dependent on the sign of one single function of the coefficients.

The case of double contact, abstraction being made of binary intersection, is a sort of intermediary state between intersection in four points and non-intersection; and accordingly, as shown in my former paper for this case, the two equal values of λ will make the three quantities

$$AB - C'^2, \quad BC - A'^2, \quad CA - B'^2$$

all real; so that two of the values of y corresponding to the equal values of λ are zero, and the criterion becomes nugatory as it ought to do.

Again, when the two conics do not intersect, I distinguished two cases according as they lie each without, or one within the other, that is, according as they have four common tangents or none.

But, as Mr Cayley has well remarked to me, a similar distinction exists when the conics intersect in four points; in that case also they may have four common tangents or not any: when they intersect in two points they have necessarily two and only two common tangents. There is no difficulty in separating these four cases.

Let the conics be written

$$(U) = \xi^2 + \eta^2 - \zeta^2,$$
$$(V) = A\xi^2 + B\eta^2 - C\zeta^2,$$

(U) and (V) being what U and V become when the coordinates are changed from x, y, z to ξ, η, ζ.

A, B, C are the three values of λ in the equation

$$\square(V - \lambda U) = 0.$$

If the curves intersect $A - C$, $B - C$ must have different signs, that is, C must be an intermediary quantity between A and B.

Again, the tangential equations to the conics expressed by the correlative system of coordinates will be

$$\xi_1^2 + \eta_1^2 - \zeta_1^2 = 0,$$
$$\frac{\xi_1^2}{A} + \frac{\eta_1^2}{B} - \frac{\zeta_1^2}{C} = 0;$$

and that these may have four real systems of roots,

$$\frac{1}{A} - \frac{1}{C}, \quad \frac{1}{C} - \frac{1}{B}$$

must have the same sign; and consequently, as $A - C$ and $C - B$ are supposed to have the same sign, A and B, and therefore all three A, B, C, have the same sign. We have therefore the following rule:

Let the equation in λ, namely, $\square(U+\lambda V)=0$, be called $\theta=0$, and the equation in y, above given, $\omega=0$. By an equation being congruent or incongruent, understand that its roots have all the same sign or not all the same sign.

Then ω congruent, θ congruent, implies that the intersections and common tangents are both real; ω congruent, θ incongruent, implies that the intersections are real, but the common tangents imaginary; ω incongruent, θ congruent, implies that the intersections and common tangents are both imaginary; ω incongruent, θ incongruent, implies that the intersections are imaginary, but the common tangents real.

In like manner, as the cases of contact of lines are limiting cases to those which relate to the relative configurations of their points of intersection, so the cases of contact of surfaces are limiting cases in which the characters which usually separate the different forms of their curve of intersection exist blended and indistinguishable. The first step therefore to the study of the particular species of the curve of the fourth degree*, in which two surfaces of the second degree intersect, is to obtain the analytical and geometrical characters of their various species of contact. Accordingly I have made an enumeration of these different species, no less than 12 in number, many of them highly curious and I believe unsuspected, which the reader may consult in the *Philosophical Magazine* for February, 1851†.

By the aid of these landmarks, I have little doubt, should time and leisure permit, of mapping out a natural arrangement of the principal distinctions of form between that class at least of lines in space of the fourth order which admit of being considered the complete intersection of two surfaces.

* I have found that the 16 points of spherical flexure in this curve are the four sets of four points in which it meets the four faces of the pyramid whose summits are the vertices of the four cones of the second degree in which the curve is completely contained, which 16 points reduce to 4 when the two surfaces have an ordinary contact, and to 1 when they have a cuspidal contact: of course in the case of contact the pyramid above described in a manner folds up and vanishes, as there are no longer 4 distinct vertices. I have found also that when the factors of $\square(U+\lambda V)$, (U and V being the characteristics of the two surfaces) are all unreal, the points of flexure are all unreal. When two factors are real and two imaginary, two of the faces of the pyramid (namely, its two real faces) will each contain one (and only one) pair of real points of flexure, and the other two planes none; and lastly, when the factors of $\square(U+\lambda V)$ are all real, then either all the points of flexure are imaginary, or else all the eight contained in a certain two of the pyramidal faces are real: and these two cases admit of being distinguished by a method analogous in its general features to that whereby I have shown in the text above how to distinguish between the cases of 4 real and 4 imaginary points of intersection of two conics. Where the two surfaces have an ordinary contact, the curve of intersection, it is well known, has a double point; and where the surfaces have a higher contact, the curve has a cusp. Thus in the fact of the 16 flexures reducing to 4 and to 1 in these respective cases, we see a beautiful analogy to what takes place with the 9 flexures of a plane curve of the third degree, which contract to 3 and 1, according as the curve has a double point or a cusp.

[† p. 219 below.]

30.

ON CERTAIN GENERAL PROPERTIES OF HOMOGENEOUS FUNCTIONS.

[*Cambridge and Dublin Mathematical Journal*, VI. (1851), pp. 1—17.]

LET χ denote the operation

$$x_1\frac{d}{da_1}+x_2\frac{d}{da_2}+\ldots+x_n\frac{d}{da_n},$$

and A the operation

$$a_1\frac{d}{dx_1}+a_2\frac{d}{dx_2}+\ldots+a_n\frac{d}{dx_n}:$$

and now suppose that ω, a homogeneous function of ι dimensions of $a_1, a_2\ldots a_n$, and not of any of the quantities $x_1, x_2\ldots x_n$, is subjected to the successive operations indicated by $A^s\chi^r$.

We have

$$A^s\chi^r\omega=A^{s-1}A\chi^r\omega,$$

$$A\chi^r\omega=\left(a_1\frac{d}{dx_1}+a_2\frac{d}{dx_2}+\ldots+a_n\frac{d}{dx_n}\right)\left(x_1\frac{d}{da_1}+x_2\frac{d}{da_2}+\ldots+x_n\frac{d}{da_n}\right)^r\omega$$

$$=r\left(a_1\frac{d}{da_1}+a_2\frac{d}{da_2}+\ldots+a_n\frac{d}{da_n}\right)\chi^{r-1}\omega$$

$$=r(\iota-r+1)\chi^{r-1}\omega,$$

for $\chi^{r-1}\omega$ is of $(r-1)$ dimensions, lower than ω (which is of ι dimensions) in $a_1, a_2\ldots a_n$.

Hence

$$A^s\chi^r\omega=r(\iota-r+1)A^{s-1}\chi^{r-1}\omega$$

$$=\&\text{c.}=\{r(r-1)\ldots(r-s+1)\}$$

$$\{(\iota-r+1)(\iota-r+2)\ldots(\iota-r+s)\}\chi^{r-s}\omega. \qquad (1)$$

Now in the expression

$$\chi^r \omega (a_1, a_2 \dots a_n),$$

suppose that we write

$$x_1 = u_1 + a_1 \epsilon,$$

$$x_2 = u_2 + a_2 \epsilon,$$

$$\dots\dots\dots\dots$$

$$x_n = u_n + a_n \epsilon,$$

we have, by Taylor's theorem,

$$\chi^r \omega = U^r \omega + A U^r \omega \epsilon + A^2 U^r \omega \frac{\epsilon^2}{1 . 2} + \dots + A^r U^r \omega \frac{\epsilon^r}{1 . 2 . 3 \dots r},$$

where $U^r \omega$ denotes what $\chi^r \omega$ becomes, on substituting u's for x's, and A now represents

$$u_1 \frac{d}{da_1} + u_2 \frac{d}{da_2} + \dots + u_n \frac{d}{da_n}.$$

This expansion stops spontaneously at the $(r+1)$th term, because $\chi^r \omega$ is only of r dimensions in $x_1, x_2 \dots x_n$.

Applying now theorem (1), we obtain

$$\chi^r \omega = U^r \omega + r (\iota - r + 1) U^{r-1} \omega \epsilon + \tfrac{1}{2} r (r-1) \{(\iota - r + 1)(\iota - r + 2)\} U^{r-2} \omega \epsilon^2$$
$$+ \dots + \{(\iota - r + 1)(\iota - r + 2) \dots \iota\} \omega \epsilon^r. \qquad (2)$$

In using this theorem in the course of the ensuing pages, it will be found convenient to assign to ϵ a specific value, and I shall suppose it equal to $\frac{x_n}{a_n}$; this gives

$$u_1 = x_1 - \frac{a_1}{a_n} x_n,$$

$$u_2 = x_2 - \frac{a_2}{a_n} x_n,$$

$$\dots\dots\dots\dots$$

$$u_n = x_n - \frac{a_n}{a_n} x_n$$
$$= 0.$$

And inasmuch as the U symbol now contains $a_1, a_2, \dots a_n$, so that UU^r no longer equals U^{r+1}, I shall write U_r for U^r. Theorem (2) will thus assume the form

$$\chi^r \omega = U_r \omega + r (\iota - r + 1) U_{r-1} \omega \frac{x_n}{a_n} + \tfrac{1}{2} r (r-1)(\iota - r + 1)(\iota - r + 2) U_{r-2} \omega \left(\frac{x_n}{a_n}\right)^2$$
$$+ \dots + \{(\iota - r + 1) \dots \iota\} \omega \left(\frac{x_n}{a_n}\right)^r, \qquad (3)$$

where U_r for all values of r denotes what

$$\left(x_1 \frac{d}{da_1} + x_2 \frac{d}{da_2} + \dots + x_{n-1} \frac{d}{da_{n-1}}\right)^r \omega$$

becomes, on substituting $u_1, u_2, \dots u_{n-1}$ for $x_1, x_2, \dots x_{n-1}$, *after* the processes of derivation have been completed: this it is essential to observe, because $u_1, u_2, \dots u_{n-1}$ now involve $a_1, a_2, \dots a_{n-1}, a_n$. The term $x_n \frac{d}{da_n}$ is omitted from the symbol of linear derivation, because in the substitutions x_n will be replaced by zero.

As an example of this last theorem, take

$$\omega = a^3 + b^3 + c^3 + kabc;$$

then

$$\chi \omega = 3a^2 x + 3b^2 y + 3c^2 z + kbcx + kcay + kabz,$$

$$\chi^2 \omega = 6ax^2 + 6by^2 + 6cz^2 + 2kcxy + 2kayz + 2kbzx,$$

$$\chi^3 \omega = 6x^3 + 6y^3 + 6z^3 + 6kxyz.$$

$$U_1 \omega = 3a^2 \left(x - \frac{az}{c}\right) + 3b^2 \left(y - \frac{bz}{c}\right) + kbc \left(x - \frac{az}{c}\right) + kca \left(y - \frac{bz}{c}\right),$$

$$U_2 \omega = 6a \left(x - \frac{az}{c}\right)^2 + 6b \left(y - \frac{bz}{c}\right)^2 + 2kc \left(x - a\frac{z}{c}\right)\left(y - \frac{bz}{c}\right),$$

$$U_3 \omega = 6 \left(x - \frac{az}{c}\right)^3 + 6 \left(y - \frac{bz}{c}\right)^3,$$

and it will be found that the equations given by theorem (3) are satisfied, namely

$$\chi \omega = U \omega + 3 \frac{z}{c} \omega,$$

$$\chi^2 \omega = U_2 \omega + 2 . 2 \frac{z}{c} U \omega + 2 . 3 \frac{z^2}{c^2} \omega,$$

$$\chi^3 \omega = U_3 \omega + 3 \frac{z}{c} U_2 \omega + 3 . 1 . 2 \frac{z^2}{c^2} U \omega + 1 . 2 . 3 \frac{z^3}{c^3} \omega.$$

Probably, as this theorem is of rather a novel character, the annexed sketch of a somewhat different course of demonstration may be not unacceptable to my readers.

