THEORIA NOVI MULTIPLICATORIS SYSTEMATI AEQUATIONUM DIFFERENTIALIUM VULGARIUM APPLICANDI.

§. 1.

Argumentum.

Propositurus sum sequentibus Euleriani Multiplicatoris extensionem, per totum calculum integralem uberrimi usus et frequentissimae applicationis, eamque ab amplificationibus ab ipso Eulero et Lagrange factis diversissimam. Quae amplificatio maxime nititur analogia, quam in alia Commentatione pluribus prosecutus sum, inter quotientes differentiales et Determinantia functionalia. Efficit Eulerianus Multiplicator, ut duae *duarum* variabilium functiones datae producant eiusdem functionis differentialia partialia. Respondent autem differentialibus partialibus Determinantia functionalia partialia, quae formari possunt, quoties variabilium numerus numerum functionum superat, variis eligendo modis variabiles, quarum respectu Determinans formetur. Ita, datis n functionibus $n+1$ variabilium, earum functionum dabuntur $n+1$ Determinantia partialia; veluti si f et φ trium variabilium x, y, z functiones sunt, tria earum functionum Determinantia partialia erunt

$$\frac{\partial f}{\partial y}\cdot\frac{\partial \varphi}{\partial z}-\frac{\partial f}{\partial z}\cdot\frac{\partial \varphi}{\partial y},\quad \frac{\partial f}{\partial z}\cdot\frac{\partial \varphi}{\partial x}-\frac{\partial f}{\partial x}\cdot\frac{\partial \varphi}{\partial z},\quad \frac{\partial f}{\partial x}\cdot\frac{\partial \varphi}{\partial y}-\frac{\partial f}{\partial y}\cdot\frac{\partial \varphi}{\partial x}.$$

Quibus considerationibus motus, ut Eulerianam theoriam amplificarem, generaliter Multiplicatorem examinavi, in quem ducendae essent $n+1$ functiones $n+1$ variabilium, ut producta haberi possent pro earundem n functionum Determinantibus functionalibus partialibus. Quemadmodum autem, proposita functione duarum variabilium, inter bina eius differentialia partialia intercedit aliqua conditio ex elementis nota, scilicet ut alterius differentiale secundum alteram variabilem sumtum alterius differentiali secundum alteram variabilem sumto aequale sit: ita inter illa $n+1$ Determinantia functionalia partialia inveni locum habere conditionem analogam. Singulis enim Determinantibus functionalibus partialibus respective secundum singulas variabiles differentiatis, aggregatum

$n+1$ quantitatum provenientium videbimus identice evanescere. Quod suppeditat aequationem differentialem partialem, cui Multiplicator ille satisfacere debeat, ei analogam, qua Eulerianus Multiplicator definitur. Et vice versa, sicuti in theoria Euleriana, quamcunque quantitatem, aequationi illi differentiali partiali satisfacientem, videbimus pro Multiplicatore haberi posse. Unde ad Multiplicatorem aliquem obtinendum non necessarium erit, ut illae n functiones ipsae innotescant.

Investigatio ipsius functionis duarum variabilium, cuius differentialia partialia datis functionibus proportionalia sint, pendet ab integratione completa aequationis differentialis vulgaris primi ordinis inter duas variabiles; quippe quae ea erit functio, quae Constanti arbitrariae aequalis evadit. Multiplicator autem, qui functiones datas *aequales* efficit binis differentialibus eius functionis partialibus, ipsius *aequationis differentialis* Multiplicator appellatur. Qui aequationis differentialis integratione completa sponte suppeditatur, et vice versa eius cognitione ipsa integratio maxime expeditur, videlicet ad solas revocatur Quadraturas. Similiter datis $n+1$ variabilium $n+1$ functionibus, ut obtineantur n functiones, quarum Determinantia partialia rationes easdem atque illae inter se habeant: facile patebit, integrandum esse systema n aequationum differentialium vulgarium primi ordinis, quo scilicet statuitur illarum $n+1$ variabilium differentialia esse in ratione ipsarum $n+1$ quantitatum propositarum. Quo complete integrato, functiones, quae Constantibus arbitrariis a se independentibus aequales evadunt, ipsae erunt n functiones quaesitae. Atque Multiplicatorem, qui $n+1$ quantitates datas Determinantibus earum functionum partialibus aequales efficit, per analogiam illius *systematis aequationum differentialium vulgarium Multiplicatorem* appello. Iam quidem complete integrato systemate aequationum differentialium vulgarium, eius facile innotescit Multiplicator; quippe ad quem inveniendum tantum opus est, ut functionum Constantibus arbitrariis aequalium, quae per integrationem completam constant, unum aliquod formetur Determinans partiale. At vice versa, cognito aliquo systematis aequationum differentialium Multiplicatore, sive, quod idem est, cognita aliqua solutione aequationis differentialis partialis, qua Multiplicator definitur, non ita patebat, utrum et quodnam inde commodum vel auxilium ad integrandum systema peti posset, ita ut nostri Multiplicatoris analogia cum Euleriano videretur in ea ipsa re deficere, qua propter olim Eulerus sui Multiplicatoris theoriam condidit. Contigit tandem usum introspicere plane singularem, quem in integrando aequationum differentialium systemate e Multiplicatoris cognitione percipere liceat, quod scilicet eius ope non prima aliqua, sed omnium ultima integratio ad Quadraturas revocetur. Hinc in theoria integrationis aequationum differentialium vulgarium novus disquisitionum aperitur campus, videlicet ultimas investigandi integrationes, dum primae non innotescunt. Quippe in vastis et luculentissimis problematis per theoriam hic propositam fit, ut ultima generaliter absolvatur integratio, dum in casibus tantum particularibus Integralia prima invenire licet.

Capite primo examinabo Multiplicatoris nostri varias formas insignioresque proprietates. In altero Capite eius monstrabo usum in integrando aequationum differentialium vulgarium systemate. In Capite tertio theoriam Multiplicatoris extendam ad systemata aequationum differentialium vulgarium cuiuslibet ordinis. In Commentationibus deinde subsequentibus mihi propositum est praecepta hic tradita variis illustrare applicationibus; e quibus est principium novum mechanicum latissime patens, nuper a me sine demonstratione divulgatum.

Caput primum.

Novi Multiplicatoris definitio et variae proprietates.

§. 2.

Lemma fundamentale eiusque varii usus; de Determinantibus functionalibus partialibus.

Aequatione inter variabiles x et y proposita

$$f(x, y) = \text{const.},$$

obtinetur differentialium dx et dy ratio

$$dx : dy = \frac{\partial f}{\partial y} : -\frac{\partial f}{\partial x} \text{*)}.$$

Si de hac ratione differentialium dx et dy sola agitur, in dextra parte aequationis antecedentis omittere licet differentialium partialium $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ factorem vel denominatorem, si quo afficiuntur, communem. Ubi vero pro quantitatibus, quae differentialibus dx et dy proportionales evadunt, ipsa sumere placet $\frac{\partial f}{\partial y}$ et $-\frac{\partial f}{\partial x}$ vel $-\frac{\partial f}{\partial y}$ et $\frac{\partial f}{\partial x}$, qualia differentiatione partiali prodeunt, nullo

*) Differentialia vulgaria ut in aliis Commentationibus charactere d, partialia charactere ∂ denoto.

factore aut denominatore communi rejecto, eam conditionem formula analytica exprimi posse constat.

Videlicet si quantitas ipsi dx proportionalis differentiatur ipsius x respectu, quantitas ipsi dy proportionalis differentiatur ipsius y respectu, quantitatum differentiatione provenientium summa identice evanescere debet. Theorema simile ad plures variabiles valet.

Aequationibus enim inter x, y, z propositis

$$f(x, y, z) = \text{Const.}, \quad \varphi(x, y, z) = \text{Const.},$$

obtinetur differentiando

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0,$$

$$\frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy + \frac{\partial \varphi}{\partial z} dz = 0.$$

E quibus aequationibus eruuntur differentialium dx, dy, dz rationes

$$dx : dy : dz = A : B : C,$$

siquidem ponitur

$$A = \frac{\partial f}{\partial y} \cdot \frac{\partial \varphi}{\partial z} - \frac{\partial f}{\partial z} \cdot \frac{\partial \varphi}{\partial y},$$

$$B = \frac{\partial f}{\partial z} \cdot \frac{\partial \varphi}{\partial x} - \frac{\partial f}{\partial x} \cdot \frac{\partial \varphi}{\partial z},$$

$$C = \frac{\partial f}{\partial x} \cdot \frac{\partial \varphi}{\partial y} - \frac{\partial f}{\partial y} \cdot \frac{\partial \varphi}{\partial x}.$$

Si tantum de rationibus differentialium dx, dy, dz agitur, factorem vel denominatorem communem quantitatum A, B, C, si quo afficiuntur, omittere licet. Ubi vero pro quantitatibus, quae differentialibus dx, dy, dz proportionales evadunt, ipsa sumere placet A, B, C, nullo factore vel denominatore communi rejecto, eam conditionem aliqua formula analytica exprimi posse videbimus. Fit enim

$$\frac{\partial A}{\partial x} = \frac{\partial \varphi}{\partial z} \cdot \frac{\partial^2 f}{\partial y \partial x} + \frac{\partial f}{\partial y} \cdot \frac{\partial^2 \varphi}{\partial z \partial x} - \frac{\partial \varphi}{\partial y} \cdot \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial f}{\partial z} \cdot \frac{\partial^2 \varphi}{\partial y \partial x},$$

$$\frac{\partial B}{\partial y} = \frac{\partial \varphi}{\partial x} \cdot \frac{\partial^2 f}{\partial z \partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial \varphi}{\partial z} \cdot \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial f}{\partial x} \cdot \frac{\partial^2 \varphi}{\partial z \partial y},$$

$$\frac{\partial C}{\partial z} = \frac{\partial \varphi}{\partial y} \cdot \frac{\partial^2 f}{\partial x \partial z} + \frac{\partial f}{\partial x} \cdot \frac{\partial^2 \varphi}{\partial y \partial z} - \frac{\partial \varphi}{\partial x} \cdot \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial f}{\partial y} \cdot \frac{\partial^2 \varphi}{\partial x \partial z}.$$

Quae expressiones additae sese mutuo destruunt, unde eruitur

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} = 0,$$

hoc est, si quantitatem ipsi dx proportionalem ipsius x respectu, quantitatem ipsi dy proportionalem ipsius y respectu, quantitatem ipsi dz proportionalem ipsius z respectu differentiamus, trium quantitatum differentiatione provenientium summa identice evanescere debet. Quae conditio prorsus analoga est ei, quae antecedentibus de duabus variabilibus tradita est atque e primis elementis constat. Antecedentia ad numerum variabilium quemcunque extendere licet, siquidem advocantur propositiones, quas in *Diario* Crell. Vol. XXII. [Cf. Vol. III. h. ed. pag. 355 et 393] de Determinantibus algebraicis et functionalibus tradidi et quarum per totam hanc Commentationem usum frequentissimum faciam. Habetur enim sequens

Lemma fundamentale:

„*Sint* A, A_1, A_2, ..., A_n *quantitates, quae in Determinante functionali*

$$\Sigma \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n}$$

respective per $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial x_1}$, $\frac{\partial f}{\partial x_2}$, ..., $\frac{\partial f}{\partial x_n}$ *multiplicatae reprehenduntur, erit*

$$\frac{\partial A}{\partial x} + \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \cdots + \frac{\partial A_n}{\partial x_n} = 0.\text{“}$$

Demonstratio.

Secundum definitionem quantitatum A, A_1 etc. fit

$$\Sigma \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n} = \frac{\partial f}{\partial x} A + \frac{\partial f}{\partial x_1} A_1 + \frac{\partial f}{\partial x_2} A_2 + \cdots + \frac{\partial f}{\partial x_n} A_n.$$

Unde Lemma demonstratu propositum sic quoque exhibere licet:

$$\Sigma \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n} = \frac{\partial (fA)}{\partial x} + \frac{\partial (fA_1)}{\partial x_1} + \frac{\partial (fA_2)}{\partial x_2} + \cdots + \frac{\partial (fA_n)}{\partial x_n}.$$

Facio, hanc formulam iam demonstratam esse pro $n-1$ functionibus n variabilium, probabo Lemma ad n functiones $n+1$ variabilium valere.

Designo per (i, k) quantitatem, quae in Determinante functionali $\Sigma \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n}$ multiplicata reprehenditur per factorem

$$\frac{\partial f}{\partial x_i} \cdot \frac{\partial f_1}{\partial x_k}.$$

Constat autem per Determinantium proprietates iam olim ab Ill°. Laplace adnotatas, *bina Aggregata, in Determinante functionali proposito resp. per*

$\frac{\partial f}{\partial x_i}\cdot\frac{\partial f_m}{\partial x_k}$ *et per* $\frac{\partial f}{\partial x_k}\cdot\frac{\partial f_m}{\partial x_i}$ *multiplicata, valoribus oppositis gaudere.* Unde sequitur

$$(i,\ k)=-(k,\ i)\quad\text{sive}\quad (i,\ k)+(k,\ i)=0.$$

Est A_i complexus terminorum eius Determinantis, qui per $\frac{\partial f}{\partial x_i}$ multiplicantur, unde fit

$$A_i=\frac{\partial f_1}{\partial x}(i,\ 0)+\frac{\partial f_1}{\partial x_1}(i,\ 1)+\frac{\partial f_1}{\partial x_2}(i,\ 2)+\cdots+\frac{\partial f_1}{\partial x_n}(i,\ n),$$

qua in formula ipsum $(i,\ i)$ aut omittendum aut $=0$ ponendum est. Est porro A_i Determinans functionum $f_1,\ f_2,\ \dots,\ f_n$ formatum respectu variabilium $x,\ x_1,\ x_2,\ \dots,\ x_{i-1},\ x_{i+1},\ \dots,\ x_n$ atque sunt $(i,\ 0)$, $(i,\ 1)$, etc. quantitates, quae in Determinante functionali A_i multiplicatae reprehenduntur per $\frac{\partial f_1}{\partial x}$, $\frac{\partial f_1}{\partial x_1}$, etc. Unde si Lemma propositum ad $n-1$ functiones n variabilium valet, erit pro indicis i valoribus $0,\ 1,\ 2,\ \dots,\ n$

$$\frac{\partial(i,0)}{\partial x}+\frac{\partial(i,1)}{\partial x_1}+\cdots+\frac{\partial(i,n)}{\partial x_n}=0,$$

ideoque etiam

$$A_1=\frac{\partial[f_1\cdot(i,0)]}{\partial x}+\frac{\partial[f_1\cdot(i,1)]}{\partial x_1}+\cdots+\frac{\partial[f_1\cdot(i,n)]}{\partial x_n}.$$

Quae formula pro quolibet ipsius i valore $0,\ 1,\ 2,\ \dots,\ n$ valet. Iam generaliter observo, *quoties ponatur*

$$H_i=\frac{\partial a_{i,0}}{\partial x}+\frac{\partial a_{i,1}}{\partial x_1}+\cdots+\frac{\partial a_{i,n}}{\partial x_n},$$

designantibus $a_{i,k}$ *quantitates quascunque, pro quibus sit*

$$a_{i,k}+a_{k,i}=0,\quad a_{i,i}=0,$$

fieri

$$\frac{\partial H}{\partial x}+\frac{\partial H_1}{\partial x_1}+\frac{\partial H_2}{\partial x_2}+\cdots+\frac{\partial H_n}{\partial x_n}=0.$$

Bina enim differentialia inter se juncta

$$\frac{\partial\frac{\partial a_{i,k}}{\partial x_k}}{\partial x_i}+\frac{\partial\frac{\partial a_{k,i}}{\partial x_i}}{\partial x_k}$$

mutuo destruuntur, unde totam expressionem $\frac{\partial H}{\partial x}+\frac{\partial H_1}{\partial x_1}+\cdots+\frac{\partial H_n}{\partial x_n}$ identice evanescere invenis. Ponendo autem $f_1\cdot(i,\ k)=a_{i,k}$, satisfit conditioni $a_{i,k}=-a_{k,i}$, porro fit $H_i=A_i$; ideoque

$$\frac{\partial A}{\partial x}+\frac{\partial A_1}{\partial x_1}+\cdots+\frac{\partial A_n}{\partial x_n}=0,$$

sive Lemma de n functionibus $n+1$ variabilium justum erit, dummodo de $n-1$ functionibus n variabilium locum habet. Unde tantum necesse est, ut Lemma pro una functione duarum variabilium constet. Pro una autem functione f_1 duarum variabilium x et y abeunt quantitates A etc. in differentialia partialia $\frac{\partial f_1}{\partial y}$ et $-\frac{\partial f_1}{\partial x}$, ideoque Lemma redit in formulam

$$\frac{\partial\frac{\partial f_1}{\partial x}}{\partial y}-\frac{\partial\frac{\partial f_1}{\partial y}}{\partial x}=0,$$

quae est differentialium partialium proprietas fundamentalis supra commemorata.

Lemma generale etiam directe demonstrari potest absque illa reductione numeri n ad numerum $n-1$. Nam cum A_i vacet differentialibus, ipsius x_i respectu sumtis, e quantitatibus $\frac{\partial A_i}{\partial x_i}$ nulla implicare potest differentialia bis secundum eandem variabilem sumta. Differentialia autem secunda, secundum variabiles diversas x_i et x_k sumta, non provenire possunt nisi e solis duobus terminis

$$\frac{\partial A_i}{\partial x_i}+\frac{\partial A_k}{\partial x_k}:$$

Unde ad probandum Lemma propositum sufficit ut demonstretur, in Aggregato $\frac{\partial A_i}{\partial x_i}+\frac{\partial A_k}{\partial x_k}$ se mutuo destruere terminos per quantitates $\frac{\partial^2 f_m}{\partial x_i\partial x_k}$ multiplicatos. Quod facile patet. Ponamus enim

$$A_i=\alpha_1\frac{\partial f_1}{\partial x_k}+\alpha_2\frac{\partial f_2}{\partial x_k}+\cdots+\alpha_n\frac{\partial f_n}{\partial x_k},$$

fit secundum Determinantium proprietatem, in priore demonstratione in usum vocatam,

$$A_k=-\left\{\alpha_1\frac{\partial f_1}{\partial x_i}+\alpha_2\frac{\partial f_2}{\partial x_i}+\cdots+\alpha_n\frac{\partial f_n}{\partial x_i}\right\}.$$

Quantitates α_1, α_2, etc. neque differentialibus secundum x_i sumtis, neque differentialibus secundum x_k sumtis afficiuntur. Unde substituendo ipsarum A_i et A_k expressiones antecedentes, de Aggregato

$$\frac{\partial A_i}{\partial x_i}+\frac{\partial A_k}{\partial x_k}$$

prorsus exulant differentialia secunda, secundum variabiles x_i et x_k sumta, terminis binis

$$+\alpha_m\frac{\partial^2 f_m}{\partial x_k\partial x_i}-\alpha_m\frac{\partial^2 f_m}{\partial x_i\partial x_k}$$

se mutuo destruentibus. Erant autem inter omnes terminos Aggregati propositi

$$\frac{\partial A}{\partial x}+\frac{\partial A_1}{\partial x_1}+\frac{\partial A_2}{\partial x_2}+\cdots+\frac{\partial A_n}{\partial x_n}$$

soli termini $\frac{\partial A_i}{\partial x_i}+\frac{\partial A_k}{\partial x_k}$, qui affici possint differentialibus $\frac{\partial^2 f_m}{\partial x_i\partial x_k}$, unde in Aggregato proposito termini differentialibus secundis secundum x_i et x_k sumtis affecti se mutuo destruunt. Unde, cum x_i et x_k binae quaecunque variabiles esse possint a se diversae, illud Aggregatum totum evanescit. Q. d. e.

Quoties numerus variabilium, quas datae functiones $f_1, f_2, \ldots, f_n$ implicant, ipsum functionum numerum n superat, proponi potest, earum functionum Determinantia respectu quarumque n variabilium formare. Quae vocabo functionum $f_1, f_2, \ldots, f_n$ *Determinantia partialia* secundum analogiam denominationis de differentialibus usitatae.

Si numerus variabilium est $n+1$ sicuti antecedentibus, erit numerus Determinantium functionalium partialium $n+1$; si numerus variabilium est $n+2$, dabuntur $\frac{1}{2}(n+2)(n+1)$ Determinantia functionalia partialia, et ita porro. Eorum Determinantium functionalium partialium signa cum in arbitrio posita sint, casu, quo variabilium numerus numerum functionum tantum unitate superat, supponam, signa omnium Determinantium ab eorum uno ita pendere, ut binorum Determinantium partialium alterum de altero deducatur, in signis differentialibus binarum variabilium independentium commutatione facta, omnium simul terminorum mutatis signis. Quem invenis esse habitum quantitatum $A, A_1, \ldots, A_n$, quae sunt functionum $f_1, f_2, \ldots, f_n$ Determinantia partialia. Videlicet de uno

$$A=\Sigma\pm\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}$$

deducitur $-A_i$, loco ipsorum

$$\frac{\partial f_1}{\partial x_i},\ \frac{\partial f_2}{\partial x_i},\ \ldots,\ \frac{\partial f_n}{\partial x_i}$$

respective scribendo

$$\frac{\partial f_1}{\partial x},\ \frac{\partial f_2}{\partial x},\ \ldots,\ \frac{\partial f_n}{\partial x}.$$

Pro una duarum variabilium x et y functione f_1 abibunt Determinantia partialia in differentialia partialia functionis f_1, alterum positivo alterum negativo signo sumtum,

$$\frac{\partial f_1}{\partial y},\ -\frac{\partial f_1}{\partial x}\quad\text{vel}\quad-\frac{\partial f_1}{\partial y},\ \frac{\partial f_1}{\partial x}.$$

Et quemadmodum inter differentialia partialia $\frac{\partial f_1}{\partial x}$ et $\frac{\partial f_1}{\partial y}$ locum habet formula fundamentalis

$$\frac{\partial\frac{\partial f_1}{\partial y}}{\partial x}-\frac{\partial\frac{\partial f_1}{\partial x}}{\partial y}=0,$$

ita, $n+1$ variabilium $x, x_1, x_2, \ldots, x_n$ propositis n functionibus $f_1, f_2, \ldots, f_n$, Lemmate antecedente constituitur inter Determinantia partialia $A, A_1, A_2, \ldots, A_n$ aequatio conditionalis fundamentalis

$$\frac{\partial A}{\partial x}+\frac{\partial A_1}{\partial x_1}+\frac{\partial A_2}{\partial x_2}+\cdots+\frac{\partial A_n}{\partial x_n}=0.$$

Quod igitur Lemma gravissimam manifestat analogiam Determinantium functionalium et quotientium differentialium partialium.

Lemma traditum dedi olim in Commentatione, *Vol. VI. Diar.* Crell. *pag.* 263 *sqq. inserta*, „*De resolutione aequationum per series infinitas.*“ Quod eo loco adhibui ad demonstrandam Propositionem, quae et ipsa luculentam analogiam Determinantium functionalium cum differentialibus constituit. Nam cum pateat seriei e solis variabilis x potestatibus conflatae quotientem differentialem vacare termino $\frac{1}{x}$, demonstravi, *serierum $f, f_1, \ldots, f_n$, conflatarum e solis variabilium $x, x_1, \ldots, x_n$ potestatibus, Determinans functionale*

$$\Sigma\pm\frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}$$

vacare termino $\frac{1}{xx_1x_2\ldots x_n}$. Quippe Determinans antecedens per Lemma nostrum aequatur quantitati

$$\frac{\partial(fA)}{\partial x}+\frac{\partial(fA_1)}{\partial x_1}+\cdots+\frac{\partial(fA_n)}{\partial x_n},$$

cuius terminus primus evolutus vacare debet termino in $\frac{1}{x}$ ducto, secundus termino in $\frac{1}{x_1}$ ducto, et ita porro, ita ut in tota quantitate evoluta non obvenire possit terminus $\frac{1}{xx_1x_2\dots x_n}$.

Quae propositio adhiberi potest ad amplificandam theoriam Cauchyanam residuorum dictam, eiusque ope radices systematis simultanei aequationum in series infinitas evolvi, quod in Commentatione citata videas.

Data occasione breviter adhuc innuam usum Lemmatis propositi in integralibus multiplicibus inter datos limites determinandis. Proponatur integrale multiplex

$$\int U df\, df_1 \dots df_n,$$

ponamusque limites, inter quos integratio afficienda sit, eo definiri, quod introducendo certas alias variabiles $x, x_1, \dots, x_n$ pro variabilibus independentibus, harum novarum variabilium limites a se invicem independentes sive constantes sint. Constat, novis variabilibus exhibitum integrale propositum fore

$$\int U df\, df_1 \dots df_n = \int U\left(\Sigma\pm\frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdots\frac{\partial f_n}{\partial x_n}\right) dx\, dx_1\dots dx_n.$$

Variabilibus propositis $f, f_1, \dots, f_n$ expressa U integrataque ipsius f respectu, prodeat Π, ita ut sit

$$\Pi=\int U df,\quad U=\frac{\partial \Pi}{\partial f},$$

erit

$$U\Sigma\pm\frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdots\frac{\partial f_n}{\partial x_n}=\Sigma\pm\frac{\partial \Pi}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdots\frac{\partial f_n}{\partial x_n}.$$

Quod patet substituendo valores

$$\frac{\partial \Pi}{\partial x_i}=\frac{\partial \Pi}{\partial f}\cdot\frac{\partial f}{\partial x_i}+\frac{\partial \Pi}{\partial f_1}\cdot\frac{\partial f_1}{\partial x_i}+\cdots+\frac{\partial \Pi}{\partial f_n}\cdot\frac{\partial f_n}{\partial x_i},$$

et observando, post substitutionem factam evanescere quantitates omnes in

$$\frac{\partial \Pi}{\partial f_1},\ \frac{\partial \Pi}{\partial f_2},\ \dots,\ \frac{\partial \Pi}{\partial f_n}$$

ductas. Fit autem e Lemmate proposito

$$\Sigma\pm\frac{\partial \Pi}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}=\frac{\partial(\Pi A)}{\partial x}+\frac{\partial(\Pi A_1)}{\partial x_1}+\cdots+\frac{\partial(\Pi A_n)}{\partial x_n}.$$

Unde eruitur formula reductionis

$$\int U df\, df_1\dots df_n$$
$$=\int[\Pi A]dx_1 dx_2\dots dx_n+\int[\Pi A_1]dx\, dx_2\dots dx_n+\cdots+\int[\Pi A_n]dx\, dx_1\dots dx_{n-1}.$$

Hic signo $[\Pi A_i]$ denoto, in functionibus $f, f_1, \dots, f_n$ ipsi x_i substituendos esse binos eius limites constantes, binasque expressiones ipsius ΠA_i provenientes alteram de altera detrahendas esse. Hinc integrale $(n+1)$-tuplex propositum videmus revocari ad $2n+2$ integralia n-tuplicia. Quae singula eadem quidem formula exhiberi possunt

$$\int \Pi df_1 df_2\dots df_n\text{*)},$$

sed pro singulis erit Π diversa ipsarum $f_1, f_2, \dots, f_n$ functio, limitesque ipsarum $f_1, f_2, \dots, f_n$ diversi erunt. Singula deinde integralia n-tuplicia eadem methodo ad $2n$ integralia $(n-1)$-tuplicia revocari possunt, eaque ratione pergere licet, usque dum tota integratio inter limites propositos perfecta sit.

Lemma traditum sub alia quoque forma proponi potest memoratu digna. Habeamus enim $x, x_1, \dots, x_n$ pro ipsarum $f, f_1, \dots, f_n$ functionibus, earumque quaeramus differentialia partialia, ipsius f respectu sumta. Quae per regulas notas inveniuntur

$$\frac{\partial x}{\partial f}=\frac{A}{R},\quad \frac{\partial x_1}{\partial f}=\frac{A_1}{R},\quad \dots,\quad \frac{\partial x_n}{\partial f}=\frac{A_n}{R},$$

siquidem R est Determinans propositum

$$R=\Sigma\pm\frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdots\frac{\partial f_n}{\partial x_n}.$$

Hinc formula nostra

$$\frac{\partial A}{\partial x}+\frac{\partial A_1}{\partial x_1}+\cdots+\frac{\partial A_n}{\partial x_n}=0,$$

si reputamus esse

*) Habendo enim x pro Constante, fit

$$\int \Pi A\, dx_1 dx_2\dots dx_n=\int \Pi df_1 df_2\dots df_n,$$

cum sit

$$A=\Sigma\pm\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n},$$

et similis formula pro reliquis integralibus valet.

$$\frac{\partial R}{\partial f} = \frac{\partial R}{\partial x}\cdot\frac{\partial x}{\partial f} + \frac{\partial R}{\partial x_1}\cdot\frac{\partial x_1}{\partial f} + \cdots + \frac{\partial R}{\partial x_n}\cdot\frac{\partial x_n}{\partial f},$$

formam induit sequentem:

$$0 = \frac{\partial R}{\partial f} + R\left\{\frac{\partial\frac{\partial x}{\partial f}}{\partial x} + \frac{\partial\frac{\partial x_1}{\partial f}}{\partial x_1} + \cdots + \frac{\partial\frac{\partial x_n}{\partial f}}{\partial x_n}\right\}$$

sive

$$0 = \frac{\partial \log R}{\partial f} + \frac{\partial\frac{\partial x}{\partial f}}{\partial x} + \frac{\partial\frac{\partial x_1}{\partial f}}{\partial x_1} + \cdots + \frac{\partial\frac{\partial x_n}{\partial f}}{\partial x_n}.$$

In his formulis supponitur, ipsas R, x, x_1, ..., x_n primum pro quantitatum f, f_1, ..., f_n functionibus haberi omnesque secundum f differentiari; deinde differentialia partialia $\frac{\partial x}{\partial f}$, $\frac{\partial x_1}{\partial f}$, etc. rursus per ipsas x, x_1, ..., x_n exprimi, et respective secundum x, x_1, ..., x_n differentiari. Commutando quantitates x, x_1, etc. cum quantitatibus f, f_1, etc., formula antecedens in aliam abit, quam in *Diar.* Crell. Vol. XXII. pag. 336 [Conf. Vol. III. h. ed. p. 412] demonstravi.

§. 3.

Novi Multiplicatoris definitio. Aequatio differentialis partialis, cui satisfacit. Variae formae, quas Multiplicatoris valor induere potest.

Sint X, X_1, ..., X_n variabilium x, x_1, ..., x_n functiones quaecunque non simul omnes identice evanescentes; proposita aequatione differentiali partiali lineari primi ordinis

$$0 = X\frac{\partial f}{\partial x} + X_1\frac{\partial f}{\partial x_1} + \cdots + X_n\frac{\partial f}{\partial x_n},$$

solutiones ejus exstant n a se invicem independentes. Quarum Determinantia partialia erunt inter se ut Coëfficientes aequationis differentialis partialis propositae X, X_1, ..., X_n. Solutionibus enim illis a se independentibus vocatis

$$f_1,\ f_2,\ \ldots,\ f_n,$$

habentur aequationes identicae

$$0 = X\frac{\partial f_1}{\partial x} + X_1\frac{\partial f_1}{\partial x_1} + \cdots + X_n\frac{\partial f_1}{\partial x_n},$$

$$0 = X\frac{\partial f_2}{\partial x} + X_1\frac{\partial f_2}{\partial x_1} + \cdots + X_n\frac{\partial f_2}{\partial x_n},$$

. . . .

$$0 = X\frac{\partial f_n}{\partial x} + X_1\frac{\partial f_n}{\partial x_1} + \cdots + X_n\frac{\partial f_n}{\partial x_n},$$

quae sunt n aequationes lineares inter $n+1$ quantitates X, X_1, ..., X_n, terminis carentes constantibus. Quibus aequationibus determinantur rationes, quas ipsae X, X_1, etc. inter se tenent. Videlicet per regulas notas algebraicas invenitur, ipsas X, X_1, ..., X_n esse inter se ut quantitates A, A_1, ..., A_n, §. pr. consideratas, quae erant complexus terminorum, in Determinante functionali

$$\Sigma \pm \frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}$$

respective per $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial x_1}$, ..., $\frac{\partial f}{\partial x_n}$ multiplicatorum, sive functionum f_1, f_2, ..., f_n Determinantia partialia. Sit M factor, per quem Coëfficientes X, X_1, ..., X_n multiplicati ipsa producant Determinantia partialia A, A_1, ..., A_n, ita ut fiat:

$$(1)\quad MX = A,\quad MX_1 = A_1,\quad \ldots,\quad MX_n = A_n.$$

Posito

$$R = \Sigma \pm \frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdots\frac{\partial f_n}{\partial x_n},$$

cum habeatur

$$R = A\frac{\partial f}{\partial x} + A_1\frac{\partial f}{\partial x_1} + \cdots + A_n\frac{\partial f}{\partial x_n},$$

sequitur

$$(2)\quad R = M\left(X\frac{\partial f}{\partial x} + X_1\frac{\partial f}{\partial x_1} + \cdots + X_n\frac{\partial f}{\partial x_n}\right).$$

Iisdem substitutis formulis (1), Lemma §. pr. demonstratum in hanc formulam abit:

$$(3)\quad 0 = \frac{\partial(MX)}{\partial x} + \frac{\partial(MX_1)}{\partial x_1} + \cdots + \frac{\partial(MX_n)}{\partial x_n}.$$

Habemus igitur Propositionem sequentem, qua Multiplicatoris M continetur definitio.

Propositio.

„Proponatur expressio

$$X\frac{\partial f}{\partial x} + X_1\frac{\partial f}{\partial x_1} + \cdots + X_n\frac{\partial f}{\partial x_n},$$

in qua sint X, X_1, ..., X_n datae variabilium x, x_1, ..., x_n functiones: functionibus f_1, f_2, ..., f_n rite determinatis, ipsa f autem indeterminata manente, semper exstabit factor M, per quem multiplicata expressio proposita formam induat Determinantis functionalis

$$M\left(X\frac{\partial f}{\partial x} + X_1\frac{\partial f}{\partial x_1} + \cdots + X_n\frac{\partial f}{\partial x_n}\right) = \Sigma \pm \frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n},$$

isque Multiplicator satisfaciet aequationi differentiali partiali

$$0 = \frac{\partial(MX)}{\partial x} + \frac{\partial(MX_1)}{\partial x_1} + \cdots + \frac{\partial(MX_n)}{\partial x_n}.$$

E valoribus ipsius M in sequentibus perpetuo excludo valorem $M = 0$. Quem patet satisfacere aequationi (2), qua Multiplicator definitur, dummodo statuatur functionum $f_1, f_2, \ldots, f_n$ unam reliquarum functionem esse; constat enim Determinans functionale evanescere, si functiones propositae non a se invicem sint independentes. Illo autem ipsius M valore excluso, Propositio antecedens inverti potest. Videlicet, *si Multiplicator M definitur conditione, ut pro functione indefinita f expressio*

$$M\left(X\frac{\partial f}{\partial x} + X_1\frac{\partial f}{\partial x_1} + \cdots + X_n\frac{\partial f}{\partial x_n}\right)$$

evadat Determinans functionale

$$R = \Sigma \pm \frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdots\frac{\partial f_n}{\partial x_n},$$

functiones $f_1, f_2, \ldots, f_n$ necessario erunt solutiones a se independentes aequationis differentialis partialis linearis

$$X\frac{\partial f}{\partial x} + X_1\frac{\partial f}{\partial x_1} + \cdots + X_n\frac{\partial f}{\partial x_n} = 0.$$

Nam pro ipsa f, quae erat functio indefinita, sumendo aliquam functionum $f_1, f_2, \ldots, f_n$, identice evanescit Determinans R. Quod cum supponatur aequale expressioni

$$M\left(X\frac{\partial f}{\partial x} + X_1\frac{\partial f}{\partial x_1} + \cdots + X_n\frac{\partial f}{\partial x_n}\right),$$

atque factor M a nihilo diversus statuatur, fieri debet ut, substituendo ipsi f functiones $f_1, f_2, \ldots, f_n$, identice habeatur

$$X\frac{\partial f}{\partial x} + X_1\frac{\partial f}{\partial x_1} + \cdots + X_n\frac{\partial f}{\partial x_n} = 0,$$

sive ut $f_1, f_2, \ldots, f_n$ ipsae sint aequationis differentialis partialis propositae solutiones. Eruntque solutiones illae $f_1, f_2, \ldots, f_n$ a se invicem independentes; si enim una reliquarum functio esset, Determinans R identice evanesceret pro functione f indefinita; unde etiam pro functione indefinita f evanescere deberet expressio

$$X\frac{\partial f}{\partial x} + X_1\frac{\partial f}{\partial x_1} + \cdots + X_n\frac{\partial f}{\partial x_n},$$

quod fieri non potest, nisi omnes X, X_1, etc. simul identice evanescunt.

Datis functionibus $f_1, f_2, \ldots, f_n$, una quaelibet ex aequationum (1) numero ad definiendum Multiplicatorem sufficit, veluti aequatio

$$MX = A = \Sigma \pm \frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n},$$

e qua sequitur

$$(4)\qquad M = \frac{1}{X}\,\Sigma \pm \frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}.$$

Qua tamen formula ut definiatur Multiplicator aequationis differentialis partialis propositae, addenda conditio est, ut X et A non evanescant.

Pro duabus variabilibus x et x_1 Multiplicator antecedentibus definitus cum Euleriano convenit. Sint enim X, X_1 datae variabilium x et x_1 functiones, atque proponatur aequatio differentialis primi ordinis inter x et x_1

$$X dx_1 - X_1 dx = 0.$$

Est Multiplicator Eulerianus eiusmodi factor M, per quem multiplicata pars laeva aequationis antecedentis abit in differentiale completum functionis alicuius f_1, ita ut sit

$$df_1 = \frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial x_1}dx_1 = M(X dx_1 - X_1 dx),$$

sive

$$MX = \frac{\partial f_1}{\partial x_1},\quad MX_1 = -\frac{\partial f_1}{\partial x}.$$

E quibus formulis sequitur, pro functione indefinita f induere expressionem

$$M\left(X\frac{\partial f}{\partial x} + X_1\frac{\partial f}{\partial x_1}\right)$$

formam Determinantis functionalis

$$\frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1} - \frac{\partial f}{\partial x_1}\cdot\frac{\partial f_1}{\partial x},$$

et Multiplicatorem M satisfacere aequationi differentiali partiali

$$\frac{\partial(MX)}{\partial x} + \frac{\partial(MX_1)}{\partial x_1} = 0.$$

Quae pro duabus variabilibus independentibus sunt eaedem proprietates characteristicae, quas Multiplicatori generali assignavi.

Problema solvendi aequationem differentialem partialem propositam

$$X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}+\cdots+X_n\frac{\partial f}{\partial x_n}=0$$

cum duobus aliis problematis arctissime coniunctum est. Designante enim Π quamcunque aequationis praecedentis solutionem, ex aequatione

$$\Pi=0$$

petatur ipsius x expressio per reliquas variabiles $x_1, x_2, \ldots, x_n$: notum est, eam fieri solutionem alterius aequationis differentialis partialis

$$X=X_1\frac{\partial x}{\partial x_1}+X_2\frac{\partial x}{\partial x_2}+\cdots+X_n\frac{\partial x}{\partial x_n}.$$

Unde haec aequatio differentialis partialis ad aequationem differentialem partialem propositam revocari potest. Porro ad aequationis differentialis partialis propositae solutionem constat revocari posse integrationem completam systematis aequationum differentialium vulgarium primi ordinis inter $n+1$ variabiles x, $x_1, \ldots, x_n$, quod repraesentemus proportionibus

$$dx:dx_1:\ldots:dx_n=X:X_1:\ldots:X_n.$$

Videlicet, si aequationis differentialis partialis propositae solutiones, a se independentes, sunt $f_1, f_2, \ldots, f_n$, obtinentur aequationes, quibus illud aequationum differentialium vulgarium systema complete integratur, aequando solutiones illas Constantibus arbitrariis. Et vice versa, si ex aequationibus integralibus completis petuntur variabilium functiones Constantibus arbitrariis a se independentibus aequales, ab iisdemque Constantibus arbitrariis ipsae vacuae: hae functiones erunt aequationis differentialis partialis propositae solutiones a se independentes. Propter hunc trium problematum consensum Multiplicatorem M ad tria illa problemata perinde refero. Qua de re *ipsum M perinde appellabo Multiplicatorem huius aequationis differentialis partialis*

$$X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}+\cdots+X_n\frac{\partial f}{\partial x_n}=0,$$

vel huius

$$0=X-X_1\frac{\partial x}{\partial x_1}-X_2\frac{\partial x}{\partial x_2}-\cdots-X_n\frac{\partial x}{\partial x_n},$$

vel etiam systematis aequationum differentialium vulgarium

$$dx:dx_1:dx_2:\ldots:dx_n=X:X_1:X_2:\ldots:X_n.$$

Ubi ad has refertur Multiplicator, quod plerumque usu venit, pro variis

formis, quibus earum aequationes integrales completae proponuntur, variae obtinentur Multiplicatoris repraesentationes. Quas sequentibus exponam.

Si aequationes integrales proponuntur ipsa forma, cuius modo mentionem iniecimus,

$$(5)\quad f_1=\alpha_1,\quad f_2=\alpha_2,\quad \ldots,\quad f_n=\alpha_n,$$

designantibus α_1 etc. Constantes arbitrarias, functiones f_1 etc. non afficientes, ideoque $f_1, f_2, \ldots, f_n$ solutiones a se independentes aequationis

$$X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}+\cdots+X_n\frac{\partial f}{\partial x_n}=0,$$

erat Multiplicator

$$(6)\quad M=\frac{1}{X}\,\Sigma\pm\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}.$$

Iam vero proponantur aequationes integrales completae hac forma maxime usitata, ut variabiles omnes per earum unam, veluti x, et Constantes arbitrarias exprimantur:

$$(7)\quad x_1=\varphi_1(x),\quad x_2=\varphi_2(x),\quad \ldots,\quad x_n=\varphi_n(x),$$

functionibus φ_1, φ_2, etc. involventibus praeter variabilem x Constantes arbitrarias α_1 etc., erit

$$(8)\quad \Sigma\pm\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}=\frac{1}{\Sigma\pm\frac{\partial\varphi_1}{\partial\alpha_1}\cdot\frac{\partial\varphi_2}{\partial\alpha_2}\cdots\frac{\partial\varphi_n}{\partial\alpha_n}},$$

D. F. §. 9 (3)*). Unde fit

$$(9)\quad M=\frac{1}{X\Sigma\pm\frac{\partial\varphi_1}{\partial\alpha_1}\cdot\frac{\partial\varphi_2}{\partial\alpha_2}\cdots\frac{\partial\varphi_n}{\partial\alpha_n}}=\frac{1}{X\Sigma\pm\frac{\partial x_1}{\partial\alpha_1}\cdot\frac{\partial x_2}{\partial\alpha_2}\cdots\frac{\partial x_n}{\partial\alpha_n}}.$$

Si vero generalius inter omnes $2n+1$ quantitates $x, x_1, \ldots, x_n, \alpha_1$, $\alpha_1, \ldots, \alpha_n$ proponuntur n aequationes integrales

$$\Pi_1=0,\quad \Pi_2=0,\quad \ldots,\quad \Pi_n=0,$$

fit (D. F. §. 10 (5))

$$(10)\quad \Sigma\pm\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}=\frac{(-1)^n\Sigma\pm\frac{\partial\Pi_1}{\partial x_1}\cdot\frac{\partial\Pi_2}{\partial x_2}\cdots\frac{\partial\Pi_n}{\partial x_n}}{\Sigma\pm\frac{\partial\Pi_1}{\partial\alpha_1}\cdot\frac{\partial\Pi_2}{\partial\alpha_2}\cdots\frac{\partial\Pi_n}{\partial\alpha_n}}.$$

*) Commentationem de Determinantibus functionalibus. Vol. XXII Diarii Crelliani insertam [Cf. Vol. III. h. ed. p. 393] designabo per D. F.

Unde obtinetur, rejecto, quod licet, signo ancipiti,

$$(11)\quad M = \frac{1}{X}\cdot\frac{\Sigma\pm\frac{\partial\Pi_1}{\partial x_1}\cdot\frac{\partial\Pi_2}{\partial x_2}\cdots\frac{\partial\Pi_n}{\partial x_n}}{\Sigma\pm\frac{\partial\Pi_1}{\partial\alpha_1}\cdot\frac{\partial\Pi_2}{\partial\alpha_2}\cdots\frac{\partial\Pi_n}{\partial\alpha_n}},$$

quae est Multiplicatoris expressio maxime generalis.

Formulae (10) ope investigatio valoris Determinantis functionalis haud raro egregie expeditur. Transponamus ex. gr. Constantes arbitrarias in alteram partem aequationum (5), atque pro quolibet ipsius i valore statuamus functionem Π_i aequalem functioni $f_i-\alpha_i$, quocunque modo per aequationes

$$f_{i+1}=\alpha_{i+1},\quad f_{i+2}=\alpha_{i+2},\quad \ldots,\quad f_n=\alpha_n$$

transformatae. Poterit loco cuiusque aequationis $f_i=\alpha_i$ adhiberi aequatio $\Pi_i=0$, unde systema aequationum sequentium

$$\Pi_1=0,\quad \Pi_2=0,\quad \ldots,\quad \Pi_n=0$$

haberi poterit pro aequationum integralium completarum systemate. Quae ita sunt comparatae aequationes, ut quaelibet functio Π_i non involvat quantitates $\alpha_1, \alpha_2, \ldots, \alpha_{i-1}$, quantitatem α_i autem in unico termino addito $-\alpha_i$. Unde erit

$$\frac{\partial\Pi_i}{\partial\alpha_1}=\frac{\partial\Pi_i}{\partial\alpha_2}=\cdots=\frac{\partial\Pi_i}{\partial\alpha_{i-1}}=0,\quad \frac{\partial\Pi_i}{\partial\alpha_i}=-1,$$

sive, quantitatibus $\frac{\partial\Pi_i}{\partial\alpha_k}$ in figuram quadratam dispositis hunc in modum:

$$\begin{array}{cccc}
\frac{\partial\Pi_1}{\partial\alpha_1}, & \frac{\partial\Pi_1}{\partial\alpha_2}, & \ldots, & \frac{\partial\Pi_1}{\partial\alpha_n},\\
\frac{\partial\Pi_2}{\partial\alpha_1}, & \frac{\partial\Pi_2}{\partial\alpha_2}, & \ldots, & \frac{\partial\Pi_2}{\partial\alpha_n},\\
\cdot & \cdot & \cdot & \cdot\\
\frac{\partial\Pi_n}{\partial\alpha_1}, & \frac{\partial\Pi_n}{\partial\alpha_2}, & \ldots, & \frac{\partial\Pi_n}{\partial\alpha_n},
\end{array}$$

quadratoque per diagonalem, a laeva ad dextram partem ductam, in duas partes diviso, termini in laeva parte positi omnes evanescunt. Quod ubi fit, abit Determinans in productum terminorum in ipsa diagonali positorum. Qui termini cum singuli fiant -1, eruitur

$$\Sigma\pm\frac{\partial\Pi_1}{\partial\alpha_1}\cdot\frac{\partial\Pi_2}{\partial\alpha_2}\cdots\frac{\partial\Pi_n}{\partial\alpha_n}=(-1)^n,$$

ideoque

$$(12)\quad \begin{cases} XM = \Sigma\pm\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}\\ \quad\;\; = \Sigma\pm\frac{\partial\Pi_1}{\partial x_1}\cdot\frac{\partial\Pi_2}{\partial x_2}\cdots\frac{\partial\Pi_n}{\partial x_n}.\end{cases}$$

Quae docet formula propositionem frequentissimae applicationis, *valentibus aequationibus*

$$f_1=\alpha_1,\quad f_2=\alpha_2,\quad \ldots,\quad f_n=\alpha_n,$$

Determinans functionale

$$\Sigma\pm\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}$$

valorem non mutare, si ante differentiationes partiales transigendas quaeque functio f_i per aequationes

$$f_{i+1}=\alpha_{i+1},\quad f_{i+2}=\alpha_{i+2},\quad \ldots,\quad f_n=\alpha_n$$

quascunque subeat mutationes. In hac propositione sunt $\alpha_1, \alpha_2, \ldots, \alpha_n$ Constantes; quae si iunguntur functionibus $f_1, f_2, \ldots, f_n$, ita ut ipsius $f_i-\alpha_i$ loco scribatur f_i, refertur propositio ad valorem, quem induit Determinans functionale, functionibus ipsis evanescentibus. In applicatione huius propositionis facienda functiones $f_1, f_2, \ldots, f_n$ sive aequationes $f_1=0,\ f_2=0,\ \ldots,\ f_n=0$ certo disponendae sunt ordine tali, ut quaequae aequatio $f_i=0$ insequentium ope formam induere possit concinnam, simulque differentialia partialia functionis f_i evadant simplicissima. Quin adeo eandem operationem indefinite repetere licet, siquidem post idoneas mutationes, pro certo functionum et aequationum ordine factas, eaedem functiones alio semperque alio ordine disponuntur et pro quaque nova dispositione mutationes vel eliminationes convenientes operantur. Quantascunque autem mutationes per varias istas dispositiones et eliminationes subire possunt functiones propositae f_1 etc., non tamen inde nascuntur functionum mutationes, quae obtineri possunt, si *eodem tempore* ad unamquamque transformandam, nullo ordinis functionum respectu habito, omnes adhibentur n aequationes, quae reliquas omnes functiones nihilo aequando proveniunt. Nam in propositione tradita unica tantum erat e $n+1$ functionibus, ad quam transformandam adhiberi poterant n aequationes; praeter hanc una tantum erat, ad quam transformandam $n-1$ aequationes adhiberi poterant, et ita porro. Functionibus in alium aliumque ordinem dispositis et pro quaque nova dispositione propositionis traditae applicatione facta, effici quidem potest, ut unaquaeque functio sua vice adiumento n aequationum transmutetur; sed differentia in eo constituitur, quod hac ratione aequationes ad transmutationes adhibendae non amplius proveniant nihilo aequando functiones propositas, sed functiones et ipsas iam transmutatas. Veluti si f per aequationem $f_1 = 0$ mutatur in φ, ac deinde f_1 per aequationem $\varphi = 0$ in φ_1: ipsa φ_1 non easdem induere potest formas, in quas mutari potest f_1 nihilo aequando ipsam functionem propositam f. Nam si valorem generalem functionis, in quam f per aequationem $f_1 = 0$ mutari potest, designamus, quod licet, per

$$\varphi = f + \lambda f_1,$$

atque similiter valorem generalem functionis, in quam f_1 per aequationem $\varphi = 0$ mutatur, per

$$\varphi_1 = f_1 + \mu\varphi = (1+\lambda\mu)f_1 + \mu f:$$

haec functio diversa erit a functione $f_1 + \mu f$, in quam f_1 per aequationem $f = 0$ mutatur. Atque Determinans functionum φ et φ_1 idem quidem erit atque functionum propositarum; functionum vero $f + \lambda f_1$, $f_1 + \mu f$ ab illo discrepabit, scilicet aequabitur Determinanti functionum f et f_1, per factorem $1 - \lambda\mu$ multiplicato. Quod pluribus illustrare placuit, ut emendarem errorem, quem in Commentatione *de Determinantibus functionalibus* commisi proponendo, Determinantis functionalis valorem, quem induat ipsis functionibus evanescentibus, immutatum manere, si unaquaeque functio mutationes subeat, quascunque nihilo aequando reliquas omnes subire possit. Generaliter si ponitur

$$\varphi_i = \lambda^{(i)} f + \lambda_1^{(i)} f_1 + \cdots + \lambda_n^{(i)} f_n,$$

demonstrabitur per Determinantium proprietates, valentibus aequationibus

$$f = 0, \quad f_1 = 0, \quad \ldots, \quad f_n = 0,$$

fieri

$$\Sigma \pm \frac{\partial \varphi}{\partial x} \cdot \frac{\partial \varphi_1}{\partial x_1} \cdots \frac{\partial \varphi_n}{\partial x_n} = \Sigma \pm \lambda \lambda_1' \lambda_2'' \ldots \lambda_n^{(n)} . \Sigma \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n}.$$

Unde ut Determinantia functionum $f, f_1, \ldots, f_n$ et $\varphi, \varphi_1, \ldots, \varphi_n$ inter se aequalia existant, habetur conditio generalis

$$\Sigma \pm \lambda \lambda_1' \ldots \lambda_n^{(n)} = 1.$$

E Propositione supra tradita, identidem pro aliis aliisque functionum dispositionibus repetita, innumera deducuntur quantitatum $\lambda_k^{(i)}$ systemata, quae conditioni illi satisfaciunt.

Inter mutationes, quas functio variabilium x, x_1, etc. per aequationes inter easdem variabiles positas subire potest, referri potest eliminatio variabilium numeri numero aequationum aequalis. Unde in formula (12) definire licet Π_i ut functionem variabilium $x, x_1, \ldots, x_i$, in quam abeat $f_i - \alpha_i$, si ope aequationum $f_{i+1} = \alpha_{i+1}$, $f_{i+2} = \alpha_{i+2}$, $\ldots$, $f_n = \alpha_n$ variabiles $x_{i+1}, x_{i+2}, \ldots, x_n$ eliminantur. Quo statuto, omnia evanescunt differentialia partialia $\frac{\partial \Pi_i}{\partial x_k}$, in quibus $k > i$; unde figura quadrata, quae a quantitatibus $\frac{\partial \Pi_i}{\partial x_k}$ formatur, ita comparata erit, ut in ea, per diagonalem divisa, rursus termini in altera parte positi evanescant, ideoque fiat

$$\Sigma \pm \frac{\partial \Pi_1}{\partial x_1} \cdot \frac{\partial \Pi_2}{\partial x_2} \cdots \frac{\partial \Pi_n}{\partial x_n} = \frac{\partial \Pi_1}{\partial x_1} \cdot \frac{\partial \Pi_2}{\partial x_2} \cdots \frac{\partial \Pi_n}{\partial x_n}.$$

Hinc formula (12) abit in hanc

$$(13) \quad XM = \frac{\partial \Pi_1}{\partial x_1} \cdot \frac{\partial \Pi_2}{\partial x_2} \cdots \frac{\partial \Pi_n}{\partial x_n},$$

sive Determinans functionale, quo Multiplicator definitur, in simplex productum redit. Forma autem aequationum integralium

$$\Pi_1 = 0, \quad \Pi_2 = 0, \quad \ldots, \quad \Pi_n = 0,$$

quae illam simplicem Determinantis functionalis expressionem suppeditat, eadem est atque per integrationem *successivam* proveniens, post quodque Integrale inventum una variabilium eliminata. Servata enim functionum $\Pi_1, \Pi_2, \ldots, \Pi_n$ significatione antecedente, si eliminatur x_n per Integrale

$$\Pi_n = f_n - \alpha_n = 0,$$

erit $\Pi_{n-1} = 0$ Integrale aequationum differentialium

$$dx : dx_1 : \ldots : dx_{n-1} = X : X_1 : \ldots : X_{n-1},$$

cuius Integralis ope eliminata x_{n-1}, erit $\Pi_{n-2} = 0$ Integrale aequationum differentialium

$$dx : dx_1 : \ldots : dx_{n-2} = X : X_1 : \ldots : X_{n-2},$$

et ita porro. Si e functione Π_i Constantes arbitrarias $\alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_n$, quas implicat, ope aequationum

$$\Pi_{i+1} = 0, \quad \Pi_{i+2} = 0, \quad \ldots, \quad \Pi_n = 0$$

eliminamus, redit aequatio $\Pi_i = 0$ in aequationum differentialium propositarum

Integrale $f_i - \alpha_i = 0$. Voco autem, ut in aliis Commentationibus, *Integrale* systematis aequationum differentialium vulgarium huiusmodi aequationem integralem, quae differentiata identica evadat per solas aequationes differentiales propositas, neque ipsa illa aequatione integrali neque ulla alia in auxilium advocata.

§. 4.

Multiplicatoris expressio generalis. Bini Multiplicatores suppeditant Integrale. Expressio generalis functionum, quarum detur Determinans.

Iam varias, quae de Multiplicatore nostro tradi possunt, proprietates exponam. Ac primum inquiram quomodo, uno cognito Multiplicatore, eruantur alii innumeri, sive Multiplicatoris investigabo formam generalem. Sit M datus Multiplicator aequationis

$$(1)\quad X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}+\cdots+X_n\frac{\partial f}{\partial x_n}=0,$$

satisfacere debet M secundum §. pr. huiusmodi aequationi

$$(2)\quad MX=\Sigma\pm\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n},$$

designantibus $f_1, f_2, \ldots, f_n$ solutiones aequationis (1) a se invicem indepentes. Sit μ alius quicunque Multiplicator, satisfaciens aequationi

$$(3)\quad \mu X=\Sigma\pm\frac{\partial F_1}{\partial x_1}\cdot\frac{\partial F_2}{\partial x_2}\cdots\frac{\partial F_n}{\partial x_n},$$

designantibus $F_1, F_2, \ldots, F_n$ aliud systema solutionum eiusdem aequationis (1) a se invicem independentium. Functiones F_1, F_2, etc. esse debent solarum $f_1, f_2, \ldots, f_n$ functiones; cognitis enim aequationis (1) solutionibus n a se invicem independentibus, quaevis alia eiusdem aequationis solutio harum n solutionum functio est. Fit autem per formulam notam (D. F. §. 11. Prop. II.):

$$(4)\quad \begin{cases} \Sigma\pm\frac{\partial F_1}{\partial x_1}\cdot\frac{\partial F_2}{\partial x_2}\cdots\frac{\partial F_n}{\partial x_n} \\ =\Sigma\pm\frac{\partial F_1}{\partial f_1}\cdot\frac{\partial F_2}{\partial f_2}\cdots\frac{\partial F_n}{\partial f_n}\cdot\Sigma\pm\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n}, \end{cases}$$

siquidem habentur $F_1, F_2, \ldots, F_n$ in laeva formulae parte pro variabilium x, $x_1, \ldots, x_n$ functionibus, in dextra parte pro functionibus ipsarum $f_1, f_2, \ldots, f_n$. E (2)—(4) autem obtinetur haec formula:

$$(5)\quad \mu=M\Sigma\pm\frac{\partial F_1}{\partial f_1}\cdot\frac{\partial F_2}{\partial f_2}\cdots\frac{\partial F_n}{\partial f_n}.$$

Unde sequitur vice versa, ipsarum $f_1, f_2, \ldots, f_n$ quibuscunque sumtis functionibus a se independentibus $F_1, F_2, \ldots, F_n$, Multiplicatorem M ductum in harum functionum Determinans

$$\Sigma\pm\frac{\partial F_1}{\partial f_1}\cdot\frac{\partial F_2}{\partial f_2}\cdots\frac{\partial F_n}{\partial f_n}$$

alterum suppeditare Multiplicatorem μ. Quaecunque enim sint $F_1, F_2, \ldots, F_n$ ipsarum $f_1, f_2, \ldots, f_n$ functiones a se independentes, ex aequationibus (2), (4), (5) sequitur formula (3), in qua $F_1, F_2, \ldots, F_n$ erunt aequationis (1) solutiones a se invicem independentes, unde secundum §. pr. quantitas μ, formula (3) determinata, aequationis (1) erit Multiplicator.

Videmus ex antecedentibus, binorum quorumque Multiplicatorum Quotientem $\frac{\mu}{M}$ aequari functioni ipsarum $f_1, f_2, \ldots, f_n$, videlicet Determinanti ipsarum $F_1, F_2, \ldots, F_n$, pro functionibus quantitatum $f_1, f_2, \ldots, f_n$ habitarum, et vice versa, Multiplicatore M ducto in Determinans quarumcunque n functionum a se independentium quantitatum $f_1, f_2, \ldots, f_n$, alterum obtineri Multiplicatorem. Semper autem quantitatum $f_1, f_2, \ldots, f_n$ functiones $F_1, F_2, \ldots, F_n$ invenire licet, quarum Determinans sit earundem quantitatum data quaecunque functio. Unde non modo binorum Multiplicatorum M et μ Quotiens functioni aequatur ipsarum $f_1, f_2, \ldots, f_n$, sed etiam vice versa, Multiplicatore M in quamcunque functionem ipsarum $f_1, f_2, \ldots, f_n$ ducto, rursus prodit Multiplicator. Et cum ipsarum $f_1, f_2, \ldots, f_n$ quaelibet functio aequationis (1) solutio sit, neque aliae aequationis (1) solutiones exstare possint, nisi quae ipsarum $f_1, f_2, \ldots, f_n$ functiones sint, sequitur ex antecedentibus haec Propositio.

Propositio.

„Designante M Multiplicatorem aequationis differentialis partialis

$$X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}+\cdots+X_n\frac{\partial f}{\partial x_n}=0,$$

erit Multiplicatoris forma generalis

$$\Pi M,$$

designante Π quamcunque aequationis propositae solutionem."

Cognita aequationis (1) solutione Π ac designante α Constantem arbitrariam, aequatione $\Pi=\alpha$ determinatur variabilium $x_1, x_2, \ldots, x_n$ functio x, satisfaciens aequationi differentiali partiali

$$(6)\quad 0=X-X_1\frac{\partial x}{\partial x_1}-X_2\frac{\partial x}{\partial x_2}-\cdots-X_n\frac{\partial x}{\partial x_n},$$

nec non erit $\Pi = \alpha$ Integrale aequationum differentialium vulgarium simultanearum

$$(7) \qquad dx : dx_1 : \ldots : dx_n = X : X_1 : \ldots : X_n .$$

Unde Propositio antecedens docet, *cognitis aequationis differentialis partialis* (6) *vel aequationum* (7) *differentialium vulgarium binis Multiplicatoribus M et M_1, non solo factore constante inter se diversis, aequationem*

$$\frac{M_1}{M} = \text{Const.}$$

fore aequationis differentialis partialis (6) *solutionem vel systematis aequationum differentialium* (7) *Integrale.*

Pluribus datis Multiplicatoribus M, M_1, $\ldots$, M_k, haec quoque quantitas

$$MF\left(\frac{M_1}{M},\ \frac{M_2}{M},\ \ldots,\ \frac{M_k}{M}\right)$$

erit Multiplicator. Designante enim F ipsarum $\frac{M_1}{M}$ etc. functionem arbitrariam, non tantum fractiones $\frac{M_1}{M}$, $\frac{M_2}{M}$, etc., sed ipsa F quoque aequationis (1) solutio fit. Unde etiam aequatione $F = 0$ sive, quod idem est, *quacunque aequatione homogenea inter datos Multiplicatores posita determinatur aequationis* (6) *solutio.* Nec non designantibus α_1, α_2, $\ldots$, α_n Constantes arbitrarias, erunt

$$\frac{M_1}{M} = \alpha_1, \quad \frac{M_2}{M} = \alpha_2, \quad \ldots, \quad \frac{M_k}{M} = \alpha_k$$

Integralia aequationum differentialium vulgarium (7).

Restat, ut paucis exponam, quomodo inveniantur functiones, quarum Determinans datae variabilium functioni aequetur, quod semper fieri posse supra innui. Immo videbimus idem innumeris modis succedere, videlicet functiones praeter unam omnes ex arbitrio sumi posse, una reliqua per solam Quadraturam determinata.

Designante Π datam quamcunque quantitatum f_1, f_2, $\ldots$, f_n functionem, simplicissima habetur solutio aequationis

$$(8) \qquad \Sigma \pm \frac{\partial F_1}{\partial f_1} \cdot \frac{\partial F_2}{\partial f_2} \cdots \frac{\partial F_n}{\partial f_n} = \Pi,$$

ponendo

$$F_2 = f_2, \quad F_3 = f_3, \quad \ldots, \quad F_n = f_n,$$

unde Determinans propositum in simplex differentiale abit

$$\frac{\partial F_1}{\partial f_1} = \Pi.$$

Quo igitur casu fit

$$F_1 = \int \Pi df_1,$$

cui integrali functionem ipsarum f_2, f_3, $\ldots$, f_n arbitrariam addere licet, quippe quae inter integrationem pro Constantibus habentur. Aequationis (8) solutio generalis obtinetur sequenti modo. Pro ipsis F_2, F_3, $\ldots$, F_n ex arbitrio sumantur ipsarum f_1, f_2, $\ldots$, f_n functiones a se independentes, atque fingatur, reliquam functionem F_1 exhiberi per quantitates

$$f_1, \quad F_2, \quad F_3, \quad \ldots, \quad F_n.$$

Functionis F_1 hoc modo repraesentatae differentialia partialia uncis includam, quo distinguantur a differentialibus eiusdem functionis per f_1, f_2, $\ldots$, f_n exhibitae, ita ut sit

$$\frac{\partial F_1}{\partial f_1} = \left(\frac{\partial F_1}{\partial f_1}\right) + \left(\frac{\partial F_1}{\partial F_2}\right)\frac{\partial F_2}{\partial f_1} + \left(\frac{\partial F_1}{\partial F_3}\right)\frac{\partial F_3}{\partial f_1} + \cdots + \left(\frac{\partial F_1}{\partial F_n}\right)\frac{\partial F_n}{\partial f_1},$$

et, quoties index i ab unitate diversus est,

$$\frac{\partial F_1}{\partial f_i} = \left(\frac{\partial F_1}{\partial F_2}\right)\frac{\partial F_2}{\partial f_i} + \left(\frac{\partial F_1}{\partial f_3}\right)\frac{\partial F_3}{\partial f_i} + \cdots + \left(\frac{\partial F_1}{\partial F_n}\right)\frac{\partial F_n}{\partial f_i}.$$

Quae ipsarum

$$\frac{\partial F_1}{\partial f_1}, \quad \frac{\partial F_1}{\partial f_2}, \quad \ldots, \quad \frac{\partial F_1}{\partial f_n}$$

expressiones si substituuntur in Determinante

$$\Sigma \pm \frac{\partial F_1}{\partial f_1} \cdot \frac{\partial F_2}{\partial f_2} \cdots \frac{\partial F_n}{\partial f_n},$$

identice evanescunt singula aggregata, per singula differentialia partialia

$$\left(\frac{\partial F_1}{\partial F_2}\right), \quad \left(\frac{\partial F_1}{\partial F_3}\right), \quad \ldots, \quad \left(\frac{\partial F_1}{\partial F_n}\right)$$

multiplicata, unde simplex formula obtinetur:

$$(9) \qquad \Sigma \pm \frac{\partial F_1}{\partial f_1} \cdot \frac{\partial F_2}{\partial f_2} \cdots \frac{\partial F_n}{\partial f_n} = \left(\frac{\partial F_1}{\partial f_1}\right) \Sigma \pm \frac{\partial F_2}{\partial f_2} \cdot \frac{\partial F_3}{\partial f_3} \cdots \frac{\partial F_n}{\partial f_n}.$$

(D. F. §. 12. (4)). E (8) et (9) sequitur

$$\left(\frac{\partial F_1}{\partial f_1}\right) = \frac{\Pi}{\Sigma \pm \frac{\partial F_2}{\partial f_2} \cdot \frac{\partial F_3}{\partial f_3} \cdots \frac{\partial F_n}{\partial f_n}};$$

quae formula, exprimendo $f_2, f_3, \ldots, f_n$ per $f_1, F_2, F_3, \ldots, F_n$, sic quoque exhiberi potest:

$$(10)\quad \left(\frac{\partial F_1}{\partial f_1}\right) = \Pi \Sigma \pm \frac{\partial f_2}{\partial F_2} \cdot \frac{\partial f_3}{\partial F_3} \cdots \frac{\partial f_n}{\partial F_n}.$$

(D. F. §. 9. (2)). Secundum hanc formulam, ut modo maxime generali variabilium $f_1, f_2, \ldots, f_n$ inveniantur functiones, quarum Determinans datae earundem variabilium functioni Π aequatur, ex arbitrio exprimantur $f_2, f_3, \ldots, f_n$ per f_1 aliasque $n-1$ quantitates $F_2, F_3, \ldots, F_n$, determinataque F_1 per formulam

$$(11)\quad F_1 = \int \Pi \Sigma \pm \frac{\partial f_2}{\partial F_2} \cdot \frac{\partial f_3}{\partial F_3} \cdots \frac{\partial f_n}{\partial F_n} df_1,$$

ipsae $F_1, F_2, \ldots, F_n$, vice versa per $f_1, f_2, \ldots, f_n$ expressae, erunt functiones quaesitae.

Ponendo $\Pi = 1$ antecedentibus innumera obtinentur systemata functionum quantitatum $f_1, f_2, \ldots, f_n$, quarum Determinans unitati aequatur. Quibus omnibus idem respondet Multiplicator. Quoties enim

$$\Sigma \pm \frac{\partial F_1}{\partial f_1} \cdot \frac{\partial F_2}{\partial f_2} \cdots \frac{\partial F_n}{\partial f_n} = 1,$$

sequitur e (5)

$$\mu = M.$$

Vice versa, si idem Multiplicator respondet binis systematis n solutionum a se independentium aequationis differentialis partialis (1), $f_1, f_2, \ldots, f_n$ atque $F_1, F_2, \ldots, F_n$, ita ut sit

$$MX = \Sigma \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n}$$
$$= \Sigma \pm \frac{\partial F_1}{\partial x_1} \cdot \frac{\partial F_2}{\partial x_2} \cdots \frac{\partial F_n}{\partial x_n}:$$

erunt $F_1, F_2, \ldots, F_n$ quantitatum $f_1, f_2, \ldots, f_n$ functiones, quarum Determinans unitati aequatur.

§. 5.

Multiplicatoris definitio per aequationem differentialem partialem. Conditio, ut Multiplicator aequari possit unitati.