We have

$$\chi \omega = \left(x_1 \frac{d}{da_1} + x_2 \frac{d}{da_2} + \dots + x_n \frac{d}{da_n}\right) \omega;$$

and by the well-known law for homogeneous functions,

$$\iota \omega = \left(a_1 \frac{d}{da_1} + a_2 \frac{d}{da_2} + \dots + a_n \frac{d}{da_n}\right) \omega.$$

Hence

$$\left(\chi - \iota\frac{x_n}{a_n}\right)\omega = \left(u_1\frac{d}{da_1} + u_2\frac{d}{da_2} + \ldots + u_{n-1}\frac{d}{da_{n-1}}\right)\omega$$

$$= U\omega.$$

Hence

$$\chi\omega = \left(U + \iota\frac{x_n}{a_n}\right)\omega,$$

$$\chi^2\omega = \left\{U + (\iota - 1)\frac{x_n}{a_n}\right\}\left(U + \iota\frac{x_n}{a_n}\right)\omega,$$

$$\chi^3\omega = \left\{U + (\iota - 2)\frac{x_n}{a_n}\right\}\left\{U + (\iota - 1)\frac{x_n}{a_n}\right\}\left(U + \iota\frac{x_n}{a_n}\right)\omega,$$

&c. = &c.

But in performing the process indicated by the several factors it must be carefully borne in mind that UU_r is not $= U_{r+1}$; this would be the case were it not for the terms $-\frac{a_1}{a_n}x_n$, $-\frac{a_2}{a_n}x_n$, &c., which enter into $u_1, u_2 \ldots u_{n-1}$. But on account of these terms, we have

$$UU_r\omega = \left(u_1\frac{d}{da_1} + u_2\frac{d}{da_2} + \ldots + u_{n-1}\frac{d}{da_{n-1}}\right)\left(u_1\frac{d}{da_1} + u_2\frac{d}{da_2} + \ldots + u_{n-1}\frac{d}{da_{n-1}}\right)^r$$

$$= U_{r+1}\omega - r\frac{x_n}{a_n}\left\{u_1\frac{d}{da_1} + u_2\frac{d}{da_2} + \ldots + u_{n-1}\frac{d}{da_{n-1}}\right\}^{r-1}\omega,$$

for

$$\frac{d}{da_1}u_1 = \frac{d}{da_2}u_2 = \ldots = \frac{d}{da_{n-1}}u_{n-1} = -\frac{x_n}{a_n}.$$

Hence

$$UU_r\omega = U_{r+1}\omega - r\frac{x_n}{a_n}U_r\omega.$$

Let $\frac{x_n}{a_n}$ be called ϵ; we find

$$\chi = U + \iota\epsilon,$$

$$\chi^2 = \{U + (\iota - 1)\epsilon\}(U + \iota\epsilon)$$

$$= UU + (2\iota - 1)\epsilon U + (\iota - 1)\iota\epsilon^2$$

$$= U_2 + 2(\iota - 1)\epsilon U + (\iota - 1)\iota\epsilon^2;$$

$$\chi^3 = \{U + (\iota - 2)\epsilon\}\chi^2$$

$$= UU_2 + 2(\iota - 1)\epsilon UU + (\iota - 1)\iota\epsilon^2 U$$

$$+ (\iota - 2)\epsilon U_2 + 2(\iota - 2)(\iota - 1)\epsilon^2 U + (\iota - 2)(\iota - 1)\iota\epsilon^3$$

$$= U_3 + 3(\iota - 2)\epsilon U_2 + 3(\iota - 2)(\iota - 1)\epsilon^2 U + (\iota - 2)(\iota - 1)\iota\epsilon^3.$$

The same process being continued will lead to results identical with those previously obtained and expressed in theorem (3).

The expansion of χ^r, treated according to this second method, appears to require the solution of the partial equation in differences

$$a_{r+1,\,s+1} = a_{r,\,s+1} + (\iota - 2r)\,a_{r,\,s},$$

$a_{0,\,s}$ being given as unity for $s = 1$ and as zero for all other values of s.

It is probable however that the solution of this equation might be evaded by some artifice peculiar to the particular case to be dealt with. I do not propose to dwell upon this inquiry, which would be foreign to the object of my present research. It may however not be out of place to make the passing remark, that the equations expressing χ^r in terms of powers of U admit easily of being reverted, as indeed may the more general form

$$\chi_r = u_r + \epsilon_r u_{r-1} + \frac{1}{1.2}\epsilon_r\epsilon_{r-1}u_{r-2} + \text{\&c.}$$

which becomes the equation of formula (3), on making

$$\epsilon_r = r(\iota + 1 - r)\frac{x_n}{a_n},\quad \chi_r = \chi^r\omega, \text{ and } u_r = U_r\omega;$$

for let

$$u_r = \epsilon_1\epsilon_2\ldots\epsilon_r v_r,$$

$$\chi_r = \epsilon_1\epsilon_2\ldots\epsilon_r y_r,$$

then

$$y_r = v_r + v_{r-1} + \frac{v_{r-2}}{1.2} + \frac{v_{r-3}}{1.2.3} + \text{\&c.};$$

whence

$$v_r = e^{-\frac{d}{dr}}y_r$$

$$= y_r - y_{r-1} + \frac{y_{r-2}}{1.2} - \frac{y_{r-3}}{1.2.3} + \text{\&c.}:$$

and therefore

$$u_r = \chi_r - \epsilon_r\chi_{r-1} + \tfrac{1}{2}\epsilon_r\epsilon_{r-1}\chi_{r-2} + \text{\&c.}$$

Thus we obtain, from equation (3),

$$U_r\omega = \chi^r\omega - r(\iota - r + 1)\chi^{r-1}\omega\frac{x_n}{a_n} + \text{\&c.}$$

As a first application of theorem (3), I shall proceed to show how Joachimsthal's equation to the surface drawn from a given point $(\alpha, \beta, \gamma, \delta)$ through the intersection of two surfaces $\phi(x, y, z, t) = 0$, $\theta(x, y, z, t) = 0$, may be expressed under the *explicit* form of the equation to a cone.

The equation in question is obtained by eliminating λ between

$$\phi\lambda^m + \chi\phi\lambda^{m-1} + \frac{1}{1.2}\chi^2\phi\lambda^{m-2} + \text{\&c.} = 0,$$

$$\theta\lambda^m + \chi\theta\lambda^{m-1} + \frac{1}{1.2}\chi^2\theta\lambda^{m-2} + \frac{1}{1.2.3}\chi^3\theta\lambda^{m-3} + \text{\&c.} = 0,$$

where

$$\phi = \phi(\alpha, \beta, \gamma, \delta), \quad \theta = \theta(\alpha, \beta, \gamma, \delta), \quad \chi = x\frac{d}{d\alpha} + y\frac{d}{d\beta} + z\frac{d}{d\gamma} + t\frac{d}{d\delta}.$$

By theorem (3), these two equations, on writing $\frac{x_n}{a_n} = \epsilon$, become

$$\phi\lambda^m + \{U\phi + m\phi\epsilon\}\lambda^{m-1} + \{U^2\phi + 2(m-1)U\phi\epsilon + (m-1)m\phi\epsilon^2\}\frac{\lambda^{m-2}}{1.2} + \&c. = 0,$$

$$\theta\lambda^n + \{U\theta + n\theta\epsilon\}\lambda^{n-1} + (U^2\theta + \&c.)\frac{\lambda^{n-2}}{1.2} + \{U^3\theta + 3(n-2)U^2\theta\epsilon + 3(n-2)(n-1)U\theta\epsilon^2 + (n-2)(n-1)n\epsilon^3\}\frac{\lambda^{n-3}}{1.2.3} + \&c.$$

Now on writing $\lambda = \mu - \epsilon$, these equations take the forms

$$\phi\mu^m + U\phi\mu^{m-1} + U^2\phi\frac{\mu^{m-2}}{1.2} + \&c. = 0,$$

$$\theta\mu^n + U\theta\mu^{n-1} + U^2\theta\frac{\mu^{n-2}}{1.2} + \&c. = 0,$$

as is easily seen by substituting back $\lambda + \epsilon$ in place of μ. Consequently ϵ no longer appears in the coefficients of the terms of the equations between which the elimination is to be performed, and the resultant will accordingly come out as a function only of ϕ, $U\phi$, $U^2\phi$, &c., that is, of α, β, γ, δ, and of

$$x - \frac{\alpha}{\delta}t, \quad y - \frac{\beta}{\delta}t, \quad z - \frac{\gamma}{\delta}t,$$

showing that the equation in x, y, z, t, is of the form of that to a cone, as we know *à priori* it ought to be. Precisely a similar method may be applied to the elucidation of the corresponding theorem for a system of rays drawn from a given point through the locus of the intersection of two curves.

Before entering upon some further and more interesting applications of theorem (3), it will be convenient to explain a nomenclature which has been employed by me on another occasion, and which is almost indispensable in inquiries of the nature we are now engaged upon. Homogeneous functions may be characterized by their degree, by the number of letters which enter into them, and lastly, by the lowest number of linear functions of the letters which may be introduced in place of the letters to represent such functions. Any such linear function I designate as an order, and am now able to discriminate between the number of letters and the number of orders which enter into a given function. The latter number, *generally* speaking, is the same as the former; it can never exceed it, but *may* be any number of units less than it.

I need scarcely observe that a pair of points becoming coincident, a conic becoming a pair of lines, a conoid becoming a cone, and so forth, for the higher realms of space, will be expressed by the homogeneous function of the second order which characterizes such loci*, losing one order, that is, having an order less than the number of letters entering therein. Calling such characteristic $\phi(x, y, z \dots t)$, it is well known that the condition of such loss of an order is the vanishing of the determinant

$$\begin{vmatrix} \frac{d^2\phi}{dx^2}, & \frac{d^2\phi}{dxdy} & \cdots & \frac{d^2\phi}{dxdt} \\ \frac{d^2\phi}{dydx}, & \frac{d^2\phi}{dy^2} & \cdots & \frac{d^2\phi}{dydt} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{d^2\phi}{dtdx}, & \frac{d^2\phi}{dtdy} & \cdots & \frac{d^2\phi}{dt^2} \end{vmatrix}.$$

A conoid becoming a pair of planes, a cone becoming a pair of coincident lines, a pair of points becoming indeterminate, will, in like manner, be denoted by their characteristic losing two orders, and so forth, for the higher degrees of degradation. In like manner, in general, a homogeneous function of three letters of any degree losing an order, typifies that the locus to which it is the characteristic will break up into a system of right lines.

Now let ω be a homogeneous function of $\alpha, \beta, \gamma \dots \delta$, and suppose that we have the equations $\omega = 0$, $\chi\omega = 0$, $\chi^2\omega = 0$, where χ as above

$$= x\frac{d}{d\alpha} + y\frac{d}{d\beta} + z\frac{d}{d\gamma} + \dots + t\frac{d}{d\delta}.$$

I say that on eliminating any of the variables $x, y, z \dots t$ between the second and third of the above equations, the resulting equation will be of one order less than the number of letters, that is, the expulsion of one letter will be attended by the expulsion of *two* orders.

For we have, by theorem (3),

$$\chi\omega = U\omega + 2\frac{x_n}{a_n}\omega = 0,$$

$$\chi^2\omega = U_2\omega + 2\frac{x_n}{a_n}U\omega + 2\left(\frac{x_n}{a_n}\right)^2\omega = 0,$$

and by hypothesis

$$\omega = 0.$$

Hence we have also

$$U\omega = 0,$$

$$U_2\omega = 0;$$

and since $U\omega$, $U_2\omega$ contain one order less than the number of letters in

* If $U = 0$ is the equation to any locus, U may be said to *characterize* the same, or to be its characteristic.

ω, the resultant of the elimination between them will contain two orders less than the number of letters in ω; and consequently, whichever of the letters $x, y, z \ldots t$ we eliminate between $\chi\omega = 0$ and $\chi^2\omega = 0$, provided that $\omega = 0$, the resultant equation will contain one order less than the number of letters remaining.

Thus we see how it is that the tangent line to a conic meets it in two coincident points, the tangent plane to a conoid in two intersecting lines, and so forth, for the higher regions of space*. For instance, if we take $\omega(x, y, z, t) = 0$, the equation to a conoid, and $\alpha, \beta, \gamma, \delta$, the coordinates to any point therein, we shall have $\omega(\alpha, \beta, \gamma, \delta) = 0$,

$$\left(x\frac{d}{d\alpha} + y\frac{d}{d\beta} + z\frac{d}{d\gamma} + t\frac{d}{d\delta}\right)\omega, \text{ that is, } \chi\omega = 0,$$

and

$$\omega(x, y, z, t), \text{ that is, } \chi^2\omega = 0,$$

x, y, z, t representing the coordinates of any point in the intersection of the conoid by the tangent plane.

Consequently, by what has been shown above, on eliminating any one of the four letters x, y, z, t, the resultant function of three letters will contain only two orders, and will thus represent a pair of lines, real or imaginary, intersecting one another at $\alpha, \beta, \gamma, \delta$.

The fact which has just been demonstrated (that the resultant of $\chi\omega = 0$, $\chi^2\omega = 0$, loses an order if $\omega = 0$), indicates that on expressing one of the quantities $x, y, z \ldots t$ in terms of the others, by means of the first equation, and then substituting this value in the second, the determinant of the equation so obtained must be zero.

Now by virtue of a theorem which was given by me in a note† to my paper in the preceding number of this *Journal*, this determinant will be equal to the squared reciprocal of the coefficient in the equation $\chi\omega = 0$ of the letter eliminated multiplied by the determinant in respect to $x, y, z \ldots t, \lambda$ of

$$\chi^2\omega + \chi\omega\lambda.$$

This latter determinant is therefore zero; but this determinant is the resultant of the equations

$$\left.\begin{array}{l}
\dfrac{d}{dx}\left(x\dfrac{d}{da} + y\dfrac{d}{db} + \text{\&c.}\right)^2\omega + \dfrac{d}{dx}\left(x\dfrac{d}{da} + y\dfrac{d}{db} + \ldots\right)\omega = 0, \\
\dfrac{d}{dy}\left(x\dfrac{d}{da} + y\dfrac{d}{db} + \text{\&c.}\right)^2\omega + \dfrac{d}{dy}\left(x\dfrac{d}{da} + y\dfrac{d}{db} + \ldots\right)\omega = 0, \\
\quad\text{\&c.} \qquad \text{\&c.} \qquad \text{\&c.} \qquad \text{\&c.} \\
\qquad\qquad \chi\omega = 0, \text{ that is, } \left(x\dfrac{d}{da} + y\dfrac{d}{db} + \ldots\right)\omega = 0,
\end{array}\right\}.$$

* Thus a tangential section of a hyperlocus of the second degree at any point cuts it in two cones.

[† p. 135 above.]

Thus we obtain the singular law, that the symmetrical determinant

$$\begin{vmatrix}
\dfrac{d}{da}\dfrac{d}{da}\omega, & \dfrac{d}{da}\dfrac{d}{db}\omega, \cdots & \dfrac{d}{da}\dfrac{d}{dl}\omega, & \dfrac{d}{da}\omega \\
\dfrac{d}{db}\dfrac{d}{da}\omega, & \dfrac{d}{db}\dfrac{d}{db}\omega, \cdots & \dfrac{d}{db}\dfrac{d}{dl}\omega, & \dfrac{d}{db}\omega \\
\dfrac{d}{dc}\dfrac{d}{da}\omega, & \dfrac{d}{dc}\dfrac{d}{db}\omega, \cdots & \dfrac{d}{dc}\dfrac{d}{dl}\omega, & \dfrac{d}{dc}\omega \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots \\
\dfrac{d}{dl}\dfrac{d}{da}\omega, & \dfrac{d}{dl}\dfrac{d}{db}\omega, \cdots & \dfrac{d}{dl}\dfrac{d}{dl}\omega, & \dfrac{d}{dl}\omega \\
\dfrac{d}{da}\omega, & \dfrac{d}{db}\omega, \cdots & \dfrac{d}{dl}\omega, & 0
\end{vmatrix}$$

is zero when ω is zero.

This is easily shown independently by means of a remarkable and I believe novel theorem, relative to homogeneous functions.

If ω be any homogeneous function of ι dimensions of $a, b, c \ldots l$, we have (by Euler's theorem already repeatedly applied), remembering that $\frac{d\omega}{da}, \frac{d\omega}{db} \cdots \frac{d\omega}{dl}$ are all homogeneous,

$$\begin{array}{l}
-\iota\omega + \left(a\dfrac{d}{da} + b\dfrac{d}{db} + \ldots\ldots\ldots\ldots + l\dfrac{d}{dl}\right)\omega = 0, \\
-(\iota - 1)\dfrac{d\omega}{da} + \left(a\dfrac{d}{da}\dfrac{d}{da} + b\dfrac{d}{da}\dfrac{d}{db} + \ldots + l\dfrac{d}{da}\dfrac{d}{dl}\right)\omega = 0, \\
-(\iota - 1)\dfrac{d\omega}{db} + \left(a\dfrac{d}{db}\dfrac{d}{da} + \ldots\ldots\ldots\ldots + l\dfrac{d}{db}\dfrac{d}{dl}\right)\omega = 0, \\
\qquad\quad \text{\&c.} \qquad\qquad \text{\&c.} \qquad\qquad \text{\&c.} \\
-(\iota - 1)\dfrac{d\omega}{dl} + \left(a\dfrac{d}{dl}\dfrac{d}{da} + \ldots\ldots\ldots\ldots\ldots + l\dfrac{d}{dl}\dfrac{d}{dl}\right)\omega = 0.
\end{array}$$

Between these equations we may eliminate *all the letters*, $a, b, c \ldots l$, and we obtain the equation

$$\begin{vmatrix}
\dfrac{d}{da}\dfrac{d}{da}\omega, & \dfrac{d}{da}\dfrac{d}{db}\omega, \cdots & \dfrac{d}{da}\dfrac{d}{dl}\omega, & \dfrac{d}{da}\omega \\
\dfrac{d}{db}\dfrac{d}{da}\omega, & \dfrac{d}{db}\dfrac{d}{db}\omega, \cdots & \dfrac{d}{db}\dfrac{d}{dl}\omega, & \dfrac{d}{db}\omega \\
\dfrac{d}{dc}\dfrac{d}{da}\omega, & \dfrac{d}{dc}\dfrac{d}{db}\omega, \cdots & \dfrac{d}{dc}\dfrac{d}{dl}\omega, & \dfrac{d}{dc}\omega \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots \\
\dfrac{d}{dl}\dfrac{d}{da}\omega, & \dfrac{d}{dl}\dfrac{d}{db}\omega, \cdots & \dfrac{d}{dl}\dfrac{d}{dl}\omega, & \dfrac{d}{dl}\omega \\
\dfrac{d}{da}\omega, & \dfrac{d}{db}\omega, \cdots & \dfrac{d}{dl}\omega, & \dfrac{\iota}{\iota - 1}\omega
\end{vmatrix} = 0.$$

As a corollary to this theorem, we see that if $\omega = 0$ the determinant obtained in the previous investigation becomes zero, agreeing with what has been already shown; in fact the last-named determinant is always equal to

$$\frac{\iota - 1}{\iota}\,\omega \times \begin{vmatrix} \frac{d}{da}\frac{d}{da}\omega, & \frac{d}{da}\frac{d}{db}\omega, \ldots & \frac{d}{da}\frac{d}{dl}\omega \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{d}{dl}\frac{d}{da}\omega, & \frac{d}{dl}\frac{d}{db}\omega, \ldots & \frac{d}{dl}\frac{d}{dl}\omega \end{vmatrix}.$$

This remarkable theorem, which I have communicated to friends nearly a twelvemonth back, is here, I believe, published for the first time*.