Vidimus §. 3, aequationis differentialis partialis

$$(1)\quad X\frac{\partial f}{\partial x} + X_1\frac{\partial f}{\partial x_1} + \cdots + X_n\frac{\partial f}{\partial x_n} = 0$$

Multiplicatorem quemcunque M alii satisfacere aequationi differentiali partiali:

$$(2)\quad \frac{\partial(MX)}{\partial x} + \frac{\partial(MX_1)}{\partial x_1} + \cdots + \frac{\partial(MX_n)}{\partial x_n} = 0.$$

Vice versa, *quaecunque habetur solutio μ aequationis differentialis partialis*

$$(3)\quad \frac{\partial(\mu X)}{\partial x} + \frac{\partial(\mu X_1)}{\partial x_1} + \cdots + \frac{\partial(\mu X_n)}{\partial x_n} = 0,$$

erit illa aequationis (1) *Multiplicator.*

Ponamus enim $\mu = \Pi . M$, abit aequatio (3) in sequentem:

$$0 = \Pi\left(\frac{\partial(MX)}{\partial x} + \frac{\partial(MX_1)}{\partial x_1} + \cdots + \frac{\partial(MX_n)}{\partial x_n}\right)$$
$$+ M\left(X\frac{\partial \Pi}{\partial x} + X_1\frac{\partial \Pi}{\partial x_1} + \cdots + X_n\frac{\partial \Pi}{\partial x_n}\right).$$

Partis dextrae Aggregatum in Π ductum secundum (2) evanescit; unde, cum supponamus ipsum M non evanescere, sequitur:

$$0 = X\frac{\partial \Pi}{\partial x} + X_1\frac{\partial \Pi}{\partial x_1} + \cdots + X_n\frac{\partial \Pi}{\partial x_n}.$$

Erit igitur Π aequationis (1) solutio ideoque secundum Propositionem §. pr. traditam, Multiplicatorem in solutionem aequationis (1) quamcunque ductum reproducere Multiplicatorem, erit $\Pi . M = \mu$ Multiplicator, q. d. e.

Cum quilibet Multiplicator sit solutio aequationis (3) et secundum antecedentia quaelibet aequationis (3) solutio sit Multiplicator, poterit aequatio (3) adhiberi ad Multiplicatorem definiendum. Habemus igitur Propositionem sequentem.

Propositio I.

„Designante M solutionem quamcunque aequationis differentialis partialis

$$\frac{\partial(MX)}{\partial x} + \frac{\partial(MX_1)}{\partial x_1} + \cdots + \frac{\partial(MX_n)}{\partial x_n} = 0,$$

semper dantur functiones $f_1, f_2, \ldots, f_n$, quae pro functione f indefinita efficiant aequationem

$$M\left(X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}+\cdots+X_n\frac{\partial f}{\partial x_n}\right)=\Sigma\pm\frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdots\frac{\partial f_n}{\partial x_n}.\text{"}$$

Videri possit parum lucri percipi e nova Multiplicatoris determinatione per aequationem differentialem partialem (3). Aequationis (3) enim solutio generalis non habetur, nisi aequationis (1) data sit solutio generalis sive eius innotescant n solutiones particulares a se invicem independentes. His autem cognitis, habetur Multiplicator per formulam (2) §. pr. At observo, ad Multiplicatorem eruendum tantum nos indigere una aliqua solutione particulari aequationis (3), et quamquam aequationis (3) solutio generalis a solutione aequationis (1) pendet et pro complicatiore habenda est, fieri tamen potest, ut aequationis (3) innotescat solutio particularis, dum aequationis (1) solutiones adhuc omnes ignoramus.

Inter solutiones aequationis differentialis partialis (1) non referenda est, quae sponte se offert, $f=$ Const. Sed e solutionibus aequationis (3), quae Multiplicatorem suggerunt, quantitates constantes non excluduntur. Fit autem Multiplicator Constanti vel, si placet, unitati aequalis, si inter ipsas X, X_1, etc. locum habet aequatio:

$$(4)\qquad \frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}+\cdots+\frac{\partial X_n}{\partial x_n}=0.$$

Eo casu ipsa expressio proposita

$$X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}+\cdots+X_n\frac{\partial f}{\partial x_n}$$

pro functione f indefinita aequivalet alicui Determinanti functionali

$$\Sigma\pm\frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n},$$

sive, adhibendo notationes §. 3 usitatas, statuere licet

$$X=A,\quad X_1=A_1,\ \ldots,\ X_n=A_n.$$

Quod, si ea tenes, quae §. 2 de Determinantibus functionalibus partialibus monui, sic quoque proponi potest.

Propositio II.

„Si $n+1$ variabilium x, x_1, …, x_n functiones X, X_1, …, X_n satisfaciant conditioni

$$\frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}+\frac{\partial X_2}{\partial x_2}+\cdots+\frac{\partial X_n}{\partial x_n}=0,$$

ipsae $n+1$ quantitates X, X_1, …, X_n haberi possunt pro certarum n functionum Determinantibus partialibus."

Haec Propositio analoga est notae elementari, si variabilium x et y functiones X et Y satisfaciant conditioni $\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}=0$, ipsas Y et $-X$ respective haberi posse pro eiusdem functionis differentialibus partialibus, variabilium x et y respectu sumtis.

Si inter quantitates X, X_1, etc. conditio (4) locum habet, aequatio differentialis partialis (3), qua Multiplicator definitur, in ipsam (1) redit. Eo igitur casu quaecunque aequationis (1) solutio eiusdem aequationis Multiplicator erit, siquidem iam unitatem vel numeros constantes inter solutiones referimus. Unde etiam patet, eo casu aequationum differentialium vulgarium

$$dx:dx_1:\ldots:dx_n=X:X_1:\ldots:X_n$$

Multiplicatorem fore quantitatem quamcunque, aut per se constantem, aut quae per aequationes integrales completas Constanti aequetur.

§. 6.

Cognito systematis aequationum differentialium vulgarium Multiplicatore quocunque, eruuntur Determinantia functionum, quae per aequationes integrales completas valoribus variabilium initialibus aequivalent.

Vidimus §. 3, designantibus f_1, f_2, …, f_n solutiones a se independentes aequationis

$$(1)\qquad X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}+\cdots+X_n\frac{\partial f}{\partial x_n}=0,$$

harum functionum Determinantia partialia A_1, A_2, …, A_n esse inter se ut aequationis (1) Coëfficientes, sive fieri

$$(2)\qquad A:A_1:\ldots:A_n=X:X_1:\ldots:X_n.$$

Unde omnia A_1, A_2, …, A_n uno determinantur A. Antecedentibus autem demonstravi, designante μ Multiplicatorem aequationis (1) quemcunque sive quamcunque solutionem aequationis

$$(3)\qquad \frac{\partial(X\mu)}{\partial x}+\frac{\partial(X_1\mu)}{\partial x_1}+\cdots+\frac{\partial(X_n\mu)}{\partial x_n}=0,$$

fieri $\mu=\Pi M$, ideoque

$$(4)\qquad \mu X=\Pi.A=\Pi.\Sigma\pm\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_n}{\partial x_n},$$

ubi $\boldsymbol{\Pi}$ certa quaedam est ipsarum $f_1, f_2, \ldots, f_n$ functio sive aequationis (1) solutio. Hinc e data quacunque aequationis (3) solutione μ cognoscitur valor Determinantis A, dummodo determinata erit functio $\boldsymbol{\Pi}$. *Eruitur autem functio* $\boldsymbol{\Pi}$, *dummodo Determinantis A innotescat valor, quem pro $x = 0$ induit.* Generaliter enim, ut functio f aequationi differentiali partiali (1) satisfaciens omnino determinata sit, poscitur et sufficit, ut aliqua cognoscatur functio, cui illa aequalis evadat, ubi inter variabiles $x, x_1, \ldots, x_n$ data aliqua aequatio locum habet, veluti si ipsius f datur valor, quem pro $x = 0$ induit. Hinc si ponimus, pro $x = 0$ abire μ, X, A in variabilium $x_1, x_2, \ldots, x_n$ functiones μ^0, X^0, A^0; functio $\boldsymbol{\Pi}$ eo determinabitur, quod esse debeat aequationis (1) solutio atque pro $x = 0$ aequalis evadat variabilium $x_1, x_2, \ldots, x_n$ functioni

$$\frac{\mu^0 X^0}{A^0}.$$

Eiusmodi solutio autem ut inveniatur, sint $f_1^0, f_2^0, \ldots, f_n^0$ variabilium $x_1, x_2, \ldots, x_n$ functiones, in quas pro $x = 0$ abeunt $f_1, f_2, \ldots, f_n$; exprimatur porro variabilium $x_1, x_2, \ldots, x_n$ functio $\frac{\mu^0 X^0}{A^0}$ per $f_1^0, f_2^0, \ldots, f_n^0$; in qua expressione ponendo ipsarum $f_1^0, f_2^0, \ldots, f_n^0$ loco ipsas $f_1, f_2, \ldots, f_n$, prodibit functio quaesita $\boldsymbol{\Pi}$. Quippe functio sic inventa erit aequationis (1) solutio et pro $x = 0$ abibit in variabilium $x_1, x_2, \ldots, x_n$ functionem $\frac{\mu^0 X^0}{A^0}$.

Functionem A^0 casu prae ceteris notando a priori assignare licet, videlicet quoties $f_1, f_2, \ldots, f_n$ tales sunt aequationis (1) solutiones, *quae pro $x = 0$ in ipsas variabiles $x_1, x_2, \ldots, x_n$ abeunt.* Tunc enim habetur

$$f_1^0 = x_1, \quad f_2^0 = x_2, \quad \ldots, \quad f_n^0 = x_n,$$

ideoque

$$A^0 = \Sigma \pm \frac{\partial f_1^0}{\partial x_1} \cdot \frac{\partial f_2^0}{\partial x_2} \cdots \frac{\partial f_n^0}{\partial x_n} = 1.$$

Hinc secundum regulam traditam functio $\boldsymbol{\Pi}$ e functione $\mu^0 X^0$ eruitur substituendo variabilibus $x_1, x_2, \ldots, x_n$ functiones $f_1, f_2, \ldots, f_n$, sive, quod idem est, substituendo in ipsa μX variabilibus $x, x_1, x_2, \ldots, x_n$ quantitates $0, f_1, f_2, \ldots, f_n$. Id quod sequentem suppeditat Propositionem.

Propositio I.

„Sint $f_1, f_2, \ldots, f_n$ solutiones aequationis

$$X \frac{\partial f}{\partial x} + X_1 \frac{\partial f}{\partial x_1} + \cdots + X_n \frac{\partial f}{\partial x_n} = 0,$$

quae pro $x = 0$ in ipsas variabiles $x_1, x_2, \ldots, x_n$ abeunt; sit μ quantitas quaecunque satisfaciens aequationi

$$\frac{\partial (X\mu)}{\partial x} + \frac{\partial (X_1 \mu)}{\partial x_1} + \cdots + \frac{\partial (X_n \mu)}{\partial x_n} = 0,$$

atque sit $\boldsymbol{\Pi}$ ipsarum $f_1, f_2, \ldots, f_n$ functio, quae e producto μX provenit substituendo variabilibus $x, x_1, x_2, \ldots, x_n$ quantitates $0, f_1, f_2, \ldots, f_n$: erit

$$\Sigma \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n} = \frac{\mu X}{\boldsymbol{\Pi}};$$

sive generalius, designante f functionem indefinitam, erit

$$\Sigma \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n} = \frac{\mu}{\boldsymbol{\Pi}} \left\{ X \frac{\partial f}{\partial x} + X_1 \frac{\partial f}{\partial x_1} + \cdots + X_n \frac{\partial f}{\partial x_n} \right\}.“$$

Observo hac occasione generaliter, datis aequationis (1) solutionibus $f_1, f_2, \ldots, f_n$, quae pro $x = 0$ in ipsas $x_1, x_2, \ldots, x_n$ abeant, quamvis aliam eiusdem aequationis solutionem $\boldsymbol{\Pi}$ per ipsas $f_1, f_2, \ldots, f_n$ absque omni eliminationis negotio exhiberi. Scilicet sufficit in functione $\boldsymbol{\Pi}$ variabilibus $x, x_1, x_2, \ldots, x_n$ substituere quantitates $0, f_1, f_2, \ldots, f_n$.

Casu speciali, quem sub finem §. pr. consideravi, posito insuper $X = 1$, e Propositione praecedente emergit haec:

Propositio II.

„Sint $f_1, f_2, \ldots, f_n$ tales solutiones aequationis

$$\frac{\partial f}{\partial x} + X_1 \frac{\partial f}{\partial x_1} + X_2 \frac{\partial f}{\partial x_2} + \cdots + X_n \frac{\partial f}{\partial x_n} = 0,$$

quae pro $x = 0$ respective in $x_1, x_2, \ldots, x_n$ abeant, sitque identice

$$\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n} = 0,$$

erit

$$\Sigma \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n} = 1,$$

atque reliqua functionum $f_1, f_2, \ldots, f_n$ Determinantia partialia $A_1, A_2, \ldots, A_n$ in ipsas redeunt quantitates $X_1, X_2, \ldots, X_n$.“

Convenit Propositiones antecedentibus inventas ad systemata aequationum differentialium vulgarium referre. Proponatur enim systema aequationum differentialium vulgarium

$$dx : dx_1 : dx_2 : \ldots : dx_n = X : X_1 : X_2 : \ldots : X_n,$$

eiusque integratione completa facta, pro Constantibus arbitrariis adhibeantur valores, quos $x_1, x_2, \ldots, x_n$ pro $x = 0$ induunt; resolutione deinde aequationum integralium erui poterunt variabilium $x, x_1, \ldots, x_n$ functiones illis Constantibus arbitrariis aequales, quae ipsae erunt functiones $f_1, f_2, \ldots, f_n$, in Propp. I. et II. consideratae. Generaliter Integralia completa sint

$$f_1 = \alpha_1, \quad f_2 = \alpha_2, \quad \ldots, \quad f_n = \alpha_n,$$

designantibus α_1, α_2, etc. Constantes arbitrarias quascunque, a quibus ipsae f_1, f_2, etc. vacuae supponuntur. Quorum Integralium ope expressis $x_1, x_2, \ldots, x_n$ per x et Constantes arbitrarias $\alpha_1, \alpha_2, \ldots, \alpha_n$, fit secundum formulas de Determinantibus functionalibus traditas:

$$A = \Sigma \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n} = \left\{ \Sigma \pm \frac{\partial x_1}{\partial \alpha_1} \cdot \frac{\partial x_2}{\partial \alpha_2} \cdots \frac{\partial x_n}{\partial \alpha_n} \right\}^{-1}.$$

Unde formula (4) docet, cognito aequationum differentialium vulgarium propositarum Multiplicatore aliquo μ, sive aequationis (3) solutione, fieri

$$\Sigma \pm \frac{\partial x_1}{\partial \alpha_1} \cdot \frac{\partial x_2}{\partial \alpha_2} \cdots \frac{\partial x_n}{\partial \alpha_n} = \frac{C}{\mu X},$$

designante C functionem Constantium arbitrariarum. Quoties sunt $\alpha_1, \alpha_2, \ldots, \alpha_n$ valores initiales variabilium $x_1, x_2, \ldots, x_n$, ipsi $x = 0$ respondentes, Determinans functionale, in laeva parte aequationis antecedentis collocatum, ponendo $x = 0$ in *unitatem* abit. Quo igitur casu Constans C ex ipsa μX eruitur ponendo variabilium $x, x_1, x_2, \ldots, x_n$ loco valores $0, \alpha_1, \alpha_2, \ldots, \alpha_n$. Casu speciali, quo Multiplicator unitatem aequat, e Propositione II. eruitur sequens prae ceteris simplex Propositio.

Propositio III.

„Proponantur aequationes differentiales vulgares simultaneae

$$\frac{dx_1}{dx} = X_1, \quad \frac{dx_2}{dx} = X_2, \quad \ldots, \quad \frac{dx_n}{dx} = X_n,$$

in quibus sint $X_1, X_2, \ldots, X_n$ tales variabilium $x, x_1, x_2, \ldots, x_n$ functiones, quae satisfaciant aequationi

$$\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n} = 0;$$

integratione completa expressis $x_1, x_2, \ldots, x_n$ per x earumque valores initiales $\alpha_1, \alpha_2, \ldots, \alpha_n$, erit non tantum pro $x = 0$, sed pro valore ipsius x indefinito

$$\Sigma \pm \frac{\partial x_1}{\partial \alpha_1} \cdot \frac{\partial x_2}{\partial \alpha_2} \cdots \frac{\partial x_n}{\partial \alpha_n} = 1.\text{“}$$

Quae licet a proposito meo aliena utile videbatur obiter adnotare.

Quo rectius intelligantur, quae supra monui de definienda solutione f aequationis differentialis partialis (1), sequentia adiicio. Sit φ functio, in quam abire debet f pro aequatione aliqua inter variabiles $x, x_1, \ldots, x_n$ data. Si φ et ipsa aequationis (1) solutio est, erit $f = \varphi$ functio quaesita, quaecunque sit illa aequatio. Si φ non est aequationis (1) solutio, fieri non debet, ut aequatio illa ad aliam inter quantitates $f_1, f_2, \ldots, f_n$ revocari possit, sive ut ex aequatione illa peti possit solutio aequationis differentialis partialis

$$X = X_1 \frac{\partial x}{\partial x_1} + X_2 \frac{\partial x}{\partial x_2} + \cdots + X_n \frac{\partial x}{\partial x_n}.$$

Nisi forte eiusmodi solutio sit *singularis* seu non redeat in aequationem inter quantitates $f_1, f_2, \ldots, f_n$, quo casu nihil impedit quominus functio f definiatur ope valoris, quem pro data illa aequatione induit. Infra autem videbimus, pro aequationis differentialis partialis antecedentis solutione singulari fieri

$$\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n} = \infty,$$

ubi ipsae X, X_1, etc. cum a factoribus communibus tum a denominatoribus purgatae supponuntur. Ita non definiri poterit f ope valoris, quem pro $x = 0$ induit, ubi pro $x = 0$ habetur $X = 0$ nec simul $\frac{\partial X}{\partial x} = \infty$. Quod obiter observo.

§. 7.

Multiplicatoris definitio per aequationem differentialem vulgarem.

Multiplicatorem, quem antecedentibus per aequationem differentialem partialem definivi, etiam per formulam differentialem vulgarem definire licet. Quae nova forma aequationis prae ceteris indagando Multiplicatori apta est.

Primum aequationem differentialem partialem, qua Multiplicator μ definitur, sic exhibeo:

$$(1) \quad 0 = X \frac{\partial \mu}{\partial x} + X_1 \frac{\partial \mu}{\partial x_1} + \cdots + X_n \frac{\partial \mu}{\partial x_n} + \mu \left\{ \frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n} \right\},$$

vel, dividendo per μ:

$$(2) \quad 0 = X \frac{\partial \log \mu}{\partial x} + X_1 \frac{\partial \log \mu}{\partial x_1} + \cdots + X_n \frac{\partial \log \mu}{\partial x_n} + \frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n}.$$

Per aequationes autem differentiales vulgares, quarum μ est Multiplicator,

$$(3)\quad dx : dx_1 : dx_2 : \ldots : dx_n = X : X_1 : X_2 : \ldots : X_n$$

aequationem praecedentem brevius sic repraesentare licet:

$$(4)\quad 0 = X\frac{d\log\mu}{dx} + \frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n}.$$

Hinc poterit aequationum differentialium vulgarium (3) Multiplicator μ definiri ut *functio, quae solarum aequationum differentialium propositarum* (3) *ope, nulla in auxilium vocata aequatione integrali, aequationi* (4) *satisfaciat.* Quippe quod fieri non potest, nisi μ *identice* satisfaciat aequationi (2), qua Multiplicator definiebatur.

Sequitur ex antecedentibus, ad investigandum Multiplicatorem circumspiciendum esse, an aequationum differentialium (3) ope contingat, expressioni

$$\left\{\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n}\right\}\frac{dx}{X}$$

formam conciliare alicuius differentialis completi dU. Quippe hoc patrato fit e (4) Multiplicator:

$$(5)\quad \mu = e^{-\int\left(\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n}\right)\frac{dx}{X}} = e^{-U}.$$

Hanc indagandi Multiplicatoris methodum infra per varia exempla illustrabo, in quibus integrationem, quae Multiplicatorem suggerit, videbimus praestari posse, aequationum differentialium vulgarium propositarum nullo Integrali cognito. Esse tamen poterit formulae (4) usus etiam, si aequationes differentiales complete integratae sunt. Tum enim formula (4) docet, formationi Determinantis functionalis, quam determinatio Multiplicatoris requirebat, substitui posse Quadraturam, minus interdum molestam. Etenim ope integrationis completae quantitas ipsi $\frac{d\log\mu}{dx}$ aequalis per solam x et Constantes arbitrarias exhiberi potest, unde ipsum $\log\mu$ per Quadraturam obtines:

$$(6)\quad \log\mu = -\int\frac{dx}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n}\right).$$

Post integrationem factam substituendo Constantibus arbitrariis variabilium x, $x_1, x_2, \ldots, x_n$ functiones aequivalentes, prodibit ipsius $\log\mu$ expressio, aequationi differentiali partiali (2) satisfaciens.

Post aequationum (3) integrationem completam expressis $x_1, x_2, \ldots, x_n$

per x et Constantes arbitrarias $\alpha_1, \alpha_2, \ldots, \alpha_n$, fit secundum §. pr.

$$(7)\quad \log\Sigma\pm\frac{\partial x_1}{\partial\alpha_1}\cdot\frac{\partial x_2}{\partial\alpha_2}\cdots\frac{\partial x_n}{\partial\alpha_n} = \log\frac{C}{\mu X},$$

designante C Constantium arbitrariarum functionem. Unde, omissa, quod licet, Constante, e formula (6) eruitur

$$(8)\quad \log\Sigma\pm\frac{\partial x_1}{\partial\alpha_1}\cdot\frac{\partial x_2}{\partial\alpha_2}\cdots\frac{\partial x_n}{\partial\alpha_n} = \log\frac{1}{X} + \int\frac{dx}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n}\right).$$

Quae formula immutata manere debet, omnibus $X, X_1, \ldots, X_n$ per factorem quemcunque communem multiplicatis. Quod ut pateat observo, per aequationes differentiales vulgares propositas aequationem (4) aucta symmetria sic proponi posse:

$$(9)\quad 0 = d\log\mu + \frac{\partial\log X}{\partial x}dx + \frac{\partial\log X_1}{\partial x_1}dx_1 + \cdots + \frac{\partial\log X_n}{\partial x_n}dx_n.$$

Unde e formula (7) eruitur:

$$\log\Sigma\pm\frac{\partial x_1}{\partial\alpha_1}\cdot\frac{\partial x_2}{\partial\alpha_2}\cdots\frac{\partial x_n}{\partial\alpha_n} = \log\frac{C}{\mu X}$$
$$= \log\frac{1}{X} + \int\left(\frac{\partial\log X}{\partial x}dx + \frac{\partial\log X_1}{\partial x_1}dx_1 + \cdots + \frac{\partial\log X_n}{\partial x_n}dx_n\right).$$

Si in hac formula simul omnes X, X_1, etc. in factorem communem ν ducuntur, augetur integrale quantitate

$$\int\left(\frac{\partial\log\nu}{\partial x}dx + \frac{\partial\log\nu}{\partial x_1}dx_1 + \cdots + \frac{\partial\log\nu}{\partial x_n}dx_n\right) = \int d\log\nu = \log\nu.$$

Eadem autem quantitate minuitur $\log\frac{1}{X}$, unde tota expressio immutata manet, q. d. e.

Si in formula (8) ponimus $X = 1$, prodit Propositio sequens.

Propositio.

„Facta integratione completa aequationum differentialium vulgarium

$$\frac{dx_1}{dx} = X_1,\quad \frac{dx_2}{dx} = X_2,\quad \ldots,\quad \frac{dx_n}{dx} = X_n,$$

exhibeantur $x_1, x_2, \ldots, x_n$ per x et Constantes arbitrarias $\alpha_1, \alpha_2, \ldots, \alpha_n$, erit

$$\log\Sigma\pm\frac{\partial x_1}{\partial\alpha_1}\cdot\frac{\partial x_2}{\partial\alpha_2}\cdots\frac{\partial x_n}{\partial\alpha_n} = \int\left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n}\right)dx,$$

quantitate sub signo integrationis et ipsa per x et Constantes arbitrarias expressa."

Si in Propositione antecedente ipsae $\alpha_1, \alpha_2, \ldots, \alpha_n$ designant variabilium valores initiales, valori $x = 0$ respondentes, integrationem inde a valore $x = 0$ fieri oportet. Ope huius Propositionis vel formulae generalioris (8) fieri potest, ut Quadratura alias satis abscondita eruatur; sicuti vice versa si Quadratura in promtu est, valor inde eruitur Determinantis functionalis.

Propositio antecedens primum a Cl° Liouville tradita est in Commentatione „*sur la variation des constantes arbitraires*“, ipsius *Diario Mathematico* (Vol. III. pag. 342) inserta. Eadem sequitur e formula iam supra citata D. F. §. 9 (1), loco f, f_1, etc. scribendo $x_1, x_2, \ldots, x_n$ atque x loco a, loco x_1, x_2, etc. autem $\alpha_1, \alpha_2, \ldots, \alpha_n$. Scilicet est ea consequentia lemmatis, quod circa variationem logarithmi Determinantis loco citato dedi. Habeantur enim n systemata aequationum linearium inter n incognitas $u_1, u_2, \ldots, u_n$, quae systemata iisdem gaudeant Coëfficientibus incognitarum et tantum terminis prorsus constantibus inter se discrepent, unde etiam omnibus idem erit Determinans. Denotentur in k^{to} aequationum linearium systemate termini constantes, in altera parte aequationum positi, respective per variationes Coëfficientium, quibus in singulis aequationibus incognita u_k afficitur, atque e primo systemate aequationum petatur valor ipsius u_1, e secundo valor ipsius u_2, et ita porro: omnium horum valorum summa aequivalebit variationi logarithmi Determinantis. In signis: sit $(u_k)_k$ valor ipsius u_k petitus e systemate aequationum

$$(10)\quad \begin{cases} a_1' \, u_1 + a_2' \, u_2 + \cdots + a_n' \, u_n = \delta a_k', \\ a_1'' \, u_1 + a_2'' \, u_2 + \cdots + a_n'' \, u_n = \delta a_k'', \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_1^{(n)} u_1 + a_2^{(n)} u_2 + \cdots + a_n^{(n)} u_n = \delta a_k^{(n)}, \end{cases}$$

erit

$$(11)\quad (u_1)_1 + (u_2)_2 + \cdots + (u_n)_n = \delta \log \Sigma \pm a_1' a_2'' \ldots a_n^{(n)}.$$

Faciamus iam, in aequationibus differentialibus

$$\frac{dx_1}{dx} = X_1, \quad \frac{dx_2}{dx} = X_2, \quad \ldots, \quad \frac{dx_n}{dx} = X_n$$

substitui variabilium $x_1, x_2, \ldots, x_n$ valores per x et Constantes arbitrarias $\alpha_1, \alpha_2, \ldots, \alpha_n$ exhibitos, qua substitutione prodire debent aequationes identicae. Quas si ipsarum α_i respectu differentiamus, obtinemus nn huiusmodi aequationes:

$$(12)\quad d\frac{\partial x_k}{\partial \alpha_i} = \left\{ \frac{\partial X_k}{\partial x_1} \cdot \frac{\partial x_1}{\partial \alpha_i} + \frac{\partial X_k}{\partial x_2} \cdot \frac{\partial x_2}{\partial \alpha_i} + \cdots + \frac{\partial X_k}{\partial x_n} \cdot \frac{\partial x_n}{\partial \alpha_i} \right\} dx.$$

Ipsi i tribuendo valores $1, 2, \ldots, n$, ex aequatione antecedente prodeunt n

aequationes lineares, in quibus habentur pro incognitis quantitates $\frac{\partial X_k}{\partial x_1} dx$, $\frac{\partial X_k}{\partial x_2} dx$, etc. Prodeunt n eiusmodi systemata aequationum linearium tribuendo ipsi quoque k valores $1, 2, \ldots, n$; in omnibusque illis aequationum linearium systematis incognitae iisdem gaudebunt Coëfficientibus. Hinc si in aequationibus (10) ponimus

$$a_k^{(i)} = \frac{\partial x_k}{\partial \alpha_i},$$

atque variationibus substituimus differentialia, aequationes (10) abeunt in aequationes (12), unde eruitur

$$(u_k)_k = \frac{\partial X_k}{\partial x_k} dx.$$

Unde e (11) sequitur

$$\left\{ \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n} \right\} dx = d \log \Sigma \pm \frac{\partial x_1}{\partial \alpha_1} \cdot \frac{\partial x_2}{\partial \alpha_2} \cdots \frac{\partial x_n}{\partial \alpha_n},$$

qua formula integrata, Propositio supra tradita obtinetur.

§. 8.

Aequationis $X - X_1 \frac{\partial x}{\partial x_1} - X_2 \frac{\partial x}{\partial x_2} - \cdots - X_n \frac{\partial x}{\partial x_n} = 0$ pars laeva Multiplicatore suo efficitur Determinans functionale completum. Pro solutione singulari Multiplicator fit infinitus. Multiplicatorem nihilo aut infinito aequando obtinetur aequatio integralis.

Quemadmodum, proposita una plurium variabilium functione, distinguimus inter differentialia eius partialia, in quibus variabiles omnes pro independentibus habentur, et differentiale completum, in quo omnes ab earum una *indefinite* pendent, ita, propositis n functionibus $n+m$ variabilium, praeter earum Determinantia partialia, de quibus supra dixi, in quibus variabiles omnes pro independentibus habentur, in considerationem venire potest *Determinans completum*, quod formatur habendo numerum m variabilium pro reliquarum n functionibus *indefinitis*. Designantibus A et B ipsarum x et y functiones, aequationem differentialem

$$A + B\frac{dy}{dx} = 0$$

docuit Eulerus semper in talem duci posse Multiplicatorem, ut altera aequationis pars evadat differentiale completum sive differentiale certae functionis variabilium x et y, in qua y pro functione ipsius x habetur *indefinita*. Similiter *aequatio differentialis partialis*

$$(1) \quad X - X_1 \frac{\partial x}{\partial x_1} - X_2 \frac{\partial x}{\partial x_2} - \cdots - X_n \frac{\partial x}{\partial x_n} = 0,$$

in qua $X, X_1, \ldots, X_n$ *designant variabilium* $x, x_1, \ldots, x_n$ *functiones, semper in talem duci potest Multiplicatorem, ut altera aequationis pars evadat Determinans functionale completum sive Determinans certarum n functionum variabilium* $x, x_1, x_2, \ldots, x_n$, *in quibus habetur* x *pro variabilium* $x_1, x_2, \ldots, x_n$ *functione indefinita.* Functio in aequationem (1) ducenda ipse est aequationis (1) *Multiplicator* supra appellatus et antecedentibus fusius explicatus. Unde nova nostri et Euleriani Multiplicatoris similitudo emergit novaque inter Determinantia functionalia et differentialia analogia.

Demonstratio Propositionis antecedentis sic patet. Designantibus rursus $f_1, f_2, \ldots, f_n$ solutiones a se independentes aequationis

$$X \frac{\partial f}{\partial x} + X_1 \frac{\partial f}{\partial x_1} + \cdots + X_n \frac{\partial f}{\partial x_n} = 0,$$

supra vidimus, semper dari Multiplicatorem M, in quem ductae ipsae $X, X_1, \ldots, X_n$ evadant functionum $f_1, f_2, \ldots, f_n$ Determinantia partialia, ita ut, ponendo pro functione f indefinita

$$\Sigma \pm \frac{\partial f}{\partial x} \cdot \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n} = A \frac{\partial f}{\partial x} + A_1 \frac{\partial f}{\partial x_1} + \cdots + A_n \frac{\partial f}{\partial x_n},$$

identice sit

$$MX = A, \quad MX_1 = A_1, \quad \ldots, \quad MX_n = A_n.$$

Hinc eruitur

$$(2) \quad \begin{cases} M\left\{X - X_1 \frac{\partial x}{\partial x_1} - X_2 \frac{\partial x}{\partial x_2} - \cdots - X_n \frac{\partial x}{\partial x_n}\right\} \\ = A - A_1 \frac{\partial x}{\partial x_1} - A_2 \frac{\partial x}{\partial x_2} - \cdots - A_n \frac{\partial x}{\partial x_n}. \end{cases}$$

At in Commentatione de Det. F. §. 17 (6) demonstravi, siquidem in functionibus $f_1, f_2, \ldots, f_n$ habeatur x pro variabilium $x_1, x_2, \ldots, x_n$ functione indefinita, fieri

$$(3) \quad \Sigma \pm \left(\frac{\partial f_1}{\partial x_1}\right)\left(\frac{\partial f_2}{\partial x_2}\right) \cdots \left(\frac{\partial f_n}{\partial x_n}\right) = A - A_1 \frac{\partial x}{\partial x_1} - A_2 \frac{\partial x}{\partial x_2} - \cdots - A_n \frac{\partial x}{\partial x_n}.$$

Qua in formula uncis innui, haberi x pro reliquarum variabilium $x_1, x_2, \ldots, x_n$

functione. Scilicet in Determinante functionali (3) substituendo ipsarum $\left(\frac{\partial f_i}{\partial x_k}\right)$ expressiones

$$\left(\frac{\partial f_i}{\partial x_k}\right) = \frac{\partial f_i}{\partial x_k} + \frac{\partial f_i}{\partial x} \cdot \frac{\partial x}{\partial x_k},$$

mutuo destruuntur termini omnes, in quibus inter se multiplicata inveniuntur differentialia partialia $\frac{\partial x}{\partial x_1}$, $\frac{\partial x}{\partial x_2}$, etc., ita ut horum differentialium non nisi ipsa expressio *linearis* remaneat, quae dextram partem aequationis (3) constituit. E (2) et (3) sequitur formula

$$(4) \quad \begin{cases} M\left\{X - X_1 \frac{\partial x}{\partial x_1} - X_2 \frac{\partial x}{\partial x_2} - \cdots - X_n \frac{\partial x}{\partial x_n}\right\} \\ \quad = \Sigma \pm \left(\frac{\partial f_1}{\partial x_1}\right)\left(\frac{\partial f_2}{\partial x_2}\right) \cdots \left(\frac{\partial f_n}{\partial x_n}\right). \end{cases}$$

Unde ducta aequatione (1) in Multiplicatorem eius M, altera eius pars identice aequatur Determinanti functionum $f_1, f_2, \ldots, f_n$, in quibus x pro variabilium $x_1, x_2, \ldots, x_n$ functione habetur indefinita. Q. d. e.

Formula (4) methodum suppeditat, ut Lagrangii appellatione utar, syntheticam ad eruendam aequationis (1) solutionem generalem. Nam secundum (4) aequatio (1) identice convenit cum sequente:

$$(5) \quad \Sigma \pm \left(\frac{\partial f_1}{\partial x_1}\right)\left(\frac{\partial f_2}{\partial x_2}\right) \cdots \left(\frac{\partial f_n}{\partial x_n}\right) = 0.$$

Quoties autem $f_1, f_2, \ldots, f_n$ sunt variabilium $x_1, x_2, \ldots, x_n$ functiones earumque Determinans identice evanescit, semper et sine ulla exceptione inter functiones $f_1, f_2, \ldots, f_n$ aliqua locum habere debet aequatio, et vice versa, si qua inter functiones $f_1, f_2, \ldots, f_n$ locum habet aequatio, earum Determinans evanescit (D. F. §. 7). Hinc docet formula (5), ut ipsius x expressio per $x_1, x_2, \ldots, x_n$ sit aequationis (1) solutio, sufficere et posci, post eius substitutionem ipsas $f_1, f_2, \ldots, f_n$ abire in tales variabilium $x_1, x_2, \ldots, x_n$ functiones, inter quas una quaecunque locum habeat aequatio. Unde vice versa dabitur solutio generalis petendo functionis quaesitae valorem ex aequatione arbitraria inter $f_1, f_2, \ldots, f_n$ posita

$$\Pi(f_1, f_2, \ldots, f_n) = 0;$$

sive, quod idem est, obtinetur aequationis (1) solutio nihilo aequando solutionem quamcunque aequationis

$$(6) \quad X \frac{\partial f}{\partial x} + X_1 \frac{\partial f}{\partial x_1} + \cdots + X_n \frac{\partial f}{\partial x_n} = 0.$$

Haec egregia methodus aequationem differentialem partialem (1) ad (6) revocandi cum ea convenit, quam olim Ill. Lagrange tradidit (*Hist. Ac. Ber.* ad a. 1779 pag. 152), ubi primum hanc quaestionem aggressus est. Quae prolixior quidem videri possit methodus quam aliae, quibus ipse Lagrange aliique postea usi sunt; qua de re ipse auctor eam ad exemplum tantum trium variabilium applicuit. Sane supponendo, aequationem inter x, x_1, $\ldots$, x_n quaesitam certe unam involvere Constantem arbitrariam α, eamque aequationem ipsius α respectu resolutam fieri $f = \alpha$, aequatio proposita (1) extemplo ad (6) reducitur. Sed eadem ratione omnes quoque inveniri solutiones a Constantibus arbitrariis prorsus vacuas, non ita bene per alias methodos constat atque illam Lagrangianam. Scilicet aequatio identica (4) docet, nullam dari exceptionem solutionis traditae, nisi forte exstet solutio, pro qua Multiplicator M evadat infinitus. Quodsi igitur more consueto solutionem eiusmodi exceptionalem seu quae generali se subducit appellamus *singularem*, methodus hic tradita rigorose demonstrat, *si qua exstet aequationis* (1) *solutio singularis, semper eam reddere Multiplicatorem aequationis infinitum.* Quod novam nostri Multiplicatoris similitudinem cum Euleriano manifestat.

Loco aequationis differentialis partialis (1) consideremus systema aequationum differentialium vulgarium cum ea connexum, atque systema aequationum integralium *singulare* appellemus, quod e completo non provenit tribuendo uni pluribusve Constantibus arbitrariis valores particulares seu unam pluresve relationes inter Constantes arbitrarias statuendo: quo facto ex antecedentibus haec eruitur Propositio.

Propositio I.

„*Proponantur aequationes differentiales*

$$dx : dx_1 : \ldots : dx_n = X : X_1 : \ldots : X_n,$$

earumque exstet systema aequationum integralium singulare, $n-1$ *Constantes arbitrarias involvens: eliminatis Constantibus arbitrariis e* n *aequationibus integralibus, prodit aequatio, quae Multiplicatorem systematis aequationum differentialium propositarum reddit infinitum.*"

Ut Propositio haec demonstretur, primum generaliter ponamus, aequationes integrales datas $n-1$ Constantibus arbitrariis affici. Quarum aequationum ubi $n-1$ resolvuntur Constantium arbitrariarum respectu, quod semper fieri posse suppono, harumque valores provenientes in n^{ta} aequatione integrali substituuntur,

obtinebitur aequatio a Constantibus arbitrariis vacua. E qua petatur unius variabilium, veluti x, valor per reliquas variabiles x_1, x_2, etc. expressus, atque in differentiali eius

$$dx = \frac{\partial x}{\partial x_1}\,dx_1 + \frac{\partial x}{\partial x_2}\,dx_2 + \cdots + \frac{\partial x}{\partial x_n}\,dx_n$$

substituantur aequationes differentiales propositae

$$(7)\quad dx : dx_1 : \ldots : dx_n = X : X_1 : \ldots : X_n;$$

eruitur

$$X = \frac{\partial x}{\partial x_1}\,X_1 + \frac{\partial x}{\partial x_2}\,X_2 + \cdots + \frac{\partial x}{\partial x_n}\,X_n,$$

sive ille ipsius x valor suppeditabit aequationis differentialis partialis (1) solutionem. Scilicet non fit, ut aequatio antecedens ex aliis $n-1$ aequationibus integralibus datis fluat, quippe e quibus supponitur non deduci posse alteram aequationem a Constantibus arbitrariis liberam. Eritque solutio illa aut particularis aut singularis, prout aequatio a Constantibus arbitrariis libera, cuius ope ipsa x per reliquas variabiles exprimebatur, in aequationem inter quantitates f_1, f_2, $\ldots$, f_n redit aut non redit. Iam demonstrabo, etiam systema aequationum integralium propositum iisdem casibus aut particulare aut singulare fore. Substituamus enim eum ipsius x valorem in $n-1$ aequationibus integralibus, quarum ope Constantes arbitrariae eliminabantur, simulque in functionibus X_1, X_2, $\ldots$, X_n: aequationibus illis, ut $n-1$ Constantes arbitrarias involventibus, *complete* integrantur aequationes differentiales

$$(8)\quad dx_1 : dx_2 : \ldots : dx_n = X_1 : X_2 : \ldots : X_n.$$

Unde quibuscunque aequationibus integralibus, $n-1$ Constantes arbitrarias involventibus, semper haec forma conciliari potest, ut earum una exhibeatur una variabilium x per reliquas variabiles x_1, x_2, etc., reliquae $n-1$ aequationes autem sint Integralia completa aequationum differentialium (8), in quibus ille ipsius x valor in functionibus X_1, X_2, $\ldots$, X_n substitutus est. Ponamus, aequationem illam a Constantibus arbitrariis vacuam, e qua valor ipsius x petitus est, redire in aequationem aliquam $F = 0$, designante F quantitatum f_1, f_2, $\ldots$, f_n functionem. Designantibus F, F_1, $\ldots$, F_{n-1} earundem f_1, f_2, $\ldots$, f_n functiones a se invicem independentes, dabitur aequationum differentialium propositarum (7) integratio completa per formulas

$$(9)\quad F = \alpha,\quad F_1 = \alpha_1,\quad \ldots,\quad F_{n-1} = \alpha_{n-1},$$

designantibus α, α_1, etc. Constantes arbitrarias. Ex aequatione $F = \alpha$ petito ipsius x valore eoque in functionibus $F_1, F_2, \ldots, F_{n-1}, X_1, X_2, \ldots, X_n$ substituto, evadunt

$$F_1 = \alpha_1, \quad F_2 = \alpha_2, \quad \ldots, \quad F_{n-1} = \alpha_{n-1}$$

Integralia completa aequationum differentialium

$$dx_1 : dx_2 : \ldots : dx_n = X_1 : X_2 : \ldots : X_n,$$

quae cum aequationibus differentialibus (8) supra consideratis conveniunt ponendo $\alpha = 0$. Unde ponendo $\alpha = 0$ in aequationum differentialium propositarum Integralibus completis (9), prodit systema aequationum integralium propositarum. Quippe quae redibant in aequationem, qua ipsa x exprimitur per reliquas variabiles et quae cum aequatione $F = 0$ conveniebat, atque in aequationum differentialium (8) Integralia completa, quae ex aequationibus $F_1 = \alpha_1$, $F_2 = \alpha_2, \ldots$, $F_{n-1} = \alpha_{n-1}$ obtinentur, eliminata x ope aequationis $F = 0$. Unde aequationibus differentialibus (7) integratis systemate aequationum, $n-1$ Constantes arbitrarias involventium, quoties aequatio eliminatione Constantium arbitrariarum proveniens redit in aequationem inter ipsas $f_1, f_2, \ldots, f_n$, illud aequationum integralium systema erit particulare, utpote e completo proveniens tribuendo Constanti arbitrariae valorem particularem. Hinc vice versa, si illud aequationum integralium systema non est particulare, aequatio eliminatione $n-1$ Constantium arbitrariarum proveniens non redit in aequationem inter quantitates $f_1, f_2, \ldots, f_n$, ideoque solutio, quam suppeditat, aequationis differentialis partialis (1) erit singularis. Cuiusmodi solutione, cum secundum antecedentibus probata efficiatur $M = \infty$, demonstratum est, quod propositum erat, *quoties systema aequationum differentialium vulgarium integretur systemate aequationum singulari, numerum Constantium arbitrariarum involvente unitate minorem quam completum involvit, Constantium arbitrariarum eliminatione provenire aequationem, qua Multiplicator systematis aequationum differentialium abeat in infinitum.* Et in hac propositione supponitur, quantitates X, X_1, etc. ita a denominatoribus purgatas esse, ut earum nulla pro illa aequatione integrali seu solutione singulari infinita evadat.

Propositionis antecedentis alia haec est demonstratio. Integratione completa exprimantur $x_1, x_2, \ldots, x_n$ per x et Constantes arbitrarias $\beta_1, \beta_2, \ldots, \beta_n$. Ponamus, aequationibus differentialibus satisfieri posse statuendo $\beta_1, \beta_2, \ldots, \beta_n$ esse ipsius x functiones; sequitur e formula

$$dx_i = \frac{\partial x_i}{\partial x} dx + \frac{\partial x_i}{\partial \beta_1} d\beta_1 + \frac{\partial x_i}{\partial \beta_2} d\beta_2 + \cdots + \frac{\partial x_i}{\partial \beta_n} d\beta_n$$

haec:

$$\frac{X_i}{X} dx = \frac{\partial x_i}{\partial x} dx + \frac{\partial x_i}{\partial \beta_1} d\beta_1 + \frac{\partial x_i}{\partial \beta_2} d\beta_2 + \cdots + \frac{\partial x_i}{\partial \beta_n} d\beta_n.$$

At eliminando quantitates $\beta_1, \beta_2, \ldots, \beta_n$ sequitur ex aequationibus integralibus positis

$$\frac{X_i}{X} = \frac{\partial x_i}{\partial x},$$

quippe quod prodire debebat ponendo $\beta_1, \beta_2, \ldots, \beta_n$ esse Constantes; illis autem eliminatis quantitatibus, perinde est sive constantes sive variabiles fuerint. Substituendo aequationem antecedentem eruitur pro singulis ipsius i valoribus

$$(10) \quad \frac{\partial x_i}{\partial \beta_1} d\beta_1 + \frac{\partial x_i}{\partial \beta_2} d\beta_2 + \cdots + \frac{\partial x_i}{\partial \beta_n} d\beta_n = 0.$$

Ut satisfiat n aequationibus, quae ponendo $i = 1, 2, \ldots, n$ ex antecedente fluunt, neque simul sit $d\beta_1 = d\beta_2 = \cdots = d\beta_n = 0$ sive $\beta_1, \beta_2, \ldots, \beta_n$ Constantes sint, evadere debet

$$(11) \quad \Sigma \pm \frac{\partial x_1}{\partial \beta_1} \cdot \frac{\partial x_2}{\partial \beta_2} \cdots \frac{\partial x_n}{\partial \beta_n} = 0.$$

Quoties poscitur, ut functiones $\beta_1, \beta_2, \ldots, \beta_n$ involvant $n-1$ Constantes arbitrarias, non fieri potest, ut aequatio (11) in relationem inter solas variabiles $\beta_1, \beta_2, \ldots, \beta_n$ redeat, sed fieri debet, ut e (11) peti possit ipsius x valor per $\beta_1, \beta_2, \ldots, \beta_n$ expressus; quo substituto in quantitatibus $\frac{\partial x_i}{\partial \beta_k}$, habebuntur e (10) $n-1$ aequationes differentiales primi ordinis inter quantitates $\beta_1, \beta_2, \ldots, \beta_n$, quibus complete integratis prodibunt $n-1$ aequationes inter quantitates $\beta_1, \beta_2, \ldots, \beta_n$, $n-1$ Constantibus arbitrariis affectae. Quibus $n-1$ aequationibus iuncta aequatione, qua x per $\beta_1, \beta_2, \ldots, \beta_n$ exprimebatur, ipsarumque β_1, β_2, etc. loco substitutis variabilium $x, x_1, \ldots, x_n$ functionibus, quibus per integrationem completam aequivalent, obtinetur systema aequationum integralium singularium, $n-1$ Constantibus arbitrariis affectum. Fit autem secundum §. 6

$$\Sigma \pm \frac{\partial x_1}{\partial \beta_1} \cdot \frac{\partial x_2}{\partial \beta_2} \cdots \frac{\partial x_n}{\partial \beta_n} = \frac{C}{X.\mu},$$

designante C quantitatum $\beta_1, \beta_2, \ldots, \beta_n$ functionem atque μ aequationum differentialium propositarum Multiplicatorem. Unde, cum supponatur, aequationem (11) non redire in relationem inter quantitates $\beta_1, \beta_2, \ldots, \beta_n$, porro ipsam X non infinitam evadere, sequitur e (11) $\mu = \infty$, q. d. e.

Secundum ea, quae §. 7 tradidi, Multiplicator M systematis aequationum differentialium post earum integrationem completam factam sic erui potest. Sint rursus Integralia completa

$$f_1 = \alpha_1, \quad f_2 = \alpha_2, \quad \ldots, \quad f_n = \alpha_n,$$

eorum ope exprimatur

$$-\frac{1}{X}\left\{\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n}\right\}$$

per $x, \alpha_1, \alpha_2, \ldots, \alpha_n$. Qua expressione integrata ipsius x respectu, prodeat

$$\varphi(x, \alpha_1, \alpha_2, \ldots, \alpha_n),$$

secundum §. 7 erit Multiplicator

$$e^{\varphi(x, f_1, f_2, \cdots, f_n)}.$$

Haec quantitas ut infinita evadat per solutionem seu aequationem integralem singularem, hoc est per solutionem seu aequationem integralem, quae non redeat in aequationem inter solas quantitates $f_1, f_2, \ldots, f_n$ (quod semper fieri vidimus, quoties omnino eiusmodi aequatio singularis exstat) ex ea aequatione talis provenire debet valor ipsius x per quantitates $f_1, f_2, \ldots, f_n$ expressus, quae quantitatem $\varphi(x, f_1, f_2, \ldots, f_n)$ reddat infinitam. A *fortiori* igitur pro eo ipsius x valore infinita evadere debet quantitas

$$\frac{\partial \varphi}{\partial x} = -\frac{1}{X}\left\{\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n}\right\},$$

cum generaliter, *quoties pro certo ipsius x valore infinita evadat functio aliqua $\varphi(x)$, pro eodem etiam infinita evadat functio $\frac{\partial \varphi}{\partial x}$ vel adeo $\frac{\partial \varphi}{\varphi \partial x}$* *). Supponimus autem, aequatione singulari non in infinitum abire quantitatem X, unde haec emergit Propositio.

Propositio II.

„Quoties exstat solutio singularis aequationis differentialis partialis

$$X = X_1 \frac{\partial x}{\partial x_1} + X_2 \frac{\partial x}{\partial x_2} + \cdots + X_n \frac{\partial x}{\partial x_n},$$

pro eadem fit

$$\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n} = \infty.\text{“}$$

*) Demonstrationem huius propositionis quivis sibi supplere potest.

Difficilius videtur solidis argumentis evincere propositionem inversam, videlicet quoties aequatio

$$\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n} = \infty$$

suppeditet aequationis differentialis partialis (1) solutionem, eam fore singularem. Neque video, solidam dari demonstrationem in casu elementari aequationis differentialis primi ordinis inter duas variabiles, cum in demonstrationibus passim traditis minus recte supponatur, functionem, quae pro $\alpha = 0$ evanescat, semper evolvi posse secundum ipsius α dignitates positivas.

Sub finem demonstretur de Multiplicatore nostro haec gravissima Propositio.

Propositio III.

„Quoties aequatio $M = 0$ aut $M = \infty$ est aequatio legitima, semper ea suppeditat solutionem aequationis differentialis partialis, seu aequationem integralem systematis aequationum differentialium vulgarium, cuius M est Multiplicator.“

Sit M aut $\frac{1}{M}$ aequale functioni u, ita ut aequatio $u = \infty$ alterutram significet aequationum $M = 0$ aut $\frac{1}{M} = 0$. Eam aequationem legitimam dico, si eius ope quaeque variabilium, quas continet, determinatur ut functio reliquarum, eiusque differentialia quoque prorsus definiantur differentialibus reliquarum variabilium. Statim patet, non esse legitimam aequationem $u = \infty$, si est $u = 1$; sed eo dicendi modo etiam non erit legitima huiusmodi aequatio $\frac{1}{x+y} = 0$, quippe qua non definitur y ut ipsius x functio, sed enunciatur tantum, $x+y$ esse functionem quamcunque per Constantem infinite magnam multiplicatam; neque definitur ipsius y incrementum, quod capit, ubi x in $x+dx$ abit, cum aequatio $x+y = \infty$ salva maneat, si x et y incrementa quaecunque a se independentia capiunt. Addo, si ex aequatione $u = \infty$ fluat variabilis x valor per $x_1, x_2, \ldots, x_n$ expressus, fractiones $\frac{\partial u}{\partial x_i} : \frac{\partial u}{\partial x}$ per aequationem $u = \infty$ infinitas evadere non posse, cum negative sumtae aequentur differentialibus partialibus functionis variabilium $x_1, x_2, \ldots, x_n$, cui x aequalis invenitur. His praeparatis, propositio tradita sic patet. Secundum aequationem differentialem partialem, qua M definitur, sequitur ex aequatione $u = \infty$

$$(12)\quad \begin{cases} X - X_1\dfrac{\partial x}{\partial x_1} - X_2\dfrac{\partial x}{\partial x_2} - \cdots - X_n\dfrac{\partial x}{\partial x_n} \\ = \pm\dfrac{1}{\dfrac{\partial \log u}{\partial x}}\left\{\dfrac{\partial X}{\partial x} + \dfrac{\partial X_1}{\partial x_1} + \cdots + \dfrac{\partial X_n}{\partial x_n}\right\}. \end{cases}$$

Iam si supponitur, uti supra, aequatione $u = \infty$ nullam quantitatum $X, X_1, \ldots, X_n$ infinitam reddi, quaelibet quantitatum ad dextram $\frac{\partial X_i}{\partial x_i} : \frac{\partial \log u}{\partial x}$ pro $u = \infty$ evanescit, etsi $\frac{\partial X_i}{\partial x_i}$ pro $u = \infty$ infinitum fiat. Quod sufficit probare de quantitate $\frac{\partial X_i}{\partial x_i} : \frac{\partial \log u}{\partial x_i}$, cum fractio $\frac{\partial u}{\partial x_i} : \frac{\partial u}{\partial x}$ valorem finitum habeat. Generale autem habetur lemma, cuius demonstrationi difficultatibus non obnoxiae hic brevitatis causa supersedeo, *si binae functiones pro certo variabilis valore altera infinita fiat, altera finita maneat, prioris differentiale pro eodem variabilis valore infinite maius fore quam posterioris differentiale.* Petendo autem ex aequatione $u = \infty$ valorem ipsius x_i, pro eo ipsius x_i valore secundum suppositionem factam X_i finita manet, dum $\log u$ infinitus evadit, unde fractiones $\frac{\partial X_i}{\partial x_i} : \frac{\partial \log u}{\partial x_i}$ ideoque etiam fractiones $\frac{\partial X_i}{\partial x_i} : \frac{\partial \log u}{\partial x}$ pro $u = \infty$ evanescunt. Unde, evanescente aequationis (12) parte dextra, aequatio $u = \infty$ suppeditat aequationis differentialis partialis (1) solutionem, ideoque etiam aequationem integralem systematis aequationum differentialium vulgarium (7).

Notione aequationis legitimae supra propositae solvitur paradoxon, quod in theoria integrationum singularium obvenit. Constat enim, rarissime aequationes differentiales gaudere integrationibus singularibus. At methodus Lagrangiana quandam prae se fert generalitatis speciem, quae in errorem inducere possit, ac si de quavis integratione completa deducere liceat singularem. Scilicet Ill. Lagrange de aequationibus $y = f(x, \alpha)$, $\frac{\partial f}{\partial \alpha} = 0$ ipsam α eliminare iubet; at in rarissimis casibus, quando $y = f(x, \alpha)$ est aequatio integralis completa, Constante arbitraria α affecta, fit $\frac{\partial f}{\partial \alpha} = 0$ aequatio legitima, qua sola hic uti licet. Idem ad methodum valet, qua supra de systemate aequationum integralium completarum deduxi aequationum integralium singularium systema, quod numerum Constantium arbitrariarum unitate minorem implicat.

Caput secundum.

De usu novi Multiplicatoris in aequationibus differentialibus integrandis. Principium ultimi Multiplicatoris.

§. 9.

De Multiplicatore aequationum differentialium transformatarum e propositarum derivando.

In aequationibus differentialibus propositis

$$(1)\quad dx : dx_1 : \ldots : dx_n = X : X_1 : \ldots : X_n$$

loco variabilium $x, x_1, \ldots, x_n$ aliae introducantur $w, w_1, \ldots, w_n$, quae supponuntur datae variabilium $x, x_1, \ldots, x_n$ functiones a se independentes, unde etiam $x, x_1, \ldots, x_n$ erunt quantitatum $w, w_1, \ldots, w_n$ functiones independentes. Cum fiat

$$dw_i = \frac{\partial w_i}{\partial x}dx + \frac{\partial w_i}{\partial x_1}dx_1 + \cdots + \frac{\partial w_i}{\partial x_n}dx_n,$$

sequitur ex aequationibus (1):

$$(2)\quad dw : dw_1 : \ldots : dw_n = W : W_1 : \ldots : W_n,$$

ponendo

$$(3)\quad W_i = \varDelta\left\{\frac{\partial w_i}{\partial x}X + \frac{\partial w_i}{\partial x_1}X_1 + \cdots + \frac{\partial w_i}{\partial x_n}X_n\right\},$$

ubi $\varDelta$ factor adhuc indeterminatus sit. Porro fit

$$\frac{\partial f}{\partial x_i} = \left(\frac{\partial f}{\partial w}\right)\frac{\partial w}{\partial x_i} + \left(\frac{\partial f}{\partial w_1}\right)\frac{\partial w_1}{\partial x_i} + \cdots + \left(\frac{\partial f}{\partial w_n}\right)\frac{\partial w_n}{\partial x_i},$$

siquidem uncis, quibus includimus differentialia partialia, innuimus functiones differentiandas per novas variabiles $w, w_1, \ldots, w_n$ exhibitas esse. Antecedente formula substituta et advocata (3), sequitur *pro quacunque functione f:*

$$(4)\quad \begin{cases} \varDelta\left\{X\dfrac{\partial f}{\partial x} + X_1\dfrac{\partial f}{\partial x_1} + \cdots + X_n\dfrac{\partial f}{\partial x_n}\right\} \\ = W\left(\dfrac{\partial f}{\partial w}\right) + W_1\left(\dfrac{\partial f}{\partial w_1}\right) + \cdots + W_n\left(\dfrac{\partial f}{\partial w_n}\right). \end{cases}$$

Aequationum (1) Multiplicator M definiebatur aequatione

$$(5)\quad M\left\{X\frac{\partial f}{\partial x} + X_1\frac{\partial f}{\partial x_1} + \cdots + X_n\frac{\partial f}{\partial x_n}\right\} = \Sigma \pm \frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdots\frac{\partial f_n}{\partial x_n}.$$

Similiter datur aequationum (2) Multiplicator N per formulam

$$(6)\quad \begin{cases} N\left\{W\left(\frac{\partial f}{\partial w}\right)+W_1\left(\frac{\partial f}{\partial w_1}\right)+\cdots+W_n\left(\frac{\partial f}{\partial w_n}\right)\right\} \\ \qquad = \Sigma\pm\left(\frac{\partial f}{\partial w}\right)\left(\frac{\partial f_1}{\partial w_1}\right)\cdots\left(\frac{\partial f_n}{\partial w_n}\right). \end{cases}$$

At secundum propositionem notam (*De Determ. Funct.* §. 11 Prop. II.) fit

$$(7)\quad \begin{cases} \qquad \Sigma\pm\frac{\partial f}{\partial x}\cdot\frac{\partial f_1}{\partial x_1}\cdots\frac{\partial f_n}{\partial x_n} \\ = \Sigma\pm\left(\frac{\partial f}{\partial w}\right)\left(\frac{\partial f_1}{\partial w_1}\right)\cdots\left(\frac{\partial f_n}{\partial w_n}\right).\Sigma\pm\frac{\partial w}{\partial x}\cdot\frac{\partial w_1}{\partial x_1}\cdots\frac{\partial w_n}{\partial x_n}. \end{cases}$$

Unde e (4), (5) obtinetur pro quacunque functione f:

$$(8)\quad \begin{cases} \frac{M}{\Delta}\left\{W\left(\frac{\partial f}{\partial w}\right)+W_1\left(\frac{\partial f_1}{\partial w_1}\right)+\cdots+W_n\left(\frac{\partial f_n}{\partial w_n}\right)\right\} \\ = \Sigma\pm\frac{\partial w}{\partial x}\cdot\frac{\partial w_1}{\partial x_1}\cdots\frac{\partial w_n}{\partial x_n}.\Sigma\pm\left(\frac{\partial f}{\partial w}\right)\left(\frac{\partial f_1}{\partial w_1}\right)\cdots\left(\frac{\partial f_n}{\partial w_n}\right). \end{cases}$$

Quam formulam comparando cum (6) sequitur, *posito in formula* (3)

$$(9)\quad \Delta = \frac{1}{\Sigma\pm\frac{\partial w}{\partial x}\cdot\frac{\partial w_1}{\partial x_1}\cdots\frac{\partial w_n}{\partial x_n}} = \Sigma\pm\left(\frac{\partial x}{\partial w}\right)\left(\frac{\partial x_1}{\partial w_1}\right)\cdots\left(\frac{\partial x_n}{\partial w_n}\right)$$

[Det. Funct. §. 9 (3)], *fieri* $N = M$, *sive aequationum differentialium propositarum* (1) *atque transformatarum* (2) *eundem fore Multiplicatorem.*

Servando factori Δ valorem (9), cum sit idem M aequationum (1) et (2) Multiplicator, fit e proprietate Multiplicatoris fundamentali

$$(10)\quad \begin{cases} 0 = X\frac{\partial M}{\partial x}+X_1\frac{\partial M}{\partial x_1}+\cdots+X_n\frac{\partial M}{\partial x_n} \\ \quad +M\left\{\frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}+\cdots+\frac{\partial X_n}{\partial x_n}\right\}, \end{cases}$$

$$(11)\quad \begin{cases} 0 = W\left(\frac{\partial M}{\partial w}\right)+W_1\left(\frac{\partial M}{\partial w_1}\right)+\cdots+W_n\left(\frac{\partial M}{\partial w_n}\right) \\ \quad +M\left\{\left(\frac{\partial W}{\partial w}\right)+\left(\frac{\partial W_1}{\partial w_1}\right)+\cdots+\left(\frac{\partial W_n}{\partial w_n}\right)\right\}. \end{cases}$$

At ponendo M pro functione indefinita f in formula (4) fit

$$X\frac{\partial M}{\partial x}+X_1\frac{\partial M}{\partial x_1}+\cdots+X_n\frac{\partial M}{\partial x_n} = \frac{1}{\Delta}\left\{W\left(\frac{\partial M}{\partial w}\right)+W_1\left(\frac{\partial M}{\partial w_1}\right)+\cdots+W_n\left(\frac{\partial M}{\partial w_n}\right)\right\}.$$

Unde de aequatione (11) per Δ divisa detrahendo aequationem (10) et dividendo

per M eruitur:

$$(12)\quad \frac{\partial X}{\partial x}+\frac{\partial X_1}{\partial x_1}+\cdots+\frac{\partial X_n}{\partial x_n} = \frac{1}{\Delta}\left\{\left(\frac{\partial W}{\partial w}\right)+\left(\frac{\partial W_1}{\partial w_1}\right)+\cdots+\left(\frac{\partial W_n}{\partial w_n}\right)\right\}.$$

Quae est formula memoratu digna, in qua X, X_1, ..., X_n sunt functiones quaecunque, ipsae autem Δ, W, W_1, ..., W_n formulis (9) et (3) definiuntur.

Si quantitates W, W_1, etc. per factorem communem Δ dividimus, per eundem multiplicandus erit aequationum (2) Multiplicator. Unde, si definimus quantitates W_i formula

$$W_i = \frac{\partial w_i}{\partial x}X+\frac{\partial w_i}{\partial x_1}X_1+\cdots+\frac{\partial w_i}{\partial x_n}X_n,$$

aequationum differentialium

$$dw : dw_1 : \ldots : dw_n = W : W_1 : \ldots : W_n$$

erit Multiplicator $\Delta.M$. Ponamus

$$t = \int\frac{dx}{X},$$

poterunt aequationes differentiales (1) sic proponi:

$$(13)\quad \frac{dx}{dt} = X,\quad \frac{dx_1}{dt} = X_1,\quad \ldots,\quad \frac{dx_n}{dt} = X_n;$$

unde sequitur

$$\frac{dw_i}{dt} = \frac{\partial w_i}{\partial x}X+\frac{\partial w_i}{\partial x_1}X_1+\cdots+\frac{\partial w_i}{\partial x_n}X_n,$$

sive

$$\frac{dw_i}{dt} = W_i.$$

Aequationum (1) *Multiplicatorem in sequentibus etiam appellabo Multiplicatorem aequationum* (13). Unde antecedentibus inventa sic poterunt enunciari:

Propositio I.

„Designantibus X, X_1, ..., X_n variabilium x, x_1, ..., x_n functiones quaslibet, proponantur aequationes differentiales

$$\frac{dx}{dt} = X,\quad \frac{dx_1}{dt} = X_1,\quad \ldots,\quad \frac{dx_n}{dt} = X_n,$$

quarum sit M Multiplicator; in quibus aequationibus ipsarum x, x_1, etc. loco aliae introducantur variabiles w, w_1, ..., w_n; quo facto si obtinentur aequationes differentiales

$$(14)\quad \frac{dw}{dt} = W, \quad \frac{dw_1}{dt} = W_1, \quad \ldots, \quad \frac{dw_n}{dt} = W_n,$$

harum aequationum Multiplicator erit $\Delta . M$, posito

$$\Delta \frac{1}{\Sigma \pm \frac{\partial w}{\partial x} \cdot \frac{\partial w_1}{\partial x_1} \cdots \frac{\partial w_n}{\partial x_n}} = \Sigma \pm \left(\frac{\partial x}{\partial w}\right)\left(\frac{\partial x_1}{\partial w_1}\right) \cdots \left(\frac{\partial x_n}{\partial w_n}\right).\text{“}$$

Ubi rursus quantitates W_i formula (3) definimus, formulam (12) sic proponere licet.

Propositio II.

„Ipsarum $x, x_1, \ldots, x_n$ loco introducendo $w, w_1, \ldots, w_n$, ponendoque

$$d\tau = \Sigma \pm \frac{\partial w}{\partial x} \cdot \frac{\partial w_1}{\partial x_1} \cdots \frac{\partial w_n}{\partial x_n} \cdot dt,$$

ex aequationibus differentialibus

$$\frac{dx}{dt} = X, \quad \frac{dx_1}{dt} = X_1, \quad \ldots, \quad \frac{dx_n}{dt} = X_n$$

proveniant sequentes:

$$\frac{dw}{d\tau} = W, \quad \frac{dw_1}{d\tau} = W_1, \quad \ldots, \quad \frac{dw_n}{d\tau} = W_n,$$

erit

$$\left\{\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n}\right\} dt = \left\{\left(\frac{\partial W}{\partial w}\right) + \left(\frac{\partial W_1}{\partial w_1}\right) + \cdots + \left(\frac{\partial W_n}{\partial w_n}\right)\right\} d\tau.\text{“}$$

In antecedentibus suppositum est, neque ipsas X, X_1, etc. implicare variabilem t neque eam variabilem afficere relationes, quae inter variabiles propositas $x, x_1, \ldots, x_n$ atque novas $w, w_1, \ldots, w_n$ intercedunt. *Si quantitates X, X_1, etc. praeter variabiles x, x_1, etc. ipsa quoque t afficiuntur, aequationum* (13) *Multiplicatorem eundem dicere placet atque aequationum*

$$(15)\quad dt : dx : dx_1 : \ldots : dx_n = 1 : X : X_1 : \ldots : X_n.$$

Designantibus x, x_1, etc. ipsarum $t, w, w_1, \ldots, w_n$, sive w, w_1, etc. ipsarum $t, x, x_1, \ldots, x_n$ functiones, ponamus rursus, ex aequationibus differentialibus (13) vel (15) sequi aequationes (14) sive aequationes

$$(16)\quad dt : dw : dw_1 : \ldots : dw_n = 1 : W : W_1 : \ldots : W_n,$$

atque aequationum (15) Multiplicatorem esse M, aequationum (16) Multiplicatorem $\Delta . M$. Quibus statutis, secundum antecedentia ad $n+2$ variabiles amplificata erit

$$\Delta = \Sigma \pm \left(\frac{\partial t}{\partial t}\right)\left(\frac{\partial x}{\partial w}\right)\left(\frac{\partial x_1}{\partial w_1}\right) \cdots \left(\frac{\partial x_n}{\partial w_n}\right).$$

Sed habetur $\left(\frac{\partial t}{\partial t}\right) = 1$, $\left(\frac{\partial t}{\partial w_i}\right) = 0$, unde

$$\Sigma \pm \left(\frac{\partial t}{\partial t}\right)\left(\frac{\partial x}{\partial w}\right)\left(\frac{\partial x_1}{\partial w_1}\right) \cdots \left(\frac{\partial x_n}{\partial w_n}\right) = \Sigma \pm \left(\frac{\partial x}{\partial w}\right)\left(\frac{\partial x_1}{\partial w_1}\right) \cdots \left(\frac{\partial x_n}{\partial w_n}\right).$$

Hinc sequitur, *Propositionem I. ad eum quoque casum valere, quo quantitates X, X_1, etc. atque functiones novis variabilibus aequandae w, w_1, etc. praeter ipsas x, x_1, etc. variabili t afficiuntur.*

Si tantum pro parte variabilium aliae introducuntur, ipsius Δ expressio simplicior evadit. Propositis enim aequationibus (13)

$$\frac{dx}{dt} = X, \quad \frac{dx_1}{dt} = X_1, \quad \ldots, \quad \frac{dx_n}{dt} = X_n,$$

quarum est M Multiplicator, si tantum loco variabilium $x, x_1, \ldots, x_\mu$ aliae introducuntur $w, w_1, \ldots, w_\mu$, ita ut aequationes differentiales transformatae fiant

$$\frac{dw}{dt} = W, \quad \frac{dw_1}{dt} = W_1, \quad \ldots, \quad \frac{dw_\mu}{dt} = W_\mu,$$
$$\frac{dx_{\mu+1}}{dt} = X_{\mu+1}, \quad \frac{dx_{\mu+2}}{dt} = X_{\mu+2}, \quad \ldots, \quad \frac{dx_n}{dt} = X_n,$$

fit harum Multiplicator $\Delta . M$, posito

$$\Delta = \Sigma \pm \left(\frac{\partial x}{\partial w}\right)\left(\frac{\partial x_1}{\partial w_1}\right) \cdots \left(\frac{\partial x_\mu}{\partial w_\mu}\right) = \frac{1}{\Sigma \pm \frac{\partial w}{\partial x} \cdot \frac{\partial w_1}{\partial x_1} \cdots \frac{\partial w_\mu}{\partial x_\mu}},$$

sicuti ex expressione generali ipsius Δ patet ponendo $w_{\mu+1} = x_{\mu+1}$, $w_{\mu+2} = x_{\mu+2}$, etc. Quae formulae variis applicationibus idoneae sunt.

§. 10.

Multiplicator aequationum differentialium ope Integralium completorum reductarum e Multiplicatore propositarum eruitur. Pro reductionibus diversis Multiplicatores alii de aliis deducuntur.

Per formulas §. pr. traditas facile solvitur quaestio, si aequationum differentialium

$$(1)\quad dx : dx_1 : \ldots : dx_n = X : X_1 : \ldots : X_n$$

inventa sint m Integralia

$$(2)\quad w=\alpha,\quad w_1=\alpha_1,\quad \ldots,\quad w_{m-1}=\alpha_{m-1},$$

designantibus $\alpha, \alpha_1, \ldots, \alpha_{m-1}$ Constantes arbitrarias, aequationum differentialium ope illorum Integralium reductarum Multiplicatorem e Multiplicatore propositarum investigandi. Sint enim $w_m, w_{m+1}, \ldots, w_n$ aliae variabilium $x, x_1, \ldots, x_n$ functiones a se ipsis et ab ipsis $w, w_1, \ldots, w_{m-1}$ independentes, inter quas propositum sit aequationes differentiales exhibere reductas. Poterunt $w, w_1, \ldots, w_n$ ipsarum $x, x_1, \ldots, x_n$ loco pro variabilibus in calculum introduci. Quo facto secundum §. pr. abeunt aequationes differentiales vulgares (1) in sequentes:

$$(3)\quad dw : dw_1 : dw_2 : \ldots : dw_n = W : W_1 : W_2 : \ldots : W_n,$$

siquidem statuitur

$$(4)\quad W_i = \Delta\left\{X\frac{\partial w_i}{\partial x}+X_1\frac{\partial w_i}{\partial x_1}+\cdots+X_n\frac{\partial w_i}{\partial x_n}\right\}.$$

Ponendo factorem Δ, quem ex arbitrio determinare licet, fieri

$$(5)\quad \Delta = \Sigma\pm\left(\frac{\partial x}{\partial w}\right)\left(\frac{\partial x_1}{\partial w_1}\right)\cdots\left(\frac{\partial x_n}{\partial w_n}\right) = \frac{1}{\Sigma\pm\frac{\partial w}{\partial x}\cdot\frac{\partial w_1}{\partial x_1}\cdots\frac{\partial w_n}{\partial x_n}},$$

vidimus §. pr., Multiplicatorem aequationum differentialium propositarum (1) eundem evadere atque Multiplicatorem aequationum transformatarum (3). Unde, designante M aequationum (1) Multiplicatorem, identice erit

$$(6)\quad \left(\frac{\partial(MW)}{\partial w}\right)+\left(\frac{\partial(MW_1)}{\partial w_1}\right)+\cdots+\left(\frac{\partial(MW_n)}{\partial w_n}\right)=0,$$

qua in formula $M, W, W_1, \ldots, W_n$ per variabiles $w, w_1, \ldots, w_n$ expressae finguntur. At cum sint (2) aequationum differentialium (1) Integralia, sequitur, esse $w, w_1, \ldots, w_{m-1}$ solutiones aequationis differentialis partialis

$$X\frac{\partial f}{\partial x}+X_1\frac{\partial f}{\partial x_1}+\cdots+X_n\frac{\partial f}{\partial x_n}=0,$$

unde patet e formula (4), identice fieri

$$(7)\quad W=0,\quad W_1=0,\quad \ldots,\quad W_{m-1}=0.$$

Unde aequatio (6) in hanc reducitur:

$$(8)\quad \left(\frac{\partial(MW_m)}{\partial w_m}\right)+\left(\frac{\partial(MW_{m+1})}{\partial w_{m+1}}\right)+\cdots+\left(\frac{\partial(MW_n)}{\partial w_n}\right)=0.$$

In aequatione antecedente expressae sunt MW_m, MW_{m+1}, etc. per $w, w_1, \ldots, w_n$,

sed differentiationes partiales solarum $w_m, w_{m+1}, \ldots, w_n$ respectu transiguntur. Unde in aequatione praecedente ipsis $w, w_1, \ldots, w_{m-1}$ substituere licet Constantes arbitrarias aequivalentes $\alpha, \alpha_1, \ldots, \alpha_{m-1}$. Idem si facimus in aequationibus differentialibus (3), obtinemus aequationes differentiales per inventa Integralia (2) reductas

$$(9)\quad dw_m : dw_{m+1} : \ldots : dw_n = W_m : W_{m+1} : \ldots : W_n,$$

in quibus sunt $W_m, W_{m+1}, \ldots, W_n$ ipsarum $w_m, w_{m+1}, \ldots, w_n$ et Constantium arbitrariarum $\alpha, \alpha_1, \ldots, \alpha_{m-1}$ functiones, in quas quantitatés (4) per inventa Integralia (2) abeunt. Simulque docet aequatio identica (8), ipsum M, per w_m, $w_{m+1}, \ldots, w_n$ atque $\alpha, \alpha_1, \ldots, \alpha_{m-1}$ expressum, foré aequationum quoque reductarum (9) Multiplicatorem.

Antecedentibus valores quantitatum W_i per talem factorem Δ multiplicavi, ut aequationum differentialium (1) atque (3) Multiplicator M idem fiat. Si in formulis (4) hunc factorem omittimus sive omnes quantitates W_i per factorem Δ dividimus, ipse M per eundem multiplicari debebat, sive aequationum (3) vel (9) Multiplicator poni debebat $\Delta.M$ (§. 9). Quod si facimus, antecedentibus inventa sic proponere licet.

Propositio I.

„Aequationum differentialium

$$dx : dx_1 : \ldots : dx_n = X : X_1 : \ldots : X_n,$$

quarum sit M Multiplicator, inventa sint m Integralia

$$w=\alpha,\quad w_1=\alpha_1,\quad \ldots,\quad w_{m-1}=\alpha_{m-1},$$

quorum ope variabiles $x, x_1, \ldots, x_n$ omnes exprimantur per Constantes arbitrarias $\alpha, \alpha_1, \ldots, \alpha_{m-1}$ atque variabilium $x, x_1, \ldots, x_n$ functiones

$$w_m,\quad w_{m+1},\quad \ldots,\quad w_n,$$

ponendo

$$W_i = X\frac{\partial w_i}{\partial x}+X_1\frac{\partial w_i}{\partial x_1}+\cdots+X_n\frac{\partial w_i}{\partial x_n},$$

dabuntur inter variabiles $w_m, w_{m+1}, \ldots, w_n$ aequationes differentiales

$$dw_m : dw_{m+1} : \ldots : dw_n = W_m : W_{m+1} : \ldots : W_n,$$

harumque Multiplicator erit

$$\Delta.M,$$

siquidem ponitur

$$\varDelta = \Sigma \pm \left(\frac{\partial x}{\partial w_m}\right)\left(\frac{\partial x_1}{\partial w_{m+1}}\right)\cdots\left(\frac{\partial x_{n-m}}{\partial w_n}\right)\left(\frac{\partial x_{n-m+1}}{\partial \alpha}\right)\left(\frac{\partial x_{n-m+2}}{\partial \alpha_1}\right)\cdots\left(\frac{\partial x_n}{\partial \alpha_{m-1}}\right)$$
$$= \left\{\Sigma \pm \frac{\partial w_m}{\partial x}\cdot\frac{\partial w_{m+1}}{\partial x_1}\cdots\frac{\partial w_n}{\partial x_{n-m}}\cdot\frac{\partial w}{\partial x_{n-m+1}}\cdot\frac{\partial w_1}{\partial x_{n-m+2}}\cdots\frac{\partial w_{m-1}}{\partial x_n}\right\}^{-1}.\text{“}$$

Quae est Propositio in theoria Multiplicatoris fundamentalis. Determinans inversum, quo $\varDelta$ exprimitur, sic quoque scribi potest:

$$\left\{\Sigma \pm \frac{\partial w}{\partial x}\cdot\frac{\partial w_1}{\partial x_1}\cdots\frac{\partial w_n}{\partial x_n}\right\}^{-1},$$

cum permutatione functionum w, w_1, etc. valor Determinantis tantum signum mutare queat, quod hic non curamus.

Pro ipsis w_m, w_{m+1}, ..., w_n etiam $n-m+1$ quantitates e numero ipsarum x, x_1, ..., x_n sumere licet. Si statuimus

$$w_m = x,\quad w_{m+1} = x_1,\quad \ldots,\quad w_n = x_{n-m},$$

fit

$$(10)\quad \begin{cases} \varDelta = \Sigma \pm \left(\frac{\partial x_{n-m+1}}{\partial \alpha}\right)\left(\frac{\partial x_{n-m+2}}{\partial \alpha_1}\right)\cdots\left(\frac{\partial x_n}{\partial \alpha_{m-1}}\right) \\ \quad = \left\{\Sigma \pm \frac{\partial w}{\partial x_{n-m+1}}\cdot\frac{\partial w_1}{\partial x_{n-m+2}}\cdots\frac{\partial w_{m-1}}{\partial x_n}\right\}^{-1}. \end{cases}$$

Porro e (4) obtinetur

$$W_m = X,\quad W_{m+1} = X_1,\quad \ldots,\quad W_n = X_{n-m}.$$

Hinc eruitur haec Propositio.

Propositio II.

„Aequationum differentialium

$$dx : dx_1 : \ldots : dx_n = X : X_1 : \ldots : X_n,$$

quarum M est Multiplicator, inventis m Integralibus

$$w = \alpha,\quad w_1 = \alpha_1,\quad \ldots,\quad w_{m-1} = \alpha_{m-1},$$

si exhibentur x_{n-m+1}, x_{n-m+2}, ..., x_n per x, x_1, ..., x_{n-m} atque Constantes arbitrarias α, α_1, ..., α_{m-1}, aequationum differentialium reductarum

$$dx : dx_1 : \ldots : dx_{n-m} = X : X_1 : \ldots : X_{n-m}$$

evadit Multiplicator:

$$M\Sigma \pm \left(\frac{\partial x_{n-m+1}}{\partial \alpha}\right)\left(\frac{\partial x_{n-m+2}}{\partial \alpha_1}\right)\cdots\left(\frac{\partial x_n}{\partial \alpha_{m-1}}\right)$$
$$= M\left\{\Sigma \pm \frac{\partial w}{\partial x_{n-m+1}}\cdot\frac{\partial w_1}{\partial x_{n-m+2}}\cdots\frac{\partial w_{m-1}}{\partial x_n}\right\}^{-1}.\text{“}$$

Si eaedem aequationes differentiales propositae per diversa Integralium systemata reducuntur, Multiplicatores diversorum aequationum differentialium reductarum systematum ex eorum uno deduci possunt. Qua in re semper supponitur, unumquodque Integrale, quod reductioni inservit, sua affici Constante arbitraria, ideoque aequationes differentiales reductas omnes ingredi Constantes arbitrarias, quibus Integralia, quorum ope reductio effecta est, afficiuntur.

Sint enim rursus Integralia reductioni adhibenda

$$w = \alpha,\quad w_1 = \alpha_1,\quad \ldots,\quad w_{m-1} = \alpha_{m-1},$$

atque aequationes differentiales reductae, inter variabiles w_m, w_{m+1}, ..., w_n exhibitae,

$$(11)\quad dw_m : dw_{m+1} : \ldots : dw_n = W_m : W_{m+1} : \ldots : W_n.$$

Eaedem aequationes differentiales propositae (1) ope Integralium

$$u = \beta,\quad u_1 = \beta_1,\quad \ldots,\quad u_{k-1} = \beta_{k-1}$$

reducantur ad has, inter variabiles u_k, u_{k+1}, ..., u_n exhibitas:

$$(12)\quad du_k : du_{k+1} : \ldots : du_n = U_k : U_{k+1} : \ldots : U_n.$$

Sit M Multiplicator aequationum differentialium propositarum, sint respective N et K Multiplicatores aequationum differentialium reductarum (11) et (12): erit secundum Prop. I.

$$(13)\quad \begin{cases} N = M\left\{\Sigma \pm \frac{\partial w}{\partial x}\cdot\frac{\partial w_1}{\partial x_1}\cdots\frac{\partial w_n}{\partial x_n}\right\}^{-1}, \\ K = M\left\{\Sigma \pm \frac{\partial u}{\partial x}\cdot\frac{\partial u_1}{\partial x_1}\cdots\frac{\partial u_n}{\partial x_n}\right\}^{-1}, \end{cases}$$

unde

$$(14)\quad K = N\,\frac{\Sigma \pm \frac{\partial w}{\partial x}\cdot\frac{\partial w_1}{\partial x_1}\cdots\frac{\partial w_n}{\partial x_n}}{\Sigma \pm \frac{\partial u}{\partial x}\cdot\frac{\partial u_1}{\partial x_1}\cdots\frac{\partial u_n}{\partial x_n}}.$$

Quae formula supponit, in aequationibus differentialibus reductis (11) et (12) ita definiri quantitates differentialibus proportionales, ut fiat

$$\frac{dw_m}{W_m} = \frac{du_k}{U_k}.$$

Si ipsae $w, w_1, \ldots, w_n$ per $u, u_1, \ldots, u_n$ exprimuntur, formulam (14) notae Propositionis beneficio (D. F. §. 10 (5)) concinnius sic exhibere licet:

$$(15)\quad K = N\Sigma \pm \frac{\partial w}{\partial u}\cdot\frac{\partial w_1}{\partial u_1}\cdots\frac{\partial w_n}{\partial u_n}.$$

Quae formula generalis duos amplectitur casus particulares, alterum, quo aequationes differentiales propositae per eadem Integralia reducuntur, sed reductae inter diversas variabiles exhibentur, alterum, quo per diversa Integralia reductae inter easdem variabiles exhibentur.

Etenim ponendo $k = m$ atque

$$u = w,\quad u_1 = w_1,\quad \ldots,\quad u_{m-1} = w_{m-1}$$

sequitur e (15), si eaedem aequationes differentiales propositae per eadem Integralia

$$w = \alpha,\quad w_1 = \alpha_1,\quad \ldots,\quad w_{m-1} = \alpha_{m-1}$$

reducantur ad $n-m$ aequationes differentiales inter $n-m+1$ variabiles w_m, $w_{m+1}, \ldots, w_n$ vel ad alias inter variabiles $u_m, u_{m+1}, \ldots, u_n$, fieri

$$(16)\quad K = N\Sigma \pm \frac{\partial w_m}{\partial u_m}\cdot\frac{\partial w_{m+1}}{\partial u_{m+1}}\cdots\frac{\partial w_n}{\partial u_n},$$

ubi $w_m, w_{m+1}, \ldots, w_n$ expressae supponuntur per variabiles $u_m, u_{m+1}, \ldots, u_n$ atque Constantes arbitrarias $\alpha, \alpha_1, \ldots, \alpha_{m-1}$.

Si vero rursus $k = m$ atque

$$u_m = w_m,\quad u_{m+1} = w_{m+1},\quad \ldots,\quad u_n = w_n,$$

vel si aequationes differentiales propositae per hoc m Integralium systema

$$w = \alpha,\quad w_1 = \alpha_1,\quad \ldots,\quad w_{m-1} = \alpha_{m-1},$$

aut per hoc

$$u = \beta,\quad u_1 = \beta_1,\quad \ldots,\quad u_{m-1} = \beta_{m-1}$$

reducuntur ad $n-m$ aequationes differentiales diversas inter easdem $n-m+1$ variabiles $w_m, w_{m+1}, \ldots, w_n$: abit formula (15) in hanc:

$$(17)\quad K = N\Sigma \pm \frac{\partial w}{\partial \beta}\cdot\frac{\partial w_1}{\partial \beta_1}\cdots\frac{\partial w_{m-1}}{\partial \beta_{m-1}},$$

siquidem in formando Determinante functionali supponitur expressas esse w, $w_1, \ldots, w_{m-1}$ per variabiles $w_m, w_{m+1}, \ldots, w_n$ atque Constantes arbitrarias $\beta, \beta_1, \ldots, \beta_m$.

§. 11.

Principium ultimi Multiplicatoris sive quomodo cognito Multiplicatore systematis aequationum differentialium vulgarium ultima integratio ad Quadraturas revocatur.

Propositionum I. et II. §. pr. prae ceteris memorabilis est casus $m = n-1$, quo, omnibus praeter unum inventis Integralibus, una integranda restat aequatio differentialis primi ordinis inter duas variabiles. Eo casu Multiplicator aequationis differentialis reductae redit in Multiplicatorem **Eulerianum**, qui eam per se integrabilem reddit sive ad Quadraturas revocat. Unde ponendo $n = m-1$ e Propp. I. et II. §. pr. memorabiles prodeunt Propositiones, quae novum constituunt principium, e quo Calculus Integralis haud parum incrementi capit. Quod *principium ultimi Multiplicatoris* appellare convenit.

Propositio I.

„*Propositis aequationibus differentialibus*

$$dx : dx_1 : \ldots : dx_n = X : X_1 : \ldots : X_n,$$

habeatur Multiplicator M sive solutio quaecunque aequationis differentialis partialis

$$\frac{\partial(MX)}{\partial x} + \frac{\partial(MX_1)}{\partial x_1} + \cdots + \frac{\partial(MX_n)}{\partial x_n} = 0;$$

porro inventa sint Integralia praeter unum omnia

$$w = \alpha,\quad w_1 = \alpha_1,\quad \ldots,\quad w_{n-2} = \alpha_{n-2},$$

designantibus α, etc. Constantes arbitrarias, quibus ipsae functiones w, w_1, etc. non afficiantur; sumtis ex arbitrio duabus ipsarum $x, x_1, \ldots, x_n$ functionibus w_{n-1}, w_n, fiat

$$X\frac{\partial w_{n-1}}{\partial x} + X_1\frac{\partial w_{n-1}}{\partial x_1} + \cdots + X_n\frac{\partial w_{n-1}}{\partial x_n} = W_{n-1},$$

$$X\frac{\partial w_n}{\partial x} + X_1\frac{\partial w_n}{\partial x_1} + \cdots + X_n\frac{\partial w_n}{\partial x_n} = W_n,$$

erit ultimum Integrale

$$\int \frac{M\{W_n dw_{n-1} - W_{n-1} dw_n\}}{\Sigma \pm \frac{\partial w}{\partial x}\cdot\frac{\partial w_1}{\partial x_1}\cdots\frac{\partial w_n}{\partial x_n}} = \text{Const.}“$$

Propositio II.

„Inventis aequationum differentialium

$$dx : dx_1 : \ldots : dx_n = X : X_1 : \ldots : X_n$$

Integralibus praeter unum omnibus

$$w = \alpha, \quad w_1 = \alpha_1, \quad \ldots, \quad w_{n-2} = \alpha_{n-2},$$

ac designante M solutionem quamcunque aequationis differentialis partialis

$$0 = \frac{\partial (MX)}{\partial x} + \frac{\partial (MX_1)}{\partial x_1} + \cdots + \frac{\partial (MX_n)}{\partial x_n},$$

exprimantur

$$x_2, \quad x_3, \quad \ldots, \quad x_n, \quad X, \quad X_1, \quad M$$

per x et x_1 atque Constantes arbitrarias

$$\alpha, \quad \alpha_1, \quad \ldots, \quad \alpha_{n-2}:$$

erit ultima aequatio integralis

$$\int \frac{M\{X_1 dx - X dx_1\}}{\Sigma \pm \frac{\partial w}{\partial x_2} \cdot \frac{\partial w_1}{\partial x_3} \cdots \frac{\partial w_{n-2}}{\partial x_n}} = \text{Const.“}$$

In duabus Propositionibus antecedentibus quantitas sub integrationis signo posita evadit differentiale completum, ubi expressiones in bina differentialia ducta per easdem duas variabiles exhibentur, inter quas aequatio differentialis reducta locum habet. Similiter in sequentibus, etsi pressis verbis non adnotetur, quoties formula integralis Constanti arbitrariae aequiparatur, innuitur, sub signo integrationis haberi differentiale completum.

In Propp. antecedentibus loco divisionis per Determinantia functionalia

$$\Sigma \pm \frac{\partial w}{\partial x} \cdot \frac{\partial w_1}{\partial x_1} \cdots \frac{\partial w_n}{\partial x_n},$$

$$\Sigma \pm \frac{\partial w}{\partial x_2} \cdot \frac{\partial w_1}{\partial x_3} \cdots \frac{\partial w_{n-2}}{\partial x_n}$$

etiam multiplicatio institui potuisset per Determinantia functionalia sensu inverso formata (*Det. Funct.* §. 9). Quod ubi fit, erit in altera Propositione ultima aequatio integralis

$$(1) \quad \int M\Delta(W_n dw_{n-1} - W_{n-1} dw_n) = \text{Const.},$$

posito

$$(2) \quad \begin{cases} \Delta = \Sigma \pm \frac{\partial x_2}{\partial \alpha} \cdot \frac{\partial x_3}{\partial \alpha_1} \cdots \frac{\partial x_n}{\partial \alpha_{n-2}} \cdot \frac{\partial x}{\partial w_{n-1}} \cdot \frac{\partial x_1}{\partial w_n} \\ \quad = \left\{ \Sigma \pm \frac{\partial w}{\partial x} \cdot \frac{\partial w_1}{\partial x_1} \cdots \frac{\partial w_n}{\partial x_n} \right\}^{-1}, \end{cases}$$

vel in altera

$$(3) \quad \int M\Delta(X_1 dx - X dx_1) = \text{Const.},$$

posito

$$(4) \quad \Delta = \Sigma \pm \frac{\partial x_2}{\partial \alpha} \cdot \frac{\partial x_3}{\partial \alpha_1} \cdots \frac{\partial x_n}{\partial \alpha_{n-2}} = \left\{ \Sigma \pm \frac{\partial w}{\partial x_2} \cdot \frac{\partial w_1}{\partial x_3} \cdots \frac{\partial w_{n-2}}{\partial x_n} \right\}^{-1}.$$

In formandis Determinantibus functionalibus (2) et (4) supponitur, aut ipsa $n-2$ Integralia dari novasque quoque variabiles w_{n-1}, w_n per x, x_1, $\ldots$, x_n expressas esse, aut per integrationes transactas variabiles omnes expressas esse per binas w_{n-1}, w_n vel x, x_1 atque per Constantes arbitrarias, quae singulis integrationibus accedunt. Generalius si reductio ad aequationem differentialem primi ordinis inter duas variabiles efficitur ope $n-1$ aequationum integralium quarumcunque

$$\Pi = 0, \quad \Pi_1 = 0, \quad \ldots, \quad \Pi_{n-2} = 0,$$

quae afficiuntur totidem Constantibus arbitrariis

$$\alpha, \quad \alpha_1, \quad \ldots, \quad \alpha_{n-2},$$

poni poterit in formula (2)

$$(5) \quad \Delta = \frac{\Sigma \pm \frac{\partial \Pi}{\partial \alpha} \cdot \frac{\partial \Pi_1}{\partial \alpha_1} \cdots \frac{\partial \Pi_{n-2}}{\partial \alpha_{n-2}}}{\Sigma \pm \frac{\partial \Pi}{\partial x_2} \cdot \frac{\partial \Pi_1}{\partial x_3} \cdots \frac{\partial \Pi_{n-2}}{\partial x_n} \cdot \frac{\partial w_{n-1}}{\partial x} \cdot \frac{\partial w_n}{\partial x_1}},$$

vel in formula (4)

$$(6) \quad \Delta = \frac{\Sigma \pm \frac{\partial \Pi}{\partial \alpha} \cdot \frac{\partial \Pi_1}{\partial \alpha_1} \cdots \frac{\partial \Pi_{n-2}}{\partial \alpha_{n-2}}}{\Sigma \pm \frac{\partial \Pi}{\partial x_2} \cdot \frac{\partial \Pi_1}{\partial x_3} \cdots \frac{\partial \Pi_{n-2}}{\partial x_n}}$$

(Cf. D. F. §. 10). Formula antecedens prae ceteris cum fructu adhibetur. Aequationibus enim integralibus inventis, saepissime per varias eliminationes eiusmodi formas induere licet, pro quibus Determinantia functionalia, quae numeratorem et denominatorem fractionis antecedentis constituunt, sine molestia inveniantur. Commode etiam adhiberi potest ad Determinantia functionalia formanda Propositio, valorem Determinantium functionalium

$$\Sigma \pm \frac{\partial w}{\partial x} \cdot \frac{\partial w_1}{\partial x_1} \cdots \frac{\partial w_n}{\partial x_n}, \quad \Sigma \pm \frac{\partial w}{\partial x_2} \cdot \frac{\partial w_1}{\partial x_3} \cdots \frac{\partial w_{n-2}}{\partial x_n}$$

non mutari, si ante differentiationes partiales transigendas functio quaeque w_i ope aequationum

$$(7) \quad w = \alpha, \quad w_1 = \alpha_1, \quad \ldots, \quad w_{i-1} = \alpha_{i-1}$$

mutationes quascunque subeat. Inservire possunt aequationes (7) ad eliminandas e quaque functione w_i variabiles

$$x_n, \quad x_{n-1}, \quad \ldots, \quad x_{n-i+1}.$$

Quo facto si abit w_i in Π_i, erunt

$$\Pi - \alpha = 0, \quad \Pi_1 - \alpha_1 = 0, \quad \ldots, \quad \Pi_{n-2} - \alpha_{n-2} = 0$$

aequationes integrales, quales per integrationem et eliminationem successivam inveniuntur. Porro fit

$$(8) \quad \Sigma \pm \frac{\partial w}{\partial x_2} \cdot \frac{\partial w_1}{\partial x_3} \cdots \frac{\partial w_{n-2}}{\partial x_n} = \frac{\partial \Pi}{\partial x_n} \cdot \frac{\partial \Pi_1}{\partial x_{n-1}} \cdots \frac{\partial \Pi_{n-2}}{\partial x_2}.$$

(Cf. §. 3.) Si vero adhibentur variabilium expressiones, quales ex eliminatione successiva prodeunt, videlicet ipsius x_n expressio per $x, x_1, \ldots, x_{n-1}, \alpha$; ipsius x_{n-1} expressio per $x, x_1, \ldots, x_{n-2}, \alpha, \alpha_1$, etc., abit Determinans

$$\Sigma \pm \frac{\partial x_2}{\partial \alpha} \cdot \frac{\partial x_3}{\partial \alpha_1} \cdots \frac{\partial x_n}{\partial \alpha_{n-2}}$$

in productum

$$\left(\frac{\partial x_n}{\partial \alpha}\right)\left(\frac{\partial x_{n-1}}{\partial \alpha_1}\right) \cdots \left(\frac{\partial x_2}{\partial \alpha_{n-2}}\right),$$

ubi uncis innuo, esse x_{n-i} ipsarum $x, x_1, \ldots, x_{n-i-1}, \alpha, \alpha_1, \ldots, \alpha_i$ functionem. Quibus substitutis in (4), fit

$$(9) \quad \varDelta = \left(\frac{\partial x_n}{\partial \alpha}\right)\left(\frac{\partial x_{n-1}}{\partial \alpha_1}\right) \cdots \left(\frac{\partial x_2}{\partial \alpha_{n-2}}\right) = \frac{1}{\frac{\partial \Pi}{\partial x_n} \cdot \frac{\partial \Pi_2}{\partial x_{n-1}} \cdots \frac{\partial \Pi_{n-2}}{\partial x_2}}.$$

Hinc sequentes emergunt Propositiones.

Propositio III.

„Aequationum differentialium vulgarium

$$dx : dx_1 : \ldots : dx_n = X : X_1 : \ldots : X_n,$$

quarum M est Multiplicator, inventis per integrationem et eliminationem successivam aequationibus integralibus praeter unam omnibus

$$\Pi = \alpha, \quad \Pi_1 = \alpha_1, \quad \ldots, \quad \Pi_{n-2} = \alpha_{n-2},$$

ubi Π_i est functio variabilium $x, x_1, \ldots, x_{n-i}$ atque Constantium arbitrariarum $\alpha, \alpha_1, \ldots, \alpha_{i-1}$: fit ultima aequatio integralis

$$\int \frac{M\{X_1 dx - X dx_1\}}{\frac{\partial \Pi}{\partial x_n} \cdot \frac{\partial \Pi_1}{\partial x_{n-1}} \cdots \frac{\partial \Pi_{n-2}}{\partial x_2}} = \text{Const.}“$$

Propositio IV.

„Aequationum differentialium vulgarium

$$dx : dx_1 : \ldots : dx_n = X : X_1 : \ldots : X_n,$$

quarum M est Multiplicator, inventis per integrationem et eliminationem successivam expressionibus ipsius x_n per $x, x_1, \ldots, x_{n-1}$ atque Constantem arbitrariam α; ipsius x_{n-1} per $x, x_1, \ldots, x_{n-2}$ atque Constantes arbitrarias α, α_1, etc., denique ipsius x_2 per x, x_1 atque Constantes arbitrarias $\alpha, \alpha_1, \ldots, \alpha_{n-2}$, dabitur aequatio inter x et x_1 per formulam

$$\int \left(\frac{\partial x_n}{\partial \alpha}\right)\left(\frac{\partial x_{n-1}}{\partial \alpha_1}\right) \cdots \left(\frac{\partial x_2}{\partial \alpha_{n-2}}\right) M(X_1 dx - X dx_1) = \text{Const.}“$$

In utraque Propositione functiones sub signo integrationis ope aequationum integralium inventarum per x et x_1 exprimendae sunt.

Quod e Multiplicatore aequationum differentialium propositarum eruitur Multiplicator aequationis differentialis, in quam post inventa praeter unum omnia Integralia problema redit, id eo maioris momenti est, quia huius ultimae aequationis differentialis primi ordinis inter duas variabiles valde latere potest Multiplicator, dum systematis aequationum differentialium propositarum sponte se offert. Veluti, quod in gravissimis quaestionibus evenit, si ipsarum X, X_1, etc. expressiones ita sunt comparatae, ut identice habeatur

$$\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n} = 0,$$

aequationum differentialium propositarum Multiplicator *unitati* aequalis evadit; aequationis autem postremo integrandae Multiplicator secundum antecedentia aequatur Determinanti functionali, cui valor complicatus competere potest. Casu illo particulari in quatuor Propositionibus antecedentibus ponere licet $M = 1$; quod ubi ex gr. in Prop. IV. facimus, emergit haec Propositio:

Propositio V.

„*Proponantur aequationes differentiales simultaneae*

$$dx : dx_1 : \dots : dx_n = X : X_1 : \dots : X_n,$$

designantibus X, X_1, *etc. variabilium* x, x_1, *etc. functiones, pro quibus identice habeatur*

$$\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \dots + \frac{\partial X_n}{\partial x_n} = 0;$$

inventis aequationum propositarum $n-1$ *Integralibus,* $n-1$ *Constantes arbitrarias* α, α_1, $\dots$, α_{n-2} *involventibus, exprimantur* X *et* X_1 *atque variabiles* x_2, x_3, $\dots$, x_n *per* x, x_1 *atque istas Constantes arbitrarias* α, α_1, $\dots$, α_{n-2}: *erit ultimum Integrale*

$$\int \left(\Sigma \pm \frac{\partial x_2}{\partial \alpha} \cdot \frac{\partial x_3}{\partial \alpha_1} \cdots \frac{\partial x_n}{\partial \alpha_{n-2}} \right) \left\{ X_1 dx - X dx_1 \right\} = \text{Const.},$$

ubi expressio sub integrationis signo differentiale completum existit."

Propositionis antecedentis afferam exempla pro $n = 2$ et $n = 3$.

I. „Proponantur aequationes differentiales

$$dx : dy : dz = X : Y : Z,$$

designantibus X, Y, Z variabilium x, y, z functiones, pro quibus identice fiat

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0;$$

invento uno Integrali involvente Constantem arbitrariam α, exprimantur X, Y, z per x, y, α, erit alterum Integrale

$$\int \frac{\partial z}{\partial \alpha} \{ Y dx - X dy \} = \text{Const.}“$$

II. „Proponantur aequationes differentiales

$$dt : dx : dy : dz = T : X : Y : Z,$$

designantibus T, X, Y, Z variabilium t, x, y, z functiones, pro quibus identice fiat

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0,$$

inventis duobus Integralibus involventibus Constantes arbitrarias α et β. exprimantur T, X, y, z per t, x, α, β; erit tertium Integrale

$$\int \left(\frac{\partial y}{\partial \alpha} \cdot \frac{\partial z}{\partial \beta} - \frac{\partial y}{\partial \beta} \cdot \frac{\partial z}{\partial \alpha} \right) (X dt - T dx) = \text{Const.}“$$

Quae exempla non sine molesto calculo verificantur.

§. 12.

Quibus casibus Multiplicator aequationum differentialium per aequationes integrales *particulares* reductarum ex aequationum differentialium propositarum Multiplicatore eruitur. Principium ultimi Multiplicatoris sine Determinantium adiumento comprobatum.

Si aequationes integrales, aequationibus differentialibus reducendis adhibitae, sunt particulares, in genere non licet Multiplicatorem aequationum differentialium reductarum e Multiplicatore propositarum deducere. In Prop. II. §. 10, quae docet, quomodo aequationum differentialium propositarum et reductarum Multiplicatores a se invicem pendeant, possunt quidem Constantibus arbitrariis, quibus Integralia afficiuntur, valores particulares tribui: supponitur autem, ipsa cognita esse aequationum differentialium propositarum Integralia generalia. Quae tamen suppositio necessaria non est. Etenim si aequationes integrales reductioni adhibendae alia post aliam investigantur, sufficit, unamquamque aequationem integralem inventam ita comparatam esse, ut differentiata per aequationes differentiales propositas identica reddatur, simul omnibus *ipsam praecedentibus* aequationibus integralibus accitis. Neque vero propositum succederet, si ex aequationibus integralibus reductioni adhibitis duae pluresve ita comparatae essent, ut quaeque earum differentiata per aequationes differentiales propositas identica reddi non possit, nisi simul omnes reliquae aequationes integrales, nullo ordine observato, in auxilium vocentur.

Antecedentia cum e formulis traditis patent tum ope Propositionis elementaris directe demonstrantur, quoties aequationes integrales alia post aliam inventae ad variabiles successive eliminandas adhibentur. Sit enim aequationum differentialium propositarum primum Integrale inventum

$$F = \alpha;$$

cujus ope e quantitatibus X, X_1, $\dots$, X_{n-1} eliminetur x_n. Ponendo $m = 1$ in Prop. II. §. 10 sequitur, *Multiplicatorem aequationum differentialium reductarum*

$$(1) \quad dx : dx_1 : \dots : dx_{n-1} = X : X_1 : \dots : X_{n-1}$$

aequari Multiplicatori aequationum differentialium propositarum diviso per $\frac{\partial F}{\partial x_n}$ *sive quantitati*

$$\frac{M}{\frac{\partial F}{\partial x_n}},$$

in qua variabilis x_n per aequationem $F = \alpha$ eliminanda est. Constans α in hac Propositione fundamentali arbitraria est ideoque valor ei quicunque tribui potest particularis.

Tributo in functionibus $X, X_1, \ldots, X_{n-1}$ Constanti α, quam implicant, valore particulari, sit aequationum (1) Integrale

$$F_1 = \alpha_1.$$

Quod non erit Integrale aequationum differentialium propositarum. Quippe aequatio $dF_1 = 0$ per aequationes differentiales propositas identica non redditur, nisi simul Constans α ubique functioni F aequatur. Quae Constantis α eliminatio ubi fit in functione F_1, aequatio $F_1 = \alpha_1$ evadit Integrale aequationum differentialium propositarum. Sed ea Constantis α eliminatio fieri non potest, si ei in aequationibus differentialibus reductis (1) tribuitur valor particularis, neque igitur eo casu ex aequationum differentialium reductarum Integrali Integrale propositarum restituere licet.

Eliminata x_{n-1} ope aequationis $F_1 = \alpha_1$, obtinentur e (1) aequationes differentiales denuo reductae

$$(2) \quad dx : dx_1 : \ldots : dx_{n-2} = X : X_1 : \ldots : X_{n-2}.$$

Quarum Multiplicator secundum eandem regulam derivatur e Multiplicatore aequationum (1), atque hic e Multiplicatore aequationum differentialium propositarum erutus est, videlicet dividendo per $\frac{\partial F_1}{\partial x_{n-1}}$, unde prodit aequationum (2) Multiplicator

$$\frac{M}{\frac{\partial F}{\partial x_n} \cdot \frac{\partial F_1}{\partial x_{n-1}}},$$

quae quantitas, variabilibus x_n et x_{n-1} per aequationes $F = \alpha$, $F_1 = \alpha_1$ eliminatis, solarum $x, x_1, \ldots, x_{n-2}$ functio evadit. Unde aequationum differentialium (2) erutus est Multiplicator, quamquam reductio facta est per duas aequationes $F = \alpha$, $F_1 = \alpha_1$, quarum tantum altera est aequationum differentialium propositarum Integrale, altera non est neque ad tale revocari potest, si Constanti α tributus est valor particularis.

Rursus tributo Constanti α_1 valore particulari quocunque, aequationum (2) quaeratur Integrale, quo invento aequationes differentiales (2) ulterius reduci possunt, reductarumque per eandem regulam constabit Multiplicator. Sic pergendo successive eruantur m aequationes integrales

$$(3) \quad F = \alpha, \quad F_1 = \alpha_1, \quad \ldots, \quad F_{m-1} = \alpha_{m-1},$$

in quibus $\alpha, \alpha_1, \ldots, \alpha_{m-1}$ sint Constantes particulares quaecunque; quarum aequationum integralium ope revocatis $X, X_1, \ldots, X_{n-m}$ ad solarum $x, x_1, \ldots, x_{n-m}$ functiones, aequationum differentialium, ad quas successiva eliminatione pervenitur,

$$(4) \quad dx : dx_1 : \ldots : dx_{n-m} = X : X_1 : \ldots : X_{n-m}$$

eruitur Multiplicator

$$\frac{M}{\frac{\partial F}{\partial x_n} \cdot \frac{\partial F_1}{\partial x_{n-1}} \cdots \frac{\partial F_{m-1}}{\partial x_{n-m+1}}},$$

quae quantitas et ipsa per aequationes (3) ad solarum $x, x_1, \ldots, x_{n-m}$ functionem revocanda est. Aequationes (3) reductionibus successivis inservientes hic ita comparatae sunt, ut quaeque $F_i = \alpha_i$ sit Integrale aequationum differentialium

$$dx : dx_1 : \ldots : dx_{n-i} = X : X_1 : \ldots : X_{n-i},$$

variabilibus $x_n, x_{n-1}, \ldots, x_{n-i+1}$ e $X, X_1, \ldots, X_{n-i}$ eliminatis ope aequationum ipsam $F_i = \alpha_i$ praecedentium

$$F = \alpha, \quad F_1 = \alpha_1, \quad \ldots, \quad F_{i-1} = \alpha_{i-1}.$$

Si $m = n-1$, formula (5) suppeditat Multiplicatorem aequationis differentialis primi ordinis inter duas variabiles x et x_1

$$(6) \quad X_1 dx - X dx_1 = 0,$$

quae post inventas aequationes integrales

$$(7) \quad F = \alpha, \quad F_1 = \alpha_1, \quad \ldots, \quad F_{n-2} = \alpha_{n-2}$$

unica integranda restat. Multiplicatore sic invento

$$\frac{M}{\frac{\partial F}{\partial x_n} \cdot \frac{\partial F_1}{\partial x_{n-1}} \cdots \frac{\partial F_{n-2}}{\partial x_2}}$$

laeva pars aequationis (6) evadit differentiale completum, unde eius integratio ad Quadraturas revocatur, sive fit ultima aequatio integralis

$$(8) \quad \int \frac{M(X_1 dx - X dx_1)}{\frac{\partial F}{\partial x_n} \cdot \frac{\partial F_1}{\partial x_{n-1}} \cdots \frac{\partial F_{n-2}}{\partial x_2}} = \text{Const.}$$

Qua in formula adiumento aequationum integralium inventarum (7) quantitates, sub integrationis signo in differentialia dx et dx_1 ductae, per solas x et x_1 exprimendae sunt.

Cum antecedentibus Constantes $\alpha, \alpha_1, \ldots, \alpha_{n-2}$ sint particulares *quaecunque*, earum valorem etiam generalem seu indefinitam servare licet, quo facto formula (8) redit in Prop. III. §. pr. Vice versa Prop. III. §. pr., in qua designant $\alpha, \alpha_1, \ldots, \alpha_{n-2}$ Constantes arbitrarias, eum quoque amplectitur casum, quo post quamque novam integrationem Constanti arbitrariae, qua afficitur, valor tribuitur particularis. Quod intelligitur observando, aequationibus differentialibus Constantes arbitrarias involventibus, idem earum Integrale obtineri posse, sive ante sive post integrationem Constantibus arbitrariis illis valores particulares tribuas.

Necessarium non est, ut quaeque nova aequatio integralis inveniatur ut Integrale ipsarum aequationum differentialium, ad quas propositae reducuntur, eliminato per aequationes integrales antea inventas aequali variabilium numero; generalius ea esse poterit Integrale aequationum differentialium propositarum, per aequationes integrales ante ipsam inventas quocunque modo transformatarum. Aequationum enim differentialium propositarum per Integrale $F = \alpha$ transformatarum sit Integrale $F_1 = \alpha_1$; aequationum differentialium propositarum per binas aequationes $F = \alpha$, $F_1 = \alpha_1$ transformatarum sit Integrale $F_2 = \alpha_2$, per tres aequationes $F = \alpha$, $F_1 = \alpha_1$, $F_2 = \alpha_2$ transformatarum sit Integrale $F_3 = \alpha_3$, et ita porro, ubi Constantes α, α_1, etc. poterunt arbitrariae esse sive particulares quaecunque. Quibus positis, ex aequatione integrali $F = \alpha$ et aequationibus differentialibus propositis sequi debet $dF_1 = 0$; unde per aequationem $F = \alpha$ eliminata x_n e functionibus $X, X_1, \ldots, X_{n-1}$, fieri debet $F_1 = \alpha_1$ Integrale aequationum differentialium

$$dx : dx_1 : \ldots : dx_{n-1} = X : X_1 : \ldots : X_{n-1}.$$

Ex aequationibus integralibus $F = \alpha$, $F_1 = \alpha_1$ et aequationibus differentialibus propositis sequi debet $dF_2 = 0$; unde per aequationes $F = \alpha$, $F_1 = \alpha_1$ eliminatis x_n et x_{n-1} e functionibus $X, X_1, \ldots, X_{n-2}$, fieri debet $F_2 = \alpha_2$ Integrale aequationum differentialium

$$dx : dx_1 : \ldots : dx_{n-2} = X : X_1 : \ldots : X_{n-2},$$

et ita porro. Generaliter si primum functiones F_1, F_2, etc. ratione illa generaliori, qua eas definivi, obtinebantur, ac deinde e quaque F_i eliminantur x_n,

$x_{n-1}, \ldots, x_{n-i+1}$ per aequationes $F = \alpha$, $F_1 = \alpha_1$, $\ldots$, $F_{i-1} = \alpha_{i-1}$, eaedem functiones F, F_1, F_2, etc. prodeunt, quas in formulis (5 et 8) consideravi. Ea autem reductione adhibita, abit Determinans functionale

$$\Sigma \pm \frac{\partial F}{\partial x_n} \cdot \frac{\partial F_1}{\partial x_{n-1}} \cdots \frac{\partial F_{m-1}}{\partial x_{n-m+1}}$$

in simplex productum

$$\frac{\partial F}{\partial x_n} \cdot \frac{\partial F_1}{\partial x_{n-1}} \cdots \frac{\partial F_{m-1}}{\partial x_{n-m+1}},$$

quod formulae (5) denominatorem afficit (§. 3). Unde si functionibus F, F_1, F_2, etc. generaliorem significationem servare placet, formula (5) evadere debet

$$(9) \qquad \frac{M}{\Sigma \pm \frac{\partial F}{\partial x_n} \cdot \frac{\partial F_1}{\partial x_{n-1}} \cdots \frac{\partial F_{m-1}}{\partial x_{n-m+1}}},$$

ideoque etiam formula (8)

$$(10) \qquad \int \frac{M\{X_1 dx - X dx_1\}}{\Sigma \pm \frac{\partial F}{\partial x_n} \cdot \frac{\partial F_1}{\partial x_{n-1}} \cdots \frac{\partial F_{n-2}}{\partial x_2}} = \text{Const.}$$

Definitio functionum F, F_1, etc. amplectitur casum, quo omnes aequationes $F_i = \alpha_i$ sunt ipsarum aequationum differentialium Integralia generalia. Unde e simplice Propositione elementari tradita derivatur principium ultimi Multiplicatoris, si reductio ad aequationem differentialem primi ordinis inter duas variabiles per Integralia generalia fit, simulque monstrantur casus maxime generales, quibus invenire liceat ultimum Multiplicatorem, etsi aequationes integrales reductioni adhibitae sint particulares.

Addam demonstrationem Propositionis fundamentalis, qua antecedentibus vidimus principium ultimi Multiplicatoris via maxime elementari adeoque absque ullo Determinantium adiumento superstrui.

Propositio.

„*Sit F solutio quaecunque aequationis*

$$X \frac{\partial F}{\partial x} + X_1 \frac{\partial F}{\partial x_1} + \cdots + X_n \frac{\partial F}{\partial x_n} = 0,$$

exclusa Constante; sit porro M solutio quaecunque aequationis

$$\frac{\partial (MX)}{\partial x} + \frac{\partial (MX_1)}{\partial x_1} + \cdots + \frac{\partial (MX_n)}{\partial x_n} = 0,$$

Constante non exclusa: posito

$$N = \frac{M}{\frac{\partial F}{\partial x_n}},$$

ipsisque $N, X, X_1, \ldots, X_{n-1}$ *per* $x, x_1, \ldots, x_{n-1}, F$ *expressis, fit* N *solutio aequationis*

$$\frac{\partial(NX)}{\partial x} + \frac{\partial(NX_1)}{\partial x_1} + \cdots + \frac{\partial(NX_{n-1})}{\partial x_{n-1}} = 0.\text{“}$$

Demonstratio.

Ponatur

$$\frac{\partial F}{\partial x_n} = u;$$

differentiando variabilis x_n respectu aequationem identicam

$$X\frac{\partial F}{\partial x} + X_1\frac{\partial F}{\partial x_1} + \cdots + X_n\frac{\partial F}{\partial x_n} = 0,$$

prodit

$$X\frac{\partial u}{\partial x} + X_1\frac{\partial u}{\partial x_1} + \cdots + X_n\frac{\partial u}{\partial x_n}$$
$$+\frac{\partial X}{\partial x_n}\cdot\frac{\partial F}{\partial x} + \frac{\partial X_1}{\partial x_n}\cdot\frac{\partial F}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n}\cdot\frac{\partial F}{\partial x_n} = 0.$$

Innuendo uncis, quibus differentialia partialia includantur, exhiberi X, X_1, etc. per $x, x_1, \ldots, x_{n-1}, F$, fit

$$\frac{\partial X_i}{\partial x_n} = \left(\frac{\partial X_i}{\partial F}\right)\frac{\partial F}{\partial x_n} = \left(\frac{\partial X_i}{\partial F}\right)u.$$

Quam formulam in aequatione praecedente substituendo atque per u dividendo prodit

$$X\frac{\partial \log u}{\partial x} + X_1\frac{\partial \log u}{\partial x_1} + \cdots + X_n\frac{\partial \log u}{\partial x_n}$$
$$+\left(\frac{\partial X}{\partial F}\right)\frac{\partial F}{\partial x} + \left(\frac{\partial X_1}{\partial F}\right)\frac{\partial F}{\partial x_1} + \cdots + \left(\frac{\partial X_n}{\partial F}\right)\frac{\partial F}{\partial x_n} = 0.$$

Haec formula detrahatur de sequente, quae ex ea, qua M definitur, fluit,

$$X\frac{\partial \log M}{\partial x} + X_1\frac{\partial \log M}{\partial x_1} + \cdots + X_n\frac{\partial \log M}{\partial x_n}$$
$$+\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n} = 0,$$

simulque observetur, haberi pro indicis i valoribus $1, 2, \ldots, n-1$

$$\frac{\partial X_i}{\partial x_i} = \left(\frac{\partial X_i}{\partial x_i}\right) + \left(\frac{\partial X_i}{\partial F}\right)\frac{\partial F}{\partial x_i},$$

prodit ponendo $\frac{M}{u} = N$:

$$X\frac{\partial \log N}{\partial x} + X_1\frac{\partial \log N}{\partial x_1} + \cdots + X_n\frac{\partial \log N}{\partial x_n}$$
$$+\left(\frac{\partial X}{\partial x}\right) + \left(\frac{\partial X_1}{\partial x_1}\right) + \cdots + \left(\frac{\partial X_{n-1}}{\partial x_{n-1}}\right) = 0.$$

Fit autem

$$X\frac{\partial \log N}{\partial x} + X_1\frac{\partial \log N}{\partial x_1} + \cdots + X_n\frac{\partial \log N}{\partial x_n}$$
$$= X\left(\frac{\partial \log N}{\partial x}\right) + X_1\left(\frac{\partial \log N}{\partial x_1}\right) + \cdots + X_{n-1}\left(\frac{\partial \log N}{\partial x_{n-1}}\right)$$
$$+\frac{\partial \log N}{\partial F}\left\{X\frac{\partial F}{\partial x} + X_1\frac{\partial F}{\partial x_1} + \cdots + X_n\frac{\partial F}{\partial x_n}\right\}$$
$$= X\left(\frac{\partial \log N}{\partial x}\right) + X_1\left(\frac{\partial \log N}{\partial x_1}\right) + \cdots + X_{n-1}\left(\frac{\partial \log N}{\partial x_{n-1}}\right),$$

aggregato in $\left(\frac{\partial \log N}{\partial F}\right)$ ducto identice evanescente. Unde aequatio antecedens sic quoque exhiberi potest:

$$X\left(\frac{\partial \log N}{\partial x}\right) + X_1\left(\frac{\partial \log N}{\partial x_1}\right) + \cdots + X_{n-1}\left(\frac{\partial \log N}{\partial x_{n-1}}\right)$$
$$+\left(\frac{\partial X}{\partial x}\right) + \left(\frac{\partial X_1}{\partial x_1}\right) + \cdots + \frac{\partial X_{n-1}}{\partial x_{n-1}} = 0,$$

quae per N multiplicata suppeditat

$$\left(\frac{\partial(NX)}{\partial x}\right) + \left(\frac{\partial(NX_1)}{\partial x_1}\right) + \cdots + \left(\frac{\partial(NX_{n-1})}{\partial x_{n-1}}\right) = 0,$$

quae est formula demonstranda.

Vidimus supra, Propositione antecedente iteratis vicibus adhibita erui aequationum differentialium reductarum Multiplicatorem e Multiplicatore propositarum. Sed ad hunc finem non necesse est, ut hic ipse cognoscatur, sed sufficit eius cognoscere valorem, quem per aequationes integrales reductioni adhibitas induere potest. Si problema ad aequationem differentialem primi ordinis inter x et x_1 revocatum est, definitur M aequationibus

$$(11)\quad \begin{cases} \dfrac{d\log M}{dx} = -\dfrac{1}{X}\left(\dfrac{\partial X}{\partial x} + \dfrac{\partial X_1}{\partial x_1} + \cdots + \dfrac{\partial X_n}{\partial x_n}\right), \\ X_1 dx = X dx_1, \end{cases}$$

in quibus *post differentiationes partiales factas* eliminandae sunt $x_2, x_3, \ldots, x_n$. Si aequationes integrales, quarum ope reductiones et eliminationes propositae operantur, particulares sunt, evenire potest, ut e formulis (11) eruatur valor ipsius M ad formandum ultimum Multiplicatorem requisitus, neque tamen inveniri queat ipsius M valor generalis sive ipsarum aequationum differentialium propositarum Multiplicator. Directe aequationis differentialis

$$X_1 dx - X dx_1 = 0$$

definitur Multiplicator P per formulam

$$(12)\quad \frac{d\log P}{dx} = -\frac{1}{X}\left(\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1}\right),$$

in cuius dextra parte X et X_1 *ante differentiationes partiales transigendas* per solas x et x_1 exprimendae sunt. Potest autem evenire, ut via non pateat, qua ipsum P e (12) eruatur, dum ipsius M determinatio per formulam (11) in promptu est. Quae adeo, nullis cognitis aequationibus integralibus, in amplis gravissimisque problematis succedit, unde pro quibuscunque aequationibus integralibus reductioni adhibitis sive completis sive dicta ratione inventis particularibus ultimus Multiplicator constat.

§. 13.

De usu Multiplicatoris in integrandis systematis quibusdam aequationum differentialium specialibus.

Systema aequationum differentialium propositarum ita comparatum esse potest, ut ultima Integratio sponte in Quadraturam redeat. Quod evenit, si unius variabilis differentiale tantum, non ipsa in aequationibus differentialibus invenitur. Ponamus, ipsam x esse variabilem, a qua simul omnes functiones vacuae sint $X, X_1, \ldots, X_n$: redire constat integrationem n aequationum differentialium inter $n+1$ variabiles

$$(1)\quad dx : dx_1 : dx_2 : \ldots : dx_n = X : X_1 : X_2 : \ldots : X_n$$

in integrationem $n-1$ aequationum differentialium inter n variabiles unamque Quadraturam. Integratis enim aequationibus

$$(2)\quad dx_1 : dx_2 : \ldots : dx_n = X_1 : X_2 : \ldots : X_n,$$

quae sunt $n-1$ aequationes differentiales inter n variabiles $x_1, x_2, \ldots, x_n$, exhiberi poterunt variabiles $x_1, x_2, \ldots, x_n$ per earum unam veluti x_1: unde, expressa $\frac{X}{X_1}$ per x_1, dabit simplex Quadratura ipsius x valorem

$$(3)\quad x = \int \frac{X dx_1}{X_1} + \text{Const.}$$

Iam cognito aequationum differentialium (1) Multiplicatore quaeritur, quemnam ex eo fructum ad integrationem perficiendam percipere liceat, cum ultima integratio sua sponte in Quadraturam redeat. Quod ut cognoscatur, inter duos casus distinguendum erit, prout datus aequationum differentialium (1) Multiplicator a variabili x afficiatur sive non afficiatur.

Aequationum differentialium (2) systema vocabo *proprium*, quo distinguatur a systemate *proposito* aequationum differentialium (1), cuius integratio componitur ex integratione systematis proprii et Quadratura. Si datus systematis propositi Multiplicator M et ipse a variabili x vacuus est, idem erit systematis proprii Multiplicator. Tum enim evanescente termino $\frac{\partial (MX)}{\partial x}$, satisfaciet aequationum differentialium (1) Multiplicator aequationi

$$\frac{\partial (MX_1)}{\partial x_1} + \frac{\partial (MX_2)}{\partial x_2} + \cdots + \frac{\partial (MX_n)}{\partial x_n} = 0;$$

eadem autem aequatione definitur aequationum differentialium (2) Multiplicator. Quoties igitur datus systematis propositi (1) Multiplicator et ipse variabili x vacat, systematis proprii ultima integratio ad Quadraturas revocari potest, sive, quod idem est, *systematis aequationum differentialium propositarum duae ultimae integrationes per Quadraturas absolvuntur.*

Vice versa si datur systematis proprii (2) Multiplicator N, qui erit solarum variabilium $x_1, x_2, \ldots, x_n$ functio, idem erit systematis propositi (1) Multiplicator. Evanescente enim termino $\frac{\partial (NX)}{\partial x}$, functio N, quae huic aequationi satisfacere debet

$$0 = \frac{\partial (NX_1)}{\partial x_1} + \frac{\partial (NX_2)}{\partial x_2} + \cdots + \frac{\partial (NX_n)}{\partial x_n},$$

etiam huic satisfaciet, qua systematis propositi Multiplicator definitur,

$$0 = \frac{\partial (NX)}{\partial x} + \frac{\partial (NX_1)}{\partial x_1} + \cdots + \frac{\partial (NX_n)}{\partial x_n}.$$

Inventis autem omnibus systematis proprii Integralibus

$$(4)\quad f_1 = \alpha_1,\quad f_2 = \alpha_2,\quad \ldots,\quad f_{n-1} = \alpha_{n-1},$$

ubi Constantes arbitrariae α_1 etc. dextram aequationum partem occupant, erit aequationum (2) Multiplicator

$$(5)\quad N = \frac{1}{X_n}\,\Sigma\pm\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_{n-1}}{\partial x_{n-1}}.$$

Qui igitur systematis quoque propositi Multiplicator erit. Unde si systematis propositi datur Multiplicator M, variabilem x implicans, simulque systema proprium complete integratum est, duo innotescunt systematis propositi Multiplicatores M et N. Quibus cognitis, secundum §. 4 systematis propositi constabit Integrale

$$(6)\quad \frac{N}{M} = \frac{1}{MX_n}\,\Sigma\pm\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\cdots\frac{\partial f_{n-1}}{\partial x_{n-1}} = \text{Const.}$$

Quo Integrali dabitur x per $x_1, x_2, \ldots, x_n$, sive ope Integralium (4) expressis $x_2, x_3, \ldots, x_n$ per x_1, dabitur x per x_1. Unde *si innotescit systematis propositi Multiplicator variabili x affectus, post systematis proprii integrationem completam non amplius opus erit Quadratura, quam formula (3) poscebat ad inveniendum ipsius x valorem per x_1 expressum.*

Fieri potest, ut, solo cognito systematis propositi Multiplicatore variabili x affecto, absque ulla integratione eruantur systematis proprii unum plurave Integralia. Expressa enim per (4) quantitate $\frac{X}{X_1}$ per $x_1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$, in functione

$$\int\frac{X dx_1}{X_1}$$

post factam integrationem Constantium α_1, α_2, etc. loco restituamus functiones f_1, f_2, etc.; quo facto prodeat variabilium $x_1, x_2, \ldots, x_n$ functio

$$\xi = \int\frac{X dx_1}{X_1}:$$

erit e (3), designante α_n novam Constantem arbitrariam,

$$x - \xi = \alpha_n$$

systematis propositi Integrale. Sit rursus variabilium $x_1, x_2, \ldots, x_n$ functio N systematis proprii ideoque etiam systematis propositi Multiplicator, erit se-

cundum §. 4 expressio generalis Multiplicatoris systematis propositi

$$M = \Pi(x-\xi, f_1, f_2, \ldots, f_{n-1}).N.$$

Cognito igitur valore ipsius M, variabili x affecto, erit $\frac{\partial \log M}{\partial x}$ ipsarum $x-\xi$, $f_1, f_2, \ldots, f_{n-1}$ functio

$$\frac{\partial \log M}{\partial x} = \Phi(x-\xi, f_1, f_2, \ldots, f_{n-1}).$$

Unde ponendo

$$(7)\quad \frac{\partial \log M}{\partial x} = u,$$

atque ex hac aequatione quaerendo ipsius x valorem per $u, x_1, x_2, \ldots, x_n$ expressum, prodit

$$x = \xi + \psi(u, f_1, f_2, \ldots, f_{n-1}),$$

designante ψ certam ipsarum $u, f_1, f_2, \ldots, f_{n-1}$ functionem. Quaerendo igitur e (7) ipsius x valorem per $u, x_1, x_2, \ldots, x_n$ expressum, atque in ea expressione ipsius u loco ponendo varios valores constantes arbitrarios, differentiae quantitatum provenientium erunt solarum $f_1, f_2, \ldots, f_n$ functiones, ideoque Constantibus arbitrariis aequiparatae suppeditabunt systematis proprii Integralia. Methodus hic tradita semper succedit, si non tantum M sed etiam $\frac{\partial \log M}{\partial x}$ ipsam x involvit atque ψ non solius u vel Φ non solius $x-\xi$ functio est. Quoties autem $\Phi = \frac{\partial \log M}{\partial x}$ solius $x-\xi$ functio est, erit $\frac{\partial \Phi}{\partial x}$ ipsius Φ functio. Unde *e systematis propositi Multiplicatore cognito M semper deducere licet absque integratione systematis proprii unum plurave Integralia, quoties $\frac{\partial^2 \log M}{\partial x^2}$ non ipsius $\frac{\partial \log M}{\partial x}$ functio est.* Similiter demonstratur, *cognito systematis propositi Integrali, variabili x affecto, $v = \alpha$, designante α Constantem arbitrariam, ex eo semper derivari posse unum plurave systematis proprii Integralia, nisi $\frac{\partial v}{\partial x}$ ipsius v functio sit.* Nam cum esse debeat v quantitatum $x-\xi, f_1, f_2, \ldots, f_{n-1}$ functio, ex aequatione $v = \alpha$ sequitur huiusmodi

$$x = \xi + \psi(\alpha, f_1, f_2, \ldots, f_{n-1});$$

unde eruendo e $v = \alpha$ ipsius x valore in eoque ponendo ipsius α loco varios valores constantes arbitrarios, differentiae expressionum provenientium Constantibus arbitrariis aequiparatae suppeditabunt systematis proprii Integralia.

Ut habeatur exemplum, quo systematis propositi Multiplicator variabili x affectus innotescit ideoque post systematis proprii integrationem completam ipsa x per $x_1, x_2, \ldots, x_n$ absque Quadratura exprimitur, ponamus $X = 1$ simulque fieri

$$\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n} = c,$$

designante c quantitatem constantem; quod inter alia evenit, si X_1, X_2, etc. variabilium x_1, x_2, etc. functiones sunt lineares. Dabitur systematis propositi Multiplicator per formulam

$$\frac{d \log M}{dx} + c = 0, \quad \text{unde} \quad M = e^{-cx}.$$

Hinc sequitur e (6) sumendo logarithmos:

$$x = -\frac{1}{c} \log \left(\frac{1}{X_n} \Sigma \pm \frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_{n-1}}{\partial x_{n-1}} \right) + \text{Const.}$$

Cognitione igitur Multiplicatoris in hoc exemplo non reductionem aequationis differentialis ad Quadraturas sed Quadraturam lucramur.

Antecedentibus demonstratum est, *si aequationum differentialium* (1), *in quibus X, X_1, etc. solarum $x_1, x_2, \ldots, x_n$ functiones sunt, detur Multiplicator et ipse variabili x vacans, duas postremas integrationes per Quadraturam absolvi; si Multiplicator variabili x afficiatur, ultimam aequationem integralem ipsam sine Quadratura obtineri.* Quae Propositio sic amplificatur.

Ponamus, functiones $X_{m+1}, X_{m+2}, \ldots, X_n$ vacuas esse a variabilibus x, $x_1, \ldots, x_m$, simulque $X, X_1, \ldots, X_m$, nisi ab iisdem variabilibus vacuae sunt, certe satisfacere conditioni

$$(7) \quad \frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_m}{\partial x_m} = 0.$$

Eo casu aequationes differentiales propositae (1) sic tractabuntur, ut primum aequationum differentialium inter solas $x_{m+1}, x_{m+2}, \ldots, x_n$ locum habentium

$$(8) \quad dx_{m+1} : dx_{m+2} : \ldots : dx_n = X_{m+1} : X_{m+2} : \ldots : X_n$$

quaerantur Integralia

$$(9) \quad f_1 = \alpha_1, \quad f_2 = \alpha_2, \quad \ldots, \quad f_{n-m-1} = \alpha_{n-m-1},$$

eorumque ope exprimantur variabiles $x_{m+1}, x_{m+2}, \ldots, x_n$ per earum unam x_{m+1}: quibus factis superest, ut integrentur aequationes differentiales inter ipsas x, $x_1, \ldots, x_{m+1}$ locum habentes

$$(10) \quad dx : dx_1 : \ldots : dx_{m+1} = X : X_1 : \ldots : X_{m+1}.$$

Per conditionem (7) constat, aequationum differentialium propositarum (1) Multiplicatorem, a variabilibus $x, x_1, \ldots, x_m$ vacuum, eundem esse atque aequationum differentialium (8) Multiplicatorem, et vice versa harum Multiplicatorem ipsarum quoque aequationum differentialium (1) Multiplicatorem esse. Designante enim M quantitatem a variabilibus $x, x_1, \ldots, x_m$ vacuam, sequitur e (7)

$$\frac{\partial (MX)}{\partial x} + \frac{\partial (MX_1)}{\partial x_1} + \cdots + \frac{\partial (MX_m)}{\partial x_m} = 0,$$

unde pro eiusmodi ipsius M valore conditio, ut M aequationum (1) sit Multiplicator,

$$\frac{\partial (MX)}{\partial x} + \frac{\partial (MX_1)}{\partial x_1} + \cdots + \frac{\partial (MX_n)}{\partial x_n} = 0$$

convenit cum conditione, ut M aequationum (8) Multiplicator sit,

$$\frac{\partial (MX_{m+1})}{\partial x_{m+1}} + \frac{\partial (MX_{m+2})}{\partial x_{m+2}} + \cdots + \frac{\partial (MX_n)}{\partial x_n} = 0.$$

Aequationum differentialium (10) *semper assignare licet Multiplicatorem.* Nam cum ipsarum $x_{m+2}, x_{m+3}, \ldots, x_n$ expressiones per x_{m+1} e (9) petitae ab ipsis $x, x_1, \ldots, x_m$ vacuae sint, conditio (7) valebit etiam post harum expressionum substitutionem. Qua substitutione cum X_{m+1} in solius x_{m+1} functionem abeat, valebit etiam aequatio (7), si loco ipsarum X_i ponitur $\frac{X_i}{X_{m+1}}$. Unde sequitur, *aequationum differentialium* (10) *Multiplicatorem esse* $\frac{1}{X_{m+1}}$. Qua de re *aequationum differentialium* (10) *ultima integratio semper solis Quadraturis absolvitur.*

Si datur Multiplicator aequationum differentialium propositarum (1), variabilibus $x, x_1, \ldots, x_m$ non affectus, idem erit aequationum (8) Multiplicator, ideoque eo casu cum aequationum (8) tum aequationum (10) ultima integratio Quadraturis absolvitur. Iam vero sit aequationum differentialium propositarum (1) datus Multiplicator M variabilibus $x, x_1, \ldots, x_m$ affectus. Inventis aequationum differentialium (8) Integralibus (9), earum fit Multiplicator

$$N = \frac{1}{X_{m+1}} \Sigma \pm \frac{\partial f_1}{\partial x_{m+2}} \cdot \frac{\partial f_2}{\partial x_{m+3}} \cdots \frac{\partial f_{n-m-1}}{\partial x_n},$$

idemque ex antecedentibus fit Multiplicator aequationum differentialium propositarum (1). Quarum igitur cognitis duobus Multiplicatoribus M et N, datur absque Quadratura Integrale

$$\frac{N}{M} = \frac{1}{MX_{m+1}} \Sigma \pm \frac{\partial f_1}{\partial x_{m+2}} \cdot \frac{\partial f_2}{\partial x_{m+3}} \cdots \frac{\partial f_{n-m-1}}{\partial x_n} = \text{Const.}$$

Quod substituendo ipsarum $x_{m+2}, x_{m+3}, \ldots, x_n$ valores per x_{m+1} exhibitos in aequationum (10) Integrale abit. Harum aequationum praeterea vidimus ultimam integrationem Quadraturis absolvi. Unde propositis aequationibus differentialibus

$$dx : dx_1 : \ldots : dx_n = X : X_1 : \ldots : X_n,$$

in quibus functiones $X_{m+1}, X_{m+2}, \ldots, X_n$ variabilibus $x, x_1, \ldots, x_m$ vacant simulque fit

$$\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_m}{\partial x_m} = 0,$$

si datur Multiplicator et ipse variabilibus $x, x_1, \ldots, x_m$ vacans, duae integrationes per Quadraturas absolvuntur; si vero datus Multiplicator variabilibus x, $x_1, \ldots, x_n$ afficitur, una aliqua aequatio integralis absque omni Quadratura constabit atque altera integratio Quadraturis efficietur.

Antecedentia exemplo esse possunt, ad aequationes differentiales integrandas e Multiplicatoris cognitione semper fructum aliquem percipi, etsi ultima integratio absque eius auxilio Quadraturis absolvi possit. Neque nessarium est, ut in antecedentibus aequationes (4) sint Integralia ipsarum aequationum differentialium (2), vel aequationes (9) sint Integralia ipsarum aequationum differentialium (8). Nam secundum ea, quae §. 12 tradidi, Constanti arbitrariae post quamque novam integrationem accedenti valorem tribuere licet particularem quemcunque. Sufficit, ut quaelibet aequatio $f_i = \text{Const.}$ sit Integrale aequationum differentialium quocunque modo transformatarum per aequationes integrales ante eam inventas

$$f_1 = \alpha_1, \quad f_2 = \alpha_2, \quad \ldots, \quad f_{i-1} = \alpha_{i-1},$$

in quibus ad dextram habentur quantitates constantes quaecunque particulares.

Caput tertium.

Theoria Multiplicatoris systematis aequationum differentialium ad varia exempla applicata.

§. 14.

De Multiplicatore systematis aequationum differentialium cuiuslibet ordinis.

Aequationum differentialium systema, quo altissima quaeque variabilium dependentium differentialia per differentialia inferiora ipsasque variabiles exprimuntur, constat in systema redire aequationum differentialium primi ordinis, si cuiusque variabilis dependentis differentialia altissimo inferiora ipsis variabilibus adscribantur. Designantibus enim x, y, etc. variabilis independentis t functiones, proponantur inter t, x, y, etc. aequationes differentiales

$$(1) \quad \frac{d^p x}{dt^p} = A, \quad \frac{d^q y}{dt^q} = B, \quad \text{etc.};$$

ipsaeque A, B, etc. non altioribus afficiantur differentialibus quam $(p-1)^{\text{to}}$ ipsius x, $(q-1)^{\text{to}}$ ipsius y, etc. Patet, habendo pro novis variabilibus dependentibus differentialia, quae Lagrangiano more per indices denoto,

$$x' = \frac{dx}{dt}, \quad x'' = \frac{d^2x}{dt^2}, \quad \ldots, \quad x^{(p-1)} = \frac{d^{p-1}x}{dt^{p-1}},$$
$$y' = \frac{dy}{dt}, \quad y'' = \frac{d^2y}{dt^2}, \quad \ldots, \quad y^{(q-1)} = \frac{d^{q-1}y}{dt^{q-1}}, \quad \text{etc.},$$

aequationibus differentialibus (1) has alias substitui posse *primi* ordinis:

$$(2) \quad \left\{\begin{aligned} & dt : dx : dx' : \ldots : dx^{(p-2)} : dx^{(p-1)} \\ & \quad : dy : dy' : \ldots : dy^{(q-2)} : dy^{(q-1)} : \text{etc.} \\ = {} & 1 : x' : x'' : \ldots : x^{(p-1)} : A \\ & \quad : y' : y'' : \ldots : y^{(q-1)} : B : \text{etc.} \end{aligned}\right.$$

Quibus in aequationibus variabilium numerus summam ordinum altissimorum differentialium in (1) unitate superat.