Suppose next that $\omega(x, y, z)$ is the characteristic of a line of any degree, to which a tangent is drawn at the point α, β, γ, using U in a manner correspondent to its previous signification to denote

$$\left(x - \frac{\alpha}{\gamma}z\right)\frac{d}{d\alpha} + \left(y - \frac{\beta}{\gamma}z\right)\frac{d}{d\beta},$$

and understanding $\omega(\alpha, \beta, \gamma)$ by ω, we have for determining the point of intersection, $\omega = 0$, $\chi\omega = 0$, $\chi^n\omega = 0$; and consequently, by aid of our theorem (3), we shall obtain

$$\omega = 0,$$

$$U\omega = 0,$$

$$U_n\omega + n\frac{z}{\gamma}U_{n-1}\omega + \ldots = 0$$

By means of the two latter equations, we obtain

$$\left(x - \frac{\alpha z}{\gamma}\right)^2 F\left(x - \frac{\alpha z}{\gamma}\right) = 0,$$

$$\left(y - \frac{\beta z}{\gamma}\right)^2 G\left(y - \frac{\beta z}{\gamma}\right) = 0,$$

* Thus let z be a homogeneous function in x and y of ι dimensions, and let

$$\frac{dz}{dx}, \quad \frac{dz}{dy}, \quad \frac{d^2z}{dx^2}, \quad \frac{d^2z}{dx\,dy}, \quad \frac{d^2z}{dy^2},$$

be called p, q, r, s, t; we shall find

$$\begin{vmatrix} r, & s, & p \\ s, & t, & q \\ p, & q, & \frac{\iota}{\iota - 1}\omega \end{vmatrix} = 0,$$

that is,

$$\omega = \frac{\iota - 1}{\iota}\,\frac{rq^2 - 2pqs + tp^2}{rt - s^2}.$$

where F and G are each of only $(n-2)$ dimensions, and serve to determine the intersections of the tangent with the curve, extraneous to the two coincident ones at the point of contact.

Again, suppose that ω is a function of any degree of any number of letters α, β, γ, &c., and that we have given $\omega = 0$, $\chi\omega = 0$, $\chi^2\omega = 0, \ldots \chi^m\omega = 0$; it is evident from our fundamental theorem that these equations may be replaced by

$$\omega = 0, \quad U_1\omega = 0, \quad U_2\omega = 0, \ldots\ U_m\omega = 0;$$

and consequently that the expulsion of $(m-1)$ letters, by aid of the last m of the given equations, will be attended by the disappearance of m orders, or, in other words, the resultant will be minus an order, that is, will have one order less than the number of letters remaining in it.

In applying to space conceptions the preceding theorem, it will be convenient to use a general nomenclature for geometrical species of various dimensions.

Thus we may call a line a monotheme, a surface a ditheme, the species beyond a tritheme, and so on, *ad infinitum*.

A system of points according to the same system of nomenclature would be called a kenotheme.

An n-theme has for its characteristic a homogeneous function of $(n+2)$ letters.

Again, it will be convenient to give a general name to all themes expressed by equations of the first degree. Right lines and planes agree in conveying an idea of levelness and uniformity; they may both be said to be homalous. I shall therefore employ the word homaloid to signify in general any theme of the first degree.

Now let $\omega(x, y, z \ldots t)$ be the characteristic to an n-theme of the nth degree.

The number of letters $x, y, z \ldots t$ is $(n+2)$.

As usual, let ω represent $\omega(\alpha, \beta, \gamma \ldots \delta)$, and suppose

$$\omega = 0, \quad \chi\omega = 0, \quad \chi^2\omega = 0 \ldots \chi^n\omega = 0,$$

and consequently

$$U_1\omega = 0, \quad U_2\omega = 0 \ldots U_n\omega = 0.$$

On eliminating $(n-1)$ letters between the n last equations, the resulting function will be of three letters but of only two orders, and of the $1.2.3 \ldots n$ degree. Hence we see that at every point of an n-theme of the nth degree, and lying in the tangent homaloid thereto, $1.2\ldots.n$ right lines may be drawn coinciding throughout with the n-theme.

Thus one right line can be drawn at each point of a line of the first order lying on the line; two right lines at each point of a surface of the second order lying on the surface; six right lines at each point of a hyperlocus of the third degree, and so forth.

It is obvious that a surface may be treated as the homaloidal section of a tritheme, just as a plane curve may be regarded as a section of a surface. I shall proceed to show upon this view, how we may obtain a theorem given by Mr Salmon for surfaces of the third degree of a particular character from the law just laid down, according to which a tritheme of the third degree admits of six right lines being drawn upon it at every point*.

Let $\omega(x, y, z, t, u)$ be the characteristic of any tritheme of the third degree; $\alpha, \beta, \gamma, \delta, \epsilon$, coordinates to any point in the same. Then $\omega(\alpha, \beta, \gamma, \delta, \epsilon) = 0$, and the equation to the tangent homaloid will be $\chi\omega(\alpha, \beta, \gamma, \delta, \epsilon) = 0$, and the equation to the polar of the second degree to the given tritheme in relation to the assumed point as origin, (that is, the infinite system of homaloids that may be drawn from the point to touch the tritheme), will be $\chi^2\omega(\alpha, \beta, \gamma, \delta, \epsilon) = 0$.

But the section of any polar through its origin is the polar of the section to the same origin; hence the polar to the intersection of $\omega(x, y, z, t, u) = 0$, with $\chi\omega(\alpha, \beta, \gamma, \delta, \epsilon) = 0$, is the intersection of $\chi\omega = 0$ with $\chi^2\omega = 0$.

The projections of these intersections upon the space x, y, z, t will be found by eliminating u, and getting the correspondent two equations between x, y, z, t. Hence we see that the projection of the latter intersection upon any space x, y, z, t is a cone; or, in other words, this intersection itself, that is, the polar to the intersection of the tritheme with its tangent homaloid, is a cone; that is to say, the surface of the third degree formed by cutting a tritheme of the third degree by any tangent homaloid has a conical point at the point of contact; so that every surface of the third degree with a conical point may be considered as the intersection of a tritheme of the third degree with any tangent homaloid thereto†.

* The reduction of any equation of the sixth degree to depend upon one of the fifth may be shown by Mr Jerrard's method to be equivalent to drawing a straight line upon a tritheme of the third degree, just as the reduction of the equation of the fifth degree to a trinomial form may be shown to be dependent upon our being able to draw a straight line upon a ditheme of the second degree. Now at every point of a tritheme straight lines may be drawn, but as they keep together in groups of sixes they cannot be found *in general* at a given point without solving an equation of the sixth degree.

† So in like manner a surface of the third degree with more than one conical point may be generated by the intersection of the tritheme with a pluri-tangent plane; and so too we may get other varieties by taking homaloidal sections of trithemes whose characteristics are minus one or more orders.

Hence then we see, as an instantaneous deduction from our general theorem, that at any conical point (when one exists) of a surface of the third degree *six* right lines may be drawn lying completely upon it. This theorem is thus brought into an immediate and natural connexion with the well-known one, that at *every* point in a surface of the second degree, *two* right lines can be drawn lying wholly upon the surface*.

The last geometrical application of the theorem (3) which I shall make, refers to the equations employed by Mr Salmon in No. XXI. (New Series) of this *Journal*, to obtain the locus of the points on any surface at which tangent lines can be drawn passing through four consecutive points. I may remark in passing that these equations may be obtained by rather simpler considerations than Mr Salmon has employed so to do, and without any reference to Joachimsthal's theorem; for if we take ξ, η, ζ, θ, as the coordinates of any point in one of the tangent lines above described, and if we take the first polar to the surface with this point as origin, three out of the four original points will be found in such polar consecutive but distinct; and consequently in the second polar, referred to the same origin, two will continue consecutive but distinct, and consequently one will remain over in the third polar.

Hence writing the equation to the surface $\omega(x, y, z, t) = 0$, and using D to denote $\xi\frac{d}{dx} + \eta\frac{d}{dy} + \zeta\frac{d}{dz} + \theta\frac{d}{dt}$, we shall evidently have

$$\omega = 0, \qquad (1)$$

$$D\omega = 0, \qquad (2)$$

$$D^2\omega = 0, \qquad (3)$$

$$D^3\omega = 0, \qquad (4)$$

as obtained by Mr Salmon. And the same kind of reasoning precisely applies to the theory of points of inflexion in curves; three consecutive points in a right line in this case corresponding to four such in the case above considered.

If now we make

$$\xi - \frac{x}{t}\theta = u,$$

$$\eta - \frac{y}{t}\theta = v,$$

$$\zeta - \frac{z}{t}\theta = w,$$

* If we have an indeterminate system of algebraical equations consisting of one quadratic and another n^c function of three variables, this may be completely resolved by considering the first as an equation to a surface of the second degree, finding at any point thereof the two lines which lie upon the surface, and determining their respective intersections with the surface represented by the second equation. This will require therefore the solution only of a quadratic and an n^c equation. In like manner an indeterminate system of two equations of four variables, one of the third and the other of the nth degree, may be completely resolved (with the aid of the theorem in the text) by means of two equations, one of the sixth and the other of the nth degree.

the equations (2), (3), (4), by our theorem, may be expressed in terms of u, v, w, which being eliminated we obtain an equation between x, y, z, t, which will express the surface whose intersection with the given surface $\omega=0$ serves to determine the locus of the points in question.

Hence if we proceed in the ordinary manner to eliminate two of the four letters, as ξ and η, between the equations (2), (3), (4), the resultant will be of the form $M\times\phi(\zeta,\theta)$, where M does not contain ξ, η, ζ or θ, and where by the general laws of elimination $\phi(\zeta,\theta)$ will be an integral function of the sixth degree in respect to ζ, θ: and it is manifest that $M\times\phi(\zeta,\theta)$ will be identical with the resultant of (2), (3), (4) expressed in terms of u, v, w, when u and v are eliminated *cy-près* of an integralizing factor, showing that $\phi(\zeta,\theta)$ is w^6 integralized, that is, is equal to $(t\zeta-z\theta)^6$. Consequently as $M\phi$ is of the order $(n-1)\,2.3+(n-2)\,1.3+(n-3)\,1.2$, that is, $11n-18$ in respect to x, y, z, t, it follows that $M=0$, the equation to the second surface spoken of above, will be of the order $11n-24$, agreeable to Mr Salmon's showing.

I shall conclude this paper by showing the application of our theorem to the subject propounded by Mr Jerrard and Sir William Hamilton, of systems of equations containing a sufficient number of variable letters for effecting the solution without elevation of degree.

If we have n homogeneous equations containing a sufficient number of letters $a_1, a_2 \dots a_m$ to enable us to express the solution of $(n-1)$ of the equations under the form

$$\begin{aligned} a_1 &= \alpha_1 + \lambda\beta_1, \\ a_2 &= \alpha_2 + \lambda\beta_2, \\ &\dots\dots\dots\dots \\ a_m &= \alpha_m + \lambda\beta_m, \end{aligned}$$

where $\alpha_1, \alpha_2 \dots \alpha_m$, $\beta_1, \beta_2 \dots \beta_m$ are supposed known, and λ is indeterminate, it is evident that by substituting these values in the nth equation, λ may be found by solving an equation of the same degree as that equation contains dimensions of $a_1, a_2 \dots a_m$.

Let us then propose this question: how many letters $a_1, a_2 \dots a_r$ are needed to obtain a linear solution of a system of n equations

$$\phi_1=0,\ \phi_2=0, \dots \phi_n=0,$$

of the several degrees $\iota_1, \iota_2 \dots \iota_n$, without elevation of degree; by a linear solution being understood a solution under the form

$$\begin{aligned} a_1 &= \alpha_1 + \lambda\beta_1, \\ a_2 &= \alpha_2 + \lambda\beta_2, \\ &\dots\dots\dots\dots \\ a_r &= \alpha_r + \lambda\beta_r, \end{aligned}$$

where λ is left indeterminate.