Multiplicator aequationum differentialium primi ordinis (2), cum quibus aequationes differentiales (1) conveniunt, etiam a me in sequentibus appellabitur aequationum (1) Multiplicator. Unde ut omnia theoremata de Multiplicatore aequationum differentialium primi ordinis in duobus Capitibus praecedentibus in medium prolata ad Multiplicatores aequationum differentialium cuiuslibet ordinis (1) applicentur, sufficit, ut pro aequationibus ibi propositis

$$(3) \quad dx : dx_1 : dx_2 : \ldots : dx_n = X : X_1 : X_2 : \ldots : X_n$$

sumantur aequationes (2).

Si aequationes differentiales primi ordinis (2) et (3) inter se comparamus, videmus in illis specialitatem quandam formae locum habere, videlicet quantitatum primis differentialibus proportionalium, quae generaliter variabilium functiones sunt, maximam partem in ipsas abire variabiles, neque vero in eas, quarum differentialibus proportionales ponuntur. Quo habitu speciali fit, ut aequationum (2) Multiplicator, quem aequationum (1) quoque Multiplicatorem voco, definiatur formula, quae, tantopere licet aucto in (2) variabilium numero, non pluribus constat terminis, quam si ipsae primi ordinis fuissent aequationes differentiales propositae (1). Consideremus enim formulam ad definiendum aequationum (3) Multiplicatorem propositam §. 7 (4):

$$(4) \quad \frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n} = -X \frac{d \log M}{dx}.$$

Si pro aequationibus (3) sumimus aequationes (2), fit $x = t$, $X = 1$; porro variabilibus x_1, x_2, etc. substituendae sunt

$$\begin{array}{l} x, \ x', \ x'', \ \ldots, \ x^{(p-2)}, \ x^{(p-1)}, \\ y, \ y', \ y'', \ \ldots, \ y^{(q-2)}, \ y^{(q-1)}, \ \text{etc.}; \end{array}$$

functionibus denique X_1, X_2, etc. substituendae sunt quantitates

$$\begin{array}{l} x', \ x'', \ x''', \ \ldots, \ x^{(p-1)}, \ A, \\ y', \ y'', \ y''', \ \ldots, \ y^{(q-1)}, \ B, \ \text{etc.} \end{array}$$

Iam in (4), quoties est X_i una e variabilibus x, x_1, x_2, etc., ab ipsa x_i diversa, evanescit terminus $\frac{\partial X_i}{\partial x_i}$, uti generaliter fit, si functio X_i ipsam x_i non implicat. Unde sumendo pro (3) aequationes (2), abit aggregatum (4) in hanc expressionem simplicem:

$$\frac{\partial A}{\partial x^{(p-1)}} + \frac{\partial B}{\partial y^{(q-1)}} + \text{etc.} = -\frac{d \log M}{dt}.$$

Hac formula Multiplicator M definitur systematis aequationum differentialium cuiuslibet ordinis (1).

Sequitur e (5), quoties simul ipsum A a differentiali $(p-1)^{\text{to}}$ ipsius x, ipsum B a differentiali $(q-1)^{\text{to}}$ ipsius y, etc. vacuum sit, sive generalius, *quoties*

aggregatum

$$\frac{\partial A}{\partial x^{(p-1)}} + \frac{\partial B}{\partial y^{(q-1)}} + \text{etc.}$$

identice evanescat, statui posse $M = 1$. Si aggregatum (5) non identice evanescit, ad indagandum Multiplicatorem circumspiciendum erit differentiale completum, cui idem aggregatum sua sponte vel etiam per aequationes differentiales propositas aequetur.

§. 15.

Principium ultimi Multiplicatoris systemati aequationum differentialium cuiuslibet ordinis applicatum.

Aequationum differentialium propositarum (1) §. pr. Integralibus praeter unum omnibus inventis, quantitates

$$(A.) \quad \begin{cases} t, \ x, \ x', \ \ldots, \ x^{(p-1)}, \\ \quad\ y, \ y', \ \ldots, \ y^{(q-1)}, \ \text{etc.} \end{cases}$$

omnes exprimere licet per duas u et v, pro quibus sumere licet binas e quantitatibus (A.) vel earum functiones quaslibet. Differentialia $\frac{du}{dt}$ et $\frac{dv}{dt}$, substituendo differentialibus $x^{(p)}$, $y^{(q)}$, etc., si opus est, valores A, B, etc., et ipsa aequantur quantitatum t, x, x', etc. functionibus. Quae functiones, Integralium inventorum ope per u et v expressae, si denotantur per

$$U = \frac{du}{dt}, \quad V = \frac{dv}{dt},$$

dabitur inter u et v aequatio differentialis primi ordinis, ultima quae integranda restat,

$$(1) \quad V du - U dv = 0.$$

Secundum ea, quae §. 11 tradidi, cognito aequationum differentialium propositarum Multiplicatore M, erui potest factor N, qui eius ultimae aequationis differentialis (1) laevam partem efficiat differentiale completum, quem *ultimum Multiplicatorem* appello. *Habendo enim, quod per Integralia inventa licet, quantitates* (A.) *pro functionibus ipsarum u et v Constantiumque arbitrariarum, quas Integralia implicant, earumque functionum formando Determinans* Δ, *fit ultimus Multiplicator* $N = \Delta . M$.

Principium ultimi Multiplicatoris, quod Propositione antecedente continetur, etiam sic concipi potest:

diviso ultimae aequationis differentialis (1) *Multiplicatore per Determinans Δ, conditionem* E u l e r i a n a m *pro Multiplicatore valentem transformari in aliam conditionem ab Integralibus reductioni adhibitis independentem, cui formulandae sufficiant solae aequationes differentiales propositae.*

Videlicet aequatio conditionalis, cui aequationis (1) Multiplicator N satisfacere debet, fit

$$\frac{\partial(NU)}{\partial u}+\frac{\partial(NV)}{\partial v}=0.$$

Quae, ponendo

$$M=\frac{N}{\Delta}$$

et substituendo Constantibus arbitrariis functiones quantitatum (A.) aequivalentes, transformabitur in hanc:

$$\frac{d\log M}{dt}+\frac{\partial A}{\partial x^{(p-1)}}+\frac{\partial B}{\partial y^{(q-1)}}+\text{etc.}=0,$$

cui formandae sufficiunt aequationes differentiales propositae (1).

Sint $\Pi_1=0$, $\Pi_2=0$, etc. aequationes integrales reductioni adhibitae binaeque aequationes, quibus u et v ab ipsis t, x, x', etc. pendent, sive etiam aliae quaecunque aequationes cum illis aequivalentes: constat e Determinantium functionalium proprietatibus, *aequari Δ fractioni, cuius denominator sit functionum Π_1, Π_2, etc. Determinans formatum quantitatum* (A.) *respectu, numerator autem earundem functionum Determinans, quantitatum u et v Constantiumque arbitrariarum respectu formatum.* Si pro u et v ipsae sumuntur t et x, pro aequationibus $\Pi_1=0$, $\Pi_2=0$, etc. solae sumendae sunt aequationes integrales simulque t et x in binis Determinantibus formandis de numero variabilium tollendae sunt. Porro aequatio (1) in hanc abit:

$$dx-V\,dt=0,$$

ubi V est ipsius $\frac{dx}{dt}$ valor, Integralium inventorum ope per t et x expressus. Si aequationes $\Pi_1=0$, $\Pi_2=0$, etc. inventae sunt per integrationem successivam, ita ut in quaque aequatione insequente, in qua nova accedit Constans arbitraria, simul unius variabilis differentiale altissimum ad ordinem proxime inferiorem sit depressum, alterutrum Determinans in unicum terminum abit. Sic proposita unica aequatione differentiali n^{ti} ordinis inter t et x

$$\frac{d^nx}{dt^n}=f\left(t,\ x,\ \frac{dx}{dt},\ \dots,\ \frac{d^{n-1}x}{dt^{n-1}}\right),$$

integratione successiva inventae sint aequationes

$$(2)\quad\begin{cases}\dfrac{d^{n-1}x}{dt^{n-1}}=f_1\left(t,\ x,\ \dfrac{dx}{dt},\ \dots,\ \dfrac{d^{n-2}x}{dt^{n-2}},\ \alpha_1\right),\\[2mm] \dfrac{d^{n-2}x}{dt^{n-2}}=f_2\left(t,\ x,\ \dfrac{dx}{dt},\ \dots,\ \dfrac{d^{n-3}x}{dt^{n-3}},\ \alpha_1,\ \alpha_2\right),\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \dfrac{dx}{dt}=f_{n-1}(t,\ x,\ \alpha_1,\ \alpha_2,\ \dots,\ \alpha_{n-1}),\end{cases}$$

in quibus α_1, α_2, $\dots$, α_{n-1} sunt Constantes arbitrariae: simpliciter erit

$$\Delta=\frac{\partial f_1}{\partial\alpha_1}\cdot\frac{\partial f_2}{\partial\alpha_2}\cdots\frac{\partial f_{n-1}}{\partial\alpha_{n-1}},$$

cum alterum Determinans in ipsam unitatem abeat. Si functio f ab ipso $\frac{d^{n-1}x}{dt^{n-1}}$ vacuum est, fit aequationis differentialis propositae Multiplicator $=1$. Quo igitur casu hoc eruitur ultimum Integrale:

$$\int\frac{\partial f_1}{\partial\alpha_1}\cdot\frac{\partial f_2}{\partial\alpha_2}\cdots\frac{\partial f_{n-1}}{\partial\alpha_{n-1}}[dx-f_{n-1}(t,\ x,\ \alpha_1,\ \alpha_2,\ \dots,\ \alpha_{n-1})dt]=\text{Const.},$$

ubi quantitas sub integrationis signo, per t et x expressa, fit differentiale completum. Ut per solas t et x exprimatur valor producti

$$\frac{\partial f_1}{\partial\alpha_1}\cdot\frac{\partial f_2}{\partial\alpha_2}\cdots\frac{\partial f_{n-1}}{\partial\alpha_{n-1}},$$

sufficit, ut in eo *successive* substituantur differentialium $\frac{d^{n-2}x}{dt^{n-2}}$, $\frac{d^{n-3}x}{dt^{n-3}}$, $\dots$, $\frac{dx}{dt}$ valores f_2, f_3, $\dots$, f_{n-1}.

§. 16.

Formula symbolica, qua Multiplicator systematis aequationum differentialium impliciti definiri potest.

Aequationes differentiales, e quibus petantur altissimorum differentialium valores

$$(1)\quad x^{(p)}=A,\quad y^{(q)}=B,\quad \text{etc.},$$

ponamus forma dari implicita

$$(2)\quad \varphi=0,\quad \psi=0,\quad \text{etc.}$$

E quibus aequationibus ut eruantur valores differentialium partialium

$$\frac{\partial A}{\partial x^{(p-1)}},\quad \frac{\partial B}{\partial y^{(q-1)}},\quad \text{etc.},$$

quarum summa aequat ipsum $-\frac{d\log M}{dt}$, statuo

$$(3)\quad \begin{cases} \dfrac{\partial \varphi}{\partial x^{(p)}} = a, & \dfrac{\partial \varphi}{\partial y^{(q)}} = a_1, \quad \text{etc.}, \\ \dfrac{\partial \psi}{\partial x^{(p)}} = b, & \dfrac{\partial \psi}{\partial y^{(q)}} = b_1, \quad \text{etc.} \quad \text{etc.}, \end{cases}$$

nec non

$$(5)\quad \begin{cases} \dfrac{\partial \varphi}{\partial x^{(p-1)}} = \alpha, & \dfrac{\partial \varphi}{\partial y^{(q-1)}} = \alpha_1, \quad \text{etc.}, \\ \dfrac{\partial \psi}{\partial x^{(p-1)}} = \beta, & \dfrac{\partial \psi}{\partial y^{(q-1)}} = \beta_1, \quad \text{etc.} \quad \text{etc.}, \end{cases}$$

formoque aequationes

$$(5)\quad \begin{cases} au + a_1 u_1 + \cdots + \alpha v + \alpha_1 v_1 + \cdots = 0, \\ bu + b_1 u_1 + \cdots + \beta v + \beta_1 v_1 + \cdots = 0. \end{cases}$$

Resolutione aequationum (5) si determinantur u, u_1, etc. ut functiones lineares quantitatum v, v_1, etc., erit, quod ex elementis calculi differentialis sequitur,

$$(6)\quad \frac{\partial A}{\partial x^{(p-1)}} = \frac{\partial u}{\partial v}, \quad \frac{\partial B}{\partial y^{(q-1)}} = \frac{\partial u_1}{\partial v_1}, \quad \text{etc.},$$

unde prodit

$$(7)\quad d\log M = -\left\{\frac{\partial u}{\partial v} + \frac{\partial u_1}{\partial v_1} + \cdots\right\} dt.$$

Iam e formulis, quas de aequationum linearium resolutione et Determinantium proprietatibus tradidi, sequitur, *si in aequationibus linearibus* (5) *ponatur*

$$(8)\quad \begin{cases} \alpha dt = \delta a, & \alpha_1 dt = \delta a_1, \quad \text{etc.}, \\ \beta dt = \delta b, & \beta_1 dt = \delta b_1, \quad \text{etc.} \quad \text{etc.}, \end{cases}$$

fieri

$$(9)\quad -\left\{\frac{\partial u}{\partial v} + \frac{\partial u_1}{\partial v_1} + \cdots\right\} dt = \delta \log \Sigma \pm a b_1 \ldots$$

Unde formula, qua Multiplicator M definitur, proponi potest hac forma *symbolica:*

$$(10)\quad d\log M = \delta \log \Sigma \pm a b_1 \ldots.$$

Cui formulae ea inest significatio, ut, variando per regulas notas ipsum $\lg \Sigma \pm a b_1 \ldots$ atque elementorum variationibus singulis substituendo valores (8), obtineatur expressio ipsi $d\log M$ aequalis.

Si statuitur

$$(11)\quad \begin{cases} \alpha dt - \lambda da = \Delta a, & \alpha_1 dt - \lambda da_1 = \Delta a_1, \quad \text{etc.}, \\ \beta dt - \lambda db = \Delta b, & \beta_1 dt - \lambda db_1 = \Delta b_1, \quad \text{etc.} \quad \text{etc.}, \end{cases}$$

characteristicae δ substituendum est $\lambda d + \Delta$, unde abit (10) in hanc formulam:

$$(12)\quad d\log M = \lambda d \log \Sigma \pm a b_1 \cdots + \Delta \log \Sigma \pm a b_1 \ldots,$$

sive, designante λ Constantem:

$$(13)\quad d\log \frac{M}{\{\Sigma \pm a b_1 \ldots\}^{\lambda}} = \Delta \log \Sigma \pm a b_1 \ldots$$

Quae formula cum commodo adhibetur, quoties variationum Δa, Δb, etc. valores valoribus variationum δa, δb, etc. simpliciores sunt.

Sint n aequationes differentiales inter t et variabiles dependentes x_1, x_2, ..., x_n propositae

$$(14)\quad \varphi_1 = 0, \quad \varphi_2 = 0, \quad \ldots, \quad \varphi_n = 0,$$

sintque altissima differentialia in iis obvenientia et quorum valores ex iis petere liceat

$$x_1^{(m_1)}, \quad x_2^{(m_2)}, \quad \ldots, \quad x_n^{(m_n)}.$$

Statuendo secundum antecedentia

$$(15)\quad \begin{cases} \dfrac{\partial \varphi_i}{\partial x_k^{(m_k)}} = a_k^{(i)}, \\ \dfrac{\partial \varphi_i}{\partial x_k^{(m_k-1)}} dt = \delta a_k^{(i)} = \lambda d a_k^{(i)} + \Delta a_k^{(i)}, \end{cases}$$

fit

$$(16)\quad \begin{cases} d\log M = \delta \log \Sigma \pm a_1' a_2'' \ldots a_n^{(n)}, \\ d\log \dfrac{M}{\{\Sigma \pm a_1' a_2'' \ldots a_n^{(n)}\}^{\lambda}} = \Delta \log \Sigma \pm a_1' a_2'' \ldots a_n^{(n)}. \end{cases}$$

Accuratius examinemus casum, quo fit

$$(17)\quad a_k^{(i)} = a_i^{(k)},$$

unde elementa $a_k^{(i)}$ ad numerum $\frac{n(n+1)}{2}$ reducere licet. Differentialia partialia uncis includendo aut non includendo, prout ista reductio facta est aut non facta est, habetur, si i et k inter se diversi sunt,

$$\left(\frac{\partial R}{\partial a_k^{(i)}}\right) = \frac{\partial R}{\partial a_i^{(k)}} + \frac{\partial R}{\partial a_k^{(i)}}, \quad \left(\frac{\partial R}{\partial a_i^{(i)}}\right) = \frac{\partial R}{\partial a_i^{(i)}}.$$

Designante R Determinans

$$R = \Sigma \pm a_1' a_2'' \ldots a_n^{(n)},$$

constat per notas Determinantium proprietates, si aequationes (17) locum habeant, etiam fieri

$$(18)\qquad \frac{\partial R}{\partial a_k^{(i)}} = \frac{\partial R}{\partial a_i^{(k)}},$$

unde

$$(19)\qquad \left(\frac{\partial R}{\partial a_k^{(i)}}\right) = 2\,\frac{\partial R}{\partial a_k^{(i)}}.$$

Cum in symbolis adhibitis variationes $\delta a_k^{(i)}$ vel $\Delta a_k^{(i)}$ ab ipsis $a_k^{(i)}$ independentes sint, ex aequationibus (17) non etiam variationum aequalitas sequitur, unde in formanda Determinantis variatione pro diversis haberi debent $\delta a_k^{(i)}$ et $\delta a_i^{(k)}$ vel $\Delta a_k^{(i)}$ et $\Delta a_i^{(k)}$, ideoque post institutam ipsius R variationem demum aequationum (17) usus faciendus est. At observandum est, in Determinantis variatione binorum elementorum $a_k^{(i)}$ et $a_i^{(k)}$ variationum tantum summam obvenire, cum per (18) et (19) habeatur

$$\frac{\partial R}{\partial a_k^{(i)}}\,\delta a_k^{(i)} + \frac{\partial R}{\partial a_i^{(k)}}\,\delta a_i^{(k)} = \tfrac{1}{2}\left(\frac{\partial R}{\partial a_k^{(i)}}\right)\{\delta a_k^{(i)} + \delta a_i^{(k)}\}.$$

Quae formula docet, in Determinante R etiam ante eius variationem instituendam poni posse $a_k^{(i)} = a_i^{(k)}$, modo ipsi $\delta a_k^{(i)} = \delta a_i^{(k)}$ tribuatur valor $\frac{1}{2}\{\delta a_k^{(i)} + \delta a_i^{(k)}\}$. Quoties igitur aequationes (17) locum habent sive fit

$$\frac{\partial \varphi_i}{\partial x_k^{(m_k)}} = \frac{\partial \varphi_k}{\partial x_i^{(m_i)}},$$

valebunt adhuc aequationes (16), etsi Determinantis elementa ad numerum $\frac{n(n+1)}{2}$ inter se inaequalium revocentur, dummodo statuatur

$$(20)\qquad \tfrac{1}{2}\left\{\frac{\partial \varphi_i}{\partial x_k^{(m_k-1)}} + \frac{\partial \varphi_k}{\partial x_i^{(m_i-1)}}\right\}dt = \delta a_k^{(i)} = \lambda\, d a_k^{(i)} + \Delta a_k^{(i)}.$$

Quod si igitur aequationes differentiales propositae (14) *ita comparatae sunt, ut habeatur*

$$\frac{\partial \varphi_i}{\partial x_k^{(m_k)}} = \frac{\partial \varphi_k}{\partial x_i^{(m_i)}},$$

$$\tfrac{1}{2}\left\{\frac{\partial \varphi_i}{\partial x_k^{(m_k-1)}} + \frac{\partial \varphi_k}{\partial x_i^{(m_i-1)}}\right\}dt = \lambda d\,\frac{\partial \varphi_i}{\partial x_k^{(m_k)}},$$

designante λ Constantem, evanescet variatio Δ dabiturque Multiplicator

$$M = \left\{\Sigma \pm \frac{\partial \varphi_1}{\partial x_1^{(m_1)}} \cdot \frac{\partial \varphi_2}{\partial x_2^{(m_2)}} \cdots \frac{\partial \varphi_n}{\partial x_n^{(m_n)}}\right\}^{\lambda}.$$

Cuius Propositionis applicatio infra dabitur.

Observo ipsum R pro Determinante functionali haberi posse; erit enim R functionum $\varphi_1, \varphi_2, \ldots, \varphi_n$ Determinans, si sola altissima differentialia $x_1^{(m_1)}$, $x_2^{(m_2)}$, etc. pro variabilibus sumuntur, quarum respectu Determinans formetur. Quarum variabilium valores cum supponamus ex aequationibus (14) peti posse, non fieri potest ut Determinans R identice evanescat; alioquin enim functiones φ_1, φ_2, etc. earum variabilium respectu non a se invicem independentes forent. (V. *Comm. de Det. Funct.* §§. 3 sqq.) Si vero per ipsas (14) evanescit Determinans R, id indicio est, duo valorum variabilium systemata inter se aequalia evadere, unde aequationum praeparatione quadam opus est, qua radicibus duplicibus liberentur.

Iam praecepta generalia variis applicabo exemplis.

§. 17.

De Multiplicatore systematis aequationum differentialium linearium.

Proponantur aequationes differentiales lineares primi ordinis

$$(1)\qquad \begin{cases} \dfrac{dx_1}{dt} = A_1'\, x_1 + A_2'\, x_2 + \cdots + A_n'\, x_n = X_1, \\ \dfrac{dx_2}{dt} = A_1''\, x_1 + A_2''\, x_2 + \cdots + A_n''\, x_n = X_2, \\ \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot \\ \dfrac{dx_n}{dt} = A_1^{(n)} x_1 + A_2^{(n)} x_2 + \cdots + A_n^{(n)} x_n = X_n, \end{cases}$$

quarum Coëfficientes $A_k^{(i)}$ solius t functiones designant. Systematis aequationum (1) Multiplicator M definitur formula differentiali

$$\frac{d\log M}{dt} = -\left\{\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n}\right\}$$
$$= -\{A_1' + A_2'' + \cdots + A_n^{(n)}\},$$

unde

$$(2)\qquad M = e^{-\int\{A_1' + A_2'' + \cdots + A_n^{(n)}\}dt}.$$

Hac formula cognito M, sequitur e §. 15, *si aequationes differentiales lineares* (1) *per quascunque $n-1$ aequationes integrales, $n-1$ Constantibus arbitrariis affectas, ad unicam aequationem differentialem primi ordinis inter duas variabiles reducantur, eius quoque ultimae aequationis integrationem per Quadraturas absolvi* *posse.* Quod hactenus non constabat, nisi aequationes quoque integrales reductioni adhibitae lineares erant.

Aequationibus (1) alterum systema aequationum differentialium linearium coniugatum est

$$(3)\quad \begin{cases} \dfrac{dy_1}{dt} = -\{A_1'y_1 + A_1''y_2 + \cdots + A_1^{(n)}y_n\}, \\ \dfrac{dy_2}{dt} = -\{A_2'y_1 + A_2''y_2 + \cdots + A_2^{(n)}y_n\}, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \dfrac{dy_n}{dt} = -\{A_n'y_1 + A_n''y_2 + \cdots + A_n^{(n)}y_n\}. \end{cases}$$

Aequationibus (1) respective per $y_1, y_2, \ldots, y_n$ atque aequationibus (3) respective per $x_1, x_2, \ldots, x_n$ multiplicatis omniumque aequationum provenientium additione facta, termini ad dextram positi omnino abeunt, expressio ad laevam autem fit differentiale aggregati $x_1y_1 + x_2y_2 + \cdots + x_ny_n$; unde integratione facta eruitur

$$(4)\quad x_1y_1 + x_2y_2 + \cdots + x_ny_n = \text{Const.}$$

Quot habentur aequationum (3) solutiones particulares, tot formula (4) suppeditantur aequationum (1) Integralia, et quot habentur solutiones particulares aequationum (1), tot eadem formula suppeditantur aequationum (3) Integralia. Aequationum (3) Multiplicator invenitur

$$N = e^{\int \{A_1' + A_2'' + \cdots + A_n^{(n)}\} dt},$$

unde *binorum systematum aequationum differentialium linearium inter se coniugatorum Multiplicatores M et N valoribus reciprocis gaudent.*

Functionum $y_1, y_2, \ldots, y_n$ denotemus n systemata a se independentia per

$$y_1^{(k)},\ y_2^{(k)},\ \ldots,\ y_n^{(k)},$$

tribuendo successive indici superiori k valores $1, 2, \ldots, n$. Unde aequationum (1) proveniunt n Integralia huiusmodi

$$f_k = x_1y_1^{(k)} + x_2y_2^{(k)} + \cdots + x_ny_n^{(k)} = \alpha_k,$$

designantibus $\alpha_1, \alpha_2, \ldots, \alpha_n$ Constantes arbitrarias. Secundum Multiplicatoris definitionem, initio huius Commentationis adhibitam, fit

$$(5)\quad \begin{cases} M = \Sigma \pm \dfrac{\partial f_1}{\partial x_1} \cdot \dfrac{\partial f_2}{\partial x_2} \cdots \dfrac{\partial f_n}{\partial x_n} \\ \quad = \Sigma \pm y'y_2'' \cdots y_n^{(n)}. \end{cases}$$

Unde obtinetur formula

$$(6)\quad \Sigma \pm y_1'y_2'' \cdots y_n^{(n)} = e^{-\int \{A_1' + A_2'' + \cdots + A_n^{(n)}\} dt}.$$

Quae sic directe demonstratur.

Designante enim R Determinans ad laevam, fit

$$dR = \Sigma\Sigma \frac{\partial R}{\partial y_i^{(k)}} dy_i^{(k)}$$
$$= -\Sigma\Sigma \frac{\partial R}{\partial y_i^{(k)}} \{A_i'y_1^{(k)} + A_i''y_2^{(k)} + \cdots + A_i^{(n)}y_n^{(k)}\} dt,$$

extensa duplici summatione ad omnes indicum i et k valores $1, 2, \ldots, n$. Summando primum indicis k respectu, evanescunt termini in A_i', A_i'', etc. ducti praeter eos, qui in $A_i^{(i)}$ ducuntur,

$$-A_i^{(i)} \left\{ \frac{\partial R}{\partial y_i'} y_i' + \frac{\partial R}{\partial y_i''} y_i'' + \cdots + \frac{\partial R}{\partial y_i^{(n)}} y_i^{(n)} \right\} dt$$
$$= -A_i^{(i)}.R\,dt,$$

sicuti notis Determinantium proprietatibus patet. Hinc altera summatio indicis i respectu instituta suggerit

$$dR = -\{A_1' + A_2'' + \cdots + A_n^{(n)}\}.R\,dt,$$

cuius aequationis integratione formula (6) obtinetur.

Si aequationes differentiales lineares proponuntur, quae altiora quam prima differentialia involvunt, secundum §. 14 (5) statim earum quoque Multiplicator obtinetur. Brevitatis causa duas tantum consideremus aequationes

$$(7)\quad \begin{cases} \dfrac{d^px}{dt^p} = A\,x + A_1\dfrac{dx}{dt} + \cdots + A_{p-1}\dfrac{d^{p-1}x}{dt^{p-1}} \\ \qquad + B\,x + B_1\dfrac{dy}{dt} + \cdots + B_{q-1}\dfrac{d^{q-1}y}{dt^{q-1}}, \\ \dfrac{d^qx}{dt^q} = A'x + A_1'\dfrac{dx}{dt} + \cdots + A_{p-1}'\dfrac{d^{p-1}x}{dt^{p-1}} \\ \qquad + B'y + B_1'\dfrac{dy}{dt} + \cdots + B_{q-1}'\dfrac{d^{q-1}y}{dt^{q-1}}, \end{cases}$$

in quibus Coëfficientes A, A_1, etc. solius t functiones designant; fit earum aequationum Multiplicator

$$M = e^{-\int \{A_{p-1} + B_{q-1}'\} dt}.$$

Ponamus, addendo aequationes (7) respective per λ et μ multiplicatas produci aequationem per se integrabilem: secundum conditiones integrabilitatis fieri debet

$$(8) \quad \begin{cases} \dfrac{d^p\lambda}{dt^p} = -\dfrac{d^{p-1}(A_{p-1}\lambda+A'_{p-1}\mu)}{dt^{p-1}} + \dfrac{d^{p-2}(A_{p-2}\lambda+A'_{p-2}\mu)}{dt^{p-2}} - \cdots \pm (A\lambda+A'\mu), \\ \dfrac{d^q\mu}{dt^q} = -\dfrac{d^{q-1}(B_{q-1}\lambda+B'_{q-1}\mu)}{dt^{q-1}} + \dfrac{d^{q-2}(B_{q-2}\lambda+B'_{q-2}\mu)}{dt^{q-2}} - \cdots \pm (B\lambda+B'\mu), \end{cases}$$

quod est aequationum differentialium systema proposito coniugatum. Quod, si p et q inter se inaequales sunt, non ea gaudet forma, qua §. 14 supposui aequationes differentiales exhibitas esse, videlicet ut altissima differentialia inveniantur per inferiora ipsasque variabiles expressa. Si $p > q$, ut ea forma obtineatur, aequatio posterior $p-q-1$ vicibus iteratis differentianda est et aequationum ope provenientium eliminanda sunt e priore ipsius μ differentialia superiora $(q-1)^{\text{to}}$. Hac eliminatione priorem aequationem novi non ingrediuntur termini $(p-1)^{\text{to}}$ ipsius λ differentiali affecti, unde in ea immutatus manet unicus terminus differentiale $\frac{d^{p-1}\lambda}{dt^{p-1}}$ implicans

$$-A_{p-1}\frac{d^{p-1}\lambda}{dt^{p-1}}.$$

Porro in aequatione posteriore unicus extat terminus ipso $\frac{d^{q-1}\mu}{dt^{q-1}}$ affectus

$$-B'_{q-1}\frac{d^{q-1}\mu}{dt^{q-1}}.$$

Unde secundum §. 14 aequationum (8), dicto modo praeparatarum, eruitur Multiplicator

$$N = e^{\int\{A_{p-1}+B'_{q-1}\}dt}.$$

Videmus igitur, bina quoque systemata coniugata (7) et (8) Multiplicatoribus reciprocis gaudere. Similiter pro pluribus aequationibus demonstratur, in systemate coniugato per eliminationes, ad formam normalem obtinendam instituendas, hos non mutari terminos, qui valorem ipsius $\frac{d\log N}{dt}$ afficiunt, unde facile sequitur, binorum systematum coniugatorum Multiplicatores semper evadere inter se reciprocos.

Observo, formam normalem aequationibus (8) conciliari posse sine differentiationibus et eliminationibus, cum earum loco hoc pateat substitui posse systema aequationum differentialium linearium primi ordinis inter $p+q+1$ variabiles:

$$(9) \quad \begin{cases} \dfrac{d\lambda}{dt} = -\{A_{p-1}\lambda+A'_{p-1}\mu+\lambda_1\}, \\ \dfrac{d\lambda_1}{dt} = -\{A_{p-2}\lambda+A'_{p-2}\mu+\lambda_2\}, \\ \cdots\cdots\cdots \\ \dfrac{d\lambda_{p-1}}{dt} = -\{A\lambda+A'\mu\}, \\ \dfrac{d\mu}{dt} = -\{B_{q-1}\lambda+B'_{q-1}\mu+\mu_1\}, \\ \dfrac{d\mu_1}{dt} = -\{B_{q-2}\lambda+B'_{q-2}\mu+\mu_2\}, \\ \cdots\cdots\cdots \\ \dfrac{d\mu_{q-1}}{dt} = -\{B\lambda+B'\mu\}. \end{cases}$$

Aequationes (9) eodem gaudent Multiplicatore N supra invento. Quod adnotari meretur. Nam valor supra inventus N Multiplicatori aequationum (8) conveniebat supponendo eas locum tenere aequationum differentialium primi ordinis, in quibus praeter t, λ, μ pro variabilibus habeantur

$$(A) \quad \begin{cases} \dfrac{d\lambda}{dt}, & \dfrac{d^2\lambda}{dt^2}, & \ldots, & \dfrac{d^{p-1}\lambda}{dt^{p-1}}, \\ \dfrac{d\mu}{dt}, & \dfrac{d^2\mu}{dt^2}, & \ldots, & \dfrac{d^{q-1}\mu}{dt^{q-1}}; \end{cases}$$

dum aequationes (9) sunt inter t, λ, μ aliasque variabiles

$$(B) \quad \begin{cases} \lambda_1, & \lambda_2, & \ldots, & \lambda_{p-1}, \\ \mu_1, & \mu_2, & \ldots, & \mu_{q-1}. \end{cases}$$

Aliis autem variabilibus introductis vidimus in secundo Capite mutari Multiplicatorem, videlicet eum dividi per novarum variabilium Determinans, ipsarum formatum variabilium respectu, quarum loco introductae sunt. Unde, cum utrique aequationum systemati *idem* conveniat Multiplicator N, sequitur, si quantitates (B) per t, λ, μ et quantitates (A) exprimantur, Determinans quantitatum (B), ipsarum (A) respectu formatum, aequari Constanti, ac reapse aequale invenitur unitati.

§. 18.

Aequationes differentiales secundi ordinis, quarum assignare licet Multiplicatorem. Exempla Euleriana.

Paullo immorabor applicationi theoriae novi Multiplicatoris ad aequationes differentiales secundi ordinis inter duas variabiles, qui est casus simplicissimus post aequationes differentiales primi ordinis, ad quas Eulerianus Multiplicator refertur. Ac primum per theoremata §§. 14, 15 tradita patet,

„*si proponatur aequatio* $\frac{d^2y}{dx^2}+A\frac{dy}{dx}+B=0$, *in qua A solius x, B utriusque x et y functiones quaecunque sunt, atque integratione prima eruatur* $\frac{dy}{dx}=u$, *designante u variabilium x et y et Constantis arbitrariae α functionem, fore alterum Integrale*

$$\int e^{\int Adx}\cdot\frac{\partial u}{\partial\alpha}(dy-udx) = \text{Const.}“$$

Quantitatem sub maiore integrationis signo esse differentiale completum, sic verificari potest. Nam ut aequatio differentialis proposita proveniat differentiatione aequationis $\frac{dy}{dx}=u$, locum habere debet aequatio *identica*

$$\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial y}+Au+B = 0.$$

Qua ipsius α respectu differentiata et per $e^{\int Adx}$ multiplicata, prodit

$$\frac{\partial\left(e^{\int Adx}\frac{\partial u}{\partial\alpha}\right)}{\partial x}+\frac{\partial\left(e^{\int Adx}u\frac{\partial u}{\partial\alpha}\right)}{\partial y} = 0,$$

quae est conditio requisita, ut quantitas

$$e^{\int Adx}\frac{\partial u}{\partial\alpha}(dy-udx)$$

differentiale completum sit.

Generalius e §§. 14, 15 sequitur, *si proponatur aequatio*

$$(1)\quad \frac{d^2y}{dx^2}+\tfrac{1}{2}\frac{\partial\varphi}{\partial y}\left(\frac{dy}{dx}\right)^2+\frac{\partial\varphi}{\partial x}\cdot\frac{dy}{dx}+B = 0,$$

in qua et φ et B variabilium x et y functiones quaecunque sunt, atque integratione prima inventum sit $\frac{dy}{dx}=u$, *designante u variabilium x et y et Constantis arbitrariae α functionem, fieri aequationem inter x et y quaesitam*

$$(2)\quad \int e^{\varphi}\frac{\partial u}{\partial\alpha}(dy-udx) = \text{Const.}$$

Aequationis (1) tractavit Eulerus specimina, quibus et integratio prima successit (Cf. Calc. Integr. Vol. II. Sect. I. Cap. VI. §§. 915 sqq.). At aequationes differentiales primi ordinis, ad quas ea ratione pervenit, tanta irrationalitate erant

implicatae, ut de integratione directa desperans alia artificia circumspexerit. Atque missum facto Integrali invento contigit ei, aequationes differentiales secundi ordinis propositas differentiando alias deducere lineares, Coëfficientibus constantibus affectas, quarum nota integratio propositarum quoque ei suppeditavit integrationem completam. At per antecedentem formulam (2) illarum aequationum differentialium primi ordinis quamvis complicatarum assignare licet Multiplicatores. Adiungam ipsam variabilium separationem, qua elucescat, revera adiectis illis Multiplicatoribus aequationes sponte integrabiles fore.

Exempla Euleriana forma paullo generaliori exhibebo, quod sine calculi complicatione fieri potest.

Exemplum I.

$$y^2\frac{d^2y}{dx^2}+y\left(\frac{dy}{dx}\right)^2+by-cx = 0.$$

(*b* et *c* Constantes.)

Secundum Eulerum aequationis propositae fit Integrale primum, quod si placet differentiando comprobare licet,

$$y^3\left(\frac{dy}{dx}\right)^3+bxy^2\left(\frac{dy}{dx}\right)^2+(by-3cx)y^2\frac{dy}{dx}$$
$$+cy^3+b^2y^2x-2bcyx^2+c^2x^3 = \alpha,$$

designante α Constantem arbitrariam. Cuius aequationis resolutione eruatur

$$y\frac{dy}{dx} = yu = v,$$

designante v radicem aequationis cubicae

$$(3)\quad \begin{cases} v^3+bxv^2+y(by-3cx)v \\ \quad +cy^3+b^2y^2x-2bcyx^2+c^2x^3 = \alpha. \end{cases}$$

Comparando aequationem differentialem propositam cum (1) fit

$$\varphi = 2\log y,\quad e^{\varphi} = y^2,$$

unde secundum (2) invenitur alterum Integrale

$$\int y^2\frac{\partial u}{\partial\alpha}(dy-udx) = \int\frac{\partial v}{\partial\alpha}(ydy-vdx) = \text{Const.}$$

Fit autem e (3)

$$\frac{\partial v}{\partial\alpha} = \frac{1}{3vv+2bxv+y(by-3cx)}.$$

Quem aequationis $ydy-vdx=0$ Multiplicatorem esse, propter ipsius v irrationalitatem non facile cognoscitur, et minus adhuc separatio variabilium in promptu est. Quam sic assequor.

Aequationem (3) bene vidit Eulerus hac ratione exhiberi posse:

$$(4)\quad f.f'.f'' = a,$$

posito

$$(5)\quad \begin{cases} f \;\, = v + \lambda y + \dfrac{c}{\lambda}x, \\ f' \, = v + \lambda' y + \dfrac{c}{\lambda'}x, \\ f'' = v + \lambda'' y + \dfrac{c}{\lambda''}x, \end{cases}$$

designantibus λ, λ', λ'' radices diversas aequationis cubicae

$$(6)\quad \lambda^3 + b\lambda - c = 0,$$

unde $\lambda + \lambda' + \lambda'' = 0$, $\lambda\lambda'\lambda'' = c$. Ex aequationibus (4) et (5) sequitur

$$\frac{\partial f}{\partial a} = \frac{\partial f'}{\partial a} = \frac{\partial f''}{\partial a} = \frac{\partial v}{\partial a}$$
$$= \frac{1}{f'f'' + f''f + ff'},$$

unde expressio

$$\frac{ydy - vdx}{f'f'' + f''f + ff'}$$

fieri debet differentiale completum. Invenitur autem e (5):

$$\begin{aligned} d(f' - f'') &= (\lambda' - \lambda'')(dy - \lambda\, dx), \\ d(f'' - f\;) &= (\lambda'' - \lambda\;)(dy - \lambda'\, dx), \\ d(f\; - f'\,) &= (\lambda\; - \lambda'\,)(dy - \lambda''\, dx), \end{aligned}$$
$$\lambda f.d(f' - f'') + \lambda' f'.d(f'' - f) + \lambda'' f''.d(f - f')$$
$$= \Delta.(ydy - vdx),$$

siquidem ponitur

$$\begin{aligned} \Delta &= \lambda^2(\lambda' - \lambda'') + \lambda'^2(\lambda'' - \lambda) + \lambda''^2(\lambda - \lambda') \\ &= (\lambda - \lambda')(\lambda - \lambda'')(\lambda' - \lambda''), \end{aligned}$$

atque adnotatur fieri

$$\lambda^3(\lambda' - \lambda'') + \lambda'^3(\lambda'' - \lambda) + \lambda''^3(\lambda - \lambda')$$
$$= \Delta(\lambda + \lambda' + \lambda'') = 0.$$

Hinc substitutendo $\lambda'' = -(\lambda + \lambda')$ fit

$$\begin{aligned} \Delta(ydy - vdx) = {} & \lambda\{(f + f'')df' - d(f\,f'')\} \\ & - \lambda'\{(f' + f'')df - d(f'f'')\}, \end{aligned}$$

unde denuo substituendo, quod e (4) sequitur,

$$d(ff'') = -ff''.\frac{df'}{f'}, \quad d(f'f'') = -f'f''.\frac{df}{f},$$

eruitur

$$\frac{ydy - vdx}{f'f'' + f''f + ff'} = \frac{1}{\Delta}\left\{\frac{\lambda df'}{f'} - \frac{\lambda' df}{f}\right\}.$$

Quod per se integrabile est atque nihilo aequiparatum integratumque suppeditat:

$$\frac{\log f}{\lambda} - \frac{\log f'}{\lambda'} = \text{Const.},$$

quod alterum Integrale est.

Exemplum II.

$$2y^3\frac{d^2y}{dx^2} + y^2\left(\frac{dy}{dx}\right)^2 - ay^2 + bx^2 - c = 0.$$

(a, b, c Constantes.)

Secundum Eulerum huius aequationis integratione prima obtinetur $ydy - vdx = 0$, designante v radicem aequationis biquadraticae

$$(7)\quad (aa - 4b)y^2 - 2(abx^2 + av^2 - 4bxv) + \left(\frac{c - bx^2 + v^2}{y}\right)^2 = \alpha$$

atque α Constantem arbitrariam. Comparando aequationem differentialem propositam cum (1) fit

$$\varphi = \log y, \quad e^{\varphi} = y,$$

unde e (2) eruitur aequatio integralis inter x et y quaesita

$$\int y\frac{\partial u}{\partial \alpha}\{dy - udx\} = \int \frac{\partial v}{\partial \alpha}.\frac{ydy - vdx}{y} = \text{Const.}$$

Ponamus $a = \lambda + \lambda'$, $b = \lambda\lambda'$, abit (7) in hanc formam:

$$(8)\quad \begin{cases} (\lambda - \lambda')^2y^2 - 2\{\lambda(v - \lambda'x)^2 + \lambda'(v - \lambda x)^2\} \\ \qquad\qquad + \left\{\dfrac{c - \lambda\lambda' x^2 + v^2}{y}\right\}^2 = \alpha. \end{cases}$$

Ponatur

$$(9)\quad v - \lambda' x = (\lambda - \lambda')p, \quad v - \lambda x = (\lambda' - \lambda)p',$$

unde

$$(10)\quad \begin{cases} \qquad x = p + p', \quad v = \lambda p + \lambda' p', \\ \sqrt{\lambda}.p + \sqrt{\lambda'}.p' = \dfrac{v + \sqrt{\lambda\lambda'}.x}{\sqrt{\lambda} + \sqrt{\lambda'}}, \quad \sqrt{\lambda}.p - \sqrt{\lambda'}.p' = \dfrac{v - \sqrt{\lambda\lambda'}.x}{\sqrt{\lambda} - \sqrt{\lambda'}}; \end{cases}$$

abit (8) in hanc aequationem

$$(11)\quad \begin{cases} y^2+\left\{\frac{c}{\lambda-\lambda'}+\lambda p^2-\lambda' p'^2\right\}^2\frac{1}{y^2} \\ =2\left\{\lambda p^2+\lambda' p'^2+\frac{\alpha}{2(\lambda-\lambda')^2}\right\}. \end{cases}$$

Hinc fit

$$(12)\quad y=\sqrt{\varepsilon+\lambda pp}+\sqrt{\varepsilon'+\lambda' p'p'};$$

siquidem ponitur

$$(13)\quad \varepsilon=\frac{\alpha}{4(\lambda-\lambda')^2}+\frac{c}{2(\lambda-\lambda')},\quad \varepsilon'=\frac{\alpha}{4(\lambda-\lambda')^2}+\frac{c}{2(\lambda'-\lambda)}.$$

E formulis (9) et (13) sequitur

$$\frac{\partial p}{\partial \alpha}=-\frac{\partial p'}{\partial \alpha}=\frac{1}{\lambda-\lambda'}\cdot\frac{\partial v}{\partial \alpha},$$

$$\frac{\partial \varepsilon}{\partial \alpha}=\frac{\partial \varepsilon'}{\partial \alpha}=\frac{1}{4(\lambda-\lambda')^2};$$

unde e (12) obtinetur

$$(14)\quad \frac{1}{y}\cdot\frac{\partial v}{\partial \alpha}=\frac{1}{8(\lambda-\lambda')\{\lambda' p'\sqrt{\varepsilon+\lambda pp}-\lambda p\sqrt{\varepsilon'+\lambda' p'p'}\}},$$

qui fieri debet Multiplicator aequationis $ydy-vdx=0$. Ac reapse invenitur e (10) et (12):

$$\begin{aligned} ydy-vdx &= \left\{\frac{\lambda p dp}{\sqrt{\varepsilon+\lambda pp}}+\frac{\lambda' p' dp'}{\sqrt{\varepsilon'+\lambda' p'p'}}\right\}\{\sqrt{\varepsilon+\lambda pp}+\sqrt{\varepsilon'+\lambda' p'p'}\} \\ &\quad -\{dp+dp'\}\{\lambda p+\lambda' p'\} \\ &= \{\lambda p\sqrt{\varepsilon'+\lambda' p'p'}-\lambda' p'\sqrt{\varepsilon+\lambda pp}\}\left\{\frac{dp}{\sqrt{\varepsilon+\lambda pp}}-\frac{dp'}{\sqrt{\varepsilon'+\lambda' p'p'}}\right\}. \end{aligned}$$

Unde per factorem (14) atque substitutionem (9) aequationem differentialem $ydy-vdx=0$ in aliam mutamus, in qua variabiles separatae sunt,

$$\frac{dp}{\sqrt{\varepsilon+\lambda pp}}-\frac{dp'}{\sqrt{\varepsilon'+\lambda' p'p'}}=0.$$

Cuius integratione prodit:

$$\frac{(\sqrt{\lambda}.p+\sqrt{\varepsilon+\lambda pp})^{\sqrt{\lambda'}}}{(\sqrt{\lambda'}.p'+\sqrt{\varepsilon'+\lambda' p'p'})^{\sqrt{\lambda}}}=\text{Const.}$$

Ponendo autem

$$(\sqrt{\lambda}+\sqrt{\lambda'})y+v+\sqrt{\lambda\lambda'}.x=A,$$
$$(\sqrt{\lambda}-\sqrt{\lambda'})y+v-\sqrt{\lambda\lambda'}.x=B,$$
$$(\sqrt{\lambda}-\sqrt{\lambda'})y-v+\sqrt{\lambda\lambda'}.x=C,$$

fit e (10) et (11) post calculos faciles

$$\sqrt{\lambda}.p+\sqrt{\varepsilon+\lambda pp}=\frac{AB+c}{2(\lambda-\lambda')y},$$
$$\sqrt{\lambda'}.p'+\sqrt{\varepsilon'+\lambda' p'p'}=\frac{AC-c}{2(\lambda-\lambda')y}.$$

Unde aequatio integralis inventa sic exhiberi potest:

$$\frac{(AB+c)^{\sqrt{\lambda'}}}{(AC-c)^{\sqrt{\lambda}}}=\beta.y^{\sqrt{\lambda'}-\sqrt{\lambda}},$$

ubi β est nova Constans arbitraria atque quantitas v, quae ipsas A, B, C afficit, est radix aequationis biquadraticae (7), porro λ et λ' sunt radices diversae aequationis quadraticae $\lambda^2-a\lambda+b=0$.

Integrationem his duobus exemplis praestitam etiam assequi licuisset ponendo cum Eulero $dx=ydt$, et aequationem differentialem secundi ordinis exemplo primo propositam *semel*, exemplo secundo propositam *bis* differentiando, ita ut t pro variabili independente habeatur. Quo facto respective pervenitur ad aequationes differentiales lineares tertii et quarti ordinis, quae Coëfficientibus gaudent constantibus notisque methodis integrantur.

§. 19.

De Multiplicatore systematis aequationum differentialium vulgarium, quod mediante solutione completa unius aequationis differentialis partialis primi ordinis integratur.

Systema aequationum differentialium vulgarium proponatur hoc:

$$(1)\quad \begin{cases} \frac{dq_1}{dt}=\frac{\partial\varphi}{\partial p_1},\quad \frac{dp_1}{dt}=-\left\{\frac{\partial\varphi}{\partial q_1}+p_1\frac{\partial\varphi}{\partial V}\right\}, \\ \frac{dq_2}{dt}=\frac{\partial\varphi}{\partial p_2},\quad \frac{dp_2}{dt}=-\left\{\frac{\partial\varphi}{\partial q_2}+p_2\frac{\partial\varphi}{\partial V}\right\}, \\ \cdots\cdots\cdots\cdots \\ \frac{dq_n}{dt}=\frac{\partial\varphi}{\partial p_n},\quad \frac{dp_n}{dt}=-\left\{\frac{\partial\varphi}{\partial q_n}+p_n\frac{\partial\varphi}{\partial V}\right\}, \\ \frac{dV}{dt}=p_1\frac{\partial\varphi}{\partial p_1}+p_2\frac{\partial\varphi}{\partial p_2}+\cdots+p_n\frac{\partial\varphi}{\partial p_n}, \end{cases}$$

ubi φ est functio quaecunque quantitatum $q_1, q_2, \ldots, q_n, V, p_1, p_2, \ldots, p_n$. Designante M aequationum (1) Multiplicatorem, secundum formulas nostras generales fit

$$\frac{d\log M}{dt}=-\Sigma\frac{\partial^2\varphi}{\partial p_i\partial q_i}+\Sigma\left\{\frac{\partial^2\varphi}{\partial q_i\partial p_i}+p_i\frac{\partial^2\varphi}{\partial V\partial p_i}\right\}+n\frac{\partial\varphi}{\partial V}$$
$$-\frac{\partial\left\{p_1\frac{\partial\varphi}{\partial p_1}+p_2\frac{\partial\varphi}{\partial p_2}+\cdots+p_n\frac{\partial\varphi}{\partial p_n}\right\}}{\partial V},$$

tribuendo indici i valores $1, 2, \ldots, n$. Unde, reiectis terminis se destruentibus, obtinetur

$$(2) \quad \frac{d\log M}{dt} = n\frac{\partial\varphi}{\partial V}.$$

Quae evanescit expressio, si φ ipsa V vacat. *Quoties igitur functio φ ab ipsa V vacua est, aequationum* (1) *Multiplicatorem unitati aequare licet.*

Aequationum (1) habetur Integrale unum

$$(3) \quad \varphi = h,$$

designante h Constantem. In ea aequatione ponatur

$$(4) \quad p_1 = \frac{\partial V}{\partial q_1}, \quad p_2 = \frac{\partial V}{\partial q_2}, \quad \ldots, \quad p_n = \frac{\partial V}{\partial q_n},$$

obtinetur aequatio differentialis partialis primi ordinis, in qua V est functio quaesita atque $q_1, q_2, \ldots, q_n$ sunt variabiles independentes. Faciamus, inventam esse eius aequationis differentialis partialis solutionem *quamcunque* V, dico aequationes (4) totidem esse aequationes integrales, quibus aequationes differentiales vulgares (1) gaudere possint. Nam differentiando ex. gr. earum primam $\frac{\partial V}{\partial q_1} - p_1 = 0$ et substituendo aequationes differentiales (1) prodit

$$(5) \quad \Sigma \frac{\partial^2 V}{\partial q_1 \partial q_i} \cdot \frac{\partial\varphi}{\partial p_i} + \frac{\partial\varphi}{\partial q_1} + p_1 \frac{\partial\varphi}{\partial V} = 0.$$

Cui aequationi satisfit substituendo ipsarum p_1, p_2, etc. valores (4). Nimirum e suppositione facta aequatio (3) identica evadit substituendo (4) solutionisque V valorem, eam autem aequationem identicam ipsius q_1 respectu differentiando prodit aequatio, in quam abit (5) per aequationes (4). *Itaque aequationes* (4) *una cum ipsa aequatione, qua V per $q_1, q_2, \ldots, q_n$ definiri ponitur, constituunt systema $n+1$ aequationum integralium idque tale, e quo differentiando ipsasque aequationes differentiales propositas substituendo deducere non licet aequationes integrales novas.* Scilicet aequationes provenientes (5) per illas $n+1$ aequationes identicas fieri vidimus.

Constans h, ubi servat significationem generalem, ingredi debet solutionem quamcunque V, unde, data V, differentiale quoque partiale $\frac{\partial V}{\partial h}$ assignare licebit, quod per z designabo. Erit per (1), (3), (4)

$$(6) \quad \frac{dz}{dt} = \Sigma \frac{\partial^2 V}{\partial h \partial q_i} \cdot \frac{\partial\varphi}{\partial p_i} = \frac{\partial\varphi}{\partial h} - \frac{\partial\varphi}{\partial V} z = 1 - \frac{\partial\varphi}{\partial V} z.$$

Si solutio V aliquam involvit Constantem arbitrariam α atque ponitur $\frac{\partial V}{\partial\alpha} = y$, similiter erit

$$(7) \quad \frac{dy}{dt} = \Sigma \frac{\partial^2 V}{\partial\alpha\partial q_i} \cdot \frac{\partial\varphi}{\partial p_i} = \frac{\partial\varphi}{\partial\alpha} - \frac{\partial\varphi}{\partial V} y = -\frac{\partial\varphi}{\partial V} y.$$

Scilicet functio φ, substituendo datam solutionem V atque ponendo $p_i = \frac{\partial V}{\partial q_i}$, identice aequatur Constanti h ideoque post eam substitutionem differentiata ipsius h respectu unitati aequatur, differentiata ipsius α respectu evanescit. E (2) et (7) sequitur

$$d\log M = -n\, d\log y,$$

ideoque fit

$$(8) \quad y^n M = \left(\frac{\partial V}{\partial\alpha}\right)^n M = \beta,$$

designante β Constantem. Haec formula docet, Multiplicatori M competere valorem, qui per aequationes integrales (3) et (4) aequetur quantitati $\left\{\frac{\partial V}{\partial\alpha}\right\}^{-n}$. Observo adhuc, e binis formulis (6) et (7) sequi

$$ydz - zdy = ydt,$$

unde, designante U functionem quantitatum y et z homogeneam rationalem $(-1)^{\text{ti}}$ ordinis, assignari poterit integrale $\int Udt$. Si solutio V plures Constantes arbitrarias involvit, totidem habebuntur aequationes (8), binarumque divisione obtinebuntur aequationes integrales, inventis (3) et (4) accedentes. Si functio φ ab ipsa V vacua est ideoque $M = 1$, aequationes (8) per se sunt aequationes integrales.

Si habetur solutio *completa* $V = F$, n Constantes arbitrarias $\alpha_1, \alpha_2, \ldots, \alpha_n$ involvens, poniturque $\frac{\partial F}{\partial\alpha_i} = u_i$, fit systema aequationum integralium completarum:

$$(9) \quad \begin{cases} F - V = 0, & \frac{\partial F}{\partial q_1} - p_1 = 0, \quad \frac{\partial F}{\partial q_2} - p_2 = 0, \quad \ldots, \quad \frac{\partial F}{\partial q_n} - p_n = 0, \\ & \frac{u_1}{u_n} - \beta_1 = 0, \quad \frac{u_2}{u_n} - \beta_2 = 0, \quad \ldots, \quad \frac{u_{n-1}}{u_n} - \beta_{n-1} = 0, \end{cases}$$

designantibus $\beta_1, \beta_2, \ldots, \beta_{n-1}$ alias Constantes arbitrarias. Si ex his aequationibus petuntur valores quantitatum h, α_i, β_i, atque functionum iis aequivalentium formantur Determinantia *partialia*, in quibus una quantitatum q_i, p_i, V pro Constante, reliquae pro variabilibus habentur, ea aequare debent quantitates ad dextram aequationum differentialium (1) positas, in *Multiplicatorem* ductas. Supersedere resolutioni aequationum (9) et immediate functionum $F-V$, $\frac{\partial F}{\partial q_1}-p_1$, etc. sumere possumus Determinantia partialia, dummodo ea dividimus per earundem functionum Determinans, quantitatum h, α_i, β_i respectu formatum. Qua de re Cap. I. egi. Determinantia functionalia hic obvenientia in alia simpliciora redeunt, propterea quod quantitates $V, p_1, p_2, \dots, p_n$ tantum in $n+1$ prioribus aequationum (9), quantitates $\beta_1, \beta_2, \dots, \beta_{n-1}$ tantum in $n-1$ posterioribus, singulae in singulis reprehenduntur. Sic Determinans, quantitatum h, α_i, β_i respectu formatum, quod per ∇ designabo, aequatur Determinanti functionum ab ipsis β_i vacuarum

$$F, \ \frac{\partial F}{\partial q_1}, \ \frac{\partial F}{\partial q_2}, \ \dots, \ \frac{\partial F}{\partial q_n},$$

solarum h et $\alpha_1, \alpha_2, \dots, \alpha_n$ respectu formato. Determinans partiale, in quo q_n pro Constante habetur et quod per (q_n) designabo, aequatur Determinanti functionum

$$\frac{u_1}{u_n}, \ \frac{u_2}{u_n}, \ \dots, \ \frac{u_{n-1}}{u_n},$$

formato solarum respectu $q_1, q_2, \dots, q_{n-1}$. Per theorema autem in Comment. *de Determinantibus functionalibus* comprobato, quod Determinantia spectat functionum communi denominatore praeditarum, fit

$$(q_n) = u_n^{-n} Q_n = \left(\frac{\partial F}{\partial \alpha_n}\right)^{-n} Q_n,$$

posito

$$Q_n = \Sigma \pm \frac{\partial u_1}{\partial q_1} \cdot \frac{\partial u_2}{\partial q_2} \cdots \frac{\partial u_{n-1}}{\partial q_{n-1}} u_n,$$

ubi formantur Determinantis Q_n termini permutando omnimodis functiones u_1, $u_2, \dots, u_n$. Substituendo autem valores $u_i = \frac{\partial F}{\partial \alpha_i}$ et differentiationum ordinem invertendo sequitur, *Determinans* Q_n *fieri Determinans functionum*

$$F, \ \frac{\partial F}{\partial q_1}, \ \frac{\partial F}{\partial q_2}, \ \dots, \ \frac{\partial F}{\partial q_{n-1}},$$

quantitatum $\alpha_1, \alpha_2, \dots, \alpha_n$ *respectu formatum.* Iam aequationem identicam

$$\varphi\left(q_1, q_2, \dots, q_n, F, \frac{\partial F}{\partial q_1}, \frac{\partial F}{\partial q_2}, \dots, \frac{\partial F}{\partial q_n}\right) = h$$

differentiando respectu quantitatum $h, \alpha_1, \alpha_2, \dots, \alpha_n$, quibus ipsae F, $\frac{\partial F}{\partial q_1}$, etc.

afficiuntur, scribendoque V et p_i ipsarum F et $\frac{\partial F}{\partial q_i}$ loco, obtinentur inter incognitas $\frac{\partial \varphi}{\partial V}$ et $\frac{\partial \varphi}{\partial p_i}$ aequationes $n+1$ lineares, quarum resolutione invenitur

$$\frac{\partial \varphi}{\partial p_n} = \frac{Q_n}{\nabla},$$

unde

$$\frac{(q_n)}{\nabla} = \frac{\partial \varphi}{\partial p_n} \left\{\frac{\partial F}{\partial \alpha_n}\right\}^{-n}.$$

Eadem ratione generaliter, ubi vocamus (q_i) functionum (9) Determinans partiale, in quo q_i pro Constante habetur, invenitur

$$(10) \quad \frac{(q_i)}{\nabla} = \left\{\frac{\partial F}{\partial \alpha_n}\right\}^{-n} \cdot \frac{\partial \varphi}{\partial p_i}.$$

Vocando W functionum

$$\frac{\partial F}{\partial q_1}, \ \frac{\partial F}{\partial q_2}, \ \dots, \ \frac{\partial F}{\partial q_n}$$

Determinans, quantitatum $\alpha_1, \alpha_2, \dots, \alpha_n$ respectu formatum, earundem $n+1$ aequationum linearium resolutione eruitur

$$\frac{\partial \varphi}{\partial V} = \frac{W}{\nabla}.$$

Functionum (9) Determinans partiale (p_n), in quo p_n pro Constante habetur, aequatur Determinanti functionum

$$\frac{\partial F}{\partial q_n}, \ \frac{u_1}{u_n}, \ \frac{u_2}{u_n}, \ \dots, \ \frac{u_{n-1}}{u_n},$$

quantitatum $q_1, q_2, \dots, q_n$ respectu formato. Invertendo autem ordinem differentiationum in differentialibus ipsius $\frac{\partial F}{\partial q_n}$ atque similes adhibendo formulas earum, quibus supra (q_n) ad Q_n revocavi, redit $u_n^n(p_n)$ in differentiam Determinantis P_n functionum

$$F, \ \frac{\partial F}{\partial q_1}, \ \frac{\partial F}{\partial q_2}, \ \dots, \ \frac{\partial F}{\partial q_n},$$

quantitatum $q_n, \alpha_1, \alpha_2, \dots, \alpha_n$ respectu formati, atque Determinantis functionalis modo adhibiti W per $\frac{\partial F}{\partial q_n}$ multiplicati, sive fit

$$\left(\frac{\partial F}{\partial \alpha_n}\right)^n (p_n) = P_n - \frac{\partial F}{\partial q_n} . W = P_n - p_n W.$$

Adiiciendo autem $n+1$ aequationibus linearibus commemoratis aliam provenientem ex aequatione $\varphi = h$, quantitatis q_n respectu differentiata, eruitur per eliminationem quantitatum $\frac{\partial \varphi}{\partial V}, \frac{\partial \varphi}{\partial p_1}, \frac{\partial \varphi}{\partial p_2}, \ldots, \frac{\partial \varphi}{\partial p_n}$:

$$\nabla \frac{\partial \varphi}{\partial q_n} + P_n = 0.$$

Unde fit

$$\frac{(p_n)}{\nabla} = \left\{\frac{\partial F}{\partial \alpha_n}\right\}^{-n} \left\{\frac{P_n}{\nabla} - p_n \frac{W}{\nabla}\right\} = -\left\{\frac{\partial F}{\partial \alpha_n}\right\}^{-n} \cdot \left\{\frac{\partial \varphi}{\partial q_n} + p_n \frac{\partial \varphi}{\partial V}\right\};$$

eademque ratione obtinetur generaliter, ubi (p_i) est functionum (9) Determinans partiale, in quo habetur p_i pro Constante:

$$(11)\quad \frac{(p_i)}{\nabla} = -\left\{\frac{\partial F}{\partial \alpha_n}\right\}^{-n} \cdot \left\{\frac{\partial \varphi}{\partial q_i} + p_i \frac{\partial \varphi}{\partial V}\right\}.$$

Quae paullo difficiliora erant indagatu. Postremo functionum (9) Determinans partiale (V), in quo habetur V pro Constante, aequale erit functionum

$$F, \; \frac{u_1}{u_n}, \; \frac{u_2}{u_n}, \; \ldots, \; \frac{u_{n-1}}{u_n}$$

Determinanti, quantitatum $q_1, q_2, \ldots, q_n$ respectu formato. Quod, adhibendo notationem supra traditam, fieri patet

$$(V) = \frac{\partial F}{\partial q_1}(q_1) + \frac{\partial F}{\partial q_2}(q_2) + \cdots + \frac{\partial F}{\partial q_n}(q_n),$$

unde secundum (10) invenitur:

$$(12)\quad \frac{(V)}{\nabla} = \left\{\frac{\partial F}{\partial \alpha_n}\right\}^{-n} \left\{p_1 \frac{\partial \varphi}{\partial p_1} + p_2 \frac{\partial \varphi}{\partial p_2} + \cdots + p_n \frac{\partial \varphi}{\partial p_n}\right\}.$$

Formulae (10), (11), (12) docent, functionum ad laevam aequationum (9) positarum Determinantia partialia aequari quantitatibus ad dextram aequationum differentialium (1) positis, per factorem communem $\left\{\frac{\partial F}{\partial \alpha_n}\right\}^{-n}$ multiplicatis. Ea Determinantia partialia autem sunt ut differentialia dq_i, dp_i, dV. Unde antecedentibus continetur demonstratio directa, aequationes differentiales propositas e formulis (9) differentiatis per aequationum linearium resolutionem fluere easque Multiplicatore gaudere $\left\{\frac{\partial F}{\partial \alpha_n}\right\}^{-n}$, qualis e formula (8) obtinebatur. Quam de-

monstrationem hic breviter indicasse placuit, cum ad illustrandam Determinantium theoriam faciat.

Casu, quo φ ab ipsa V vacua est, cum cognitus sit Multiplicator, videamus, quid sit, quod ea cognitione lucremur in exemplo simplicissimo, quo $n = 2$. Tributo Constanti h valore particulari, substituamus aequationi $\varphi = h$ aliam, qua ipsius p_2 valor per q_1, q_2, p_1 exhibetur, ita ut aequationes differentiales proponantur sequentes:

$$(13)\quad dq_1 : dq_2 : dp_1 = \frac{\partial p_2}{\partial p_1} : -1 : -\frac{\partial p_2}{\partial q_1}.$$

Quarum Multiplicatorem patet *unitati* aequari, cum summa differentialium quantitatum ad dextram, respective secundum q_1, q_2, p_1 sumtorum, evanescat. Unde si post primam integrationem exprimitur p_1 per q_1, q_2 et Constantem arbitrariam α, secundum principium ultimi Multiplicatoris fit alterum Integrale:

$$(14)\quad \int \frac{\partial p_1}{\partial \alpha}\left\{dq_1 + \frac{\partial p_2}{\partial p_1} dq_2\right\} = \text{Const.}$$

Sub integrationis signo haberi differentiale completum, e Lagrangiana aequationum differentialium partialium theoria sic probatur. Nam cum, expressis p_1 et p_2 per q_1 et q_2, fieri debeat $p_1 dq_1 + p_2 dq_2$ differentiale completum atque p_2 per q_1, q_2, p_1 expressum detur, pro p_1 talis sumi debet quantitatum q_1 et q_2 functio, quae satisfaciat conditioni

$$\frac{\partial p_1}{\partial q_2} - \frac{\partial p_2}{\partial p_1} \cdot \frac{\partial p_1}{\partial q_1} - \frac{\partial p_2}{\partial q_1} = 0.$$

Qualem functionem, e theoria aequationum differentialium partialium primi ordinis *linearium* constat, e quocunque Integrali aequationum differentialium vulgarium (13) erui. Quod ubi Constantem arbitrariam α implicat, eandem implicabunt valores ipsarum p_1 et p_2 per q_1 et q_2 exhibiti, qui expressionem $p_1 dq_1 + p_2 dq_2$ integrabilem reddebant. Qua secundum Constantem α differentiata, rursus prodire debet expressio integrabilis, sive expressio

$$\frac{\partial p_1}{\partial \alpha} dq_1 + \frac{\partial p_2}{\partial p_1} \cdot \frac{\partial p_1}{\partial \alpha} dq_2 = \frac{\partial p_1}{\partial \alpha}\left\{dq_1 + \frac{\partial p_2}{\partial p_1} dq_2\right\}$$

evadere debet differentiale completum. Q. D. E. Simul videmus, Integrale (14) obtineri aequiparando novae Constanti arbitrariae differentiale partiale solutionis $V = \int\{p_1 dq_1 + p_2 dq_2\}$, ipsius α respectu sumtum, id quod cum supra expositis convenit.

§. 20.

De Multiplicatore aequationum differentialium vulgarium systematis, quod mediante solutione completa problematis Pfaffiani integratur.

Problema Pfaffianum voco integrationem singularis aequationis differentialis linearis primi ordinis inter numerum variabilium parem per semissem aequationum finitarum numerum. Sit aequatio differentialis singularis proposita

$$(1)\quad 0 = X_1 dx_1 + X_2 dx_2 + \cdots + X_{2m} dx_{2m},$$

designantibus X_1, X_2, etc. variabilium x_1, x_2, ..., x_{2m} functiones quascunque. Qua integrata per numerum m aequationum, totidem Constantibus arbitrariis affectarum, demonstravi *Diar. Crell. Vol. XVII. pgg.* 148 *sqq.* (cf. h. Vol. p. 112 sqq.), praestari integrationem completam systematis aequationum differentialium sequentis:

$$(2)\quad \begin{cases} X_1 dt = \quad \cdot \quad + a_{1,2} dx_2 + a_{1,3} dx_3 + \cdots + a_{1,2m} dx_{2m}, \\ X_2 dt = -a_{1,2} dx_1 \quad \cdot \quad + a_{2,3} dx_3 + \cdots + a_{2,2m} dx_{2m}, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ X_{2m} dt = -a_{1,2m} dx_1 - a_{2,2m} dx_2 - \cdots\cdots \quad \cdot \quad , \end{cases}$$

ubi

$$(3)\quad a_{i,k} = -a_{k,i} = \frac{\partial X_i}{\partial x_k} - \frac{\partial X_k}{\partial x_i}, \quad a_{i,i} = 0.$$

Dedi in *Diario Crell. Vol. II. pgg.* 354 *sqq.* (cf. h. Vol. p. 26 sqq.) resolutionem algebraicam generalem aequationum linearium ad instar aequationum (2) formatarum. Cuius ope exhibitis aequationibus differentialibus forma proportionum nobis usitata

$$(4)\quad dx_1 : dx_2 : \ldots : dx_{2m} = A_1 : A_2 : \ldots : A_{2m},$$

investigemus formulam, qua aequationum (4) Multiplicator definiatur, sive valorem expressionis

$$(5)\quad \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \cdots + \frac{\partial A_{2m}}{\partial x_{2m}} = -A_i \frac{d \log M}{dx_i}.$$

Auspicabor ab aequationum linearium (2) resolutione, quae sic proponi potest.

Deriventur de producto

$$a_{1,2} a_{3,4} \cdots a_{2m-1,2m}$$

alii similes termini, mutando indices 2, 3, ..., $2m-1$, $2m$ respective in 3, 4, ..., $2m$, 2, eandemque indicum commutationem repetendo, donec ad terminum primitivum reditur, id quod suggerit $2m-1$ terminos diversos. Ea ratione, indicum certo ordine proposito, si quisque eorum in proxime sequentem, ultimus in primum mutatur idque repetitur, dum ad ordinem indicum primitivum reditur, dicam *indices cyclum percurrere*. Postquam e producto proposito $2m-1$ termini deducti sunt per cyclum, quem indices 2, 3, ..., $2m$ fecimus percurrere, rursus in eorum terminorum unoquoque ponamus indices $2m-3$ postremos cyclum percurrere, unde nanciscimur terminorum numerum $(2m-1)(2m-3)$. In eorum terminorum unoquoque rursus ponamus indices $2m-5$ postremos cyclum percurrere, erit terminorum diversorum provenientium numerus totalis $(2m-1)(2m-3)(2m-5)$. Ita pergendo, donec postremo soli tres indices postremi cyclum percurrant, producta $3.5\ldots(2m-1)$ ex uno proposito deducta erunt, quorum omnium aggregatum R vocemus. Sit ex. gr. $m=3$, erit R aggregatum *quindecim* terminorum

$$\begin{aligned} & a_{1,2}a_{3,4}a_{5,6} + a_{1,2}a_{3,5}a_{6,4} + a_{1,2}a_{3,6}a_{4,5} \\ + & a_{1,3}a_{4,5}a_{6,2} + a_{1,3}a_{4,6}a_{2,5} + a_{1,3}a_{4,2}a_{5,6} \\ + & a_{1,4}a_{5,6}a_{2,3} + a_{1,4}a_{5,2}a_{3,6} + a_{1,4}a_{5,3}a_{6,2} \\ + & a_{1,5}a_{6,2}a_{3,4} + a_{1,5}a_{6,3}a_{4,2} + a_{1,5}a_{6,4}a_{2,3} \\ + & a_{1,6}a_{2,3}a_{4,5} + a_{1,6}a_{2,4}a_{5,3} + a_{1,6}a_{2,5}a_{3,4}, \end{aligned}$$

quorum quinque in prima verticali ex eorum uno derivantur, identidem mutando indices 2, 3, 4, 5, 6 in 3, 4, 5, 6, 2; terni iuxta positi, indicibus tribus posterioribus cyclum percurrentibus, ex uno eorum fluunt. Aggregatum R fit denominator communis expressionum algebraicarum, quibus valores incognitarum exhibentur. Numeratorum autem Coëfficientes, qui ducuntur in terminos ad laevam aequationum linearium constitutos, sunt ipsius R differentialia, quantitatum $a_{i,k}$ respectu sumta, ita ut aequationum (2) resolutione proveniant valores

$$(5^*)\quad \begin{cases} R\dfrac{dx_1}{dt} = \quad \cdot \quad - \dfrac{\partial R}{\partial a_{1,2}} X_2 - \cdots - \dfrac{\partial R}{\partial a_{1,2m}} X_{2m}, \\ R\dfrac{dx_2}{dt} = \dfrac{\partial R}{\partial a_{1,2}} X_1 \quad \cdot \quad - \cdots - \dfrac{\partial R}{\partial a_{2,2m}} X_{2m}, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ R\dfrac{dx_{2m}}{dt} = \dfrac{\partial R}{\partial a_{1,2m}} X_1 + \dfrac{\partial R}{\partial a_{2,2m}} X_2 + \cdots\cdots \quad \cdot \quad . \end{cases}$$

Aggregatum R gaudet proprietatibus plane analogis earum, quae de Determinantibus circumferuntur. Quarum gravissima ea est, ut *binis indicum* 1, 2, ..., $2m$ *inter se permutatis simul omnes ipsius R termini valores oppositos induant ideoque* *ipsum R in valorem oppositum abeat.* Porro fit

$$(6)\quad R = a_{1,i}\frac{\partial R}{\partial a_{1,i}} + a_{2,i}\frac{\partial R}{\partial a_{2,i}} + \cdots + a_{2m,i}\frac{\partial R}{\partial a_{2m,i}},$$

et quoties i et k inter se diversi sunt,

$$(7)\quad 0 = a_{1,i}\frac{\partial R}{\partial a_{1,k}} + a_{2,i}\frac{\partial R}{\partial a_{2,k}} + \cdots + a_{2m,i}\frac{\partial R}{\partial a_{2m,k}},$$

ubi terminus in $a_{k,i}$ ductus ommittendus est. Designantibus i, i', i'', etc. indices inter se diversos, si sumuntur differentialia partialia

$$\frac{\partial R}{\partial a_{i,i'}},\quad \frac{\partial^2 R}{\partial a_{i,i'}\partial a_{i'',i'''}},\quad \text{etc.:}$$

ea erunt aggregata ad instar aggregati R formata, respective reiectis Coëfficientium binis, quatuor etc. seriebus cum horizontalibus tum verticalibus, eritque

$$(8)\quad \frac{\partial^2 R}{\partial a_{i,i'}\partial a_{i'',i'''}} = \frac{\partial^2 R}{\partial a_{i,i''}\partial a_{i''',i'}} = \frac{\partial^2 R}{\partial a_{i,i'''}\partial a_{i',i''}}.$$

His rebus praemissis, quarum demonstrationem aliis relinquo vel ad alium locum relego, Multiplicator quaesitus sic invenitur. Sequitur e (5*), siquidem signo summatorio subscribuntur indices, quorum respectu summatio instituenda est,

$$(9)\quad R\frac{dx_i}{dt} = A_i = \sum_{\alpha}\frac{\partial R}{\partial a_{\alpha,i}}X_\alpha,$$

unde

$$(10)\quad \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \cdots + \frac{\partial A_{2m}}{\partial x_{2m}} = \sum_{\alpha,i}\frac{\partial\frac{\partial R}{\partial a_{\alpha,i}}}{\partial x_i}\cdot X_\alpha + \sum_{\alpha,i}\frac{\partial R}{\partial a_{\alpha,i}}\cdot\frac{\partial X_\alpha}{\partial x_i},$$

ubi indicibus α et i tribuuntur valores $1, 2, \ldots, 2m$, solis ommissis valoribus $i = \alpha$. Examinemus formulae (10) summam priorem. Aggregati $\frac{\partial R}{\partial a_{\alpha,i}}$ cum terminus nullus afficiatur elemento, cuius alter index est α aut i, fit

$$\frac{\partial\frac{\partial R}{\partial a_{\alpha,i}}}{\partial x_i} = \sum_{k,l}\frac{\partial^2 R}{\partial a_{\alpha,i}\partial a_{k,l}}\cdot\frac{\partial a_{k,l}}{\partial x_i},$$

summatione duplici ad omnes $\frac{(2m-2)(2m-3)}{1.2}$ combinationes extensa, quibus indices k et l valores obtinent et inter se et ab ipsis α et i diversos. E formula antecedente sequitur

$$\sum_i\frac{\partial\frac{\partial R}{\partial a_{\alpha,i}}}{\partial x_i} = \sum_{i,k,l}\frac{\partial^2 R}{\partial a_{\alpha,i}\partial a_{k,l}}\cdot\frac{\partial a_{k,l}}{\partial x_i},$$

ubi indicum i, k, l valores in quoque termino sub signo summatorio et inter se et ab indice α diversi sunt, ipsi i valores $1, 2, \ldots, 2m$ conveniunt, binorum k et l valores non inter se permutari debent. Unde triplex summa conflatur e $\frac{(2m-1)(2m-2)(2m-3)}{1.2.3}$ terminis huiusmodi

$$\frac{\partial^2 R}{\partial a_{\alpha,i}\partial a_{k,l}}\left\{\frac{\partial a_{k,l}}{\partial x_i} + \frac{\partial a_{l,i}}{\partial x_k} + \frac{\partial a_{i,k}}{\partial x_l}\right\},$$

qui obtinentur sumendo pro indicibus i, k, l ternos diversos ex indicibus $1, 2, \ldots, \alpha-1, \alpha+1, \ldots, 2m$. At substituendo quantitatum $a_{i,k}$ valores (3), ternorum terminorum uncis inclusorum summa

$$\frac{\partial a_{k,l}}{\partial x_i} + \frac{\partial a_{l,i}}{\partial x_k} + \frac{\partial a_{i,k}}{\partial x_l}$$

identice evanescit, ideoque pro quoque ipsius α valore fit

$$(11)\quad \sum_i\frac{\partial\frac{\partial R}{\partial a_{\alpha,i}}}{\partial x_i} = 0,$$

sive formulae (10) prior summa evanescit. Alterius summae valor facile invenitur permutando indices α et i formulamque (6) in auxilium vocando, quae summata pro omnibus indicis i valoribus suppeditat

$$\sum_{\alpha,i} a_{\alpha,i}\frac{\partial R}{\partial a_{\alpha,i}} = 2m.R.$$

Hinc enim fit

$$\sum_{\alpha,i}\frac{\partial R}{\partial a_{\alpha,i}}\cdot\frac{\partial X_\alpha}{\partial x_i} = \tfrac{1}{2}\sum_{\alpha,i}\frac{\partial R}{\partial a_{\alpha,i}}\left\{\frac{\partial X_\alpha}{\partial x_i} - \frac{\partial X_i}{\partial x_\alpha}\right\} = \tfrac{1}{2}\sum_{\alpha,i}\frac{\partial R}{\partial a_{\alpha,i}}a_{\alpha,i} = mR.$$

Unde iam formula (10) in hanc abit:

$$(12)\quad \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \cdots + \frac{\partial A_{2m}}{\partial x_{2m}} = mR.$$

Cuius formulae pars laeva cum secundum (5) et (9) ipsi $-R\frac{d\log M}{dt}$ aequetur, aequationum differentialium (4) Multiplicatorem statuere licet

$$(13)\quad M = e^{-mt}.$$

Docet ea formula, aequationibus differentialibus (4) complete integratis, ad eruendam relationem inter t et variabiles x_i nulla amplius opus esse Quadratura, sed valorem integralis

$$\int \frac{R\,dx_i}{A_i} = t + \text{Const.}$$

exhiberi posse per logarithmum Determinantis functionum, quae Constantibus arbitrariis aequantur.

Ponamus, quod semper licet, $X_{2m} = -1$ sintque Coëfficientes reliqui omnes $X_1, X_2, \ldots, X_{2m-1}$ a variabili x_{2m} vacui, redit problema Pfaffianum in hoc, *ut expressio differentialis* $2m-1$ *variabilium*

$$X_1 dx_1 + X_2 dx_2 + \cdots + X_{2m-1} dx_{2m-1}$$

per $m-1$ *aequationes finitas reddatur differentiale completum* dx_{2m}. Scilicet ea re effecta, obtinetur m^{ta} aequatio per solas Quadraturas

$$x_{2m} + \text{Const.} = \int \{X_1 dx_1 + X_2 dx_2 + \cdots + X_{2m-1} dx_{2m-1}\}.$$

Eo casu evanescunt omnes quantitates $a_{i,2m}$ ideoque ipsum quoque dt, unde aequationes differentiales (2) in has abeunt:

$$(14)\quad \begin{cases} 0 = \quad \cdot \quad + a_{1,2}\, dx_2 + a_{1,3}\, dx_2 + \cdots + a_{1,2m-1}\, dx_{2m-1}, \\ 0 = a_{2,1}\, dx_1 \quad \cdot \quad + a_{2,3}\, dx_3 + \cdots + a_{2,2m-1}\, dx_{2m-1}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = a_{2m-1,1} dx_1 + a_{2m-1,2} dx_2 + a_{2m-1,3} dx_3 + \cdots\cdots \quad \cdot \quad . \end{cases}$$

Quarum una e reliquis fluit, sicuti sequitur summando aequationes respective per $\frac{\partial R}{\partial a_{1,2m}}, \frac{\partial R}{\partial a_{2,2m}}, \ldots, \frac{\partial R}{\partial a_{2m-1,2m}}$ multiplicatas. Evanescentibus $a_{i,2m}$, evanescunt et ipsum R et omnia ipsius R differentialia $\frac{\partial R}{\partial a_{i,k}}$, in quibus neuter indicum i et k ipsi $2m$ aequatur. Unde e (9) fit

$$A_1 = \frac{\partial R}{\partial a_{1,2m}},\quad A_2 = \frac{\partial R}{\partial a_{2,2m}},\quad \ldots,\quad A_{2m-1} = \frac{\partial R}{\partial a_{2m-1,2m}},$$
$$A_{2m} = X_1 A_1 + X_2 A_2 + \cdots + X_{2m-1} A_{2m-1}.$$

Cum A_{2m} a variabili x_{2m} vacua sit, formula (12) abit in hanc:

$$(15)\quad \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \cdots + \frac{\partial A_{2m-1}}{\partial x_{2m-1}} = 0.$$

Quae docet, *aequationum differentialium, quae e* (14) *proveniunt,*

$$(16)\quad dx_1 : dx_2 : \ldots : dx_{2m-1} = A_1 : A_2 : \ldots : A_{2m-1}$$

Multiplicatorem aequari unitati.

Principium ultimi Multiplicatoris applicemus exemplo simplicissimo, quo $m = 2$ sive quo aequationes differentiales proponuntur

$$(17)\quad dx_1 : dx_2 : dx_3 = \frac{\partial X_2}{\partial x_3} - \frac{\partial X_3}{\partial x_2} : \frac{\partial X_3}{\partial x_1} - \frac{\partial X_1}{\partial x_3} : \frac{\partial X_1}{\partial x_2} - \frac{\partial X_2}{\partial x_1}.$$

Inventa per primam integrationem variabilis x_3 expressione per x_1, x_2 et Constantem arbitrariam α, secundum principium illud fit altera aequatio integralis

$$(18)\quad \int \frac{\partial x_3}{\partial \alpha} \left\{ \left(\frac{\partial X_3}{\partial x_1} - \frac{\partial X_1}{\partial x_3} \right) dx_1 + \left(\frac{\partial X_3}{\partial x_2} - \frac{\partial X_2}{\partial x_3} \right) dx_2 \right\} = \text{Const.}$$

Quantitatem sub integrationis signo differentiale completum esse, sic verificari potest. Substituta variabilis x_3 expressione per integrationem primam inventa in formula $X_1 dx_1 + X_2 dx_2 + X_3 dx_3$, obtinetur

$$\left(X_1 + X_3 \frac{\partial x_3}{\partial x_1} \right) dx_1 + \left(X_2 + X_3 \frac{\partial x_3}{\partial x_2} \right) dx_2.$$

Eadem expressione substituta in aequationibus differentialibus, prodit aequatio

$$\frac{\partial X_1}{\partial x_2} - \frac{\partial X_2}{\partial x_1} = \frac{\partial x_3}{\partial x_1} \left\{ \frac{\partial X_2}{\partial x_3} - \frac{\partial X_3}{\partial x_2} \right\} + \frac{\partial x_3}{\partial x_2} \left\{ \frac{\partial X_3}{\partial x_1} - \frac{\partial X_1}{\partial x_3} \right\},$$

quae est conditio, ut formula differentialis antecedens sit differentiale aliquod completum dx_4. Si ipsius x_3 expressio implicat Constantem arbitrariam α, fit

$$\begin{aligned} d\,\frac{\partial x_4}{\partial \alpha} &= \frac{\partial \left\{ X_1 + X_3 \frac{\partial x_3}{\partial x_1} \right\}}{\partial \alpha} dx_1 + \frac{\partial \left\{ X_2 + X_3 \frac{\partial x_3}{\partial x_2} \right\}}{\partial \alpha} dx_2 \\ &= \frac{\partial x_3}{\partial \alpha} \left\{ \left(\frac{\partial X_1}{\partial x_3} + \frac{\partial X_3}{\partial x_3} \cdot \frac{\partial x_3}{\partial x_1} \right) dx_1 + \left(\frac{\partial X_2}{\partial x_3} + \frac{\partial X_3}{\partial x_3} \cdot \frac{\partial x_3}{\partial x_2} \right) dx_2 \right\} \\ &\qquad + X_3 \left\{ \frac{\partial^2 x_3}{\partial x_1 \partial \alpha} dx_1 + \frac{\partial^2 x_3}{\partial x_2 \partial \alpha} dx_2 \right\} \\ &= \frac{\partial x_3}{\partial \alpha} \left\{ \left(\frac{\partial X_1}{\partial x_3} - \frac{\partial X_3}{\partial x_1} \right) dx_1 + \left(\frac{\partial X_2}{\partial x_3} - \frac{\partial X_3}{\partial x_2} \right) dx_2 \right\} \\ &\qquad + \frac{\partial x_3}{\partial \alpha} dX_3 + X_3\, d\,\frac{\partial x_3}{\partial \alpha}. \end{aligned}$$

Unde sequitur, quod propositum erat, quantitatem sub integrationis signo aequari differentiali completo, videlicet differentiali

$$d\left(X_3 \frac{\partial x_3}{\partial \alpha} \right) - d\,\frac{\partial x_4}{\partial \alpha}.$$

Quod si igitur functio x_4 inventa est, aequationem integralem (18) sic quoque repraesentare licet:

$$(19)\quad X_3\frac{\partial x_3}{\partial \alpha}-\frac{\partial x_4}{\partial \alpha}=\text{Const.}$$

Quae de formulis quoque generalibus deduci potuit, quas loco citato tradidi de aequationum differentialium (2) systemate per solutionem completam aequationis (1) integrando. Qua de integratione hac occasione novas addam Propositiones novasque demonstrationes sequentes.

§. 21.

Conditiones ut aequatio differentialis vulgaris linearis primi ordinis inter p variabiles per pauciores quam $\frac{1}{2}p$ aequationes integrari possit.

Ac primum comprobabo Propositionem, *si aequatio differentialis singularis*

$$(20)\quad X_1dx_1+X_2dx_2+\cdots+X_pdx_p=0$$

integretur per m aequationes quascunque, earum ope fieri, ut de quibusque m e numero p aequationum differentialium sequentium:

$$(21)\quad \begin{cases} X_1dt = \quad\cdot\quad +a_{1,2}dx_2+a_{1,3}dx_3+\cdots+a_{1,p}dx_p, \\ X_2dt = a_{2,1}dx_1 \quad\cdot\quad +a_{2,3}dx_3+\cdots+a_{2,p}dx_p, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ X_pdt = a_{p,1}dx_1+a_{p,2}dx_2+a_{p,3}dx_3+\cdots\cdots\quad\cdot \end{cases}$$

reliquae $p-m$ sponte fluant, ipsis $a_{k,k'}$ designantibus quantitates $\frac{\partial X_k}{\partial x_{k'}}-\frac{\partial X_{k'}}{\partial x_k}$.

Cuius Propositionis demonstrationem sic adorno.

Designo

per h, h' etc. indices $1, 2, \ldots, m$,
per i, i' etc. indices $m+1, m+2, \ldots, p$,
per k, k' etc. indices $1, 2, 3, \ldots, p$.

Aequando $x_1, x_2, \ldots, x_m$ *quibuscunque* reliquarum variabilium $x_{m+1}, x_{m+2}, \ldots, x_p$ functionibus, abeunt aequationes (21) in sequentes:

$$(22)\quad 0=u_k=X_kdt-\sum_i b_{k,i}dx_i,$$

siquidem statuitur

$$(23)\quad \begin{cases} b_{k,i}=a_{k,1}\frac{\partial x_1}{\partial x_i}+a_{k,2}\frac{\partial x_2}{\partial x_i}+\cdots+a_{k,m}\frac{\partial x_m}{\partial x}+a \\ \quad =a_{k,i}+\sum_h a_{k,h}\frac{\partial x_h}{\partial x_i}. \end{cases}$$

Ponamus porro

$$(24)\quad v_i=X_1\frac{\partial x_1}{\partial x_i}+X_2\frac{\partial x_2}{\partial x_i}+\cdots+X_m\frac{\partial x_m}{\partial x_i}+X_i,$$

erit substituendo (22):

$$(25)\quad \frac{\partial x_1}{\partial x_{i'}}u_1+\frac{\partial x_2}{\partial x_{i'}}u_2+\cdots+\frac{\partial x_m}{\partial x_{i'}}u_m+u_{i'}=v_{i'}dt-\sum_i c_{i',i}dx_i,$$

posito

$$(26)\quad \begin{cases} c_{i',i}=\frac{\partial x_1}{\partial x_{i'}}b_{1,i}+\frac{\partial x_2}{\partial x_{i'}}b_{2,i}+\cdots+\frac{\partial x_m}{\partial x_i}b_{m,i}+b_{i',i} \\ \quad =b_{i',i}+\sum_h\frac{\partial x_h}{\partial x_{i'}}b_{h,i}. \end{cases}$$

Substituendo ipsarum $b_{k,i}$ valores (23), induit $c_{i',i}$ valorem sequentem:

$$(27)\quad c_{i',i}=a_{i',i}+\sum_h a_{i',h}\frac{\partial x_h}{\partial x_i}+\sum_{h'}a_{h',i}\frac{\partial x_{h'}}{\partial x_{i'}}+\sum_{h,h'}a_{h',h}\frac{\partial x_h}{\partial x_i}\cdot\frac{\partial x_{h'}}{\partial x_{i'}},$$

sive reponendo quantitatum $a_{k,k'}$ valores:

$$(28)\quad c_{i',i}=\frac{\partial X_{i'}}{\partial x_i}-\frac{\partial X_i}{\partial x_{i'}}+\sum_h\left\{\frac{\partial X_{i'}}{\partial x_h}-\frac{\partial X_h}{\partial x_{i'}}\right\}\frac{\partial x_h}{\partial x_i}+\sum_{h'}\left\{\frac{\partial X_{h'}}{\partial x_i}-\frac{\partial X_i}{\partial x_{h'}}\right\}\frac{\partial x_{h'}}{\partial x_{i'}}$$
$$+\sum_{h,h'}\left\{\frac{\partial X_{h'}}{\partial x_h}-\frac{\partial X_h}{\partial x_{h'}}\right\}\frac{\partial x_h}{\partial x_i}\cdot\frac{\partial x_{h'}}{\partial x_{i'}}.$$

Includamus uncis differentialia partialia, in quibus solae x_i sive $x_{m+1}, x_{m+2}, \ldots, x_p$ pro independentibus habentur atque quantitates x_h sive $x_1, x_2, \ldots, x_m$ pro earum functionibus: erit

$$(29)\quad \left(\frac{\partial X_k}{\partial x_i}\right)=\frac{\partial X_k}{\partial x_i}+\sum_h\frac{\partial X_k}{\partial x_h}\cdot\frac{\partial x_h}{\partial x_i},$$

unde

$$(30)\quad c_{i',i}=\left(\frac{\partial X_{i'}}{\partial x_i}\right)-\left(\frac{\partial X_i}{\partial x_{i'}}\right)+\sum_h\left\{\left(\frac{\partial X_h}{\partial x_i}\right)\frac{\partial x_h}{\partial x_{i'}}-\left(\frac{\partial X_h}{\partial x_{i'}}\right)\frac{\partial x_h}{\partial x_i}\right\}.$$

Id quod sequitur, indicibus h et h' in summa duplici $\sum_{h,h'}\frac{\partial X_{h'}}{\partial x_h}\cdot\frac{\partial x_h}{\partial x_i}\cdot\frac{\partial x_{h'}}{\partial x_{i'}}$ inter se permutatis nec non in (29) scripto h' ipsius h loco. Inventam autem ipsius $c_{i',i}$ expressionem (30) ope formulae (24) sic exhibere licet:

$$(31)\quad c_{i',i}=\left(\frac{\partial v_{i'}}{\partial x_i}\right)-\left(\frac{\partial v_i}{\partial x_{i'}}\right),$$

reiectis qui se mutuo destruunt terminis:

$$X_h\frac{\partial^2 x_h}{\partial x_{i'}\partial x_i}-X_h\frac{\partial^2 x_h}{\partial x_i\partial x_{i'}}.$$

Quo ipsius $c_{i',i}$ valore substituto in (25), eruimus formulam, quae valet, *quaecunque sint quantitates x_k reliquarum x_i functiones:*

$$(32)\quad \frac{\partial x_1}{\partial x_{i'}}u_1+\frac{\partial x_2}{\partial x_{i'}}u_2+\cdots+\frac{\partial x_m}{\partial x_{i'}}u_m+u_{i'} = v_{i'}dt+\sum_i\left\{\left(\frac{\partial v_i}{\partial x_{i'}}\right)-\left(\frac{\partial v_{i'}}{\partial x_i}\right)\right\}dx_i.$$

Quantitatibus x_k per variabiles x_i expressis, cum fiat e (24)

$$(33)\quad X_1dx_1+X_2dx_2+\cdots+X_pdx_p = v_{m+1}dx_{m+1}+v_{m+2}dx_{m+2}+\cdots+v_pdp,$$

si per m aequationes, quibus quantitates x_k per variabiles $x_{m+1}, x_{m+2}, \ldots, x_p$ determinantur, aequatio differentialis (20) integratur, singuli termini ad dextram formulae (33) per se evanescere debent, sive fieri debet

$$(34)\quad v_{m+1}=v_{m+2}=\cdots=v_p=0.$$

Unde etiam aequationis (32) pars laeva evanescere debet sive, scribendo i ipsius i' loco, pro quolibet ipsius i valore fieri debet

$$(34^*)\quad \frac{\partial x_1}{\partial x_i}u_1+\frac{\partial x_2}{\partial x_i}u_2+\cdots+\frac{\partial x_m}{\partial x_i}u_m+u_i = 0.$$

Quae formula docet, si per m aequationes integretur aequatio differentialis (20), earum aequationum ope fieri, ut ex aequationibus

$$u_1=0,\quad u_2=0,\quad \ldots,\quad u_m=0$$

reliquae

$$u_{m+1}=0,\quad u_{m+2}=0,\quad \ldots,\quad u_p=0$$

sponte fluant. Q. D. E.

Si $p>2m$, inter coëfficientes X_1, X_2, etc. certae quaedam locum habere debent relationes, cum determinando m functiones $x_1, x_2, \ldots, x_m$ satisfieri debeat pluribus conditionibus, videlicet $p-m$ aequationibus

$$0=v_i=X_1\frac{\partial x_1}{\partial x_i}+X_2\frac{\partial x_2}{\partial x_i}+\cdots+X_m\frac{\partial x_m}{\partial x_i}+X_i.$$

Quae relationes obtineri possunt e formula (32). Nam secundum eam formulam aequationibus differentialibus (21) sive aequationibus

$$u_1=0,\quad u_2=0,\quad \ldots,\quad u_p=0$$

satisfit per numerum $2m$ aequationum, videlicet per m aequationes, quibus x_1, $x_2, \ldots, x_m$ per reliquas variabiles determinantur, atque m aequationes differentiales $u_1=0$, $u_2=0, \ldots, u_m=0$. Unde inter quantitates X_1, X_2, etc. tales locum habere debent relationes, ut de p aequationum (21) numero $2m$ reliquae

$p-2m$ sponte fluant sive, ope $2m$ aequationum differentialium $u_1=0$, $u_2=0, \ldots,$ $u_{2m}=0$ eliminatis $2m$ differentialibus $dx_1, dx_2, \ldots, dx_{2m}$, reliquae $p-2m$ aequationes differentiales $u_{2m+1}=0$, $u_{2m+2}=0, \ldots, u_p=0$ identicae evadant. Secundum observationem olim a me factam in Diar. Crell. Vol. II. pag. 352 (cf. h. Vol. p. 24) hae $p-2m$ aequationes post eam eliminationem formam induunt eandem atque propositae (21), videlicet formam huiusmodi:

$$\begin{aligned}
F_1dt &= \quad\cdot\quad + f_{1,2}dx_{2m+2}+f_{1,3}dx_{2m+3}+\cdots+f_{1,p-2m}dx_p,\\
F_2dt &= f_{2,1}dx_{2m+1}\quad\cdot\quad +f_{2,3}dx_{2m+3}+\cdots+f_{2,p-2m}dx_p,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
F_{p-2m}dt &= f_{p-2m,1}dx_{2m+1}+f_{p-2m,2}dx_{2m+2}+\cdots\cdots\quad\cdot\quad,
\end{aligned}$$

ubi $f_{i,k}=-f_{k,i}$. Quae aequationes ut identicae evadant, evanescere debent et $p-2m$ quantitates F_i et $\frac{(p-2m)(p-2m-1)}{2}$ quantitates $f_{i,k}$. Unde *locum habere debent* $\frac{(p-2m)(p-2m+1)}{1.2}$ *conditiones, ut aequatio differentialis linearis primi ordinis inter p variabiles* (20) *per $m<\frac{1}{2}p$ aequationes integrari possit, eaedemque sunt conditiones, quibus efficitur, ut p aequationes lineares* (21) *ex earum numero $2m$ fluant.* Si $p=2m+1$, prodit una conditio iam a Cl. Pfaff olim exhibita, quae, si $m=1$, notam conditionem integrabilitatis suppeditat. Si $p=2m+2$, locum habere debent tres conditiones, quas pro $m=1$ accuratius examinemus.

Sit igitur propositum indagare conditiones, ut aequatio differentialis linearis inter *quatuor* variabiles

$$(35)\quad X_1dx_1+X_2dx_2+X_3dx_3+X_4dx_4=0$$

unica aequatione integrari possit. Qua aequatione si exprimitur una variabilium x_4 per x_1, x_2, x_3, proposita (35) identica fieri debet, id quod aequationes poscit sequentes:

$$(36)\quad \frac{\partial x_4}{\partial x_1}=-\frac{X_1}{X_4},\quad \frac{\partial x_4}{\partial x_2}=-\frac{X_2}{X_4},\quad \frac{\partial x_4}{\partial x_3}=-\frac{X_3}{X_4}.$$

Secunda et tertia earum aequationum suppeditant

$$\begin{aligned}
X_4^2\frac{\partial^2 x}{\partial x_2\partial x_3} &= X_2\left\{\frac{\partial X_4}{\partial x_3}-\frac{X_3}{X_4}\cdot\frac{\partial X_4}{\partial x_4}\right\}-X_4\left\{\frac{\partial X_2}{\partial x_3}-\frac{X_3}{X_4}\cdot\frac{\partial X_2}{\partial x_4}\right\}\\
&= X_3\left\{\frac{\partial X_4}{\partial x_2}-\frac{X_2}{X_4}\cdot\frac{\partial X_4}{\partial x_4}\right\}-X_4\left\{\frac{\partial X_3}{\partial x_2}-\frac{X_2}{X_4}\cdot\frac{\partial X_3}{\partial x_4}\right\}.
\end{aligned}$$

Unde, ponendo $a_{i,k}=\frac{\partial X_i}{\partial x_k}-\frac{\partial X_k}{\partial x_i}$ similesque aequationes de tertia et prima, de prima et secunda aequationum (36) deducendo, obtinentur tres primae aequationum sequentium, quibus duas alias addidi ex iis provenientes:

$$
(37)\quad \begin{cases} 0 = \;\cdot\; + a_{3,4}X_2 + a_{4,2}X_3 + a_{2,3}X_4, \\ 0 = a_{4,3}X_1 \;\cdot\; + a_{1,4}X_3 + a_{3,1}X_4, \\ 0 = a_{2,4}X_1 + a_{4,1}X_2 \;\cdot\; + a_{1,2}X_4, \\ 0 = a_{3,2}X_1 + a_{1,3}X_2 + a_{2,1}X_3 \;\cdot\;, \\ 0 = a_{2,3}a_{1,4} + a_{3,1}a_{2,4} + a_{1,2}a_{3,4}. \end{cases}
$$

Ad easdem autem relationes secundum Propositionem generalem supra conditam pervenire debemus, si quaerimus conditiones, ut quatuor aequationum linearium

$$
\begin{aligned} X_1 dt &= \;\cdot\; + a_{1,2}dx_2 + a_{1,3}dx_3 + a_{1,4}dx_4, \\ X_2 dt &= a_{2,1}dx_1 \;\cdot\; + a_{2,3}dx_3 + a_{2,4}dx_4, \\ X_3 dt &= a_{3,1}dx_1 + a_{3,2}dx_2 \;\cdot\; + a_{3,4}dx_4, \\ X_4 dt &= a_{4,1}dx_1 + a_{4,2}dx_2 + a_{4,3}dx_3 \;\cdot \end{aligned}
$$

binae e duabus reliquis fluant. Quod re vera fieri, facile comprobatur. Aequationum (37) quatuor primae sunt notae conditiones integrabilitatis aequationis differentialis linearis primi ordinis inter tres variabiles, ex eadem aequatione (35) provenientis, si successive x_1, x_2, x_3, x_4 constantes ponuntur. Quatuor illarum aequationum ternae cum quartam secum ducant, sequitur, *si tres aequationes*

$$
\begin{aligned} X_2dx_2 + X_3dx_3 + X_4dx_4 &= 0, \\ X_1dx_1 + X_3dx_3 + X_4dx_4 &= 0, \\ X_1dx_1 + X_2dx_2 + X_4dx_4 &= 0, \end{aligned}
$$

habitis respective x_1, x_2, x_3 pro Constantibus, conditioni integrabilitatis satisfaciant, hanc quoque aequationem

$$X_1dx_1 + X_2dx_2 + X_3dx_3 = 0,$$

si in ea x_4 pro Constante habeatur, conditioni integrabilitatis satisfacturam esse, nec non aequationem $X_1dx_1 + X_2dx_2 + X_3dx_3 + X_4dx_4 = 0$, in qua omnes quatuor quantitates x_1, x_2, x_3, x_4 variabiles sunt, unica aequatione integrari posse. Ut ipsa absolvatur integratio, opus erit integratione completa trium aequationum differentialium primi ordinis inter duas variabiles, id quod simili ratione demonstratur atque in tractatibus Calculi Integralis probatur, ad integrandam aequationem differentialem linearem primi ordinis inter tres variabiles, conditioni integrabilitatis satisfacientem, requiri integrationem completam duarum aequationum differentialium primi ordinis inter duas variabiles. Quae res in tractatibus ita

proponi solet, ut alteram ne condere quidem liceat aequationem differentialem, nisi iam antea altera complete integrata habeatur. At observo, si aequatio differentialis inter tres variabiles x_1, x_2, x_3, conditioni integrabilitatis satisfaciens, est $X_1dx_1 + X_2dx_2 + X_3dx_3 = 0$, pro duabus aequationibus inter duas variabiles integrandis sumi posse has, quae *separatim* tractari possint:

$$X_1dx_1 + X_2dx_2 = 0, \quad X_2^0dx_2 + X_3^0dx_3 = 0,$$

quae e proposita proveniunt, prima habendo x_3 pro Constante, secunda ponendo $x_1 = 0$. Scilicet post integrationem secundae in locum ipsius x_2 substituenda est ea quantitatum x_1, x_2, x_3 functio, quae per integrationem primae aequiparatur valori variabilis x_2, qui ipsi $x_1 = 0$ respondet. Similiter, si proponitur integrare aequationem inter quatuor variabiles:

$$X_1dx_1 + X_2dx_2 + X_3dx_3 + X_4dx_4 = 0,$$

conditionibus (37) locum habentibus, pro tribus aequationibus inter duas variabiles, quae integrandae sunt, sumi possunt sequentes separatim tractandae:

$$X_1dx_1 + X_2dx_2 = 0, \quad X_2^0dx_2 + X_3^0dx_3 = 0, \quad X_3^{00}dx_3 + X_4^{00}dx_4 = 0,$$

in quibus designant X_2^0 et X_3^0 valores, in quos X_2 et X_3 abeunt pro $x_1 = 0$, porro X_3^{00} et X_4^{00} valores, in quos X_3 et X_4 pro $x_1 = x_2 = 0$ abeunt; deinde in prima aequatione x_3 et x_4, in secunda x_4 pro Constantibus habendae sunt. Integrata tertia aequatione, ipsi x_3 ea substituenda est quantitatum x_2, x_3, x_4 functio, quae per integrationem secundae aequat variabilis x_3 valorem ipsi $x_2 = 0$ respondentem; ac deinde ipsi x_2 ea quantitatum x_1, x_2, x_3, x_4 functio substituenda est, quae per aequationis primae integrationem aequat variabilis x_2 valorem ipsi $x_1 = 0$ respondentem.

Propositis p aequationibus differentialibus vulgaribus inter $p+1$ variabiles quibuscunque, aequationes m inter ipsas variabiles sunt integrales propositarum, si efficiunt, ut harum numerus m e reliquis $p-m$ fluat; porro tale constituunt aequationum integralium systema, e quo per differentiationem aequationumque differentialium substitutionem aliae novae non obtineantur, si earum adiumento non plures quam m aequationes differentiales e reliquis fluunt. Antecedentibus vidimus, per m aequationes, quibus integretur aequatio differentialis vulgaris linearis inter p variabiles (20), fieri ut e p aequationum differentialium vulgarium (21) numero m reliquae $p-m$ sponte fluant. Unde si $p-m = m$ sive $p = 2m$, qui est casus problematis Pfaffiani, sequitur, *quascunque m aequa-*

tiones, quibus integretur aequatio differentialis linearis primi ordinis inter $2m$ variabiles

$$0 = X_1 dx_1 + X_2 dx_2 + \cdots + X_{2m} dx_{2m},$$

haberi posse pro integralibus systematis $2m$ aequationum differentialium vulgarium

$$X_k dt = a_{k,1} dx_1 + a_{k,2} dx_2 + \cdots + a_{k,2m} dx_{2m},$$

ex iisque per differentiationem novas deduci non posse aequationes integrales. Si $m < \frac{1}{2}p$ atque aequatio (20) integrari potest m aequationibus, vidimus p aequationum (21) tantum $2m$ a se independentes esse, reliquas $p-2m$ ex iis sponte fluere; unde ex arbitrio iis addere licet $p-2m$ aequationes differentiales, ut habeatur systema p aequationum differentialium inter $p+1$ variabiles. Eo casu aequationes m, quibus aequatio (20) integrari supponitur, rursus haberi possunt pro aequationibus eius systematis integralibus, quaecunque sint $p-2m$ aequationes differentiales ipsis (21) ex arbitrio adiectae, cum illae m aequationes efficiant, quod e (32) sequebatur, ut m aequationes differentiales $u_{m+1} = 0$, $u_{m+2} = 0, \ldots, u_{2m} = 0$ ex aliis systematis aequationibus differentialibus $u_1 = 0$, $u_2 = 0, \ldots, u_m = 0$ obtineantur.

Designantibus A_1, A_2, etc. quascunque variabilium $x_1, x_2, \ldots, x_p$ functiones, quoties aequationum differentialium

$$dx_1 : dx_2 : \ldots : dx_p = A_1 : A_2 : \ldots : A_p$$

dantur aequationes integrales m, quarum differentiatione aliae novae non prodeunt, earumque ope exprimuntur $x_1, x_2, \ldots, x_m$ ut functiones variabilium x_{m+1}, $x_{m+2}, \ldots, x_p$, eas functiones satisfacere constat systemati aequationum differentialium partialium linearium primi ordinis sequenti:

$$(38)\quad \begin{cases} A_1 = A_{m+1}\dfrac{\partial x_1}{\partial x_{m+1}} + A_{m+2}\dfrac{\partial x_1}{\partial x_{m+2}} + \cdots + A_p \dfrac{\partial x_1}{\partial x_p}, \\ A_2 = A_{m+1}\dfrac{\partial x_2}{\partial x_{m+1}} + A_{m+2}\dfrac{\partial x_2}{\partial x_{m+2}} + \cdots + A_p \dfrac{\partial x_2}{\partial x_p}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A_m = A_{m+1}\dfrac{\partial x_m}{\partial x_{m+1}} + A_{m+2}\dfrac{\partial x_m}{\partial x_{m+2}} + \cdots + A_p \dfrac{\partial x_m}{\partial x_p}. \end{cases}$$

Qua de re pluribus egi in alia Commentatione *Diar.* Crell. *Vol. XXIII.* inserta (cf. h. Vol. p. 230 sqq.). Systema (38) ita est comparatum, ut in quaque aequatione eiusdem functionis reperiantur differentialia partialia secundum diversas variabiles independentes sumta, atque differentialia partialia diversarum functionum secundum eandem

variabilem independentem in diversis aequationibus sumta eodem afficiantur Coëfficiente. Eiusmodi systematis hoc, a cuius solutione problema Pfaffianum pendet,

$$(39)\quad v_{m+1} = 0,\quad v_{m+2} = 0,\quad \ldots,\quad v_{2m} = 0$$

quodammodo inversum est, sicuti e functionis v_i expressione (24) patet; quippe in quaque huius systematis aequatione diversarum functionum differentialia reprehenduntur secundum eandem variabilem sumta, atque eiusdem functionis differentialia, secundum diversas variabiles independentes in diversis aequationibus sumta, eodem afficiuntur Coëfficiente. Secundum antecedentia e systemate (39) sequitur aliud eius inversum formae systematis (38). Nam ubi aequationes (2) ad formam aequationum (9) revocamus, sequitur ex antecedentibus, m aequationes, quae systemati (39) satisfaciant sive quibus (1) integretur, ipsarum (9) fieri aequationes integrales, quarum differentiatione aliae novae non prodeant, ideoque easdem systemati aequationum (38) satisfacere. Unde haec obtinetur Propositio.

Propositio.

„E systemate aequationum differentialium partialium linearium primi ordinis huiusmodi

$$(39^*)\quad \begin{cases} -X_{m+1} = X_1\dfrac{\partial x_1}{\partial x_{m+1}} + X_2\dfrac{\partial x_2}{\partial x_{m+1}} + \cdots + X_m \dfrac{\partial x_m}{\partial x_{m+1}}, \\ -X_{m+2} = X_1\dfrac{\partial x_1}{\partial x_{m+2}} + X_2\dfrac{\partial x_2}{\partial x_{m+2}} + \cdots + X_m \dfrac{\partial x_m}{\partial x_{m+2}}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -X_{2m} = X_1\dfrac{\partial x_1}{\partial x_{2m}} + X_2\dfrac{\partial x_2}{\partial x_{2m}} + \cdots + X_m \dfrac{\partial x_m}{\partial x_{2m}} \end{cases}$$

hoc sequitur alterum formae quodammodo inversae

$$\begin{aligned} A_1 &= A_{m+1}\frac{\partial x_1}{\partial x_{m+1}} + A_{m+2}\frac{\partial x_1}{\partial x_{m+2}} + \cdots + A_{2m}\frac{\partial x_1}{\partial x_{2m}}, \\ A_2 &= A_{m+1}\frac{\partial x_2}{\partial x_{m+1}} + A_{m+2}\frac{\partial x_2}{\partial x_{m+2}} + \cdots + A_{2m}\frac{\partial x_2}{\partial x_{2m}}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A_m &= A_{m+1}\frac{\partial x_m}{\partial x_{m+1}} + A_{m+2}\frac{\partial x_m}{\partial x_{m+2}} + \cdots + A_{2m}\frac{\partial x_m}{\partial x_{2m}}, \end{aligned}$$

ubi, posito $a_{k,k'} = \dfrac{\partial X_k}{\partial x_{k'}} - \dfrac{\partial X_{k'}}{\partial x_k}$ *ac designante R aggregatum, e* $1.3\ldots(2m-1)$ *terminis huiusmodi*

$$a_{1,2}\,a_{3,4}\ldots a_{2m-1,2m}$$

ratione supra descripta conflatum, fit

$$A_k = \frac{\partial R}{\partial a_{1,k}} X_1 + \frac{\partial R}{\partial a_{2,k}} X_2 + \cdots + \frac{\partial R}{\partial a_{2m,k}} X_{2m},$$

omisso termino in X_k ducto."

Huius memorabilis Propositionis si demonstrationem cupis ab aequationum differentialium vulgarium consideratione independentem, rem sic adornare licet.

Sit rursus

$$v_i = X_1 \frac{\partial x_1}{\partial x_i} + X_2 \frac{\partial x_2}{\partial x_i} + \cdots + X_m \frac{\partial x_m}{\partial x_i} + X_i,$$

ac designantibus

$$y, \; y_1, \; \ldots, \; y_{2m}$$

quantitates *indefinitas*, ponatur

$$U_k = X_k . y - a_{k,1} y_1 - a_{k,2} y_2 - \cdots - a_{k,2m} y_{2m},$$

$$Y_h = y_h - \frac{\partial x_h}{\partial x_{m+1}} y_{m+1} - \frac{\partial x_h}{\partial x_{m+2}} y_{m+2} - \cdots - \frac{\partial x_h}{\partial x_{2m}} y_{2m},$$

$$u_k = U_k + a_{k,1} Y_1 + a_{k,2} Y_2 + \cdots + a_{k,m} Y_m.$$

Eodem modo, atque (32) probavimus, demonstratur, quaecunque sint x_1, $x_2, \ldots, x_m$ reliquarum variabilium $x_{m+1}, x_{m+2}, \ldots, x_{2m}$ functiones, fieri

$$\frac{\partial x_1}{\partial x_i} u_1 + \frac{\partial x_2}{\partial x_i} u_2 + \cdots + \frac{\partial x_m}{\partial x_i} u_m + u_i = v_i y + \sum_{i'} \left\{ \left(\frac{\partial v_{i'}}{\partial x_i} \right) - \left(\frac{\partial v_i}{\partial x_{i'}} \right) \right\} y_{i'}.$$

Partes ad dextram signi aequalitatis evanescunt, ubi pro $x_1, x_2, \ldots, x_m$ sumuntur functiones satisfacientes m aequationibus $v_i = 0$, quae sunt ipsae functiones in theoremate tradito propositae, quas a se independentes esse subintelligo. Hinc si quantitatum u_k expressiones substituuntur atque statuitur

$$L_{i,h} = \frac{\partial x_1}{\partial x_i} a_{1,h} + \frac{\partial x_2}{\partial x_i} a_{2,h} + \cdots + \frac{\partial x_m}{\partial x_i} a_{m,h} + a_{i,h},$$

sequitur, per m aequationes $v_i = 0$ obtineri m sequentes:

$$(40) \quad \begin{cases} 0 = \dfrac{\partial x_1}{\partial x_i} U_1 + \dfrac{\partial x_2}{\partial x_i} U_2 + \cdots + \dfrac{\partial x_m}{\partial x_i} U_m + U_i \\ \qquad + L_{i,1} Y_1 + L_{i,2} Y_2 + \cdots + L_{i,m} Y_m. \end{cases}$$

Supponamus, quantitatum indefinitarum y, y_1, etc. functiones lineares $U_1, U_2, \ldots,$ U_{2m} a se independentes esse, sive quantitatem, supra per R designatam,

$$\Sigma a_{1,2} a_{3,4} \ldots a_{2m-1,2m}$$

neque per se neque substituendo functionum x_k valores evanescere. Quae secundum supra tradita est conditio, ut aequatio

$$X_1 dx_1 + X_2 dx_2 + \cdots + X_{2m} dx_{2m} = 0$$

non paucioribus quam m aequationibus integrari possit. Eo casu etiam m functiones ipsarum $Y_1, Y_2, \ldots, Y_m$ lineares, quas per H_i designabo,

$$L_{i,1} Y_1 + L_{i,2} Y_2 + \cdots + L_{i,m} Y_m = H_i$$

a se independentes erunt, sive non dabuntur factores ab ipsis y_k independentes λ_1, λ_2, etc., qui efficiant

$$\lambda_1 H_{m+1} + \lambda_2 H_{m+2} + \cdots + \lambda_m H_{2m} = 0.$$

Nam si eiusmodi dantur factores, secundum (40) aut $x_1, x_2, \ldots, x_i$ non a se independentes sunt aut datur aequatio inter functiones lineares $U_1, U_2, \ldots, U_{2m}$, quod utrumque contra suppositionem est. Functiones autem a se independentes $H_{m+1}, H_{m+2}, \ldots, H_{2m}$ omnes simul evanescere non possunt, nisi simul evanescunt omnes $Y_1, Y_2, \ldots, Y_m$. Iam igitur cum pro ipsarum y, y_1, etc. valoribus

$$y = R, \; y_1 = A_1, \; y_2 = A_2, \; \ldots, \; y_{2m} = A_{2m}$$

omnes simul evanescant $U_1, U_2, \ldots, U_{2m}$, siquidem quantitatum A_k, R valores sunt ipsi in Propositione tradita assignati, ideoque omnes secundum (40) evanescant H_i, pro valoribus illis omnes quoque $Y_1, Y_2, \ldots, Y_m$ evanescere debent, sive pro ipsius h valoribus $1, 2, \ldots, m$ fieri debet

$$0 = A_h - \frac{\partial x_h}{\partial x_{m+1}} A_{m+1} - \frac{\partial x_h}{\partial x_{m+2}} A_{m+2} - \cdots - \frac{\partial x_h}{\partial x_{2m}} A_{2m},$$

quae est Propositio demonstranda.

Propositionis antecedentis pro casu simplicissimo $m = 2$ hoc addam exemplum:

„Ubi semper ponitur $a_{\alpha,\beta} = \dfrac{\partial X_\alpha}{\partial x_\beta} - \dfrac{\partial X_\beta}{\partial x_\alpha}$, ex aequationibus

$$-X_3 = X_1 \frac{\partial x_1}{\partial x_3} + X_2 \frac{\partial x_2}{\partial x_3},$$

$$-X_4 = X_1 \frac{\partial x_1}{\partial x_4} + X_2 \frac{\partial x_2}{\partial x_4}$$

fluunt sequentes:

$$a_{3,4}X_2+a_{4,2}X_3+a_{2,3}X_4$$
$$=(a_{2,4}X_1+a_{4,1}X_2+a_{1,2}X_4)\frac{\partial x_1}{\partial x_3}+(a_{3,2}X_1+a_{1,3}X_2+a_{2,1}X_3)\frac{\partial x_1}{\partial x_4},$$
$$a_{4,3}X_1+a_{1,4}X_3+a_{3,1}X_4$$
$$=(a_{2,4}X_1+a_{4,1}X_2+a_{1,2}X_4)\frac{\partial x_2}{\partial x_3}+(a_{3,2}X_1+a_{1,3}X_2+a_{2,1}X_3)\frac{\partial x_2}{\partial x_4}.\text{“}$$

Si $p > 2m$ atque variabilium independentium $x_{m+1}, x_{m+2}, \dots, x_p$ functiones $x_1, x_2, \dots, x_m$ ita determinari possunt, ut $p-m$ aequationibus $v_i = 0$ satisfaciant, habentur *complura systemata* aequationum differentialium partialium, ad instar aequationum (38) formata. Videlicet e numero m aequationum

$$v_{p-m+1}=0,\quad v_{p-m+2}=0,\quad \dots,\quad v_p=0$$

per Propositionem antecedentem deducere licet alterum m aequationum differentialium partialium systema (38), eaque ratione aliud aliudque systema (38) obtinebitur, prout aliae $p-2m$ e $p-m$ variabilibus independentibus Constantium loco habentur.

Ponamus iam, esse $x_1, x_2, \dots, x_m$ variabilium $x_{m+1}, x_{m+2}, \dots, x_p$ functiones *involventes Constantem arbitrariam* α, sitque

$$(41)\quad w = X_1\frac{\partial x_1}{\partial \alpha}+X_2\frac{\partial x_2}{\partial \alpha}+\dots+X_m\frac{\partial x_m}{\partial \alpha},$$

porro

$$v_i = X_1\frac{\partial x_1}{\partial x_i}+X_2\frac{\partial x_2}{\partial x_i}+\dots+X_m\frac{\partial x_m}{\partial x_i}+X_i,$$
$$u_k = X_k dt-|a_{k,1}dx_1+a_{k,2}dx_2+\dots+a_{k,p}dx_p|$$
$$= X_k dt - dX_k+\frac{\partial X_1}{\partial x_k}dx_1+\frac{\partial X_2}{\partial x_k}dx_2+\dots+\frac{\partial X_p}{\partial x_k}dx_p$$
$$= X_k dt - dX_k+\sum_i\frac{\partial X_i}{\partial x_k}dx_i+\sum_{hi}\frac{\partial X_h}{\partial x_k}\cdot\frac{\partial x_h}{\partial x_i}dx_i.$$

Quae ubi substituuntur in formula

$$dw-\left\{\frac{\partial x_1}{\partial \alpha}dX_1+\frac{\partial x_2}{\partial \alpha}dX_2+\dots+\frac{\partial x_m}{\partial \alpha}dX_m\right\}$$
$$= X_1 d\frac{\partial x_1}{\partial \alpha}+X_2 d\frac{\partial x_2}{\partial \alpha}+\dots+X_m d\frac{\partial x_m}{\partial \alpha}$$
$$= \sum_{hi} X_h\frac{\partial^2 x_h}{\partial \alpha\,\partial x_i}dx_i,$$

obtinetur

$$(42)\quad \begin{cases} dw - w\,dt+\dfrac{\partial x_1}{\partial \alpha}u_1+\dfrac{\partial x_2}{\partial \alpha}u_2+\dots+\dfrac{\partial x_m}{\partial \alpha}u_m \\ \quad = \sum\limits_i\left\{\left(\dfrac{\partial X_i}{\partial \alpha}\right)+\sum\limits_h\left[\left(\dfrac{\partial X_h}{\partial \alpha}\right)\dfrac{\partial x_h}{\partial x_i}+X_h\dfrac{\partial^2 x_h}{\partial \alpha\,\partial x_i}\right]\right\}dx_i \\ \quad = \sum\limits_i\left(\dfrac{\partial v_i}{\partial \alpha}\right)dx_i, \end{cases}$$

siquidem uncis differentialia partialia includendo innuitur, ante differentiationes substitutos esse functionum $x_1, x_2, \dots, x_m$ valores. Si m aequationibus, quibus $x_1, x_2, \dots, x_m$ determinantur, integratur aequatio

$$0 = X_1 dx_1+X_2 dx_2+\dots+X_p dp,$$

locum habere debent $p-m$ aequationes $v_i = 0$, unde aequationis (42) dextra pars evanescit sive fit

$$(43)\quad dw - w\,dt+\frac{\partial x_1}{\partial \alpha}u_1+\frac{\partial x_2}{\partial \alpha}u_2+\dots+\frac{\partial x_m}{\partial \alpha}u_m = 0.$$

Si $p \geqq 2m$, vidimus supra, m aequationibus illis fieri, ut de m aequationibus differentialibus $u_h = 0$ fluant $p-m$ reliquae $u_i = 0$, ita ut m aequationes illae sint aequationes integrales systematis aequationum differentialium $u_k = 0$, quarum $p-2m$ e reliquis fluunt. Formula (43) docet, si insuper inter variabiles $t, x_{m+1}, x_{m+2}, \dots, x_p$ statuatur aequatio $w = \beta e^t$ sive

$$(44)\quad X_1\frac{\partial x_1}{\partial \alpha}+X_2\frac{\partial x_2}{\partial \alpha}+\dots+X_m\frac{\partial x_m}{\partial \alpha} = \beta e^t,$$

designante β Constantem arbitrariam, ipsas m aequationes differentiales $u_h = 0$ in earum $m-1$ redire, ideoque (44) esse novam eiusdem systematis $u_k = 0$ aequationem integralem. Si m aequationes, quibus aequatio

$$X_1 dx_1+X_2 dx_2+\dots+X_p dx_p = 0$$

integratur, plures involvunt Constantes arbitrarias, per (44) totidem obtinentur systematis $u_k = 0$ aequationes integrales, quas diversae ingrediuntur Constantes arbitrariae β, et e quarum binis per solam divisionem eliminatur t. Quae manent aequationes integrales, quaecunque $p-2m$ aequationes differentiales adiiciantur systemati $u_k = 0$, quippe quod tantum $2m$ aequationum differentialium vices gerit. Ubi Constantes arbitrariae sunt numero m, habetur problematis Pfaffiani solutio completa, simulque m aequationes (44) iunctae m aequationibus, quibus aequatio (20) integratur, suppeditant systematis aequationum differentialium (21) integrationem completam.

Si $p = 2m$, aequationes Constantem arbitrariam α involventes, quibus aequatio

$$X_1 dx_1 + X_2 dx_2 + \cdots + X_{2m} dx_{2m} = 0$$

integratur et quibus determinabantur functiones $x_1, x_2, \ldots, x_m$, sunt aequationes integrales systematis aequationum differentialium (2), sive resolutione earum provenientium (4):

$$dx_1 : dx_2 : \ldots : dx_{2m} = A_1 : A_2 : \ldots : A_{2m}.$$

Quarum Multiplicatorem, docent formulae (13) et (44), per illas m aequationes integrales induere valorem

$$M = \left\{ X_1 \frac{\partial x_1}{\partial \alpha} + X_2 \frac{\partial x_2}{\partial \alpha} + \cdots + X_m \frac{\partial x_m}{\partial \alpha} \right\}^{-m}.$$

Si $X_{2m} = -1$ atque omnes $X_1, X_2, \ldots, X_{2m-1}$ variabili x_{2m} vacant, vidimus supra Multiplicatorem Constanti aequari. Ac reapse eo casu evanescente dt, e (44) eruitur

$$X_1 \frac{\partial x_1}{\partial \alpha} + X_2 \frac{\partial x_2}{\partial \alpha} + \cdots + X_m \frac{\partial x_m}{\partial \alpha} = \beta,$$

quae ipsarum (4) aequatio integralis est. Quae pro $m = 2$ cum formula (19) convenit, quam supra alia via erui.

Methodum ad solvendum problema Pfaffianum ab ipso autore adhibitam, data occasione observo, per plures et altiores procedere integrationes quam methodus vera et genuina poscat. Quam novam methodum exemplo simplice explicabo. Ad aequationem differentialem

$$X_1 dx_1 + X_2 dx_2 + X_3 dx_3 + X_4 dx_4 = 0$$

per duas aequationes integrandam poscit Pfaffiana methodus integrationem completam systematis trium aequationum differentialium primi ordinis inter quatuor variabiles ac deinde unius aequationis differentialis primi ordinis inter duas variabiles. Illius igitur systematis Integrali uno invento, secundum illam methodum restat integratio completa duarum aequationum differentialium primi ordinis inter tres variabiles sive unius aequationis differentialis secundi ordinis inter duas variabiles ac deinde aequationis differentialis primi ordinis inter duas variabiles. At observo, si Integrali illo invento exprimatur x_4 per x_1, x_2, x_3, aequationem differentialem propositam abire in aliam linearem primi ordinis inter tres variabiles, conditioni integrabilitatis satisfacientem; cuius integrationem

vidimus absolvi posse per integrationes separatas duarum aequationum differentialium primi ordinis inter duas variabiles. Unde loco aequationis differentialis secundi ordinis tantum integrandae sunt duae aequationes differentiales separatae primi ordinis, quae est reductio maxime insignis; integrationi autem aequationis differentialis primi ordinis postremo praestandae omnino supersedetur. Tractatio huius rei gravissimae completa ac generalis alii Commentationi reservanda est.

§. 22.

Novum Principium generale Mechanicum, quod e Principio ultimi Multiplicatoris fluit.

Sint x_i, y_i, z_i Coordinatae orthogonales puncti massa m_i praediti; sint vires massam m_i secundum directiones Coordinatarum sollicitantes X_i, Y_i, Z_i. Ubi systema n punctorum materialium $m_1, m_2, \ldots, m_n$ prorsus liberum est, inter tempus t atque Coordinatas punctorum habentur $3n$ aequationes differentiales secundi ordinis

$$(1) \quad \begin{cases} \dfrac{d^2 x_i}{dt^2} = \dfrac{1}{m_i} X_i, \\ \dfrac{d^2 y_i}{dt^2} = \dfrac{1}{m_i} Y_i, \\ \dfrac{d^2 z_i}{dt^2} = \dfrac{1}{m_i} Z_i. \end{cases}$$

Vires X_i, Y_i, Z_i suppositione maxime generali erunt functiones $3n$ Coordinatarum x_i, y_i, z_i, temporis t atque differentialium primorum Coordinatarum

$$x_i' = \frac{dx_i}{dt}, \quad y_i' = \frac{dy_i}{dt}, \quad z_i' = \frac{dz_i}{dt},$$

quae sunt punctorum velocitates in Coordinatarum directiones proiectae. Secundum (5) §. 14 systematis aequationum differentialium dynamicarum (1) Multiplicator definitur formula

$$(2) \quad \frac{d \log M}{dt} + \Sigma \frac{1}{m_i} \left(\frac{\partial X_i}{\partial x_i'} + \frac{\partial Y_i}{\partial y_i'} + \frac{\partial Z_i}{\partial z_i'} \right) = 0,$$

indice i valente ad omnia puncta materialia systematis.

Quoties vires sollicitantes a solis massarum positionibus in spatio pendent sive praeterea etiam a tempore t, quantitates X_i, Y_i, Z_i ipsa x_i', y_i', z_i' omnino non involvunt, ideoque evanescente expressione

$$\Sigma \frac{1}{m_i}\left(\frac{\partial X_i}{\partial x_i'}+\frac{\partial Y_i}{\partial y_i'}+\frac{\partial Z_i}{\partial z_i'}\right),$$

statuere licet

$$M = 1.$$

Hinc secundum principium ultimi Multiplicatoris sequitur, si systema punctorum materialium liberum sit atque vires mobilia propellentes ab eorum velocitatibus non pendeant, ultimam integrationem, vel si vires etiam a tempore non explicite pendeant, *duas ultimas integrationes* revocari posse ad Quadraturas. Videlicet posteriore casu constat tempus t prorsus separari posse et post alias omnes integrationes transactas per Quadraturam inveniri.

Idem iam demonstrabo pro casu generali, quo systema n punctorum materialium non est liberum, sed certis obnoxium est conditionibus, quae exprimantur per aequationes inter Coordinatas x_i, y_i, z_i locum habentes

$$(3)\quad \Pi = 0,\quad \Pi_1 = 0,\quad \text{etc.}$$

Aequationes differentiales dynamicas pro motu sic impedito praecepit Ill. Lagrange haberi sequentes:

$$(4)\quad \begin{cases} \dfrac{d^2x_i}{dt^2} = \dfrac{1}{m_i}\left\{X_i+\lambda\dfrac{\partial \Pi}{\partial x_i}+\lambda_1\dfrac{\partial \Pi_1}{\partial x_i}+\text{ etc.}\right\}, \\ \dfrac{d^2y_i}{dt^2} = \dfrac{1}{m_i}\left\{Y_i+\lambda\dfrac{\partial \Pi}{\partial y_i}+\lambda_1\dfrac{\partial \Pi_1}{\partial y_i}+\text{ etc.}\right\}, \\ \dfrac{d^2z_i}{dt^2} = \dfrac{1}{m_i}\left\{Z_i+\lambda\dfrac{\partial \Pi}{\partial z_i}+\lambda_1\dfrac{\partial \Pi_1}{\partial z_i}+\text{ etc.}\right\}, \end{cases}$$

factoribus λ, λ_1, etc. determinatis per aequationes lineares, quae obtinentur substituendo aequationes differentiales (4) in aequationibus conditionalibus bis differentiatis

$$\frac{d^2\Pi}{dt^2}=0,\quad \frac{d^2\Pi_1}{dt^2}=0,\quad \text{etc.}$$

Ad eas aequationes lineares formandas pono

$$(5)\quad \begin{cases} U = \Sigma\left\{x_i'\dfrac{d\dfrac{\partial \Pi}{\partial x_i}}{dt}+y_i'\dfrac{d\dfrac{\partial \Pi}{\partial y_i}}{dt}+z_i'\dfrac{d\dfrac{\partial \Pi}{\partial z_i}}{dt}\right\}, \\ U_1 = \Sigma\left\{x_i'\dfrac{d\dfrac{\partial \Pi_1}{\partial x_i}}{dt}+y_i'\dfrac{d\dfrac{\partial \Pi_1}{\partial y_i}}{dt}+z_i'\dfrac{d\dfrac{\partial \Pi_1}{\partial z_i}}{dt}\right\}, \\ \qquad\text{etc.}\qquad\text{etc.,} \end{cases}$$

fit

$$0=\frac{d^2\Pi}{dt^2}=\Sigma\left\{\frac{\partial \Pi}{\partial x_i}\cdot\frac{d^2x_i}{dt^2}+\frac{\partial \Pi}{\partial y_i}\cdot\frac{d^2y_i}{dt^2}+\frac{\partial \Pi}{\partial z_i}\cdot\frac{d^2z_i}{dt^2}\right\}+U,$$

$$0=\frac{d^2\Pi_1}{dt^2}=\Sigma\left\{\frac{\partial \Pi_1}{\partial x_i}\cdot\frac{d^2x_i}{dt^2}+\frac{\partial \Pi_1}{\partial y_i}\cdot\frac{d^2y_i}{dt^2}+\frac{\partial \Pi_1}{\partial z_i}\cdot\frac{d^2z_i}{dt^2}\right\}+U_1,$$

$$\text{etc.}\qquad\qquad\text{etc.}$$

Ubi in his aequationibus substituuntur formulae (4) atque ponitur

$$(6)\quad \begin{cases} V = U+\Sigma\dfrac{1}{m_i}\left\{\dfrac{\partial \Pi}{\partial x_i}X_i+\dfrac{\partial \Pi}{\partial y_i}Y_i+\dfrac{\partial \Pi}{\partial z_i}Z_i\right\}, \\ V_1 = U_1+\Sigma\dfrac{1}{m_i}\left\{\dfrac{\partial \Pi_1}{\partial x_i}X_i+\dfrac{\partial \Pi_1}{\partial y_i}Y_i+\dfrac{\partial \Pi_1}{\partial z_i}Z_i\right\}, \\ \qquad\text{etc.}\qquad\qquad\text{etc.,} \end{cases}$$

porro

$$(7)\quad (\alpha,\beta)=(\beta,\alpha)=\Sigma\frac{1}{m_i}\left\{\frac{\partial \Pi_\alpha}{\partial x_i}\cdot\frac{\partial \Pi_\beta}{\partial x_i}+\frac{\partial \Pi_\alpha}{\partial y_i}\cdot\frac{\partial \Pi_\beta}{\partial y_i}+\frac{\partial \Pi_\alpha}{\partial z_i}\cdot\frac{\partial \Pi_\beta}{\partial z_i}\right\},$$

aequationes, quibus λ, λ_1, etc. determinantur, evadunt sequentes:

$$(8)\quad \begin{cases} 0 = V+(0,0)\lambda+(0,1)\lambda_1+\text{ etc.}, \\ 0 = V_1+(1,0)\lambda+(1,1)\lambda_1+\text{ etc.}, \\ \qquad\text{etc.}\qquad\text{etc.} \end{cases}$$

His de factorum λ, λ_1, etc. valoribus praemissis, aequationum Lagrangianarum (4) investigabo Multiplicatorem.