Let us suppose that $\alpha_1, \alpha_2 \dots \alpha_r$, substituted respectively for $a_1, a_2 \dots a_r$, satisfy the given system of equations. The determination of these values without elevation of degree will, from what has been said before, depend upon the linear solution of a system of equations differing from the given system by the omission of any one of them at pleasure.

Now make

$$D = \alpha_1\frac{d}{da_1} + \alpha_2\frac{d}{da_2} + \dots + \alpha_r\frac{d}{da_r},$$

and then write

$$\left.\begin{aligned} &D\phi_1=0,\ D^2\phi_1=0 \dots D^{\iota_1}\phi_1=0 \\ &D\phi_2=0,\ D^2\phi_2=0 \dots D^{\iota_2}\phi_2=0 \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &D\phi_n=0,\ D^2\phi_n=0 \dots D^{\iota_n}\phi_n=0 \end{aligned}\right\}. \qquad (\theta)$$

The values of $a_1, a_2 \dots a_r$ derived from this system, say $(a)_1, (a)_2 \dots (a)_r$, give

$$a_1=\alpha_1+\lambda(a)_1,\ a_2=\alpha_2+\lambda(a)_2, \dots a_r=\alpha_r+\lambda(a)_r,$$

a solution under the required form, where λ is left indeterminate.

The solution of this new system without elevation of degree depends on the *linear solution* of all but one of them; this excepted one may be taken the one whose dimensions ι_r are the highest or as high as any of the quantities $\iota_1, \iota_2 \dots \iota_n$.

Consequently, if we use the symbol $(k_1, k_2 \dots k_r)$ to denote the number of letters required for the linear solution (without elevation of degree) of k_1 equations of the first degree, k_2 of the second, k_3 of the third, ..., k_r of the rth, it would at first sight appear from the preceding reduction that we must have

$$(k_1, k_2 \dots k_r) = \{K_1, K_2 \dots K_{r-1}, K_r'\},$$

where

$$\begin{aligned} K_1 &= k_1 + k_2 + \dots + k_{r-1} + k_r, \\ K_2 &= k_2 + \dots + k_{r-1} + k_r, \\ &\dots\dots\dots\dots\dots\dots \\ K_{r-1} &= k_{r-1} + k_r, \\ K_r' &= k_r - 1. \end{aligned}$$

But now steps in our theorem (3), and shows that the system (θ) may be superseded by another, in which the variables, instead of being $a_1, a_2 \dots a_n$, will be

$$a_1 - \frac{\alpha_1}{\alpha_n}a_n,\ a_2 - \frac{\alpha_2}{\alpha_n}a_n, \dots a_{n-1} - \frac{\alpha_{n-1}}{\alpha_n}a_n;$$

consequently the number of really independent variables is only $(n-1)$; we must therefore have

$$(k_1, k_2 \dots k_r) = 1 + \{K_1, K_2 \dots K_r'\}.$$

Since the introduction of a new simple equation is equivalent to the requirement of one more disposable letter, we may write the above more symmetrically under the form

$$(k_1, k_2 \dots k_r) = ('K_1, K_2 \dots K_{r-1}, K_r'),$$

where

$$'K_1 = 1 + k_1 + k_2 + \dots + k_r,$$
$$K_r' = \qquad\qquad k_r - 1.$$

By means of this formula of reduction $(k_1, k_2 \dots k_r)$ may be finally brought down to the form (L), and the value of (L) being the number of letters required for the linear solution of a system of L linear equations is evidently $L+2$.

Thus, to determine the number of letters required for the linear solution of a single quadratic, we write

$$(0, 1) = (2) = 4.$$

For two quadratics, we write

$$(0, 2) = (3, 1) = (5) = 7;$$

for a quadratic and a cubic,

$$(0, 1, 1) = (3, 2) = (6, 1) = (8) = 10;$$

for two cubics,

$$(0, 0, 2) = (3, 2, 1) = (7, 3) = (11, 2) = (14, 1) = (16) = 18.$$

These results coincide with those obtained by Sir William Hamilton in his Report on Mr Jerrard's Transformation of the Equation of the Fifth Degree in the *Transactions* of the British Association. I have much more to say on the subject of the linear solution of a system of indeterminate equations, and am, I believe, able to present the subject in a more general light than has hitherto been done; but my observations on this matter must be deferred until a subsequent communication.

31.

REPLY TO PROFESSOR BOOLE'S OBSERVATIONS ON A THEOREM CONTAINED IN THE LAST NOVEMBER NUMBER OF THE JOURNAL.

[*Cambridge and Dublin Mathematical Journal*, VI. (1851), pp. 171—174.]

THE restricted space that can be spared for discussion in these pages, necessitates me to compress within the narrowest limit the remarks which I feel bound to make on Mr Boole's extraordinary observations* in the February number of this *Journal*, on my theorem contained in the antecedent number thereof†, which statements I cannot, in the interests of truth and honesty, suffer to pass unchallenged. The object of that theorem was to show how the determinant of the quadratic function resulting from the elimination of any set of the variables between a given quadratic function and a number of linear functions of the same variables, could be represented *without performing* the actual elimination by a fraction, of which the numerator would be constant whichever set of the variables might be selected for elimination, and the denominator the square of the determinant corresponding to the coefficients of the variables so eliminated. The numerator itself is a determinant, obtained by forming the square corresponding to the determinant of the given quadratic function, and bordering it horizontally and vertically with the lines and columns corresponding to the coefficients of all the variables in the given linear equations. An *immediate corollary* from this theorem leads to Mr Boole's. Conversely upon the principle that "tout est dans tout" Mr Boole devotes a page and a half of close print merely to indicate the steps of a method by which from his theorem mine is capable of being deduced, ending with the announcement, that the numerator in question is equal to the quantity

$$\phi_1 \phi_2 \dots \phi_r \theta(Q),$$

(the symbols above employed being Mr Boole's own), and concludes with assuring his readers that "he has ascertained that Mr Sylvester's result is reducible to the above form." Mr Sylvester would be very sorry to put his

[* *Cambr. and Dublin Math. Jour.* VI. (1851), pp. 90, 284.] [† p. 135 above.]

result under any such form. Mr Boole could scarcely have reflected upon the effect of his words when he indulged in the remark which follows—"*there cannot be a doubt* that for the discovery of the actual relation in question, the above theorem is far more convenient than Mr Sylvester's." Of the value to be attached to this assertion the annexed comparison of results is submitted as a specimen.

Let the quadratic function be

$$ax^2 + by^2 + cz^2 + dt^2 + 2exy + 2\epsilon zt + 2gxz + 2\gamma yt + 2hyz + 2\eta xt,$$

and the linear functions (taken two in number)

$$lx + my + nz + pt,$$
$$l'x + m'y + n'z + p't.$$

My numerator will be the determinant (hereinafter cited as the *extended* determinant),

$$\begin{vmatrix} a & e & g & \eta & l & l' \\ e & b & h & \gamma & m & m' \\ g & h & c & \epsilon & n & n' \\ \eta & \gamma & \epsilon & d & p & p' \\ l & m & n & p & 0 & 0 \\ l' & m' & n' & p' & 0 & 0 \end{vmatrix}.$$

To find the numerator of Mr Boole's fraction, we must form the symbolical operator

$$\left\{\begin{array}{l} l^2 \dfrac{d}{da} + m^2 \dfrac{d}{db} + n^2 \dfrac{d}{dc} + p^2 \dfrac{d}{dd} \\ \quad + 2lm \dfrac{d}{de} + 2np \dfrac{d}{d\epsilon} + 2ln \dfrac{d}{dg} + 2mp \dfrac{d}{d\gamma} + 2lp \dfrac{d}{dh} + 2mn \dfrac{d}{d\eta} \end{array}\right\}$$
$$\times \left\{\begin{array}{l} l'^2 \dfrac{d}{da} + m'^2 \dfrac{d}{db} + n'^2 \dfrac{d}{dc} + p'^2 \dfrac{d}{dd} \\ \quad + 2l'm' \dfrac{d}{de} + 2n'p' \dfrac{d}{d\epsilon} + 2l'n' \dfrac{d}{dg} + 2m'p' \dfrac{d}{d\gamma} + 2l'p' \dfrac{d}{dh} + 2m'n' \dfrac{d}{d\eta} \end{array}\right\}$$

and after expanding the determinant hereunder written

$$\begin{vmatrix} a & e & g & \eta \\ e & b & h & \gamma \\ g & h & c & \epsilon \\ \eta & \gamma & \epsilon & d \end{vmatrix},$$

perform the operations above indicated upon the result so obtained.

These are the operations and processes which, on Professor Boole's authority, we are to accept "*as without doubt far more convenient*" than the one simple process of forming, and when necessary, calculating the extended determinant above given. Here for the present I leave the case between Mr Boole and myself to the judgment of the readers of this *Journal*.

In the April Number of the *Philosophical Magazine**, I have shown that the extended determinant serves, not only to represent the full and complete determinant of the reduced quadratic function, but likewise all the minor determinants thereof; the last *set* of which will be evidently no other than the coefficients themselves. For instance, in the example above given, if we wish to find the coefficient of x^2 after z and t have been eliminated, we have only to strike out the line and column $e\ b\ h\ \gamma\ m\ m'$ from the extended determinant; if we wish to find the coefficient of y^2, we must strike out the line and column $a\ e\ g\ \eta\ l\ l'$; to find the coefficient of xy, we must strike out the line $a\ e\ g\ \eta\ l\ l'$ and the column $e\ b\ h\ \gamma\ m\ m'$, or *vice versâ*.

In each of these cases the determinant so obtained is the numerator of the equivalent fraction; the denominator remaining always the same function of the coefficients of transformation as in the original theorem.

Again, if there be taken only one linear equation, and by aid of it x is supposed to be eliminated; and if the reduced quadratic function be called

$$Ly^2 + Mz^2 + Nt^2 + 2Pzt + 2Qyt + 2Rzy,$$

the same extended determinant as before given will serve, when stripped of its outer border, consisting of the line and column $l'\ m'\ n'\ p'$, to produce the various equivalent fractions: thus form the square

$$\begin{matrix} L & R & Q \\ R & M & P \\ Q & P & N. \end{matrix}$$

The numerator of the fraction equivalent to $\begin{vmatrix} L & R \\ R & M \end{vmatrix}$, that is, to $LM - R^2$, may be found by striking out from the form of the extended determinant the line and column $\eta\ \gamma\ \epsilon\ d\ p$; that corresponding to $\begin{vmatrix} L & Q \\ R & P \end{vmatrix}$, that is, $LP - RQ$, will be found by striking out the line $g\ h\ c\ \epsilon\ n$ and the column $\eta\ \gamma\ \epsilon\ d\ p$, or *vice versâ*; and so forth for all the first minor determinants; and similarly the second minors, that is, L, M, N, P, Q, R, may be obtained by striking out in each case a correspondent pair of lines and pair of columns. Thus, to find the numerator of L the same pair of lines and columns, namely, $(g\ h\ c\ \epsilon\ n)$, $(\eta\ \gamma\ \epsilon\ d\ p)$, must be elided. To find the numerator of R, the pair of lines $(g\ h\ c\ \epsilon\ n)$, $(\eta\ \gamma\ \epsilon\ d\ p)$, and the pair of columns $(e\ b\ h\ \gamma\ m)$, $(\eta\ \gamma\ \epsilon\ d\ p)$, or *vice versâ*, will have to be elided; and so forth for the remaining second minors. I may conclude with observing, that the theorem contested by Mr Boole is an immediate corollary from the general Theory of Relative Determinants alluded† to in the "Sketch" inserted in the present number of the *Journal*.

[* p. 241 below.] [† p. 188 below.]

32.

SKETCH OF A MEMOIR ON ELIMINATION, TRANSFORMATION, AND CANONICAL FORMS.

[*Cambridge and Dublin Mathematical Journal*, VI. (1851), pp. 186—200.]

THERE exists a peculiar system of analytical logic, founded upon the properties of zero, whereby, from dependencies of equations, transition may be made to the relations of functional forms, and *vice versâ*: this I call the logic of characteristics.

The resultant of a given system of homogeneous equations of as many variables, is the function whose nullity implies and is implied by the possibility of their coexistence, that is, is the characteristic of such possibility; but inasmuch as any numerical product of any power of a characteristic is itself an equivalent characteristic, in order to give definiteness to the notion of a resultant, it must further be restricted to signify the characteristic taken in the *lowest form* of which it *in general* admits.

The following very general and important proposition for the change of the independent variables in the process of elimination, is an immediate consequence of the doctrine of characteristics.

Let there be two sets of homogeneous forms of function;

the 1st, $\phi_1,\ \phi_2 \dots \phi_n$,

the 2nd, $\psi_1,\ \psi_2 \dots \psi_n$.

Let the results of applying these forms to any sets of n variables be called

$$(\phi_1),\ (\phi_2) \dots (\phi_n),$$
$$(\psi_1),\ (\psi_2) \dots (\psi_n);$$

then will the resultant (in respect to those variables) of

$$\phi_1\,\{(\psi_1),\ (\psi_2) \dots (\psi_n)\},$$
$$\phi_2\,\{(\psi_1),\ (\psi_2) \dots (\psi_n)\},$$
$$\dots\dots\dots\dots$$
$$\phi_n\,\{(\psi_1),\ (\psi_2) \dots (\psi_n)\},$$

be the product of powers (assignable by the law of homogeneity) of the separate resultants of the two systems,

$$\{(\phi_1),\ (\phi_2) \dots (\phi_n)\},$$
$$\{(\psi_1),\ (\psi_2) \dots (\psi_n)\}.$$

By means of the doctrine of characteristics the following general problem may be resolved.

Given any number of functions of as many letters, and an inferior number of functions of the same inferior number of letters, obtained by combining, *inter se*, in a known manner, the given functions, to determine the factor by which, the resultant of the reduced system being divided, the resultant of the original system may be obtained.

If in the theorem for the change of the independent variables both sets of forms of functions be taken linear, we obtain the common rule for the multiplication of determinants: if we take one set linear and the other not, we deduce two rules, namely, That the resultant of a given set of functional forms of a given set of variables, enters as a factor into the resultant,

1st, of linear functions of the given functions of the given variables;

2nd, of the given functions of linear functions of the given variables:

the extraneous factor in each case being a power of what may be conveniently termed the *modulus of transformation*, that is, the resultant of the imported linear forms of functions.

From the second of these rules we obtain the law first stated I believe for functions beyond the second degree by Mr Boole, to wit, that the determinant of any homogeneous algebraical function (meaning thereby the resultant of its first partial differential coefficients) is unaltered by any linear transformations of the variables, except so far as regards the introduction of a power of the modulus of transformation. This is also abundantly apparent from the fact, that the nullity of such determinant implies an immutable, that is, a fixed and inherent, property of a certain corresponding geometrical locus.

There exist (as is now well known) other functions besides the determinant, called by their discoverer (Mr Cayley) hyperdeterminants, gifted with a similar property of immutability. I have discovered a process for finding hyperdeterminants of functions of any degree of any number of letters, by means of a process of Compound Permutation. All Mr Cayley's forms for functions of two letters may be obtained in this manner by the aid of one of the two processes (to wit, that one which will hereafter be called the derivational process), for passing from immutable constants to immutable forms. Such constants and forms, derived from given forms, may be best termed adjunctive; a term slightly varied from that employed by M. Hermite in a more restricted sense.

The two processes alluded to may be termed respectively appositional and derivational. The appositional is founded upon the properties of the binary function $x\xi + y\eta + z\zeta + \ldots$; in which, whether we substitute linear functions of x, y, z, &c., or linear functions of ξ, η, ζ, &c., in place of x, y, z, &c., or ξ, η, ζ, &c., the result is the same.

Consequently, if we apply the form ϕ to $\xi, \eta \ldots \zeta$, and take any constant (in respect to $\xi, \eta \ldots \zeta$) adjunctive to

$$\phi(\xi, \eta \ldots \zeta) + (x\xi + y\eta + \ldots + z\zeta + kt)\, t^{n-1},$$

calling this quantity $\psi(x, y \ldots z, t)$, the form ψ is evidently adjunctive to the form ϕ: and if we expand so as to obtain

$$\psi(x, y \ldots z, t) = \psi_1(x, y \ldots z)\, t^{\alpha} + \psi_2(x, y \ldots z)\, t^{\beta} + \&c.,$$

it is evident ψ_1, ψ_2, &c. will be each separately adjunctive to ϕ. These forms, when ψ is obtained by finding the determinant in respect to $\xi, \eta \ldots \zeta$ of S, are, in fact, identical with Hermite's "formes adjointes."

The derivational mode of generating forms from constants depends upon the property of the operative symbol

$$\chi = \xi \frac{d}{dx} + \eta \frac{d}{dy} + \ldots + \zeta \frac{d}{dz},$$

applied to ϕ a function of $x, y \ldots z$; namely, that if in ϕ, in place of these letters, we write linear functions thereof, to wit $x', y' \ldots z'$, we may write

$$\chi = \xi' \frac{d}{dx'} + \eta' \frac{d}{dy'} + \ldots + \zeta' \frac{d}{dz'},$$

where $\xi', \eta' \ldots \zeta'$ will be the same functions of $\xi, \eta \ldots \zeta$ that $x', y' \ldots z'$ are of $x, y \ldots z$.

Suppose now, in the first place, that in regard to $\xi, \eta \ldots \zeta$, $\psi(x, y \ldots z)$ is adjunctive to $\chi^r \phi(x, y \ldots z)$; then is the form ψ adjunctive to the form ϕ, for on changing $x, y \ldots z$ to $x', y' \ldots z'$,

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy} + \ldots + \zeta \frac{d}{dz}\right)^r \phi(x', y' \ldots z')$$

becomes

$$\left(\xi' \frac{d}{dx'} + \eta' \frac{d}{dy'} + \ldots + \zeta' \frac{d}{dz'}\right) \phi(x', y' \ldots z');$$

and consequently $\psi(x, y \ldots z)$ becomes $\psi(x', y' \ldots z')$, multiplied by a power of the modulus of transformation, the modulus of that transformation, be it well observed, whereby $x', y' \ldots z'$ would be replaced by $x, y \ldots z$, and not as in the appositional mode of that converse transformation according to which

$x, y \ldots z$ would be replaced by $x', y' \ldots z'$. It is on account of this converseness of the modes of transformation that the appositional and derivational modes of generating forms cannot except for a certain class of *restricted* linear transformations be combined in a single process. More generally, if instead of a single function $\chi^r \phi(x, y \ldots z)$, we take as many such with different indices to χ as there are variables, and form either the resultant in respect to $\xi, \eta \ldots \zeta$, or any other immutable constant in regard to those variables, (presuming in extension of the hyperdeterminant theory and as no doubt is the case, that such exist), every such resultant or other constant will give a form of function of $x, y \ldots z$ adjunctive to the given form ϕ.

It may be shown that every such resultant so formed will contain ϕ as a factor.

Again, in the former more available determinant mode of generation, if we take the determinant in respect to $\xi, \eta \ldots \zeta$, it may be shown that all the adjunctive functions so obtained will be algebraical derivees of the partial differential coefficients of ϕ in respect to $x, y \ldots z$; that is to say, if these be respectively zero, all such adjunctive functions so derived, as last aforesaid, will be zero, or in other words, each such adjunctive is a syzygetic function of the partial differential coefficients of the primitive function.

To Mr Boole is due the high praise of discovering and announcing, under a somewhat different and more qualified form and mode of statement, this marvel-working process of derivational generation of adjunctive forms. I was led back to it, in ignorance of what Mr Boole had done, by the necessity which I felt to exist of combining Hesse's so-called functional determinant, under a common point of view with the common constant determinant of a function; under pressure of which sense of necessity, it was not long before I perceived that they formed the two ends of a chain of which Hesse's end exists for all homogeneous functions, but the other only when such functions are algebraical.

In fact, if we give to r every value from 2 upwards, the successive determinants in respect to $\xi, \eta \ldots \zeta$ of

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy} + \ldots + \zeta \frac{d}{dz}\right)^r \phi(x, y, z),$$

will produce the chain in question, which, when ϕ is algebraical and of n dimensions, comes to a natural termination when $r = n - 1$. The last member of and the number of terms in this chain are identical with the last member of and the number of terms in Sturm's auxiliary functions, when the variables are reduced to two. There is some reason to anticipate that this chain of functions may be made available in superseding Sturm's chain of auxiliaries; and if so, then the fatal hindrance to progress, arising from the unsymmetrical nature of the latter, is overcome, and we shall be able to pass from Sturm's theorem, which relates to the theory of Keno-themes, or Point-systems, to certain corresponding but much higher theories for lines, surfaces, and n-themes generally.

The restriction of space allowed to me in the present number of the *Journal* will permit me only to allude in the briefest terms to the theory of Relative Determinants, which, as it will be seen, plays an important part in the effectuation of the reductions of the higher algebraical functions to their simplest forms. Nor can the effect of the processes to be indicated be correctly appreciated without a knowledge of the circumstances under which the resultant of a *given* system of equations can sink in degree below the resultant of the *general* type of such system. Abstracting from the case when the equations separately, or in combination, subdivide into factors, this lowering of degree, as may be shown by the doctrine of characteristics, can only happen in one of two ways. Either the particular resultant obtained is a rational root of the general resultant, or the general resultant becomes zero for the case supposed, and the particular resultant is of a distinct character from the general resultant, being in fact the characteristic of the possibility not of the given system of equations being merely able to coexist (for that is already supposed), but of their being able to coexist for a certain system of values *other than* a given system or given systems. Such a resultant may be termed a Sub-resultant; the lowest resultant in the former case may be termed a Reduced-resultant. The theory of Sub-resultants is one altogether remaining to be constructed, and is well worthy equally of the attention of geometers and of analysts.