Ac primum observo, secundum ea, quae de viribus sollicitantibus statuta sunt, in dextris partibus aequationum (4) solos factores λ, λ_1, etc. implicare differentialia prima x_i', y_i', z_i'. Unde e (5) §. 14 Multiplicator M definietur formula

$$-\frac{d\log M}{dt}=\Sigma\frac{1}{m_i}\left\{\frac{\partial \Pi}{\partial x_i}\cdot\frac{\partial \lambda}{\partial x_i'}+\frac{\partial \Pi}{\partial y_i}\cdot\frac{\partial \lambda}{\partial y_i'}+\frac{\partial \Pi}{\partial z_i}\cdot\frac{\partial \lambda}{\partial z_i'}\right\}$$
$$+\Sigma\frac{1}{m_i}\left\{\frac{\partial \Pi_1}{\partial x_i}\cdot\frac{\partial \lambda_1}{\partial x_i'}+\frac{\partial \Pi_1}{\partial y_i}\cdot\frac{\partial \lambda_1}{\partial y_i'}+\frac{\partial \Pi_1}{\partial z_i}\cdot\frac{\partial \lambda_1}{\partial z_i'}\right\}$$
$$+\qquad\text{etc.}\qquad\qquad\text{etc.,}$$

quam, posito

$$(9)\quad \Lambda_{\alpha,\beta}=\Sigma\frac{1}{m_i}\left\{\frac{\partial \Pi_\alpha}{\partial x_i}\cdot\frac{\partial \lambda_\beta}{\partial x_i'}+\frac{\partial \Pi_\alpha}{\partial y_i}\cdot\frac{\partial \lambda_\beta}{\partial y_i'}+\frac{\partial \Pi_\alpha}{\partial z_i}\cdot\frac{\partial \lambda_\beta}{\partial z_i'}\right\},$$

sic exhibere licet

$$(10)\quad d\log M = -\{\Lambda_{0,0}+\Lambda_{1,1}+\text{etc.}\}\,dt.$$

Ad quantitates $\Lambda_{0,0}$, $\Lambda_{1,1}$, etc. determinandas, aequationes (8)

$$0 = V_\beta+(\beta,0)\lambda+(\beta,1)\lambda_1+\text{ etc.},$$

quarum Coëfficientes $(\beta, 0)$, $(\beta, 1)$, etc. solarum x_i, y_i, z_i functiones sunt, secundum omnes quantitates x_i', y_i', z_i' differentientur, aequationesque differentiationibus provenientes respective per quantitates

$$\frac{1}{m_i}\cdot\frac{\partial \Pi_\alpha}{\partial x_i},\quad \frac{1}{m_i}\cdot\frac{\partial \Pi_\alpha}{\partial y_i},\quad \frac{1}{m_i}\cdot\frac{\partial \Pi_\alpha}{\partial z_i}$$

multiplicatae consummentur: prodit

$$(11)\quad 0 = u_{\alpha,\beta}+(\beta,0)\Lambda_{\alpha,0}+(\beta,1)\Lambda_{\alpha,1}+\text{etc.},$$

siquidem statuitur

$$u_{\alpha,\beta} = \Sigma\frac{1}{m_i}\left\{\frac{\partial \Pi_\alpha}{\partial x_i}\cdot\frac{\partial V_\beta}{\partial x_i'}+\frac{\partial \Pi_\alpha}{\partial y_i}\cdot\frac{\partial V_\beta}{\partial y_i'}+\frac{\partial \Pi_\alpha}{\partial z_i}\cdot\frac{\partial V_\beta}{\partial z_i'}\right\}.$$

Cum secundum (6) habeatur

$$\frac{\partial V_\beta}{\partial x_i'} = \frac{\partial U_\beta}{\partial x_i'},\quad \frac{\partial V_\beta}{\partial y_i'} = \frac{\partial U_\beta}{\partial y_i'},\quad \frac{\partial V_\beta}{\partial z_i'} = \frac{\partial U_\beta}{\partial z_i'},$$

quantitates $u_{\alpha,\beta}$ sic repraesentare licet:

$$u_{\alpha,\beta} = \Sigma\frac{1}{m_i}\left\{\frac{\partial \Pi_\alpha}{\partial x_i}\cdot\frac{\partial U_\beta}{\partial x_i'}+\frac{\partial \Pi_\alpha}{\partial y_i}\cdot\frac{\partial U_\beta}{\partial y_i'}+\frac{\partial \Pi_\alpha}{\partial z_i}\cdot\frac{\partial U_\beta}{\partial z_i'}\right\}.$$

At e (5) obtinetur, evolutione differentialium $d\frac{\partial \Pi_\beta}{\partial x_i}$ etc. facta,

$$(12)\quad \begin{cases} \dfrac{\partial U_\beta}{\partial x_i'} = 2\,\dfrac{d\frac{\partial \Pi_\beta}{\partial x_i}}{dt}, \\[2ex] \dfrac{\partial U_\beta}{\partial y_i'} = 2\,\dfrac{d\frac{\partial \Pi_\beta}{\partial y_i}}{dt}, \\[2ex] \dfrac{\partial U_\beta}{\partial z_i'} = 2\,\dfrac{d\frac{\partial \Pi_\beta}{\partial z_i}}{dt}, \end{cases}$$

quibus valoribus substitutis fit

$$(13)\quad u_{\alpha,\beta} = 2\Sigma\frac{1}{m_i}\left\{\frac{\partial \Pi_\alpha}{\partial x_i}\cdot\frac{d\frac{\partial \Pi_\beta}{\partial x_i}}{dt}+\frac{\partial \Pi_\alpha}{\partial y_i}\cdot\frac{d\frac{\partial \Pi_\beta}{\partial y_i}}{dt}+\frac{\partial \Pi_\alpha}{\partial z_i}\cdot\frac{d\frac{\partial \Pi_\beta}{\partial z_i}}{dt}\right\}.$$

Cuius aequationis beneficio obtinentur quantitatum (α, β) per formulam (7) definitarum differentialia

$$(14)\quad \frac{d(\alpha,\beta)}{dt} = \frac{d(\beta,\alpha)}{dt} = \tfrac{1}{2}\{u_{\alpha,\beta}+u_{\beta,\alpha}\}.$$

In aequatione (11) indici β valores 0, 1, 2, etc. tribuendo obtinentur aequationes lineares, quibus quantitas $\Lambda_{\alpha,\alpha}$ determinatur. At quantitatum omnium sic inventarum $\Lambda_{\alpha,\alpha}$ aggregatum docui per formulam symbolicam concinnam exhiberi posse, quaecunque sint quantitates $u_{\alpha,\beta}$. Vocetur enim R earum aequationum linearium Determinans sive sit

$$\Sigma\pm(00)(11)(22)\ldots = R,$$

atque statuatur

$$\tfrac{1}{2}\{u_{\alpha,\beta}+u_{\beta,\alpha}\}dt = \delta(\alpha,\beta) = \delta(\beta,\alpha):$$

sequitur per ratiocinia similia atque §. 16 adhibui:

$$-\{\Lambda_{0,0}+\Lambda_{1,1}+\text{etc.}\}dt = \delta\log R.$$

Unde cum secundum (14) sit

$$\delta(\alpha,\beta) = d(\alpha,\beta)\quad\text{ideoque}\quad \delta\log R = d\log R,$$

eruitur e (10)

$$-\{\Lambda_{0,0}+\Lambda_{1,1}+\text{etc.}\}dt = d\log M = d\log R,$$

id quod suppeditat

$$(15)\quad M = R = \Sigma\pm(00)(11)(22)\ldots$$

qui est Multiplicatoris quaesiti valor.

Operae pretium est adnotare, aequationem inventam $M = R$ non tantum ad casum valere, quo functiones X_i, Y_i, Z_i, viribus sollicitantibus aequales, tempus t explicite continent, sed ad hunc quoque casum, *quo tempus t ipsas explicite afficit aequationes conditionales* $\Pi = 0$, $\Pi_1 = 0$, etc. Eo casu aequationes dynamicae Lagrangianae (4) eandem servant formam, sed factoribus λ, λ_1, etc. alii competunt valores; quippe quantitatibus U, U_1, etc. ideoque etiam quantitatibus V, V_1, etc., quae aequationum linearium (8), quibus factores λ, λ_1, etc. determinantur, terminos constantes constituunt, respective addendi sunt termini

$$2\frac{d\frac{\partial \Pi}{\partial t}}{dt},\quad 2\frac{d\frac{\partial \Pi_1}{\partial t}}{dt},\quad \text{etc.}$$

At patet, inde non mutari aequationes (12); unde aequationes quoque (13) et (14) immutatae manebunt ideoque formula pro aggregato $\Delta_{0,0}+\Delta_{1,1}+$ etc. inventa ideoque etiam ipsius Multiplicatoris valor R.

Si vires sollicitantes X_i, Y_i, Z_i solarum functiones sunt Coordinatarum x_i, y_i, z_i, atque inter has solas dantur aequationes conditionales $\Pi = 0$, $\Pi_1 = 0$, etc., valor $M = R$ inventus secundum principium ultimi Multiplicatoris hoc suppeditat theorema:

Novum Principium Generale Mechanicum.

„Proponatur motus systematis n punctorum materialium, quae in datis superficiebus vel curvis aut dato quocunque modo inter se connexa manere debent, ita ut inter Coordinatas eorum locum habeant k aequationes conditionales; porro vires sollicitantes et magnitudine et directione solis punctorum positionibus datae sint: semper duas ultimas integrationes absolvere licet Quadraturis. Sint enim

punctorum massae $m_1, m_2, \dots, m_n$;

massae m_i *Coordinatae orthogonales* x_i, y_i, z_i, *earumque differentialia prima*

$$x_i' = \frac{dx_i}{dt},\quad y_i' = \frac{dy_i}{dt},\quad z_i' = \frac{dz_i}{dt};$$

sint aequationes conditionales $\Pi = 0$, $\Pi_1 = 0$, $\dots$, $\Pi_{k-1} = 0$ *et differentiatione prima ex iis provenientes* $\Pi' = 0$, $\Pi_1' = 0$, $\dots$, $\Pi_{k-1}' = 0$, *ubi*

$$\Pi_\alpha' = \Sigma\left\{\frac{\partial \Pi_\alpha}{\partial x_i}x_i' + \frac{\partial \Pi_\alpha}{\partial y_i}y_i' + \frac{\partial \Pi_\alpha}{\partial z_i}z_i'\right\};$$

inter $6n$ *quantitates* $x_i, y_i, z_i, x_i', y_i', z_i'$ *praeter* $2k$ *aequationes* $\Pi_\alpha = 0$, $\Pi_\alpha' = 0$, *inventa sint* $6n-2k-2 = \mu$ *Integralia* $F_1 = \alpha_1$, $F_2 = \alpha_2$, $\dots$, $F_\mu = \alpha_\mu$, *designantibus* $\alpha_1, \alpha_2, \dots, \alpha_\mu$ *Constantes arbitrarias; restabit integratio unius aequationis differentialis primi ordinis inter duas quantitates u et v*

$$v'du - u'dv = 0,$$

ubi u et v esse possunt ipsarum $x_i, y_i, z_i, x_i', y_i', z_i'$ *functiones quaecunque atque* u' *et* v' *designant valores differentialium* $\frac{du}{dt}$ *et* $\frac{dv}{dt}$, *adiumento aequationum datarum et integratione inventarum nec non ipsarum aequationum differentialium dynamicarum per ipsas u et v expressos. His praemissis, ponatur*

$$(\alpha,\beta) = \Sigma\frac{1}{m_i}\left\{\frac{\partial \Pi_\alpha}{\partial x_i}\cdot\frac{\partial \Pi_\beta}{\partial x_i} + \frac{\partial \Pi_\alpha}{\partial y_i}\cdot\frac{\partial \Pi_\beta}{\partial y_i} + \frac{\partial \Pi_\alpha}{\partial z_i}\cdot\frac{\partial \Pi_\beta}{\partial z_i}\right\},$$

atque kk quantitatum (α, β) *formetur Determinans R; porro si vocatur* Δ *Determinans functionale* $6n$ *functionum*

$$\Pi,\ \Pi_1,\ \dots,\ \Pi_{k-1},\ \Pi',\ \Pi_1',\ \dots,\ \Pi_{k-1}',$$
$$F_1,\ F_2,\ \dots,\ F_{6n-2k-2},\ u,\ v,$$

$6n$ *quantitatum* $x_i, y_i, z_i, x_i', y_i', z_i'$ *respectu formatum, exprimantur R et* Δ *et ipsa per solas u et v; erit aequationis* $v'du - u'dv = 0$ *Multiplicator* $\frac{R}{\Delta}$, *unde nova habetur aequatio integralis*

$$\int\frac{R}{\Delta}(v'du - u'dv) = \text{Const.},$$

ubi expressio sub integrationis signo est differentiale completum; denique si nova illa aequatione integrali exprimitur v per u, unde evadit etiam u' *solius u functio, invenitur simplice Quadratura*

$$t + \text{Const.} = \int\frac{du}{u'}.\text{“}$$

Sub forma antecedente principium novum mechanicum ante hos tres annos cum illustri Academia *Petropolitana* communicavi. Alias eiusdem formas infra tradam. Ultimam integrationem, qua t per Coordinatas exprimatur, Quadraturis absolvi, res erat nota et sponte patens. At inventum novum, penultimam quoque integrationem Quadraturis perfici posse, constituere mihi videbatur principium mechanicum.

Si tempus t vires sollicitantes sive etiam aequationes conditionales afficit, non amplius ipsum t a reliquis variabilibus separare licet, unde eo casu principium nostrum tantum omnium ultimam integrationem per Quadraturas absolvere docet. Supponendo, inventa esse $6n-2k-1$ Integralia

$$F_1 = \alpha_1,\quad F_2 = \alpha_2,\quad \dots,\quad F_{6n-2k-1} = \alpha_{6n-2k-1},$$

atque u et v esse ipsius t et $6n$ quantitatum $x_i, y_i, z_i, x_i', y_i', z_i'$ functiones, Determinans Δ formandum est $6n+1$ functionum

$$F_1,\ F_2,\ \dots,\ F_{6n-2k-1},\ \Pi,\ \Pi_1,\ \dots,\ \Pi_{k-1},\ \Pi',\ \Pi_1',\ \dots,\ \Pi_{k-1}',\ u,\ v,$$

$6n+1$ quantitatum $t, x_i, y_i, z_i, x_i', y_i', z_i'$ respectu; eadem manente ipsius R significatione, rursus exprimenda erunt R, Δ, $u' = \frac{du}{dt}$, $v' = \frac{dv}{dt}$ per u et v, eritque aequatio integralis ultima

$$\int \frac{R}{\Delta}(v'du - u'dv) = \text{Const.},$$

ubi expressio sub integrationis signo est differentiale completum.

Habemus hic exemplum, quo ad reductionem aequationum differentialium propositarum adhibentur Integralia *particularia;* nam ex aequationibus differentialibus (4) sequuntur Integralia completa $\Pi'_\alpha = C_\alpha$, $\Pi_\alpha = C_\alpha t + C'_\alpha$, designantibus C_α, C'_α Constantes arbitrarias. Neque tamen sunt $\Pi'_\alpha = 0$, $\Pi_\alpha = 0$ aequationes integrales particulares *quaecunque*, sed tales, pro quibus secundum §. 12 fit, ut Multiplicator, quo aequationes differentiales earum beneficio reductae gaudent, e Multiplicatore propositarum (4) deduci possit. Scilicet aequatio quidem integralis particularis est $\Pi'_\alpha = 0$, at functio Π'_α ita comparata est, ut Constanti arbitrariae aequiparata suppeditet Integrale completum; porro si reductioni adhibetur aequatio integralis particularis $\Pi'_\alpha = 0$ ex eaque nova deducitur aequatio integralis $\Pi_\alpha = 0$, rursus innotescit functio Π_α, quae Constanti arbitrariae aequiparata non quidem aequationum differentialium propositarum (4), sed reductarum tamen Integrale completum suppeditat. Quod secundum §. 12 poscitur et sufficit.

Designentur $3n$ quantitates $x_i\sqrt{m_i}$, $y_i\sqrt{m_i}$, $z_i\sqrt{m_i}$ per

$$\xi_1,\ \xi_2,\ \ldots,\ \xi_{3n},$$

fit e (7)

$$(\alpha,\beta) = \frac{\partial \Pi_\alpha}{\partial \xi_1}\cdot\frac{\partial \Pi_\beta}{\partial \xi_1} + \frac{\partial \Pi_\alpha}{\partial \xi_2}\cdot\frac{\partial \Pi_\beta}{\partial \xi_2} + \cdots + \frac{\partial \Pi_\alpha}{\partial \xi_{3n}}\cdot\frac{\partial \Pi_\beta}{\partial \xi_{3n}}.$$

Unde secundum Propositionem notam, in Commentatione *de formatione atque proprietatibus Determinantium* §. 14 (cf. h. edit. Vol. III p. 385) probatam, quantitatum (α,β) Determinans exhibere licet ut aggregatum quadratorum Determinantium functionum Π, Π_1, ..., Π_{k-1}, formatorum respectu quarumque k e numero quantitatum ξ_1, ξ_2, ..., ξ_{3n} sumtarum, sive ponere licet

$$(16)\quad R = M = S\left\{\Sigma \pm \frac{\partial \Pi}{\partial \xi_{m'}}\cdot\frac{\partial \Pi_1}{\partial \xi_{m''}}\cdots\frac{\partial \Pi_{k-1}}{\partial \xi_{m^{(k)}}}\right\}^2,$$

siquidem m', m'', ..., $m^{(k)}$ designant quoscunque k diversos ex indicibus 1, 2, ..., $3n$. Ex. gr. pro uno puncto, massa $= 1$ praedito, cuius Coordinatae orthogonales sunt x, y, z, et quod moveri debet in superficie, cuius aequatio $\Pi = 0$, fit

$$M = R = \left(\frac{\partial \Pi}{\partial x}\right)^2 + \left(\frac{\partial \Pi}{\partial y}\right)^2 + \left(\frac{\partial \Pi}{\partial z}\right)^2;$$

si punctum moveri debet in curva, cuius aequationes sunt $\Pi = 0$, $\Pi_1 = 0$, fit

$$\begin{aligned} M = R = &\left\{\frac{\partial \Pi}{\partial y}\cdot\frac{\partial \Pi_1}{\partial z} - \frac{\partial \Pi}{\partial z}\cdot\frac{\partial \Pi_1}{\partial y}\right\}^2 \\ &+\left\{\frac{\partial \Pi}{\partial z}\cdot\frac{\partial \Pi_1}{\partial x} - \frac{\partial \Pi}{\partial x}\cdot\frac{\partial \Pi_1}{\partial z}\right\}^2 \\ &+\left\{\frac{\partial \Pi}{\partial x}\cdot\frac{\partial \Pi_1}{\partial y} - \frac{\partial \Pi}{\partial y}\cdot\frac{\partial \Pi_1}{\partial x}\right\}^2. \end{aligned}$$

Erat R Determinans aequationum linearium, quibus factores Lagrangiani λ, λ_1, etc. determinantur, qui igitur factores indeterminati aut infiniti evadere nequeunt, nisi evanescat R. At docet formula (16), non evanescere posse R, nisi singula evanescant Determinantia functionalia

$$\Sigma \pm \frac{\partial \Pi}{\partial \xi_{m'}}\cdot\frac{\partial \Pi_1}{\partial \xi_{m''}}\cdots\frac{\partial \Pi_{k-1}}{\partial \xi_{m^{(k)}}}.$$

Id quod ubi *identice* fit, ipsarum Π, Π_1, ..., Π_{k-1} una reliquarum functio est, quo casu aequationes conditionales aut sibi contradicunt aut una, quae e reliquis sequitur, est superflua. Singula Determinantia illa si non quidem identice evanescunt sed ipsarum aequationum $\Pi = 0$, $\Pi_1 = 0$, ..., $\Pi_{k-1} = 0$ adiumento, id indicio est, earum aequationum unam reliquarum ope formam *Quadrati* induere. Eo casu per certas eliminationes et radicis extractionem transformari debent aequationes $\Pi = 0$ etc.; quam praeparationem semper factam esse supponi debet, ut aequationum dynamicarum Lagrangianarum usus esse possit.

Si ex antecedentibus semper supponere licet, Determinans R non indefinite evanescere, fieri tamen potest, ut R evanescat pro punctorum materialium positionibus particularibus determinatis. Quemadmodum si inter tres puncti Coordinatas una vel duae habentur aequationes conditionales repraesentantes superficiem aut curvam apice praeditam, evanescit R, si punctum in eo apice collocatur. Ubi agitur de aequilibrio systematis punctorum materialium in eiusmodi positionibus particularibus collocatorum, pro quibus Determinans R evanescit, praecepta statica generalia aut deficiunt aut accuratioribus explicationibus indigent. Nec non si in certo temporis momento systema in motu suo ad tales positiones particulares pervenit, velocitatum intensitates et directiones mutationem finitam in temporis intervallo infinite parvo subeunt. Si, ut in rerum natura fieri solet, conditiones, quibus systema subiicitur, non exprimuntur per aequationes, sed per inaequalitates $\Pi > 0$, $\Pi_1 > 0$, etc., inde ab eo temporis momento ipsae plerumque aequationes differentiales (4) cum aliis commutari debent.

§. 23.

De Multiplicatore aequationum differentialium dynamicarum forma Lagrangiana secunda exhibitarum.

III. Lagrange aequationes differentiales dynamicas generales alia quoque forma memorabili exhibuit, Coordinatarum $3n$ loco, k aequationibus conditionalibus satisfacientium, introducendo $3n-k$ quantitates a se independentes

$$q_1, \quad q_2, \quad \ldots, \quad q_{3n-k}.$$

Quarum ipsae Coordinatae x_i, y_i, z_i tales esse debent functiones, quae substitutae in aequationibus conditionalibus $\Pi = 0$, $\Pi_1 = 0$, etc. sponte iis satisfaciant. Unde etiam aequationem $\Pi_a = 0$ cuiuslibet variabilis q_m respectu differentiando habetur

$$(1) \quad \sum_i \left\{ \frac{\partial \Pi_a}{\partial x_i} \cdot \frac{\partial x_i}{\partial q_m} + \frac{\partial \Pi_a}{\partial y_i} \cdot \frac{\partial y_i}{\partial q_m} + \frac{\partial \Pi_a}{\partial z_i} \cdot \frac{\partial z_i}{\partial q_m} \right\} = 0.$$

Statuatur

$$(2) \quad \sum_i \left\{ X_i \frac{\partial x_i}{\partial q_m} + Y_i \frac{\partial y_i}{\partial q_m} + Z_i \frac{\partial z_i}{\partial q_m} \right\} = Q_m;$$

consummando $3n$ aequationes (4) §. pr. respective per $m_i \frac{\partial x_i}{\partial q_m}$, $m_i \frac{\partial y_i}{\partial q_m}$, $m_i \frac{\partial z_i}{\partial q_m}$ multiplicatas, evanescunt secundum (1) aggregata in factores λ, λ_1, etc. ducta, unde prodit

$$(3) \quad \sum_i m_i \left\{ \frac{d^2 x_i}{dt^2} \cdot \frac{\partial x_i}{\partial q_m} + \frac{d^2 y_i}{dt^2} \cdot \frac{\partial y_i}{\partial q_m} \cdot \frac{d^2 z_i}{dt^2} \cdot \frac{\partial z_i}{\partial q_m} \right\} = Q_m.$$

Ponendo $q'_m = \frac{dq_m}{dt}$ et considerando quantitates x'_i ut quantitatum q_m, q'_m functiones, quae dantur formula

$$x'_i = \frac{\partial x_i}{\partial q_1} q'_1 + \frac{\partial x_i}{\partial q_2} q'_2 + \cdots + \frac{\partial x_i}{\partial q_{3n-k}} q'_{3n-k},$$

sequitur

$$\frac{\partial x'_i}{\partial q'_m} = \frac{\partial x_i}{\partial q_m}.$$

Porro

$$\frac{\partial x'_i}{\partial q_m} = \frac{\partial^2 x_i}{\partial q_m \partial q_1} q'_1 + \frac{\partial^2 x_i}{\partial q_m \partial q_2} q'_2 + \cdots + \frac{\partial^2 x_i}{\partial q_m \partial q_{3n-k}} q'_{3n-k} = \frac{d \frac{\partial x_i}{\partial q_m}}{dt}.$$

Eodem modo pro omnibus tribus Coordinatis fit

$$(4) \quad \begin{cases} \dfrac{\partial x'_i}{\partial q'_m} = \dfrac{\partial x_i}{\partial q_m}, & \dfrac{\partial y'_i}{\partial q'_m} = \dfrac{\partial y_i}{\partial q_m}, & \dfrac{\partial z'_i}{\partial q'_m} = \dfrac{\partial z_i}{\partial q_m}, \\ \dfrac{\partial x'_i}{\partial q_m} = \dfrac{d \frac{\partial x_i}{\partial q_m}}{dt}, & \dfrac{\partial y'_i}{\partial q_m} = \dfrac{d \frac{\partial y_i}{\partial q_m}}{dt}, & \dfrac{\partial z'_i}{\partial q_m} = \dfrac{d \frac{\partial z_i}{\partial q_m}}{dt}. \end{cases}$$

Unde aequatio (3) sic exhiberi potest:

$$Q_m = \sum_i m_i \left\{ \frac{dx'_i}{dt} \cdot \frac{\partial x'_i}{\partial q'_m} + \frac{dy'_i}{dt} \cdot \frac{\partial y'_i}{\partial q'_m} + \frac{dz'_i}{dt} \cdot \frac{\partial z'_i}{\partial q'_m} \right\}$$

$$= \frac{d \sum_i m_i \left\{ x'_i \frac{\partial x'_i}{\partial q'_m} + y'_i \frac{\partial y'_i}{\partial q'_m} + z'_i \frac{\partial z'_i}{\partial q'_m} \right\}}{dt} - \sum_i m_i \left\{ x'_i \frac{\partial x'_i}{\partial q_m} + y'_i \frac{\partial y'_i}{\partial q_m} + z'_i \frac{\partial z'_i}{\partial q_m} \right\},$$

sive ponendo

$$T = \tfrac{1}{2} \sum_i m_i \{ x'_i x'_i + y'_i y'_i + z'_i z'_i \},$$

fit

$$Q_m = \frac{d \frac{\partial T}{\partial q'_m}}{dt} - \frac{\partial T}{\partial q_m}.$$

Qua in formula ubi T et quantitates Q_m per $6n-2k$ quantitates $q_1, q_2, \ldots, q_{3n-k}, q'_1, q'_2, \ldots, q'_{3n-k}$ exprimuntur atque indici m tribuuntur valores $1, 2, \ldots, 3n-k$, obtinentur $3n-k$ aequationes differentiales secundi ordinis inter tempus t atque $3n-k$ variabiles a se independentes q_m:

$$(5) \quad \begin{cases} \dfrac{d \frac{\partial T}{\partial q'_1}}{dt} - \dfrac{\partial T}{\partial q_1} - Q_1 = 0, \\ \dfrac{d \frac{\partial T}{\partial q'_2}}{dt} - \dfrac{\partial T}{\partial q_2} - Q_2 = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \dfrac{d \frac{\partial T}{\partial q'_{3n-k}}}{dt} - \dfrac{\partial T}{\partial q_{3n-k}} - Q_{3n-k} = 0, \end{cases}$$

quae altera est forma Lagrangiana aequationum differentialium dynamicarum. Aequationum (5) iam investigabo Multiplicatorem.

Sint aequationes dynamicae

$$\varphi_1 = 0, \quad \varphi_2 = 0, \quad \ldots, \quad \varphi_{3n-k} = 0,$$

ubi φ_1, φ_2, etc. designent laevas partes aequationum (5). Statuamus

$$(6)\quad T = \tfrac{1}{2}\sum_{i,i'} a_{i,i'} q'_i q'_{i'},$$

utroque i et i' ad omnes indices 1, 2, ..., $3n-k$ valente et designantibus quantitatibus $a_{i,i'} = a_{i',i}$ solarum $q_1, q_2, \ldots, q_{3n-k}$ functiones. Hinc fit e (5)

$$\varphi_m = \frac{d\sum_i a_{i,m} q'_i}{dt} - \tfrac{1}{2}\sum_{i,i'} \frac{\partial a_{i,i'}}{\partial q_m} q'_i q'_{i'} - Q_m,$$

unde, ponendo $q''_i = \frac{d^2 q_i}{dt^2}$, eruitur

$$(7)\quad \frac{\partial \varphi_m}{\partial q''_h} = a_{h,m} \quad \text{ideoque} \quad \frac{\partial \varphi_m}{\partial q''_h} = \frac{\partial \varphi_h}{\partial q''_m}.$$

Porro si vires sollicitantes X_i, Y_i, Z_i a quantitatibus x'_i, y'_i, z'_i non pendent ideoque etiam quantitates Q_m ipsa q'_1, q'_2, etc. non implicant, fit

$$\frac{\partial \varphi_m}{\partial q'_h} = \frac{da_{h,m}}{dt} + \sum_i \frac{\partial a_{i,m}}{\partial q_h} q'_i - \sum_i \frac{\partial a_{i,h}}{\partial q_m} q'_i,$$

unde, reiectis terminis se mutuo destruentibus, fit

$$\tfrac{1}{2}\left\{\frac{\partial \varphi_m}{\partial q'_h} + \frac{\partial \varphi_h}{\partial q'_m}\right\} = \frac{da_{h,m}}{dt},$$

sive

$$(8)\quad \tfrac{1}{2}\left\{\frac{\partial \varphi_m}{\partial q'_h} + \frac{\partial \varphi_h}{\partial q'_m}\right\} = \frac{d\frac{\partial \varphi_m}{\partial q''_h}}{dt} = \frac{d\frac{\partial \varphi_h}{\partial q''_m}}{dt}.$$

At e Propositione generali, quam sub finem §. 16 tradidi, ponendo $\lambda = 1$ sequitur, ubi formulae (8) locum habeant, aequationum differentialium (5) fieri Multiplicatorem

$$(9)\quad M_1 = \Sigma \pm \frac{\partial \varphi_1}{\partial q''_1} \cdot \frac{\partial \varphi_2}{\partial q''_2} \cdots \frac{\partial \varphi_{3n-k}}{\partial q''_{3n-k}} = \Sigma \pm a_{1,1} a_{2,2} \cdots a_{3n-k,3n-k}.$$

Si rursus $3n$ quantitatum $x_i\sqrt{m_i}$, $y_i\sqrt{m_i}$, $z_i\sqrt{m_i}$ loco ponimus $\xi_1, \xi_2, \ldots, \xi_{3n}$, fit

$$(10)\quad T = \tfrac{1}{2}\{\xi'_1\xi'_1 + \xi'_2\xi'_2 + \cdots + \xi'_{3n}\xi'_{3n}\},$$

qua expressione in formula (6) substituta, obtinetur

$$(11)\quad a_{i,i'} = \frac{\partial \xi_1}{\partial q_i} \cdot \frac{\partial \xi_1}{\partial q_{i'}} + \frac{\partial \xi_2}{\partial q_i} \cdot \frac{\partial \xi_2}{\partial q_{i'}} + \cdots + \frac{\partial \xi_{3n}}{\partial q_i} \cdot \frac{\partial \xi_{3n}}{\partial q_{i'}}.$$

Harum quantitatum Determinans, secundum eandem Propositionem, quam §. pr. allegavi (*De form. et propr. Determ.* §. 14), aequatur aggregato quadratorum

Determinantium functionalium quarumque $3n-k$ e numero functionum ξ_1, $\xi_2, \ldots, \xi_{3n}$, quantitatum $q_1, q_2, \ldots, q_{3n-k}$ respectu formatorum, sive fit

$$(12)\quad \begin{cases} M_1 = \Sigma \pm a_{1,1} a_{2,2} \cdots a_{3n-k,3n-k} \\ \quad\;\; = S\left\{\Sigma \pm \frac{\partial \xi_{m'}}{\partial q_1} \cdot \frac{\partial \xi_{m''}}{\partial q_2} \cdots \frac{\partial \xi_{m^{(3n-k)}}}{\partial q_{3n-k}}\right\}^2, \end{cases}$$

designantibus m', m'', etc. quoscunque $3n-k$ ex indicibus 1, 2, ..., $3n$.

In deducendis aequationibus differentialibus (5) supposui, aequationes conditionales tempus t non explicite continere. Quod ubi fit, statuendum erit, functiones, quibus $3n$ quantitates x_i, y_i, z_i aequantur, praeter $3n-k$ quantitates q_m etiam ipsum t continere. At hinc non mutabuntur formulae (1), (3), (4), ideoque ipsae aequationes (5) immutatae manebunt. Unde altera quoque forma Lagrangiana aequationum differentialium dynamicarum ad hunc valet casum, quo aequationes conditionales tempus explicite continent. Neque eo casu mutationem subeunt formulae (7) et (8), unde etiam valor Multiplicatoris inventus immutatus manet. Quod breviter adnotare sufficiat.

§. 24.

De Multiplicatore aequationum differentialium dynamicarum forma tertia exhibitarum. Multiplicatores trium formarum aequationum differentialium dynamicarum inter se comparantur. Principium ultimi Multiplicatoris ad tertiam formam relatum.

Quantitatum $q'_1, q'_2, \ldots, q'_{3n-k}$ respectu functio T homogenea erat secundi gradus, unde fit

$$2T = q'_1 \frac{\partial T}{\partial q'_1} + q'_2 \frac{\partial T}{\partial q'_2} + \cdots + q'_{3n-k} \frac{\partial T}{\partial q'_{3n-k}},$$

sive

$$T = q'_1 \frac{\partial T}{\partial q'_1} + q'_2 \frac{\partial T}{\partial q'_2} + \cdots + q'_{3n-k} \frac{\partial T}{\partial q'_{3n-k}} - T.$$

Si variamus quantitates omnes, quarum T functio est, ponimusque

$$(1)\quad \frac{\partial T}{\partial q'_i} = p_i,$$

sequitur e valore ipsius T praecedente

$$(2)\quad \begin{cases} \delta T = q'_1\,\delta p_1 + q'_2\,\delta p_2 + \cdots + q'_{3n-k}\,\delta p_{3n-k} \\ \qquad - \left\{\frac{\partial T}{\partial q_1}\delta q_1 + \frac{\partial T}{\partial q_2}\delta q_2 + \cdots + \frac{\partial T}{\partial q_{3n-k}}\delta q_{3n-k}\right\}, \end{cases}$$

ubi in dextra parte bini termini se mutuo destruentes $\frac{\partial T}{\partial q'_i}\delta q'_i - \frac{\partial T}{\partial q'_i}\delta q'_i$ omissi sunt. Formula (2) docet, si per $3n-k$ aequationes, e (6) §. pr. fluentes,

$$(3)\quad p_i = a_{i,1}q'_1 + a_{i,2}q'_2 + \cdots + a_{i,3n-k}q'_{3n-k}$$

quantitates q'_i per quantitates p_i et q_i exprimantur earumque valores in functione T substituantur, fore ipsius T differentialia partialia quantitatum q_i et p_i respectu sumta, quae uncis includendo distinguamus ab ipsius T differentialibus partialibus quantitatum q_i et q'_i respectu sumtis,

$$(4)\quad \left(\frac{\partial T}{\partial q_i}\right) = -\frac{\partial T}{\partial q_i},\quad \left(\frac{\partial T}{\partial p_i}\right) = q'_i.$$

Harum formularum ope aequationes differentiales (5) §. pr. exhibere licet ut systema $6n-2k$ aequationum differentialium primi ordinis inter t et quantitates $q_1, q_2, \ldots, q_{3n-k}, p_1, p_2, \ldots, p_{3n-k}$:

$$(5)\quad \frac{dq_i}{dt} = \left(\frac{\partial T}{\partial p_i}\right),\quad \frac{dp_i}{dt} = -\left(\frac{\partial T}{\partial q_i}\right) + Q_i.$$

Hae formulae *tertiam* formam aequationum differentialium dynamicarum constituunt. Quas, pro casu, quo $3n$ quantitates X_i, Y_i, Z_i sunt differentialia partialia eiusdem functionis U respective secundum x_i, y_i, z_i sumta, primus condidit Celeb. Hamilton, Astronomus Regius Hibernensis. Eo casu fit e (2) §. pr. $Q_i = \frac{\partial U}{\partial q_i}$, unde statuendo $T-U=H$, si vires non a velocitatibus pendent ideoque U ab ipsis p_i vacua est, aequationes differentiales dynamicae evadunt

$$(6)\quad \frac{dq_i}{dt} = \left(\frac{\partial H}{\partial p_i}\right),\quad \frac{dp_i}{dt} = -\left(\frac{\partial H}{\partial q_i}\right).$$

Iam olim quidem Ill. Poisson in celeberrimo opere de Constantium arbitrariarum variatione id egerat, ut quantitatum q'_i loco in aequationibus differentialibus dynamicis Lagrangianis secundis introduceret quantitates p_i; quae aequationes si ea substitutione abeunt in

$$(7)\quad \frac{dq_i}{dt} = A_i,\quad \frac{dp_i}{dt} = B_i,$$

bene idem cognoverat fore

$$\left(\frac{\partial A_i}{\partial q_k}\right) = -\left(\frac{\partial B_k}{\partial p_i}\right),\quad \left(\frac{\partial A_i}{\partial p_k}\right) = \left(\frac{\partial A_k}{\partial p_i}\right),\quad \left(\frac{\partial B_i}{\partial q_k}\right) = \left(\frac{\partial B_k}{\partial q_i}\right),$$

unde sequebatur, omnes $6n-2k$ quantitates A_i et $-B_i$ esse differentialia partialia eiusdem functionis, ipsarum p_i et q_i respectu sumta. At meritum, eam

functionem $H = T-U$ ipsam assignavisse eaque re aequationibus differentialibus dynamicis formam perfectissimam conciliavisse, Celeb. Hamilton debetur.

Casu, quo mobilium Coordinatae functionibus aequantur, quae praeter quantitates q_i ipsum tempus t implicant, forma simplex aequationum (5) perit, qua de re hoc quidem loco transformationem Hamiltonianam ad eum casum non applicabo.

Facile invenitur aequationum (5) Multiplicator M_2. Etenim si aequationes (5) per formulas (7) designamus, fit

$$\frac{d \log M_2}{dt} + \Sigma\left\{\left(\frac{\partial A_i}{\partial q_i}\right) + \left(\frac{\partial B_i}{\partial p_i}\right)\right\} = 0.$$

At ponendo

$$A_i = \left(\frac{\partial T}{\partial p_i}\right),\quad B_i = -\left(\frac{\partial T}{\partial q_i}\right) + Q_i,$$

sequitur, si vires sollicitantes a velocitatibus non pendent ideoque functiones Q_i quantitates p_1, p_2, etc. non implicant,

$$\left(\frac{\partial A_i}{\partial q_i}\right) + \left(\frac{\partial B_i}{\partial p_i}\right) = 0,$$

ideoque

$$(8)\quad M_2 = 1.$$

Si functiones Q_i quoque implicant quantitates p_i, definitur M_2 per formulam

$$(9)\quad \frac{d \log M_2}{dt} + \frac{\partial Q_1}{\partial p_1} + \frac{\partial Q_2}{\partial p_2} + \cdots + \frac{\partial Q_{3n-k}}{\partial p_{3n-k}} = 0.$$

Iam tres Multiplicatores M, M_1, M_2, pro tribus aequationum differentialium dynamicarum formis inventos, inter se comparemus.

Forma secunda aequationum differentialium dynamicarum proveniebat e prima reducta per $2k$ aequationes integrales

$$(10)\quad \begin{cases} \Pi = 0, & \Pi_1 = 0, \quad \ldots, \quad \Pi_{k-1} = 0, \\ \Pi' = 0, & \Pi'_1 = 0, \quad \ldots, \quad \Pi'_{k-1} = 0. \end{cases}$$

Quae aequationes integrales, licet non completae, ita tamen sunt comparatae, ut aequationum differentialium reductarum Multiplicator e Multiplicatore propositarum per eandem formulam obtineatur ac si reductio per aequationes integrales completas facta esset (cf. §§. 10 et 12). Cum per aequationes (10) revocentur $6n$ variabiles $x_i, y_i, z_i, x'_i, y'_i, z'_i$ ad $6n-2k$ variabiles q_i et q'_i, secundum ea, quae l. c. tradidi, duorum Multiplicatorum Quotiens $\frac{M}{M_1}$ aequatur Determinanti $6n$ functionum

$$\Pi,\quad \Pi_1,\quad \ldots,\quad \Pi_{k-1},\quad q_1,\quad q_2,\quad \ldots,\quad q_{3n-k},$$
$$\Pi',\quad \Pi_1',\quad \ldots,\quad \Pi_{k-1}',\quad q_1',\quad q_2',\quad \ldots,\quad q_{3n-k}',$$

formato respectu $6n$ quantitatum $x_i, y_i, z_i, x_i', y_i', z_i'$. Expressiones novarum variabilium q_1, q_2, etc. per x_i, y_i, z_i per aequationes (10) diversas subire possunt mutationes, quibus tamen illius Determinantis valor non mutatur (cf. §. 3 (12)). Ponamus rursus, ut supra, $3n$ quantitates ξ_i loco quantitatum $x_i\sqrt{m_i}$, $y_i\sqrt{m_i}$, $z_i\sqrt{m_i}$, atque $3n$ quantitates ξ_i' loco quantitatum $x_i'\sqrt{m_i}$, $y_i'\sqrt{m_i}$, $z_i'\sqrt{m_i}$, valor ipsius $\frac{M}{M_1}$ etiam aequari poterit Determinanti earundem $6n$ functionum, formato quantitatum ξ_i et ξ_i' respectu, quippe quod ab illo Determinante functionali tantum discrepat factore constante (cubo producti massarum). Cum $3n$ quantitates ξ_i' non reprehendantur in $3n$ functionibus Π_m et q_m, Determinans Quotienti $\frac{M}{M_1}$ aequale induit formam producti

$$\Sigma \pm \frac{\partial \Pi}{\partial \xi_1} \cdot \frac{\partial \Pi_1}{\partial \xi_2} \cdots \frac{\partial \Pi_{k-1}}{\partial \xi_k} \cdot \frac{\partial q_1}{\partial \xi_{k+1}} \cdot \frac{\partial q_2}{\partial \xi_{k+2}} \cdots \frac{\partial q_{3n-k}}{\partial \xi_{3n}}$$
$$\times \Sigma \pm \frac{\partial \Pi'}{\partial \xi_1'} \cdot \frac{\partial \Pi_2'}{\partial \xi_2'} \cdots \frac{\partial \Pi_{k-1}'}{\partial \xi_k'} \cdot \frac{\partial q_1'}{\partial \xi_{k+1}'} \cdot \frac{\partial q_2'}{\partial \xi_{k+2}'} \cdots \frac{\partial q_{3n-k}'}{\partial \xi_{3n}'}.$$

Cum vero insuper sit

$$\frac{\partial \Pi_m'}{\partial \xi_i'} = \frac{\partial \Pi_m}{\partial \xi_i}, \quad \frac{\partial q_m'}{\partial \xi_i'} = \frac{\partial q_m}{\partial \xi_i},$$

utrumque in se ductum Determinans aequale evadit, unde eruitur

$$(11) \quad \frac{M}{M_1} = \left\{\Sigma \pm \frac{\partial \Pi}{\partial \xi_1} \cdot \frac{\partial \Pi_1}{\partial \xi_2} \cdots \frac{\partial \Pi_{k-1}}{\partial \xi_k} \cdot \frac{\partial q_1}{\partial \xi_{k+1}} \cdot \frac{\partial q_2}{\partial \xi_{k+2}} \cdots \frac{\partial q_{3n-k}}{\partial \xi_{3n}}\right\}^2.$$

Sint

$$m',\quad m'',\quad \ldots,\quad m^{(3n-k)}$$

indices diversi ex ipsorum $1, 2, \ldots, 3n$ numero, supponere licet, ipsas $q_1, q_2, \ldots, q_{3n-k}$ expressas esse per solas $3n-k$ quantitates

$$\xi_{m'},\quad \xi_{m''},\quad \ldots,\quad \xi_{m^{(3n-k)}};$$

tum autem Quotientis $\frac{M}{M_1}$ valor formam simpliciorem induit

$$(12) \quad \begin{cases} \frac{M}{M_1} = \left\{\Sigma \pm \frac{\partial q_1}{\partial \xi_{m'}} \cdot \frac{\partial q_2}{\partial \xi_{m''}} \cdots \frac{\partial q_{3n-k}}{\partial \xi_{m^{(3n-k)}}}\right\}^2 \\ \qquad \times \left\{\Sigma \pm \frac{\partial \Pi}{\partial \xi_{m^{(3n-k+1)}}} \cdot \frac{\partial \Pi_1}{\partial \xi_{m^{(3n-k+2)}}} \cdots \frac{\partial \Pi_{k-1}}{\partial \xi_{m^{(3n)}}}\right\}^2, \end{cases}$$

siquidem $m^{(3n-k+1)}$, $m^{(3n-k+2)}$, $\ldots$, $m^{(3n)}$ designant k reliquos indicum $1, 2, \ldots, 3n$. Unde tandem per formulam notam (*Determ. Funct.* §. 9 (3)) sequitur

$$(13) \quad \begin{cases} M\left\{\Sigma \pm \frac{\partial \xi_{m'}}{\partial q_1} \cdot \frac{\partial \xi_{m''}}{\partial q_2} \cdots \frac{\partial \xi_{m^{(3n-k)}}}{\partial q_{3n-k}}\right\}^2 \\ = M_1\left\{\Sigma \pm \frac{\partial \Pi}{\partial \xi_{m^{(3n-k+1)}}} \cdot \frac{\partial \Pi_1}{\partial \xi_{m^{(3n-k+2)}}} \cdots \frac{\partial \Pi_{k-1}}{\partial \xi_{m^{(3n)}}}\right\}^2. \end{cases}$$

Quod antecedentibus suppositum est, novas variabiles $q_1, q_2, \ldots, q_{3n-k}$ per totidem quantitates $\xi_{m'}, \xi_{m''}$, etc. expressas esse, id fieri non potest, quoties ex aequationibus conditionalibus $\Pi = 0$ etc. aequatio inter easdem $3n-k$ quantitates $\xi_{m'}$ etc. sequitur; nam cum $3n-k$ quantitates q_1, q_2, etc. a se independentes sint, etiam $3n-k$ quantitates $\xi_{m'}$ etc., per quas exprimantur, a se independentes esse debent. Nihilo tamen minus pro eo quoque casu formula (13) valet. Quoties enim ex aequationibus $\Pi = 0$ etc. fluit aequatio inter solas $3n-k$ quantitates $\xi_{m'}, \xi_{m''}, \ldots, \xi_{m^{(3n-k)}}$, hae aequabuntur $3n-k$ functionibus quantitatum $q_1, q_2, \ldots, q_{3n-k}$ non a se independentibus, quarum functionum Determinans evanescere constat. (*Determ. Funct.* §. 6.) Porro si e k aequationibus $\Pi = 0$ etc. obtineri potest aequatio inter solas $3n-k$ quantitates $\xi_{m'}, \xi_{m''}, \ldots, \xi_{m^{(3n-k)}}$, fieri debet, ut ex iisdem reliquae k quantitates $\xi_{m^{(3n-k+1)}}$ etc. eliminari possint. At *si de k aequationibus $\Pi = 0$ etc. totidem quantitates eliminari possunt, functionum Π etc. Determinans earum quantitatum respectu formatum per ipsas aequationes evanescit**). Unde casu, de quo agitur, utroque Determinante ad dextram et laevam signi aequalitatis posito evanescente, aequatio (13) iusta manet.

Si, quod secundum antecedentia licet, in aequatione (13) pro systemate

*) Ponamus enim, ex aequatione $\Pi = 0$ eliminari posse k quantitates ope reliquarum aequationum $\Pi_1 = 0$, $\Pi_2 = 0, \ldots, \Pi_{k-1} = 0$, per easdem induere debet Π formam producti μF, designante F functionem a k quantitatibus vacuam, ut ex aequationibus conditionalibus sequatur inter reliquas quantitates aequatio $F = 0$. Secundum §. 3 (12) in Determinante functionum $\Pi, \Pi_1, \ldots, \Pi_{k-1}$ ipsum μF substituere licet functioni Π. Quoties autem $F = 0$, differentialia prima ipsius μF ita formare licet, ac si factor μ constans esset, unde etiam in formando Determinante functionum $\mu F, \Pi_1, \Pi_2, \ldots, \Pi_{k-1}$ habere licet μ pro Constante. Quod igitur Determinans aequivalebit factori μ ducto in Determinans functionum $F, \Pi_1, \Pi_2, \ldots, \Pi_{k-1}$, ideoque evanescet, cum F ab ipsis quantitatibus vacua sit, quarum respectu Determinans functionale formatur.

indicum $m', m'', \ldots, m^{(3n-k)}$ sumuntur quique $3n-k$ diversi indicum $1, 2, \ldots, 3n$, omnesque $\frac{3n(3n-1)\ldots(3n-k+1)}{1.2\ldots k}$ aequationes provenientes consummantur, prodit aequatio

$$M\,S\left\{\Sigma\pm\frac{\partial\xi_{m'}}{\partial q_1}\cdot\frac{\partial\xi_{m''}}{\partial q_2}\cdots\frac{\partial\xi_{m^{(3n-k)}}}{\partial q_{3n-k}}\right\}^2$$
$$= M_1 S\left\{\Sigma\pm\frac{\partial\Pi}{\partial\xi_{m'}}\cdot\frac{\partial\Pi_1}{\partial\xi_{m''}}\cdots\frac{\partial\Pi_{k-1}}{\partial\xi_{m^{(k)}}}\right\}^2,$$

ubi in altera summa loco indicum $m^{(3n-k+1)}, m^{(3n-k+2)}, \ldots, m^{(3n)}$, quippe qui aliam non habent significationem quam quorumque k diversorum ex indicibus $1, 2, \ldots, 3n$, scripsi $m', m'', \ldots, m^{(k)}$. Aequatio antecedens perfecte congruit cum supra inventis. Nam secundum formulam (16) §. 22 aequatur M summae ad dextram, secundum formulam (12) §. 23 aequatur M_1 summae ad laevam signi aequalitatis positae.

Aequationum dynamicarum forma secunda in tertiam mutabatur introducendo variabilium $q'_1, q'_2, \ldots, q'_{3n-k}$ loco totidem alias $p_1, p_2, \ldots, p_{3n-k}$. Unde secundum §. 9 tertiae formae Multiplicatore M_2 e secundae Multiplicatore M_1 obtinetur formula

$$\frac{M_1}{M_2} = \Sigma\pm\frac{\partial p_1}{\partial q'_1}\cdot\frac{\partial p_2}{\partial q'_2}\cdots\frac{\partial p_{3n-k}}{\partial q'_{3n-k}}.$$

Dantur autem novae quantitates p_i aequationibus linearibus

$$p_i = a_{i,1}q'_1 + a_{i,2}q'_2 + \cdots + a_{i,3n-k}q'_{3n-k},$$

posito secundum (11) §. 23

$$a_{i,i'} = \frac{\partial\xi_1}{\partial q_i}\cdot\frac{\partial\xi_1}{\partial q_{i'}} + \frac{\partial\xi_2}{\partial q_i}\cdot\frac{\partial\xi_2}{\partial q_{i'}} + \cdots + \frac{\partial\xi_{3n}}{\partial q_i}\cdot\frac{\partial\xi_{3n}}{\partial q_{i'}},$$

unde fit

$$\frac{M_1}{M_2} = \Sigma\pm a_{1,1}a_{2,2}\ldots a_{3n-k,3n-k}.$$

Quod rursus cum supra inventis congruit, cum secundum (9) §. pr. aequetur M_1 Determinanti ad dextram, secundum (8) autem M_2 *unitati*. Per considerationes antecedentes videmus, e valore $M_2 = 1$, qui sponte patet, inveniri potuisse M_1 et M, supra via diversissima inventos. Qua methodorum diversitate cum Multiplicatoris tum Determinantium functionalium theoria haud parum illustratur.

Principium ultimi Multiplicatoris ad formam aequationum differentialium dynamicarum tertiam relatum sic enunciari potest:

„Punctorum materialium systema subiectum sit conditionibus et sollicitetur viribus quibuscunque, a sola positione systematis in spatio pendentibus; qua positione determinata per μ quantitates independentes q_i, semisumma virium vivarum T exprimatur per quantitates q_i et $q'_i = \frac{dq_i}{dt}$; ad motum systematis definiendum, eliminato tempore, integrandae erunt $2\mu-1$ aequationes differentiales primi ordinis, quarum inventa sint $2\mu-2$ Integralia, totidem Constantes arbitrarias involventia, ita ut integranda restet unica aequatio differentialis primi ordinis inter duas variabiles u et v

$$v'du - u'dv = 0,$$

designantibus in hac aequatione u' et v' ipsarum u et v functiones, quibus quotientes differentiales $\frac{du}{dt}$ et $\frac{dv}{dt}$ ope Integralium inventorum aequantur; erit huius aequationis differentialis primi ordinis inter duas variabiles ultimo loco integrandae Multiplicator aequalis Determinanti functionali 2μ quantitatum q_i et $\frac{\partial T}{\partial q'_i}$, ipsarum u, v atque $2\mu-2$ Constantium arbitrariarum respectu formato.

Iam novum principium generale mechanicum exemplis applicabo.

§. 25.

De motu puncti versus centrum fixum attracti.

Pro motu libero puncti in plano ex ultimi Multiplicatoris principio generali fluit haec

Propositio.

Proponantur pro motu puncti in plano aequationes differentiales

$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = Y,$$

designantibus X et Y Coordinatarum puncti orthogonalium x et y functiones quascunque; si habentur aequationum differentialium propositarum duo Integralia

$$f(x, y, x', y') = \alpha, \quad \varphi(x, y, x', y') = \beta,$$

ubi α et β sunt Constantes arbitrariae, dabitur orbita puncti formula

$$\int\left(\frac{\partial x'}{\partial\alpha}\cdot\frac{\partial y'}{\partial\beta} - \frac{\partial x'}{\partial\beta}\cdot\frac{\partial y'}{\partial\alpha}\right)(y'dx - x'dy) = \gamma,$$

sive etiam formula

$$\int \frac{y'dx - x'dy}{\frac{\partial f}{\partial x'} \cdot \frac{\partial \varphi}{\partial y'} - \frac{\partial f}{\partial y'} \cdot \frac{\partial \varphi}{\partial x'}} = \gamma,$$

ubi, duorum Integralium inventorum ope exhibitis x' et y' per x, y, α, β, quantitates sub integrationis signo differentialia completa fiunt atque γ tertiam Constantem arbitrariam designat.

Aliam Propositionem, qua puncti liberi in plano moti orbita Quadraturis definiri potest, si puncti velocitatis intensitas et directio per duo Integralia inventa determinatae sunt, iam ante multos annos cum illustri *Academia Parisiensi* communicavi [cf. h. Vol. p. 37], sed ea Propositio tantum respiciebat casum, quo vires Coordinatis parallelae X et Y eiusdem quantitatum x et y functionis aequantur differentialibus ipsarum x et y respectu sumtis, dum in Propositione antecedente X et Y quantitatum x et y functiones quaecunque esse possunt.

Pro motu puncti in dato plano versus centrum fixum attracti duo constant Integralia principiis conservationis vis vivae et conservationis areae, quibus si principium ultimi Multiplicatoris addis, per tria illa principia generalia a priori constat, eius motus determinationem solis Quadraturis absolvi. Quod facto calculo sic comprobatur.

Pro motu proposito habentur aequationes differentiales

$$\frac{d^2x}{dt^2} = -\frac{xF(r)}{r}, \quad \frac{d^2y}{dt^2} = -\frac{yF(r)}{r},$$

ubi x et y Coordinatae orthogonales sunt, quarum initium in centro attractionis est; porro $r = \sqrt{xx+yy}$ atque $F(r)$ intensitas vis attractivae pro distantia r. Posito

$$R = \int F(r)dr,$$

e principiis generalibus mechanicis conservationis vis vivae et areae statim habentur duo Integralia

$$f = \tfrac{1}{2}(x'x'+y'y') + R = \alpha,$$
$$\varphi = \quad xy' - yx' \quad = \beta,$$

designantibus α et β Constantes arbitrarias. Unde fit

$$\frac{\partial f}{\partial x'} \cdot \frac{\partial \varphi}{\partial y'} - \frac{\partial f}{\partial y'} \cdot \frac{\partial \varphi}{\partial x'} = xx'+yy'.$$

E duobus Integralibus appositis sequitur

$$xx'+yy' = \sqrt{\varrho},$$

posito

$$\varrho = 2r^2(\alpha - R) - \beta\beta.$$

Unde secundum principium ultimi Multiplicatoris dabitur puncti orbita per aequationem

$$\int \frac{y'dx - x'dy}{\frac{\partial f}{\partial x'} \cdot \frac{\partial \varphi}{\partial y'} - \frac{\partial f}{\partial y'} \cdot \frac{\partial \varphi}{\partial x'}} = \int \frac{y'dx - x'dy}{\sqrt{\varrho}} = \gamma,$$

designante γ novam Constantem arbitrariam. Ex aequationibus

$$xy' - yx' = \beta, \quad xx'+yy' = \sqrt{\varrho}$$

sequitur

$$x' = \frac{x\sqrt{\varrho} - \beta y}{rr}, \quad y' = \frac{y\sqrt{\varrho} + \beta x}{rr};$$

unde substituendo $xdx + ydy = rdr$ fit

$$\frac{y'dx - x'dy}{\sqrt{\varrho}} = \frac{ydx - xdy}{rr} + \frac{\beta dr}{r\sqrt{\varrho}}.$$

Posito igitur $x = r\cos\vartheta$, $y = r\sin\vartheta$, unde $ydx - xdy = -rrd\vartheta$, dabitur orbita per formulam

$$\vartheta + \gamma = \beta \int \frac{dr}{r\sqrt{2r^2(\alpha - R) - \beta\beta}}.$$

Si lex attractionis est *Neutoniana*, ponendum est $F(r) = \frac{k^2}{rr}$, $R = -\frac{k^2}{r}$, designante k^2 vim attractivam pro unitate distantiae, institutaque integratione prodit aequatio sectionis conicae inter Coordinatas polares r, $\vartheta + \gamma$.

Aequationum differentialium antecedentium dextrae parti addamus *Coordinatarum x et y functiones homogeneas $(-3)^{tae}$ dimensionis* X et Y, aequationum differentialium provenientium

$$\frac{d^2x}{dt^2} = -x\frac{F(r)}{r} + X,$$
$$\frac{d^2y}{dt^2} = -y\frac{F(r)}{r} + Y$$

semper aliquod obtineri poterit Integrale. Nam ex his aequationibus eruitur

$$\tfrac{1}{2}d\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)^2 = (xdy - ydx)(xY - yX) = x^2(xY - yX)\,d\left(\frac{y}{x}\right).$$

At est $x^2(xY - yX)$ functio variabilium x et y homogenea *nullae* dimensionis ideoque functio ipsius $\frac{y}{x}$, unde aequationis antecedentis pars utraque est differentiale cempletum, factaque integratione prodit

$$\varphi = \tfrac{1}{2}(xy'-yx')^2 - V = \tfrac{1}{2}\beta^2,$$

siquidem β Constans arbitraria est atque

$$V = \int x^2(xY-yX)\,d\left(\frac{y}{x}\right).$$

Si X et Y sunt differentialia partialia functionis homogeneae $(-2)^{\text{tae}}$ dimensionis U, ipsarum x et y respectu sumta, principium conservationis vis vivae alterum suppeditat Integrale

$$f = \tfrac{1}{2}(x'x'+y'y')+R-U = \alpha,$$

siquidem α est altera Constans arbitraria atque rursus

$$R = \int F(r)\,dr.$$

Functiones f et φ inventas substituendo fit

$$\frac{\partial f}{\partial x'}\cdot\frac{\partial \varphi}{\partial y'} - \frac{\partial f}{\partial y'}\cdot\frac{\partial \varphi}{\partial x'} = (xx'+yy')(xy'-yx').$$

At ex Integralibus inventis eruitur

$$(xx'+yy')(xy'-yx') = \sqrt{2r^2(\alpha-R+U)-(2V+\beta^2)}.\sqrt{2V+\beta^2},$$

quippe ponendo

$$2r^2(\alpha-R+U)-(2V+\beta^2) = \varrho,$$

fit

$$xy'-yx' = \sqrt{2V+\beta^2}, \quad xx'+yy' = \sqrt{\varrho}.$$

Hinc sequitur

$$\frac{\partial f}{\partial x'}\cdot\frac{\partial \varphi}{\partial y'} - \frac{\partial f}{\partial y'}\cdot\frac{\partial \varphi}{\partial x'} = \sqrt{\varrho}\cdot\sqrt{2V+\beta^2};$$

$$x' = \frac{x\sqrt{\varrho}-y\sqrt{2V+\beta^2}}{r^2},$$

$$y' = \frac{y\sqrt{\varrho}+x\sqrt{2V+\beta^2}}{r^2},$$

Quibus formulis substitutis in tertio Integrali, quod principio ultimi Multiplicatoris suppeditatur,

$$\int \frac{y'dx-x'dy}{\frac{\partial f}{\partial x'}\cdot\frac{\partial \varphi}{\partial y'} - \frac{\partial f}{\partial y'}\cdot\frac{\partial \varphi}{\partial x'}} = \gamma,$$

obtinetur formula, quae puncti orbitam determinat,

$$\int\left(\frac{ydx-xdy}{rr\sqrt{2V+\beta^2}} + \frac{dr}{r\sqrt{\varrho}}\right) = \gamma,$$

sive, ponendo rursus $x = r\cos\vartheta$, $y = r\sin\vartheta$,

$$\int\left(\frac{dr}{r\sqrt{\varrho}} - \frac{d\vartheta}{\sqrt{2V+\beta^2}}\right) = \gamma,$$

semper designante γ tertiam Constantem arbitrariam. Cum sit U functio homogenea $(-2)^{\text{ti}}$ ordinis, erit

$$2U = -\left\{x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}\right\} = -\{xX+yY\},$$

unde

$$\begin{aligned} d(r^2U) &= -\{xX+yY\}(xdx+ydy)+\{xx+yy\}(Xdx+Ydy) \\ &= \quad (xY-yX)(xdy-ydx). \end{aligned}$$

Eadem quantitas aequabatur ipsi dV, unde in formulis antecedentibus statuere licet

$$V = rrU,$$
$$\varrho = 2r^2(a-R)-\beta^2.$$

Secundum suppositionem factam fit $r^2U = V$ ipsius $\frac{y}{x} = \operatorname{tang}\vartheta$ functio, unde in aequatione orbitae

$$\int\frac{dr}{r\sqrt{2r^2(a-R)-\beta^2}} = \int\frac{d\vartheta}{\sqrt{2V+\beta^2}}+\gamma$$

alterum Integrale solius r, alterum solius ϑ functio est. Temporis expressio habetur per formulam

$$t+\tau = \int\frac{rdr}{xx'+yy'} = \int\frac{rdr}{\sqrt{\varrho}} = \int\frac{r^2d\vartheta}{\sqrt{2V+\beta^2}},$$

in qua τ est nova Constans arbitraria.

In motu antecedentibus considerato vis $F(r)$, qua punctum versus centrum fixum attrahitur, aucta est alia vi, quae secundum axes orthogonales disposita differentialibus partialibus $\frac{\partial U}{\partial x}$ et $\frac{\partial U}{\partial y}$ aequatur. Eadem vis secundum radii vectoris directionem eique perpendiculariter disposita evadit

$$P = \frac{1}{r}\left\{x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}\right\}, \quad Q = \frac{1}{r}\left\{y\frac{\partial U}{\partial x}-x\frac{\partial U}{\partial y}\right\}.$$

Secundum suppositionem de functionis U indole factam statui potest

$$r^2U = V = \Psi(\vartheta),$$

designante $\Psi(\vartheta)$ functionem anguli ϑ, quem radius vector cum axe fixo format. Qua expressione substituta positoque $\frac{d\Psi(\vartheta)}{d\vartheta} = \Psi'(\vartheta)$, eruitur

$$P = -\frac{2}{r^3}\Psi(\vartheta), \quad Q = -\frac{1}{r^2}\Psi'(\vartheta).$$

Si iam ponitur

$$\beta\int\frac{dr}{r\sqrt{\varrho}} = \beta\int\frac{dt}{r^2} = \beta\int\frac{d\vartheta}{\sqrt{2\Psi(\vartheta)+\beta^2}} = \Theta,$$

docent formulae antecedentibus inventae, illis viribus P et Q ad vim attractivam $F(r)$ accedentibus orbitae aequationem polarem eam mutationem subire, ut angulus ϑ in angulum Θ mutetur. At simul videmus, *illa virium P et Q accessione relationem inter radium vectorem et tempus omnino immutatam manere.* Quae curiosa Propositio valet etiam, si non, quod antecedentibus supposui, motus in plano fit. Sit enim U ipsarum x, y, z functio homogenea $(-2)^{\text{tae}}$ dimensionis, ac proponantur aequationes differentiales

$$\frac{d^2x}{dt^2} = -\frac{x}{r}F(r)+\frac{\partial U}{\partial x},$$
$$\frac{d^2y}{dt^2} = -\frac{y}{r}F(r)+\frac{\partial U}{\partial y},$$
$$\frac{d^2z}{dt^2} = -\frac{z}{r}F(r)+\frac{\partial U}{\partial z};$$

rursus $\int F(r)dr = R$ ponendo sequitur

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2 = 2(-R+U+\alpha),$$
$$x\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}+z\frac{d^2z}{dt^2} = -rF(r)-2U.$$

Quibus additis fit

$$d\left\{x\frac{dx}{dt}+y\frac{dy}{dt}+z\frac{dz}{dt}\right\} = d\left(r\frac{dr}{dt}\right) = \{2(\alpha-R)-rF(r)\}dt,$$

unde multiplicando per $2r\frac{dr}{dt}$ et integrando prodit

$$r^2\left(\frac{dr}{dt}\right)^2 = 2r^2(\alpha-R)+\varepsilon,$$

ideoque

$$t+\tau = \int\frac{rdr}{\sqrt{2r^2(\alpha-R)+\varepsilon}},$$

qua in formula τ et ε Constantes arbitrariae sunt. Patet autem, quod demonstrandum erat, in hac formula nullum functionis U vestigium remansisse. Addo, si U gaudeat forma particulari

$$U = \frac{1}{r^2}\left\{f\left(\frac{x}{r}\right)+\varphi\left(\frac{y}{r}\right)\right\},$$

designantibus f et φ functiones quascunque, eum ipsum motum, qui in plano non continetur, totum Quadraturis determinari posse.

Motus puncti in spatio pendet a *quinque* aequationibus differentialibus primi ordinis inter *sex* quantitates x, y, z, x', y', z'; unde *quatuor* Integralibus egemus, ut problema ad aequationem differentialem primi ordinis inter duas variabiles revocetur, quae ope principii ultimi Multiplicatoris per solas Quadraturas integrabitur. At quoties vires sollicitantes diriguntur versus axem fixum viriumque intensitates non pendent ab angulo, quem planum per axem et mobile ductum cum plano fixo per eundem axem transeunte facit, problema ad motum puncti in plano revocari potest, et nonnisi *duobus* Integralibus opus erit, ut totum absolvatur Quadraturis. Designantibus enim x, v, ζ puncti Coordinatas orthogonales positoque

$$vv+\zeta\zeta = yy,$$

sint aequationes differentiales, quibus motus puncti definitur,

$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2v}{dt^2} = Y\frac{v}{y}, \quad \frac{d^2\zeta}{dt^2} = Y\frac{\zeta}{y},$$

ubi secundum suppositionem factam et X et Y solarum x et y functiones esse debent: erit

$$v\frac{d^2\zeta}{dt^2}-\zeta\frac{d^2v}{dt^2} = 0,$$

unde sequitur

$$v\frac{d\zeta}{dt}-\zeta\frac{dv}{dt} = \alpha,$$

designante α Constantem arbitrariam. Fit autem

$$\frac{d^2y}{dt^2} = \frac{d^2\sqrt{vv+\zeta\zeta}}{dt^2} = \frac{1}{\sqrt{(vv+\zeta\zeta)^3}}\cdot\left(v\frac{d\zeta}{dt}-\zeta\frac{dv}{dt}\right)^2+\frac{1}{\sqrt{vv+\zeta\zeta}}\cdot\left(v\frac{d^2v}{dt^2}+\zeta\frac{d^2\zeta}{dt^2}\right),$$

ideoque

$$\frac{d^2y}{dt^2} = \frac{\alpha\alpha}{y^3}+Y.$$

Unde aequationes differentiales propositae evadunt sequentes

$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = \frac{\alpha\alpha}{y^3}+Y.$$

(Cf. *Diar.* Crell. *Vol. XXIV. pag.* 16 *sqq.; h. Vol. p.* 277). Ponendo

$$v = y\cos f, \quad \zeta = y\sin f,$$

fit

$$v\frac{d\zeta}{dt}-\zeta\frac{dv}{dt}=yy\frac{df}{dt}=\alpha,$$

unde Constans α aequabitur plani per punctum mobile et axem fixum ducti velocitati rotatoriae initiali, multiplicatae per quadratum distantiae initialis puncti ab axe. Duobus Integralibus inter x, y, x', y' inventis, tertium Integrale principio ultimi Multiplicatoris suppeditatur. Quorum Integralium ope si $y'=\frac{dy}{dt}$ per y exprimitur, cum rotationis angulus f tum tempus t Quadraturis determinantur ope formularum

$$f=\alpha\int\frac{dt}{y^2}=\alpha\int\frac{dy}{y^2y'}, \quad t=\int\frac{dy}{y'}.$$

Unde in casu proposito cognitis *duobus* Integralibus tria reliqua a solis Quadraturis pendent. Consideretur ex. gr. motus puncti versus centrum fixum attracti; posito $r=\sqrt{xx+yy}$, secundum antecedentia erit

$$\frac{d^2x}{dt^2}=-\frac{x}{r}F(r); \quad \frac{d^2y}{dt}=-\frac{y}{r}F(r)+\frac{\alpha\alpha}{y^3}.$$

Quae aequationes in eas redeunt, quas supra integravi, ponendo

$$Y=\frac{\alpha\alpha}{y^3}, \quad U=-\frac{\alpha\alpha}{2yy}=-\frac{\alpha\alpha}{2rr\sin^2\vartheta},$$

unde

$$V=\Psi(\vartheta)=-\frac{\alpha\alpha}{2\sin^2\vartheta},$$

$$\Theta=\int\frac{\beta.d\vartheta}{\sqrt{2\Psi(\vartheta)+\beta^2}}=\int\frac{\beta.\sin\vartheta d\vartheta}{\sqrt{\beta^2\sin^2\vartheta-\alpha^2}},$$

ideoque

$$\cos\Theta=\frac{\beta}{\sqrt{\beta^2-\alpha^2}}\cos\vartheta.$$

Si r et ϑ sunt puncti attracti Coordinatae polares in plano fixo, in quo illud re vera movetur, in aequatione orbitae, quam in hoc plano describit, angulus Θ loco ipsius ϑ substitui debet, ut eruatur orbita descripta in plano mobili per axem ipsarum x ducto. Relationem inter r et t pro motu in utroque plano eandem manere, ex ipsa natura rei patet. Plani angulus rotatorius f datur per formulam

$$df=\frac{\alpha dt}{yy}=\frac{\alpha dt}{rr\sin^2\vartheta}=\frac{\alpha}{\beta}\cdot\frac{d\Theta}{\sin^2\vartheta}=\frac{\alpha\beta.d\Theta}{\alpha^2\cos^2\Theta+\beta^2\sin^2\Theta},$$

unde, designante ε Constantem arbitrariam,

$$\mathrm{tang}(f+\varepsilon)=\frac{\beta}{\alpha}\mathrm{tang}\,\Theta.$$

Si per centrum attractionis ex arbitrio axis fixus ducitur, in formulis antecedentibus axem Coordinatarum x pro axe fixo sumendo motus puncti attracti componitur e motu puncti in plano per ipsum et axem fixum ducto eiusque plani rotatione circa axem fixum. Statuatur $\alpha=\beta\sin\delta$, erit

$$\cos\vartheta=\cos\delta.\cos\Theta, \quad \mathrm{tang}\,\Theta=\sin\delta.\mathrm{tang}(f+\varepsilon), \quad \sin\vartheta.\sin(f+\varepsilon)=\sin\Theta.$$

E centro attractionis describatur superficies sphaerica, cuius intersectio cum axe fixo, cum radio vectore et cum plano orbitae puncti attracti sit A, P et circulus maximus PQ; porro in sphaera e A ad circulum maximum PQ demittatur perpendicularis AO: in triangulo rectangulo sphaerico AOP erit

$$AO=\delta, \quad AP=\vartheta, \quad PO=\Theta, \quad OAP=f+\varepsilon.$$

Cuius constructionis ope formulae antecedentes geometrice comprobari possunt.