As to the theory of Relative Determinants, the object of this theory is to obtain the determinant resulting from eliminating as many variables as can be eliminated, chosen at pleasure from a set of variables greater in number than the equations containing them; and the mode of effecting this object is through the method of the indeterminate multiplier. To avoid the discussion of the theory of sub-resultants and other particularities, I shall content myself with giving the rule applicable to the case (the only one of which as yet a practical application has offered itself to me in the course of my present inquiries) when all but one of the functions are linear.

If U, L_1, $L_2 \ldots L_m$ be the first an n^c and the others linear functions of n variables, and it be desired to find the determinant of the resultant arising from the elimination of any m out of the n variables, the following is the rule:

Find the determinant, that is, the resultant of the partial differential coefficients in respect to the given variables, and of $\lambda_1, \lambda_2 \ldots \lambda_m$ of

$$U + L_1\lambda_1 + L_2\lambda_2 + \ldots + L_m\lambda_m.$$

This resultant, in its lowest form, will be always a rational $(n-1)$th root of the resultant of the homogeneous system of equations to which the system above given can be referred as its type; and this reduced resultant divided by a power (determinable by the law of homogeneity) of the resultant of $L_1, L_2 \ldots L_m$, when all but the selected variables are made zero, will be the resultant determinant required*. As regards what has been said concerning the reducibility of the general typical resultant in the case before us, this is a consequence of, and may be brought into connexion with, the following theorem, which is easily demonstrable by the theory of characteristics. If $Q_1, Q_2 \ldots Q_m$ be m homogeneous functions of m variables of the same degree, r of which enter in each equation only as simple powers uncombined with any of the other variables, then the degree of the reduced resultant is equal to the number of the equations multiplied by the $(m-r-1)$th power of the number of units in the degree of each, subject to the obvious exception that when r is m, (there being in fact but *one* step from $r=m-2$ to $r=m$), instead of r, $(r-1)$ must be employed in the above formula. As an example of a sub-resultant as distinguished from a reduced-resultant, I instance the case of three quadratics U, V, W, functions of x, y, z, in each of which no squared power of z is supposed to enter: it may easily be shown by my dialytic method that instead of six equations, between which to eliminate x^2, y^2, z^2, xy, xz, yz, we shall have only 5, the three original ones and two instead of three auxiliaries between which to eliminate x^2, y^2, xy, xz, yz, the *apparent* resultant is accordingly of the 9th instead of the 12th degree. But this is not the true characteristic of the possibility of the coexistence of the given systems, which in fact is zero, as is evidenced by the fact that they always *do* coexist, since they are always satisfiable by only *two* relations between the variables, to wit $x=0$, $y=0$. The apparent resultant is then something different, and what has been termed by the above a Sub-resultant.

I take this opportunity of entering my simple protest against the appropriation of my method of finding the resultant of any set of three equations of degrees equal or differing only by a unit, one from those of the other two, by Dr Hesse, so far as regards quadratic functions, without acknowledgment, four years after the publication of my memoir in the *Philosophical Magazine*: the fundamental idea of Dr Hesse's partial method is identical with that of my general one. Still more unjustifiable is the subsequent use of the *dialytic* principle, by the same author, equally without acknowledgment, and in cases where there is no peculiarity of form of procedure to give even a plausible ground for evading such acknowledgment. It is capable of moral proof that

* The same method applies not only to the Final or Constant Determinant, but likewise to all the Functional Determinants in the chain above described, extending upwards from this to the Hessian, or as it ought to be termed, the first Boolian Determinant.

what I had written on the matter was sufficiently known in Berlin and at Königsberg, at each epoch of Dr Hesse's use of the method.

I now proceed to the consideration of the more peculiar branch of my inquiry, which is as to the mode of reducing Algebraical Functions to their simplest and most symmetrical, or as my admirable friend M. Hermite well proposes to call them, their Canonical forms. Every quadratic function of any number of variables may always be linearly transformed into any other quadratic functions of the same, and that too in an infinite variety of ways; but in every other instance there will be only a limited number of ways, whereby, when possible, one form will admit of being transmuted into any other: and with the sole exception of a cubic function of two letters, such transmutation will never be possible, unless a certain condition, or certain conditions, be satisfied between the constants of the forms proposed for transmutation. The number of such conditions is the number of parameters entering into the canonical form, and is of course equal to the number of terms in the general form of the function diminished by the square of the number of letters. Thus there is one parameter in the canonical form for the biquadratic function of two and the cubic function of three letters, and no parameter in the cubic function of two letters. Hitherto no canonical forms have been studied beyond the cases above cited, but I have succeeded, as will presently be shown, in obtaining methods for reducing to their canonical forms functions with *two* and *four* parameters respectively. Owing to what has been remarked above, the theory of quadratic functions is a theory apart. Simultaneous transformation gives definiteness to that theory, but has no existence for any useful purpose for functions of the higher degrees. Where the theory of simultaneous transformation ends, that of canonical forms properly begins; and in what follows, the case of quadratic forms is to be understood as entirely excluded. Such exclusion being understood, there is no difficulty in assigning the canonical, that is, the simplest and most symmetrical general, form to which every function of two letters admits of being reduced by linear transformations. If the degree be odd, say $2m+1$, the canonical form will be

$$u_1^{2m+1} + u_2^{2m+1} + \dots + u_{m+1}^{2m+1};$$

if the degree be even, say $2m$, the canonical form will be

$$u_1^{2m} + u_2^{2m} + \dots + u_m^{2m} + K(u_1 u_2 \dots u_m)^2,$$

all the u's being linear functions of the two given variables. It is easy to extend an analogous mode of representation to functions of any number of letters. From the above we see that for cubic, biquadratic, and quintic functions of two letters, the canonical forms will be respectively

$$u^3+v^3, \quad u^4+v^4+Ku^2v^2, \quad u^5+v^5+w^5,$$

with a linear relation in the last-named case between u, v, w.

First as to the reduction of any 4^c function to Cayley's form

$$u^4+v^4+Ku^2v^2.$$

This may be effected in a great variety of ways, of which the following is not the simplest as regards the calculations required, but the most obvious. Let the modulus of transformation, whereby the given biquadratic function, say $F(x, y)$, becomes transmuted into its canonical form, be called M; let the determinant of F be called D_1, and the determinant of the determinant in respect to ξ and η of

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^2 F(x, y),$$

which latter, for brevity's sake, may be termed the Hessian of F, (although in stricter justice the Boolian would be the more proper designation), be called D_2. Then, by examining the canonical form itself (which is as it were the very *palpitating heart* of the function laid bare to inspection), we shall obtain without difficulty the two equations

$$(1-9m^2)^2 = M^{12} D_1 \frac{1}{4^6},$$

$$m^2(1-9m^2)^2(m^2-1)^2 = M^{24} D_2 \frac{1}{12^{12}4^6}.$$

Eliminating the unknown quantity M, we obtain

$$\frac{m^2(m^2-1)^2}{(1-9m^2)^2} = c, \quad \text{or} \quad \frac{m^3-m}{1-9m^2} = c^{\frac{1}{2}},$$

where c is a known quantity.

This *cubic* equation for finding m is of a peculiar form; it being easy to show *à priori*, by going back to the canonical form, that its three roots are m, $\theta(m)$, $\theta^2(m)$, where

$$\theta(m) = \frac{m-1}{3m+1},$$

θ being a periodical form of function such that $\theta^3(m) = m$.

This it is which accounts for the simple expression for m, that may be obtained by solving the cubic above given. A better practical mode is to take, instead of the determinant of the given function and its Hessian, the two hyperdeterminants and eliminate as before: a cubic equation having precisely the same properties, and in fact virtually identical with the former, will result. When m and consequently M are found, there is no difficulty whatever, calling the given function F and its Hessian $H(F)$, in forming linear functions of the two, as

$$\left.\begin{aligned}\phi(m)F + \psi(m)H(F)\\ \phi_1(m)F + \psi_1(m)H(F)\end{aligned}\right\},$$

which shall be equal to, that is, identical with, $(u^2+v^2)^2$ and u^2v^2, whence u and v are completely determined.

Another and interesting mode of solution is to take, besides the given function F and its Hessian, either the *second* Hessian or the post-Hessian of the given function, by the post-Hessian understanding the determinant in respect of ξ and η of

$$\left(\xi \frac{d}{dx} + \eta \frac{d}{dy}\right)^3 F:$$

any three of the four functions will be linearly related, and it may be shown that, calling either the second Hessian (that is, the Hessian of the Hessian) or the post-Hessian H', we shall have

$$H'(F) + aH(F) + bF = 0,$$

where a and b will be *rational* and *integer* functions of the coefficients of F, and numerical multiples of two quantities R and S, such that the determinant of F will be equal $R^3 + S^2$; and this, be it observed, without any previous knowledge of the existence of these hyperdeterminants R and S.

If now we go to Hesse's form for a cubic function of three letters, we shall find that precisely similar modes of investigation apply step for step. Calling the function F and its Hessian $H(F)$, and the post-Hessian or second Hessian at choice $H'(F)$, we shall find

$$H'(F) + mSH(F) + nR^2F = 0,$$

where m and n are numerical quantities and $R^3 + S^2$ equal the determinant of F. It is interesting to contrast this equation with the one previously mentioned as applicable to the 4^c functions of two letters, namely,

$$H'(F) + mRH(F) + nSF = 0.$$

In both instances there is no difficulty in assigning the relations between the original R and S, and the R and S of any adjunctive form. All Aronhold's results may be thus obtained and further extended without the slightest difficulty. As regards the equation for finding the parameter in Hesse's canonical form for the cubic of three letters, this will be of the 4th degree in respect to the cube of the parameter, and the roots will be functionally representable as

$$x; \quad \theta(x); \quad \phi(x); \quad \psi(x),$$

where

$$\theta^2(x) = \phi^2(x) = \psi^2(x) = x;$$
$$\theta\phi(x) = \phi\theta(x) = \psi(x),$$
$$\phi\psi(x) = \psi\phi(x) = \theta(x),$$
$$\psi\theta(x) = \theta\psi(x) = \phi(x);$$

owing to which property the equation is soluble under the peculiar form observed by Aronhold.