Si punctum versus centra fixa quotcunque in eadem recta disposita secundum Neutonianam sive aliam quamcunque legem attrahitur, quibus attractionis viribus accedere potest vis constans rectae parallela, e duobus Integralibus, quae antecedentibus poscebantur, ut reliquae integrationes omnes Quadraturis absolverentur, alterum conservationis vis vivae principio suppeditatur. Si abest vis constans atque duo tantum sunt centra attrahentia lexque attractionis est Neutoniana, alterum Integrale Eulerus invenit. Eo igitur casu motus ille principio conservationis areae certi cuiusdam axis respectu valentis, principio conservationis vis vivae, Integrali Euleriano, tandem principio ultimi Multiplicatoris ad Quadraturas revocatur. Quod iam accuratius exponam.

§ 26.

Motus puncti versus duo centra fixa secundum legem Neutonianam attracti.

Punctum inter utrumque centrum medium sumatur pro initio Coordinatarum, recta centra iungens pro axe Coordinatarum x, sit porro y distantia mobilis ab hoc axe. Si massae centrorum sunt m et m' atque a semidistantia centrorum, secundum antecedentia valebunt inter x et y aequationes differentiales sequentes:

$$(1)\quad \begin{cases} \dfrac{d^2x}{dt^2}=-\dfrac{m(x-a)}{\{(x-a)^2+y^2\}^{\frac{3}{2}}}-\dfrac{m'(x+a)}{\{(x+a)^2+y^2\}^{\frac{3}{2}}}, \\ \dfrac{d^2y}{dt^2}=-\dfrac{my}{\{(x-a)^2+y^2\}^{\frac{3}{2}}}-\dfrac{m'y}{\{(x+a)^2+y^2\}^{\frac{3}{2}}}+\dfrac{\alpha^2}{y^3}, \end{cases}$$

designante α Constantem arbitrariam. Porro angulus rotationis plani per axem et mobile ducti datur formula

$$(2)\quad df = \frac{\alpha dt}{y^2}.$$

A principio conservationis vis vivae Integrale suppeditatur hoc:

$$(3)\quad \tfrac{1}{2}(x'x'+y'y') = \frac{m}{\{(x-a)^2+y^2\}^{\frac{1}{2}}} + \frac{m'}{\{(x+a)^2+y^2\}^{\frac{1}{2}}} - \frac{\alpha^2}{2y^2} + \beta,$$

designante β alteram Constantem arbitrariam. Integrale Eulerianum invenitur deducendo ex aequationibus (1) sequentem:

$$d(xy'-yx') = -\frac{maydt}{\{(x-a)^2+y^2\}^{\frac{3}{2}}} + \frac{m'aydt}{\{(x+a)^2+y^2\}^{\frac{3}{2}}} + \frac{\alpha^2 x dt}{y^3},$$

unde fit

$$\tfrac{1}{2}d(xy'-yx')^2 = -\frac{may\{(x-a)dy-ydx\}}{\{(x-a)^2+y^2\}^{\frac{3}{2}}} + \frac{m'ay\{(x+a)dy-ydx\}}{\{(x+a)^2+y^2\}^{\frac{3}{2}}}$$
$$+\frac{\alpha^2 x(xdy-ydx)}{y^3} - \frac{ma^2ydy}{\{(x-a)^2+y^2\}^{\frac{3}{2}}} - \frac{m'a^2ydy}{\{(x+a)^2+y^2\}^{\frac{3}{2}}}.$$

Hinc aequationum (1) alteram substituendo fluit

$$\tfrac{1}{2}d(xy'-yx')^2 = -\tfrac{1}{2}ma\frac{d\left(\frac{y}{x-a}\right)^2}{\left\{1+\left(\frac{y}{x-a}\right)^2\right\}^{\frac{3}{2}}} + \tfrac{1}{2}m'a\frac{d\left(\frac{y}{x+a}\right)^2}{\left\{1+\left(\frac{y}{x+a}\right)^2\right\}^{\frac{3}{2}}}$$
$$-\tfrac{1}{2}\alpha^2 d\left(\frac{x}{y}\right)^2 + a^2y'dy' - \alpha^2a^2\frac{dy}{y^3}.$$

Cuius aequationis termini singuli cum differentialia completa sint, obtinetur Integrale

$$(4)\quad \begin{cases} (xy'-yx')^2+\text{Const.} \\ = \dfrac{2ma(x-a)}{\{(x-a)^2+y^2\}^{\frac{1}{2}}} - \dfrac{2m'a(x+a)}{\{(x+a)^2+y^2\}^{\frac{1}{2}}} - \dfrac{\alpha^2x^2}{y^2} + a^2y'y' + \dfrac{\alpha^2a^2}{y^2}. \end{cases}$$

Si ponitur

$$(5)\quad \begin{cases} L = \dfrac{2m}{\{(x-a)^2+y^2\}^{\frac{1}{2}}} + \dfrac{2m'}{\{(x+a)^2+y^2\}^{\frac{1}{2}}} - \dfrac{\alpha^2}{y^2} + 2\beta, \\ M = \dfrac{2ma(x-a)}{\{(x-a)^2+y^2\}^{\frac{1}{2}}} - \dfrac{2m'a(x+a)}{\{(x+a)^2+y^2\}^{\frac{1}{2}}} + \dfrac{\alpha^2}{y^2}(a^2-x^2+y^2)+\gamma, \end{cases}$$

duo Integralia inventa evadunt

$$(6)\quad x'x'+y'y' = L,\quad (xy'-yx')^2-a^2y'y' = M,$$

sive

$$\psi = \beta,\quad \varphi = \gamma,$$

siquidem statuitur

$$\psi = \tfrac{1}{2}(x'x'+y'y') - \tfrac{1}{2}L + \beta,$$
$$\varphi = (xy'-yx')^2 - a^2y'y' - M + \gamma.$$

Si duorum Integralium ope et x' et y' per x et y exhibentur, secundum principium ultimi Multiplicatoris obtinetur tertium Integrale

$$\int \frac{y'dx-x'dy}{\frac{\partial\psi}{\partial x'}\cdot\frac{\partial\varphi}{\partial y'} - \frac{\partial\psi}{\partial y'}\cdot\frac{\partial\varphi}{\partial x'}} = \text{Const.}$$

At cum et L et M ab ipsis x' et y' vacua sint, fit

$$\frac{\partial\psi}{\partial x'} = x',\qquad \frac{\partial\psi}{\partial y'} = y',$$
$$\frac{\partial\varphi}{\partial x'} = -2y(xy'-yx'),\quad \frac{\partial\varphi}{\partial y'} = 2x(xy'-yx')-2a^2y'.$$

Quibus formulis substitutis, eruitur

$$\frac{\partial\psi}{\partial x'}\cdot\frac{\partial\varphi}{\partial y'} - \frac{\partial\psi}{\partial y'}\cdot\frac{\partial\varphi}{\partial x'} = 2(xx'+yy')(xy'-yx')-2a^2x'y'$$
$$= -2\{xy(x'x'-y'y')+(a^2-x^2+y^2)x'y'\}.$$

Unde tertium Integrale evadit

$$(7)\quad \int \frac{y'dx-x'dy}{xy(x'x'-y'y')+(a^2-x^2+y^2)x'y'} = \varepsilon,$$

designante ε Constantem arbitrariam.

In formula antecedente expressio sub integrationis signo posita, quantitatum x' et y' valoribus substitutis, evadere debet differentiale completum. Qui valores ut eruantur et commoda substitutio fiat, adhibeo methodum in calculis algebraicis usitatam, videlicet addo aequationes (6), altera multiplicata factore λ, quem hac conditione determino, ut aequationis provenientis pars laeva evadat quadratum functionis ipsarum x' et y' linearis. Ea ratione venit

$$(8)\quad (x^2-a^2+\lambda)y'y' - 2xyx'y' + (y^2+\lambda)x'x' = M+\lambda L,$$

quantitate λ determinata per aequationem

$$(9)\quad \begin{cases} 0 = (\lambda+y^2)(\lambda+x^2-a^2)-x^2y^2 \\ = \lambda^2+\lambda(x^2+y^2-a^2)-a^2y^2. \end{cases}$$

Huius aequationis quadraticae radices vocemus λ' et λ'', erit

$$(10)\quad a^2y^2 = -\lambda'\lambda'',\quad x^2+y^2 = a^2-\lambda'-\lambda'',\quad a^2x^2 = (a^2-\lambda')(a^2-\lambda'').$$

Hinc quadrata distantiarum puncti mobilis a centris attractionum fiunt

$$(x\pm a)^2+y^2 = 2a^2-\lambda'-\lambda''\pm 2\sqrt{(a^2-\lambda')(a^2-\lambda'')},$$

ideoque ipsae distantiae

$$(11)\quad \{(x\pm a)^2+y^2\}^{\frac{1}{2}} = \sqrt{a^2-\lambda'}\pm\sqrt{a^2-\lambda''}.$$

Porro fit

$$\lambda'-a^2\pm ax = -\sqrt{a^2-\lambda'}\,\{\sqrt{a^2-\lambda'}\mp\sqrt{a^2-\lambda''}\},$$
$$\lambda''-a^2\pm ax = \pm\sqrt{a^2-\lambda''}\,\{\sqrt{a^2-\lambda'}\mp\sqrt{a^2-\lambda''}\},$$

ideoque

$$(12)\quad \begin{cases} \dfrac{\lambda'-a^2\pm ax}{\{(x\mp a)^2+y^2\}^{\frac{1}{2}}} = -\sqrt{a^2-\lambda'}, \\[2ex] \dfrac{\lambda''-a^2\pm ax}{\{(x\mp a)^2+y^2\}^{\frac{1}{2}}} = \pm\sqrt{a^2-\lambda''}. \end{cases}$$

Si has formulas substituimus in (5), sequitur, quantitatem $M+\lambda'L$ solius λ', quantitatem $M+\lambda''L$ solius λ'' functionem esse. Etenim si advocamus formulas e (10) fluentes

$$(13)\quad \begin{cases} \dfrac{a^2-x^2-\lambda'}{y^2} = \dfrac{y^2+\lambda''}{y^2} = 1-\dfrac{a^2}{\lambda'}, \\[2ex] \dfrac{a^2-x^2-\lambda''}{y^2} = \dfrac{y^2+\lambda'}{y^2} = 1-\dfrac{a^2}{\lambda''}, \end{cases}$$

e (5), (12), (13) eruitur:

$$(14)\quad \begin{cases} \tfrac{1}{2}(M+\lambda'L) = -(m+m')\sqrt{a^2-\lambda'}+\alpha^2\left(1-\dfrac{a^2}{2\lambda'}\right)+\beta\lambda'+\tfrac{1}{2}\gamma, \\[2ex] \tfrac{1}{2}(M+\lambda''L) = \quad (m-m')\sqrt{a^2-\lambda''}+\alpha^2\left(1-\dfrac{a^2}{2\lambda''}\right)+\beta\lambda''+\tfrac{1}{2}\gamma. \end{cases}$$

Ipsae quibus x' et y' determinantur aequationes e (8) prodeunt substituendo ipsius λ valores λ' et λ''. Quae aequationes per $-a^2$ multiplicatae, formulis (10) substitutis, evadunt

$$\lambda''(a^2-\lambda')y'y'+2\sqrt{-\lambda'\lambda''(a^2-\lambda')(a^2-\lambda'')}.x'y'-\lambda'(a^2-\lambda'')x'x'$$
$$= -a^2(M+\lambda'L),$$
$$\lambda'(a^2-\lambda'')y'y'+2\sqrt{-\lambda'\lambda''(a^2-\lambda')(a^2-\lambda'')}.x'y'-\lambda''(a^2-\lambda')x'x'$$
$$= -a^2(M+\lambda''L),$$

sive extractis radicibus

$$(15)\quad \begin{cases} \sqrt{\lambda''(a^2-\lambda')}.y'+\sqrt{-\lambda'(a^2-\lambda'')}.x' = a\sqrt{-(M+\lambda'L)}, \\ \sqrt{-\lambda'(a^2-\lambda'')}.y'-\sqrt{\lambda''(a^2-\lambda')}.x' = a\sqrt{M+\lambda''L}. \end{cases}$$

Easdem aequationes (10) differentiando sequitur

$$2a(y'dx-x'dy) = 2y'd\sqrt{(a^2-\lambda')(a^2-\lambda'')}-2x'd\sqrt{-\lambda'\lambda''}$$
$$= \frac{-d\lambda'}{\sqrt{-\lambda'(a^2-\lambda')}}\cdot\{\sqrt{-\lambda'(a^2-\lambda'')}.y'-\sqrt{\lambda''(a^2-\lambda')}.x'\}$$
$$\frac{-d\lambda''}{\sqrt{\lambda''(a^2-\lambda'')}}\cdot\{\sqrt{\lambda''(a^2-\lambda')}.y'+\sqrt{-\lambda'(a^2-\lambda'')}.x'\}.$$

Unde formulas (15) substituendo prodit:

$$(16)\quad 2(y'dx-x'dy) = -\frac{\sqrt{M+\lambda''L}.d\lambda'}{\sqrt{-\lambda'(a^2-\lambda')}}-\frac{\sqrt{-(M+\lambda'L)}.d\lambda''}{\sqrt{\lambda''(a^2-\lambda'')}}$$

Aequationibus (15) in se ductis et rursus (10) advocatis, eruitur

$$(17)\quad xy(y'y'-x'x')+(x^2-y^2-a^2)x'y' = \sqrt{-(M+\lambda'L)(M+\lambda''L)}.$$

Per hanc formulam ubi dividimus antecedentem (16), prodit

$$(18)\quad \begin{cases} \dfrac{y'dx-x'dy}{xy(x'x'-y'y')+(a^2-x^2+y^2)x'y'} \\[2ex] = \dfrac{-d\lambda'}{2\sqrt{\lambda'(a^2-\lambda')(M+\lambda'L)}}+\dfrac{d\lambda''}{2\sqrt{\lambda''(a^2-\lambda'')(M+\lambda''L)}}. \end{cases}$$

Hanc supra vidimus expressionem secundum principium ultimi Multiplicatoris fieri debere differentiale completum. Ac revera, quantitatum $\frac{1}{2}(M+\lambda'L)$ et $\frac{1}{2}(M+\lambda''L)$ valoribus (14) substitutis, in ea expressione differentiale $d\lambda'$ per solius λ', differentiale $d\lambda''$ per solius λ'' functionem multiplicatum reprehenditur. Unde, formula (18) substituta in (7), tertium Integrale per duas Quadraturas obtinetur.

Si formulas adiicere placet, quibus t et f per λ' et λ'' solarum ope Quadraturarum determinantur, differentietur aequatio (9), posito $\lambda = \lambda'$, unde prodit

$$0 = (\lambda'-\lambda'')d\lambda'+2\lambda'xdx-2(a^2-\lambda')ydy$$
$$= (\lambda'-\lambda'')d\lambda'-\frac{2}{a}\sqrt{-\lambda'(a^2-\lambda')}.[\sqrt{-\lambda'(a^2-\lambda'')}dx+\sqrt{\lambda''(a^2-\lambda')}dy]$$
$$= (\lambda'-\lambda'')d\lambda'+2\sqrt{\lambda'(a^2-\lambda')(M+\lambda'L)}dt.$$

Hinc, si aequationem differentialem

$$(19)\quad \frac{d\lambda'}{\sqrt{\lambda'(a^2-\lambda')(M+\lambda'L)}} = \frac{d\lambda''}{\sqrt{\lambda''(a^2-\lambda'')(M+\lambda''L)}}$$

advocamus, obtinemus

$$(20)\quad dt = -\tfrac{1}{2}\frac{\sqrt{\lambda'}.d\lambda'}{\sqrt{(a^2-\lambda')(M+\lambda'L)}}+\tfrac{1}{2}\frac{\sqrt{\lambda''}.d\lambda''}{\sqrt{(a^2-\lambda'')(M+\lambda''L)}},$$

$$(21)\quad \begin{cases} df = \dfrac{-\alpha a^2 dt}{\lambda'\lambda''} \\ = \tfrac{1}{2}\alpha a^2\left\{-\dfrac{1}{\sqrt{\lambda'^3}}\cdot\dfrac{d\lambda'}{\sqrt{(a^2-\lambda')(M+\lambda' L)}}+\dfrac{1}{\sqrt{\lambda''^3}}\cdot\dfrac{d\lambda''}{\sqrt{(a^2-\lambda'')(M+\lambda'' L)}}\right\}. \end{cases}$$

His formulis videmus, ad variabilium t et f valores per Quadraturas inveniendos non opus esse, ut antea variabilium λ' et λ'' altera per alteram expressa habeatur.

§. 27.

De corporis solidi ictu impulsi rotatione circa punctum fixum.

Exemplum applicationis principii ultimi Multiplicatoris ad motum non liberum suppeditet rotatio solidi circa punctum eius fixum, si corpus solo ponitur ictu impulsum esse, nulla accedente vi acceleratrice. Valet pro eo motu principium conservationis virium vivarum nec non cuiuslibet plani respectu principium conservationis arearum. Quibus si additur principium ultimi Multiplicatoris, per sola principia generalia problema olim difficillimum ad Quadraturas reducetur.

Sint ξ, v, ζ Coordinatae orthogonales ad axes relatae in solido fixos, in spatio mobiles, quorum initium punctum fixum sit, circa quod solidum rotatur. Sint x, y, z Coordinatae orthogonales eodem initio gaudentes, ad axes in spatio fixos relatae. In aequationibus, quae inter utrasque Coordinatas locum habent,

$$(1)\quad x=\alpha\xi+\beta v+\gamma\zeta,\quad y=\alpha_1\xi+\beta_1 v+\gamma_1\zeta,\quad z=\alpha_2\xi+\beta_2 v+\gamma_2\zeta$$

sunt ξ, v, ζ Constantes, novem Coëfficientes α, β, etc. variabiles, inter quas relationes notae intercedunt, quibus illae ad quantitates tres revocari possunt*). Adhibita differentialium notatione Lagrangianae (1) sequitur

$$x'=\alpha'\xi+\beta' v+\gamma'\zeta,\quad y'=\alpha_1'\xi+\beta_1' v+\gamma_1'\zeta,\quad z'=\alpha_2'\xi+\beta_2' v+\gamma_2'\zeta.$$

Ponamus

$$\beta\gamma'+\beta_1\gamma_1'+\beta_2\gamma_2' = -\{\gamma\beta'+\gamma_1\beta_1'+\gamma_2\beta_2'\} = a,$$
$$\gamma\alpha'+\gamma_1\alpha_1'+\gamma_2\alpha_2' = -\{\alpha\gamma'+\alpha_1\gamma_1'+\alpha_2\gamma_2'\} = b,$$
$$\alpha\beta'+\alpha_1\beta_1'+\alpha_2\beta_2' = -\{\beta\alpha'+\beta_1\alpha_1'+\beta_2\alpha_2'\} = c;$$

*) Formulae (1) si Coordinatarum orthogonalium transformationem exprimunt, fit

$$\beta_1\gamma_2-\gamma_1\beta_2=\pm\alpha \text{ etc.},\quad \alpha(\beta_1\gamma_2-\gamma_1\beta_2)+\beta(\gamma_1\alpha_2-\alpha_1\gamma_2)+\gamma(\alpha_1\beta_2-\beta_1\alpha_2)=\pm 1.$$

At in hac rotationis quaestione, iam alibi adnotavi, semper signum + sumendum esse. Ponamus enim inter binorum corporum puncta correlationem dari talem, ut alterius corporis puncto, cuius Coordinatae sunt ξ, v, ζ, respondeat alterius corporis punctum, cuius Coordinatae ad *eosdem* axes relatae valoribus x, y, z gaudent: prout in illis formulis signum + aut — locum habet, erunt corpora aut *congruentia* aut uti dicitur *symmetrica*. Casu posteriore autem fieri non potest, ut alterum corpus in alterius positione collocetur, neque igitur rotatione alterum in alterius locum pervenire potest.

ex aequationibus

$$\alpha\alpha'+\alpha_1\alpha_1'+\alpha_2\alpha_2'=0,\quad \beta\alpha'+\beta_1\alpha_1'+\beta_2\alpha_2'=-c,\quad \gamma\alpha'+\gamma_1\alpha_1'+\gamma_2\alpha_2'=b,$$

quarum prima e formula $\alpha\alpha+\alpha_1\alpha_1+\alpha_2\alpha_2=1$ sequitur, fluit

$$\alpha'=-\beta c+\gamma b,\quad \alpha_1'=-\beta_1 c+\gamma_1 b,\quad \alpha_2'=-\beta_2 c+\gamma_2 b,$$

eodemque modo obtinetur

$$\beta'=-\gamma a+\alpha c,\quad \gamma'=-\alpha b+\beta a,$$
$$\beta_1'=-\gamma_1 a+\alpha_1 c,\quad \gamma_1'=-\alpha_1 b+\beta_1 a,$$
$$\beta_2'=-\gamma_2 a+\alpha_2 c,\quad \gamma_2'=-\alpha_2 b+\beta_2 a.$$

Quibus valoribus substitutis, eruitur

$$x' = \alpha(cv-b\zeta)+\beta(a\zeta-c\xi)+\gamma(b\xi-av),$$
$$y' = \alpha_1(cv-b\zeta)+\beta_1(a\zeta-c\xi)+\gamma_1(b\xi-av),$$
$$z' = \alpha_2(cv-b\zeta)+\beta_2(a\zeta-c\xi)+\gamma_2(b\xi-av).$$

Unde sequitur

$$(2)\quad x'x'+y'y'+z'z' = (cv-b\zeta)^2+(a\zeta-c\xi)^2+(b\xi-av)^2.$$

Porro e (1) proveniunt formulae

$$\alpha_2 y-\alpha_1 z=\beta\zeta-\gamma v,\quad \alpha z-\alpha_2 x=\beta_1\zeta-\gamma_1 v,\quad \alpha_1 x-\alpha y=\beta_2\zeta-\gamma_2 v,$$
$$\beta_2 y-\beta_1 z=\gamma\xi-\alpha\zeta,\quad \beta z-\beta_2 x=\gamma_1\xi-\alpha_1\zeta,\quad \beta_1 x-\beta y=\gamma_2\xi-\alpha_2\zeta,$$
$$\gamma_2 y-\gamma_1 z=\alpha v-\beta\xi,\quad \gamma z-\gamma_2 x=\alpha_1 v-\beta_1\xi,\quad \gamma_1 x-\gamma y=\alpha_2 v-\beta_2\xi.$$

Unde, substitutis ipsarum x', y', z' valoribus, eruitur

$$(3)\quad \begin{cases} yz'-zy' = (\beta\zeta-\gamma v)(cv-b\zeta)+(\gamma\xi-\alpha\zeta)(a\zeta-c\xi)+(\alpha v-\beta\xi)(b\xi-av), \\ zx'-xz' = (\beta_1\zeta-\gamma_1 v)(cv-b\zeta)+(\gamma_1\xi-\alpha_1\zeta)(a\zeta-c\xi)+(\alpha_1 v-\beta_1\xi)(b\xi-av), \\ xy'-yx' = (\beta_2\zeta-\gamma_2 v)(cv-b\zeta)+(\gamma_2\xi-\alpha_2\zeta)(a\zeta-c\xi)+(\alpha_2 v-\beta_2\xi)(b\xi-av). \end{cases}$$

Axes Coordinatarum ξ, v, ζ semper ita in ipso solido disponere licet, ut, designante dm solidi elementum, cuius Coordinatae sunt ξ, v, ζ, sit

$$Sv\zeta dm=0,\quad S\zeta\xi dm=0,\quad S\xi v dm=0,$$

summis ad omnia elementa materialia corporis extensis. Unde, ponendo

$$A=S(vv+\zeta\zeta)dm,\quad B=S(\zeta\zeta+\xi\xi)dm,\quad C=S(\xi\xi+vv)dm,$$

fit e (2) et (3):

$$(4)\quad T=\tfrac{1}{2}S\{x'x'+y'y'+z'z'\}dm=\tfrac{1}{2}\{Aaa+Bbb+Ccc\},$$

$$(5)\quad \begin{cases} L=S(yz'-zy')dm=-\{\alpha.Aa+\beta.Bb+\gamma.Cc\}, \\ M=S(zx'-xz')dm=-\{\alpha_1.Aa+\beta_1.Bb+\gamma_1.Cc\}, \\ N=S(xy'-yx')dm=-\{\alpha_2.Aa+\beta_2.Bb+\gamma_2.Cc\}. \end{cases}$$

Quibus in formulis secundum principia conservationis virium vivarum et arearum quatuor quantitates T, L, M, N aequantur Constantibus arbitrariis.

Novem Coëfficientes α, β, etc. per tres angulos q_1, q_2, q_3 exprimamus ope formularum notissimarum, quas olim Eulerus in *Introductione in Anal. Infin.* dedit:

$$(6)\quad \begin{cases} \alpha \;= \cos q_1 \sin q_2 \sin q_3 + \cos q_2 \cos q_3, \\ \alpha_1 = \cos q_1 \cos q_2 \sin q_3 - \sin q_2 \cos q_3, \\ \alpha_2 = -\sin q_1 \sin q_3; \\ \beta \;= \cos q_1 \sin q_2 \cos q_3 - \cos q_2 \sin q_3, \\ \beta_1 = \cos q_1 \cos q_2 \cos q_3 + \sin q_2 \sin q_3, \\ \beta_2 = -\sin q_1 \cos q_3; \\ \gamma \;= \sin q_1 \sin q_2, \\ \gamma_1 = \sin q_1 \cos q_2, \\ \gamma_2 = \cos q_1. \end{cases}$$

E quibus formulis sequitur:

$$\begin{aligned} \alpha' &= -\gamma \sin q_3 . q_1' + \alpha_1 . q_2' + \beta . q_3', \\ \alpha_1' &= -\gamma_1 \sin q_3 . q_1' - \alpha . q_2' + \beta_1 . q_3', \\ \alpha_2' &= -\gamma_2 \sin q_3 . q_1' \qquad\quad + \beta_2 . q_3', \\ \beta' &= -\gamma \cos q_3 . q_1' + \beta_1 . q_2' - \alpha . q_3', \\ \beta_1' &= -\gamma_1 \cos q_3 . q_1' - \beta . q_2' - \alpha_1 . q_3', \\ \beta_2' &= -\gamma_2 \cos q_3 . q_1' \qquad\quad - \alpha_2 . q_3', \\ \gamma' &= \cos q_1 \sin q_2 . q_1' + \gamma_1 . q_2', \\ \gamma_1' &= \cos q_1 \cos q_2 . q_1' - \gamma . q_2', \\ \gamma_2' &= -\sin q_1 . q_1'. \end{aligned}$$

Unde eruitur

$$(7)\quad \begin{cases} a = \quad \beta\gamma' + \beta_1\gamma_1' + \beta_2\gamma_2' \;= \quad \cos q_3 . q_1' - \sin q_1 \sin q_3 . q_2', \\ b = -\{\alpha\gamma' + \alpha_1\gamma_1' + \alpha_2\gamma_2'\} = -\sin q_3 . q_1' - \sin q_1 \cos q_3 . q_2', \\ c = \quad \alpha\beta' + \alpha_1\beta_1' + \alpha_2\beta_2' \;= \quad \cos q_1 . q_2' - q_3'. \end{cases}$$

Quas quantitates in aequatione (4) substituendo evadit virium vivarum semisumma T quantitatum $q_1, q_2, q_3, q_1', q_2', q_3'$ functio. Quam ipsarum q_1', q_2', q_3' respectu differentiando prodit

$$(8)\quad \begin{cases} \dfrac{\partial T}{\partial q_1'} = p_1 = \cos q_3 . Aa - \sin q_3 . Bb, \\ \dfrac{\partial T}{\partial q_2'} = p_2 = -\sin q_1 \sin q_3 . Aa - \sin q_1 \cos q_3 . Bb + \cos q_1 . Cc, \\ \dfrac{\partial T}{\partial q_3'} = p_3 = -Cc. \end{cases}$$

Hae quantitates autem aequantur sequentibus:

$$(9)\quad \begin{cases} p_1 = -L \cos q_2 + M \sin q_2, \\ p_2 = -N, \\ p_3 = (L \sin q_2 + M \cos q_2) \sin q_1 + N \cos q_1, \end{cases}$$

sicuti patet substituendo quantitatum L, M, N expressiones (5) et Coëfficientium α, β, etc. valores (6). Ponendo

$$(10)\quad \frac{p_3 \cos q_1 + p_2}{\sin q_1} = u,$$

e formulis (8) fluunt sequentes:

$$\begin{aligned} Aa &= \quad \cos q_3 . p_1 - \sin q_3 . u, \\ Bb &= -\sin q_3 . p_1 - \cos q_3 . u, \\ Cc &= -p_3. \end{aligned}$$

Quibus formulis quadratis ac respective per A, B, C divisis consummatisque, obtinetur post faciles reductiones:

$$(11)\quad \begin{cases} 2T = \tfrac{1}{2}\left(\dfrac{1}{A} + \dfrac{1}{B}\right)(p_1 p_1 + uu) + \dfrac{1}{C} p_3 p_3 \\ \qquad + \tfrac{1}{2}\left(\dfrac{1}{A} - \dfrac{1}{B}\right)\{(p_1 p_1 - uu)\cos 2q_3 - 2p_1 u \sin 2q_3\}. \end{cases}$$

Cum T, L, M, N Constantibus aequentur, per quatuor aequationes (9) et (11) sex variabiles $q_1, q_2, q_3, p_1, p_2, p_3$ ad duas revocare licet. Quomodocunque hae duae variabiles eligantur, aequatio differentialis primi ordinis inter eas locum habens principio ultimi Multiplicatoris ad Quadraturas revocabitur. At duas variabiles eligere convenit tales, per quas reliquae commode exprimantur, quales sunt p_1 et p_3. Cum solidum *nullis* viribus acceleratricibus sollicitetur, aequationum dynamicarum forma tertia §. 24 tradita suppeditat

$$\frac{dp_1}{dt} = -\frac{\partial T}{\partial q_1}, \quad \frac{dp_3}{dt} = -\frac{\partial T}{\partial q_3},$$

unde aequatio differentialis inter p_1 et p_3, quae integranda restat, fit

$$(12)\quad \frac{\partial T}{\partial q_3} dp_1 - \frac{\partial T}{\partial q_1} dp_3 = 0.$$

Partibus dextris aequationum (9) et (11) in laevam translatis, aequationem (11) denotemus per $\Pi = 0$, aequationes (9) per $\Pi_1 = 0$, $\Pi_2 = 0$, $\Pi_3 = 0$, erit secundum theoremata generalia §§. 24 et 11 tradita aequationis differentialis (12) Multiplicator

$$\mu = \frac{\Sigma \pm \frac{\partial \Pi}{\partial T} \frac{\partial \Pi_2}{\partial N} \frac{\partial \Pi_1}{\partial L} \frac{\partial \Pi_3}{\partial M}}{\Sigma \pm \frac{\partial \Pi}{\partial q_3} \frac{\partial \Pi_1}{\partial q_2} \frac{\partial \Pi_2}{\partial p_2} \frac{\partial \Pi_3}{\partial q_1}}.$$

Cuius fractionis ipsorumque $\frac{\partial T}{\partial q_3}$ et $\frac{\partial T}{\partial q_1}$ valores sic determino.

Cum sit $\frac{\partial \Pi}{\partial T} = 2$, $\frac{\partial \Pi_2}{\partial N} = 1$, numerator fractionis antecedentis eruitur

$$2\Sigma \pm \frac{\partial \Pi_1}{\partial L} \cdot \frac{\partial \Pi_3}{\partial M} = -2 \sin q_1.$$

E variabilibus p_2, q_1, q_2, q_3 functio Π_2 unicam p_2 implicat, functio Π_1 unicam q_2, functio Π_3 solas q_1 et q_2; porro fit $\frac{\partial \Pi_2}{\partial p_2} = 1$, unde fractionis antecedentis denominator evadit

$$\frac{\partial \Pi_1}{\partial q_2} \cdot \frac{\partial \Pi_3}{\partial q_1} \cdot \frac{\partial \Pi}{\partial q_3}.$$

Fit autem

$$\frac{\partial \Pi_1}{\partial q_2} = -\{L \sin q_2 + M \cos q_2\},$$

$$\frac{\partial \Pi_3}{\partial q_1} = -\{L \sin q_2 + M \cos q_2\} \cos q_1 + N \sin q_1 = -u,$$

$$\frac{\partial \Pi}{\partial q_3} = -2 \frac{\partial T}{\partial q_3}.$$

Unde aequationis differentialis (12) Multiplicator fit

$$(13) \quad \mu = \frac{\sin q_1}{(L \sin q_2 + M \cos q_2) u} \cdot \left(\frac{1}{\frac{\partial T}{\partial q_3}} \right).$$

At e (9) et (10), brevitatis causa posito

$$h = LL + MM + NN,$$

sequitur

$$(14) \quad \begin{cases} L \sin q_2 + M \cos q_2 = \sqrt{LL + MM - p_1 p_1} = \sqrt{h - NN - p_1 p_1}, \\ u = (L \sin q_2 + M \cos q_2) \cos q_1 - N \sin q_1 = \sqrt{h - p_1 p_1 - p_3 p_3}, \\ (h - p_1 p_1) \sin q_1 = (L \sin q_2 + M \cos q_2) p_3 - Nu. \end{cases}$$

Quibus in ipsius μ valore (13) substitutis sequitur

$$(15) \quad \mu \cdot \frac{\partial T}{\partial q_3} = \frac{1}{h - p_1 p_1} \left\{ \frac{p_3}{\sqrt{h - p_1 p_1 - p_3 p_3}} - \frac{N}{\sqrt{h - NN - p_1 p_1}} \right\}.$$

Restat ut quantitates $\frac{\partial T}{\partial q_3}$ et $\frac{\partial T}{\partial q_1}$ solis p_1 et p_3 exhibeantur.

Quantitatis u valor (10) cum quantitatem q_3 non implicet, e (11) sequitur

$$(16) \quad 2 \frac{\partial T}{\partial q_3} = \left(\frac{1}{B} - \frac{1}{A} \right) \{(p_1 p_1 - uu) \sin 2q_3 + 2 p_1 u \cos 2q_3\}.$$

Eius quantitatis quadratum e (11) fit

$$\left(\frac{1}{B} - \frac{1}{A} \right)^2 (p_1 p_1 + uu)^2 - \left\{ 4T - \left(\frac{1}{A} + \frac{1}{B} \right) (p_1 p_1 + uu) - \frac{2}{C} p_3 p_3 \right\}^2.$$

Unde ponendo

$$K = 2T - \frac{1}{A}(p_1 p_1 + uu) - \frac{1}{C} p_3 p_3,$$

$$K_1 = \frac{1}{B}(p_1 p_1 + uu) + \frac{1}{C} p_3 p_3 - 2T,$$

sive

$$(17) \quad \begin{cases} K = 2T - \frac{h}{A} + \left(\frac{1}{A} - \frac{1}{C} \right) p_3 p_3, \\ K_1 = \frac{h}{B} - 2T + \left(\frac{1}{C} - \frac{1}{B} \right) p_3 p_3, \end{cases}$$

sequitur

$$(18) \quad \frac{\partial T}{\partial q_3} = -\sqrt{KK_1}.$$

Cum elementum dt natura temporis numquam regredientis semper positivum sit docet formula $dp_3 = -\frac{\partial T}{\partial q_3} dt$, radicale $\sqrt{KK_1}$ negativo signo afficiendum esse uti in (18), quamdiu p_3 crescat, positivo quam diu p_3 decrescat.

Ipsum $\frac{\partial T}{\partial q_1}$ e (11) eruimus

$$(19) \quad \frac{\partial T}{\partial q_1} = \tfrac{1}{2} \frac{\partial u}{\partial q_1} \left\{ \left(\frac{1}{A} + \frac{1}{B} \right) u + \left(\frac{1}{B} - \frac{1}{A} \right) (u \cos 2q_3 + p_1 \sin 2q_3) \right\}.$$

Fit autem e (10) et (9)

$$\frac{\partial u}{\partial q_1} = -\frac{p_3 + p_2 \cos q_1}{\sin^2 q_1} = -\frac{L \sin q_2 + M \cos q_2}{\sin q_1},$$

ideoque e (13) et (18) obtinetur

$$(20) \quad \mu \frac{\partial u}{\partial q_1} = -\frac{1}{u \frac{\partial T}{\partial q_3}} = \frac{1}{u \sqrt{KK_1}}.$$

Porro ex aequationibus (11), (16), (18) fit

$$4T-\frac{2}{C}p_3p_3 = \left(\frac{1}{B}-\frac{1}{A}\right)\{(uu-p_1p_1)\cos 2q_3+2p_1u\sin 2q_3\}+\left(\frac{1}{A}+\frac{1}{B}\right)(p_1p_1+uu),$$

$$-2\sqrt{KK_1} = \left(\frac{1}{B}-\frac{1}{A}\right)\{2p_1u\cos 2q_3+(p_1p_1-uu)\sin 2q_3\},$$

unde

$$\frac{u\left(4T-\frac{2}{C}p_3p_3\right)-2p_1\sqrt{KK_1}}{uu+p_1p_1} = \left(\frac{1}{A}+\frac{1}{B}\right)u+\left(\frac{1}{B}-\frac{1}{A}\right)(u\cos 2q_3+p_1\sin 2q_3).$$

Hinc valore $uu+p_1p_1 = h-p_3p_3$ substituto, e (19) et (20) eruitur

$$(21)\quad \mu\frac{\partial T}{\partial q_1} = \frac{u\left(2T-\frac{p_3p_3}{C}\right)-p_1\sqrt{KK_1}}{(h-p_3p_3)u\sqrt{KK_1}}.$$

Unde iam aequatio differentialis

$$\mu\frac{\partial T}{\partial q_3}dp_1-\mu\frac{\partial T}{\partial q_1}dp_3 = 0,$$

quae per se integrabilis esse debet, per formulas (15) et (21) evadit

$$(22)\quad \begin{cases} 0 = -\dfrac{Ndp_1}{(h-p_1p_1)(h-NN-p_1p_1)^{\frac{1}{2}}}+\dfrac{p_3dp_1}{(h-p_1p_1)(h-p_1p_1-p_3p_3)^{\frac{1}{2}}} \\ \quad +\dfrac{p_1dp_3}{(h-p_3p_3)(h-p_1p_1-p_3p_3)^{\frac{1}{2}}}-\dfrac{\left(2T-\frac{p_3p_3}{C}\right)dp_3}{(h-p_3p_3)\sqrt{KK_1}}. \end{cases}$$

Quatuor terminorum dextrae partis primum et quartum differentialia completa esse patet, cum primus solam p_1, quartus secundum (17) solam p_3 implicet. Ponendo $p_1 = \sqrt{h-NN}.\sin\varphi$, primus terminus fit

$$\frac{-Nd\varphi}{h\cos^2\varphi+N^2\sin^2\varphi} = -\frac{1}{\sqrt{h}}d\operatorname{arctg}\frac{N\operatorname{tg}\varphi}{\sqrt{h}},$$

unde valorem $\operatorname{tg}\varphi = \frac{p_1}{\{h-NN-p_1p_1\}^{\frac{1}{2}}}$ restituendo evadit primus terminus

$$(23)\quad \frac{-Ndp_1}{(h-p_1p_1)(h-NN-p_1p_1)^{\frac{1}{2}}} = -\frac{1}{\sqrt{h}}d\operatorname{arctg}\frac{Np_1}{\sqrt{h}\sqrt{h-NN-p_1p_1}}.$$

Si in dextra parte huius formulae in locum Constantis N ponitur quantitas p_3, prodit expressio, utriusque p_1 et p_3 respectu symmetrica; unde si ipsam quoque

quantitatem p_3 pro variabili habemus atque utriusque p_1 et p_3 respectu differentiationem instituimus, provenire debet aggregatum duorum terminorum, qui de expressione ad laevam aequationis (23) posita derivantur, alter ponendo p_3 ipsius N loco, alter ponendo p_1 ipsius N simulque p_3 ipsius p_1 loco; unde de formula (23) deducitur haec:

$$(24)\quad \begin{cases} \left(\dfrac{p_3dp_1}{h-p_1p_1}+\dfrac{p_1dp_3}{h-p_3p_3}\right)\dfrac{1}{(h-p_3p_3-p_1p_1)^{\frac{1}{2}}} \\ = \dfrac{1}{\sqrt{h}}d\operatorname{arctg}\dfrac{p_1p_3}{\sqrt{h}\sqrt{h-p_1p_1-p_3p_3}}. \end{cases}$$

Quae docet, aequationis (22) terminos secundum et tertium iuxta sumtos et ipsos differentiale completum constituere. Formulas (17), (23) et (24) in aequatione differentiali (22) substituendo et integrando prodit Integrale *quintum:*

$$(25)\quad \begin{cases} \text{Const.} = -\dfrac{1}{\sqrt{h}}\operatorname{arctg}\dfrac{Np_1}{\sqrt{h}\sqrt{h-NN-p_1p_1}}+\dfrac{1}{\sqrt{h}}\operatorname{arctg}\dfrac{p_1p_3}{\sqrt{h}\sqrt{h-p_1p_1-p_3p_3}} \\ -\displaystyle\int\frac{\left(2T-\frac{p_3p_3}{C}\right)dp_3}{(h-p_3p_3)\sqrt{2T-\frac{h}{A}+\left(\frac{1}{A}-\frac{1}{C}\right)p_3p_3}\sqrt{\frac{h}{B}-2T+\left(\frac{1}{C}-\frac{1}{B}\right)p_3p_3}}. \end{cases}$$

Tempus t, quod unice determinandum restat, per p_3 exprimitur ope formulae

$$(26)\quad \begin{cases} t+\text{Const.} = -\displaystyle\int\frac{dp_3}{\frac{\partial T}{\partial q_3}} = \int\frac{dp_3}{\sqrt{KK_1}} \\ = \displaystyle\int\frac{dp_3}{\sqrt{2T-\frac{h}{A}+\left(\frac{1}{A}-\frac{1}{C}\right)p_3p_3}\sqrt{\frac{h}{B}-2T+\left(\frac{1}{C}-\frac{1}{B}\right)p_3p_3}}. \end{cases}$$

Ita problema rotationis propositum iam *sine plani invariabilis usu* perfecte integratum est.

Quod planum si adhibere placet atque pro Coordinatarum x et y plano sumere, fit

$$L = 0,\quad M = 0.$$

Unde e (10), (9) et (11) fit $u = -N\sin q_1$, porro

$$p_1 = 0,\quad p_2 = -N = -\sqrt{h},\quad p_3 = N\cos q_1,$$

$$\frac{2T}{N^2} = \frac{1}{A}\sin^2 q_1\sin^2 q_3+\frac{1}{B}\sin^2 q_1\cos^2 q_3+\frac{1}{C}\cos^2 q_1.$$

In dextra parte formulae (25) terminus secundus evanescit, tertius immutatus manet, primus autem *indeterminati* speciem induit. At observo, e (9) haberi

$$\frac{Np_1}{\sqrt{h}\sqrt{h-NN-p_1p_1}} = \frac{N\operatorname{tg}(q_2-\alpha)}{\sqrt{N^2+L^2+M^2}},$$

siquidem ponitur $\frac{L}{M} = \operatorname{tg}\alpha$. Hinc si ponimus $L = 0$, $M = 0$ atque Constantem $\frac{\alpha}{\sqrt{h}}$ Constanti arbitrariae adiicimus, formula (25) evadit:

$$\text{Const.} = \frac{q_2}{N} + \int \frac{\left(2T - \frac{p_3p_3}{C}\right)dp_3}{(h-p_3p_3)\sqrt{KK_1}},$$

ubi K et K_1 valores (17) immutatos servant. Nec non temporis t expressio immutata manet

$$t + \text{Const.} = \int \frac{dp_3}{\sqrt{KK_1}}.$$

Formularum antecedentium ope variabiles omnes maxima concinnitate exhiberi possunt per functiones ellipticas, quarum argumentum tempori t proportionale est. Quod egregie expositum invenis in Commentatione inaugurali Cl. A. S. Rueb Roterodamensis „de motu gyratorio corporis rigidi“, Traiecti ad Rhenum a. 1834 publicata.

In his quaestionibus de rotatione solidi atque de motu puncti versus duo centra fixa attracti data opera analysi usus sum inelegantiore, ut demonstretur, ea problemata ope principii ultimi Multiplicatoris etiam absque artificiis, quae non ita in promptu sunt, ad finem perduci posse.

§. 28.

De problemate trium corporum in eadem recta motorum. Substitutio Euleriana. Theoremata de viribus homogeneis.

Paucis adhuc agam de tribus corporibus se mutuo attrahentibus in eademque recta motis, quippe quod problema varia de Multiplicatore proposita exemplo illustrandi occasionem commodam praebebit. Ope principii conservationis virium vivarum quaestio in aequationis differentialis secundi ordinis integrationem redit. At Eulerus olim absque Integrali ab illo principio suppeditato reductionem problematis ad aequationem differentialem secundi ordinis per substitutionem memorabilem effecit. (Cf. *Nov. Comm. Ac. Petrop. Vol. XI pg. 144 sqq.*, *Nova*

Acta Vol. III pg. 126—141.) Quam rem hic ita repetam, ut simul per idoneam variabilium electionem formularum symmetriae consulam.

Sint m, m', m'' tria eiusdem rectae puncta massis m, m', m'' praedita sitque m' inter m et m''. Designante O rectae punctum fixum, ponatur

$$Om = x,\quad Om' = x_1,\quad Om'' = x_2.$$

Si directionem motus, qua punctum a m ad m', a m' ad m'' fertur, positivam, directionem oppositam, qua punctum a m'' ad m', a m' ad m movetur, negativam dicimus, statuo x, x_1, x_2 quantitates positivas aut negativas esse, prout a puncto fixo O ad puncta m, m', m'' directio positiva aut negativa est. Ubi massae m, m', m'' se mutuo secundum legem Neutonianam attrahunt, fit

$$(1)\quad \begin{cases} \dfrac{d^2x}{dt^2} = \quad\cdot\quad + \dfrac{m'}{(x_1-x)^2} + \dfrac{m''}{(x_2-x)^2}, \\ \dfrac{d^2x_1}{dt^2} = -\dfrac{m}{(x_1-x)^2} \quad\cdot\quad + \dfrac{m''}{(x_2-x_1)^2}, \\ \dfrac{d^2x_2}{dt^2} = -\dfrac{m}{(x_2-x)^2} - \dfrac{m'}{(x_2-x_1)^2} \quad\cdot\quad . \end{cases}$$

Trium massarum se mutuo attrahentium centrum gravitatis statuamus in quiete manere, quod salva generalitate licet, ipsumque ponamus centrum gravitatis esse punctum fixum O. Hinc tres quantitates x, x_1, x_2 duabus aliis u et v exprimi possunt per substitutiones lineares

$$(2)\quad x = \alpha u+\beta v,\quad x_1 = \alpha' u+\beta' v,\quad x_2 = \alpha'' u+\beta'' v,$$

in quibus α, β, etc. designant Constantes quascunque satisfacientes duabus aequationibus

$$(3)\quad m\alpha+m'\alpha'+m''\alpha'' = 0,\quad m\beta+m'\beta'+m''\beta'' = 0.$$

Quibus ex arbitrio addamus tertiam

$$(4)\quad m\alpha\beta+m'\alpha'\beta'+m''\alpha''\beta'' = 0;$$

porro ponamus

$$m\alpha\alpha+m'\alpha'\alpha'+m''\alpha''\alpha'' = \mu,$$
$$m\beta\beta+m'\beta'\beta'+m''\beta''\beta'' = \nu.$$

Substitutis (2) in aequationibus differentialibus (1) et additis tribus aequationibus respective per $m\alpha$, $m'\alpha'$, $m''\alpha''$ vel per $m\beta$, $m'\beta'$, $m''\beta''$ multiplicatis, obtinetur:

$$(5)\quad \begin{cases} \mu\dfrac{d^2u}{dt^2} = \dfrac{m'm''(\alpha'-\alpha'')}{(x_2-x_1)^2} + \dfrac{m''m(\alpha-\alpha'')}{(x_2-x)^2} + \dfrac{mm'(\alpha-\alpha')}{(x_1-x)^2}, \\ \nu\dfrac{d^2v}{dt^2} = \dfrac{m'm''(\beta'-\beta'')}{(x_2-x_1)^2} + \dfrac{m''m(\beta-\beta'')}{(x_2-x)^2} + \dfrac{mm'(\beta-\beta')}{(x_1-x)^2}. \end{cases}$$

Sit

$$(6)\quad \begin{cases} \alpha''-\alpha' = a, & \alpha''-\alpha = a', \quad \alpha'-\alpha = a'', \\ \beta''-\beta' = b, & \beta''-\beta = b', \quad \beta'-\beta = b'', \end{cases}$$

unde

$$(7)\quad \begin{cases} a+a'' = a', \quad b+b'' = b', \\ m'm''.ab+m''m.a'b'+mm'.a''b'' = 0^{*}); \end{cases}$$

obtinentur inter u et v aequationes differentiales:

$$(8)\quad \begin{cases} \mu\dfrac{d^2u}{dt^2} = -\dfrac{m'm''a}{(au+bv)^2}-\dfrac{m''ma'}{(a'u+b'v)^2}-\dfrac{mm'a''}{(a''u+b''v)^2}, \\ \nu\dfrac{d^2v}{dt^2} = -\dfrac{m'm''b}{(au+bv)^2}-\dfrac{m''mb'}{(a'u+b'v)^2}-\dfrac{mm'b''}{(a''u+b''v)^2}. \end{cases}$$

Aequationibus (8) respective per $\frac{du}{dt}$ et $\frac{dv}{dt}$ multiplicatis et additis factaque integratione obtinetur aequatio, conservationem virium vivarum exprimens:

$$(9)\quad \tfrac{1}{2}\left\{\mu\left(\frac{du}{dt}\right)^2+\nu\left(\frac{dv}{dt}\right)^2\right\} = \frac{m'm''}{au+bv}+\frac{m''m}{a'u+b'v}+\frac{mm'}{a''u+b''v}-h,$$

designante h Constantem arbitrariam.

Quantitates μ et ν ipsis a, b, etc. determinantur per formulas

$$(10)\quad \begin{cases} (m+m'+m'')\mu = m'm''a^2+m''ma'^2+mm'a''^2, \\ (m+m'+m'')\nu = m'm''b^2+m''mb'^2+mm'b''^{2}\,{}^{**}). \end{cases}$$

Ponamus

$$(11)\quad \mu = \nu = 1,$$

inter quatuor quantitates a, b, a'', b'' locum habebunt tres aequationes:

$$(12)\quad \begin{cases} m+m'+m'' = m''(m+m')a^2+2m''maa''+m(m'+m'')a''^2, \\ m+m'+m'' = m''(m+m')b^2+2m''mbb''+m(m'+m'')b''^2, \\ 0 = m''(m+m')ab+m''m(ab''+a''b)+m(m'+m'')a''b''. \end{cases}$$

Quae demonstrant, quantitates a et a'', b et b'' haberi posse pro Coordinatis punctorum in terminis positorum quarumcunque binarum semidiametrorum coniugatarum sectionis conicae, cuius aequatio est

$$m+m'+m'' = m''(m+m')x^2+2m''mxy+m(m'+m'')y^2.$$

---

*) Haec aequatio sequitur e formula identica

$$(m+m'+m'')(m\alpha\beta+m'\alpha'\beta'+m''\alpha''\beta'')-(m\alpha+m'\alpha'+m''\alpha'')(m\beta+m'\beta'+m''\beta'') = m'm''ab+m''ma'b'+mm'a''b''.$$

**) Hae aequationes sequuntur e formulis identicis

$$(m+m'+m'')(m\alpha^2+m'\alpha'^2+m''\alpha''^2)-(m\alpha+m'\alpha'+m''\alpha'')^2 = m'm''.a^2+m''m.a'^2+mm'.a''^2,$$
$$(m+m'+m'')(m\beta^2+m'\beta'^2+m''\beta''^2)-(m\beta+m'\beta'+m''\beta'')^2 = m'm''.b^2+m''m.b'^2+mm'.b''^2.$$

Si pro diametris coniugatis axes principales sumere placet, quantitates a, b, etc. determinandae erunt per aequationes:

$$(13)\quad a = A\cos\varepsilon,\quad a'' = A\sin\varepsilon,\quad b = B\sin\varepsilon,\quad b'' = -B\cos\varepsilon,$$

ubi, posito br. c.

$$m''(m+m')+m(m''+m') = n,$$

et nova quantitate M introducta, angulus ε et quantitates A et B dantur per formulas:

$$(14)\quad \begin{cases} M\cos 2\varepsilon = m'(m''-m), & M\sin 2\varepsilon = 2mm'', \\ A = \sqrt{\dfrac{m+m'+m''}{\frac{1}{2}(n+M)}}, & B = \sqrt{\dfrac{m+m'+m''}{\frac{1}{2}(n-M)}}. \end{cases}$$

Determinatis a, b, etc., invenitur

$$(15)\quad \begin{cases} \alpha = \alpha'-a'',\quad \alpha'' = \alpha'+a,\quad \beta = \beta'-b'',\quad \beta'' = \beta'+b, \\ \alpha' = \dfrac{ma''-m''a}{m+m'+m''},\quad \beta' = \dfrac{mb''-m''b}{m+m'+m''}. \end{cases}$$

De substitutione hic a me adhibita pluribus egi in Commentatione „*sur l'élimination des noeuds dans le problème des trois corps.*“ [Cf. o. h. Vol. p. 297.]

His de Coëfficientibus substitutionis linearis (2) obiter adnotatis, iam novas variabiles r, φ, s, η introduco ope substitutionis

$$(16)\quad \begin{cases} u = r\cos\varphi, \qquad v = r\sin\varphi, \\ s = \sqrt{r}\,.\,\dfrac{dr}{dt} = \dfrac{u\frac{du}{dt}+v\frac{dv}{dt}}{\sqrt[4]{u^2+v^2}}, \\ \eta = \sqrt{r^3}\,.\,\dfrac{d\varphi}{dt} = \dfrac{u\frac{dv}{dt}-v\frac{du}{dt}}{\sqrt[4]{u^2+v^2}}. \end{cases}$$

Ex aequationibus differentialibus (8), posito $\mu = \nu = 1$, sequitur

$$(17)\quad \begin{cases} \sqrt{r^3}\,.\,\dfrac{ds}{dt} = \tfrac{1}{2}s^2+\eta^2-\Phi, \\ \sqrt{r^3}\,.\,\dfrac{d\eta}{dt} = -\tfrac{1}{2}\eta s+\Phi', \end{cases}$$

siquidem ponitur $\Phi' = \frac{d\Phi}{d\varphi}$ atque

$$(18)\quad \Phi = \frac{m'm''}{a\cos\varphi+b\sin\varphi}+\frac{m''m}{a'\cos\varphi+b'\sin\varphi}+\frac{mm'}{a''\cos\varphi+b''\sin\varphi}.$$

E formulis (16) et (17) patet, *determinationem motus propositi pendere ab integratione duarum aequationum differentialium primi ordinis inter tres variabiles* φ, s, η:

$$(19)\quad d\varphi : ds : d\eta = \eta : \tfrac{1}{2}s^2+\eta^2-\Phi : -\tfrac{1}{2}s\eta+\Phi'.$$

Quas aequationes differentiales, quia a Constante generali h vacuae sunt, simpliciores censere licet iis, quae, non adhibitis substitutionibus (16) aut earum similibus, adiumento aequationis (9) per unius variabilis eliminationem obtinentur. Integratis (19), suppeditabit formula (9) valorem ipsius r. Nimirum cum sit

$$\left(\frac{du}{dt}\right)^2+\left(\frac{dv}{dt}\right)^2=\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\varphi}{dt}\right)^2=\frac{1}{r}\{s^2+\eta^2\},$$

fit e (9)

$$(20)\quad r = \frac{1}{h}\{\Phi-\tfrac{1}{2}(s^2+\eta^2)\}.$$

Denique tempus t invenitur formula

$$(21)\quad dt = \frac{\sqrt{r}}{s}\,dr = \frac{\sqrt{r^3}}{\eta}\,d\varphi.$$

Iam aequationum differentialium (19) investigabo Multiplicatorem N.

Si adhibemus formulam differentialem, qua generaliter Multiplicatorem definivi, fit

$$-\frac{\eta\, d\log N}{d\varphi} = \frac{\partial\eta}{\partial\varphi}+\frac{\partial(\frac{1}{2}s^2+\eta^2-\Phi)}{\partial s}+\frac{\partial(-\frac{1}{2}s\eta+\Phi')}{\partial\eta}=\tfrac{1}{2}s,$$

ideoque e (16)

$$(22)\quad d\log N = -\tfrac{1}{2}\frac{s}{\eta}\,d\varphi = -\tfrac{1}{2}\frac{dr}{r};\quad N=\frac{1}{\sqrt{r}}\cdot$$

Unde substituendo (20) et factorem constantem $\sqrt{h}$ reiiciendo, *fit aequationum differentialium* (19) *Multiplicator*

$$N = \frac{1}{\sqrt{\Phi-\frac{1}{2}(s^2+\eta^2)}}\cdot$$

Qui Multiplicatoris valor valet, quaecunque anguli φ sit functio Φ, qua aequationes differentiales (19) afficiuntur.

Multiplicatorem etiam per praecepta generalia Cap. II. tradita hoc modo indagare licet. Scilicet aequationum differentialium (8) Multiplicator est *unitas*. Unde aequationum differentialium

$$(23)\quad \begin{cases} \dfrac{dr}{dt}=\dfrac{s}{\sqrt{r}},\quad \dfrac{d\varphi}{dt}=\dfrac{\eta}{\sqrt{r^3}}, \\ \dfrac{ds}{dt}=\dfrac{1}{\sqrt{r^3}}\{\tfrac{1}{2}s^2+\eta^2-\Phi\}, \\ \dfrac{d\eta}{dt}=\dfrac{1}{\sqrt{r^3}}\{-\tfrac{1}{2}\eta s+\Phi'\} \end{cases}$$

Multiplicator aequatur unitati divisae per quantitatum r, φ, s, η Determinans, variabilium u, v, $\frac{du}{dt}$, $\frac{dv}{dt}$ respectu formatum. Quod Determinans, cum quantitatum r et φ valores ab ipsis $\frac{du}{dt}$ et $\frac{dv}{dt}$ vacui sint, aequatur producto Determinantis quantitatum r et φ ipsarum u et v respectu et Determinantis quantitatum s et η ipsarum $\frac{du}{dt}$ et $\frac{dv}{dt}$ respectu formati. Quorum Determinantium alterum fit $\frac{1}{r}$, alterum r, unde aequationum (23) Multiplicator et ipse $=1$ invenitur. Deinde si Integralis (20) ope eliminatur variabilis r simulque de aequationibus differentialibus (23) prima reiicitur, Multiplicator aequationum differentialium, ea eliminatione ad minorem numerum paucioresque variabiles reductarum, secundum §. 10 aequatur differentiali partiali $\frac{\partial r}{\partial h}$, designante h Constantem arbitrariam, qua Integrale (20) afficitur. Quod differentiale partiale e (20) fit $-\frac{r}{h}\cdot$ Denique aequationum differentialium (19) Multiplicator invenitur dividendo per $\sqrt{r^3}$, quippe per quod multiplicandum erat, ut quantitates ad dextram aequationum (19) prodirent; unde, factore constante $-\frac{1}{h}$ reiecto, prodit aequationum (19) Multiplicator $\frac{1}{\sqrt{r}}$, uti supra.

Cognito ipsius N valore, si aequationum differentialium (19) integratione prima exprimitur variabilis η per φ, s et Constantem arbitrariam α, principio ultimi Multiplicatoris obtinetur alterum Integrale

$$\int\frac{\partial\eta}{\partial\alpha}\cdot\frac{\eta\, ds+\{\Phi-\frac{1}{2}s^2-\eta^2\}d\varphi}{\sqrt{\Phi-\frac{1}{2}(s^2+\eta^2)}}=\beta,$$

ubi sub integrationis signo post valorem ipsius η substitutum differentiale completum habetur atque β Constantem arbitrariam designat. Eulerus integrationem primam, etsi succederet, in hac quaestione parvi adiumenti fore putavit, cum de ulteriore integratione desperandum esset. At novo principio generali ultimi

Multiplicatoris ipsam ulteriorem integrationem absolvere licuit, dum de prima integratione nihil constat.

Evanescente h, habetur aequatio integralis particularis

$$(24)\quad \Phi = \tfrac{1}{2}(s^2+\eta^2),$$

unde una tantum integranda manet aequatio differentialis primi ordinis inter duas variabiles s et φ:

$$(25)\quad \frac{ds}{d\varphi} - \tfrac{1}{2}\sqrt{2\Phi - s^2} = 0.$$

Cuius aequationis differentialis Multiplicator M definitur formula

$$\frac{d\log M}{d\varphi} = -\tfrac{1}{2}\frac{\partial\sqrt{2\Phi-s^2}}{\partial s} = \tfrac{1}{2}\frac{s}{\sqrt{2\Phi-s^2}} = \frac{s}{2\eta} = \tfrac{1}{2}\frac{d\log r}{d\varphi},$$

unde $\beta M = \sqrt{r}$. Invento aequationis differentialis (25) Integrali eiusque ope expressa φ per s et α, fit $M^{-1} = \frac{\partial s}{\partial \alpha}$, ideoque

$$(26)\quad r = \frac{\beta^2}{\left(\frac{\partial s}{\partial \alpha}\right)^2},$$

designantibus α et β Constantes arbitrarias.

Formulae prorsus analogae habentur, si mutuae attractiones non distantiarum quadratis inversis, sed aliis quibuscunque potestatibus proportionales sunt. Observo tamen, casu, quo trium corporum, quae in eadem recta moventur, mutuae attractiones *cubis* distantiarum inverse proportionales sint, motum totum tantum ab *unica Quadratura* pendere.

Si vires sollicitantes in motu systematis liberi functiones Coordinatarum homogeneae quaecunque sunt, generaliter per substitutiones antecedentibus similes systematis aequationum differentialium ordinem *unitate* diminuere licet, quantitate, cui Coordinatae proportionales statuuntur, eliminata. Quam, docet theoria nostra, aequationum differentialium iis substitutionibus reductarum Multiplicatore determinari, ideoque, si illae complete integratae sint, Determinante functionali, quo earum Multiplicator detur, variabilis quoque eliminatae valorem absque Quadratura suppeditari. Si principium conservationis virium vivarum valet, eo ipso variabilis eliminata determinari potest, unde vice versa aequationum differentialium reductarum Multiplicatorem eruere earumque ultimam integrationem reducere licet ad Quadraturas. Excipiendus est casus particularis, quo Constans arbitraria, quae valori semisummae virium vivarum accedit, nihilo

aequiparatur. Eo casu aequationum differentialium reductarum habetur Integrale particulare, unde ordinem systematis earum denuo unitate diminuere licet; quantitas eliminata autem rursus determinabitur Multiplicatore systematis aequationum differentialium bis reductarum. Hinc sequens nanciscimur theorema:

„Sint vires, quibus systema liberum n punctorum materialium sollicitatur, functiones Coordinatarum homogeneae, valeatque principium conservationis virium vivarum; casu particulari, quo Constans arbitraria valori virium vivarum adiicienda nihilo aequatur, systematis aequationum differentialium ordo *duabus* unitatibus diminui sive problema revocari potest ad integrationem $6n-3$ aequationum differentialium primi ordinis inter $6n-2$ variabiles; quibus complete integratis, obtinetur valor $(6n-1)^{\text{tae}}$ variabilis *per differentiationes* secundum Constantes arbitrarias institutas, qui valor in novam Constantem arbitrariam ducitur; $6n^{\text{ta}}$ variabilis principio conservationis virium vivarum determinatur, postremo tempus, ut semper, obtinetur Quadratura."

Quae hac Analysi demonstrantur.

Sit x una $3n$ Coordinatarum, sit m massa puncti, ad quod ea pertinet, ponatur $\frac{dx}{dt} = x'$, habeanturque $3n$ aequationes differentiales $m\frac{dx'}{dt} = X$, designante X functionem $3n$ Coordinatarum homogeneam i^{ti} ordinis. Ad quantitates analogas denotandas indices subscriptos adhibebo. Summationibus semper ad omnes $3n$ Coordinatas extensis, pono

$$\Sigma m x^2 = r^2,\quad x = rq,\quad x' = p\sqrt{r^{i+1}},\quad r' = \varrho\sqrt{r^{i+1}},\quad X = r^i Q,$$

unde quantitates Q erunt solarum quantitatum q functiones et ipsae homogeneae i^{ti} ordinis. His statutis obtinetur

$$(27)\quad \begin{cases} q' = \frac{dq}{dt} = \frac{x'}{r} - \frac{xr'}{r^2} = \sqrt{r^{i-1}}.(p - q\varrho), \\ p' = \frac{dp}{dt} = \frac{X}{m\sqrt{r^{i+1}}} - \frac{i+1}{2}\cdot\frac{x'r'}{\sqrt{r^{i+3}}} = \sqrt{r^{i-1}}.\left(\frac{Q}{m} - \tfrac{1}{2}(i+1)p\varrho\right), \\ \Sigma mqp = \varrho. \end{cases}$$

Hinc inter variabilem r et $6n$ variabiles q et p obtinentur $6n$ aequationes differentiales primi ordinis:

$$(28)\quad \begin{cases} dr : dq : dq_1 : \ldots : dp : dp_1 : \ldots \\ = r\varrho : p - q\varrho : p - q_1\varrho : \ldots : \frac{Q}{m} - \frac{i+1}{2}p\varrho : \frac{Q_1}{m_1} - \frac{i+1}{2}p_1\varrho : \ldots, \end{cases}$$

in quibus suppono ipsius ϱ substitutum esse valorem Σmqp. Si de parte dextra $r\varrho$, de laeva dr reiicitur, abeunt formulae (28) in $6n-1$ aequationes differentiales inter $6n$ variabiles q et p.

Sequitur e (28):

$$dr : \tfrac{1}{2} d\Sigma mqq = r : 1 - \Sigma mqq,$$

unde, designante c Constantem arbitrariam, fit

$$(29) \quad r^2(1-\Sigma mqq) = c.$$

Valente principio virium vivarum, designet K functionem ipsarum q homogeneam $(i+1)^{ti}$ ordinis $= \frac{1}{i+1}\Sigma q'Q = \int \Sigma Q dq$, atque h alteram Constantem arbitrariam, obtinetur

$$(30) \quad r^{i+1}(K-\tfrac{1}{2}\Sigma mpp) = h.$$

Vocemus M Multiplicatorem aequationum differentialium (28), erit

$$d\log M + \frac{U dr}{r\varrho} = 0,$$

siquidem U designat summam quantitatum $r\varrho$, $p-q\varrho$, etc., $\frac{Q}{m} - \frac{i+1}{2} p\varrho$ etc., respective secundum variabiles r, q, etc., p etc. differentiatarum. Quae summa, cum sit $\frac{\partial\varrho}{\partial q} = mp$, $\frac{\partial\varrho}{\partial p} = mq$, evadit

$$U = \varkappa\varrho, \quad \text{ubi} \quad \varkappa = 1 - \tfrac{1}{2}(i+3)(3n+1),$$

unde sequitur

$$(31) \quad d\log M = -\varkappa d\log r, \quad M = r^{-\varkappa}.$$

In quaestione proposita non adhibendum est Integrale completum (29), sed particulare, pro quo fit $c = 0$; substitutiones enim adhibitae suppeditant aequationem

$$\Sigma mqq = 1,$$

cuius ope $3n$ variabiles q ad alias $3n-1$ variabiles w reducere licet. Vocemus H Determinans functionale $3n-1$ quantitatum w et quantitatis $1-\Sigma mqq$, $3n$ variabilium q respectu formatum, sintque aequationes differentiales reductae

$$(32) \quad \begin{cases} dr : dw : dw_1 : \ldots : dp : dp_1 : \ldots \\ = r\varrho : W : W_1 : \ldots : P : P_1 : \ldots, \end{cases}$$

secundum regulas generales fit aequationum (32) Multiplicator

$$N = \frac{M}{Hr^2} = \frac{1}{Hr^{\varkappa+2}}.$$

Qui satisfacere debet aequationi

$$(33) \quad d\log N + \frac{d\log r}{\varrho}\left\{\varrho + \frac{\partial W}{\partial w} + \frac{\partial W_1}{\partial w_1} + \cdots + \frac{\partial P}{\partial p} + \frac{\partial P_1}{\partial p_1} + \cdots\right\} = 0.$$

Si vocamus L Multiplicatorem $6n-2$ aequationum differentialium primi ordinis inter $3n-1$ variabiles w et $3n$ variabiles p locum habentium,

$$(34) \quad dw : dw_1 : \ldots : dp : dp_1 : \ldots = W : W_1 : \ldots : P : P_1 : \ldots,$$

determinatur L formula

$$0 = d\log L + \frac{dw}{W}\left\{\frac{\partial W}{\partial w} + \frac{\partial W_1}{\partial w_1} + \cdots + \frac{\partial P}{\partial p} + \frac{\partial P_1}{\partial p_1} + \cdots\right\};$$

unde, cum e (32) sit $\frac{dw}{W} = \frac{d\log r}{\varrho}$, e (33) sequitur

$$d\log L = d\log Nr,$$

ideoque aequationum (34) fit Multiplicator

$$(35) \quad L = rN = \frac{1}{H.r^{\varkappa+1}} = \frac{1}{H.r^{2-\frac{1}{2}(i+3)(3n+1)}}.$$

Aequationibus (34) complete integratis, quantitas L per theoremata initio huius Commentationis proposita obtinetur formatione Determinantis functionalis, ideoque variabilis r ope aequationis (35) absque Quadratura per variabiles w et p determinabitur. Si conservatio virium vivarum valet, dabitur r aequatione (30), unde eo casu dato variabilis r valore vice versa aequationum differentialium (34) suppeditatur Multiplicator

$$(36) \quad L = \frac{1}{H.(K-\frac{1}{2}\Sigma mpp)^{\frac{3n-1}{i+1}+\frac{3n+1}{2}}}.$$

Seorsim examinemus casum particularem $h = 0$, quo fieri non potest, ut ipsius r per quantitates w et p determinatio ex aequatione (30) petatur. Eo casu ope aequationum

$$\Sigma mqq = 1, \quad \tfrac{1}{2}\Sigma mpp = K$$

poterunt $6n$ quantitates q et p ad $6n-2$ alias quantitates v reduci. Sint aequationes differentiales reductae

$$(37) \quad dr : dv_1 : dv_2 : \ldots : dv_{6n-2} = r\varrho : V_1 : V_2 : \ldots : V_{6n-2},$$

sitque G Determinans functionale $6n-2$ quantitatum v duarumque Σmqq et $K-\frac{1}{2}\Sigma mqq$, $6n$ variabilium q et p respectu formatum: secundum regulas generales Cap. II. traditas erit aequationum differentialium reductarum (37) Multiplicator

$$\mu = \frac{M}{G.r^{i+3}} = \frac{1}{G.r^{\varkappa+i+3}},$$

denominatore r^{i+3} inde proveniente, quod in aequationibus (29) et (30) functiones Constantibus arbitrariis c et h aequatae per r^2 et r^{i+1} multiplicantur. Eadem ratione, qua supra Multiplicatorem L e N deduxi, sequitur, $6n-3$ aequationum differentialium primi ordinis, inter $6n-2$ variabiles v locum habentium,

$$(38)\quad dv_1 : dv_2 : \ldots : dv_{6n-2} = V_1 : V_2 : \ldots : V_{6n-2}$$

Multiplicatorem fieri

$$(39)\quad \nu = \mu r = \frac{1}{G.r^{\varkappa+i+2}} = \frac{1}{G}\cdot r^{\frac{1}{2}(i+3)(3n-1)}.$$

Aequationibus (38) complete integratis, Multiplicator ν Determinante functionali datur, ideoque ope formulae (39) variabilis r valor per quantitates v sine Quadratura determinatur. Qui insuper in Constantem arbitrariam ducendus est, quippe proportionalis est potestati Multiplicatoris, quem factore constante arbitrario afficere licet.