I pass on now to a brief account of the method, or rather of a method (for I doubt not of being able to discover others more practical), of reducing a function of the 5th degree of two letters (say of x and y) to its canonical form $u^5 + v^5 + w^5$, subject to the linear relation $au + bv + cw = 0$, where the ratios $a:b:c$, and the linear relations between u, v, w and the two given variables are the objects of research. Here I have found great aid from the method of Relative Determinants; and I may notice that the successful application of more compendious methods to the question would be greatly facilitated were there in existence a theory of Relative Hyperdeterminants, which is still all to form, but which I little doubt, with the blessing of God, to be able to accomplish. It may some little facilitate the comprehension of what follows, if c be considered as representing unity.

Calling as before the given quintic function F, the modulus of transformation M, the Hessian and post-Hessian of F, H and H', and its ordinary or constant determinant D, we shall find

$$a^2v^3w^3 + b^2w^3u^3 + c^2u^3v^3 = M^2H,$$

and

$$P_1P_2P_3P_4 = M^6H',$$

where

$$P_1 = \quad a^{\frac{3}{2}}vw + b^{\frac{3}{2}}wu + c^{\frac{3}{2}}uv,$$
$$P_2 = \quad a^{\frac{3}{2}}vw - b^{\frac{3}{2}}wu - c^{\frac{3}{2}}uv,$$
$$P_3 = - a^{\frac{3}{2}}vw + b^{\frac{3}{2}}wu - c^{\frac{3}{2}}uv,$$
$$P_4 = - a^{\frac{3}{2}}vw - b^{\frac{3}{2}}wu + c^{\frac{3}{2}}uv;$$

also $D = M^{20}$ multiplied by the product of the sixteen values of

$$a^{\frac{1}{4}} + b^{\frac{1}{4}}(1)^{\frac{1}{4}} + c^{\frac{1}{4}}(1)^{\frac{1}{4}}.$$

From the above equations it may be shown that H' (a known function of the 8th degree of the given variables x, y) must be capable of being thrown under the form

$$L\{(x - a_1y)(x - a_2y) \times (x - a_3y)(x - a_4y)$$
$$\times (x - a_5y)(x - a_6y) \times (x - a_7y)(x - a_8y)\},$$

where

$$(a_1 - a_2)^2 \times (a_3 - a_4)^2 \times (a_5 - a_6)^2 \times (a_7 - a_8)^2$$
$$= \frac{D}{L^2} = K,$$

so that K is a known quantity*. Accordingly the said equation of the 8th degree, considered as an algebraical equation in $\frac{x}{y}$, may by known methods be

* Or in other words, the post-Hessian determinant of a given function in two letters of the second degree, may be divided into four quadratic factors in such a way that the product of the determinants of these several factors shall be equal to the determinant of the given function.

found by means of equations not exceeding the 4th or even the 3rd degree: in fact, to do this it is only necessary to form the equation to the squares of the differences of the roots of $\frac{x}{y}$ in the equation $H' \div y^8 = 0$, which new equation will be of the 28th degree. If we then form two other equations of the 378th degree, one having its roots equal to $\sqrt{K}$ multiplied by the binary products of the twenty-eight roots of the equation last named, the other to $\sqrt{K}$ multiplied by the reciprocal of such binary products, the left-hand members of these two equations expressed under the usual form will have a factor in common, which may be found by the process of common measure and will be of the 6th degree, whose roots consisting of three pairs of reciprocals may be found by the solution of cubics only.

In this way, by means of cubics and quadratics,

$$(a_1 - a_2)^2, \quad (a_3 - a_4)^2, \quad (a_5 - a_6)^2, \quad (a_7 - a_8)^2,$$

can be found, which being known,

$$a_1 a_2, \quad a_3 a_4, \quad a_5 a_6, \quad a_7 a_8,$$

can be determined in pairs by means of quadratics from the equation $H' \div y^8 = 0$. This being supposed to be done, we have

$$P_1 = fL_1,$$
$$P_2 = gL_2,$$
$$P_3 = hL_3,$$
$$P_4 = kL_4,$$

where L_1, L_2, L_3, L_4, are *known* quadratic functions of x and y. To determine the ratios of f, g, h, k, we have three equations* obtained from the identity

$$fL_1 + gL_2 + hL_3 + kL_4 (= P_1 + P_2 + P_3 + P_4) = 0;$$

$f : g : h : k$ being known, $fL_1 : gL_2 : hL_3 : kL_4$ are known ratios.

But
$$P_1 + P_2 = 2a^{\frac{3}{2}} vw,$$
$$P_1 + P_3 = 2b^{\frac{3}{2}} wu,$$
$$P_1 + P_4 = 2c^{\frac{3}{2}} uv.$$

Hence
$$a^{\frac{3}{2}} vw = \lambda P,$$
$$b^{\frac{3}{2}} wu = \lambda Q,$$
$$c^{\frac{3}{2}} uv = \lambda R,$$

where P, Q, R are known quadratic functions of x, y.

* For we must have the coefficients of x^2, xy and y^2 in

$$fL_1 + gL_2 + hL_3 + kL_4,$$

of all them zero.

Hence $a : b : c$ may be found by means of the identical equation

$$a^2 w^3 v^3 + b^2 u^3 w^3 + c^2 v^3 u^3 = H(F),$$

whereby the ratios $a^{-\frac{5}{2}} : b^{-\frac{5}{2}} : c^{-\frac{5}{2}}$ can be obtained without any further extraction of roots, showing that there is but one single true system of ratios $a^5 : b^5 : c^5$ applicable to the problem; $a : b : c$ being thus found, λ is easily determined, and thus finally u, v, w are found in terms of x and y*.

I have little doubt that a more expeditious mode of solution than the foregoing† will be afforded by an examination of the properties and relations of the *quadratic and cubic forms*, adjunctive to the general quintic functions, and indeed to every $(4n+1)^c$ function of two letters hereinbefore adverted to.

Sufficient space does not remain for detailing the steps whereby the general cubic function of *four* letters may, by aid of equations *not transcending the fifth degree*, be reduced to its canonical form $u^3 + v^3 + w^3 + p^3 + q^3$, wherein u, v, w, p, q are connected by a linear equation

$$au + bv + cw + dp + eq = 0;$$

the four ratios of whose coefficients $a : b : c : d : e$ give the necessary number $\frac{4.5.6}{1.2.3} - 4^2$ parameters furnished by the general rule. Suffice it for the present to say, that the analytical mode of solution depends upon a circumstance capable of the following geometrical statement: Every surface of the 4th degree represented by a function which is the Hessian to any given cubic function whatever of four letters, has lying upon it ten straight lines meeting three and three in ten points, and these ten points are the only points which enjoy the following property in respect to the surface of the 3rd degree denoted by equating to zero the cubical function in question, to wit, that the cone drawn from any one of them as vertex to envelop the surface, will meet it not in a continuous double curve of the 6th degree, but in two curves each of the 3rd degree, lying in *planes* which intersect in the ten lines respectively above named; so that to each of the ten points corresponds one of the ten lines: these ten points and lines are the intersections taken respectively three with three, and two with two, of *a single and unique system* of five principal planes appurtenant to every surface of the 3rd degree, and these planes are no other than those denoted by

$$u = 0, \quad v = 0, \quad w = 0, \quad p = 0, \quad q = 0.$$

* The problem thus solved may be stated as consisting in reducing the general function $ax^5 + bx^4y + cx^3y^2 + dx^2y^3 + exy^4 + fy^5$ to the form

$$(lx + my)^5 + (l'x + m'y)^5 + (l''x + m''y)^5.$$

† The coefficients in the reducing recurrent equation of the 6th degree in the process above detailed may rise to be of 541632 dimensions in respect to the original coefficients in F.

I have found also by the theory of Sub-resultants, that the analogy between lines and surfaces of the third degree, in regard to the existence of double and conical points, is preserved in this wise: that in the same way as a double point on a curve of the 3rd degree commands the existence of a double point on its Hessian, so does a conical point in a surface of the 3rd degree command over and above the 10 necessary, and so to speak natural conical points, at least one extra, that is to say an 11th conical point on *its* Hessian. And here for the present I must quit my brief and imperfect notice of this subject, composed amidst the interruptions and distractions of an official and professional life.

Observation. It may be somewhat interesting and instructive to my readers, to have a table of the successive scalar* determinants of a quintic function of two letters presented to them at a single glance. Preserving the notation above [page 193], we have the following expressions:

The given function $= u^5 + v^5 + w^5$,

its Hessian $= M^2 (a^2v^3w^3 + b^2w^3u^3 + c^2u^3v^3)$,

its post-Hessian $= M^6 \times$ the product of the *four* forms of

$$a^{\frac{3}{2}}vw + b^{\frac{3}{2}}(1)^{\frac{1}{2}}wu + c^{\frac{3}{2}}(1)^{\frac{1}{2}}uv;$$

its præter-post-Hessian $= M^{12} \times$ the product of the *nine* forms of

$$a^{\frac{4}{3}}v^{\frac{1}{3}}w^{\frac{1}{3}} + b^{\frac{4}{3}}(1)^{\frac{1}{3}}w^{\frac{1}{3}}u^{\frac{1}{3}} + c^{\frac{4}{3}}(1)^{\frac{1}{3}}u^{\frac{1}{3}}v^{\frac{1}{3}},$$

and the final determinant $= M^{20} \times$ the product of the *sixteen* forms of

$$a^{\frac{5}{4}} + (1)^{\frac{1}{4}}b^{\frac{5}{4}} + (1)^{\frac{1}{4}}c^{\frac{5}{4}}.$$

The success of the method applied depends (as above shown) upon the fact of a certain function of the roots of the post-Hessian (which is an octavic function of the variables) being known, which fact *hinges* upon the circumstance that

$$(M^6)^2 \times (M^2)^4 = M^{20}.$$

P.S. I have much pleasure in subjoining the cubical hyperdeterminant of the 12th degree function of two letters, worked out upon the principle of Compound Permutation hinted at in the foregoing pages, for which I am indebted to the kindness and skill of my friend Mr Spottiswoode.

* By which I mean the determinants in respect to ξ, η of

$$\left(\xi\frac{d}{dx} + \eta\frac{d}{dy}\right)^r F(xy).$$

The function being called

$$ax^{12} + 12bx^{11}y + \frac{12.11}{2}cx^{10}y^2 + \&c. \ldots + ly^{12},$$

the following is* its cubical hyperdeterminant:

$$\begin{aligned}
&agm - 6ahl + 15aik + 10aj^2 - 6bfm,\\
&- 24bhk + 30bgl + 20bij - 24cfl + 114cgk,\\
&- 145ci^2 + 50chj + 15cem + 20cgi + 20ch^2,\\
&- 400dgj + 280dhi + 20del + 50dfe + 10d^2k,\\
&+ 385egi - 135e^2k - 290eh^2 + 705fgh,\\
&- 330f^2i - 50g^3.
\end{aligned}$$

Mr Spottiswoode will I hope publish the work itself in the next number of the *Journal*, in which I shall also show how the hyperdeterminants of the cubical function of three letters, Aronhold's S and T, may be similarly obtained.

[* See below, p. 202.]