§. 29.

Principium ultimi Multiplicatoris applicatur ad systema liberum punctorum materialium in medio resistente motum. De cometa in aethere resistente circa solem moto.

Determinatio Multiplicatoris etiam in quibusdam problematis mechanicis succedit, in quibus viribus sollicitantibus aliae accedunt e medii resistentia natae, veluti in motu puncti in medio resistente circa centrum fixum, versus quod secundum legem Neutonianam attrahitur.

Sint rursus puncti massa m_i praediti Coordinatae orthogonales x_i, y_i, z_i, sit $x_i' = \frac{dx_i}{dt}$, $y_i' = \frac{dy_i}{dt}$, $z_i' = \frac{dz_i}{dt}$, atque puncti velocitas

$$v_i = \sqrt{x_i'x_i' + y_i'y_i' + z_i'z_i'}.$$

Si puncta moventur in medio, quod cuiusque motui in directione tangentis orbitae eius resistit, viribus massam m_i secundum Coordinatarum directiones sollicitantibus X_i, Y_i, Z_i, quae solarum Coordinatarum et, si placet, temporis t functiones esse supponuntur, accedunt vires resistentia medii provenientes

$$-m_i f_i V_i \cdot \frac{x_i'}{v_i},\quad -m_i f_i V_i \cdot \frac{y_i'}{v_i},\quad -m_i f_i V_i \cdot \frac{z_i'}{v_i},$$

ubi V_i est solius v_i functio resistentiae legem exprimens atque f_i, si forma cor-

poris m_i non respicitur, est solarum x_i, y_i, z_i functio aequalis densitati medii in puncto m_i, divisae per massam m_i et multiplicatae per Constantem superficiei corporis m_i proportionalem. Est igitur motus systematis liberi punctorum materialium determinandus per systema aequationum differentialium secundi ordinis huiusmodi:

$$(1)\quad \begin{cases} \dfrac{d^2x_i}{dt^2} = \dfrac{1}{m_i}X_i - f_iV_i\cdot\dfrac{x_i'}{v_i}, \\[2mm] \dfrac{d^2y_i}{dt^2} = \dfrac{1}{m_i}Y_i - f_iV_i\cdot\dfrac{y_i'}{v_i}, \\[2mm] \dfrac{d^2z_i}{dt^2} = \dfrac{1}{m_i}Z_i - f_iV_i\cdot\dfrac{z_i'}{v_i}. \end{cases}$$

Quarum aequationum differentialium Multiplicator M, cum functiones X_i, Y_i, Z_i, f_i ab ipsis x_i', y_i', z_i' vacuae supponantur, definitur per formulam differentialem

$$\frac{d\log M}{dt} = \Sigma f_i\left\{\frac{\partial(V_iv_i^{-1}.x_i')}{\partial x_i'} + \frac{\partial(V_iv_i^{-1}.y_i')}{\partial y_i'} + \frac{\partial(V_iv_i^{-1}.z_i')}{\partial z_i'}\right\},$$

sive

$$(2)\quad \frac{d\log M}{dt} = \Sigma f_i\left\{2V_iv_i^{-1} + \frac{dV_i}{dv_i}\right\}.$$

Si motus in plano fit, aequationis (2) loco habetur

$$\frac{d\log M}{dt} = \Sigma f_i\left\{V_iv_i^{-1} + \frac{dV_i}{dv_i}\right\}.$$

Si motus in eadem recta fit, habetur

$$\frac{d\log M}{dt} = \Sigma f_i\frac{dV_i}{dv_i},$$

unde fit $M = 1$, si V_i est constans.

Sit $V_i = v_i$ sitque medium uniforme ideoque quantitates f_i constantes; sequitur e (2):

$$(3)\quad M = e^{3\Sigma f_i\cdot t}\,{}^{*)}.$$

Haec docet formula, *si motus fiat in medio uniformi, cuius resistentia velocitati directe proportionalis sit, atque vires sollicitantes* X_i, Y_i, Z_i *a solis Coordinatis pendeant, post omnia inter quantitates* x_i, y_i, z_i, x_i', y_i', z_i' *inventa Integralia ultimo loco* t *per Coordinatam aliquam sine nova Quadratura exprimi posse.*

*) Pro motu in plano fit eo casu $M = e^{2\Sigma f_i\cdot t}$, pro motu in eadem recta $M = e^{\Sigma f_i\cdot t}$.

Sint enim pro numero n punctorum materialium $6n-1$ Integralia inventa

$$F_1 = \alpha_1, \quad F_2 = \alpha_2, \quad \ldots, \quad F_{6n-1} = \alpha_{6n-1},$$

ubi α_1, α_2, etc. sunt Constantes arbitrariae; sit x una quaecunque Coordinatarum atque Δ Determinans functionum F_1, F_2, etc., quantitatum respectu omnium $x_i, y_i, z_i, x_i', y_i', z_i'$ praeter x formatum: sequitur e (3) secundum Multiplicatoris definitionem initio huius Commentationis traditam:

$$(4) \quad 3t\Sigma f_i + \tau = \log \frac{\Delta}{x'},$$

designante τ novam Constantem arbitrariam. Si virium sollicitantium expressiones X_i, Y_i, Z_i praeter mobilium Coordinatas ipsam quoque variabilem t continent, hanc non amplius separare licet; at docet formula (3), *constante Multiplicatore M ultimam integrationem absolvi Quadraturis.*

Ponamus, systema punctorum materialium sive liberum sive certis conditionibus subiectum, si in medio non resistente moveretur, *conservatione arearum* gaudere, valebunt pro motu in medio resistente tres aequationes:

$$(5) \quad \begin{cases} d\Sigma m_i(y_i z_i' - z_i y_i') = -\Sigma m_i f_i \dfrac{V_i}{v_i}(y_i z_i' - z_i y_i')dt, \\[2ex] d\Sigma m_i(z_i x_i' - x_i z_i') = -\Sigma m_i f_i \dfrac{V_i}{v_i}(z_i x_i' - x_i z_i')dt, \\[2ex] d\Sigma m_i(x_i y_i' - y_i x_i') = -\Sigma m_i f_i \dfrac{V_i}{v_i}(x_i y_i' - y_i x_i')dt. \end{cases}$$

Hinc si rursus $V_i = v_i$ et quantitates f_i omnes eidem Constanti f aequantur, sequitur

$$(6) \quad \begin{cases} \Sigma m_i(y_i z_i' - z_i y_i') = a e^{-ft}, \\ \Sigma m_i(z_i x_i' - x_i z_i') = b e^{-ft}, \\ \Sigma m_i(x_i y_i' - y_i x_i') = c e^{-ft}, \end{cases}$$

designantibus a, b, c Constantes arbitrarias. Patet e formulis (6), *si elementa omnia sphaerica eiusdemque densitatis et magnitudinis supponantur, atque systema eorum in motu in vacuo conservatione arearum gauderet, eandem locum habere, si motus fiat in medio uniformi, cuius resistentia velocitati proportionalis est, eandemque fore plani invariabilis positionem; summam arearum autem inde a tempore $t = 0$ descriptarum et per massas multiplicatarum non sicuti in vacuo proportionalem fore tempori t, sed quantitati*

$$1 - \frac{1}{e^{ft}},$$

designante f Constantem positivam, ideoque, tempore in infinitum crescente, ad limitem crescere finitum. Ubi systema liberum est ideoque e (6) et (3) constat ipsius M valor per quantitates $x_i, y_i, z_i, x_i', y_i', z_i'$ expressus, docet principium ultimi Multiplicatoris, praeter tria cognita Integralia prima (6) adhuc *ultimum Integrale, inter quantitates $x_i, y_i, z_i, x_i', y_i', z_i'$ locum habens, Quadraturis absolvi posse.*

Iam unius puncti liberi consideremus motum planum in medio resistente. Qui motus definitur duabus aequationibus differentialibus secundi ordinis

$$(7) \quad \begin{cases} \dfrac{d^2x}{dt^2} = X - f \cdot \dfrac{x'V}{v}, \\[2ex] \dfrac{d^2y}{dt^2} = Y - f \cdot \dfrac{y'V}{v}, \end{cases}$$

ubi X, Y, f Coordinatarum orthogonalium x et y, atque V velocitatis $v = \sqrt{x'x' + y'y'}$ functiones supponuntur. Aequationum (7) Multiplicator M definitur formula differentiali

$$(8) \quad \frac{d\log M}{dt} = f\left\{\frac{\partial(x'v^{-1}V)}{\partial x'} + \frac{\partial(y'v^{-1}V)}{\partial y'}\right\} = f\left\{v^{-1}V + \frac{dV}{dv}\right\}.$$

Ponamus, vim sollicitantem constanter dirigi versus centrum fixum, quod sit initium Coordinatarum, sive esse $X : Y = x : y$, sequitur e (7):

$$(9) \quad \frac{d\log(xy' - yx')}{dt} = -f \cdot \frac{V}{v}.$$

Unde, si $V = v^n$, e (8) et (9) eruitur, quaecunque sit functio f,

$$(10) \quad M = \frac{1}{(xy' - yx')^{n+1}}.$$

Si vis attractiva est functio radii vectoris r sive distantiae a centro attractionis, quam functionem designemus per

$$F'(r) = \frac{dF(r)}{dr} = -\frac{Xdx + Ydy}{dr},$$

Multiplicatorem pro lege resistantiae adhuc generaliore assignare licet. Scilicet eo casu e (7) sequitur formula

$$(11) \quad d\{\tfrac{1}{2}vv + F(r)\} = -f.vV.dt.$$

Qua iuncta aequationi (9) patet, si a et b Constantes sint, assignari posse Integrale expressionis

$$fV\left(av + \frac{b}{v}\right)dt = -ad\{\tfrac{1}{2}vv + F(r)\} - bd\log(xy' - yx').$$

Expressione ad laevam aequiparata huic

$$f\left(\frac{V}{v}+\frac{dV}{dv}\right)dt = d\log M,$$

eruitur

$$(12)\quad V = v^{b-1}e^{\frac{1}{2}avv}.$$

Qua resistentiae lege supposita, fit

$$(13)\quad M = \frac{e^{-a[\frac{1}{2}vv+F(r)]}}{(xy'-yx')^b}.$$

Pro motibus incitatissimis, sicuti sunt cometarum, resistentiae lex formula (12) expressa non a rerum natura adhorrere videtur, praesertim si Constanti a valor perparvus tribuitur.

Introducendo Coordinatas polares sit

$$x = r\cos\varphi,\quad y = r\sin\varphi,\quad r' = \frac{dr}{dt},\quad \varphi' = \frac{d\varphi}{dt},$$

unde

$$vv = r'r'+rr\varphi'\varphi',$$
$$xy'-yx' = rr\varphi' = r\sqrt{vv-r'r'}.$$

Ponamus

$$\tfrac{1}{2}vv+F(r) = \tfrac{1}{2}(x'x'+y'y')+F(r) = \alpha,$$
$$xy'-yx' = rr\varphi' = \beta,$$

fit

$$\alpha = \tfrac{1}{2}r'r'+\tfrac{1}{2}\frac{\beta\beta}{rr}+F(r),$$

unde

$$(14)\quad r' = \sqrt{2\alpha-\frac{\beta\beta}{rr}-2F(r)}.$$

Hinc, cum sit $r'dt = dr$, sequitur e (9) et (11):

$$(15)\quad \begin{cases} \dfrac{d\alpha}{dr} = -\dfrac{fvV}{\sqrt{2\alpha-\frac{\beta\beta}{rr}-2F(r)}}, \\[2ex] \dfrac{d\beta}{dr} = -\dfrac{\beta fv^{-1}V}{\sqrt{2\alpha-\frac{\beta\beta}{rr}-2F(r)}}. \end{cases}$$

Si motus propositus est motus cometae circa solem, atque densitas aetheris solem circumdantis functioni distantiae a sole aequatur, fit f solius r functio. Porro cum sit V solius v functio, ope aequationis

$$v = \sqrt{2\alpha-2F(r)}$$

quantitates vV et $v^{-1}V$ per α et r exprimere licet. Unde idonea variabilium electione effectum est, *ut motus cometae circa solem in aethere resistente tantum pendeat ab integratione duarum aequationum differentialium primi ordinis inter tres variabiles α, β, r; qua transacta si determinantur α et β per r, obtinentur φ et t per Quadraturas:*

$$(16)\quad \begin{cases} \varphi = \displaystyle\int \frac{\beta dr}{rr\sqrt{2\alpha-\frac{\beta\beta}{rr}-2F(r)}}, \\[2ex] t = \displaystyle\int \frac{dr}{\sqrt{2\alpha-\frac{\beta\beta}{rr}-2F(r)}}. \end{cases}$$

Antecedentia valent, quaecunque sit resistentiae lex sive quaecunque sit V ipsius v functio. *Ubi autem aetheris, in quo cometa circa solem movetur, resistentia potestati velocitatis cuicunque proportionalis est sive etiam legem generaliorem sequitur expressam formula $V = v^{b-1}e^{\frac{1}{2}avv}$, in qua a et b Constantes quascunque designant, sive aether uniformis sive cum distantia a sole secundum quamcunque legem variabilis sit, quaecunque sit vis attractiva solis, unico cognito Integrali reliquae tres integrationes per Quadraturas absolvuntur.* Nimirum determinata V per formulam (12), constat per formulam (13) aequationum differentialium propositarum (7) Multiplicator M; eo autem cognito, etiam dabitur Multiplicator M_1 aequationum differentialium, quae e (7) obtinentur loco ipsarum x, y, x', y' quantitates r, φ, α, β introducendo,

$$\frac{dr}{dt} = \frac{1}{\sqrt{2\alpha-\frac{\beta\beta}{rr}-2F(r)}},$$

$$\frac{d\varphi}{dt} = \frac{\beta}{rr},\quad \frac{d\alpha}{dt} = -fvV,\quad \frac{d\beta}{dt} = -\beta fv^{-1}V.$$

Etenim aequatur $\frac{M_1}{M}$ Determinanti quantitatum x, y, x', y', variabilium r, φ, α, β respectu formato, unde, si reputamus, ipsarum x et y expressiones quantitates α et β non continere, fit

$$M_1 = \left(\frac{\partial x}{\partial r}\cdot\frac{\partial y}{\partial\varphi}-\frac{\partial x}{\partial\varphi}\cdot\frac{\partial y}{\partial r}\right)\left(\frac{\partial x'}{\partial\alpha}\cdot\frac{\partial y'}{\partial\beta}-\frac{\partial x'}{\partial\beta}\cdot\frac{\partial y'}{\partial\alpha}\right)\cdot M$$
$$= \frac{rM}{\frac{\partial\alpha}{\partial x'}\cdot\frac{\partial\beta}{\partial y'}-\frac{\partial\alpha}{\partial y'}\cdot\frac{\partial\beta}{\partial x'}} = \frac{rM}{xx'+yy'} = \frac{M}{r'}.$$

Si uti in (15) variabilem r loco ipsius t pro independente adhibemus, Multiplicator antecedens in r' ducendus est, unde in ipsum M redimus, qui ponendo $V = v^{b-1}e^{\frac{1}{2}avv}$ secundum (13) invenitur

$$(17)\quad M = \beta^{-b}e^{-a\alpha}.$$

Qui valor cum non afficiatur variabilibus φ et t iisque non magis afficiantur differentialium $\frac{d\alpha}{dr}$ et $\frac{d\beta}{dr}$ valores (15), erit $M = \beta^{-b}e^{-a\alpha}$ etiam Multiplicator duarum aequationum differentialium primi ordinis (15), inter tres variabiles r, α, β locum habentium.

Quod ut directe pateat, pono

$$(18)\quad r\gamma = \frac{\beta}{v} = \frac{\beta}{\sqrt{2\alpha - 2F(r)}},$$

unde

$$r' = \sqrt{2\alpha - 2F(r) - \frac{\beta\beta}{rr}} = v\sqrt{1-\gamma\gamma},$$

$$r\frac{\partial\gamma}{\partial\alpha} = \frac{-\beta}{v^3},\quad r\frac{\partial\gamma}{\partial\beta} = \frac{1}{v}.$$

Ubi insuper brevitatis causa vocamus R solius r functionem

$$(19)\quad r^{-(b-1)}f.e^{-aF(r)} = R,$$

fit

$$(20)\quad rvf.MV = rv^b\beta^{-b}.fe^{\frac{1}{2}avv-a\alpha} = R.\gamma^{-b}.$$

Quibus substitutis si elementum independens dr Multiplicatori M proportionale statuimus, aequationes differentiales (9) evadunt:

$$(21)\quad dr:d\alpha:d\beta = \beta^{-b}e^{-a\alpha}: -R\frac{\gamma^{-b}}{\sqrt{1-\gamma\gamma}}\cdot\frac{\partial\gamma}{\partial\beta} : R\frac{\gamma^{-b}}{\sqrt{1-\gamma\gamma}}\cdot\frac{\partial\gamma}{\partial\alpha}.$$

Quam patet ita comparatam esse formulam, ut, dextris partibus vocatis A, B, C, fiat

$$(22)\quad \frac{\partial A}{\partial r} + \frac{\partial B}{\partial\alpha} + \frac{\partial C}{\partial\beta} = \frac{\partial B}{\partial\alpha} + \frac{\partial C}{\partial\beta} = 0,$$

sicuti fieri debet.

Sint u et w duae quaecunque variabilium r, α, β functiones, atque obtineatur e (15) sive e (21)

$$dr:du:dw = \beta^{-b}e^{-a\alpha}:D:E.$$

Sit porro inventum aequationum differentialium (15) sive (21) Integrale, Constante arbitraria c affectum, cuius ope exprimantur r, α, β per c, u, w, ponaturque

$$\frac{\partial r}{\partial c}\left\{\frac{\partial\alpha}{\partial u}\cdot\frac{\partial\beta}{\partial w} - \frac{\partial\alpha}{\partial w}\cdot\frac{\partial\beta}{\partial u}\right\} + \frac{\partial r}{\partial u}\left\{\frac{\partial\alpha}{\partial w}\cdot\frac{\partial\beta}{\partial c} - \frac{\partial\alpha}{\partial c}\cdot\frac{\partial\beta}{\partial w}\right\} + \frac{\partial r}{\partial w}\left\{\frac{\partial\alpha}{\partial c}\cdot\frac{\partial\beta}{\partial u} - \frac{\partial\alpha}{\partial u}\cdot\frac{\partial\beta}{\partial c}\right\} = \Delta;$$

sequitur e principio ultimi Multiplicatoris altera aequatio integralis

$$\int\Delta\{E\,du - D\,dw\} = \text{Const.},$$

ubi, et ipsis D et E per u, w, c expressis, sub integrationis signo differentiale completum subest.

§. 30.

De Multiplicatore aequationum differentialium isoperimetricarum.

Sit U data functio variabilis independentis t, dependentium x, y, z, etc. et quotientium earum differentialium x', x'', etc., y', y'', etc., z', z'', etc. etc. Si proponitur problema, functiones x, y, z, etc. ita determinandi, ut Integrale

$$\int U\,dt$$

maximum minimumve evadat seu generalius, ut eius Integralis variatio evanescat, constat, problematis solutionem pendere ab integratione systematis aequationum differentialium:

$$0 = \frac{\partial U}{\partial x} - \frac{d\frac{\partial U}{\partial x'}}{dt} + \frac{d^2\frac{\partial U}{\partial x''}}{dt^2} - \text{etc.},$$

$$0 = \frac{\partial U}{\partial y} - \frac{d\frac{\partial U}{\partial y'}}{dt} + \frac{d^2\frac{\partial U}{\partial y''}}{dt^2} - \text{etc.},$$

$$0 = \frac{\partial U}{\partial z} - \frac{d\frac{\partial U}{\partial z'}}{dt} + \frac{d^2\frac{\partial U}{\partial z''}}{dt^2} - \text{etc. etc.}$$

Quas in sequentibus vocabo *aequationes differentiales isoperimetricas*, cum problema, quod ab earum integratione pendet, nomine licet improprio isoperimetrici appellari soleat. Quaeram aequationum differentialium isoperimetricarum Multiplicatorem.

Inchoabo a casu, quo ipsa U praeter variabilem independentem t unicam continet functionem incognitam x una cum eius differentialibus x', x'', ..., $x^{(n)}$. Eo casu unica integranda est aequatio differentialis $2n^{ti}$ ordinis

$$(1)\quad 0 = V = \frac{\partial U}{\partial x} - \frac{d\frac{\partial U}{\partial x'}}{dt} + \frac{d^2\frac{\partial U}{\partial x''}}{dt^2} - \cdots \pm \frac{d^n\frac{\partial U}{\partial x^{(n)}}}{dt^n}.$$

Ex aequatione (1) si eruitur quantitatis $x^{(2n)}$ valor

$$x^{(2n)} = A,$$

huius aequationis Multiplicator M secundum (5) §. 14 definitur formula differentiali

$$\frac{d\log M}{dt} = -\frac{\partial A}{\partial x^{(2n-1)}} = \frac{\frac{\partial V}{\partial x^{(2n-1)}}}{\frac{\partial V}{\partial x^{(2n)}}}.$$

E $n+1$ expressionis V terminis bini ultimi soli continent quantitatem $x^{(2n-1)}$, solus ultimus quantitatem $x^{(2n)}$, unde fit

$$(2)\quad \begin{cases} (-1)^n \dfrac{\partial V}{\partial x^{(2n-1)}} = \dfrac{\partial \dfrac{d^n \frac{\partial U}{\partial x^{(n)}}}{dt^n}}{\partial x^{(2n-1)}} - \dfrac{\partial \dfrac{d^{n-1} \frac{\partial U}{\partial x^{(n-1)}}}{dt^{n-1}}}{\partial x^{(2n-1)}}, \\ (-1)^n \dfrac{\partial V}{\partial x^{(2n)}} = \dfrac{\partial \dfrac{d^n \frac{\partial U}{\partial x^{(n)}}}{dt^n}}{\partial x^{(2n)}}. \end{cases}$$

Quantitatum ad dextram valores suppeditat formula generalis, quam in variis occasionibus utilem hic apponam.

Sit W functio quaecunque variabilis independentis t, dependentis x atque ipsius x quotientium differentialium x', x'', etc.; fit

$$\delta \frac{d^m W}{dt^m} = \frac{d^m(\delta W)}{dt^m} = \frac{d^m\left\{\frac{\partial W}{\partial t}\delta t + \frac{\partial W}{\partial x}\delta x + \frac{\partial W}{\partial x'}\delta x' + \frac{\partial W}{\partial x''}\delta x'' + \text{ etc.}\right\}}{dt^m}.$$

Factis differentiationibus et ubique substituta formula

$$\frac{d^i(\delta x^{(h)})}{dt^i} = \delta \frac{d^i x^{(h)}}{dt^i} = \delta x^{(h+i)},$$

eruitur quantitas in $\delta x^{(\varkappa)}$ ducta:

$$(3)\quad \frac{\partial \frac{d^m W}{dt^m}}{\partial x^{(\varkappa)}} = \frac{d^m \frac{\partial W}{\partial x^{(\varkappa)}}}{dt^m} + m\cdot \frac{d^{m-1} \frac{\partial W}{\partial x^{(\varkappa-1)}}}{dt^{m-1}} + \frac{m(m-1)}{1.2}\cdot \frac{d^{m-2} \frac{\partial W}{\partial x^{(\varkappa-2)}}}{dt^{m-2}} + \text{etc.},$$

quae formula, si $m \geqq \varkappa$, usque ad terminum

$$\frac{m(m-1)(m-2)\dots(m-\varkappa+1)}{1.2\dots\varkappa}\cdot \frac{d^{m-\varkappa} \frac{\partial W}{\partial x}}{dt^{m-\varkappa}},$$

si $m \leqq \varkappa$, usque ad terminum

$$\frac{\partial W}{\partial x^{(\varkappa-m)}}$$

continuanda est. Posteriore casu formula (3) etiam hoc modo exhiberi potest:

$$(4)\quad \frac{\partial \frac{d^m W}{dt^m}}{\partial x^{(\varkappa)}} = \frac{\partial W}{\partial x^{(\varkappa-m)}} + m\cdot \frac{d \frac{\partial W}{\partial x^{(\varkappa-m+1)}}}{dt} + \frac{m(m-1)}{1.2}\cdot \frac{d^2 \frac{\partial W}{\partial x^{(\varkappa-m+2)}}}{dt^2} + \text{ etc.}$$

Formulae antecedentes (3) et (4) immutatae manent, si functio W praeter variabilem dependentem x eiusque quotientes differentiales alias dependentes y, z, etc. earumque quotientes differentiales continet. Si functionem W plures variabiles independentes dependentesque earumque differentialia partialia afficiunt, eamque secundum diversas variabiles independentes diversis vicibus iteratis complete differentiamus, huius quoque differentialis completi differentialia partialia simili ratione inveniuntur.

Ponamus, ipsius x differentiale n^{tum} altissimum esse, quod in expressione W obveniat, sequitur e (4), si $\varkappa = m+n$,

$$(5)\quad \frac{\partial \frac{d^m W}{dt^m}}{\partial x^{(m+n)}} = \frac{\partial W}{\partial x^{(n)}},$$

si $\varkappa = m+n-1$,

$$(6)\quad \frac{\partial \frac{d^m W}{dt^m}}{\partial x^{(m+n-1)}} = \frac{\partial W}{\partial x^{(n-1)}} + m\frac{d \frac{\partial W}{\partial x^{(n)}}}{dt}.$$

Unde ponendo $m = n$, $m = n-1$ prodit

$$\frac{\partial \frac{d^n \frac{\partial U}{\partial x^{(n)}}}{dt^n}}{\partial x^{(2n)}} = \frac{\partial^2 U}{\partial x^{(n)}\partial x^{(n)}}, \quad \frac{\partial \frac{d^{n-1} \frac{\partial U}{\partial x^{(n-1)}}}{dt^{n-1}}}{\partial x^{(2n-1)}} = \frac{\partial^2 U}{\partial x^{(n)}\partial x^{(n-1)}},$$

$$\frac{\partial \frac{d^n \frac{\partial U}{\partial x^{(n)}}}{dt^n}}{\partial x^{(2n-1)}} = \frac{\partial^2 U}{\partial x^{(n)}\partial x^{(n-1)}} + n\frac{d \frac{\partial^2 U}{\partial x^{(n)}\partial x^{(n)}}}{dt}.$$

Quibus valoribus in formulis (2) substitutis, eruitur

$$(-1)^n \frac{\partial V}{\partial x^{(2n)}} = \frac{\partial^2 U}{\partial x^{(n)}\partial x^{(n)}},$$

$$(-1)^n \frac{\partial V}{\partial x^{(2n-1)}} = n\frac{d \frac{\partial^2 U}{\partial x^{(n)}\partial x^{(n)}}}{dt},$$

unde iam

$$\frac{d\log M}{dt} = n\frac{d\log\dfrac{\partial^2 U}{\partial x^{(n)}\partial x^{(n)}}}{dt}; \quad M = \left\{\frac{\partial^2 U}{\partial x^{(n)}\partial x^{(n)}}\right\}^n.$$

Multiplicatoris M valore invento, principio ultimi Multiplicatoris ultima integratio Quadraturis absolvi potest. Sit ex. gr.

$$U = \sqrt{E+2Fx'+Gx'x'},$$

ubi E, F, G ipsarum t et x datae functiones sunt, unde eruitur

$$\frac{\partial^2 U}{\partial x'\partial x'} = \frac{EG-FF}{\{E+2Fx'+Gx'x'\}^{\frac{3}{2}}}.$$

Hinc, proposita aequatione differentiali

$$\frac{d\dfrac{\partial U}{\partial x'}}{dt} - \frac{\partial U}{\partial x} = 0,$$

si per primam integrationem x' per t, x et Constantem arbitrariam α expressa datur, altera integratio dabitur formula

$$\int \frac{\dfrac{\partial x'}{\partial \alpha}(EG-FF)(x'dt-dx)}{\{E+2Fx'+Gx'x'\}^{\frac{3}{2}}} = \text{Const.},$$

ubi sub integrationis signo differentiale completum subest.

Iam statuamus, functionem U praeter variabilem independentem t pluribus affici dependentibus earumque quotientibus differentialibus, omnium autem variabilium differentialia altissima ad eundem n^{tum} ordinem ascendere. Sint variabiles dependentes tres x, y, z; tres integrandae sunt aequationes differentiales

$$(7) \quad X = 0, \quad Y = 0, \quad Z = 0,$$

posito

$$(8) \quad \begin{cases} (-1)^n X = \dfrac{\partial U}{\partial x} - \dfrac{d\dfrac{\partial U}{\partial x'}}{dt} + \dfrac{d^2\dfrac{\partial U}{\partial x''}}{dt^2} - \cdots + (-1)^n \dfrac{d^n\dfrac{\partial U}{\partial x^{(n)}}}{dt^n}, \\ (-1)^n Y = \dfrac{\partial U}{\partial y} - \dfrac{d\dfrac{\partial U}{\partial y'}}{dt} + \dfrac{d^2\dfrac{\partial U}{\partial y''}}{dt^2} - \cdots + (-1)^n \dfrac{d^n\dfrac{\partial U}{\partial y^{(n)}}}{dt^n}, \\ (-1)^n Z = \dfrac{\partial U}{\partial z} - \dfrac{d\dfrac{\partial U}{\partial z'}}{dt} + \dfrac{d^2\dfrac{\partial U}{\partial z''}}{dt^2} - \cdots + (-1)^n \dfrac{d^n\dfrac{\partial U}{\partial z^{(n)}}}{dt^{(n)}}. \end{cases}$$

Ex aequationibus (7) altissimorum, quibus afficiuntur, differentialium $x^{(2n)}$, $y^{(2n)}$, $z^{(2n)}$

petantur valores, per differentialia inferiora ipsasque variabiles x, y, z, t expressi; quibus respective secundum quantitates $x^{(2n-1)}$, $y^{(2n-1)}$, $z^{(2n-1)}$ differentiatis, fiat

$$(9) \quad \frac{\partial x^{(2n)}}{\partial x^{(2n-1)}} = u, \quad \frac{\partial y^{(2n)}}{\partial y^{(2n-1)}} = v_1, \quad \frac{\partial z^{(2n)}}{\partial z^{(2n-1)}} = w_2,$$

unde aequationum differentialium (7) Multiplicator M secundum (5) §. 14 erit

$$(10) \quad \frac{d\log M}{dt} = -\{u+v_1+w_2\}.$$

Quantitates u, v_1, w_2 determinandae sunt ternis aequationum linearium systematis, quae solis terminis ad dextram positis inter se differunt:

$$(11) \quad \begin{cases} \dfrac{\partial X}{\partial x^{(2n)}}u + \dfrac{\partial X}{\partial y^{(2n)}}v + \dfrac{\partial X}{\partial z^{(2n)}}w = -\dfrac{\partial X}{\partial x^{(2n-1)}}, \\ \dfrac{\partial Y}{\partial x^{(2n)}}u + \dfrac{\partial Y}{\partial y^{(2n)}}v + \dfrac{\partial Y}{\partial z^{(2n)}}w = -\dfrac{\partial Y}{\partial x^{(2n-1)}}, \\ \dfrac{\partial Z}{\partial x^{(2n)}}u + \dfrac{\partial Z}{\partial y^{(2n)}}v + \dfrac{\partial Z}{\partial z^{(2n)}}w = -\dfrac{\partial Z}{\partial x^{(2n-1)}}, \\ \dfrac{\partial X}{\partial x^{(2n)}}u_1 + \dfrac{\partial X}{\partial y^{(2n)}}v_1 + \dfrac{\partial X}{\partial z^{(2n)}}w_1 = -\dfrac{\partial X}{\partial y^{(2n-1)}}, \\ \dfrac{\partial Y}{\partial x^{(2n)}}u_1 + \dfrac{\partial Y}{\partial y^{(2n)}}v_1 + \dfrac{\partial Y}{\partial z^{(2n)}}w_1 = -\dfrac{\partial Y}{\partial y^{(2n-1)}}, \\ \dfrac{\partial Z}{\partial x^{(2n)}}u_1 + \dfrac{\partial Z}{\partial y^{(2n)}}v_1 + \dfrac{\partial Z}{\partial z^{(2n)}}w_1 = -\dfrac{\partial Z}{\partial y^{(2n-1)}}, \\ \dfrac{\partial X}{\partial x^{(2n)}}u_2 + \dfrac{\partial X}{\partial y^{(2n)}}v_2 + \dfrac{\partial X}{\partial z^{(2n)}}w_2 = -\dfrac{\partial X}{\partial z^{(2n-1)}}, \\ \dfrac{\partial Y}{\partial x^{(2n)}}u_2 + \dfrac{\partial Y}{\partial y^{(2n)}}v_2 + \dfrac{\partial Y}{\partial z^{(2n)}}w_2 = -\dfrac{\partial Y}{\partial z^{(2n-1)}}, \\ \dfrac{\partial Z}{\partial x^{(2n)}}u_2 + \dfrac{\partial Z}{\partial y^{(2n)}}v_2 + \dfrac{\partial Z}{\partial z^{(2n)}}w_2 = -\dfrac{\partial Z}{\partial z^{(2n-1)}}. \end{cases}$$

Ponamus

$$(12) \quad \begin{cases} \dfrac{\partial^2 U}{\partial x^{(n)}\partial x^{(n)}} = A, \quad \dfrac{\partial^2 U}{\partial y^{(n)}\partial y^{(n)}} = B, \quad \dfrac{\partial^2 U}{\partial z^{(n)}\partial z^{(n)}} = C, \\ \dfrac{\partial^2 U}{\partial y^{(n)}\partial z^{(n)}} = D, \quad \dfrac{\partial^2 U}{\partial z^{(n)}\partial x^{(n)}} = E, \quad \dfrac{\partial^2 U}{\partial x^{(n)}\partial y^{(n)}} = F, \\ \dfrac{\partial^2 U}{\partial y^{(n-1)}\partial z^{(n)}} - \dfrac{\partial^2 U}{\partial z^{(n-1)}\partial y^{(n)}} = a, \\ \dfrac{\partial^2 U}{\partial z^{(n-1)}\partial x^{(n)}} - \dfrac{\partial^2 U}{\partial x^{(n-1)}\partial z^{(n)}} = b, \\ \dfrac{\partial^2 U}{\partial x^{(n-1)}\partial y^{(n)}} - \dfrac{\partial^2 U}{\partial y^{(n-1)}\partial x^{(n)}} = c. \end{cases}$$

In formulis (5) et (6) ipsi W substituendo sex functiones $\frac{\partial U}{\partial x^{(n)}}$, $\frac{\partial U}{\partial y^{(n)}}$, $\frac{\partial U}{\partial z^{(n)}}$, $\frac{\partial U}{\partial x^{(n-1)}}$, $\frac{\partial U}{\partial y^{(n-1)}}$, $\frac{\partial U}{\partial z^{(n-1)}}$, pro ipsa x autem functiones x, y, z sumendo, sequitur:

$$
(13)\quad \begin{cases}
\frac{\partial X}{\partial x^{(2n)}} = A, \quad \frac{\partial Y}{\partial x^{(2n)}} = F, \quad \frac{\partial Z}{\partial x^{(2n)}} = E, \\
\frac{\partial X}{\partial y^{(2n)}} = F, \quad \frac{\partial Y}{\partial y^{(2n)}} = B, \quad \frac{\partial Z}{\partial y^{(2n)}} = D, \\
\frac{\partial X}{\partial z^{(2n)}} = E, \quad \frac{\partial Y}{\partial z^{(2n)}} = D, \quad \frac{\partial Z}{\partial z^{(2n)}} = C, \\
\frac{\partial X}{\partial x^{(2n-1)}} = n\frac{dA}{dt}, \quad \frac{\partial Y}{\partial x^{(2n-1)}} = n\frac{dF}{dt}+c, \quad \frac{\partial Z}{\partial x^{(2n-1)}} = n\frac{dE}{dt}-b, \\
\frac{\partial X}{\partial y^{(2n-1)}} = n\frac{dF}{dt}-c, \quad \frac{\partial Y}{\partial y^{(2n-1)}} = n\frac{dB}{dt}, \quad \frac{\partial Z}{\partial y^{(2n-1)}} = n\frac{dD}{dt}+a, \\
\frac{\partial X}{\partial z^{(2n-1)}} = n\frac{dE}{dt}+b, \quad \frac{\partial Y}{\partial z^{(2n-1)}} = n\frac{dD}{dt}-a, \quad \frac{\partial Z}{\partial z^{(2n-1)}} = n\frac{dC}{dt}.
\end{cases}
$$

Hos valores substituendo, tria systemata aequationum linearium (11) evadunt:

$$
(14)\quad \begin{cases}
Au+Fv+Ew = -n\frac{dA}{dt}, \\
Fu+Bv+Dw = -n\frac{dF}{dt}-c, \\
Eu+Dv+Cw = -n\frac{dE}{dt}+b, \\
Au_1+Fv_1+Ew_1 = -n\frac{dF}{dt}+c, \\
Fu_1+Bv_1+Dw_1 = -n\frac{dB}{dt}, \\
Eu_1+Dv_1+Cw_1 = -n\frac{dD}{dt}-a, \\
Au_2+Fv_2+Ew_2 = -n\frac{dE}{dt}-b, \\
Fu_2+Bv_2+Dw_2 = -n\frac{dD}{dt}+a, \\
Eu_2+Dv_2+Cw_2 = -n\frac{dC}{dt}.
\end{cases}
$$

Quorum systematum Determinans commune si vocatur

$$(15)\quad R = ABC-AD^2-BE^2-CF^2+2DEF,$$

eorum resolutione algebraica obtinetur:

$$
(16)\quad \begin{cases}
-Ru = n\left\{ \frac{\partial R}{\partial A}\cdot\frac{dA}{dt} + \tfrac{1}{2}\frac{\partial R}{\partial F}\cdot\frac{dF}{dt} + \tfrac{1}{2}\frac{\partial R}{\partial E}\cdot\frac{dE}{dt}\right\} + \tfrac{1}{2}\frac{\partial R}{\partial F}c - \tfrac{1}{2}\frac{\partial R}{\partial E}b, \\
-Rv_1 = n\left\{ \tfrac{1}{2}\frac{\partial R}{\partial F}\cdot\frac{dF}{dt} + \frac{\partial R}{\partial B}\cdot\frac{dB}{dt} + \tfrac{1}{2}\frac{\partial R}{\partial D}\cdot\frac{dD}{dt}\right\} + \tfrac{1}{2}\frac{\partial R}{\partial D}a - \tfrac{1}{2}\frac{\partial R}{\partial F}c, \\
-Rw_2 = n\left\{ \tfrac{1}{2}\frac{\partial R}{\partial E}\cdot\frac{dE}{dt} + \tfrac{1}{2}\frac{\partial R}{\partial D}\cdot\frac{dD}{dt} + \frac{\partial R}{\partial C}\cdot\frac{dC}{dt}\right\} + \tfrac{1}{2}\frac{\partial R}{\partial E}b - \tfrac{1}{2}\frac{\partial R}{\partial D}a.
\end{cases}
$$

Quibus formulis additis, termini per a, b, c multiplicati se mutuo destruunt, unde prodit

$$\frac{d\log M}{dt} = -\{u+v_1+w_2\} = n\frac{dR}{R\,dt},$$

ideoque

$$M = R^n = \{ABC-AD^2-BE^2-CF^2+2DEF\}^n.$$

Quo valore invento, si per omnia praeter unum Integralia inventa problema in aequationem differentialem primi ordinis inter duas variabiles redit, huius quoque Multiplicator constabit.

Adiumento theorematum generalium in fine §. 16 propositorum antecedentia extendere licet ad casum, quo functio U praeter variabilem independentem numerum quemlibet dependentium continet, singularum differentialibus altissimis omnibus ad eundem ordinem ascendentibus. At si diversarum variabilium dependentium differentialia altissima in functione U non omnia ad eundem ordinem ascendunt, Multiplicatoris aequationum differentialium isoperimetricarum determinatio difficilior est. Scilicet nascitur difficultas eo, quod casu, quem innui, aequationes differentiales isoperimetricae formam normalem exuant, qua altissima diversarum variabilium differentialia per differentialia inferiora ipsasque variabiles determinantur. Reductio ad formam normalem cum molestissima ac saepe inextricabilibus difficultatibus obnoxia sit, demonstrabo sequentibus, quomodo generaliter eruere liceat formulam differentialem, qua Multiplicator definiatur, etiamsi ipsa reductio effecta non supponatur. Quae formula in problemate isoperimetrico generali proposito ipsum Multiplicatoris valorem suppeditabit.

§. 31.

De reductione aequationum differentialium ad formam normalem et formula symbolica, qua reductarum Multiplicator definiatur. Aequationum differentialium isoperimetricarum ad formam normalem reductarum Multiplicator.

Datae sint inter variabilem independentem t atque n dependentes x_1, x_2, ..., x_n totidem aequationes differentiales

$$(1)\quad F_1 = 0, \quad F_2 = 0, \quad \ldots, \quad F_n = 0,$$

non ea forma normali praeditae, quae permittat, ut differentialium singularum variabilium altissimorum valores per differentialia inferiora ipsasque variabiles exprimantur. Cuiusmodi habentur aequationes, si in earum una pluribusve altissima differentialia sive omnino desunt sive ex iis reliquarum adiumento aequationum eliminari possunt. Eo casu iteratis aequationum (1) differentiationibus formandum est systema *aequationum auxiliarium*, quarum ope totidem differentialia eliminando forma normalis eruatur. Varios modos, quibus ea operatio institui potest, in alia Commentatione tradam, quippe quae quaestio multis egregiis theorematis nititur, quae uberiorem expositionem poscunt. Hic observare sufficiat, si ad aequationes auxiliares formandas aequatio $F_i = 0$ sit λ_i vicibus iteratis differentianda, ponaturque

$$\frac{d^{\lambda_i} F_i}{dt^{\lambda_i}} = \varphi_i,$$

numeros λ_i ita comparatos esse debere, ut ex aequationibus

$$(2) \quad \varphi_1 = 0, \quad \varphi_2 = 0, \quad \ldots, \quad \varphi_n = 0$$

altissimorum differentialium in iis obvenientium

$$x_1^{(p_1)}, \quad x_2^{(p_2)}, \quad \ldots, \quad x_n^{(p_n)}$$

peti possint valores per differentialia inferiora ipsasque variabiles expressi. Unde aequationes (2) per se consideratae constituere debent aequationum differentialium systema forma normali gaudens, multo tamen altioris ordinis quam qui systemati aequationum differentialium propositarum proprius est. Aequationes enim propositas atque auxiliares praeter ipsas (2) omnes habere licet pro aequationum (2) Integralibus earum reductioni inservientibus. Quae Integralia, licet particularia, talia sunt, ut aequationum differentialium eorum ope reductarum Multiplicator e Multiplicatore aequationum (2) erui possit. Etenim si tantum aequationes (2) proponerentur, loco aequationum

$$\frac{d^{\lambda_i - 1} F_i}{dt^{\lambda_i - 1}} = 0, \quad \frac{d^{\lambda_i - 2} F_i}{dt^{\lambda_i - 2}} = 0, \quad \ldots, \quad F_i = 0$$

ad reductionem adhiberi possent aequationum (2) Integralia completa

$$\frac{d^{\lambda_i - 1} F_i}{dt^{\lambda_i - 1}} = c_1^{(i)}, \quad \frac{d^{\lambda_i - 2} F_i}{dt^{\lambda_i - 2}} = c_1^{(i)} t + c_2^{(i)}, \quad \text{etc.},$$

designantibus $c_1^{(i)}$, $c_2^{(i)}$, etc. Constantes arbitrarias. Multiplicator autem aequationum reductarum secundum §. 12 obtinetur dividendo aequationum (2) Multiplicatorem per Determinans $\lambda_1 + \lambda_2 + \cdots + \lambda_n$ functionum

$$\frac{d^{\lambda_i - 1} F_i}{dt^{\lambda_i - 1}}, \quad \frac{d^{\lambda_i - 2} F_i}{dt^{\lambda_i - 2}}, \quad \ldots, \quad F_i,$$

formatum respectu differentialium eliminandorum, idque sive Constantibus arbitrariis $c_1^{(i)}$, $c_2^{(i)}$, etc. valores generales servantur, sive iis valores tribuuntur particulares, uti in quaestione proposita, in qua omnes statuuntur evanescere.

Aequationum (2) Multiplicator definitur formula symbolica §. 16 tradita

$$(3) \quad d \log M = \delta \log \Sigma \pm A_1' A_2'' \ldots A_n^{(n)},$$

posito

$$(4) \quad A_x^{(i)} = \frac{\partial \varphi_i}{\partial x_x^{(p_x)}}, \quad \delta A_x^{(i)} = \frac{\partial \varphi_i}{\partial x_x^{(p_x - 1)}} dt.$$

Has quantitates secundum formulas (5) et (6) §. 30 sic exhibere licet:

$$(5) \quad A_x^{(i)} = \frac{\partial F_i}{\partial x_x^{(p_x - \lambda_i)}}, \quad \delta A_x^{(i)} = \frac{\partial F_i}{\partial x_x^{(p_x - \lambda_i - 1)}} dt + \lambda_i d A_x^{(i)}.$$

Unde ad condendam formulam (3) sufficiunt datae aequationes (1) numerorumque λ_1, λ_2, ..., λ_n cognitio. Observo, si ponatur

$$\Delta A_x^{(i)} = \frac{\partial F_i}{\partial x_x^{(p_x - \lambda_i - 1)}} dt + (\lambda_i - \alpha) d A_x^{(i)},$$

designante α numerum quemcunque, formulam (3) abire in hanc:

$$d \log \frac{M}{\{\Sigma \pm A_1' A_2'' \ldots A_n^{(n)}\}^{\alpha}} = \Delta \log \Sigma \pm A_1' A_2'' \ldots A_n^{(n)},$$

unde obtineri potest variationis formandae simplificatio.

In problemate isoperimetrico, quod aequatione $\delta \int U dt = 0$ continetur, expressio U praeter variabilem independentem t contineat n dependentes x_1, x_2, ..., x_n atq uedifferentialia ipsius x_1 usque ad m_1^{tum}, ipsius x_2 usque ad m_2^{tum}, etc.: erunt aequationes differentiales integrandae:

$$(6) \quad \begin{cases} 0 = F_1 = \dfrac{d^{m_1} \dfrac{\partial U}{\partial x_1^{(m_1)}}}{dt^{m_1}} - \dfrac{d^{m_1 - 1} \dfrac{\partial U}{\partial x_1^{(m_1 - 1)}}}{dt^{m_1 - 1}} + \cdots, \\ 0 = F_2 = \dfrac{d^{m_2} \dfrac{\partial U}{\partial x_2^{(m_2)}}}{dt^{m_2}} - \dfrac{d^{m_2 - 1} \dfrac{\partial U}{\partial x_2^{(m_2 - 1)}}}{dt^{m_2 - 1}} + \cdots, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ 0 = F_n = \dfrac{d^{m_n} \dfrac{\partial U}{\partial x^{(m_n)}}}{dt^{m_n}} - \dfrac{d^{m_n - 1} \dfrac{\partial U}{\partial x_n^{(m_n - 1)}}}{dt^{m_n - 1}} + \cdots. \end{cases}$$

Si m_1 omnium numerorum $m_1, m_2, \ldots, m_n$ maximus est, aequationum auxiliarium systema facile constat obtineri differentiando aequationes $F_2=0$, $F_3=0$, etc. respective m_1-m_2, m_1-m_3, etc. vicibus, unde fit

$$\lambda_1=0,\quad \lambda_2=m_1-m_2,\quad \lambda_3=m_1-m_3,\quad \ldots,\quad \lambda_n=m_1-m_n,$$
$$p_1=2m_1,\quad p_2=m_1+m_2,\quad p_3=m_1+m_3,\quad \ldots,\quad p_n=m_1+m_n.$$

Hinc eruitur:

$$(7)\quad \begin{cases} 0=\varphi_1=\dfrac{d^{m_1}\dfrac{\partial U}{\partial x_1^{(m_1)}}}{dt^{m_1}}-\dfrac{d^{m_1-1}\dfrac{\partial U}{\partial x_1^{(m_1-1)}}}{dt^{m_1-1}}+\cdots, \\ 0=\varphi_2=\dfrac{d^{m_1}\dfrac{\partial U}{\partial x_2^{(m_2)}}}{dt^{m_1}}-\dfrac{d^{m_1-1}\dfrac{\partial U}{\partial x_2^{(m_2-1)}}}{dt^{m_1-1}}+\cdots, \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ 0=\varphi_n=\dfrac{d^{m_1}\dfrac{\partial U}{\partial x_n^{(m_n)}}}{dt^{m_1}}-\dfrac{d^{m_1-1}\dfrac{\partial U}{\partial x_n^{(m_n-1)}}}{dt^{m_1-1}}+\cdots. \end{cases}$$

Unde per formulas §. 30 sequitur

$$(8)\quad \begin{cases} A_x^{(i)}=\dfrac{\partial\varphi_i}{\partial x_x^{(m_1+m_x)}}=\dfrac{\partial^2 U}{\partial x_i^{(m_i)}\,\partial x_x^{(m_x)}}, \\ \delta A_x^{(i)}=\dfrac{\partial\varphi_i}{\partial x_x^{(m_1+m_x-1)}}\,dt=m_1\dfrac{dA_x^{(i)}}{dt}+B_{i,x}\,dt, \end{cases}$$

siquidem ponitur

$$B_{i,x}=\frac{\partial^2 U}{\partial x_i^{(m_i)}\,\partial x_x^{(m_x-1)}}-\frac{\partial^2 U}{\partial x_i^{(m_i-1)}\,\partial x_x^{(m_x)}}.$$

Cum sit

$$A_x^{(i)}=A_i^{(x)}\ \text{ ideoque }\ \frac{\partial\Sigma\pm A_1'A_2''\ldots A_n^{(n)}}{\partial A_x^{(i)}}=\frac{\partial\Sigma\pm A_1'A_2''\ldots A_n^{(n)}}{\partial A_i^{(x)}},$$
$$B_{i,k}=-B_{k,i},\quad B_{i,i}=0$$

in formanda variatione (3) binorum terminorum aggregata

$$\left\{\frac{\partial\Sigma\pm A_1'A_2''\ldots A_n^{(n)}}{\partial A_x^{(i)}}B_{i,x}+\frac{\partial\Sigma\pm A_1'A_2''\ldots A_n^{(n)}}{\partial A_i^{(x)}}B_{x,i}\right\}dt$$

evanescunt, unde ipsius $d\log M$ valor (3) eruitur

$$\delta\log\Sigma\pm A_1'A_2''\ldots A_n^{(n)}=m_1\,d\log\Sigma\pm A_1'A_2''\ldots A_n^{(n)},$$

ideoque

$$(9)\quad M=\{\Sigma\pm A_1'A_2''\ldots A_n^{(n)}\}^{m_1}.$$

Qua in formula ipsis $A_x^{(i)}$ valores (8) substituendo patet, *si m_1 maximus omnium m_1, m_2, etc., aequari M potestati m_1^{tae} Determinantis functionum*

$$\frac{\partial U}{\partial x_1^{(m_1)}},\quad \frac{\partial U}{\partial x_2^{(m_2)}},\quad \ldots,\quad \frac{\partial U}{\partial x_n^{(m_n)}},$$

ipsarum $x_1^{(m_1)}$, $x_2^{(m_2)}$, etc. respectu formati.

Reductio ad formam normalem reductarumque aequationum differentialium Multiplicator sic obtinetur.

Quoniam aequationibus (2) valores quantitatum

$$x_1^{(p_1)},\quad x_2^{(p_2)},\quad \ldots,\quad x_n^{(p_n)}$$

determinantur, his quantitatibus expressiones $\varphi_1, \varphi_2, \ldots, \varphi_n$ aliae aliis afficiantur necesse est, ita ut eliminatio successiva locum habere possit. Sint

$$x_1,\quad x_2,\quad \ldots,\quad x_n$$

ipsi numeri $1, 2, \ldots, n$ inter se permutati, positoque

$$p_{x_i}=q_i,$$

statuamus, quantitates $x_{x_1}^{(q_1)}$ ipsam φ_1, $x_{x_2}^{(q_2)}$ ipsam φ_2, ..., $x_{x_n}^{(q_n)}$ ipsam φ_n afficere, quo nihil impeditur, quin functio φ_i praeter $x_{x_i}^{(q_i)}$ quantitatum $x_{x_1}^{(q_1)}$, $x_{x_2}^{(q_2)}$, etc. alias vel etiam omnes contineat. Supponamus

$$\lambda_1\leqq\lambda_2\leqq\lambda_3\ldots\leqq\lambda_{n-1}\leqq\lambda_n,$$

atque fieri

$$\lambda_1=\lambda_2=\cdots=\lambda_a=\alpha;\quad \lambda_{a+1}=\lambda_{a+2}=\cdots=\lambda_b=\beta;\quad \text{etc.},$$
$$\lambda_{q+1}=\lambda_{q+2}=\cdots=\lambda_r=\varrho;\quad \lambda_{r+1}=\lambda_{r+2}=\cdots=\lambda_n=\sigma.$$

Porro, designante μ numerum ipso λ_i non maiorem, statuamus

$$\frac{d^{\lambda_i-\mu}F_i}{dt^{\lambda_i-\mu}}=\varphi_i^{(-\mu)},\quad F_i=\varphi_i^{(-\lambda_i)}.$$

Iam ex aequationibus propositis et auxiliaribus eligamus haec $a+1$ systemata n aequationum:

$$(10)\quad \begin{cases} \varphi_1 = 0, & \varphi_2 = 0, & \ldots, & \varphi_n = 0, \\ \varphi_1^{(-1)} = 0, & \varphi_2^{(-1)} = 0, & \ldots, & \varphi_n^{(-1)} = 0, \\ \varphi_1^{(-2)} = 0, & \varphi_2^{(-2)} = 0, & \ldots, & \varphi_n^{(-2)} = 0, \\ \cdots & \cdots & \cdots & \cdots \\ F_1 = 0, & F_2 = 0, & \ldots, & F_a = 0, \; \varphi_{a+1}^{(-\alpha)} = 0, \; \varphi_{a+2}^{(-\alpha)} = 0, \; \ldots, \; \varphi_n^{(-\alpha)} = 0. \end{cases}$$

Systemate primo, secundo, etc., ultimo respective determinantur quantitates

$$\begin{array}{llll} x_{x_1}^{(q_1)}, & x_{x_2}^{(q_2)}, & \ldots, & x_{x_n}^{(q_n)}, \\ x_{x_1}^{(q_1-1)}, & x_{x_2}^{(q_2-1)}, & \ldots, & x_{x_n}^{(q_n-1)}, \\ \cdots & \cdots & \cdots & \cdots \\ x_{x_1}^{(q_1-\alpha)}, & x_{x_2}^{(q_2-\alpha)}, & \ldots, & x_{x_n}^{(q_n-\alpha)}. \end{array}$$

Unde aequationibus (10) differentialia omnia exprimuntur per alia his postremis inferiora. Eadem ratione aequationibus

$$\begin{array}{l} \varphi_{a+1}^{(-\alpha-1)} = 0, \quad \varphi_{a+2}^{(-\alpha-1)} = 0, \quad \ldots, \quad \varphi_n^{(-\alpha-1)} = 0, \\ \varphi_{a+1}^{(-\alpha-2)} = 0, \quad \varphi_{a+2}^{(-\alpha-2)} = 0, \quad \ldots, \quad \varphi_n^{(-\alpha-2)} = 0, \\ \cdots\cdots\cdots\cdots \\ F_{a+1} = 0, \quad F_{a+2} = 0, \quad \ldots, \quad F_b = 0, \quad \varphi_{b+1}^{(-\beta)} = 0, \quad \ldots, \quad \varphi_n^{(-\beta)} = 0 \end{array}$$

differentialia omnia revocantur ad alia ipsis

$$x_{x_1}^{(q_1-\alpha)}, \quad x_{x_2}^{(q_2-\alpha)}, \quad \ldots, \quad x_{x_a}^{(q_a-\alpha)}, \quad x_{x_{a+1}}^{(q_{a+1}-\beta)}, \quad \ldots, \quad x_{x_n}^{(q_n-\beta)}$$

inferiora et ita porro. Postremo advocatis aequationibus

$$\begin{array}{l} \varphi_{r+1}^{(-\varrho-1)} = 0, \quad \varphi_{r+2}^{(-\varrho-1)} = 0, \quad \ldots, \quad \varphi_n^{(-\varrho-1)} = 0, \\ \varphi_{r+1}^{(-\varrho-2)} = 0, \quad \varphi_{r+2}^{(-\varrho-2)} = 0, \quad \ldots, \quad \varphi_n^{(-\varrho-2)} = 0, \\ \cdots\cdots\cdots\cdots \\ F_{r+1} = 0, \quad F_{r+2} = 0, \quad \ldots, \quad F_n = 0, \end{array}$$

fit, ut differentialia omnia ad alia revocentur inferiora ipsis

$$(11)\quad x_{x_1}^{(q_1-\lambda_1)}, \quad x_{x_2}^{(q_2-\lambda_2)}, \quad \ldots, \quad x_{x_n}^{(q_n-\lambda_n)}.$$

Formulae, quibus ista differentialia (11) per inferiora exprimuntur, ipsum constituunt aequationum differentialium systema forma normali gaudens, ad quod propositae (1) revocari possunt. Cuius Multiplicator secundum theoremata Cap. II. proposita eruitur $\frac{M}{D}$, designante D omnium functionum

$$\begin{array}{llll} \varphi_1^{(-1)}, & \varphi_1^{(-2)}, & \ldots, & \varphi_1^{(-\lambda_1)}, \\ \varphi_2^{(-1)}, & \varphi_2^{(-2)}, & \ldots, & \varphi_2^{(-\lambda_2)}, \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_n^{(-1)}, & \varphi_n^{(-2)}, & \ldots, & \varphi_n^{(-\lambda_n)} \end{array}$$

Determinans sumtum respectu quantitatum

$$\begin{array}{llll} x_{x_1}^{(q_1-1)}, & x_{x_1}^{(q_1-2)}, & \ldots, & x_{x_1}^{(q_1-\lambda_1)}, \\ x_{x_2}^{(q_2-1)}, & x_{x_2}^{(q_2-2)}, & \ldots, & x_{x_2}^{(q_2-\lambda_2)}, \\ \cdots & \cdots & \cdots & \cdots \\ x_{x_n}^{(q_n-1)}, & x_{x_n}^{(q_n-2)}, & \ldots, & x_{x_n}^{(q_n-\lambda_n)}. \end{array}$$

Functiones enim illas nihilo aequiparando obtinemus aequationes reducendis (2) adhibitas; quantitates illae autem sunt ipsae harum aequationum ope eliminandae. Quae eliminationes vidimus successive institui posse, ita ut aequationes, quas in eadem linea horizontali posui, per se constituant systema totidem quantitatibus eliminandis sufficiens. Unde fit, ut Determinans D productum evadat σ sive λ_n Determinantium functionalium simpliciorum:

$$\begin{aligned} D = \prod_1^{\alpha} \Sigma \pm \frac{\partial \varphi_1^{(-h)}}{\partial x_{x_1}^{(q_1-h)}} \cdot \frac{\partial \varphi_2^{(-h)}}{\partial x_{x_2}^{(q_2-h)}} \cdots \frac{\partial \varphi_n^{(-h)}}{\partial x_{x_n}^{(q_n-h)}} \\ \times \prod_{\alpha+1}^{\beta} \Sigma \pm \frac{\partial \varphi_{a+1}^{(-h)}}{\partial x_{x_{a+1}}^{(q_{a+1}-h)}} \cdot \frac{\partial \varphi_{a+2}^{(-h)}}{\partial x_{x_{a+2}}^{(q_{a+2}-h)}} \cdots \frac{\partial \varphi_n^{(-h)}}{\partial x_{x_n}^{(q_n-h)}} \\ \cdots\cdots\cdots\cdots \\ \times \prod_{\varrho+1}^{\sigma} \Sigma \pm \frac{\partial \varphi_{r+1}^{(-h)}}{\partial x_{x_{r+1}}^{(q_{r+1}-h)}} \cdot \frac{\partial \varphi_{r+2}^{(-h)}}{\partial x_{x_{r+2}}^{(q_{r+2}-h)}} \cdots \frac{\partial \varphi_n^{(-h)}}{\partial x_{x_n}^{(q_n-h)}}, \end{aligned}$$

siquidem in hac formula, designante h indicem in functione aliqua f obvenientem, ipso $\prod_{\mu}^{\nu} f(h)$ designatur productum $f(\mu) f(\mu+1) f(\mu+2) \ldots f(\nu)$. Iam in formula antecedente singula Determinantia functionalia, quae idem signum Π amplectatur, observo inter se aequalia evadere eademque fore ac si ubique index $-h$ omitteretur. Unde si ponimus

$$A_f^{(i)} = \frac{\partial \varphi_i^{(-h)}}{\partial x_{x_f}^{(q_f-h)}} = \frac{\partial \varphi_i}{\partial x_{x_f}^{(q_f)}} = \frac{\partial F_i}{\partial x_{x_f}^{(q_f-\lambda_i)}},$$

obtinetur:

$$(12)\quad \begin{cases} D = \{\Sigma \pm A_1' A_2'' \ldots A_n^{(n)}\}^{\alpha} \\ \quad \times \{\Sigma \pm A_{a+1}^{(a+1)} A_{a+2}^{(a+2)} \ldots A_n^{(n)}\}^{\beta-\alpha} \\ \quad \times \{\Sigma \pm A_{b+1}^{(b+1)} A_{b+2}^{(b+2)} \ldots A_n^{(n)}\}^{\gamma-\beta} \\ \quad \cdots\cdots\cdots \\ \quad \times \{\Sigma \pm A_{r+1}^{(r+1)} A_{r+2}^{(r+2)} \ldots A_n^{(n)}\}^{\sigma-\varrho}. \end{cases}$$

Posito

$$(13)\quad \Sigma \pm A_{i+1}^{(i+1)} A_{i+2}^{(i+2)} \dots A_n^{(n)} = R_i,$$

valor antecedens fit $R_0^{a} R_a^{\beta-a} R_b^{\gamma-\beta} \dots R_r^{\sigma-\varrho}$, qui etiam sic exhiberi potest:

$$(14)\quad D = R_0^{\lambda_1} R_1^{\lambda_2-\lambda_1} R_2^{\lambda_3-\lambda_2} \dots R_{n-1}^{\lambda_n-\lambda_{n-1}},$$

qua de formula, si bini numeri se proxime insequentes λ_i et λ_{i+1} inter se aequales existunt, potestatem $R_i^{\lambda_{i+1}-\lambda_i}$ unitati aequalem reiicere licet.

Reductiones, quibus aequationes differentiales propositae ad formas normales antecedentibus assignatas revocantur, eae sunt, quae omnium simplicissimo modo efficiuntur. Pro quibus supponere licet $\alpha = 0$ sive simul de omnibus numeris $\lambda_1, \lambda_2, \dots, \lambda_n$ eorum minimum detrahere licet; nam aequationum auxiliarium (10) nonnisi ultima series ad reductionem adhibebatur. Formae normales illis reductionibus simplicissimis erutae tot existunt inter se diversae, quot modis numeri 1, 2, ..., n in talem ordinem $\varkappa_1, \varkappa_2, \dots, \varkappa_n$ disponi possunt, ut quantitates

$$x_{\varkappa_1}^{(q_1)}, \quad x_{\varkappa_2}^{(q_2)}, \quad \dots, \quad x_{\varkappa_a}^{(q_a)} \quad \text{aequationibus} \quad \varphi_1 = 0, \quad \varphi_2 = 0, \quad \dots, \quad \varphi_a = 0,$$

$$x_{\varkappa_{a+1}}^{(q_{a+1})}, \quad x_{\varkappa_{a+2}}^{(q_{a+2})}, \quad \dots, \quad x_{\varkappa_b}^{(q_b)} \quad \text{aequationibus} \quad \varphi_{a+1} = 0, \quad \varphi_{a+2} = 0, \quad \dots, \quad \varphi_b = 0,$$

. . . .

$$x_{\varkappa_{r+1}}^{(q_{r+1})}, \quad x_{\varkappa_{r+2}}^{(q_{r+2})}, \quad \dots, \quad x_{\varkappa_n}^{(q_n)} \quad \text{aequationibus} \quad \varphi_{r+1} = 0, \quad \varphi_{r+2} = 0, \quad \dots, \quad \varphi_n = 0$$

determinentur, siquidem in aequationibus illis quantitates illae solae pro incognitis, reliquae pro datis habentur. Reductiones ad has formas pauciores poscunt aequationes auxiliares eliminationesque ac si proponeretur reductio ad ullam aliam formam normalem, ex. gr. reductio vulgaris ad unicam aequationem differentialem inter duas variabiles, quae vel omnium maxime prolixa est. Neque pro aliis formis normalibus Determinans, per quod M dividendum est, concinnitate expressionis (12) gaudet.

Antecedentia ad problema isoperimetricum propositum applicemus. Aequationum differentialium (7) unaquaeque simul omnibus altissimis differentialibus

$$x_1^{(2m_1)}, \quad x_2^{(m_1+m_2)}, \quad \dots, \quad x_n^{(m_1+m_n)}$$

afficiatur; unde ipsi $\varkappa_1, \varkappa_2, \dots, \varkappa_n$ designare possunt numeros 1, 2, ..., n *quocunque modo* permutatos. Fit

$$\lambda_i = m_1 - m_i, \quad q_i = p_{\varkappa_i} = m_1 + m_{\varkappa_i}, \quad q_i - \lambda_i = m_i + m_{\varkappa_i},$$

unde n quantitates (11) abeunt in quantitates $x_{\varkappa_i}^{(m_i+m_{\varkappa_i})}$; porro fit

$$A_f^{(i)} = \frac{\partial F_i}{\partial x_{\varkappa_f}^{(m_{\varkappa_f}+m_i)}} = \frac{\partial^2 U}{\partial x_{\varkappa_f}^{(m_{\varkappa_f})} \partial x_i^{(m_i)}}.$$

Hinc, collectis formulis (9) et (14), fluit sequens theorema.

Theorema.

„*Proponatur integrale* $\int U dt$ *Maximum Minimumve reddere, expressione* U *praeter variabilem independentem* t *continente* n *dependentes* $x_1, x_2, \dots, x_n$ *una cum earum differentialibus, respective usque ad* m_1^{tum}, m_2^{tum}, *etc.,* m_n^{tum} *ordinem ascendentibus; designantibus* $\varkappa_1, \varkappa_2, \dots, \varkappa_n$ *numeros* 1, 2, ..., n *quocunque ordine dispositos, integrandae erunt* n *aequationes differentiales*

$$x_{\varkappa_1}^{(m_1+m_{\varkappa_1})} = L_1, \quad x_{\varkappa_2}^{(m_2+m_{\varkappa_2})} = L_2, \quad \dots, \quad x_{\varkappa_n}^{(m_n+m_{\varkappa_n})} = L_n,$$

in quibus $L_1, L_2, \dots, L_n$ *ipsis differentialibus altissimis ad laevam positis non afficiuntur; si* $m_1 \geqq m_2 \geqq m_3 \dots \geqq m_{n-1} \geqq m_n$, *poniturque*

$$A_{\varkappa_f}^{(i)} = \frac{\partial^2 U}{\partial x_{\varkappa_f}^{(m_{\varkappa_f})} \partial x_i^{(m_i)}}, \quad R_i = \Sigma \pm A_{i+1}^{(i+1)} A_{i+2}^{(i+2)} \dots A_n^{(n)},$$

illarum n *aequationum differentialium habetur Multiplicator*

$$\frac{R_0^{m_1}}{R_1^{m_1-m_2} R_2^{m_2-m_3} \dots R_{n-1}^{m_{n-1}-m_n}}.\text{“}$$

Integralibus omnibus praeter unum inventis eorumque ope totidem quantitatibus variabilibus eliminatis, si aequationes $x_{\varkappa_i}^{(m_i+m_{\varkappa_i})} = L_i$ etc. ad aequationem differentialem primi ordinis inter duas variabiles reducuntur, huius quoque Multiplicator, cuius ope ea solis Quadraturis integrabilis fiat, constabit multiplicando valorem praecedentem per quantitatum eliminatarum Determinans, Constantium respectu arbitrariarum, quibus Integralia afficiuntur, formatum.

Berol. d. 26 Julii 1845.